Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Typical Dimension of the Graph of Certain Functions

J. Schmeling[1] and R. Winkler[2]

submitted: 14th October 1993

[1] Institut für Angewandte Analysis
und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany

[2] Institut für Algebra und
Diskrete Mathematik der TU Wien
Wiedner Hauptstraße 8–10
A – 1040 Wien
Österreich

Preprint No. 67
Berlin 1993

Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39
D – 10117 Berlin

Fax: + 49 30 2004975
e–mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e–mail (Internet): preprint@iaas-berlin.d400.de

Typical Dimension of the Graph of Certain Functions

Jörg Schmeling Reinhard Winkler

Abstract

Most functions from the unit interval to itself have a graph with Hausdorff and lower entropy dimension 1 and upper entropy dimension 2. The same holds for several other Baire spaces of functions. In this paper it will be proved that this is the case also in the spaces of all mappings that are Lebesque measurable, Borel measurable, integrable in the Riemann sense, continuous, uniform distribution preserving (and continuous).

1 Introduction

A result of P. M. Gruber (vgl. [G]) says that in the Baire space of all continuous functions $f : [0,1] \mapsto \mathbf{R}$ with the metric

$$d(f,g) = |f-g| = \sup_{x\in[0,1]} |f(x)-g(x)|$$

for the graph

$$\Gamma(f) = \{(x, f(x))|x \in [0,1]\} \subseteq \mathbf{R}^2$$

the typical situation is

$$\dim^H(\Gamma(f)) = \dim_E(\Gamma(f)) = 1 < \dim^E(\Gamma(f)) = 2.$$

This means the following: We say that a property is typical for an $f \in F$ if those f that do not have the property form a meager subset of the topological space F. This statement is interesting if F is a Baire space (i.e. a space where every nonempty open subset is of second category), for instance a complete metric space. A subset $G \subseteq X$ is called residual if its complement in F is meager.

The symbols $\dim^H(A)$, $\dim_E(A)$ and $\dim^E(A)$ are denoting Hausdorff, lower and upper entropy dimension of a subset A of a metric space (X, d). (We will have $X = \mathbf{R}^2$ and take the maximum metric, since balls then are squares in the plane and more convenient to handle.) First we recall the definition of the Hausdorff dimension of A. We are using the following notations: $\{U_i|i \in I\}$ is called an η-covering of A if A is contained in the union of all U_i and each U_i has diameter

$$\operatorname{diam}(U_i) = \sup\{d(x, y) | x, y \in U_i\} \leq \eta.$$

For open balls (squares) we will use the notation

$$B(x, \eta) = \{y \in X | d(x, y) < \eta\}.$$

Call a real function φ admissible if it is non-decreasing, continuous and satisfies $\varphi(0) = 0$, $\varphi(\eta) > 0$ for $\eta > 0$. For an admissible φ consider

$$\mu_{(\varphi,\eta)}(A) = \inf \sum_{i=1}^{\infty} \varphi(\operatorname{diam}(U_i)),$$

where the infimum is taken over all countable η-coverings of A.

$$\mu_\varphi(A) = \lim_{\eta \to +0} \mu_{(\varphi,\eta)}(A)$$

defines an outer measure on X. For $\varphi(t) = t^\alpha$, $\alpha > 0$, we also write μ_α instead of μ_φ. The unique number $\delta \in [0, \infty]$ such that $\mu_\alpha(A) = \infty$ for all $\alpha < \delta$ and $\mu_\alpha(A) = 0$ for all $\alpha > \delta$ is the Hausdorff dimension $\dim^H(A)$ of A.

Upper and lower entropy dimension are quite similar concepts. If there exists a finite η-covering for the set A we denote by $N(\eta, A)$ the minimal cardinality of such a covering. In the case $A = \Gamma(f)$ we will use the shorter notation $N(\eta, f)$ instead of $N(\eta, \Gamma(f))$. We say that A has upper entropy dimension $\dim^E(A) = \delta$ if δ is the supremum of all α with the property that $\eta^\alpha N(\eta, A)$ does not tend to 0 for $\eta \to 0$, i.e.

$$\lim_{\eta \to +0} \sup \eta^\alpha N(\eta, A) > 0.$$

Similarly the lower entropy dimension $\dim_E(A)$ is defined to be the supremum δ of all α such that

$$\lim_{\eta \to +0} \inf \eta^\alpha N(\eta, A) > 0.$$

For the statements

1. $\lim_{\eta \to +0} \sup \varphi(\eta) N(\eta, A) = lim_{\eta \to +0} \varphi(\eta) N(\eta, A) = 0$
2. $\lim_{\eta \to +0} \inf \varphi(\eta) N(\eta, A) = 0$
3. $\mu_\varphi(A) = \lim_{\eta \to +0} \mu_{\varphi,\eta}(A) = 0$

we have the implications 1.$\Longrightarrow$2. and 2.$\Longrightarrow$3. which gives the relation between upper entropy dimension, lower entropy dimension and Hausdorff dimension. The first implication is trivial. The second one follows from the fact that in the definition of $\mu_{\varphi,\eta}(A)$ the infimum also respects all finite coverings by η-squares, i.e.

$$\mu_{\varphi,\eta}(A) \leq \varphi(\eta) N(\eta, A).$$

The inequality

$$\dim^H(A) \leq \dim_E(A) \leq \dim^E(A)$$

is an immediate consequence.

To state our results in a stronger version we use the following notations. Consider (instead of the power functions $\varphi(\eta) = \eta^\alpha$, $\alpha > 0$) any admissible φ and define for $\mathcal{K} \subseteq \mathcal{F}$ the sets

$$\begin{aligned}
\mathcal{K}_\varphi^H &= \{f \in \mathcal{K} \mid \mu_\varphi(\Gamma(f)) = 0\}, \\
\mathcal{K}_\varphi &= \{f \in \mathcal{K} \mid \lim_{\eta \to +0} \inf \varphi(\eta) N(\eta, f) = 0\} \subseteq \mathcal{K}_\varphi^H \quad \text{and} \\
\mathcal{K}^\varphi &= \{f \in \mathcal{K} \mid \lim_{\eta \to +0} \sup \varphi(\eta) N(\eta, f) > 0\}.
\end{aligned}$$

For $\varphi(\eta) = \eta^\alpha$ we have $\dim^H(\Gamma(f)) \leq \alpha$ if $f \in \mathcal{K}_\varphi^H$, $\dim_E(\Gamma(f)) \leq \alpha$ if $f \in \mathcal{K}_\varphi$ and $\dim^E(\Gamma(f)) \geq \alpha$ if $f \in \mathcal{K}^\varphi$.

