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Numerical Methods for Stochastic Differential Equations

P.E. KLOEDEN[1] and E. PLATEN[2),3)]

Abstract. Numerical methods for stochastic differential equations, including Taylor expansion approximations, Runge-Kutta like methods and implicit methods, are summarized. Important differences between simulation techniques with respect to the strong (pathwise) and the weak (distributional) approximation criteria are discussed. Applications to the visualization of nonlinear stochastic dynamics, the computation of Lyapunov exponents and stochastic bifurcations are also presented.

1. Introduction.

In recent years a whole new spectrum of numerical methods for (ordinary) stochastic differential equations (SDEs) has been developed. Many concepts from the numerics of ordinary (deterministic) differential equations (ODEs) can also be applied to SDEs, but simplistic attempts to adapt deterministic methods to SDEs can lead to difficulties due to differences between deterministic and stochastic calculi. Here we shall indicate stochastic methods that have been found to be successful or have potential for future development and illustrate their implementation and use in a variety of applications involving stochastic dynamical systems.

Let us briefly recall some basic facts about stochastic differential equations. As references on SDEs and their numerical solution we refer the reader to Kloeden and Platen (1992), which includes an extensive bibliography, and to Kloeden, Platen and Schurz (1993), which contains a diskette with computer programs for practical simulations.

A solution $X = \{X_t, t \geq 0\}$ of an Ito stochastic differential equation

$$dX_t = a(X_t)dt + b(X_t)dW_t \tag{1}$$

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driven by a Wiener process $W = \{W_t, t \geq 0\}$ is a stochastic process starting at $X_0 = x_0$ when $t = 0$. The function $a(x)$ characterizing the local trend is called the drift coefficient, while the diffusion coefficient $b(x)$ characterizes the average size of the fluctuations of X. The Wiener process W has independent Gaussian increments and is often called Brownian motion. To define X properly one has to introduce the Ito stochastic integral. This looks like a Riemann-Stieltjes integral, but in the approximating sums its integrand is always evaluated at the left end point of each discretization subinterval and the mean-square limit is used. (The Stratonovich stochastic integral results if the integrand is always evaluated at the subinterval midpoint; it often differs in value from the corresponding Ito integral). This and consequent differences between deterministic and stochastic calculi are due the unbounded variation of the paths of the Wiener process. It should always be remembered that stochastic calculus is thus particularly sensitive to how approximations are made.

2. Discrete Time Approximation of SDEs.

Explicitly solvable SDEs are rare in practical applications, so efficient and stable numerical methods are required for the simulation of sample paths or functionals such as moments of their solutions. Here we shall focus on discrete time stochastic numerical methods which are based on finite discretizations of a time interval $[0, T]$ and generate approximate values of the sample paths step by step at the discretization times. The simulated paths can then be analysed by usual statistical methods.

The simplest time discretization of an interval $[0, T]$ consists of equidistant time points

$$\tau_n = n\,\Delta. \tag{2}$$

$n = 0, 1, \ldots, N$ with step size

$$\Delta = \frac{T}{N}. \tag{3}$$

where $N = 1, 2, \ldots$. Variable time steps are also possible but will not be considered here.

The stochastic generalization of the *Euler scheme* (see Maruyama (1955)), which is sometimes called the Euler-Maruyama scheme, has the form

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n \tag{4}$$

for $n = 0, 1, \ldots, N-1$ with initial value $Y_0 = x_0$ and Wiener process increments

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}. \tag{5}$$

The random variables ΔW_n are independent Gaussian distributed with zero mean and variance Δ and can be generated using appropriate random number generators and transformations of their output. The recursive scheme (4) obviously gives values of the Euler approximation Y^{Δ} at the discretization times only, but these can be interpolated in a variety of ways for theoretical and graphical purposes.

3. Convergence Criteria.

More so than for ODEs, numerical methods for SDEs depend significantly on the type of problem to be solved. In applications involving direct simulations of solutions of SDEs such as in simulating paths of stochastic vibrations or stochastic flows, in filtering or in testing statistical estimators, it is important that the calculated values are close to the corresponding values of the desired solution. This suggests that schemes appropriate for these tasks should be classified according to some kind of strong convergence. We say that an approximation Y^Δ *converges with strong order* $\gamma > 0$ if there exists a finite constant K such that

$$E\left(|X_T - Y_N^\Delta|\right) \le K\,\Delta^\gamma \tag{6}$$

for all step sizes $\Delta \in (0,1)$. The expectation of the absolute global error here is a straightforward generalization of the usual deterministic criterion to which it reduces in the absence of driving noise. It has been shown that the stochastic Euler scheme (4) has only strong order $\gamma = 0.5$ which is less than the order 1.0 of the deterministic Euler scheme. This is a consequence of the volatile nature of the driving Wiener process. It does not contradict the preceding remark since the order is with respect to a general class of SDEs, while for a specific SDE the scheme may in fact achieve a higher rate of convergence. Note that with these strong approximations it is essential to work with the original increments of the driving Wiener process.

In many practical situations it is not necessary to simulate good pathwise approximations of an Ito process, but only some functional of it such as a moment, a probability density or a Lyapunov exponent. That is, expectations $E(g(X_T))$ of the solution process X at time T for certain types of functions g are of interest. The above strong convergence criterion is not appropriate in these cases for which, in principle, it is only necessary to approximate the probability measure induced by the solution of the SDE. Weakly converging approximations are all that is required here. To classify them we shall say that a discrete time approximation Y^Δ *converges with weak order* $\beta > 0$ if for any polynomial g there exists a finite constant K_g such that

$$\left|E(g(X_T)) - E(g(Y_N^\Delta))\right| \le K_g\,\Delta^\beta \tag{7}$$

for all step sizes $\Delta \in (0,1)$. Note that the convergence of all moments with the same order is implied by this criterion.

Under sufficient regularity conditions it has been shown that the stochastic Euler scheme (4) converges with weak order $\beta = 1.0$, the same as that of the deterministic Euler scheme. The same weak order still holds if simpler random variables are substituted for the Gaussian ΔW_n in (4), such as two point distributed independent random variables $\Delta \hat{W}_n$ with

$$P(\Delta \hat{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}. \tag{8}$$

This indicates a freedom in designing weak schemes which can use random variables not constructed directly from the driving Wiener process and are more efficiently generated.

In principle most tasks requiring numerical simulations of SDEs fall into either of these strong or weak convergence categories. For computational efficience it is well worth checking whether or not a weak scheme could be used.

