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Stability of Weak Numerical Schemes for Stochastic Differential Equations

Norbert Hofmann[1] and Eckhard Platen[1,2]

Abstract. The paper considers numerical stability and convergence of weak schemes solving stochastic differential equations. A relatively strong notion of stability for a special type of test equations is proposed. These are stochastic differential equations with multiplicative noise. For different explicit and implicit schemes the regions of stability are also examined.

Keywords

numerical stability

stochastic differential equations

weak numerical schemes

implicit schemes

regions of stability

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1 Introduction

In many fields of applications, e.g. physics, finance, economics and biology stochastic differential equations are increasingly used for modelling stochastic dynamical phenomena. The corresponding stochastic differential equations are often multi-dimensional and in many cases non-linear. As for instance in filtering (see [11]) or finance (see [7]) the models necessarily involve multiplicative noise terms, that is the diffusion coefficient depends also on state variables. For the above mentioned classes of models it is inavoidable to apply appropriate numerical methods to solve practically relevant problems, e.g. to compute functionals as moments, probabilities, etc..

A selfcontained presentation of numerical methods for the solution of stochastic differential equations can be found in the monography [10]. One main issue concerning numerical schemes for stochastic differential equations is connected with their convergence and order of convergence. The schemes which we will discuss here are based for simplicity on an equidistant time discretization

$$0 = t_0 < t_1 < ... < t_N = T \quad \text{with } t_i = i\Delta,\ \Delta = T/N,\ N = 1, 2, \ldots .$$

We say a discrete time approximation $Y = \{Y_n,\ n = 0, ..., N\}$ (also called scheme) *converges strongly* for $\Delta \to 0$ with order $\gamma > 0$ towards a solution $X = \{X_t,\ 0 \le t \le T\}$ of a given stochastic differential equation if there exists a constant K (independent on Δ) and $\delta_0 > 0$ such that for all $\Delta \in (0, \delta_0)$

$$E|X_T - Y_N| \le K\Delta^{\gamma}. \tag{1}$$

This criterion corresponds to a pathwise approximation which is needed, e.g. for direct simulations to visualize a given dynamic or in filtering to solve the Zakai equation.

Often in practical situations it is only necessary to compute some functional of a solution of a stochastic differential equation. As discussed in [10] it turns out that it is much easier to construct numerical schemes which in principle approximate the underlying probability measure induced by the given stochastic dynamics than to provide strong approximations. We say that a discrete time approximation $Y = \{Y_n,\ n = 0, 1, ..., N\}$ (also called scheme) *converges weakly* for $\Delta \to 0$ with order $\beta > 0$ towards a solution $X = \{X_t,\ 0 \le t \le T\}$ of a given stochastic differential equation if for each real valued polynomial g there exists a constant K_g (independent of Δ) and $\delta_0 > 0$ such that for all $\Delta \in (0, \delta_0)$

$$|Eg(X_T) - Eg(Y_N)| \le K_g\Delta^{\beta}.$$

We note that we have under this criterion the weak convergence of order β for all moments because any moment refers to a specific polynomial g.

In [10] wide classes of strong and weak numerical schemes with different orders of convergence are presented and corresponding convergence theorems are proved there on the basis of stochastic Taylor expansions. Within this paper we will not consider the questions of convergence but we will study another practically important issue that is the numerical stability of schemes solving stochastic differential equations. The problem of stability is not primarily related to the question whether we have a strong or a weak scheme. Nevertheless we will mainly discuss in this paper weak schemes because they allow more easily to introduce implicitness in stochastic terms.

Stability of numerical schemes is strongly related to the phenomenon of stiffnes which intuitively means that a given dynamics contains at least two components which evolve with extremely different speeds. As a consequence one observes already in the case of a deterministic stiff ordinary differential equation that explicit numerical scheme cannot handle such equations properly, but implicit schemes easily do. In the stochastic case we have a similar but more complicated situation. It seems to be impossible to give a general definition for stiffness. Following [10] one could say that a general d - dimensional, $d \geq 2$, autonomous stochastic differential equation is stiff if its largest and smallest Lyapunov exponents (see [5]) differ extremely. This refers to the fact that there are widely differing time scales present in the solution. But in practical numerics one can meet also other situations which one should also call stiff. For instance the above definition does not cover the case that a dynamics involves at the same time very fast and also slow rotations. In the numerical practice it is most recommended to study schemes of interest at test equations which involve typical features of the practically important dynamics. After recalling the classical complex valued test equation for deterministic equations and its generalization to the stochastic case with additive noise we will propose a test equation for a case of multiplicative noise. Such dynamics with multiplicative noise are extremely important in financial and economical models but also in filtering and several physical applications. We will generalize the notion well - known of A-stability in a relatively strong sense and study the regions of stability for a number of stochastic numerical schemes.

2 Stability for Additive Noise

If we want to apply numerical methods to Ito stochastic differential equations of the form

$$dX_t = a(t, X_t)dt + b(t, X_t)\,dW_t$$

it is wise to examine their regions of stability. The knowledge about the stability of a numerical method for a given stochastic differential equation is most crucial for the decision whether the method is appropriate or not. The existence of the noise in the stochastic case provides a number of difficulties which we do not face for ordinary differential equations. Before we consider the situation for stochastic differential equations we recall the most basic concept of stability for deterministic numerical schemes, see [2], [4]. In the deterministic sense numerical stability of a one - step method

$$Y_{n+1} = Y_n + \Psi(t_n, Y_n, Y_{n+1}, \Delta)\Delta \tag{2}$$

with an increment function $\Psi = \Psi(t, x, Y, \Delta)$ means roughly that the propagation of an initial error will remain bounded for a given ordinary differential equation

$$\frac{dx}{dt} = a(t, x) \tag{3}$$

where the drift function $a(t, x)$ satisfies a Lipschitz condition. More precisely we say that a deterministic one - step method (2) is called numerically stable if for each time interval $[t_0, T]$ and given differential equation (3) there exist positive constants Δ_0 and M such that

$$|Y_n - \tilde{Y}_n| \leq M|Y_0 - \tilde{Y}_0| \tag{4}$$

for all $n = 0, 1, ..., N, \Delta < \Delta_0$, and any two solutions Y, $\tilde{Y}$ of (2) corresponding to the initial values $Y_0, \tilde{Y}_0$, respectively. Here the constant M can be quite large.

