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About loss of regularity and "blow up" of solutions for quasilinear parabolic systems

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CONTENTS

ABSTRACT. Starting from sufficient conditions for regularity of weak solutions to quasilinear parabolic systems, necessary conditions for loss of regularity are formulated. It is shown numerically that in some situations loss of regularity ("blow up") really happens accordingly to these conditions.

1. PREFACE

This paper is devoted to the problem of nonregularity of solutions to quasilinear parabolic systems. For one second order parabolic equation the "blow up" of solutions was considered in many papers and a lot of very important results were obtained. There is no need to give in this short paper a survey of these results. The only thing we would like to mention is that almost all of these results are based mainly on some variant of the maximum principle for one second order parabolic equation. For general parabolic systems this method can not be applied.
Therefore we choose the following way: the sufficient conditions, which one of the authors [5] had received for the regularity of weak solutions we apply to get necessary conditions for the loss of regularity. Since it was proved that these regularity conditions are sharp or unavoidable we shall see numerically that in some cases the necessary conditions become sufficient.
We show numerically that loss of regularity really happens for solutions of the so–called chemotaxis system. It is clear that the necessary conditions can not be sufficient for all systems. The last situation takes place, for example, for the solutions of the two–dimensional semiconductor system where under natural conditions the weak solutions are regular for all times.

2. NECESSARY CRITERION FOR LOSS OF REGULARITY

At first we shall remind the results concerning the regularity of weak solutions for quasilinear parabolic systems.
Consider a quasilinear parabolic system

$$\partial_t u - \sum_{i=0}^{m} D_i a_i(x,t,Du) = 0 \tag{2.1}$$

inside a cylinder $Q = (0,T) \times \Omega, t \in (0,T))$ and $x \in \Omega$, where $\Omega \subset R^n$ is a bounded m–dimensional domain with a sufficiently smooth boundary $\partial\Omega$ and T is an arbitrary positive number. Here $u(x,t) = \left\{u^{(1)}(x,t), \dots, u^{(N)}(x,t)\right\}$ is an unknown N–dimensional vector–function, defined on Q. The so–called coefficients

$$a_i(x,t,p) = \left\{a_i^{(1)}(x,t;p), \dots, a_i^{(N)}(x,t;p)\right\}$$

are also N–dimensional vector–functions depending on $(x,t) \in Q$ and

$$p_j = \left(p_j^{(1)}, \dots, p_j^{(N)}\right) \quad (i,j = 0,1,\dots,m).$$

The expression $Du = (D_0 u, D_1 u, \dots, D_m u)$ denotes an ordered system of unknown functions $u = D_0 u$ and its first derivatives $D_k u = u_k$ with respect to x_k $(k = 1, \dots, m)$.
We suppose that the following conditions are satisfied:

I. All the coefficients satisfy the Carathéodory conditions, i.e. they are measurable with respect to $(x,t) \in Q$ and continuously differentiable with respect to p.

II. There exists such a big number $q(m) > 1$ that the relations
 1.) $\forall u \in L_q\left\{[0,T); W_q^{(1)}(\Omega)\right\} \Rightarrow \forall a_i \in L_q(Q)$

2.) $\forall u \in L_q\left\{[0,T); W_q^{(2)}(\Omega)\right\} \Rightarrow L(u) \in L_q(Q)$
hold.

III. Denote

$$A = \left\{ \frac{\partial a_i^{(k)}}{\partial p_j^{(l)}} \right\}_{(i,j=0,1,\dots,m;\ k,l=1,\dots,N)} \tag{2.2}$$

and let A^+ and A^- be correspondingly the symmetric and skew–symmetric parts of A. Let $\{\lambda_i\}$ be the spectrum of A^+ and $\{\sigma_i\}$ the spectrum of the symmetric matrix

$$C = A^+A^- - A^-A^+ - (A^-)^2. \tag{2.3}$$

Suppose that

$$\inf_{i,x,t,p} \lambda_i = \lambda > 0, \quad \sup_{i,x,t,p} \Lambda_i = \Lambda < +\infty \tag{2.4}$$

and

$$\sup_{i,x,t,p} \sigma_i = \sigma < +\infty. \tag{2.5}$$

Denote

$$K = \begin{cases} \frac{\sigma}{\sigma+\lambda^2} & \text{for } \sigma \geq \frac{\lambda(\Lambda-\lambda)}{2} \\ \frac{(\Lambda-\lambda)^2+4\sigma}{\Lambda+\lambda)^2} & \text{for } \sigma \leq \frac{\lambda(\Lambda-\lambda)}{2} \end{cases} . \tag{2.6}$$