4. Stochastic Taylor Expansions.

The general stochastic Taylor expansion derived by Wagner and Platen (1978) (see also Platen (1982)) provides the key to the systematic derivation of higher order strong and weak schemes for SDEs. To give the reader an indication of what these expansions look like, we present a simple example for the solution of the SDE (1) expanded about the initial value X_0, namely

$$X_t = X_0 + a(X_0)\int_0^t ds + b(X_0)\int_0^t dW_s + b(X_0)\,b'(X_0)\int_0^t\int_0^{s_2} dW_{s_1}\,dW_{s_2} + R, \tag{9}$$

where R is the remainder. The multiple stochastic integral term in (9) is typical of the terms in stochastic Taylor expansions, with constant coefficients involving the drift and diffusion coefficients a and b and their higher order derivatives evaluated at the expansion point X_0. Similar types of functions occur as the integrands of the next higher order multiple integrals that constitute the remainder. Such multiple stochastic integrals, which provide more information about the behaviour of the Wiener process paths, that is about the Wiener chaos, must be present if higher accuracy is to be obtained.

5. Taylor Schemes.

Special classes of stochastic Taylor expansions required for higher order strong and weak numerical schemes for SDEs are described in Platen (1982) and in Kloeden and Platen (1991). The Taylor schemes are derived by applying truncated versions of these Taylor expansions on each time discretization subinterval.

For example, the stochastic *Euler scheme* (4), which represents the *order 0.5 strong Taylor scheme*, can be obtained by truncating the Taylor expansion with only the terms on the first line of (9) as its non-remainder terms. From the expansion (9) itself we obtain a scheme which has strong order $\gamma = 1.0$. This was proposed by Milstein (1974) and will be written as

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W_n + \frac{1}{2}\,bb'\int_{\tau_n}^{\tau_{n+1}}\int_{\tau_n}^{s_2} dW_{s_1}\,dW_{s_2}, \tag{10}$$

where (and later too) we abbreviate $a(Y_n)$ by a, $b(Y_n)$ by b, and so on. In implementing this scheme we use the fact that

$$\int_{\tau_n}^{\tau_{n+1}}\int_{\tau_n}^{s_2} dW_{s_1}\,dW_{s_2} = \frac{1}{2}\Big\{(\Delta W_n)^2 - \Delta\Big\}. \tag{11}$$

The *Milstein Scheme* (10) represents the *order 1.0 strong Taylor scheme.*

The additional terms from stochastic Taylor expansions that should be included to form a Taylor scheme of any strong order $\gamma = 1.5, 2.0, 2.5, \ldots$ are described in Kloeden and Platen (1992). For instance, the *order 1.5 strong Taylor scheme*

$$
\begin{aligned}
(12) \qquad Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta W_n + \frac{1}{2}\,bb'\left\{(\Delta W_n)^2 - \Delta\right\} \\
& + a'b\,\Delta Z_n + \frac{1}{2}\left(aa' + \frac{1}{2}\,b^2a''\right)\Delta^2 \\
& + \left(ab' + \frac{1}{2}\,b^2b''\right)\left\{\Delta W_n\,\Delta - \Delta Z_n\right\} \\
& + \frac{1}{2}\,b\left(bb'' + (b')^2\right)\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta\right\}\Delta W_n
\end{aligned}
$$

requires the additional random variable

$$
(13) \qquad \Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2,
$$

which is Gaussian distributed with mean zero, variance $\frac{1}{3}\,\Delta^3$ and correlation $E(\Delta W_n\,\Delta Z_n) = \frac{1}{2}\,\Delta^2$ and is easily generated on a computer.

Higher order strong Taylor approximations and those for SDEs involving several independent Wiener processes also contain multiple stochastic integrals which have to be approximated, for example by random Fourier series as described in Milstein (1988) and in Kloeden, Platen and Wright (1992). For special classes of SDEs many of the coefficients of such multiple integrals vanish or satisfy algebraic relationships which allow identities between integrals of higher and lower multiplicities to be used, so there is then no need to compute these higher multiple integrals. These classes, which include SDEs with additive or commutative noise, are encountered in many practical applications. While higher order strong Taylor schemes for $\gamma \geq 2$ are rather complicated in general, they simplify considerable and become computationally viable for these special SDEs.

A higher order of convergence is more easily obtained for the weak Taylor schemes as it takes only integer values and simpler random variables can be used. The stochastic *Euler scheme* (4) with two-point distributed random variables as in (8) is an *order 1.0 weak Taylor scheme*. The following *order 2.0 weak Taylor scheme* was proposed by Milstein (1978) (see also Talay (1984) for a proof of its convergence order):

$$
\begin{aligned}
(14) \qquad Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta\hat{W}_n + \frac{1}{2}\,bb'\left\{\left(\Delta\hat{W}_n\right)^2 - \Delta\right\} \\
& + \frac{1}{2}\left(a'b + ab' + \frac{1}{2}\,b''b^2\right)\Delta\hat{W}_n\,\Delta \\
& + \frac{1}{2}\left(aa' + \frac{1}{2}\,a''b^2\right)\Delta^2,
\end{aligned}
$$

where $\Delta\hat{W}_n$ can be chosen as a three-point distributed random variable with

$$(15) \qquad P\left(\Delta\hat{W}_n = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\hat{W}_n = 0\right) = \frac{2}{3}.$$

The general weak Taylor schemes of orders $\beta = 1.0, 2.0, 3.0, \ldots$ are characterized in Platen (1984) and include all multiple stochastic integrals up to multiplicity β. These multiple integrals can, however, be represented by more simply generated random variables than those in the corresponding strong schemes.

Both strong and weak Taylor schemes for SDEs are now quite well understood. They open the door to other schemes such as the derivative free Runge-Kutta type schemes for which the convergence order can be established by comparison with the corresponding Taylor schemes. The important point to be learned from the Taylor schemes is that to achieve higher order strong or weak convergence it is necessary to include in the schemes sufficient information about the driving Wiener processes over the discretization subintervals in form of multiple stochastic integrals and that in the weak schemes simpler random variables can often be substituted for these multiple integrals.

6. Strong Runge-Kutta Type Schemes.

It is natural to search for stochastic Runge-Kutta type schemes which avoid the derivatives in the coefficients of the higher order Taylor schemes. First attempts were made by Rümelin (1982) who showed that it is only possible to achieve a strong order 0.5 (or 1.0 for special SDEs) when only the increments (5) of the driven Wiener processes are used. Later Platen (1984) derived the *strong order 1.0 explicit scheme*

$$(16) \qquad Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W_n + \frac{1}{2}\left\{b\left(\hat{Y}_n\right) - b\right\}\left\{(\Delta W_n)^2 - \Delta\right\}\Delta^{-\frac{1}{2}}$$

with internal stage

$$\hat{Y}_n = Y_n + a\,\Delta + b\,\Delta^{1/2}.$$

To achieve a higher strong order we know from the strong Taylor schemes that we have to include the random variable ΔZ_n (13) in an *order 1.5 strong Runge-Kutta scheme*. An example of such a scheme has the form

