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THE SINGULARITY SPECTRUM OF SELF–AFFINE FRACTALS WITH A BERNOULLI MEASURE

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1 Introduction

Since the eighties an important idea to understand the long–time behavior of orbits was that the characteristic invariant sets (for instance attractors) arising in dynamical systems should be regarded as the supports of some invariant measures and these measures should be characterized by certain singularities. Considering a compact set $F \subset R^d$ equipped with a measure μ we are interested in subsets $K_\alpha \subseteq F$ with a given scaling law

$$K_\alpha = \left\{ x \in F : \text{ there exists } \lim_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon} \text{ and it equals } \alpha \right\}$$

($B_\varepsilon(x)$ denotes the ball with radius ε and centre x. If the centre is the origin we write simply B_ε).
Let $f(\alpha)$ denote the Hausdorff dimension $\dim_H$ of K_α. This function is the so called singularity– or $f(\alpha)$–spectrum.
Another characterization can be given by the Renyi dimension spectrum, which to each real q associates a dimensionlike value D_q.
A heuristical approach suggests that α, $f(\alpha)$ and q, D_q should be related by the Legendre transform. Our goal is to verify these heuristics in the self–affine case described below.
We consider a finite set of non–singular linear contractions $T_1, \ldots, T_k$ of some

euclidean space $\mathbf{R}^d$. So there are numbers a, a' in $(0,1)$ with

$$a'|\mathbf{x}| \leq |T_i\mathbf{x}| \leq a|\mathbf{x}|, \qquad \mathbf{x} \in \mathbf{R}^d, 1 \leq i \leq k \ .$$

To each k–tupel $(\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathbf{R}^{dk}$ we assign the collection of affine mappings $\{S_i\}_{1,2,\ldots,k} := \{\mathbf{a}_i + T_i\}_{i=1,2,\ldots,k}$ and we define a subset $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ of $\mathbf{R}^d$ by the set–up

$$\bigcup_{\mathbf{i} \in \mathcal{J}_\infty} \{\mathbf{a}_{i_1} + T_{i_1}\mathbf{a}_{i_2} + T_{i_1}T_{i_2}\mathbf{a}_{i_3} + \ldots\} \ .$$

Let us denote by $\mathcal{J}_\infty$ the set $\{1,2,\ldots,k\}^{\mathbf{N}}$ of all infinite sequences of symbols in $\{1,2,\ldots,k\}$. The set of finite sequences of length n we denote by $\mathcal{J}_n$, and finally we write $\mathcal{J}$ for the set of all finite sequences, i.e. $\mathcal{J} = \bigcup_{n \geq 0} \mathcal{J}_n$. Here $\mathcal{J}_0$ denotes the set consisting only of the empty sequence. If $\mathbf{i}, \mathbf{j}$ are two sequences, where the first one is finite, we write $\mathbf{i} \cdot \mathbf{j}$ to denote their combination, and $\mathbf{i} \leq \mathbf{j}$ in the case where $\mathbf{j}$ has $\mathbf{i}$ as starting sequence. If i is in $\{1,2,\ldots,k\}$, we write $i \cdot \mathbf{i}$ for the combination of the starting element i and the sequence $\mathbf{i}$. For two sequences $\mathbf{i}, \mathbf{j}$ we write $\mathbf{i} \wedge \mathbf{j}$ to denote their common starting sequence, which is simply the empty sequence in case $\mathbf{i}$ and $\mathbf{j}$ have different starting elements. If $\mathbf{i}$ is a sequence, we denote its (possibly infinite) length by $|\mathbf{i}|$. If $n \in \{0,1,2,\ldots,|\mathbf{i}|\}$, we denote by $\mathbf{i}(n)$ the starting sequence of length n of $\mathbf{i}$.
We equip $\mathcal{J}_\infty$ with the natural Tychonov product topology. This makes $\mathcal{J}_\infty$ to a Cantor set.
Consequently, $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ is the set of all points

$$\pi(\mathbf{i}) := \mathbf{a}_{i_1} + T_{i_1}\mathbf{a}_{i_2} + T_{i_1}T_{i_2}\mathbf{a}_{i_3} + \ldots, \qquad i \in \mathcal{J}_\infty \ ,$$

where the existence of $\pi(\mathbf{i})$ is a trivial consequence of our assumptions, and the mapping $\pi(.)$ is continuous from $\mathcal{J}_\infty$ onto $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$. Of course, π depends upon $\mathbf{a} := (\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathbf{R}^{dk}$, and if necessary we write $\pi_{\mathbf{a}}(\mathbf{i})$. From the continuity of π and the compactness of $\mathcal{J}_\infty$ we derive that $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ is a compact. So it is a measurable set.
Let us extend this definition of π to finite sequences $\mathbf{i} \in \mathcal{J}$ by

$$\pi(\mathbf{i}) := \mathbf{a}_{i_1} + T_{i_1}\mathbf{a}_{i_2} + \ldots + T_{i_1}T_{i_2}\ldots T_{i_{n-1}}\mathbf{a}_{i_n} = S_{\mathbf{i}}(0), \qquad \mathbf{i} \in \mathcal{J}_n \ .$$

We have an alternative characterization of $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$. Take some radius $\gamma = \gamma(\mathbf{a})$ such that $S_i(B_\gamma) = \mathbf{a}_i + T_i(B_\gamma) \subseteq B_\gamma$, $i = 1,2,\ldots,k$. It is easy to show that sufficiently large γ have this property. Then we have

$$F(\mathbf{a}_1,\ldots,\mathbf{a}_k)=\bigcap_{n=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{J}_n} S_{i_1}\cdot\ldots\cdot S_{i_n}(B_\gamma)\ .$$

So $F(\mathbf{a}_1,\ldots,\mathbf{a}_k)$ is a subset of $B_\gamma\subseteq\mathbf{R}^d$. $F(\mathbf{a}_1,\ldots,\mathbf{a}_k)$ is self–affine in the following sense

$$F(\mathbf{a}_1,\ldots,\mathbf{a}_k)=\bigcup_{\mathbf{i}\in\mathcal{J}_n} S_{i_1}\cdot\ldots\cdot S_{i_n}(F(\mathbf{a}_1,\ldots,\mathbf{a}_k))\ ,\qquad n=1,2,\ldots\ .$$

Additionally, let there be given a fixed probability distribution on $\{1,\ldots,k\}$ determined by a vector $(p_1,\ldots,p_k)$ with $0<p_i\leq 1, \sum p_i=1$. With respect to this distribution we consider $\mathcal{J}_\infty$ as space of i.i.d. sequences of symbols from $\{1,\ldots,k\}$. The corresponding Bernoulli (product) measure on $\mathcal{J}_\infty$ we denote by ν. The image of ν under π (i.e. $\mu:=\nu\pi^{-1}$) is a measure on $F(\mathbf{a}_1,\ldots,\mathbf{a}_k)$, which is the main object of our interest here. Especially we are interested in the following question: For given α, what is the Hausdorff dimension of that part K_α of points x in $F(\mathbf{a}_1,\ldots,\mathbf{a}_k)$ which have the property that the local dimension of μ in x is α, i.e. a small ball $B_\varepsilon(\mathbf{x})$ has a μ–measure of the order $\mu(B_\varepsilon(\mathbf{x}))\sim\varepsilon^\alpha$? We give a precise formulation only later.
Of course, we can expect the answer to depend on the special choice of the parameter vector $(\mathbf{a}_1,\ldots,\mathbf{a}_k)$. The reason for us not to fix this vector is that we intend to find an "almost sure"–type answer: For almost all $(\mathbf{a}_1,\ldots,\mathbf{a}_k)$ with respect to the dk–dimensional Lebesgue measure the Hausdorff dimension of K_α is some $f(\alpha)$.
This is the same situation as in [Fa], where a number $\Delta=d(T_1,T_2,\ldots)$ was evaluated such that $\dim_H(F(\mathbf{a}_1,\ldots,\mathbf{a}_k))=\Delta$ for almost all parameters $\mathbf{a}$, supposed that the contraction number fulfils $a\leq 1/3$.
We conclude this section with a standard fact about the $f(\alpha)$–spectrum.
Let μ be an arbitrary finite measure defined on the σ–field of Borel sets of $\mathbf{R}^d$. For an arbitrary non–negative α we consider the set

$$K_\alpha^{\leq}:=\left\{\mathbf{x}\in\mathbf{R}^d:\underline{\lim}_{\varepsilon\to 0}\frac{\log\mu(B_\varepsilon(\mathbf{x}))}{\log\varepsilon}\leq\alpha\right\}\ .$$