Theorem 2.1. *Let the conditions I., II., III. be satisfied and the inequalities*

$$\left| \frac{\partial a_i}{\partial x_k} \right| < C(1+|p|) \quad (i,k=1,\dots,m) \tag{2.7}$$

take place.
If the inequalities

$$\left. \begin{array}{ll} K^2 \frac{m}{2}\left(1+\frac{m-2}{m+1}\right)[1+(m-2)(m-1)] < 1, & m \geq 3 \\ 2K^2 < 1 & m = 2 \end{array} \right\} \tag{2.8}$$

hold then any weak solution of the system (2.1) $u \in L_2\left\{[0,T),\ W_2^{(1)}(\Omega)\right\}$ will satisfy the Hölder condition both in t and x with some small positive exponents in Q.

The condition (2.7) is a very restrictive one. Without this condition we have the following.

Theorem 2.2. *Let the assumption I., II., III. be fulfilled, $2-m < \alpha \leq 0$ for $m > 2$ and $\alpha = 0$ for $m = 2$. If the inequality*

$$K \frac{-\alpha+m-2}{\alpha+m-2} \sqrt{1 - \frac{\alpha(m-2)}{m-1}} < 1 \tag{2.9}$$

takes place then for any weak solution $u \in L_2\left\{[0,T),\ W_2^{(1)}(\Omega)\right\}$ *the estimate*

$$\sup_{x_0\in\Omega}\int_\Omega |u|^2\,|x-x_0|^\alpha\,dx\bigg|_{t=T}+\sup_{x_0\in\Omega}\int_\Omega |Du|^2\,|x-x_0|^\alpha\,dxdt<+\infty \tag{2.10}$$

is true.

Therefore a necessary condition for the loss of the Hölder continuity for the weak solution of (2.1) under the assumptions of the Theorem 2.1 will be the following

$$\left.\begin{array}{lr} K^{2\frac{m}{2}}\left(1+\frac{m-2}{m+1}\right)[1+(m-2)(m-1)]\geq 1, & m\geq 3 \\ 2K^2\geq 1 & m=2 \end{array}\right\}. \tag{2.11}$$

A necessary condition for the unboundedness of the expression

$$E[u]=\sup_{x_0\in\Omega}\int_\Omega |u|^2\,|x-x_0|^\alpha\,dx\bigg|_{t=T}+\sup_{x_0\in\Omega}\int_\Omega |Du|^2\,|x-x_0|^\alpha\,dxdt \tag{2.12}$$

under the assumption of (2.9) the Theorem 2.2 will be the following

$$\left.\begin{array}{lr} K^{\frac{-\alpha+m-2}{\alpha+m-2}}\sqrt{1-\frac{\alpha(m-2)}{m-1}}\geq 1 & m\geq 3 \\ K^2=1 & m=2 \end{array}\right\}. \tag{2.13}$$

Remark. From (2.6) always follows that $K\leq 1$.