$$\begin{aligned}(17) \qquad Y_{n+1} \;=\; & Y_n + b\,\Delta W_n + \frac{1}{2\sqrt{\Delta}}\left\{a(\bar{\Upsilon}_+) - a(\bar{\Upsilon}_-)\right\}\Delta Z_n \\ & + \frac{1}{4}\left\{a(\bar{\Upsilon}_+) + 2a + a(\bar{\Upsilon}_-)\right\}\Delta \\ & + \frac{1}{4\sqrt{\Delta}}\left\{b(\bar{\Upsilon}_+) - b(\bar{\Upsilon}_-)\right\}\left\{(\Delta W_n)^2 - \Delta\right\} \\ & + \frac{1}{2\Delta}\left\{b(\bar{\Upsilon}_+) - 2b + b(\bar{\Upsilon}_-)\right\}\left\{\Delta W_n\Delta - \Delta Z_n\right\} \\ & + \frac{1}{4\Delta}\left[b(\bar{\Phi}_+) - b(\bar{\Phi}_-) - b(\bar{\Upsilon}_+) + b(\bar{\Upsilon}_-)\right]\end{aligned}$$

$$\times \left\{ \frac{1}{3}(\Delta W_n)^2 - \Delta \right\} \Delta W_n$$

with stages

$$\bar{\Upsilon}_{\pm} = Y_n + a\,\Delta \pm b\sqrt{\Delta}$$

and

$$\bar{\Phi}_{\pm} = \bar{\Upsilon}_{+} \pm b(\bar{\Upsilon}_{+})\sqrt{\Delta}.$$

Other generalizations and interesting specializations of this scheme can be found in Kloeden and Platen (1992). Admittedly it looks rather complicated, but simple strong Runge-Kutta schemes of higher strong order are still not known except for special kinds of SDEs. For example, for additive noise where $b(x) \equiv b$, const.. Chang (1987) obtained the *strong order 2.0 method*

$$Y_{n+1} = Y_n + \frac{1}{2}\left\{\underline{a}\left(\bar{\Upsilon}_{+}\right) + \underline{a}\left(\bar{\Upsilon}_{-}\right)\right\}\Delta + b\,\Delta W_n \tag{18}$$

with stages

$$\bar{\Upsilon}_{\pm} = Y_n + \frac{1}{2}\,\underline{a}\,\Delta + \frac{1}{\Delta}\,b\left\{\Delta Z_n \pm \sqrt{2J_{(1,1,0)}\,\Delta - (\Delta Z_n)^2}\right\},$$

where

$$J_{(1,1,0)} = \frac{1}{2}\int_{\tau_n}^{\tau_{n+1}} (W_s)^2\,ds \geq \frac{1}{2\Delta}(\Delta Z_n)^2$$

and $\underline{a} = a - \frac{1}{2}bb'$. Note that an additional random variable $J_{1,1,0}$ is needed here.

Newton (1991) proposed order 1.0 Runge-Kutta methods which are asymptotically the best within the class of strong order 1.0 schemes with the increments ΔW_n of the driving Wiener process as their only random variables.

Just as stiff ODEs require implicit methods to provide numerical stability, so too do stiff SDEs (see Kloeden and Platen (1991)). In the stochastic case such schemes must be constructed with some care. A simple example is the *order 1.0 implicit strong Runge-Kutta method*

$$\begin{aligned} Y_{n+1} &= Y_n + a\left(Y_{n+1}\right)\Delta + b\,\Delta W_n \\ &\quad + \frac{1}{2\sqrt{\Delta}}\left(b\left(\bar{\Upsilon}_n\right) - b\right)\left\{(\Delta W_n)^2 - \Delta\right\} \end{aligned} \tag{19}$$

with stage

$$\bar{\Upsilon}_n = Y_n + a\,\Delta + b\sqrt{\Delta}.$$

Higher order examples are the following *strong order 1.5 implicit method*

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}\left\{a\left(Y_{n+1}\right) + a\right\}\Delta + b\,\Delta W_n \\ &\quad + \frac{1}{4\sqrt{\Delta}}\left\{b\left(\bar{\Upsilon}_{+}\right) - b\left(\bar{\Upsilon}_{-}\right)\right\}\left\{(\Delta W_n)^2 - \Delta\right\} \\ &\quad + \frac{1}{2\Delta}\left\{b\left(\bar{\Upsilon}_{+}\right) - 2b + b\left(\bar{\Upsilon}_{-}\right)\right\}\left\{\Delta W_n \Delta - \Delta Z_n\right\} \end{aligned} \tag{20}$$

$$+\frac{1}{2\sqrt{\Delta}}\left\{a\left(\bar{\Upsilon}_+\right)-a\left(\bar{\Upsilon}_-\right)\right\}\left\{\Delta Z_n-\frac{1}{2}\Delta W_n\Delta\right\}$$
$$+\frac{1}{4\Delta}\left\{b\left(\bar{\Phi}_+\right)-b\left(\bar{\Phi}_-\right)-b\left(\bar{\Upsilon}_+\right)+b\left(\bar{\Upsilon}_-\right)\right\}$$
$$\times\left\{\frac{1}{3}(\Delta W_n)^2-\Delta\right\}\Delta W_n$$

with stages

$$\bar{\Upsilon}_\pm = Y_n + a\Delta \pm b\sqrt{\Delta}$$

and

$$\bar{\Phi}_\pm = \bar{\Upsilon}_+ \pm b\left(\bar{\Upsilon}_+\right)\sqrt{\Delta}.$$

and the *implicit strong order 2.0 Runge-Kutta scheme*

$$Y_{n+1} = Y_n + \left\{\underline{a}\left(\bar{\Upsilon}_+\right)+\underline{a}\left(\bar{\Upsilon}_-\right)-\frac{1}{2}\left(\underline{a}\left(Y_{n+1}\right)+\underline{a}\right)\right\}\Delta + b\Delta W_n \tag{21}$$

with stages

$$\bar{\Upsilon}_\pm = Y_n + \frac{1}{2}\underline{a}\Delta + \frac{1}{\Delta}b\left(\Delta\tilde{Z}\pm\tilde{\zeta}\right),$$

where

$$\Delta\tilde{Z}_n = \frac{1}{2}\Delta Z_n + \frac{1}{4}\Delta W_n\Delta$$

and

$$\tilde{\zeta} = \sqrt{J_{(1,1,0)}\Delta - \frac{1}{2}(\Delta Z_n)^2 + \frac{1}{8}\left((\Delta W_n)^2 + \frac{1}{2}\left(2\Delta Z_n\Delta^{-1}-\Delta W_n\right)^2\right)\Delta^2}.$$

Strong implicit Runge-Kutta type methods have also been investigated for instance in Rümelin (1982), Milstein (1988), Saito and Mitsui (1992), Hernandez and Spigler (1991), and Klauder and Petersen (1985). A very recent strong method of order 0.5 in which stochastic terms are also implicit is the balanced method

$$Y_{n+1} = Y_n + a\Delta + b\Delta W_n + (Y_n - Y_{n+1})(a\Delta + b|\Delta W_n|) \tag{22}$$

proposed by Milstein, Platen and Schurz (1992).