In this paper we are going to prove that P. M. Gruber's result holds in several Baire spaces of functions. We shall investigate the following spaces of transformations of the unit interval $I = [0,1]$ with the sup-metric.
$\mathcal{F} = \{f \mid f : I \to I\}$, the set of all maps from I to itself.
$\mathcal{L} \subseteq \mathcal{F}$, the set of all Lebesque measurable selfmaps of I.
$\mathcal{B} \subseteq \mathcal{L}$, the set of all Borel measurable selfmaps of I.
$\mathcal{R} \subseteq \mathcal{L}$, the set of all $f \in \mathcal{F}$ that are integrable in the Riemann sense.
$\mathcal{U} \subseteq \mathcal{R}$, the set of all maps that preserve uniform distribution of sequences on I (u.d.p.-maps).
$\mathcal{C} \subseteq \mathcal{R}$, the set of all continuous transformations of I.
$\mathcal{UC} = \mathcal{U} \cap \mathcal{C}$, the set of all continuous u.d.p.-maps.

Note that a map $f : I \to I$ is called u.d.p. ($f \in \mathcal{U}$) if for every uniformly distributed sequence $(x_n)_{n \in \mathbf{N}}$ the induced sequence $(f(x_n))_{n \in \mathbf{N}}$ is uniformly distributed too. From [PSS] it follows that f is u.d.p. if and only if it is Riemann integrable and preserves the Lebesque measure λ, i.e.

$$\lambda(f^{-1}(A)) = \lambda(A)$$

for all intervals and hence for all measurable sets A. Therefore $\mathcal{U}$ (as all other considered classes) is closed under uniform limits, hence, beeing a complete metric space with respect to the uniform metric, indeed a Baire space.

In [PSS] many results about piecewise linear or continuously differentiable u.d.p. maps are presented. Continuing investigations of [TW] our result on $\mathcal{U}$ and $\mathcal{UC}$ show in a more precise way that, in the sense of Baire categories, piecewise smooth functions f (fulfilling $\dim^E(\Gamma(f)) = 1$) do not represent the typical situation in these spaces of functions.

2 The main results

Lemma 1 *Let $f \in \mathcal{F}$ and $0 < \eta_1 < \eta_2$ be arbitrary. Then there exists a $\delta > 0$*

such that

1. $N(\eta_2, g) \leq N(\eta_1, f)$ *and*
2. $N(\eta_1, g) \geq N(\eta_2, f)$

for all $g \in \mathcal{F}$ *with* $|g - f| < \delta$.

Proof:

1. The first statement is trivial with $\delta_1 = \frac{\eta_2 - \eta_1}{2}$, since for every η_1-covering U_i, $i = 1, \ldots, N$, of $\Gamma(f)$ we find p_i such that $U_i \subseteq B(p_i, \frac{\eta_1}{2})$ to get an η_2-covering $B(p_i, \frac{\eta_2}{2})$, $i = 1, \ldots, N$, for all g close to f.

2. For the second statement we assume, by contradiction, that for every $\delta_2 > 0$ there is a $g \in \mathcal{F}$ such that $|g - f| < \delta_2$ and $N(\eta_1, g) < N(\eta_2, f) = N$. We are going to construct an η_2-covering $U_1, \ldots, U_{N-1}$ of $\Gamma(f)$, contradicting $N(\eta_2, f) = N$. For every $n \in \mathbf{N}$ we get a g_n with $|g_n - f| < \frac{1}{n}$ and an η_1-covering of $\Gamma(g_n)$ by balls

$$B_i^{(n)} = B(p_i^{(n)}, \frac{\eta_1}{2}),\ p_i^{(n)} = (x_i^{(n)}, y_i^{(n)}),\ i = 1, \ldots, N-1.$$

W.l.o.g. we may assume $B_i^{(n)} \cap \Gamma(g_n) \neq \phi$. The sequence $(p_1^{(n)}, \ldots, p_{N-1}^{(n)})$ of points in the compact set $[0,1]^{2(N-1)}$ has a subsequence converging to an $(p_1, \ldots, p_{N-1})$, $p_i = (x_i, y_i)$, for $n \to \infty$. For some large n we have for every $x \in [0,1]$, $p = (x, f(x))$ and some $i \in \{1, \ldots, N-1\}$

$$\begin{aligned} d(p, p_i) &\leq d(p, (x, g_n(x))) + d((x, g_n(x)), p_i^{(n)}) + d(p_i^{(n)}, p_i) \\ &\leq |f - g_n| + \tfrac{\eta_1}{2} + d(p_i^{(n)}, p_i) < \tfrac{\eta_2}{2}, \end{aligned}$$

proving that $U_i = B(p_i, \frac{\eta_2}{2})$, $i = 1, \ldots, N-1$, is the desired covering of $\Gamma(f)$. Hence there is a δ_2 as stated in the assertion.

Now take $\delta = \min(\delta_1, \delta_2)$ to complete the proof. **q.e.d.**

Theorem 1 *Let $\mathcal{K}$ be any subclass of $\mathcal{F}$.*

1. *Let φ and ψ be admissible functions satisfying*

$$\psi(\eta) = o(\varphi(\eta)) \text{ for } \eta \to 0.$$

Furthermore assume that for every $f \in \mathcal{K}$, $\varepsilon > 0$ and $c > 0$ there is a $g \in \mathcal{K}$ and a positive $\eta < c$ such that $|g - f| < \varepsilon$ and $N(\eta, g)\varphi(\eta) < 1$. Then $\mathcal{K}_\psi$ and $\mathcal{K}_\psi^H$ are residual. Therefore, if this assumption is satisfied for every φ of the form $\varphi(\eta) = \eta^\alpha$, $\alpha > 1$,

$$\dim^H(\Gamma(f)) = \dim_E(\Gamma(f)) = 1$$

for a typical $f \in \mathcal{K}$.

2. *Let φ be an admissible function. Furthermore assume the existence of an $a > 0$ such that for every $f \in \mathcal{K}$, $\varepsilon > 0$ and $c > 0$ there is a $g \in \mathcal{K}$ and a positive $\eta < c$ such that $|g - f| < \varepsilon$ and $N(\eta, g)\varphi(\eta) > a$. Then $\mathcal{K}^\varphi$ is residual. Therefore, if this assumption is satisfied for every φ of the form $\varphi(\varepsilon) = \varepsilon^\alpha$, $\alpha < 2$,*
$$\dim^E(\Gamma(f)) = 2$$
for a typical $f \in \mathcal{K}$.