Then we have

Lemma 1

$$\dim_H(K_\alpha^{\leq}) \leq \alpha\ .$$

Proof. Let $K_\alpha^{\leq}(n) := K_\alpha^{\leq} \cap B_n$. It is sufficient to prove that $\dim_H(K_\alpha^{\leq}(n)) \leq \alpha$ for each n. Choose an arbitrary positive δ. Obviously we find a covering $\mathcal{C}$ of $K_\alpha^{\leq}(n)$ by balls with centers in $K_\alpha^{\leq}$ such for each of these balls B the relation

$$\mu(B) \geq (\mathrm{diam}(B))^{\alpha+\delta}$$

is fulfilled. The maximum of the radii of these balls can be chosen arbitrary small. Now by the covering lemma 1.9. in [Fal] we find a subset $\mathcal{C}'$ of $\mathcal{C}$ consisting of disjoint balls such that the set $\mathcal{C}''$ which is obtained from $\mathcal{C}'$ by blowing up each ball in $\mathcal{C}'$ with factor three is in turn a covering of $K_\alpha^{\leq}(n)$. This gives the following estimate

$$\begin{aligned}
&\sum_{B\in\mathcal{C}''}(\mathrm{diam}(B))^{\alpha+2\delta} = 3^{\alpha+2\delta}\sum_{B\in\mathcal{C}'}(\mathrm{diam}(B))^{\alpha+2\delta}\\
\leq\ & 3^{\alpha+2\delta}\cdot\sup_{B\in\mathcal{C}}(\mathrm{diam}(B))^{\delta}\sum_{B\in\mathcal{C}'}(\mathrm{diam}(B))^{\alpha+\delta}\\
\leq\ & 3^{\alpha+2\delta}\cdot\sup_{B\in\mathcal{C}}(\mathrm{diam}(B))^{\delta}\sum_{B\in\mathcal{C}'}\mu(B) \leq 3^{\alpha+2\delta}\cdot\sup_{B\in\mathcal{C}}(\mathrm{diam}(B))^{\delta}\cdot\mu(\mathbf{R}^d)\ .
\end{aligned}$$

So the Hausdorff dimension of $K_\alpha^{\leq}(n)$ is at most $\alpha+2\delta$. Since δ was arbitary we proved the assertion.

■

2 A Self–Affine Fractal with Measure

First let us consider an easier problem.
Let, for $\mathbf{i} \in \mathcal{J}_n$, $p_{\mathbf{i}}$ denote the probability $p_{i_1}\cdot\ldots\cdot p_{i_n}$, and let $T_{\mathbf{i}} := T_{i_1}\cdot\ldots\cdot T_{i_n}$, $S_{\mathbf{i}} := S_{i_1}\cdot\ldots\cdot S_{i_n}$. We put $t_{\mathbf{i}} := \|T_{\mathbf{i}}\|$.
Now let, for given $\mathbf{i} \in \mathcal{J}_\infty$, $\alpha^-(\mathbf{i})$ denote the expression

$$\alpha^-(\mathbf{i}) := \varliminf_{n\to\infty}\frac{\log p_{\mathbf{i}(n)}}{\log t_{\mathbf{i}(n)}}\ ,$$

and the corresponding upper limit we denote by $\alpha^+(\mathbf{i})$. In case that these values coincide we denote the limit by $\alpha(\mathbf{i})$.
Observe that the set of accumulation points of the sequence

$$\frac{\log p_{\mathbf{i}(n)}}{\log t_{\mathbf{i}(n)}}$$

coincides with the interval $[\alpha^-(\mathbf{i}), \alpha^+(\mathbf{i})]$.
We define, for $\alpha \geq 0$

$$\mathcal{J}_{(\alpha)} := \{\mathbf{i} \in \mathcal{J}_\infty : \alpha(\mathbf{i}) = \alpha\} \ .$$

Let us introduce a metric ρ in $\mathcal{J}_\infty$ by means of

$$\rho(\mathbf{i}, \mathbf{j}) := t_{\mathbf{i}\wedge\mathbf{j}} \qquad \mathbf{i} \neq \mathbf{j} \ .$$

It is easy to check that the triangle inequality is fulfilled. This metric generates the natural topology of the Cantor set $\mathcal{J}_\infty$.
Observe that, taking this metric, $\mathcal{J}_{(\alpha)}$ is the set of those points $\mathbf{i}$, for which the local dimension of ν exists and is α.
With this metric the mapping π becomes Lipschitz continuous with Lipschitz constant 2γ. Our motivation to choose this metric is that for given $\mathbf{i}, \mathbf{j} \in \mathcal{J}_\infty$ the value $t_{\mathbf{i}\wedge\mathbf{j}}$ is the magnitude of $|\pi_{\mathbf{a}}(\mathbf{i}) - \pi_{\mathbf{a}}(\mathbf{j})|$ for almost all parameters $\mathbf{a}$, see [Fa], Lemma 3.1. We have some hope that with this metric $\mathcal{J}_\infty$ becomes 'nearly isometric' to $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ and π preserves the Hausdorff and local dimensions we are interested in. We will see that this hope is partially justified.
We have to cope with the problem that the Lipschitz mapping π in general has not a Lipschitz continuous inverse or is not invertible at all. So the local dimension of $\pi(\mathbf{i})$ with respect to μ may be less than the local dimension of $\mathbf{i}$ with respect to ν. The simplest reason that this may happen is the case where π is not injective. If $\pi(\mathbf{i}) = \pi(\mathbf{j})$ and the local dimension in $\mathbf{j}$ is less than in $\mathbf{i}$, then clearly the application of π diminishes the local dimension of $\mathbf{i}$.
Although we do not employ the injectivity of π in the sequel, since this property is not strong enough to ensure that dimensions remain unchanged with the application of π, the following assertion should be of some independent interest. For the definition of Δ see [Fa] or chapter 2 of this paper. It is the Hausdorff dimension of $\mathcal{J}_\infty$ with respect to ρ. .