7. Weak Runge-Kutta Type and Extrapolation Schemes.

The construction of higher weak order Runge-Kutta schemes turns out to be much easier. There were proposals for weak second order methods in Milstein (1978) and Talay (1984), but these still contained a derivative of the diffusion coefficient b. This is avoided in the following *weak order 2.0 Runge-Kutta scheme*

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}\left(a\left(\bar{\Upsilon}\right)+a\right)\Delta \\ &\quad + \frac{1}{4}\left(b\left(\bar{\Upsilon}^+\right)+b\left(\bar{\Upsilon}^-\right)+2b\right)\Delta\hat{W}_n \\ &\quad + \frac{1}{4}\left(b\left(\bar{\Upsilon}^+\right)-b\left(\bar{\Upsilon}^-\right)\right)\left\{\left(\Delta\hat{W}_n\right)^2-\Delta\right\}\Delta^{-1/2} \end{aligned} \tag{23}$$

with three stages

$$\bar{\Upsilon} = Y_n + a\,\Delta + b\,\Delta\hat{W}_n,$$

and

$$\bar{\Upsilon}^{\pm} = Y_n + a\,\Delta \pm b\sqrt{\Delta},$$

(see Platen (1984)), where $\Delta\hat{W}_n$ can be chosen as a three-point distributed random variable defined in (15). An implicit method of the same order is obtained by writing Y_{n+1} instead of $\bar{\Upsilon}$ in (23) and a class of second order weak Runge-Kutta approximations of a specific form using at least five stages is described in Mackevicius (1992).

The derivation of general higher weak order Runge-Kutta methods has thus far proved difficult, with proposals only in special cases, that is for additive noise or for linear equations; see Milstein (1988) and Kloeden and Platen (1992). The latter reference contains the following *explicit third weak order Runge-Kutta type scheme,* which has six stages, for SDEs with additive noise:

$$\begin{aligned}(24)\qquad Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta\hat{W}_n \\ &\quad + \frac{1}{2}\left(a_\zeta^+ + a_\zeta^- - \frac{3}{2}a - \frac{1}{4}\left(\tilde{a}_\zeta^+ + \tilde{a}_\zeta^-\right)\right)\Delta \\ &\quad + \sqrt{\frac{2}{\Delta}}\left(\frac{1}{\sqrt{2}}\left(a_\zeta^+ - a_\zeta^-\right) - -\frac{1}{4}\left(\tilde{a}_\zeta^+ - \tilde{a}_\zeta^-\right)\right)\zeta\,\Delta\hat{Z}_n \\ &\quad + \frac{1}{6}\left[a\left(Y_n + \left(a + a_\zeta^+\right)\Delta + (\zeta+\rho)\,b\sqrt{\Delta}\right) - a_\zeta^+ - a_\rho^+ + a\right] \\ &\qquad \times\left[(\zeta+\rho)\,\Delta\hat{W}_n\sqrt{\Delta} + \Delta + \zeta\rho\left\{\left(\Delta\hat{W}_n\right)^2 - \Delta\right\}\right]\end{aligned}$$

with

$$a_\phi^{\pm} = a\left(Y_n + a\,\Delta \pm b\sqrt{\Delta}\,\phi\right)$$

and

$$\tilde{a}_\phi^{\pm} = a\left(Y_n + 2a\,\Delta \pm b\sqrt{2\Delta}\,\phi\right),$$

where ϕ is either ζ or ρ. Here two correlated Gaussian random variables $\Delta\hat{W}_n \sim N(0;\Delta)$ and $\Delta\hat{Z}_n \sim N(0;\frac{1}{3}\Delta^3)$ with $E(\Delta\hat{W}_n\Delta\hat{Z}_n) = \frac{1}{2}\Delta^2$ are used, together with two independent two-point distributed random variables ζ and ρ with

$$P(\zeta = \pm 1) = P(\rho = \pm 1) = \frac{1}{2}.$$

One suspects that there should be more compact and elegant weak Runge-Kutta type schemes of third and fourth order, especially since there is some freedom in the types of random variables that can be used in weak schemes. There is already evidence from a paper by Talay and Tubaro (1990) that extrapolation methods can be used to construct higher convergence from lower order schemes.

The simplest extrapolation method, the Romberg or Richardson extrapolation, is based on two applications of the Euler scheme to evaluate the functional

$$V_{1,1}^{\Delta} = E\left(g(Y^{\Delta}(T))\right)$$

for the time step sizes $\Delta = \delta$ and $\Delta = 2\delta$. Their linear combination

$$V_{1,2}^{\delta} = 2\,V_{1,1}^{\delta} - V_{1,1}^{2\,\delta} \tag{25}$$

then provides a weak order 2.0 method. Higher weak order extrapolations based on the Euler and other higher order weak schemes have also been contructed in Kloeden, Platen and Hofmann (1993). Many implicit weak schemes can be extrapolated in a similar way for stiff SDEs.

8. Visualization of Stochastic Dynamics.

It is often instructive to be able to visualize the trajectories of a dynamical system such as a noisy oscillator that is governed by a stochastic differential equation. Computer plots of numerical approximations of different sample paths for the same initial value or for the same sample path but different initial values provide an effective means of visualizing the dynamical behaviour of such a stochastic system. Strong schemes are required for this direct simulation of trajectories, while weak schemes should be used to calculate frequency histograms, moments or the invariant measure of a limiting stationary solution.

As an example let us study a Bonhoeffer-Van der Pol oscillator, the equations for which are the 2-dimensional system of Ito stochastic differential equations

$$\begin{aligned} dX_t^1 &= c\left\{X_t^1 + X_t^2 - \frac{1}{3}(X_t^1)^3 + z\right\}dt + \sigma dW_t \\ dX_t^2 &= -\frac{1}{c}\left\{X_t^1 + bX_t^2 - a\right\}dt. \end{aligned} \tag{26}$$

where $\{W_t, t \geq 0\}$ is a Wiener process and $\sigma > 0$ is the intensity of the noise. Since real dynamical systems need to function effectively under variable conditions, the behaviour of the stochastic system (26) for small σ should be similar to that of the corresponding deterministic system, which has a limit cycle, if the latter is to be a realistic model. That is, system (26) should possess a *noisy limit cycle*, evidence for which can be suggested by the direct simulation of solution paths of (26) by a strong scheme. To be able to draw reliable conclusions, however, a large number of sample paths must be examined for a variety of representative initial values.