Proof:

1. Since
$$\mathcal{K}_\psi = \bigcap_{n\in\mathbf{N}} \bigcap_{k\in\mathbf{N}} \mathcal{K}_{\psi,n,k} \subseteq K_\psi^H$$
with
$$\mathcal{K}_{\psi,n,k} = \bigcup_{\eta<\frac{1}{n}} \{f \in \mathcal{K} \mid N(\eta, f)\psi(\eta) < \frac{1}{k}\}$$
we have to prove that every $\mathcal{K}_{\psi,n,k}$, $n, k \in \mathbf{N}$, is a) dense and b) open in $\mathcal{K}$, yielding that $\mathcal{K}_\psi$ and $\mathcal{K}_\psi^H$ are residual.
ad a): For arbitrary $f \in \mathcal{K}$ and $\varepsilon > 0$ we have to find a $g \in \mathcal{K}_{\psi,n,k}$ with $|g - f| < \varepsilon$. By $\psi = o(\varphi)$ there is an $n_0 \in \mathbf{N}$ such that $\psi(\eta) < \frac{1}{k}\varphi(\eta)$ for all $\eta < \frac{1}{n_0}$. Put $c = \min(\frac{1}{n}, \frac{1}{n_0})$ and take g and η as in the assumption. By
$$N(\eta, g)\psi(\eta) < \frac{1}{k}N(\eta, g)\varphi(\eta) < \frac{1}{k}$$
this g does the job.
ad b): Pick $f \in \mathcal{K}_{\psi,n,k}$, i.e.
$$N(\eta, f)\psi(\eta) < \frac{1}{k}$$
for some $\eta < \frac{1}{n}$. By contiunuity of ψ there is an η' satisfying $\eta < \eta' < \frac{1}{n}$ and
$$N(\eta, f)\psi(\eta') < \frac{1}{k}.$$
By Lemma 1.1 there is a $\delta > 0$ such that $|g - f| < \delta$ implies $N(\eta', g) \leq N(\eta, f)$, hence
$$N(\eta', g)\psi(\eta') < \frac{1}{k}, \text{ i.e. } g \in \mathcal{K}_{\psi,n,k},$$
proving that $\mathcal{K}_{\psi,n,k}$ is open in $\mathcal{K}$.

The second assertion of the first part of the Theorem now is an easy consequence: By the first assertion every $\mathcal{K}_{\varphi_n}$, $\varphi_n(\eta) = \eta^{1+\frac{1}{n}}$, is residual, hence the countable intersection

$$\mathcal{K}^{(1)} = \bigcap_{n \in \mathbf{N}} \mathcal{K}_{\varphi_n}$$

is residual too. Since every $f \in \mathcal{K}^{(1)}$ satisfies $\dim_E(\Gamma(f)) \leq 1$ everything follows from $1 \leq \dim^H(\Gamma(f)) \leq \dim_E(\Gamma(f))$.

2. Since

$$\mathcal{K}^{\varphi} = \bigcup_{a>0} \bigcap_{n \in \mathbf{N}} \mathcal{K}^{\varphi}_{n,a}$$

with

$$\mathcal{K}^{\varphi}_{n,a} = \bigcup_{\eta < \frac{1}{n}} \{f \in \mathcal{K} \mid N(\eta, f)\varphi(\eta) > a\}$$

it suffices to prove that every $\mathcal{K}^{\varphi}_{n,a}$, $n \in \mathbf{N}$, is a) dense and b) open in $\mathcal{K}$, yielding that $\mathcal{K}^{\varphi}$ is residual.

ad a): For arbitrary $f \in \mathcal{K}$ and $\varepsilon > 0$ we have to find a $g \in \mathcal{K}^{\varphi}_{n,a}$ with $|g - f| < \varepsilon$. Put $c = \frac{1}{n}$ and take g and η as in the assumption. By

$$N(\eta, g)\varphi(\eta) > a$$

this g does the job.

ad b): Pick $f \in \mathcal{K}^{\varphi}_{n,a}$, i.e.

$$N(\eta, f)\varphi(\eta) > a$$

for some $\eta < \frac{1}{n}$. By the continuity of φ there is an $\eta' < \eta$ such that $N(\eta, f)\varphi(\eta') > a$. Hence Lemma 1.2 guarantees the existence of some $\delta > 0$ with

$$N(\eta', g)\varphi(\eta') \geq N(\eta, f)\varphi(\eta') > a, \quad i.e. \;\; g \in \mathcal{K}^{\varphi}_{n,a},$$

for all $g \in \mathcal{K}$ with $|g - f| < \delta$. This proves that $\mathcal{K}^{\varphi}_{n,a}$ is open.

The second assertion of the second part of the Theorem follows as in the first part: By the first assertion every $\mathcal{K}^{\varphi_n}$, $\varphi_n(\eta) = \eta^{2-\frac{1}{n}}$, is residual, hence the countable intersection

$$\mathcal{K}^{(2)} = \bigcap_{n \in \mathbf{N}} \mathcal{K}^{\varphi_n}$$

is residual too. Since every $f \in \mathcal{K}^{(2)}$ satisfies $\dim^E(\Gamma(f)) = 2$ the proof is complete. **q.e.d.**

Theorem 2 *For $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}, \mathcal{R}, \mathcal{U}, \mathcal{C}$ or $\mathcal{UC}$ the following holds:*

1. *For every admissible φ satisfying $\varphi(\eta) = o(\eta)$ (for $\eta \to 0$) $\mathcal{K}_\varphi$ and $\mathcal{K}_\varphi^H$ are residual. Hence*
$$\dim^H(\Gamma(f)) = \dim_E(\Gamma(f)) = 1$$
for a typical $f \in \mathcal{K}$.

2. *For every admissible φ satisfying $\eta^2 = o(\varphi(\eta))$ (for $\eta \to 0$) $\mathcal{K}^\varphi$ is residual. Hence*
$$\dim^E(\Gamma(f)) = 2$$
for a typical $f \in \mathcal{K}$.

Proof:

1. Pick $f \in \mathcal{K}$, $\varepsilon > 0$, $c > 0$ and an arbitrary admissible $\varphi(\eta) = o(\eta)$ for $\eta \to 0$. Theorem 1.1 implies the assertion when we find a $g \in \mathcal{K}$ and a positive $\eta < c$ satisfying $|g - f| < \varepsilon$ and $N(\eta, g)\varphi(\eta) < 1$. We are going to treat the different cases for $\mathcal{K}$.