3. Coupled systems

In this paragraph we shall consider systems of the type

$$\partial_t u^{(k)}-\left[\sum_{i=0}^m D_i a_i^{(k)}(x,t,Du)+\sum_{i,j=0}^m\sum_{k,l=1}^N b_{i,j}^{h,k,l}D_i\left[u^{(h)}D_j u^{(l)}\right]\right]=0, \tag{3.1}$$

where $b_{i,j}^{h,k,l}$ are constants.
First we apply the criterium (2.13). Suppose that there exists a weak solution $U(t,x)$ of the system (3.1) for which the expression $E[U]$ (2.12) is finite for the cylinder $Q_0(\eta)=(0,T_0-\eta)\times\Omega$, where η is an arbitrary sufficiently small positive number. If we'll have that

$$\lim_{\eta\to 0}E[U]=\infty, \tag{3.2}$$

then T_0 will be the blow up time for the fixed solution $U(t,x)$. It is clear that for this T_0 the necessary criterium (2.13) should take place. From the other side if we substitute $U(t,x)$ in the part of the last term of the left–hand side of (3.1) we get

$$\partial_t u^{(k)}-\left[\sum_{i=0}^m D_i a_i^{(k)}(x,t,Du)+\sum_{i,j=0}^m\sum_{k,l=1}^N b_{i,j}^{h,k,l}D_i\left[U^{(h)}D_j u^{(l)}\right]\right]=0. \tag{3.3}$$

It is clear, that if

$$\sup_Q\left|U(t,x)\right|<\epsilon, \tag{3.4}$$

where ϵ is a sufficiently small number then all conditions of the criterium (2.9) for the system (3.3) are satisfied. So, to find a T_0, for which (3.2) takes place we have to construct the matrix A (2.2) for the system (3.3) and to apply (2.13). T_0 will be found from the growth rate of the left–hand side of (3.4).
The same consideration can be applied for the criterium (2.11). But in this case the module of Hölder continuity for $u(t,x)$ should tend to infinity. In this case for the system (3.3) it should be taken in account that the coefficients (3.3) should satisfy the condition (2.7). It means that $U(t,x)$ should be differentiable with respect to x. Therefore if the solution $U(t,x)$ is loosing the differentiability at some point (t_0,x_0) the value t_0 gives also the first value of t, at which the loss of regularity happens.

4. The system of chemotaxis and semiconductor type

As a special realization of (2.1) we consider the system

$$\begin{aligned} \partial_t u &= \Delta u - \nabla \cdot (u \nabla v), \\ \delta \partial_t v &= \alpha \Delta v - \beta v + \gamma u \quad \text{in } \Omega, \\ u(0,\cdot) &= u_0 > 0,\ v(0,\cdot) = v_0 \geq 0, \\ \nu \cdot \nabla u &= \nu \cdot \nabla v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Here α, β and δ are positive constants, ν is the outward normal to Ω.
With $\gamma > 0$, the system (4.1) was introduced in [4] for modelling the dynamics of a population (concentration u) moving in Ω and driven by the gradient of a chemotactic agens (concentration v) produced by the population itself.
If $\gamma < 0$, the system (4.1) can be interpreted as a special variant of a semiconductor model for the drift of electrons (concentration u) under the electric field given by the antigradient of the electrostatic potential v [6].
The behaviour of solutions to (4.1) depends strongly on the sign of the paramter γ. In the semiconductor case ($\gamma < 0$) (4.1) has a unique global solution for arbitrary smooth $u_0 > 0$ [2]. Moreover, this solution decays exponentially in time to the unique equilibrium state

$$u^* = \overline{u}_0,\ v^* = \frac{\gamma}{\beta}\overline{u}_0, \tag{4.2}$$

where

$$\overline{w} = \frac{1}{|\Omega|}\int w d\Omega.$$

The situation turns out to be more complicated if $\gamma > 0$. It is clear (see for example [5]) that for small $\gamma\overline{u}_0$ there exists a smooth solution. However it can be expected that for sufficiently large $\gamma\overline{u}_0$ the solutions may explode in finite time. A hint for this situation was given in [3].
We want to illustrate the blow up situation by some numerical evidence. Our main aim is to check the criterium (2.13) empirically.
In order to calculate K^2 from (2.6) we have to specify the matrix (2.2). Since we are interested in necessary conditions for loss of regularity, we can consider a simplified

matrix, yielding upper resp. lower bounds for λ resp. Λ and σ. Thus, omitting $i, j = 0$ in (2.2), we find