Lemma 2 *If $a \leq 1/3$, $\Delta < 1$ and $\Delta < d - \frac{\log k}{\log a^{-1}}$, then for almost all* $\mathbf{a} \in \mathbf{R}^{dk}$ *(with respect to Lebesgue measure) the mapping $\pi_{\mathbf{a}}$ is injective.*

Proof. It is obvious from the property of $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ to be self–affine that we can confine ourselves to prove that for almost all $\mathbf{a}$ we have $\pi(\mathbf{i}) \neq \pi(\mathbf{j})$ for $\mathbf{i}, \mathbf{j} \in \mathcal{J}_\infty$ supposed that $\mathbf{i}(1) \neq \mathbf{j}(1)$, i.e. already the first members of $\mathbf{i}$ and $\mathbf{j}$ are different.
Observe that $\pi_{\mathbf{a}}(\mathbf{i})$ is linear as function of $\mathbf{a}$. So we prove that π is injective for almost all $\mathbf{a} \in B_\theta$, θ being chosen such that the volume of the ball is one. So we show that π is injective with probalility one. Denote the (probability) Lebesgue measure on B_θ by P.
Observe that we find a common value of the number $\gamma = \gamma(\mathbf{a})$, that was defined in the introduction, which does not depend on $\mathbf{a} \in B_\theta$.
Choose $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$ and $n \in \mathbf{N}$.Consider the event

$$E_n := \left\{ \mathbf{a} \in B_\theta : \left(\bigcup_{\mathbf{i} \in \mathcal{J}_n} S_{i \cdot \mathbf{i}}(B_\gamma) \right) \bigcap \left(\bigcup_{\mathbf{j} \in \mathcal{J}_n} S_{j \cdot \mathbf{j}}(B_\gamma) \right) \neq \emptyset \right\} .$$

Then

$$\begin{aligned} P(E_n) &\leq \sum_{\mathbf{i},\mathbf{j} \in \mathcal{J}_n} P\left(S_{i \cdot \mathbf{i}}(B_\gamma) \cap S_{j \cdot \mathbf{j}}(B_\gamma) \neq \emptyset\right) \\ &\leq \sum_{\mathbf{i},\mathbf{j} \in \mathcal{J}_n} P\left(B_{\gamma t_{i \cdot \mathbf{i}}}(\pi(i \cdot \mathbf{i})) \cap B_{\gamma t_{j \cdot \mathbf{j}}}(\pi(j \cdot \mathbf{j})) \neq \emptyset\right) \\ &\leq \sum_{\mathbf{i},\mathbf{j} \in \mathcal{J}_n} P\left(|\pi(i \cdot \mathbf{i})) - \pi(j \cdot \mathbf{j}))| \leq \gamma(t_{i \cdot \mathbf{i}} + t_{j \cdot \mathbf{j}})\right) . \end{aligned}$$

In order to make things compatible to [Fa], Lemma 3.1. we choose an arbitrary $i' \in \mathcal{J}_\infty$ and continue as follows, where $s \in (\Delta, d)$ is non–integer and c_s is a positive constant depending on s,

$$\begin{aligned} &\leq \sum_{\mathbf{i},\mathbf{j} \in \mathcal{J}_n} P\left(|\pi(i \cdot \mathbf{i} \cdot \mathbf{i}')) - \pi(j \cdot \mathbf{j} \cdot \mathbf{i}'))| \leq 2\gamma(t_{i \cdot \mathbf{i}} + t_{j \cdot \mathbf{j}})\right) \\ &\leq \sum_{\mathbf{i},\mathbf{j} \in \mathcal{J}_n} P\left(|\pi(i \cdot \mathbf{i} \cdot \mathbf{i}')) - \pi(j \cdot \mathbf{j} \cdot \mathbf{i}'))|^{-s} \geq (2\gamma(t_{i \cdot \mathbf{i}} + t_{j \cdot \mathbf{j}}))^{-s}\right) \\ &\leq \sum_{\mathbf{i},\mathbf{j} \in \mathcal{J}_n} E_{\mathbf{a}} |\pi(i \cdot \mathbf{i} \cdot \mathbf{i}')) - \pi(j \cdot \mathbf{j} \cdot \mathbf{i}'))|^{-s} \cdot (2\gamma(t_{i \cdot \mathbf{i}} + t_{j \cdot \mathbf{j}}))^{s} \\ &\leq c_s \sum_{\mathbf{i},\mathbf{j} \in \mathcal{J}_n} (t_{i \cdot \mathbf{i}} + t_{j \cdot \mathbf{j}})^s . \end{aligned}$$

Here in the third step we used Čebyshev's inequality. We continue with some positive constant c_s'

$$\leq c_s' \sum_{\mathbf{i},\mathbf{j}\in\mathcal{J}_n} (t_{\mathbf{i}}^s + t_{\mathbf{j}}^s) \leq 2c_s' \cdot k^{n'} \sum_{\mathbf{i}\in\mathcal{J}_n} t_{\mathbf{i}}^s \ .$$

We apply Proposition 4.1. in [Fa] to continue (observe $\Delta < 1$)

$$\leq 2c_s' \cdot k^n \cdot \max_{\mathbf{j}\in\mathcal{J}_n} t_{\mathbf{j}}^{s-\Delta} \sum_{\mathbf{i}\in\mathcal{J}_n} t_{\mathbf{i}}^{\Delta} \leq 2c_s' \cdot k^n \cdot a^{(s-\Delta)n} \cdot h(n) \ ,$$

where $h(n)$ is a positive function of n such that $n^{-1}\log h(n)$ tends to zero as n tending to infinity, and we may continue for some $C > 0$ and $0 < \tau < 1$

$$\leq C \cdot \tau^n \ ,$$

in view of our assumptions, supposed we choose s close enough to d.
Observe that in view of $S_i(B_\gamma) \subseteq B_\gamma$, $i = 1, 2, \ldots, k$, the E_n form a descending sequence of events. So we get

$$P\left(\bigcap_{n\geq 0} E_n\right) = 0 \ ,$$

i.e. for almost all $\mathbf{a}$ there is some n with $\mathbf{a} \notin E_n$, which in view of the remark at the beginning of the proof implies that $\pi_{\mathbf{a}}(\mathbf{i}) \neq \pi_{\mathbf{a}}(\mathbf{j})$ for all $\mathbf{i} \neq \mathbf{j} \in \mathcal{J}_\infty$. ■

Next we try to find the Hausdorff dimensions of the $\mathcal{J}_{(\alpha)}$.
A remarkable tool to attack our problem are some special measures on $\mathcal{J}_\infty$. The idea to consider these measures is adapted, on the one hand, from [CM], where the special case of similarity mappings instead of general linear contractions was treated. On the other hand, we follow closely the considerations in [Fa], where the Hausdorff dimension of $F(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ was investigated. For any subset $\mathcal{I}$ of $\mathcal{J}_\infty$ we consider countable coverings by ρ–disks. Any such disk is given by a natural number n and by an element $\mathbf{i}$ of $\mathcal{J}_n$. In fact, in order to specify the disk it is enough to give the centre and the number of elements, up to which the starting sequence of this centre determines the starting sequence of any other element of the disk. So we denote the disk by $D_{\mathbf{i}}$. A covering of $\mathcal{I}$ by ρ–disks is given by a subset $\mathcal{K}$ of $\mathcal{J}$ with the property

that any element in $\mathcal{I}$ has some element of $\mathcal{K}$ as starting sequence. The set of all coverings of $\mathcal{I}$ we denote by $C(\mathcal{I})$. Let us write $C_r(\mathcal{I})$ for the subset of all coverings $\mathcal{K}$, such that any $\mathbf{i} \in \mathcal{K}$ has length $|\mathbf{i}| \geq r$.
We define for any pair of real numbers q, s and for any natural number r

$$\nu_{(r)}^{q,s}(\mathcal{I}) := \inf_{\mathcal{K} \in C_r(\mathcal{I})} \sum_{\mathbf{i} \in \mathcal{K}} p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s = \inf_{\mathcal{K} \in C_r(\mathcal{I})} \sum_{\mathbf{i} \in \mathcal{K}} p_{\mathbf{i}}^q \cdot (\text{diam } D_{\mathbf{i}})^s$$

and

$$\nu^{q,s}(\mathcal{I}) := \lim_{r \to \infty} \nu_{(r)}^{q,s}(\mathcal{I}) .$$

This is the standard construction of a net measure (see [Ro], [Fa1]). It is not difficult to verify that $\nu^{q,s}$ is an outer measure on $\wp(\mathcal{J}_\infty)$ which restricts to a measure on the Borel sets of the metric space $(\mathcal{J}_\infty, \rho)$. Obverse that ρ–disks have the 'net–property': Two ρ–disks are either disjoint or one of them is contained in the other one.
We have the following