In order to ensure that the error does not grow considerably over an infinite time horizon one can introduce the notion of asymptotic numerical stability. A deterministic one - step method (2) is called asymptotically numerically stable for a given differential equation if there exist positive constants Δ_0 and M such that

$$\lim_{n\to\infty} |Y_n - \tilde{Y}_n| \leq M|Y_0 - \tilde{Y}_0| \tag{5}$$

for any two solutions Y, $\tilde{Y}$ of (2) defined as above corresponding to any time discretization with $\Delta < \Delta_0$.
From the practical point of view one is not only interested in the problem whether a method is numerically stable or not. Moreover one asks for appropriate step sizes Δ which one can choose for a given scheme applied to a specific differential equation. For this purpose one considers conveniently classes of test equations. Well - known is the complex valued test equation

$$\frac{dx}{dt} = \lambda x \tag{6}$$

with $\lambda = \lambda_1 + \lambda_2 i$ where (6) is equivalent to the 2 - dimensional differential equation

$$d\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} dt \tag{7}$$

whith $x = x^1 + x^2 i$. To decide which step sizes one can use for a given scheme applied to (7) it is helpful to study the region of stability of the scheme. If one can write a numerical scheme applied to (7) in a recursive form as

$$Y_{n+1} = G(\lambda\Delta)Y_n, \tag{8}$$

where G is a complex valued function, then the set of all complex numbers $\lambda\Delta$ with $|G(\lambda\Delta)| < 1$ describes the region of absolute stability of the scheme. For instance in the case of the explicit Euler scheme

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta \tag{9}$$

we have

$$Y_{n+1} = (1 + \lambda\Delta)Y_n$$

and the region of absolute stability is an open unit disc centered at the point $-1 + 0i$.
On the other hand for the implicit Euler scheme

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta \tag{10}$$

we obtain a region of stability which covers at least the left complex half plane. Such a deterministic scheme is called A-stable.
Let us now start our discussion for the stochastic case. Including real valued additive noise in the test equation (6) leads to a simple stochastic generalization of the concept of asymptotic numerical stability. For the resulting class of test equations

$$dX_t = \lambda X_t dt + dW_t, \tag{11}$$

where the parameter λ is a complex number with $Re(\lambda) < 0$ and W is a real valued standard Wiener process, the regions of stability for some stochastic numerical schemes were considered in [9]. Similar investigations can be found in [12], [6] or [8]. Under the assumption that a given scheme with equidistant step size Δ applied to the test equation (11) with $Re(\lambda) < 0$ allows a representation in the form

$$Y_{n+1} = G(\lambda\Delta)Y_n + Z_n \tag{12}$$

for $n = 0, 1, ...,$ where G is a complex valued function and $Z_0, Z_1, ...$ represent random variables which do not depend on λ or $Y_0, ..., Y_{n+1}$, analogously as above the set of complex numbers $\lambda\Delta$ with $\lambda_1 = Re(\lambda) < 0$ and $|G(\lambda\Delta)| < 1$ is called the region of absolute stability of the scheme . For example we know from [9] that the region of absolute stability for the explicit Euler scheme

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta + b(t_n, Y_n)\Delta W_n \tag{13}$$

is the same as in the deterministic case, namely the interior of a unit circle with the centre in the point $-1 + 0i$. Similar as in the deterministic case one also has the notion of A-stability for stochastic schemes. We say that a stochastic scheme is A-stable if its region of absolute stability is the whole left half of the complex plane. Of course an A-stable stochastic scheme is also A-stable in the deterministic sense for an ordinary differential equation.

If we want to use stochastic numerical schemes to solve applied problems we have to solve only in rare cases such simple equations as (11). The underlying stochastic dynamics is completely different if the diffusion coefficient is more complicated and depends on the state. Then it is recommended to introduce another reasonable notion of stability closely related to the given stochastic differential equation. This can be achieved by the study of stability regions for specific schemes with respect to a well chosen test equation. The aim of this paper is to provide such a notion for stochastic differential equations with multiplicative noise and to use it to find the regions of stability for given numerical methods with respect to an appropriate class of test equations. This specific class of stochastic differential equations will be complex valued generalizing the above deterministic test equation (6) and involving the effect of multiplicative noise. A similar problem is considered in [13], [14].

3 A Notion of Strong Stability for Multiplicative Noise

In this section we introduce a concept of numerical stability for stochastic differential equations with multiplicative noise. For this purpose we consider the class of complex valued test equations of Stratonovich type

$$dX_t = (1 - \alpha)\lambda X_t dt + \sqrt{\alpha}\gamma X_t \circ dW_t\,, \tag{14}$$

where $\lambda = \lambda_1 + \lambda_2 i$ and $\gamma = \gamma_1 + \gamma_2 i$ are complex numbers, W is a real standard Wiener process and the parameter α is a real positive number, $\alpha \in [0, 2]$. The stochastic differential on the right hand side of (14) shall be understood as Stratonovich differential (see [10]). Changing the parameter α shifts the weights between deterministic and stochastic integrals in the equation. For $\alpha = 0$ we have the purely deterministic test equation (6). For $\alpha = 1$ we have a Stratonovich equation without drift. In order to simplify the description of regions of stability we will consider only such equations (14) for which there is a suitable connection between the parameters λ and γ. It turns out to be convenient to choose

$$\gamma_1 = \lambda_2 \cdot \frac{1}{2\gamma_2} \quad \text{and} \quad \gamma_2 = \sqrt{\frac{1}{2}(|\lambda| + \lambda_1)}\, i. \tag{15}$$