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -U & 0 & \alpha & 0 \\ 0 & -U & 0 & \alpha \end{pmatrix}, U = U(t) = ||U(t,\cdot)||_\infty$$

and

$$\lambda = \frac{1}{2}(\alpha + 1 + \varrho),\ \Lambda = \frac{1}{2}(\alpha + 1 - \varrho),\ \varrho = \sqrt{(\alpha - 1) + U^2},\ \sigma = \frac{U}{2}\left(\frac{U}{2} + \varrho\right).$$

For numerical solving the system (4.1) in the chemotaxis case ($\gamma > 0$) we used a modified version of the well–tried code ToSCA [1] which was originally designed to solve van Roosbroeck's semiconductor equations ($\gamma < 0$) in two–dimensional domains Ω with the help of the finite elements method.
During our numerical calculations we fixed the data as follows:

$$\begin{gathered} m = 2,\ \Omega = (0,a) \times (0,a),\ a = \pi; \\ \alpha = 1,\ \beta = 0.1,\ \delta = 1,\ \gamma = 1; \\ v_0 = 0. \end{gathered} \tag{4.3}$$

We take

$$u_0 = \mu^* + \epsilon\left(e^{-(x^2+y^2)} - 1\right),\ \epsilon = 0.1, \mu^* = \alpha\left(\frac{\pi}{a}\right)^2 + \beta = 1.1 \tag{4.4}$$

or

$$u_0 = \mu^* + \epsilon\, e^{-(x^2+y^2)}. \tag{4.5}$$

From the homogeneous Neumann boundary conditions follows that the solutions of (4.1), realizing the initial values (4.4) resp. (4.5), satisfy

$$\overline{u}(t) = \overline{u}_0 = u^*,\ \mu = \gamma\overline{u}_0 = 1.0557$$

resp.

$$\overline{u}(t) = \overline{u}_0,\ \mu = 1.1557.$$

Our numerical results can be summarized as follows:

(i) the solution of the problem (4.1), with the parameters (4.3), (4.4) exists globally and satisfies the following relation

$$\lim_{t\to\infty} ||u(t) - u^*||_2 \to 0,$$

where $||\cdot||_2$ denotes the norm in $L_2(\Omega)$;

(ii) the solution of the problem (4.1), with the parameters (4.3), (4.5) blows up in finite time T_0 so that

$$\lim_{t\to T_0} ||u(t) - u^*||_2 \to \infty.$$

These statements are illustrated by some pictures. Fig. 1 shows $K^2(t)(t = T)$ and $||u(t) - u^*||_2$ corresponding to the initial value (4.4) ($\mu = mu = 1.0557$). Fig. 2 shows the same quantities for (4.5) ($\mu = mu = 1.1557$). Fig. 3 shows the solution $u = u(t, \cdot)$ to (4.1), (4.3), (4.5) at different times t_i.