   (a) $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}$: Take an $n \in \mathbb{N}$ with $n > \frac{1}{\varepsilon}$ and define $g(x) = \frac{k}{n}$, $k \in \{1, \ldots, n\}$, for $\frac{k-1}{n} \leq f(x) < \frac{k}{n}$, hence $g \in \mathcal{K}$ and $|g - f| < \varepsilon$. Since
   $$\Gamma(g) \subseteq \{(x, \frac{k}{n}) \mid x \in I, k \in \mathbb{N}, 1 \leq k \leq n\}$$
   we get
   $$N(\eta, g) \leq n(\frac{1}{\eta} + 1).$$
   Furthermore $\varphi(\eta) < \frac{\eta}{2n}$ for sufficiently small $\eta < c$ (w.l.o.g $\eta < 1$), hence indeed
   $$N(\eta, g)\varphi(\eta) < \frac{1}{2}(1 + \eta) < 1.$$

   (b) $\mathcal{K} = \mathcal{R}$: Again fix an $n \in \mathbb{N}$ with $n > \frac{1}{\varepsilon}$. Call an interval $J \subseteq [0, 1]$ regular if the following conditions hold:

      i. $J = [\frac{l}{2^m}, \frac{l+1}{2^m}]$ for some nonnegative integers l and m.
      ii. $\sup_{x \in J} f(x) - \inf_{x \in J} f(x) < \varepsilon$.
      iii. J is maximal with the properties (i) and (ii).

   Every J satisfying i. and ii. is contained in a unique regular one. On each regular J define g to be constant taking a value $\frac{k}{n}$, $k \in \{1, \ldots, n\}$ such that $|f(x) - \frac{k}{n}| < \varepsilon$ for all $x \in J$. Note that this is possible by property ii. If x is not contained in a regular J define $g(x) = \frac{k}{n}$ such that $|f(x) - \frac{k}{n}| < \varepsilon$, hence $|f - g| < \varepsilon$. As in the case (a) we get
   $$N(\eta, g) \leq n(\frac{1}{\eta} + 1) = O(\frac{1}{\eta}),$$

hence $N(\eta, g)\varphi(\eta) < 1$ for sufficiently small $\eta > 0$. It remains to prove that $g \in \mathcal{R}$. Consider the sets D_f and D_g of all $x \in I$ such that f resp. g is not continuous in x. If $x \notin D_f$, x is contained in a regular $J = [\frac{l}{2^m}, \frac{l+1}{2^m}]$. If such an x is not an end point of J then $x \notin D_g$. This reasoning proves $D_g \subseteq D_f \cup D$,

$$D = \{\frac{l}{2^m} \mid l, m \in \mathbb{N}_0\}.$$

$f \in \mathcal{R}$, hence D_f has Lebesgue measure $\lambda(D_f) = 0$, D is countable, hence $\lambda(D) = 0$ and therefore $\lambda(D_g) = 0$, proving $g \in \mathcal{R}$.

(c) $\mathcal{K} = \mathcal{U}$: Again fix an $n \in \mathbb{N}$ with $n > \frac{1}{n}$. Modifying the construction of (b) we now call an interval $J \subseteq [0,1]$ regular if the following conditions hold:

   i. $J = [\frac{l}{2^m}, \frac{l+1}{2^m}]$ for some nonnegative integers l and m.
   ii. $f(J) \subseteq (\frac{i-1}{n}, \frac{i}{n})$ for some $i \in \{1, \ldots, n\}$.
   iii. J is maximal with the properties (i) and (ii).

Again every J satisfying i. and ii. is contained in a unique regular one. We are going to prove

$$\lambda(M) = 1 \quad \text{for} \quad M = \bigcup\{J \mid J \text{ regular}\}.$$

We have $I \setminus M \subseteq D_f \cup D \cup f^{-1}(S)$ where D_f and D are defined as in (b) and

$$S = \{\frac{i}{n} \mid i \in \{0, \ldots, n\}\}.$$

Since $f \in \mathcal{U} \subseteq \mathcal{R}$ we have $\lambda(f^{-1}(S)) = \lambda(S) = 0$ and $\lambda(D_f) = 0$. This implies

$$\lambda(I \setminus M) \leq \lambda(D_f) + \lambda(D) + \lambda(f^{-1}(S)) = 0, \ \lambda(M) = 1.$$

Now define the functions g_i, $i = 1, \ldots, n$, by

$$g_i(x) = \frac{i-1}{n} + \lambda\left([0,x] \cap f^{-1}([\frac{i-1}{n}, \frac{i}{n}])\right).$$

Observe that g_i is monotonous non-decreasing, $g_i(0) = \frac{i-1}{n}$, $g_i(1) = \frac{i}{n}$ and

$$|g_i(y) - g_i(x)| \leq |y - x|.$$

In particular g_i is continuous. Now define $g(x) = g_i(x)$ for $\frac{i-1}{n} < f(x) < \frac{i}{n}$ or $g(x) = f(x) = \frac{i}{n}$. By construction we have $|g - f| \leq \frac{1}{n} < \varepsilon$.

The set D_g of discontinuities of g is contained in $I \setminus M$, hence $\lambda(D_g) = 0$ and $g \in \mathcal{R}$. To prove $g \in \mathcal{U}$ it remains to show that g preserves the Lebesque measure. Consider an arbitrary regular $J = (a, b)$, say $f(J) \subseteq (\frac{i-1}{n}, \frac{i}{n})$. g is linear on J with constant derivative $g' \equiv 1$, hence the restriction

$$g|_J : (a, b) \to g(J) = (c, d)$$

preserves the Lebesque measure. By construction g is injective on M. Thus we conclude that

$$g|_M : M \to g(M)$$

is a measure preserving injection. Since

$$\lambda(I \setminus M) = \lambda(I \setminus g(M)) = 0$$

also g is measure preserving, hence $g \in \mathcal{U}$.
The last step is to show $N(\eta, g)\varphi(\eta) < 1$ for small $\eta > 0$. For every $\Gamma(g_i)$ we find an η-covering

$$U_j,\ j = 1, \ldots, K,\ K = [\frac{1}{\eta}] + 1,$$

by taking

$$U_j = B(p_j, \frac{\eta}{2}),\ p_j = (x_j, g_i(x_j)),\ x_j = \frac{2j-1}{2K}.$$

Here we use the Lipschitz condition $|g(y) - g(x)| \leq |y - x|$. By construction

$$\Gamma(g) \subseteq \bigcup_{i=1}^{n} \Gamma(g_i) \cup \bigcup_{i=0}^{n} \Gamma(c_i)$$

where $c_i \equiv \frac{i}{n}$ are constant functions, trivially admitting an η-covering of cardinality K. Thus we get

$$N(\eta, g) \leq (2n+1)K = (2n+1)([\frac{1}{\eta}] + 1) = O(\frac{1}{\eta})$$

for $\eta \to 0$, hence $N(\eta, g)\varphi(\eta) < 1$ for sufficiently small $\eta > 0$.