With this choice which we assume in the following we have

$$\gamma^2 = \lambda. \tag{16}$$

We remark that it follows in this case for $\alpha = 2$ that (14) corresponds to an Ito equation with no drift component. A systematic case study (which we omit here) shows that most other choices of γ fulfilling (16) lead to unproper stability regions and one can interpret (15) as a natural choice for γ.
Suppose that we can write a given stochastic scheme with equidistant step size Δ applied to a test equation which belongs to the class (14) in the recursive form

$$Y_{n+1} = G(\lambda\Delta,\, \alpha)Y_n, \tag{17}$$

where G is a complex valued function which is random and which does not depend on $Y_0, ..., Y_{n+1}$. We suppress in the mapping (17) the dependence on $\gamma\sqrt{\Delta}$ because γ is related to λ according to (16). Then we shall say for a given $\alpha \in [0,2]$ that the subset Γ_α of the complex plane with

$$\Gamma_\alpha = \left\{\lambda\Delta \in C : \text{Re}(\lambda) < 0,\ \text{ess}_\omega \sup |G(\lambda\Delta, \alpha)|^2 < 1\right\} \tag{18}$$

forms the region of *strong stability* of the scheme. Here $\text{ess}_\omega \sup$ denots the essential supremum with respect to all $\omega \in \Omega$. We introduce regions $\Gamma = \{\Gamma_\alpha : 0 \leq \alpha \leq 2\}$ as the family of stability regions. If for a given $\alpha \in [0,2]$ the region of strong stability Γ_α contains the whole left half plane, then we call the scheme strongly A-stable for this α. Usually we will have strong A-stability only for some α forming subintervals of $[0,2]$. This definition generalizes the notion of A-stability for deterministic ordinary differential equations which correspond to the case $\alpha = 0$. The main difference between this multiplicative noise case characterized by the test equation (14) and the additive noise case described by (11) is that we cannot easily express the recursive representation of a given scheme in terms of a deterministic complex mapping and a random variable which is separated from the mapping. That means it remains a complex mapping which involves random variables. So, in some sense we have to consider all possible realizations of this random function. By using here the essential supremum of the mapping to characterize the region of strong stability we consider the worst case. But also other weaker stability concepts can be useful which we do not consider here.
Now, let us investigate whether the choice of our class of test equations is reasonable. For this we have to examine first the stability of the test dynamics itself (see [5]). Obviously, it is helpful to check whether for every $t \in [0,\infty)$ and $\alpha \in [0,2]$ the p th moments of X_t remain finite. To discuss this we refer at first to the fact that the explicit solution of (14) is

$$X_t = \exp\{(1-\alpha)\lambda t + \sqrt{\alpha}\gamma W_t\}X_0\,. \tag{19}$$

We can understand X_t^p, $p \geq 0$, as solution of the Stratonovich equation

$$X_t^p = X_0^p + \int_0^t p(1-\alpha)\lambda X_s^p ds + \int_0^t p\sqrt{\alpha}\gamma X_s^p \circ dW_s. \tag{20}$$

Rewriting (20) in the corresponding Ito form leads to

$$X_t^p = X_0^p + \int_0^t \left(p(1-\alpha)\lambda + \frac{1}{2} p^2 \alpha \gamma^2 \right) X_s^p ds + \int_0^t p\sqrt{\alpha}\gamma X_s^p dW_s. \tag{21}$$

With the help of the equations (21) and (16) we can derive the following expression for the p th moment

$$E(X_t^p) = \exp\left\{ p\lambda \left(1 - \alpha + \frac{1}{2} p\alpha\right) t \right\} E(X_0^p). \tag{22}$$

From (22) it follows

$$|E(X_t^p)| = |E(X_0^p)| \exp\left\{ \lambda_1 tp \left(1 + \alpha\left(\frac{p}{2} - 1\right)\right)\right\}. \tag{23}$$

Thus, we get under the condition $\lambda_1 = Re(\lambda) \leq 0$ for $p \geq 0$ and $\alpha \in [0,1]$ the estimate

$$|E(X_t^p)| \leq |E(X_0^p)|\,.$$

That means in this case the test equation is stable for all moments. This shows that the restriction $Re(\lambda) < 0$ in (18) is reasonable for $\alpha \leq 1$. In the case $\alpha \in [1,2]$ we have stability for the moments of order $p \geq 2(1 - \frac{1}{\alpha})$.

4 Stability of the Explicit Euler Scheme

In this section we want to derive the family of stability regions for the explicit Euler scheme (13). To be more precise we consider the simplified Euler scheme which is suitable for weak approximation. It has the form

$$Y_{n+1} = Y_n + a\,(t_n, Y_n)\Delta + b\,(t_n, Y_n)\sqrt{\Delta}\,\xi_n, \tag{24}$$

where ξ_n is an independent two - point distributed random variable with $P(\xi_n = \pm 1) = \frac{1}{2}$. Applied on the test equation (14) one obtains

$$Y_{n+1} = \left(1 + \left(1 - \frac{\alpha}{2}\right)\lambda\Delta + \sqrt{\alpha}\,\gamma\sqrt{\Delta}\,\xi_n\right) Y_n. \tag{25}$$

So, we have a recursive representation of the scheme involving a complex random mapping G with

$$G(\lambda\Delta, \alpha) = 1 + \left(1 - \frac{\alpha}{2}\right)\lambda\Delta + \sqrt{\alpha}\,\gamma\sqrt{\Delta}\,\xi_n$$

corresponding to the nth time step. We have chosen γ in the form (15) and it follows for fixed $\alpha \in [0,2]$

$$\begin{aligned} G(\lambda\Delta, \alpha) &= 1 + \left(1 - \frac{\alpha}{2}\right)(\lambda_1\Delta + \lambda_2\Delta\, i) \\ &\quad - \sqrt{\alpha}\left(\sqrt{\frac{1}{2}(|\lambda| + \lambda_1)} + \frac{\lambda_2\, i}{\sqrt{2(|\lambda| + \lambda_1)}}\right)\sqrt{\Delta}\,\xi_n. \end{aligned} \tag{26}$$

Then looking at the conditions on the set Γ_α in (18) leads to

$$
\begin{aligned}
H_\alpha(\lambda_1\Delta, \lambda_2\Delta) &:= \operatorname{ess}_\omega \sup |G(\lambda\Delta, \alpha)|^2 \\
&= \operatorname{ess}_\omega \sup \left[\left(1 + \left(1 - \frac{\alpha}{2}\right) \lambda_1\Delta - \sqrt{\frac{\alpha\Delta}{2}(|\lambda| + \lambda_1)}\, \xi_n \right)^2 \right. \\
&\quad \left. + \left(\left(1 - \frac{\alpha}{2}\right) \lambda_2\Delta - \lambda_2 \sqrt{\frac{\alpha\Delta}{2(|\lambda| + \lambda_1)}}\, \xi_n \right)^2 \right] \\
&= \left(1 + \left(1 - \frac{\alpha}{2}\right) \lambda_1\Delta\right)^2 + \left(1 - \frac{\alpha}{2}\right)^2 (\lambda_2\Delta)^2 \\
&\quad + \alpha|\lambda|\Delta + \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)} \left(1 + \left(1 - \frac{\alpha}{2}\right) |\lambda|\Delta\right) \qquad (27)
\end{aligned}
$$

and for a given $\alpha \in [0, 2]$ we have to find
$\Gamma_\alpha = \{\lambda\Delta \in C : \lambda_1\Delta < 0,\ H_\alpha(\lambda_1\Delta, \lambda_2\Delta) < 1\}$.
Figure 1 shows the family of stability regions for the explicit Euler scheme, where we denoted $\lambda_1\Delta$ by x, $\lambda_2\Delta$ by y and α by z. This notation of the axes will be used also for each of the following figures.