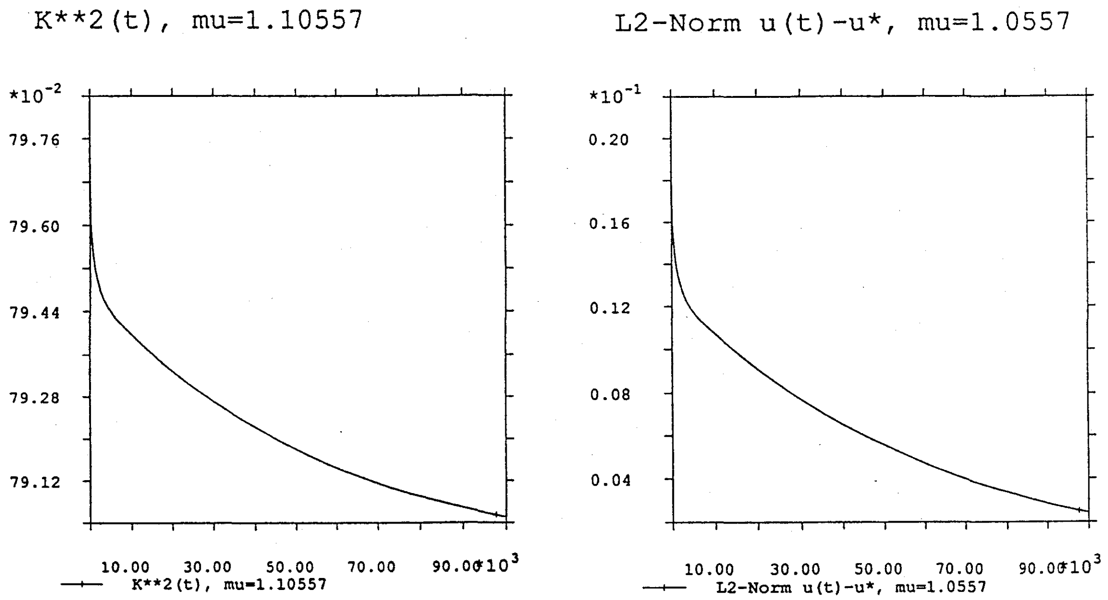

Fig. 1

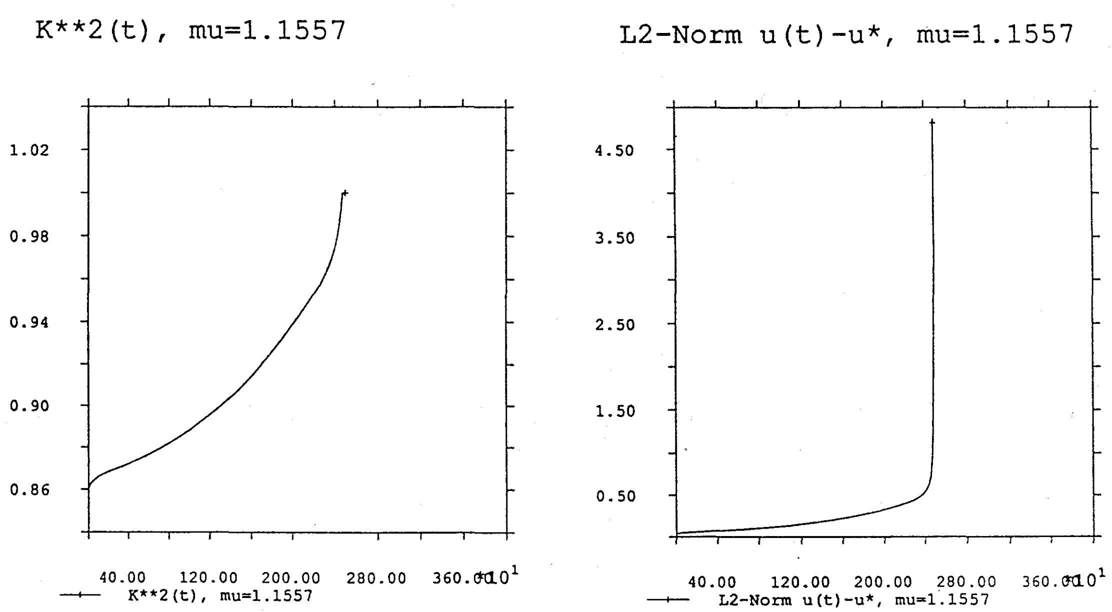

Fig. 2

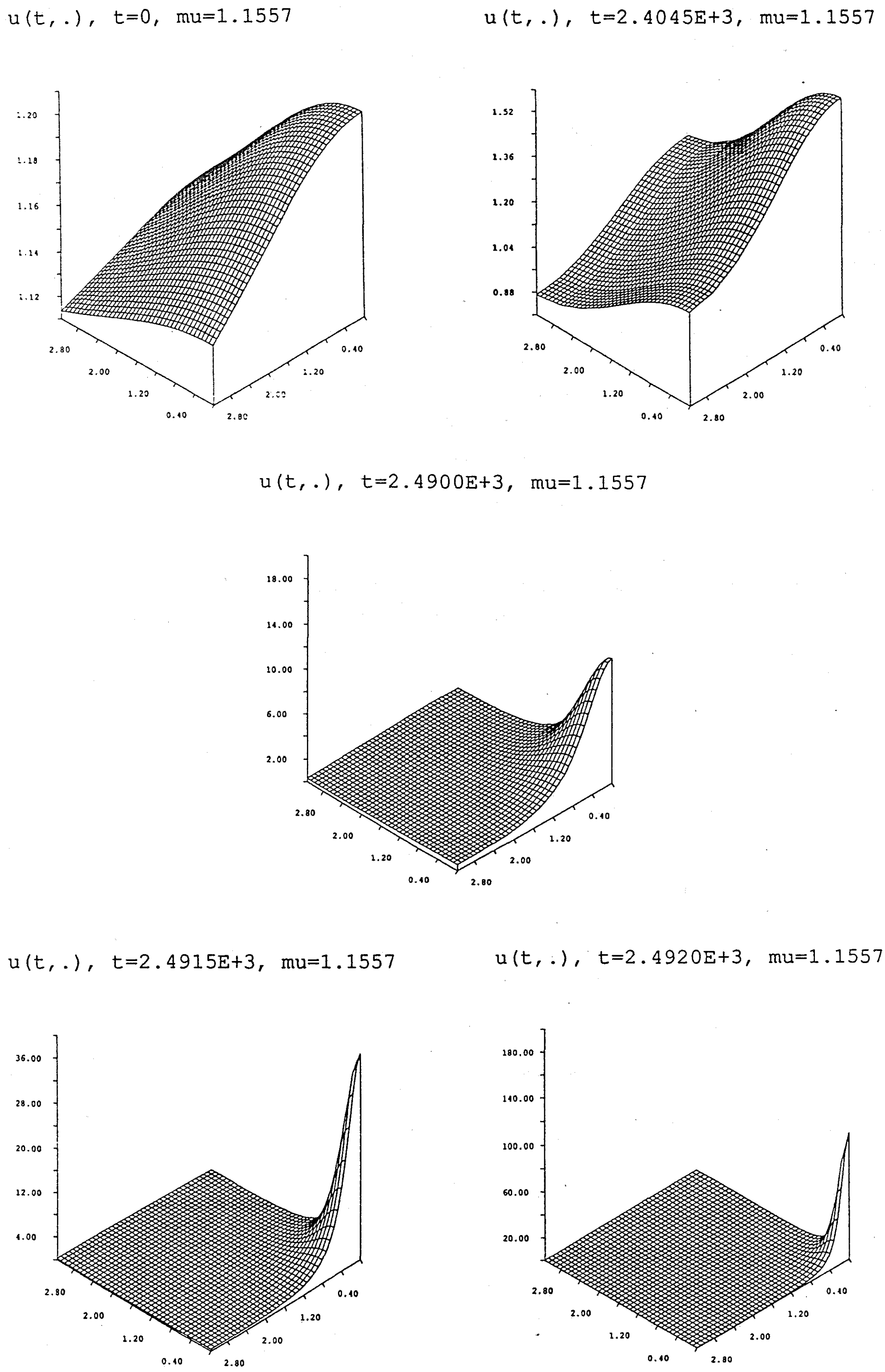
u(t,.), t=0, mu=1.1557
1.20
1.18
1.16
1.14
1.12
2.80
2.00
1.20
0.40
0.40
1.20
2.00
2.80
u(t,.), t=2.4045E+3, mu=1.1557
1.52
1.36
1.20
1.04
0.88
2.80
2.00
1.20
0.40
0.40
1.20
2.00
2.80
u(t,.), t=2.4900E+3, mu=1.1557
18.00
14.00
10.00
6.00
2.00
2.80
2.00
1.20
0.40
0.40
1.20
2.00
2.80
u(t,.), t=2.4915E+3, mu=1.1557
36.00
28.00
20.00
12.00
4.00
2.80
2.00
1.20
0.40
0.40
1.20
2.00
2.80
u(t,.), t=2.4920E+3, mu=1.1557
180.00
140.00
100.00
60.00
20.00
2.80
2.00
1.20
0.40
0.40
1.20
2.00
2.80

Fig. 3

Remark. We also considered the case $\delta = 0$ and found no qualitative differences compared with $\delta = 1$. Only the blow up time T_0 turns out to be smaller.