(d) $\mathcal{K} = \mathcal{C}$: Take any continuous and piecewise linear approximation g of f with $|g - f| < \varepsilon$. Since g satisfies a Lipschitz condition we conclude $N(\eta, g) = O(\frac{1}{\eta})$ and hence $N(\eta, g)\varphi(\eta) < 1$ for sufficiently small η.

(e) $\mathcal{K} = \mathcal{UC}$: By [TW], Theorem 3.2, there is a piecewise linear $g \in \mathcal{UC}$ with $|g - f| < \varepsilon$. Therefore the same argument works as for $\mathcal{K} = \mathcal{C}$.

2. Pick $f \in \mathcal{K}$, $\varepsilon > 0$, $c > 0$ and an arbitrary admissible φ with $\eta^2 = o(\varphi(\eta))$. Theorem 1.2 implies the assertion when we find a $g \in \mathcal{K}$ and a positive $\eta < c$ satisfying $|g - f| < \varepsilon$ and $N(\eta, g)\varphi(\eta) > 1$. We are going to treat the different cases for $\mathcal{K}$.

(a) $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}$: For irrational $x \notin \mathbf{Q}$ define $g(x) = f(x)$. To construct $g(x)$ for $x \in \mathbf{Q}$ fix an integer $n > \frac{1}{\varepsilon}$. Consider the factor groups $\mathbf{Q}/(\frac{1}{n})$ and $\mathbf{Q}/D$,

$$(\frac{1}{n}) = \{\frac{k}{n} | k \in \mathbf{Z}\} \subseteq D = \{\frac{k}{2^l n} | l, k \in \mathbf{Z}, l \geq 0\}.$$

In a natural sense $D + (\frac{1}{n})$ may be considered as a dense subgroup of $\mathbf{Q}/(\frac{1}{n})$.

The factor group

$$(\mathbf{Q}/(\frac{1}{n}))/(D + (\frac{1}{n})) \cong \mathbf{Q}/D$$

is infinite: Consider two odd primes $p > q > n$. The hypothesis $\frac{1}{p} + D = \frac{1}{q} + D$ leads to $\frac{1}{p} = \frac{1}{q} + \frac{k}{n2^l}$, k odd, $l \geq 0$ integer, hence

$$n 2^l (q - p) = kpq.$$

The right hand side is odd, hence $l = 0$, $|\frac{1}{p} - \frac{1}{q}| \geq \frac{1}{n}$, contradicting $0 < \frac{1}{p} < \frac{1}{q} < \frac{1}{n}$. Therefore all classes $\frac{1}{p} + D$, $p > n$ prime, are distinct, $\mathbf{Q}/D$ is indeed infinite.

Since all involved sets are countable there is a bijection

$$\beta : \mathbf{Q}/D \to \mathbf{Q}/(\frac{1}{n}).$$

Let $\kappa : \mathbf{Q} \to \mathbf{Q}/D$, $x \mapsto x + D$, denote the canonical map. For $x \in \mathbf{Q}$ we define $g(x)$ in such a way that

$$g(x) \in \beta(\kappa(x)) \doteq \alpha + (\frac{1}{n})$$

and $|g(x) - f(x)| < \varepsilon$.

Now g is defined on the whole interval I and satisfies $|g - f| < \varepsilon$. g and f differ only on a subset of $\mathbf{Q}$, i.e. on a countable set, hence $f \in \mathcal{K}$ implies $g \in \mathcal{K}$ for all cases $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}$. Thus it remains to find a positive $\eta < c$ such that $N(\eta, g)\varphi(\eta) > 1$. We are going to deduce $N(\eta, g) \geq \frac{1}{n\eta^2}$ for all $\eta > 0$. Then $\varphi(\eta) > n\eta^2$ for small η yields the assertion.

Consider an arbitrary finite covering of $\Gamma(g)$ by squares $U_1, \ldots, U_N$ of length η, i.e.

$$\lambda(U) \leq N\eta^2$$

for $U = \bigcup_{i=1}^N U_i$ (λ = two-dimensional Lebesque measure). We are going to consider the topological closure $\overline{\Gamma(g)}$ of the graph of g. Since $\overline{\Gamma(g)} \subseteq \overline{U}$ we have

$$\lambda(\overline{\Gamma(g)}) \leq N\eta^2.$$

Claim: For every point $(x, y) \in I \times [0, \frac{1}{n}]$ there is a $k \in \{0, \ldots, n-1\}$ with

$$(x, y + \frac{k}{n}) \in \overline{\Gamma(g)}.$$

To prove this claim we construct a sequence $(x_l, y_l) \in \Gamma(g)$, $l \in \mathbf{N}$, with $|x_l - x| < \frac{1}{l}$ and $|y_l - y - \frac{k(l)}{n}| < \frac{1}{l}$ for some $k(l) \in \{0, \ldots, n-1\}$. First take any rational $y_l' \in [0, \frac{1}{n}]$ with the property $|y_l' - y| < \frac{1}{l}$. There is an $x_l' + D \in \mathbf{Q}/D$ and a $k(l) \in \{0, \ldots, n-1\}$ with $g(x_l) = y_l = y_l' + \frac{k(l)}{n}$. Since D is dense there is an $x_l \in x_l' + D$ satisfying $|x_l - x| < \frac{1}{l}$. Among the $k(l)$, $l \in \mathbf{N}$, there is at least one k occuring infinitely many times. Hence the sequence $(x_l, g(x_l) = y_l = y_l' + \frac{k(l)}{n})$ has the cluster point $(x, y + \frac{k}{n}) \in \overline{\Gamma(g)}$.