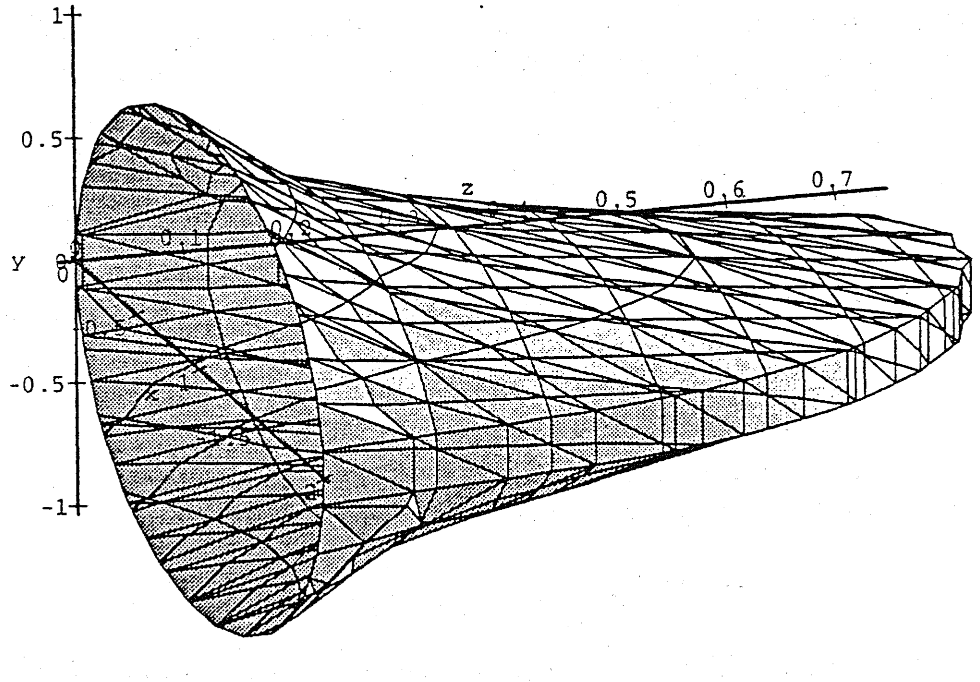

Figure 1 Family of stability regions for the explicit Euler scheme.

For the purely deterministic case $\alpha = 0$ we have Γ_α as the interior of the unit circle centered at $-1 + 0i$. For $\alpha \in (0, 1)$ we obtain that Γ_α is the interior of a subset of Γ_0, an ellipse which is symmetrical with regard to the $\lambda_1\Delta$- axis (x - axis). We note that the explicit Euler scheme becomes unstable in our sense for $\alpha \geq 1$ because there is no step size for which we have any strong stability. Thus, if the noise intensity is too large we cannot apply the explicit Euler scheme if we like to have a scheme which is strongly stable.

5 Drift Implicit Euler Scheme

Now, we want to investigate under which conditions the application of the drift implicit Euler scheme

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta + b(t_n, Y_n)\sqrt{\Delta}\,\xi_n \tag{28}$$

increases the strong stability, with respect to our test equation (14). Here the random variables ξ_n are again chosen as above. This method applied to equation (14) yields

$$Y_{n+1} = Y_n + \left((1-\alpha)\lambda Y_{n+1} + \frac{1}{2}\alpha\gamma^2 Y_{n+1}\right)\Delta + \sqrt{\alpha}\gamma Y_n \sqrt{\Delta}\xi_n.$$

It follows with (16)

$$\left(1 - \left(1 - \frac{\alpha}{2}\right)\lambda\Delta\right) Y_{n+1} = (1 + \sqrt{\alpha}\,\gamma\,\sqrt{\Delta}\,\xi_n)Y_n$$

that is

$$Y_{n+1} = \left(1 - \left(1 - \frac{\alpha}{2}\right)\lambda\Delta\right)^{-1} \cdot (1 + \sqrt{\alpha}\,\gamma\,\sqrt{\Delta}\,\xi_n)Y_n. \tag{29}$$

The recursive representation (29) of the scheme involves the complex mapping G with

$$G(\lambda\Delta, \alpha) = \frac{\left(1 - \sqrt{\alpha}\left(\sqrt{\frac{1}{2}(|\lambda| + \lambda_1)} + \frac{\lambda_2 i}{\sqrt{2(|\lambda|+\lambda_1)}}\right)\cdot\sqrt{\Delta}\xi_n\right)}{(1 - (1 - \frac{\alpha}{2})(\lambda_1\Delta + \lambda_2\Delta i))} \tag{30}$$

where γ was choosen as in (15).
Hence, for fixed $\alpha \in [0,2]$ we obtain

$$\begin{aligned} H_\alpha(\lambda_1\Delta, \lambda_2\Delta) &= \operatorname{ess}_\omega \sup \left[\frac{1 - \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)}\,\xi_n + \alpha|\lambda|\Delta}{(1 - (1 - \frac{\alpha}{2})\lambda_1\Delta)^2 + (1 - \frac{\alpha}{2})^2(\lambda_2\Delta)^2}\right] \\ &= \frac{1 + \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)} + \alpha|\lambda|\Delta}{(1 - (1 - \frac{\alpha}{2})\lambda_1\Delta)^2 + (1 - \frac{\alpha}{2})^2(\lambda_2\Delta)^2}. \end{aligned} \tag{31}$$

For $\alpha \in [0,2]$ we find the corresponding region of strong stability Γ_α according to (18) as already described in section 4. In the deterministic case $\alpha = 0$ the region of strong stability for the drift implicit Euler scheme is the exterior of the unit disc centered at $1+0i$. Thus in this case the scheme is strongly stable in the whole left half of the complex plane, that is the scheme is strongly A-stable. The family of strong stability regions for $\alpha \in [0,1]$ represents the exterior of the conical object visualized in Figure 2.