Lemma 3 *For each $q \in \mathbf{R}$ there is a $\beta(q)$ such that $\nu^{q,s}(\mathcal{J}_\infty)$ is zero for $s > \beta(q)$ and is infinite for $s < \beta(q)$. This β is a decreasing function of q.*

Proof. $\nu^{q,s}(\mathcal{J}_\infty)$ is a decreasing function of s and q. If one of the arguments q, s is fixed, for large values of the other argument $\nu^{q,s}(\mathcal{J}_\infty)$ is zero (take the $\mathcal{J}_n$ as coverings of $\mathcal{J}_\infty$), and for small values of the other argument it is infinite, since we may manage $p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s$ to grow exponentially fast with growing length of $\mathbf{i}$ in that case. Finally, observe that there cannot be two values $s < s'$ such that $\nu^{q,s}(\mathcal{J}_\infty)$ and $\nu^{q,s'}(\mathcal{J}_\infty)$ are both non–zero and finite, since the quotient $t_{\mathbf{i}}^{s-s'}$ of corresponding terms in the defining sums tends to zero exponentially fast with growing length of $\mathbf{i}$ (uniformly in $\mathbf{i}$). So we would get $\nu^{q,s}(\mathcal{J}_\infty)/\nu^{q,s'}(\mathcal{J}_\infty) = 0$, which is a contradiction. Hence the threshold between 0 and $+\infty$ is a single value of s.

■

The expression $p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s$ is a convex function of (q, s) for fixed $\mathbf{i}$. So we have

Lemma 4 *For any $\mathcal{I} \subseteq \mathcal{J}_\infty$ the expression $\nu^{q,s}(\mathcal{I})$ is a convex function of q and s.*

By Lemma 3 we obtain

Lemma 5 *The threshold function β is decreasing and convex. In particular it is continuous.*

Let us denote, for each $q \in \mathbf{R}$, by $\alpha^-(q)$ and $\alpha^+(q)$ the lower and upper derivatives of the concave function $-\beta$ at q. If β is smooth at q we define $\alpha(q) := -\beta'(q)$. Let

$$\lambda^+ := \alpha(-\infty) ,$$

$$\lambda^- := \alpha(+\infty) .$$

Note that these values are well–defined.
Before proving the main result of this section, we give the following alternative characterizations of the threshold function β, which is the adaptation of the corresponding result in [Fa]:

Lemma 6 *For each $q \in \mathbf{R}$, the following numbers exist and are all equal to $\beta(q)$*

(a) $\inf\{s : \nu^{q,s}(\mathcal{J}_\infty) = 0\} = \sup\{s : \nu^{q,s}(\mathcal{J}_\infty) = +\infty\}$,

(b) *the unique s such that* $\lim_{r\to\infty} \left[\sum_{\mathbf{i}\in\mathcal{J}_r} p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s\right]^{1/r} = 1$,

(c) $\inf\{s : \sum_{\mathbf{i}\in\mathcal{J}} p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s < +\infty\} = \sup\{s : \sum_{\mathbf{i}\in\mathcal{J}} p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s = +\infty\}$.

The measure $\nu^{q,\beta(q)}$ has total mass not less than one for $q \leq 1$ and not greater than one for $q \geq 1$.

Proof. 1. *(a)* is simply the definition of $\beta(q)$.
2. For a finite subset $\mathcal{I}$ of $\mathcal{J}$ we introduce the notation

$$S(q,s,\mathcal{I}) := \sum_{\mathbf{i}\in\mathcal{I}} p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s .$$

From the fact that the matrix norm is submultiplicative we get that $S(q,s,\mathcal{I})$ is a submultiplicative or supermultiplicative function of $\mathcal{I}$ (in the sense that $\mathcal{I}_1 \cdot \mathcal{I}_2 = \{\mathbf{i}\cdot\mathbf{j} : \mathbf{i} \in \mathcal{I}_1, \mathbf{j} \in \mathcal{I}_2\}$), depending on whether s is positive or not. So $\log S(q,s,\mathcal{J}_r)$ is sub– or superadditive as function of r. As is well–known, in either case there exists the limit for $r \to \infty$ of $(S(q,s,\mathcal{J}_r))^{1/r}$ which we shall denote by $U(q,s)$. In our situation this expression is finite, continuous

and strictly decreasing in s (observe that $a' \le t_{\mathbf{i}(n)}/t_{\mathbf{i}(n-1)} \le a$ holds for any $n \in \mathbf{N}$, $\mathbf{i} \in \mathcal{J}_\infty$). $U(q,s)$ is greater than one for small values of s and less than one for large values of s, so there is a unique s_0 for which the limit is one. Now it is obvious that this value is the threshold value appearing in *(c)*.
3. Since each $\mathcal{J}_r$ is a covering for $\mathcal{J}_\infty$, we have the relation $\beta(q) \le s_0$. On the other hand, assume that $\nu^{q,s}(\mathcal{J}_\infty) = 0$, but $U(q,s) > 1$. Then to each ε and each r we find a finite covering $\mathcal{K}$ of the compact $\mathcal{J}_\infty$ with $|\mathbf{i}| \ge r$ for each $\mathbf{i} \in \mathcal{K}$ and such that the sum $S(q,s,\mathcal{K})$ over that covering is less than ε. To come to a contradiction, we consider the two cases $s < 0$ and $s \ge 0$ separately.
4. First let $s < 0$. Assume without any loss of generality that the covering $\mathcal{K}$ is non–reducible in the sense that it does not include an $\mathbf{i}$ which could be omitted. We define the *section of level* r', $r' \in \mathbf{N}$, of that covering by

$$\mathcal{K}(r') := \{\mathbf{j} \in \mathcal{J}_{r'} : \textit{there is some } \mathbf{i} \in \mathcal{K} \textit{ with } \mathbf{j} \le \mathbf{i}\} \; .$$

Choose some large number M and assume that $S(q,s,\mathcal{K}(r')) > M$. Fix some n large enough to ensure that $\tau := S(q,s,\mathcal{J}_n) > 1$. Now we have the following relation

$$\mathcal{K}(r'+n) = \mathcal{K}(r') \cdot \mathcal{J}_n \;\setminus\; \mathcal{K}(r', r'+n)$$

where

$$\begin{aligned} &\mathcal{K}(r', r'+n) \\ := \;& \{\mathbf{j} \in \mathcal{J}_{r'+n} : \textit{there is some } \mathbf{k} \in \mathcal{K} \textit{ with } r' \le |\mathbf{k}| < r'+n \textit{ and } \mathbf{k} < \mathbf{j}\} \; . \end{aligned}$$

Here we used the property of $\mathcal{K}$ to be a non–reducible covering of $\mathcal{J}_\infty$. We get by the supermultiplicativity of S

$$S(q,s,\mathcal{K}(r'+n)) \ge \tau \cdot M - S(q,s,\mathcal{K}(r', r'+n)) \; .$$

Denote the set $\{\mathbf{i} \in \mathcal{K} : r' \le |\mathbf{i}| < r'+n\}$ by $\mathcal{K}_{r',n}$ and observe that there is some constant c depending on q, s, n but not on r', such that

$$S(q,s,\mathcal{K}(r', r'+n)) < c \cdot S(q,s,\mathcal{K}_{r',n}) \; .$$

Since $\mathcal{K}_{r',n}$ is a subset of $\mathcal{K}$ we get

$$S(q,s,\mathcal{K}(r'+n)) \ge \tau \cdot M - c \cdot \varepsilon \; .$$

Hence, if M is large enough, from the property $S(q,s,\mathcal{K}(r')) > M$ we get $S(q,s,\mathcal{K}(r'+n)) > M$, too. But from the property $|\mathbf{i}| \geq r$ for $\mathbf{i} \in \mathcal{K}$ we get that

$$\mathcal{K}(r) = \mathcal{J}_r \ ,$$

which has a measure growing exponentially with growing r. So for r large enough all the sections $\mathcal{K}(r)$, $\mathcal{K}(r+n)$, $\mathcal{K}(r+2n), \ldots$ have a measure not less than M. This contradicts the finiteness of $\mathcal{K}$, yielding that $s_0 = \beta(q)$.