We now consider the sets

$$\Gamma_k = \Gamma(g) \cap I \times [\frac{k}{n}, \frac{k+1}{n}],$$

their outer Lebesque measures $\lambda^*(\Gamma_k)$ and the translations

$$\tau_k : (x, y) \mapsto (x, y - \frac{k}{n}),$$

$k = 0, \ldots, n-1$. By the claime proved above

$$\bigcup_{k=1}^{n-1} \tau_k(\overline{\Gamma_k}) = \overline{\bigcup_{k=1}^{n-1} \tau_k(\Gamma_k)} = [0, 1] \times [0, \frac{1}{n}].$$

λ^* is subadditive and translation invariant, hence

$$\begin{aligned} \frac{1}{n} &= \lambda^* \left(\textstyle\bigcup_{k=1}^{n-1} \tau_k(\overline{\Gamma_k}) \right) \leq \textstyle\sum_{k=0}^{n-1} \lambda^*(\tau_k(\overline{\Gamma_k})) = \\ &= \lambda^*(\overline{\Gamma(g)}) = \lambda(\overline{\Gamma(g)}) \leq N\eta^2, \end{aligned}$$

yielding $N \geq \frac{1}{n\eta^2}$.

(b) $\mathcal{K} = \mathcal{R}, \mathcal{C}$: Since f is integrable in the Riemann sense there is an interval $J = [a, b] \subseteq I$ of length $\lambda = b - a > 0$ and an y_0 such that

$$f(J) \subseteq (y_0, y_0 + \varepsilon).$$

For $x \notin J$ define $g(x) = f(x)$. To define g on J let K be an integer determined later and $\alpha = \frac{\lambda}{2K+1}$. Let $\Gamma(g)$ on J be a zigzag line connecting the points $p_0, \ldots, p_{2K+1}$, where $p_0 = (a, f(a))$, $p_i = (a + i\alpha, y_0 + \varepsilon)$ for odd $i = 1, \ldots, 2K-1$, $p_i = (a + i\alpha, y_0)$ for even $i = 2, \ldots, 2K$ and $p_{2K+1} = (b, f(b))$.

We will show $N(\eta, g)\varphi(\eta) > 1$ for appropriate K and small $\eta > 0$. To cover the line segment g_j connecting p_{2j-1} and p_{2j} one needs at least $\frac{\varepsilon}{\eta}$ squares of length η. For $\eta < \frac{\lambda}{2K+1}$ none of these squares intersects with any other $g_{j'}$, hence $N(\eta, g) \geq K\frac{\varepsilon}{\eta}$ in this case. Now take any positive $\eta < c$, w.l.o.g. $\eta < \frac{\lambda}{6} \leq \frac{1}{6}$, such that

$$\varphi(\eta) > \frac{4\eta^2}{\varepsilon\lambda}$$

and the unique integer $K \in [\beta - 1, \beta)$, $\beta = \frac{1}{2}(\frac{\lambda}{\eta} - 1)$, then indeed

$$N(\eta, g)\varphi(\eta) > (\beta - 1)\frac{\varepsilon}{\eta}\frac{4\eta^2}{\varepsilon\lambda} > 1.$$

(c) $\mathcal{K} = \mathcal{U}$: $\mathcal{U} \subseteq \mathcal{R}$, hence we may define $J = [a, b]$, y_0 and $g(x) = f(x)$ for $x \notin J$ as in the case $\mathcal{K} = \mathcal{R}$. On J we have to give a construction for g such that $g \in \mathcal{U}$, i.e. such that g is integrable in the Riemann sense and satisfies

$$\lambda(g^{-1}(M) \cap J) = \lambda(f^{-1}(M) \cap J)$$

for all measurable M. This can be done by using the unique non-decreasing function $h : I \to (y_0, y_0 + \varepsilon)$ defined by the relation

$$(b - a)\lambda(h^{-1}(M)) = \lambda(f^{-1}(M) \cap J)$$

for all intervals $M \subseteq (y_0, y_0 + \varepsilon)$ (and hence for all measurable M). For any integer K we may define

$$g(x) = h(\frac{x - a - k\alpha}{\alpha}), \quad \alpha = \frac{b - a}{K},$$

if $a + k\alpha \leq x < a + (k + 1)\alpha$, $k \in \{0, \ldots, K - 1\}$, and get a g that is integrable in the Riemann sense, measure preserving, i.e. $g \in \mathcal{U}$, and satisfying $|g - f| < \varepsilon$.
Let g_k denote the restriction of g to the interval $J_k = [a + k\alpha, a + (k + 1)\alpha]$, $k = 0, \ldots, K - 1$. Suppose there is an η-covering $U_i = (u_i, u_i + \eta) \times (v_i, v_i + \eta)$, $i = 1, \ldots, l$ of $\Gamma(g_k)$ with $l < \frac{b-a}{\eta}$. Consider the set

$$V = \bigcup_{i=1}^{l} (v_i, v_i + \eta),$$

satisfying $\lambda(V) \leq l\eta < b - a$. The contradiction

$$b - a = \lambda(J) \leq \lambda(g^{-1}(V)) = \lambda(V) \leq l\eta < b - a$$

proves $N(\eta, g_k) \geq \frac{b-a}{\eta}$ for $k = 0, \ldots, K-1$. Now a reasoning similar as in the case $\mathcal{K} = \mathcal{R}$ shows that indeed

$$N(\eta, g)\varphi(\eta) > 1$$

for appropriate K and $\eta < c$.

(d) $\mathcal{K} = \mathcal{UC}$: By [TW], Theorem 3.2, there is a piecewise linear $f' \in \mathcal{UC}$ satisfying $|f - f'| < \frac{\varepsilon}{2}$. Now apply the construction of the case $\mathcal{K} = \mathcal{C}$ on f' instead of f, $\frac{\varepsilon}{2}$ instead of ε and an interval $[a, b]$ where f' is linear. Note that the resulting g satisfies

$$\lambda(g^{-1}(M)) = \lambda(f^{-1}(M)) = \lambda(M)$$

for each measurable set $M \subseteq I$, hence $g \in \mathcal{U}$. Obviously g is continuous, therefore $g \in \mathcal{UC}$. Furthermore

$$|g - f| \leq |g - f'| + |f' - f| < \varepsilon$$

and $N(\eta, g)\varphi(n) > 1$ for some $\eta < c$.

q.e.d.

Remark: Inspecting the proof of Theorem 2.2 one sees that it holds also for the restriction of f to any nontrivial subinterval $J \subseteq I$. Since each J contains at least one of the countably many nontrivial intervals with rational end points we even get the stronger result that the local upper entropy dimension which is defined by

$$\dim_{\text{loc}}^{E}(\Gamma(f)) = \inf_{J} \dim^{E}(\Gamma(f|_J))$$

equals 2 for a typical $f \in \mathcal{K}$, $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}, \mathcal{R}, \mathcal{U}, \mathcal{C}, \mathcal{UC}$. (The infimum in the definition is taken over all nontrivial subintervals $J \subseteq I$.)