We note, that increasing noise intensity destroys more and more the strong stability of the scheme. If we consider our result more precisely, then we find that the scheme is strongly A-stable for $\alpha \in [0, 0.2]$, while for $\alpha > 0.2$ the region where the scheme is not strongly stable grows into the left half of the complex plane. Figure 3 shows also what happens with the strong stability regions of the scheme for $1 \leq \alpha \leq 2$.

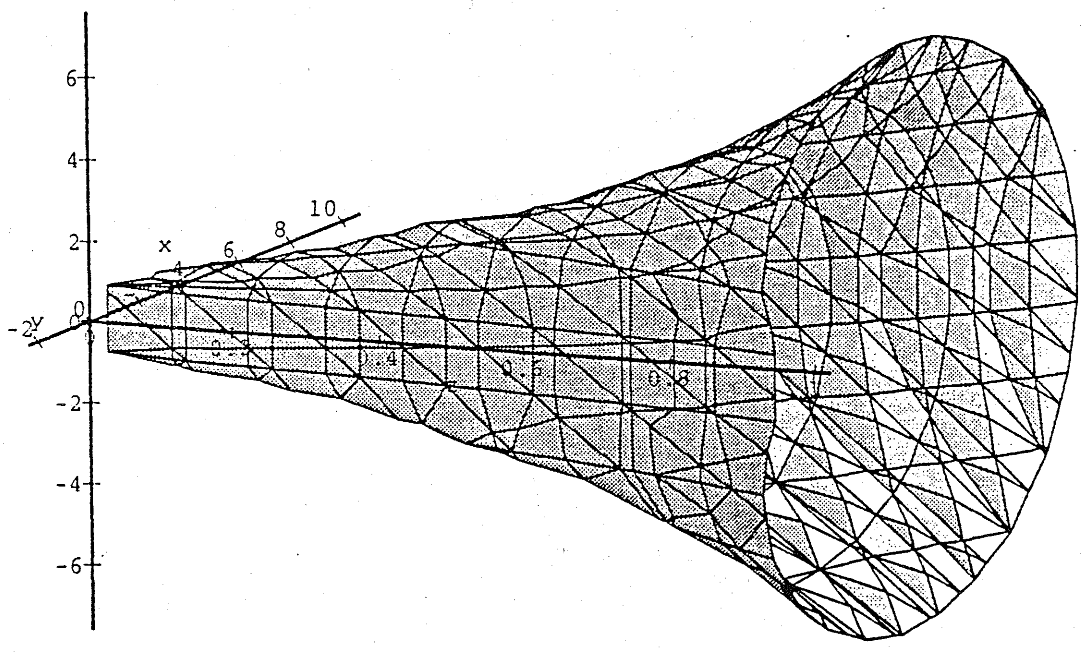

Figure 2 Family of stability regions for the drift implicit Euler scheme ($\alpha \in [0,1]$).

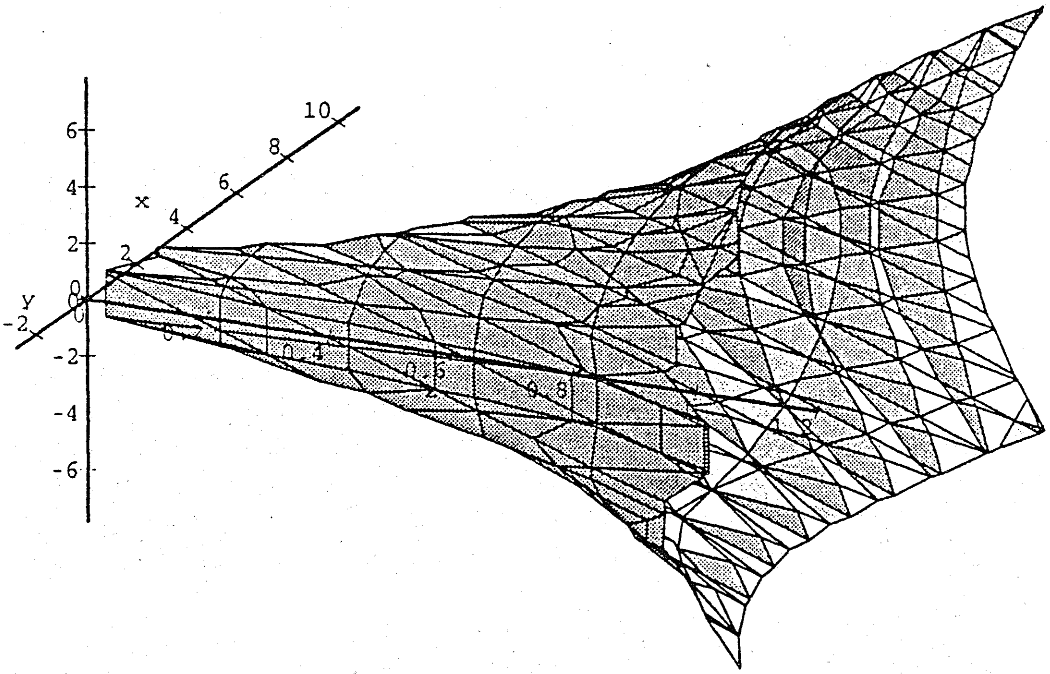

Figure 3 Family of stability regions for the drift implicit Euler scheme ($\alpha \in [0,2]$).

We observe in Figure 3 for increasing α a considerable growth of the conical object. That means we obtain strong stability only for very large step sizes. Thus the drift implicit Euler scheme becomes with respect to the strong stability criterion unsuitable.