5. The case $s \geq 0$ can be treated in exactly the same way as in [Fa]. Assume that $S(q,s,\mathcal{K}) \leq 1$. Let $p := \max\{|\mathbf{i}| : \mathbf{i} \in \mathcal{K}\}$. Define coverings $\mathcal{K}_n$, $n > p$, by

$$\mathcal{K}_n := \{\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_l : \mathbf{i}_j \in \mathcal{K}, |\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_l| \geq n \textit{ and } |\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_{l-1}| < n\} \ .$$

Then the submultiplicativity of S yields

$$\begin{aligned} S(q,s,\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_l \cdot \mathcal{K}) &\leq p^q_{\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_l} \cdot t^s_{\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_l} \cdot S(q,s,\mathcal{K}) \\ &\leq p^q_{\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_l} \cdot t^s_{\mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \ldots \cdot \mathbf{i}_l} \ . \end{aligned}$$

Using this we inductively get

$$S(q,s,\mathcal{K}_n) \leq 1 \ .$$

Now observe that $\mathcal{K}_n$ is a covering with all elements having a length between n and $n+p$. So we conclude that there is some c depending on q, s, p but not on n such that

$$S(q,s,\mathcal{J}_n) \leq c \cdot S(q,s,\mathcal{K}_n) \leq c \ .$$

So $U(q,s) \leq 1$ which contradicts the assumption, yielding that $s_0 = \beta(q)$.

6. For $q \leq 1$ we obviously have $\beta(q) \geq 0$, so that we are in the submultiplicative case treated in the 5th step. If $\nu^{q,\beta(q)}(\mathcal{J}_\infty)$ would be less than one, then we would find a finite covering $\mathcal{K}$ of $\mathcal{J}_\infty$ with $S(q,\beta(q),\mathcal{K}) < 1$. But the same argument as applied in 5. yields that this would urge the sums $S(q,\beta(q),\mathcal{J}_n)$ to go to zero exponentially fast, in contradiction to $U(q,\beta(q)) = 1$. The case $q \geq 1$ is trivial: If $\nu^{q,\beta(q)}(\mathcal{J}_\infty)$ would be greater than one, there should be $S(q,\beta(q),\mathcal{J}_n) > 1$ for n sufficiently large, so that by the supermultiplicativity this sum would tend to infinity exponentially fast, again in contradiction to $U(q,\beta(q)) = 1$.

■

In general we cannot expect that the Hausdorff type measures $\nu^{q,s}$ considered here have finite non–zero total mass at the critical value $s = \beta(q)$. If $\nu^{q,s}$ is the zero measure, it yields no information about nothing. Compared with this situation, the case of infinite total mass is clearly better, but anyway we would like to have to do with a probability measure. Fortunately, there is always a modification of $\nu^{q,\beta(q)}$, which has finite non–zero total mass. In fact, to each $q \in \mathbf{R}$ we find some function $h_q : \mathcal{J} \to \mathbf{R}_+$ such that

$$\lim_{n\to\infty} \exp(-bn) \cdot \sup_{|\mathbf{i}|=n} h_q(\mathbf{i}) = 0 \textit{ and } \lim_{n\to\infty} \exp(-bn) \cdot \sup_{|\mathbf{i}|=n} (h_q(\mathbf{i}))^{-1} = 0$$

for each positive real b, and such that the measure $\overline{\nu}^{q,s}$ defined by modifying the definition of $\nu^{q,s}$ as follows:

$$\overline{\nu}^{q,s}_{(r)}(\mathcal{I}) := \inf_{\mathcal{K}\in C_r(\mathcal{I})} \sum_{\mathbf{i}\in\mathcal{K}} h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^s, \qquad \mathcal{I} \subseteq \mathcal{J}_\infty ,$$

and

$$\overline{\nu}^{q,s}(\mathcal{I}) := \lim_{r\to\infty} \overline{\nu}^{p,s}_{(r)}(\mathcal{I}), \qquad \mathcal{I} \subseteq \mathcal{J}_\infty ,$$

has total mass one for $s = \beta(q)$. The fact that this defines a measure can be seen in exactly the same way as above. We give the proof of the property $\overline{\nu}^{q,\beta(q)}(\mathcal{J}_\infty) = 1$ in the appendix.
We write ν^q for the measure $\overline{\nu}^{q,\beta(q)}$ and define

$$\mathcal{J}(q) := \{\mathbf{i} \in \mathcal{J}_\infty : \alpha^-(q) \le \alpha^-(\mathbf{i}) \le \alpha^+(\mathbf{i}) \le \alpha^+(q)\} .$$

As main result of this section we get

Proposition 1 *For each $q \in \mathbf{R}$, the probability measure ν^q is concentrated on the set $\mathcal{J}(q)$, i.e. we have*

$$\nu^q(\mathcal{J}(q)) = 1 .$$

Remark. The statement of this proposition is that, for any q, each accumulation point of the sequence $\alpha(\mathbf{i}(n))$, where for $\mathbf{i} \in \mathcal{J}$

$$\alpha(\mathbf{i}) := \frac{\log p_{\mathbf{i}}}{\log t_{\mathbf{i}}} ,$$

is within the interval $[\alpha^-(q), \alpha^+(q)]$ for almost all $\mathbf{i}$ with respect to ν^q, so that in case of β being smooth at q the local dimension $\alpha(\mathbf{i})$ exists a.s. and is

$\alpha(q)$. Since β is convex, there is at most a countable number of q, where this function is not smooth. But we cannot exclude, that with the exception of a finite number of local dimensions α the whole spectrum of these dimensions is covered by the points, where β is not smooth. Then the identification of the set of points with a certain local dimension α with the support of the measure ν^q, where $\alpha = -\beta'(q)$, is impossible for the most part of values α.