3 Some additional remarks

The results of section 2 said, for instance, that for several Baire spaces $\mathcal{K} \subseteq \mathcal{F}$ and every admissible φ, $\eta^2 = o(\varphi(\eta))$ for $\eta \to 0$, the set $\mathcal{K}^{\varphi}$ is residual. The following question arises: How big is the set of all $f \in \mathcal{K}$ lying in all $\mathcal{K}^{\varphi}$. Since all admissible sequences form a set that is not countable we are not allowed to conclude that

$$\mathcal{K}^* = \bigcap_{\varphi} \mathcal{K}^{\varphi} \quad \text{or} \quad \mathcal{K}_* = \bigcap_{\varphi} \mathcal{K}_{\varphi}$$

is residual. (The intersection is taken over all admissible φ satisfying $\eta^2 = o(\varphi(\eta))$ resp. $\varphi(\eta) = o(\eta)$ for $\eta \to 0$.) The following Theorem shows that, in general, this is indeed wrong.

Theorem 3 *Let $f \in \mathcal{R}$ arbitrary. Then there is an admissible φ, $\eta^2 = o(\varphi(\eta))$, such that $f \notin \mathcal{R}^\varphi$. Thus*

$$\mathcal{K}^* = \bigcap_\varphi \mathcal{K}^\varphi = \phi$$

for $\mathcal{K} = \mathcal{R}, \mathcal{U}, \mathcal{C}, \mathcal{UC}$.

Proof: The assertion follows if, for arbitrary $c > 0$, we can find an $\eta \in (0, c)$ such that $\eta^2 N(\eta, f) \leq c$ for all positive $\eta < \eta_0$. This relation would give rise to an admissible φ with the desired properties.

For any $N \in \mathbf{N}$ consider

$$\Delta_N^{(j)} = \sup_{\frac{j}{N} \leq x, y \leq \frac{j+1}{N}} f(x) - f(y)$$

for $j = 0, \ldots, N-1$. We have

$$\eta^2 N(\eta, f) \leq \eta^2 \sum_{j=0}^{N-1} (\frac{1}{N\eta} + 1)(\frac{\Delta_N^{(j)}}{\eta} + 1) \leq$$

$$\leq \frac{1}{N} \sum_{j=0}^{N-1} \Delta_N^{(j)} + \eta + \eta \sum_{j=0}^{N-1} \Delta_N^{(j)} + N\eta^2.$$

Since $f \in \mathcal{R}$ the first term is $< \frac{c}{4}$ for sufficiently large N. For such an N

$$\eta < \eta_0 = \min\left(\frac{c}{4}, \frac{c}{4}\left(\sum_{j=0}^{N-1} \Delta_N^{(j)}\right)^{-1}, \sqrt{\frac{c}{4N}}\right)$$

implies $\eta^2 N(\eta, f) < c$. **q.e.d.**

For $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}$ Theorem 3 is wrong. This follows from the construction in the proof of Theorem 2.2. There, for arbitrary $f \in \mathcal{K}$, $\varepsilon > 0$, a function $g \in \mathcal{K}$ was constructed satisfying $|g - f| < \varepsilon$ and

$$\limsup_{\eta \to 0} \eta^2 N(\eta, g) > 0.$$

This proves that $\mathcal{K}^*$ is dense in $\mathcal{K}$, i.e. $\overline{\mathcal{K}^*} = \mathcal{K}$ for $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}$. The constructions in the proof of Theorem 2.1 show that the analogous result $\overline{\mathcal{K}_*} = \mathcal{K}$ holds for $\mathcal{K} = \mathcal{F}, \mathcal{L}, \mathcal{B}, \mathcal{R}, \mathcal{U}, \mathcal{C}$ and $\mathcal{UC}$.

The question whether there are nontrivial Baire subspaces $\mathcal{K} \subseteq \mathcal{F}$ for which Theorem 2 does not hold can be answered in the affirmative easily by examples: $\dim^E(\Gamma(f)) = 1$ for all $f \in \mathcal{K}$ holds if $\mathcal{K}$ consists of all constant functions or all functions that are linear in every intervall $(0, x_1)$, (x_1, x_2), ..., $(x_n, 1)$, where $0 \leq x_1, \leq \ldots \leq x_n \leq 1$ are fixed.

$\dim^H(\Gamma(f)) = 2$ for all $f \in \mathcal{K}$ holds if

$$\mathcal{K} = \{cf \mid c \in [\varepsilon, 1]\},$$

$\varepsilon > 0$ and $f \in \mathcal{F}$ arbitrary but fixed with $\dim^H(\Gamma(f)) = 2$ (such f do exist).

References

[B] W. Bosch, *Functions that preserve the uniform distribution of sequences*, Trans. Amer. Math. Soc. 307 (1988), 143-152

[F] K. J. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics 85. Cambridge University Press 1985.

[G] P. M. Gruber, *Dimension and Structure of Typical Compact Sets, Continua and Curves*, Mh. Math. 108, 149-164 (1989)

[PSS] S. Purubský, T. Šalát and O. Strauch, *Transformations that preserve uniform distribution*, Acta Arith. 49, 459-479 (1988)

[TW] R. F. Tichy and R. Winkler, *Uniform distribution preserving mappings*, Acta Arith. 50.2, 177-189 (1991)

Jörg Schmeling
Institut für Angewandte Analysis und Stochastik, Berlin
Mohrenstraße 39, D-10117
Berlin, Germany
e-mail: schmeling@iaas-berlin.dbp.de
Reinhard Winkler
Institut für Algebra und Diskrete Mathematik der TU Wien
Wiedner Hauptstraße 8-10, A-1040
or
Kommission für Mathematik der Österreichischen Akademie der Wissenschaften
Dr. Ignaz Seipelplatz 2, A-1010
Vienna, Austria
e-mail: rwin@lezvax.oeaw.ac.at

Veröffentlichungen des Instituts für Angewandte Analysis und Stochastik

Preprints 1992

1. D.A. Dawson, J. Gärtner: Multilevel large deviations.

2. H. Gajewski: On uniqueness of solutions to the drift–diffusion–model of semiconductor devices.

3. J. Fuhrmann: On the convergence of algebraically defined multigrid methods.

4. A. Bovier, J.-M. Ghez: Spectral properties of one–dimensional Schrödinger operators with potentials generated by substitutions.

5. D.A. Dawson, K. Fleischmann: A super–Brownian motion with a single point catalyst.

6. A. Bovier, V. Gayrard: The thermodynamics of the Curie–Weiss model with random couplings.

7. W. Dahmen, S. Prößdorf, R. Schneider: Wavelet approximation methods for pseudodifferential equations I: stability and convergence.