To have a specific case, we apply the Milstein scheme (10) with equidistant step size $\Delta = 2^{-9}$ to the noisy Bonhoeffer-Van der Pol equations (26) with parameters $a = 0.7$, $b = 0.8$, $c = 3.0$, $z = -0.34$ over the time interval $0 \leq t \leq T = 1$ for the initial value $X_0^1 = -1.9$, $X_0^2 = 1.2$ and noise intensity $\sigma = 0.1$. Linearly interpolated paths of each component for a typical simulation have been plotted against time t and on the (X^1, X^2)-plane in Figure 1, from which it appears that the noisy

Bonhoeffer-Van der Pol equations do have a noisy limit cycle. Increasing the noise intensity destroys the phase relationship between the two components of the system. To exclude the possibility that the results observed are just some peculiar effect of the particular step size or numerical scheme used, the calculations were repeated with other step sizes and schemes to see if similar behaviour is obtained.

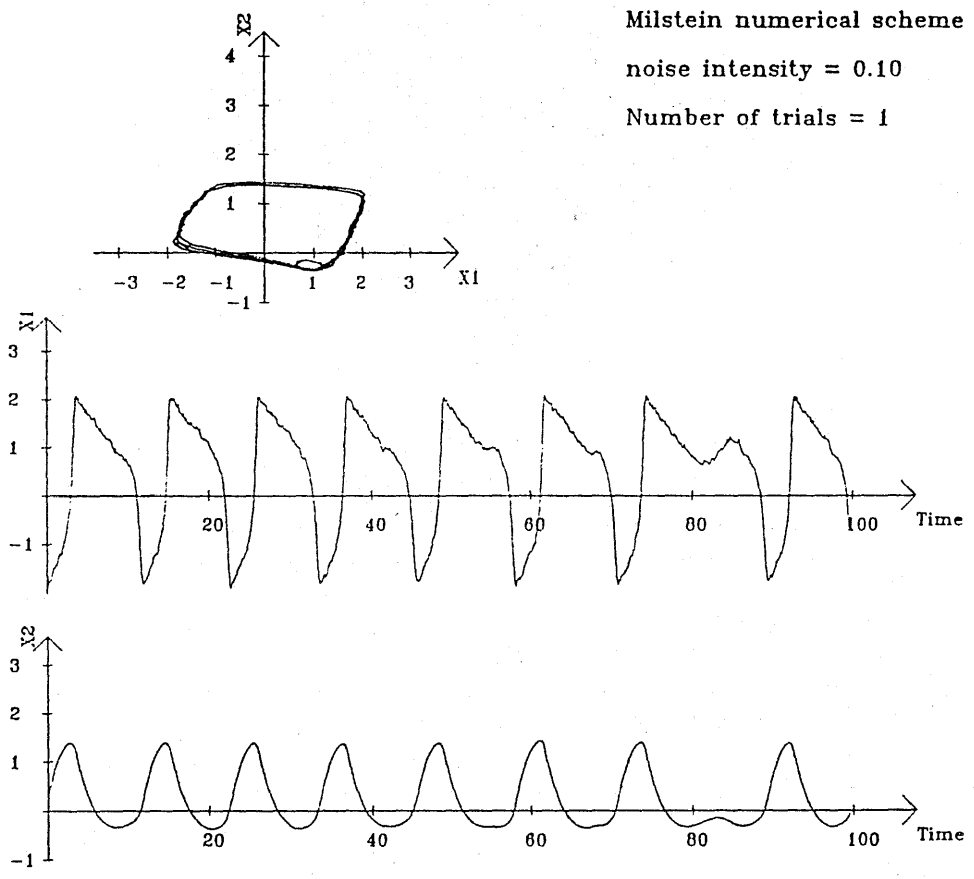

Figure 1 Bonhoeffer-Van der Pol oscillator.

The output of a weak numerical scheme can also provide useful visual information about the behaviour of a stochastic dynamical system. For example, a frequency histogram can indicate the shape and support of the density of an invariant measure associated with an asymptotically stable stationary solution such as the noisy limit cycle of the noisy Bonhoeffer-Van der Pol system (26). To illustrate the possibilities an order 2.0 weak Taylor scheme with step size $\Delta = 2^{-9}$ was applied to the equations (26) over the time interval $[0, 10]$ and a simple box counting procedure was used to construct the histogram in Figure 2. In particular, a 160×160 grid was used with the reference (1,1) cell located on the lower left of the solution field. Peaked regions of this histogram indicate slower passage time in these parts of the noisy limit cycle.

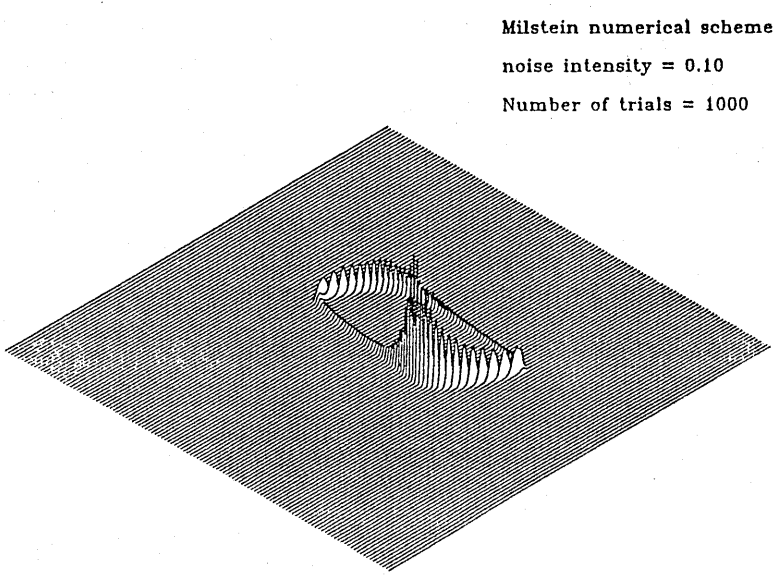

Figure 2 Frequency histogram for the noisy Bonhoeffer-Van der Pol equations with noise intensity $\sigma = 0.1$.

For each ω and fixed time instant t the mapping $T_{t,\omega}$ defined by $T_{t,\omega}(x) = X_t^x(\omega)$ for the solution X^x of a stochastic differential equation with deterministic initial value $X_0^x(\omega) = x$ is a diffeomorphism, even though the sample paths of X^x themselves are only continuous in time t. This property underlies a method of visualization of global dynamical behaviour in which a large number of solutions corresponding to the same noise sample path but starting in a grid of deterministic initial values is followed simultaneously. In view of the diffeomorphism property none of these paths can intersect each other. The ability of a strong numerical scheme to preserve this property provides an indication of its accuracy and effectiveness.