5. The stationary problem

The Figures 1 and 2 indicate the relevance of criterium (2.11) for the global behaviour of solutions to parabolic systems like (4.1). However, in fact (2.11) is an a posteriori criterium, since $U = ||u(t)||_\infty$ enters the matrix A and hence the quantity K^2. Of course, it would be desirable to have an a priori criterium involving u_0 instead of $u(t, \cdot)$.

A first step in this direction was taken [3], where the authors proved the existence of a positive number c^* with the property, that radially symmetric positive initial values u_0 with $\gamma \overline{u}_0 > c^*$ can be constructed such that the solution u to (4.1) explodes in finite time, provided $\delta = 0, m = 2, \Omega = disk$.

Our numerical experiments confirm these analytical results and suggest that the first positive eigenvalue μ^* of the problem

$$\begin{aligned} -\alpha \Delta h + \beta h = \mu h \quad \text{in } \Omega,\ \overline{h} = 0, \\ \nu \cdot \nabla h = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

is a candidate to be a lower bound for c^*. Unfortunately, we have no rigorous prove for this conjecture. To make it plausible we start from the stationary problem in its usual form

$$\begin{aligned} \nabla \cdot (\nabla u - u \nabla v) = 0,\ u = \overline{u}_0, \\ -\alpha \Delta v + \beta v = \gamma u \quad \text{in } \Omega, \\ \nu \cdot \nabla u = \nu \cdot \nabla v = 0 \text{ on } \partial\Omega. \end{aligned} \tag{5.2}$$

Since we are looking for positive solutions u, we can take

$$u = e^{v-w}.$$

From the first equation in (5.2) we find

$$\nabla \cdot \left(e^{v-w} \nabla w \right) = 0,\ \int e^{v-w} d\Omega = \overline{u}_0 \left| \Omega \right|$$

and consequently

$$w = \text{const},\ e^{-w} \int e^v d\Omega = \overline{u}_0 \left| \Omega \right|.$$

Hence the second equation in (5.2) yields

$$\begin{aligned} -\alpha \Delta v + \beta v = \gamma \overline{u}_0 \left| \Omega \right| \frac{e^v}{\int e^v d\Omega} \quad \text{in } \Omega \\ \nu \cdot \nabla v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.3}$$

Evidently, $v = v^* = \frac{\gamma}{\beta} \overline{u}_0$ is a solution to (5.3).

To find a possibly bifurcation solution v besides this trivial one we set

$$v = v^* + h,\ \overline{h} = 0.$$

Inserting v into (5.3) and linearizing with respect to h, we get just the eigenvalue problem (5.1) with $\mu = \gamma \overline{u}_0$.

Remark. Clearly, it holds

$$\gamma\overline{u}_0 \leq 0 < \mu^*, \quad \text{if } \gamma < 0.$$

This corresponds to the global attractivity of the equilibrium in the semiconductor case [2].
By the way, it is easy to see, that (5.2) has only the equilibrium solution $v^* = \frac{\gamma}{\beta}\overline{u}_0$, if $\gamma < 0$.
Indeed, let v be a solution of (5.3). Then, testing (5.3) with the negative part $z = (v - v^*)_-$ and using Jensen's inequality, we find

$$\begin{aligned} 0 &= \int \left[\alpha|\nabla z|^2 + \beta\left(v - \frac{v^*|\Omega|e^v}{\int e^v d\Omega}\right) z\right] d\Omega \\ &\geq \beta \int \left(v - \frac{v^*|\Omega|e^{v^*}}{\int e^v d\Omega}\right) z\, d\Omega \\ &\geq \beta \int z^2 d\Omega \end{aligned}$$

and thus $v \geq v^*$. Because of $\overline{v} = v^*$ this implies

$$v = v^*.$$

Remark. It would be interesting to know more about solutions of (5.3) in the chemotaxis case $\gamma > 0$.

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