Proof of Proposition 1. Consider first the case of q being positive.
1. Assume that $\alpha(\mathbf{i}(n))$ has, with a positive ν^q–probability, an accumulation point outside of the given interval. Then there is an α outside of it such that each ε–neighbourhood of α contains an accumulation point of $\alpha(\mathbf{i}(n))$ for a set of $\mathbf{i} \in \mathcal{J}_\infty$ with a positive measure. We choose ε small enough so that $[\alpha-\varepsilon, \alpha+\varepsilon]$ does not intersect with $[\alpha^-(q), \alpha^+(q)]$.
2. Define

$$\mathcal{K}_n(\alpha,\varepsilon) := \{\mathbf{i} \in \mathcal{J} : \alpha(\mathbf{i}) \in (\alpha-\varepsilon, \alpha+\varepsilon), |\mathbf{i}| \geq n\} ,$$

$$\mathcal{J}(\alpha,\varepsilon) := \{\mathbf{i} \in \mathcal{J}_\infty : \alpha(\mathbf{i}(n)) \textit{ has an accumulation point in } (\alpha-\varepsilon, \alpha+\varepsilon)\} .$$

Then, for each n, $\mathcal{K}_n(\alpha,\varepsilon)$ is a countable covering of $\mathcal{J}(\alpha,\varepsilon)$. Since

$$\nu^q(\mathcal{J}(\alpha,\varepsilon)) > 0 ,$$

we find, for sufficiently large n

$$0 < 1/2 \cdot \nu^q(\mathcal{J}(\alpha,\varepsilon)) \leq \sum_{\mathbf{i}\in\mathcal{K}_n(\alpha,\varepsilon)} h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)} .$$

Since the family $\{\mathcal{K}_n(\alpha,\varepsilon)\}$ of coverings is descending, we even get

$$+\infty = \sum_{\mathbf{i}\in\mathcal{K}_n(\alpha,\varepsilon)} h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)} .$$

We get by the definition of $\mathcal{K}_n(\alpha,\varepsilon)$,

$$+\infty = \sum_{\mathbf{i}\in\mathcal{K}_n(\alpha,\varepsilon)} h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)} \leq \sum_{\mathbf{i}\in\mathcal{K}_n(\alpha,\varepsilon)} h_q(\mathbf{i}) \cdot t_{\mathbf{i}}^{q(\alpha-\varepsilon)+\beta(q)} .$$

We conclude from the convexity of β and from the fact that α does not belong to $[\alpha^-(q), \alpha^+(q)]$, that we can find some $q' > 0$ and may choose ε so small that $q(\alpha-\varepsilon)+\beta(q) \geq q'(\alpha-\varepsilon)+\beta(q')+(2q'+1)\varepsilon$. So we have

$$\begin{aligned} +\infty &= \sum_{\mathbf{i}\in\mathcal{K}_n(\alpha,\varepsilon)} h_q(\mathbf{i}) \cdot t_{\mathbf{i}}^{q'(\alpha+\varepsilon)+\beta(q')+\varepsilon} \\ &\leq \sum_{\mathbf{i}\in\mathcal{K}_n(\alpha,\varepsilon)} h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^{q'} \cdot t_{\mathbf{i}}^{\beta(q')+\varepsilon} . \end{aligned}$$

Hence

$$\sum_{\mathbf{i}\in\mathcal{J}} h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^{q'} \cdot t_{\mathbf{i}}^{\beta(q')+\varepsilon} = +\infty .$$

So

$$\sum_{\mathbf{i}\in\mathcal{J}} p_{\mathbf{i}}^{q'} \cdot t_{\mathbf{i}}^{\beta(q')+\varepsilon/2} = +\infty ,$$

too. This is in contradiction to the property (c) in Lemma 6.
If $q<0$ we can do the same, but we have to replace $\alpha-\varepsilon$ by $\alpha+\varepsilon$ and vice versa, and we have to choose q' negative, too. Finally, for $q=0$ the signs have to be chosen according to on which side of the interval $[\alpha^-(0), \alpha^+(0)]$ the value α is situated.

■

We conclude this section with the following

Lemma 7 *For ν^q–almost all $\mathbf{i}$ we have*

$$\overline{\lim_{n\to\infty}} \frac{\nu^q(D_{\mathbf{i}(n)})}{h_q(\mathbf{i}(n)) \cdot p_{\mathbf{i}(n)}^q \cdot t_{\mathbf{i}(n)}^{\beta(q)}} \leq 1 .$$

Proof. Assume that there is some $c>1$ such that the upper limit is greater than c for a set $\mathcal{I}_c$ of positive ν^q–measure. We may choose $\mathcal{I}_c$ to be compact. We define coverings of $\mathcal{I}_c$ by means of

$$\mathcal{K}_n := \{\mathbf{i} \in \mathcal{J} : \frac{\nu^q(D_{\mathbf{i}})}{h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)}} \geq c,\ |\mathbf{i}| \geq n\} .$$

For each n we select a non–reducible finite subcovering $\mathcal{K}'_n$ of the compact $\mathcal{I}_c$. We define inductively a sequence of natural numbers: Once n_j is given, we choose n_{j+1} greater than the greatest length of any element $\mathbf{i}$ in $\mathcal{K}'_{n_j}$. Then the sequence $\{\mathcal{W}_j\} := \{\mathcal{K}'_{n_j}\}$ is a sequence of non–reducible finite coverings of $\mathcal{I}_c$. We make use of the fact that two ρ–disks are either disjoint or one is completely contained in the other one, to see that the sequence of sets

$$\mathcal{H}_j := \bigcup_{\mathbf{i}\in\mathcal{W}_j} D_{\mathbf{i}}$$

is descending and has as limit a set $\mathcal{H} \supseteq \mathcal{I}_c$.
So we get by the definition of ν^q

$$\begin{aligned} 0 &< \nu^q(\mathcal{H}) \leq \varliminf_{j\to\infty} \sum_{\mathbf{i}\in\mathcal{W}_j} h_q(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)} \leq \varliminf_{j\to\infty} \sum_{\mathbf{i}\in\mathcal{W}_j} c^{-1}\nu^q(D_{\mathbf{i}}) \\ &= \varliminf_{j\to\infty} c^{-1}\nu^q(\mathcal{H}_j) = c^{-1}\nu^q(\mathcal{H}) \ . \end{aligned}$$

This contradicts $c > 1$. We are through.

■

3 The Singularity Spectrum of the Fractal

We turn from the coding space $\mathcal{J}_\infty$ to the fractal itself. We denote it by F for short. Remember that it depends on $(\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathbf{R}^{dk}$.
The mapping π transforms $\mathcal{J}_\infty$ to F, which is a measurable subset of $\mathbf{R}^d$, the image of ν^q under π is a measure on F which we denote by μ^q.
We introduce the notations (for $q \in \mathbf{R}$ and $\alpha \geq 0$)

$$\mathcal{J}^-(q) := \begin{cases} \{\mathbf{i} \in \mathcal{J}_\infty : \alpha^-(\mathbf{i}) \leq \alpha^+(q)\} & for \quad q \geq 0 \\ \{\mathbf{i} \in \mathcal{J}_\infty : \alpha^+(\mathbf{i}) \geq \alpha^-(q)\} & for \quad q < 0 \end{cases}$$

$$\mathcal{J}^-_{(\alpha)} := \begin{cases} \{\mathbf{i} \in \mathcal{J}_\infty : \alpha^-(\mathbf{i}) \leq \alpha\} & for \quad \alpha \leq \alpha^+(0) \\ \{\mathbf{i} \in \mathcal{J}_\infty : \alpha^+(\mathbf{i}) \geq \alpha\} & for \quad \alpha > \alpha^+(0) \end{cases}$$

$$\begin{aligned} F(q) &:= \pi(\mathcal{J}(q)) \\ F^-(q) &:= \pi(\mathcal{J}^-(q)) \\ F_{(\alpha)} &:= \pi(\mathcal{J}_{(\alpha)}) \\ F^-_{(\alpha)} &:= \pi(\mathcal{J}^-_{(\alpha)}) \\ \underline{f}(\alpha) &:= \inf_q \max\{0, \alpha q + \beta(q)\} \ . \end{aligned}$$

This is essentially the Legendre transform of β. We express it in terms of the variable q, too, taking into account some abiguity for the case that β is not smooth at q.