8. A. Rathsfeld: Piecewise polynomial collocation for the double layer potential equation over polyhedral boundaries. Part I: The wedge, Part II: The cube.

9. G. Schmidt: Boundary element discretization of Poincaré–Steklov operators.

10. K. Fleischmann, I. Kaj: Large deviation probability for some rescaled superprocesses.

11. P. Mathé: Random approximation of finite sums.

12. C.J. van Duijn, P. Knabner: Flow and reactive transport in porous media induced by well injection: similarity solution.

13. G.B. Di Masi, E. Platen, W.J. Runggaldier: Hedging of options under discrete observation on assets with stochastic volatility.

14. J. Schmeling, R. Siegmund–Schultze: The singularity spectrum of self-affine fractals with a Bernoulli measure.

15. A. Koshelev: About some coercive inequalities for elementary elliptic and parabolic operators.

16. P.E. Kloeden, E. Platen, H. Schurz: Higher order approximate Markov chain filters.

17. H.M. Dietz, Y. Kutoyants: A minimum–distance estimator for diffusion processes with ergodic properties.

18. I. Schmelzer: Quantization and measurability in gauge theory and gravity.

19. A. Bovier, V. Gayrard: Rigorous results on the thermodynamics of the dilute Hopfield model.

20. K. Gröger: Free energy estimates and asymptotic behaviour of reaction–diffusion processes.

21. E. Platen (ed.): Proceedings of the 1^{st} workshop on stochastic numerics.

22. S. Prößdorf (ed.): International Symposium "Operator Equations and Numerical Analysis" September 28 – October 2, 1992 Gosen (nearby Berlin).

23. K. Fleischmann, A. Greven: Diffusive clustering in an infinite system of hierarchically interacting diffusions.

24. P. Knabner, I. Kögel–Knabner, K.U. Totsche: The modeling of reactive solute transport with sorption to mobile and immobile sorbents.

25. S. Seifarth: The discrete spectrum of the Dirac operators on certain symmetric spaces.

26. J. Schmeling: Hölder continuity of the holonomy maps for hyperbolic basic sets II.

27. P. Mathé: On optimal random nets.

28. W. Wagner: Stochastic systems of particles with weights and approximation of the Boltzmann equation. The Markov process in the spatially homogeneous case.

29. A. Glitzky, K. Gröger, R. Hünlich: Existence and uniqueness results for equations modelling transport of dopants in semiconductors.

30. J. Elschner: The *h-p*–version of spline approximation methods for Mellin convolution equations.

31. R. Schlundt: Iterative Verfahren für lineare Gleichungssysteme mit schwach besetzten Koeffizientenmatrizen.

32. G. Hebermehl: Zur direkten Lösung linearer Gleichungssysteme auf Shared und Distributed Memory Systemen.

33. G.N. Milstein, E. Platen, H. Schurz: Balanced implicit methods for stiff stochastic systems: An introduction and numerical experiments.

34. M.H. Neumann: Pointwise confidence intervals in nonparametric regression with heteroscedastic error structure.

35. M. Nussbaum: Asymptotic equivalence of density estimation and white noise.

Preprints 1993

36. B. Kleemann, A. Rathsfeld: Nyström's method and iterative solvers for the solution of the double layer potential equation over polyhedral boundaries.

37. W. Dahmen, S. Prössdorf, R. Schneider: Wavelet approximation methods for pseudodifferential equations II: matrix compression and fast solution.

38. N. Hofmann, E. Platen, M. Schweizer: Option pricing under incompleteness and stochastic volatility.

39. N. Hofmann: Stability of numerical schemes for stochastic differential equations with multiplicative noise.

40. E. Platen, R. Rebolledo: On bond price dynamics.

41. E. Platen: An approach to bond pricing.

42. E. Platen, R. Rebolledo: Pricing via anticipative stochastic calculus.

43. P.E. Kloeden, E. Platen: Numerical methods for stochastic differential equations.

44. L. Brehmer, A. Liemant, I. Müller: Ladungstransport und Oberflächenpotentialkinetik in ungeordneten dünnen Schichten.

45. A. Bovier, C. Külske: A rigorous renormalization group method for interfaces in random media.

46. G. Bruckner: On the regularization of the ill–posed logarithmic kernel integral equation of the first kind.

47. H. Schurz: Asymptotical mean stability of numerical solutions with multiplicative noise.

48. J.W. Barrett, P. Knabner: Finite element approximation of transport of reactive solutes in porous media. Part I: Error estimates for non–equilibrium adsorption processes.

49. M. Pulvirenti, W. Wagner, M.B. Zavelani Rossi: Convergence of particle schemes for the Boltzmann equation.

50. J. Schmeling: Most β shifts have bad ergodic properties.

51. J. Schmeling: Self normal numbers.

52. D.A. Dawson, K. Fleischmann: Super–Brownian motions in higher dimensions with absolutely continuous measure states.

53. A. Koshelev: Regularity of solutions for some problems of mathematical physics.

54. J. Elschner, I.G. Graham: An optimal order collocation method for first kind boundary integral equations on polygons.

55. R. Schlundt: Iterative Verfahren für lineare Gleichungssysteme auf Distributed Memory Systemen.

56. D.A. Dawson, K. Fleischmann, Y. Li, C. Müller: Singularity of super–Brownian local time at a point catalyst.

57. N. Hofmann, E. Platen: Stability of weak numerical schemes for stochastic differential equations.

58. H.G. Bothe: The Hausdorff dimension of certain attractors.

59. I.P. Ivanova, G.A. Kamenskij: On the smoothness of the solution to a boundary value problem for a differential–difference equation.

60. A. Bovier, V. Gayrard: Rigorous results on the Hopfield model of neural networks.

61. M.H. Neumann: Automatic bandwidth choice and confidence intervals in nonparametric regression.

62. C.J. van Duijn, P. Knabner: Travelling wave behaviour of crystal dissolution in porous media flow.

63. J. Förste: Zur mathematischen Modellierung eines Halbleiterinjektionslasers mit Hilfe der Maxwellschen Gleichungen bei gegebener Stromverteilung.

64. A. Juhl: On the functional equations of dynamical theta functions I.

65. J. Borchardt, I. Bremer: Zur Analyse großer strukturierter chemischer Reaktionssysteme mit Waveform–Iterationsverfahren.

66. G. Albinus, H.-Ch. Kaiser, J. Rehberg: On stationary Schrödinger–Poisson equations.