6 Fully Implicit Euler Scheme

Once again we start with equation (14) and now use the fully implicit Euler scheme

$$\begin{aligned} Y_{n+1} = \quad & Y_n + \Big(a\,(t_{n+1}, Y_{n+1}) - b\,(t_{n+1}, Y_{n+1}) \\ & \cdot \tfrac{\partial}{\partial y} b(t_{n+1}, Y_{n+1})\Big)\,\Delta + b(t_{n+1}, Y_{n+1})\sqrt{\Delta}\xi_n \end{aligned} \tag{32}$$

to obtain

$$Y_{n+1} = Y_n + \left(1 - \frac{3}{2}\alpha\right)\lambda Y_{n+1}\Delta + \sqrt{\alpha}\gamma Y_{n+1}\sqrt{\Delta}\xi_n.$$

So we have

$$\left(1 - \left(1 - \frac{3}{2}\alpha\right)\lambda\Delta - \sqrt{\alpha}\gamma\sqrt{\Delta}\xi_n\right) Y_{n+1} = Y_n$$

that is

$$Y_{n+1} = \left(1 - \left(1 - \frac{3}{2}\alpha\right)\lambda\Delta - \sqrt{\alpha}\gamma\sqrt{\Delta}\xi_n\right)^{-1} Y_n. \tag{33}$$

Here we also choose ξ_n as two - point distributed random variable with $P(\xi_n = \pm 1) = \frac{1}{2}$. By using (15) we proceed in the same manner as above to find the family of stability regions. Then for $\alpha \in [0, 2]$ we get

$$\begin{aligned} H_\alpha(\lambda_1\Delta, \lambda_2\Delta) \quad = \quad & \text{ess}_\omega \sup \Bigg[\Bigg(\left(1 - \left(1 - \frac{3}{2}\alpha\right)\lambda_1\Delta\right)^2 + \left(1 - \frac{3}{2}\alpha\right)^2 (\lambda_2\Delta)^2 \\ & + \alpha|\lambda|\Delta + \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)}\left(1 - \left(1 - \frac{3}{2}\alpha\right)|\lambda|\Delta\right)\cdot\xi_n\Bigg)^{-1}\Bigg] \\ = \quad & \Bigg|\left(1 - \left(1 - \frac{3}{2}\alpha\right)\lambda_1\Delta\right)^2 + \left(1 - \frac{3}{2}\alpha\right)^2 (\lambda_2\Delta)^2 \\ & + \alpha|\lambda|\Delta - \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)}\,|1 - \left(1 - \frac{3}{2}\alpha\right)|\lambda|\Delta\,|\Bigg|^{-1}. \end{aligned} \tag{34}$$

Finally applying (18) provides the corresponding Γ_α. If α runs through the interval $[0, 1]$, then the fully implicit Euler scheme is stable outside the geometrical object that is plotted in Figure 4.

Of course, in the purely deterministic case $\alpha = 0$ the region of strong stability is the exterior of the unit disc centered at $1 + 0i$ as for the drift implicit Euler scheme. Furthermore, we find out that the scheme remains strongly A-stable until $\alpha \approx 0.3$. After that for increasing $\alpha \leq 1$ we observe again a destabilizing effect of the noise, so that it is not recommended to choose small step sizes. In the case $\alpha = 1$ the scheme is only strongly stable outside the shown relatively large cardioid. In Figure 5 we observe that the scheme becomes more stable again for $\alpha > 1$.

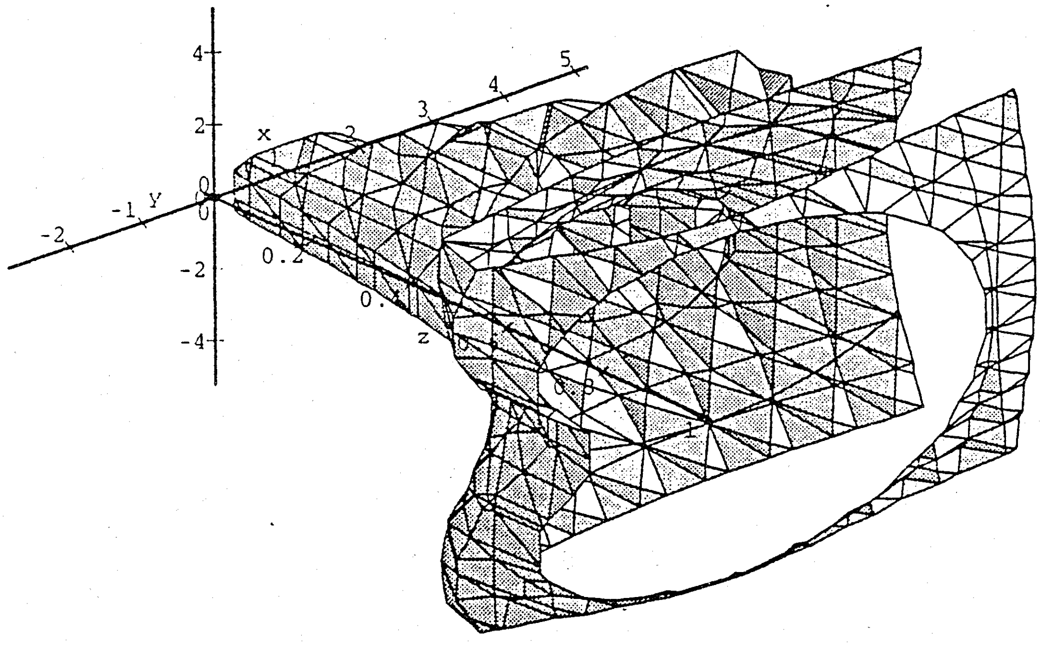

Figure 4 Family of stability regions for the fully implicit Euler scheme ($\alpha \in [0,1]$).

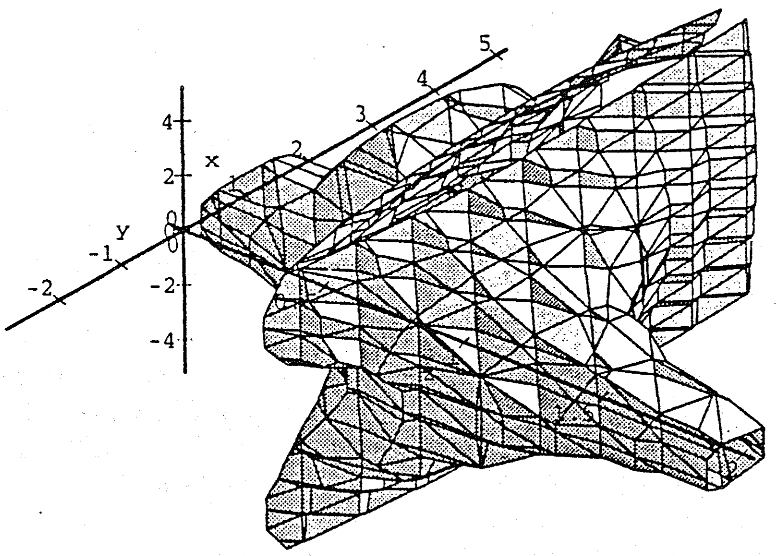

Figure 5 Family of stability regions for the fully implicit Euler scheme ($\alpha \in [0,2]$).