$$f^-(q) := \begin{cases} \alpha^-(q)\cdot q + \beta(q) & for \quad q \geq 0 \\ \alpha^+(q)\cdot q + \beta(q) & for \quad q < 0 \ . \end{cases}$$

$$f^+(q) := \begin{cases} \alpha^+(q)\cdot q + \beta(q) & for \quad q \geq 0 \\ \alpha^-(q)\cdot q + \beta(q) & for \quad q < 0 \ . \end{cases}$$

If β is smooth at q, the two values $f^+(q)$ and $f^-(q)$ coincide and we denote this number by $f(q)$. Clearly at $q = 0$ both values always coincide. We will denote $f(0)$ by Δ. In case $\Delta \leq 1$ and if some additional assumption is fulfilled, Δ is the Hausdorff dimension of F for almost all parameters $(\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathbf{R}^{dk}$, see [Fa]. We have

Proposition 2 *The Hausdorff dimension of $F^-_{(\alpha)}$ is not greater than $\underline{f}(\alpha)$,*

$$\dim_H(F^-_{(\alpha)}) \leq \underline{f}^+(\alpha) \ .$$

Consequently,

$$\dim_H(F^-(q)) \leq f^+(q) \ .$$

Proof. We define, in correspondence to the proof of Proposition 1,

$$\mathcal{K}^-_n(\alpha,\varepsilon) := \{\mathbf{i} \in \mathcal{J} : \alpha(\mathbf{i}) \leq \alpha + \varepsilon, \ |\mathbf{i}| \geq n\}, \qquad n \in \mathbf{N}, \varepsilon > 0 \ .$$

Consider the case $\alpha \leq \alpha^+(0)$.
Remember the definition of γ in the introduction. We put

$$\mathcal{C}^-_n(\alpha,\varepsilon) := \{B_{\gamma t_{\mathbf{i}}}(\pi(\mathbf{i})) : \mathbf{i} \in \mathcal{K}^-_n(\alpha,\varepsilon)\} \ .$$

Since $\mathcal{K}^-_n(\alpha,\varepsilon)$ is a covering of $\mathcal{J}^-_{(\alpha)}$, we derive from the definition of γ and of the fractal that $\mathcal{C}^-_n(\alpha,\varepsilon)$ is a covering of $F^-_{(\alpha)}$. Obviously the diameter of that covering tends to zero as n is growing. Fix some $\delta > 0$. Then we have for some $q \geq 0$ the relation $\underline{f}(\alpha) \geq \alpha q + \beta(q) - \delta$, and consequently

$$\sum_{\mathbf{i}\in\mathcal{K}^-_n(\alpha,\varepsilon)} (t_{\mathbf{i}})^{\underline{f}(\alpha)+2\delta} \leq \sum_{\mathbf{i}\in\mathcal{K}^-_n(\alpha,\varepsilon)} (t_{\mathbf{i}})^{\alpha q+\beta(q)+\delta}$$
$$\leq \sum_{\mathbf{i}\in\mathcal{K}^-_n(\alpha,\varepsilon)} p^q_{\mathbf{i}} \cdot t_{\mathbf{i}}^{\beta(q)+\delta-q\varepsilon} \leq \sum_{\substack{\mathbf{i}\in\mathcal{J} \\ |\mathbf{i}|\geq n}} p^q_{\mathbf{i}} \cdot t_{\mathbf{i}}^{\beta(q)+\delta-q\varepsilon} \ .$$

Choosing ε such that $q\varepsilon < \delta$, this last expression tends to zero as $n \to \infty$ by Lemma 6. So for $\alpha \leq \alpha^+(0)$ we proved the assertion of our proposition. The proof for $\alpha > \alpha^+(0)$ is completely analogous.

■

Since $F_{(\alpha)} \subseteq F^-_{(\alpha)}$, and $F(q) \subseteq F^-(q)$, we have an upper estimate for the Hausdorff dimensions of $F_{(\alpha)}$ and $F(q)$, too.
To derive a lower estimate, some restrictions will be made.

Proposition 3 *Assume* $\Delta = f(0) = \max \underline{f}(\alpha) \leq 1$ *and* $a < 1/3$. *Then for each* $s \in (0, f^-(q))$ *and each* $\kappa > 0$ *we have the relation*

$$\int_{B_\kappa} \int_{\mathcal{J}_\infty} \frac{d\mathbf{a}\; d\nu^q(\mathbf{j})}{|\pi_{\mathbf{a}}(\mathbf{i}) - \pi_{\mathbf{a}}(\mathbf{j})|^s} < +\infty$$

for ν^q*–almost all* $\mathbf{i}$.

Proof. The assumptions of the proposition allow to apply Falconer's Lemma 3.1., [Fa] which yields, for some constant $c_1 > 0$

$$\int_{B_\kappa} \int_{\mathcal{J}_\infty} \frac{d\mathbf{a}\; d\nu^q(\mathbf{j})}{|\pi_{\mathbf{a}}(\mathbf{i}) - \pi_{\mathbf{a}}(\mathbf{j})|^s} \leq c_1 \int_{\mathcal{J}_\infty} \frac{d\nu^q(\mathbf{j})}{t^s_{\mathbf{i}\wedge\mathbf{j}}} \leq c_1 \sum_{n=0}^{\infty} t^{-s}_{\mathbf{i}(n)} \nu^q(D_{\mathbf{i}(n)}) .$$

Now by Lemma 7 there is ν^q–almost surely a finite number $c(\mathbf{i})$ such that we may continue

$$\leq c(\mathbf{i}) c_1 \sum_{n=0}^{\infty} t^{-s}_{\mathbf{i}(n)} \cdot h_q(\mathbf{i}(n)) \cdot p^q_{\mathbf{i}(n)} \cdot t^{\beta(q)}_{\mathbf{i}(n)} .$$

Assume $q \geq 0$ and choose some $\delta > 0$. Then by Proposition 1 we get a.s. for some finite $c'(\mathbf{i})$

$$\begin{aligned} &\leq c'(\mathbf{i}) c(\mathbf{i}) c_1 \sum_{n=0}^{\infty} t^{-s}_{\mathbf{i}(n)} \cdot h_q(\mathbf{i}(n)) \cdot t^{\alpha^-(q)q - \delta q}_{\mathbf{i}(n)} \cdot t^{\beta(q)}_{\mathbf{i}(n)} \\ &= c'(\mathbf{i}) c(\mathbf{i}) c_1 \sum_{n=0}^{\infty} h_q(\mathbf{i}(n)) \cdot t^{f^-(q) - s - \delta|q|}_{\mathbf{i}(n)} . \end{aligned}$$

If $q < 0$ we finally get the same, using α^+ instead of α^-. This last expression is clearly finite if δ is small enough, since $t_{\mathbf{i}(n)}$ decreases exponentially.

■

The next step is to show that the Hausdorff dimension of $F(q)$ is not less than $f^-(q)$ (for almost all parameters $\mathbf{a}$ with respect to the dk–dimensional Lebesgue measure).