As an example we consider a simplified version of a *Duffing-Van der Pol oscillator* driven by multiplicative white noise. The Ito stochastic differential equation here is 2-dimensional, with components X^1 and X^2 representing the displacement x and speed $\dot{x}$, respectively, namely

$$(27) \qquad \begin{aligned} dX_t^1 &= X_t^2\, dt \\ dX_t^2 &= \left\{X_t^1\left(\alpha - (X_t^1)^2\right) - X_t^2\right\} dt + \sigma X_t^1\, dW_t \end{aligned}$$

where $W = \{W_t,\ t \geq 0\}$ is a Wiener process, α is a real-valued parameter and $\sigma \geq 0$ controls the intensity of the multiplicative noise. The deterministic version of (27), that is with $\sigma \equiv 0$, has the steady states

$$X^1 = 0, \qquad X^2 = 0 \quad \text{for all} \quad \alpha$$

and

$$X^1 = \pm\sqrt{\alpha}, \qquad X^2 = 0 \quad \text{for} \quad \alpha \geq 0,$$

the first of which is also a stationary state of the stochastic differential equation (27). A typical trajectory starting with nonzero displacement and zero speed is oscillatory and attracted to one of the nontrivial steady states $(\pm\sqrt{\alpha}, 0)$. The regions of attraction of these two steady states could be determined by appropriately marking each initial value on the phase plane according to the steady solution which attracts the trajectory starting there. For weak noise we might expect similar behaviour to the deterministic case just described, but as the noise is multiplicative here stronger noise may lead to substantial changes, particularly over a long period of time.

The Milstein scheme was used for the case $\alpha = 1.0$ and noise parameter $\sigma = 0.2$ using the same sample path of the Wiener process for each trajectory but starting at different initial values. The paths plotted in Figure 3 are random in appearance and remain near to each other until they come close to the origin $(0, 0)$, after which they separate and are attracted into the neighbourhood of either $(-1, 0)$ or $(1, 0)$.

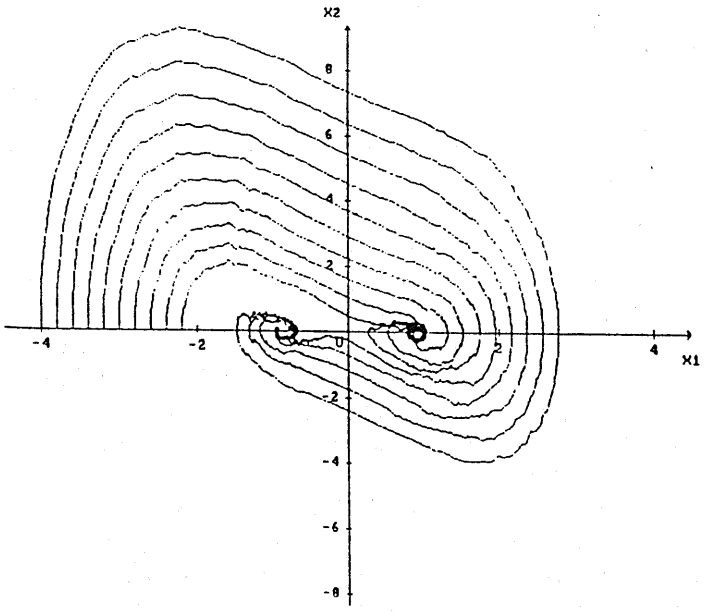

Figure 3 The Duffing-Van der Pol oscillator with weak noise.

Finally, we examine the effect of a stronger multiplicative noise in system (27) over a long period of time and plot in Figure 4 the displacement component X_t^1 against time t. While the noisy trajectories are initially attracted by one or the other of the points $(\pm 1, 0)$, not all of them remain indefinitely in the vicinity of the same point when the noise intensity is high. Instead, after spending a period of time near one of the points, the trajectories may switch over to the other point. This is suggestive of a tunneling phenomenon. To convince ourselves of the reliability of the above results, we repeated the calculations using a smaller step size and other strong schemes. While the quantitative details may then differ, the qualitative picture should be much the same. Such a check ables us to avoid results that are only an artifice of a particular numerical scheme or simulation.

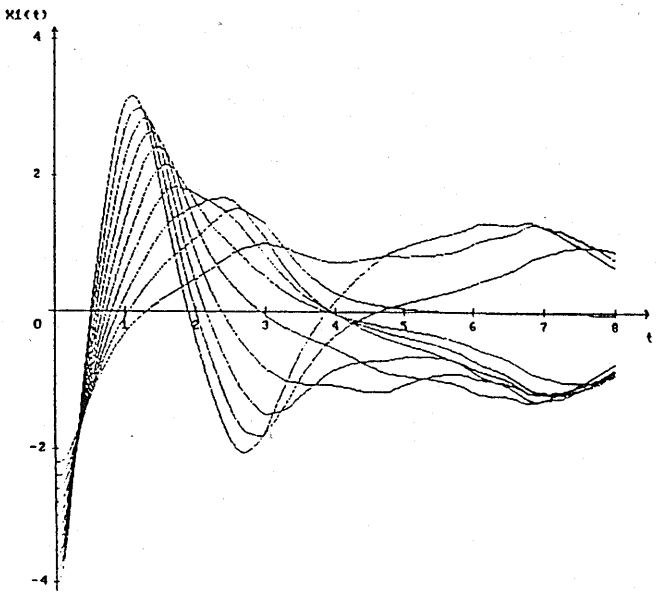

Figure 4 Displacement X_t^1 versus t for the Duffing-Van der Pol equation.

9. Lyapunov Exponents.

The asymptotic stability of a stationary solution of a linear stochastic differential equation is characterized by the sign of its largest Lyapunov exponent, with the change from negative to positive as a parameter is varied indicating the loss of stability. Since Lyapunov exponents can rarely be determined explicitly, they usually have to be evaluated numerically. A systematic, theoretical and practical discussion of the approximation and computation of Lyapunov exponents can be found in Talay (1989).

Lyapunov exponents measure the asymptotic rates of expansion or contraction of the solutions of a linear system, thus generalizing the real parts of the eigenvalues of an autonomous deterministic linear system. For a d-dimensional linear Stratonovich stochastic differential equation

$$dZ_t = AZ_t\,dt + \sum_{k=1}^{m} B^k Z_t \circ dW_t^k, \tag{28}$$

where $d \geq 2$ and $W = (W^1, \ldots, W^m)$ is an m-dimensional Wiener process, the *Lyapunov exponents* are defined as

$$\lambda(z,\omega) = \limsup_{t\to\infty} \frac{1}{t} \ln |Z_t^z(\omega)| \tag{29}$$

for each solution with initial value $Z_0^z = z$. Under appropriate assumptions on the matrices A, B^1, ..., B^m in (28), the Multiplicative Ergodic Theorem of Oseledets assures the existence of just d *nonrandom* Lyapunov exponents

$$\lambda_d \leq \ldots \leq \lambda_2 \leq \lambda_1,$$

some of which may coincide, and in fact the existence of the limits rather than just upper limits in (29). The counterparts of the eigenspaces for these Lyapunov exponents are, however, generally random.