Thus increasing α on $(1,2]$ stabilizes the scheme. As we mentioned already the case $\alpha = 2$ is connected with an Ito stochastic differential equation without drift. For this case it turns out to be useful to apply the fully implicit Euler scheme because it shows good strong stability properties.

7 Symmetrical Implicit Euler Schemes

Similar as in deterministic numerics implicit schemes with symmetries are of special interest for solving stochastic differential equations. These are schemes for which the degree of implicitness is $\frac{1}{2}$.
At first we consider the symmetrical drift implicit Euler scheme

$$Y_{n+1} = Y_n + \frac{1}{2}(a(t_n, Y_n) + a(t_{n+1}, Y_{n+1}))\Delta + b(t_n, Y_n)\sqrt{\Delta}\xi_n, \tag{35}$$

where the random variables ξ_n are chosen as above. Applying (35) to the test equation (14) yields

$$Y_{n+1} = Y_n + \frac{1}{2}\left(1 - \frac{\alpha}{2}\right)\lambda(Y_n + Y_{n+1})\Delta + \sqrt{\alpha}\gamma Y_n\sqrt{\Delta}\xi_n.$$

It follows

$$Y_{n+1} = \frac{\left(1 + \frac{1}{2}\left(1 - \frac{1}{2}\alpha\right)\lambda\Delta + \sqrt{\alpha}\gamma\sqrt{\Delta}\xi_n\right)Y_n}{\left(1 - \frac{1}{2}\left(1 - \frac{1}{2}\alpha\right)\lambda\Delta\right)}. \tag{36}$$

We continue in the familiar way to obtain

$$\begin{aligned} H_\alpha(\lambda_1\Delta, \lambda_2\Delta) &= \left[(1 - \frac{1}{2}(1 - \frac{1}{2}\alpha)\lambda_1\Delta)^2 + \frac{1}{4}(1 - \frac{1}{2}\alpha)^2(\lambda_2\Delta)^2\right]^{-1} \\ & \quad \left[(1 + \frac{1}{2}(1 - \frac{1}{2}\alpha)\lambda_1\Delta)^2 + \frac{1}{4}(1 - \frac{1}{2}\alpha)^2(\lambda_2\Delta)^2\right. \\ & \quad \left. + \alpha|\lambda|\Delta + \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)}\,(1 + \frac{1}{2}(1 - \frac{1}{2}\alpha)|\lambda|\Delta)\right] \end{aligned} \tag{37}$$

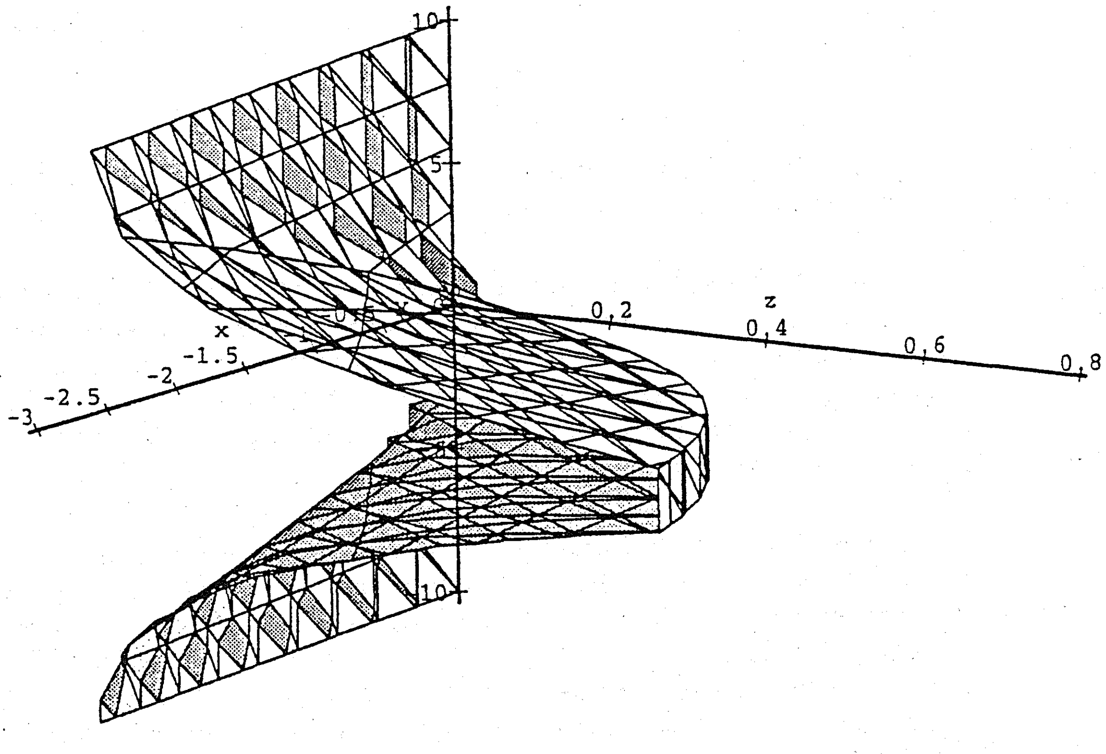

Figure 6 Family of stability regions for the symmetrical drift implicit Euler scheme.