There is an explicit formula for the top Lyapunov exponent λ_1. To determine it equation (28) must first be transformed to spherical coordinates $r = |z|$ and $s = z/|z|$ for $z \in \Re^d \setminus \{0\}$ for which it becomes

$$dR_t = R_t\, q(S_t)\, dt + \sum_{k=1}^{m} R_t\, q^k(S_t) \circ dW_t^k \tag{30}$$

$$dS_t = h(S_t, A)\, dt + \sum_{k=1}^{m} h(S_t, B^k) \circ dW_t^k \tag{31}$$

on $\Re^+ \times S^{d-1}$, where S^{d-1} is the unit sphere in $\Re^d$ and

$$q(s) = s^\top A s + \sum_{k=1}^{m} \left(\frac{1}{2} s^\top \left(B^k + \left(B^k\right)^\top \right) s - \left(s^\top B^k s \right)^2 \right),$$

$$q^k(s) = s^\top B^k s. \quad h(s, A) = \left(A - (s^\top A s) I \right) s.$$

Equation (31) does not involve the radial variable R_t, which from (30) has the explicit solution

$$\ln R_t = \ln |Z_t| = \int_0^t q(S_u)\, du + \sum_{k=1}^{m} \int_0^t q^k(S_u) \circ dW_u^k,$$

from which it follows that

$$\lim_{t\to\infty} \frac{1}{t} \ln |Z_t| = \lim_{t\to\infty} \frac{1}{t} \int_0^t q(S_u)\, du. \tag{32}$$

Consequently, the top Lyapunov exponent λ_1 of system (28) is given by the formula

$$\lambda_1 = \int_{S^{d-1}} q(s)\, d\bar{\mu}(s) \tag{33}$$

where $\bar{\mu}$ is the invariant probability measure of an ergodic solution process $S = \{S_t,\, t \geq 0\}$ of (28) on S^{d-1}. In most cases, however, it is not easy to determine $\bar{\mu}$ directly, so the limit (32) is used to approximate the λ_1 instead.

For example, for the stochastic differential equation

$$dZ_t = AZ_t\, dt + BZ_t \circ dW_t \tag{34}$$

with matrices

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -\sigma \\ \sigma & 0 \end{bmatrix}$$

and real-valued parameters a, b, σ Baxendale (1986) has shown that the top Lyapunov exponent is

$$\lambda_1 = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)\, \frac{\displaystyle\int_0^{2\pi} \cos(2\theta) \exp\left(\frac{a-b}{2\sigma^2} \cos(2\theta) \right) d\theta}{\displaystyle\int_0^{2\pi} \exp\left(\frac{a-b}{2\sigma^2} \cos(2\theta) \right) d\theta}. \tag{35}$$

Since $d = 2$ here the projected process S_t lives on the unit circle S^1 and thus can be expressed in terms of the polar angle

$$\phi_t = \arctan\left(\frac{S_t^2}{S_t^1}\right)$$

satisfying the stochastic equation

$$d\,\phi_t = \frac{1}{2}(b-a)\sin(2\phi_t)\,dt + \sigma dW_t, \tag{36}$$

which is interpreted modulo 2π, with

$$q(S_t) = \tilde{q}(\phi_t) = a(\cos\phi_t)^2 + b(\sin\phi_t)^2. \tag{37}$$

We shall use the functional

$$F_T^\delta = \frac{1}{n_T}\sum_{n=0}^{n_T-1} \tilde{q}\left(Y_n^\delta\right), \tag{38}$$

where n_T is the largest integer n such that $n\,\delta \le T$, for a discrete time approximation Y^δ of the solution of equation (36) to approximate the top Lyapunov exponent λ_1. From (35) that $\lambda_1 \simeq -0.489\ldots$ for the parameters $a = 1.0$, $b = -2.0$ and $\sigma = 10$. We plot the linearly interpolated values of F_t^δ against t for $0 \le t \le T$ in Figure 5 for the weak order 2.0 Runge-Kutta scheme (23) with $\delta = \Delta = 2^{-9}$, $T = 512$ and $Y_0 = \phi_0 = 0.0$ for these parameters, noting that F_t^δ tends towards the true value of λ_1.

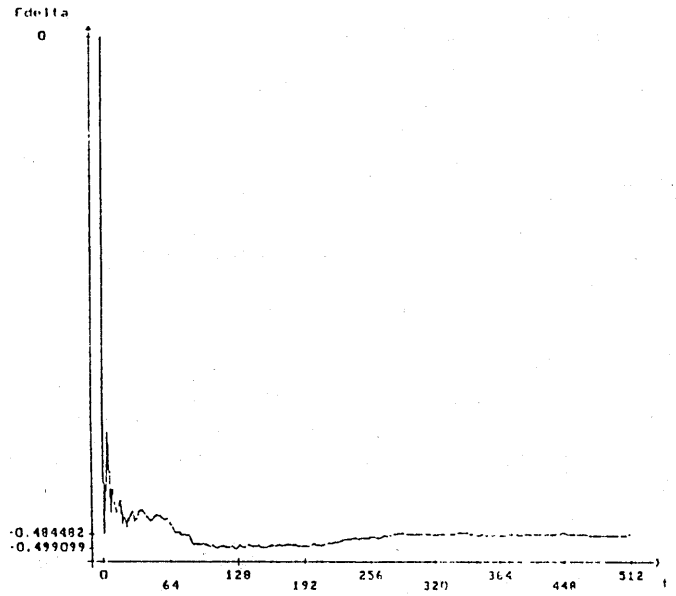

Figure 5 The top Lyapunov exponent.

The performance of the numerical scheme above was enhanced by the asymptotic stability of the null solution of the system of equations (33) as indicated by the negativity if its Lyapunov exponents. For stiff and unstable systems an implicit weak scheme may be required to provide satisfactory results.

In higher dimensions the stochastic differential equation (31) for S_t on S^{d-1} does not simplify so nicely and difficulties may be encountered in trying to solve it numerically, particularly in ensuring that the successive iterates remain on S^{d-1}. To circumvent these difficulties the first limit in (32) could be approximated directly, that is by the functional

$$\tilde{L}_T^\delta = \frac{1}{n_T \delta} \ln |Y_{n_T}^\delta| \tag{39}$$

where Y^δ is now a discrete time approximate solution of the original stochastic differential equation (28), or by the fuctional

$$L_T^\delta = \frac{1}{n_T \delta} \sum_{n=1}^{n_T} \ln \left(\frac{|Y_n^\delta|}{|Y_{n-1}^\delta|} \right) \tag{40}$$

which is computationally preferable since the logarithms in (39) will become very large in magnitude as $|Y_n^\delta|$ tends to zero or becomes very large.