The resulting object in Figure 6 extends into the left half plane. We have strong A-stability only for $\alpha = 0$. Increasing α contracts the region of strong stability more and more until it becomes an empty set.
If we introduce symmetry also in the diffusion coefficient, then we obtain the symmetrical implicit Euler scheme

$$\begin{aligned} Y_{n+1} = \; & Y_n + \tfrac{1}{2}(\underline{a}(t_n, Y_n) + \underline{a}(t_{n+1}, Y_{n+1}))\Delta \\ & + \tfrac{1}{2}(b(t_n, Y_n) + b(t_{n+1}, Y_{n+1}))\sqrt{\Delta}\xi_n. \end{aligned} \tag{38}$$

Other than in the scheme (35) we have here to use the corrected Stratonovich drift $\underline{a} = a - \frac{1}{2}bb'$. The scheme can be applied directly to our test equation (14), because it is already a Stratonovich equation. We get

$$\begin{aligned} Y_{n+1} = \; & Y_n + \tfrac{1}{2}(1-\alpha)(Y_n + Y_{n+1})\lambda\Delta \\ & + \tfrac{1}{2}\sqrt{\alpha}\gamma(Y_n + Y_{n+1})\sqrt{\Delta}\xi_n. \end{aligned}$$

Hence we have

$$\begin{aligned} Y_{n+1} = \; & (1 - \tfrac{1}{2}((1-\alpha)\lambda\Delta + \sqrt{\alpha}\gamma\sqrt{\Delta}\xi_n))^{-1} \\ & (1 + \tfrac{1}{2}((1-\alpha)\lambda\Delta + \sqrt{\alpha}\gamma\sqrt{\Delta}\xi_n))Y_n \end{aligned} \tag{39}$$

and finally for fixed $\alpha \in [0, 2]$ it follows

$$\begin{aligned} H_\alpha(\lambda_1\Delta, \lambda_2\Delta) \; = \; & \Big|(1 - \frac{1}{2}(1-\alpha)\lambda_1\Delta)^2 + \frac{1}{4}(1-\alpha)^2(\lambda_2\Delta)^2 + \alpha|\lambda|\Delta \\ & - \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)}\; | \, 1 - \frac{1}{2}(1-\alpha)|\lambda|\Delta| \Big|^{-1} \cdot \\ & \Big((1 + \frac{1}{2}(1-\alpha)\lambda_1\Delta)^2 + \frac{1}{4}(1-\alpha)^2(\lambda_2\Delta)^2 + \alpha|\lambda|\Delta \\ & + \sqrt{2\alpha\Delta(|\lambda| + \lambda_1)}\; (1 + \frac{1}{2}(1-\alpha)|\lambda|\Delta)\Big). \end{aligned} \tag{40}$$

By using (40) we obtain under the well - known condition from (18) the family of strong stability regions for the symmetrical implicit Euler scheme which is plotted in Figure 7.

Concerning the strong stability of this scheme we only find a small improvement compared with the symmetrical drift implicit Euler scheme. The figure shows strong A-stability in the deterministic case $\alpha = 0$ again and a fast shrinking of the region of strong stability if α tends to 1. If one increases the parameter α, then it becomes more difficult to find a sufficiently small step size for which the scheme is strongly stable. For $\alpha \geq 1$ the scheme is not strongly stable in the whole complex plane.

Compared with the fully implicit Euler scheme neither a symmetrical nor an explicit method is better suited to improve the strong stability for large α. Obviously, we could not mention any stochastic numerical scheme related to the test equation (14) that is strongly A-stable for every $\alpha \in [0, 2]$. If one increases the intensity of the noise, then the considered schemes became unstable at least for small step sizes. On the other hand choosing large step sizes reduces the accuracy of the approximation. It remains an open problem to develop methods for which increasing noise does not destroy the strong stability. Here the fully implicit Euler scheme showed the best strong stability properties.

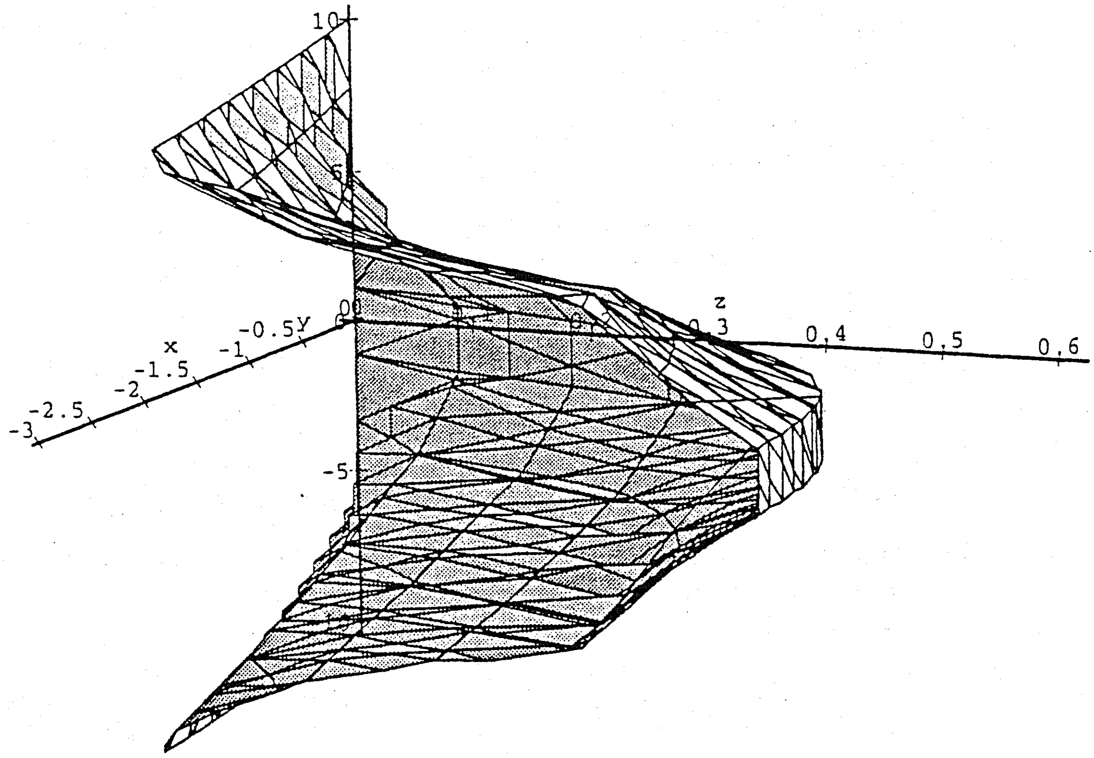

Figure 7 Family of stability regions for the symmetrical implicit Euler scheme.

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