10. Stochastic Stability and Bifurcation.

The reference solution for a stability analysis of a nonlinear stochastic system is typically a statistically stationary solution, which, to simplify matters, will be supposed to be the zero solution $X_t \equiv 0$. Consequently, the coefficients of the nonlinear (Stratonovich) stochastic differential equation

$$dX_t = \underline{a}(X_t)\, dt + \sum_{j=1}^{m} b^k(X_t) \circ dW_t^k \tag{41}$$

under consideration need to satisfy

$$\underline{a}(0) = b^1(0) = \ldots = b^m(0) = 0.$$

There is an extensive array of tests for stochastic asymptotic stability of a nonlinear system. This often follows from that of the zero solution $Z_t \equiv 0$ of the corresponding linearized stochastic differential equation

$$dZ_t = A\, Z_t\, dt + \sum_{k=1}^{m} B^k Z_t \circ dW_t^k \tag{42}$$

where $A = \nabla \underline{a}(0)$, $B^1 = \nabla b^1(0)$, ..., $B^m = \nabla b^m(0)$, which is characterized by the negativity of its top Lyapunov exponent λ_1. When the coefficients of (42) depend on parameters, so too will λ_1 and its sign may change as the parameters change, thus changing the stability of the zero solution and possibly resulting in a stochastic bifurcation. Since the theory of stochastic bifurcation is still in its

infancy numerical simulations are often the only means available to see what might happen.

As an example we consider the deterministic system of ordinary differential equations

$$\frac{dx_1}{dt} = (a-1)\,x_1 + a\,x_1^2 + (x_1+1)^2 x_2 \tag{43}$$

$$\frac{dx_2}{dt} = -a\,x_1 - a\,x_1^2 - (x_1+1)^2 x_2$$

with parameter a which is based on the Brusselator equations which model unforced periodic oscillations in certain chemical reactions. When $a < 2$ the zero solution $(x_1, x_2) \equiv (0,0)$ is globally asymptotically stable, but loses stability in a Hopf bifurcation at $a = 2$ to give rise to a limit cycle for $a > 2$. Supposing that the parameter a is noisy, that is of the form $\alpha + \sigma\,\xi_t$ where ξ_t is Gaussian white noise, leads to the system of Ito stochastic differential equations with scalar multiplicative noise

$$dX_t^1 = \{(\alpha-1)X_t^1 + \alpha(X_t^1)^2 + (X_t^1+1)^2 X_t^2\}\,dt + \sigma\,X_t^1(1+X_t^1)dW_t \tag{44}$$

$$dX_t^2 = \{-\alpha\,X_t^1 - \alpha(X_t^1)^2 - (X_t^1+1)^2 X_t^2\}\,dt - \sigma\,X_t^1(1+X_t^1)dW_t$$

for which $(X_t^1, X_t^2) \equiv (0,0)$ is a solution. The corresponding linearized system is then

$$d\begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix} = \begin{bmatrix} \alpha-1 & 1 \\ -\alpha & -1 \end{bmatrix}\begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix} dt + \sigma \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix} dW_t \tag{45}$$

in its Ito version, with the Stratonovich version having the same form but with α replaced by $\alpha - \sigma^2/2$. Its top Lyapunov exponent $\lambda_1 = \lambda_1(\alpha,\sigma)$ will depend on the two parameters α and σ. In the deterministic case $\sigma = 0$ we have $\lambda_1(\alpha,0) = \frac{1}{2}(\alpha-2)$, the real part of the complex conjugate eigenvalues of the drift coefficient matrix. When noise is present we need to evaluate $\lambda_1(\alpha,\sigma)$ numerically which can be done similarly as in the previous section. Results of such computations of the functional (40) given in Figure 6 show that the presence of noise stabilizes the system for $\alpha > 1$ with loss of stability occuring for larger α values. A similar effect also occurs in the original nonlinear system (44).

Figure 7 shows the phase diagram for the noisy Brusselator equations (44) with initial value $X_0^1 = -0.1$, $X_0^2 = 0.0$ over the time interval $0 \le t \le T = 1$ for the parameters $\alpha = 2.1$ and $\sigma = 0.1$. It suggests the presence of a noisy limit cycle, which is obviously not a closed, periodic curve as in the deterministic case, but nevertheless appears to have a characteristic mean radius, or magnitude, and mean period.

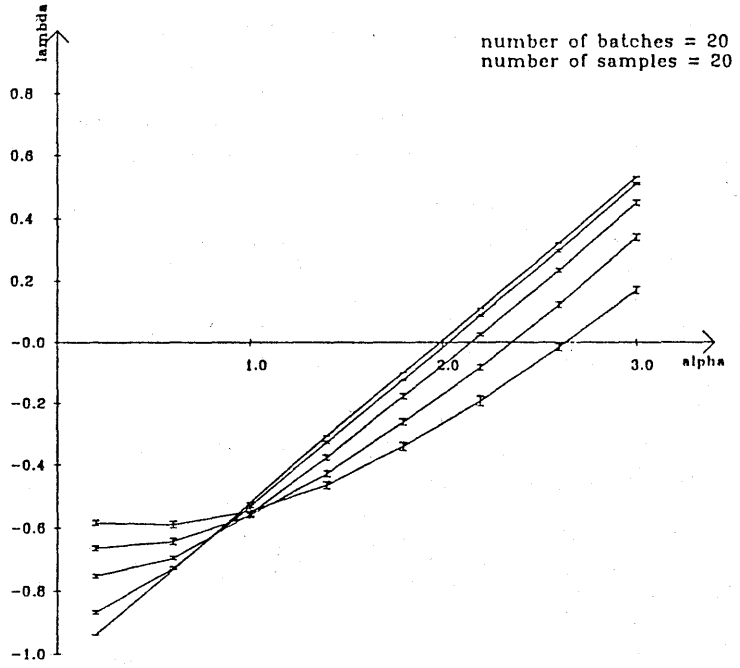

Figure 6 Top Lyapunov exponent $\lambda_1(\alpha, \sigma)$ of the noisy Brusselator.

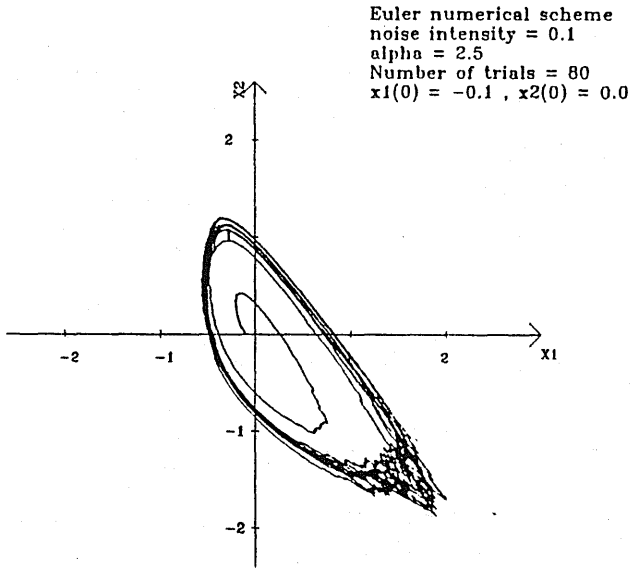

Figure 7 Noisy Brusselator phase diagram.

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