Proposition 4 *Assume* $\Delta = f(0) = \max \underline{f}(\alpha) \leq 1$ *and* $a < 1/3$. *For almost all parameters* $\mathbf{a}$ *we have*

$$\dim_H(F(q)) \geq f^-(q) \; .$$

Consequently, if β *is smooth at* q, *for* $\alpha = \alpha(q)$ *we have*

$$\dim_H(F_{(\alpha)}) = \underline{f}(\alpha) \; .$$

Proof. 1. By Proposition 3, if $s < f^-(q)$, for μ^q–a.e. $x \in \mathbf{R}^d$ and each $\kappa > 0$ we have

$$\int_{B_\kappa} \int_F \frac{d\mathbf{a} \; \mu^q(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^s} = \int_{B_\kappa} \int_{\mathcal{J}_\infty} \frac{d\mathbf{a} \; d\nu^q(\mathbf{j})}{|\mathbf{x}-\pi_{\mathbf{a}}(\mathbf{j})|^s} < +\infty \; .$$

So for almost all parameters $\mathbf{a}$ the integral

$$\int_F \frac{d\mu^q(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^s}$$

is finite μ^q–a.s. Fix $\mathbf{a}$ and choose some N such that

$$\mu^q(F_N) > 0$$

where

$$F_N := \left\{ \mathbf{x} \in F : \int_F \frac{d\mu^q(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^s} < N \right\} \; .$$

Denote by μ^q_N the restriction of μ^q to F_N. Then we have

$$\int\int \frac{d\mu^q_N(\mathbf{x}) \; d\mu^q_N(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^s} < \infty \; .$$

Hence by the potential–theoretic characterization of the Hausdorff dimension any set supporting μ^q_N has at least dimension s, see [Fal], Corollary 6.6. Obviously, this is true for μ^q, too.

2. We do not know, whether $F(q)$ is measurable. But obviously by Proposition 1 its outer μ^q–measure is the total mass of μ^q. It is easy to see that this is enough to conclude that $\dim_H(F(q)) \geq s$. In fact, let us assume the opposite. Then there would be some $s' < s$ such that to each pair n, n' of positive integers there would be a countable covering $\mathcal{K}_{n.n'}$ of $F(q)$ with open balls of diameter less than n^{-1} and such

$$\sum_{B \in \mathcal{K}_{n,n'}} (\mathrm{diam}(B))^{s'} < (n')^{-1} .$$

Consider the measurable set

$$G := \bigcap_{n,n'} \bigcup_{\mathcal{K}_{n,n'}} \supseteq F(q) .$$

Obviously all the $\mathcal{K}_{n,n'}$, are coverings of G, too. So G has a dimension less than s. But since G is a measurable set containing $F(q)$ it supports μ^q and so it should have a Hausdorff dimension of at least s. This is the desired contradiction.
This proves the proposition, since s was arbitrary in $(0, f^-(q))$.

■

Remark. So far, we estimated the Hausdorff dimension of sets with scaling properties which are defined in terms of the $\mathcal{J}_\infty$–coding of F. The remaining part of this section deals with those sets, the scaling properties of which are defined with respect to the fractal itself and the measure μ.
We define in similar way as above

$$K(q) = K(q, \mathbf{a}) := \left\{ \mathrm{x} \in F : \alpha^-(q) \leq \underline{\lim}_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(\mathrm{x}))}{\log \varepsilon} \leq \overline{\lim}_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(\mathrm{x}))}{\log \varepsilon} \leq \alpha^+(q) \right\} ,$$

$$K^-(q) = K^-(q, \mathbf{a}) := \begin{cases} \left\{ \mathrm{x} \in F : \underline{\lim}_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(\mathrm{x}))}{\log \varepsilon} \leq \alpha^+(q) \right\} & for \quad q \geq 0 \\ \left\{ \mathrm{x} \in F : \alpha^-(q) \leq \overline{\lim}_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(\mathrm{x}))}{\log \varepsilon} \right\} & for \quad q < 0 \end{cases}$$

$$K_\alpha^- = K_\alpha^-(\mathbf{a}) := \begin{cases} \left\{ \mathrm{x} \in F : \underline{\lim}_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(\mathrm{x}))}{\log \varepsilon} \leq \alpha \right\} & for \quad \alpha \leq \alpha^+(0) \\ \left\{ \mathrm{x} \in F : \alpha \leq \overline{\lim}_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(\mathrm{x}))}{\log \varepsilon} \right\} & for \quad \alpha > \alpha^+(0) . \end{cases}$$

Proposition 5 *If* $\mathbf{x} = \pi(\mathbf{i})$ $(\mathbf{x} \in \mathbf{R}^d, \mathbf{i} \in \mathcal{J}_\infty)$, *then we have*

$$a) \qquad \underline{\lim}_{\varepsilon\to 0} \frac{\log\ \mu(B_\varepsilon(\mathbf{x}))}{\log\ \varepsilon} \leq \underline{\lim}_{n\to\infty} \frac{\log\ p_{\mathbf{i}(n)}}{\log\ t_{\mathbf{i}(n)}} ,$$

$$b) \qquad \overline{\lim}_{\varepsilon\to 0} \frac{\log\ \mu(B_\varepsilon(\mathbf{x}))}{\log\ \varepsilon} \leq \overline{\lim}_{n\to\infty} \frac{\log\ p_{\mathbf{i}(n)}}{\log\ t_{\mathbf{i}(n)}} .$$

Proof. For each $\mathbf{j} \in \mathcal{J}$ we define $\varepsilon(\mathbf{j}) := 2\gamma t_{\mathbf{j}}$, where, as in the introduction, γ denotes a radius such that $\mathbf{a}_i + T_i(B_\gamma) \subseteq B_\gamma$, $i = 1, 2, \ldots, k$. Then we have

$$\mathbf{x} \in \pi(\mathbf{i}(n) \cdot \mathcal{J}_\infty) = S_{\mathbf{i}(n)}(F) \subseteq S_{\mathbf{i}(n)}(B_\gamma) \subseteq B_{\gamma t_{\mathbf{i}(n)}}(\pi(\mathbf{i}(n))) \subseteq B_{\varepsilon(\mathbf{i}(n))}(\mathbf{x})$$

so that

$$\mu(B_{\varepsilon(\mathbf{i}(n))}(\mathbf{x})) \geq \mu(S_{\mathbf{i}(n)}(F)) = \mu(\pi(\mathbf{i}(n) \cdot \mathcal{J}_\infty)) = p_{\mathbf{i}(n)} .$$

From this we immediately conclude the validity of a). On the other hand, the sequence $\log\ t_{\mathbf{i}(n)}$ decreases in steps of bounded size, so that we may invert the definition of $\varepsilon(.)$ yielding to each $\varepsilon > 0$ some $n(\varepsilon)$ such that $2\gamma t_{\mathbf{i}(n(\varepsilon))} \leq \varepsilon \leq K \cdot t_{\mathbf{i}(n(\varepsilon))}$ for some constant K and consequentely

$$\mu(B_\varepsilon(\mathbf{x})) \geq \mu(S_{\mathbf{i}(n(\varepsilon))}(F)) = \mu(\pi(\mathbf{i}(n(\varepsilon)) \cdot \mathcal{J}_\infty)) = p_{\mathbf{i}(n(\varepsilon))} .$$

This proves b).

■

The next two propositions show that for almost all parameters $\mathbf{a}$ in a) of Proposition 5 we have even the equality. Remember that by $\alpha^-(\mathbf{i})$ we denoted the expression

$$\underline{\lim}_{n\to\infty} \frac{\log\ p_{\mathbf{i}(n)}}{\log\ t_{\mathbf{i}(n)}} .$$

Proposition 6 *Assume* $\Delta = f(0) = \max \underline{f}(\alpha) \leq 1$ *and* $a < 1/3$. *If* $\mathbf{i} \in \mathcal{J}_\infty$, *then for each* $s \in (0,1)$ *with* $s < \alpha^-(\mathbf{i})$ *and each* $\kappa > 0$ *we have the estimate*

$$\int_{B_\kappa} \int_{\mathcal{J}_\infty} \frac{d\mathbf{a}\ d\nu(\mathbf{j})}{|\pi_{\mathbf{a}}(\mathbf{i}) - \pi_{\mathbf{a}}(\mathbf{j})|^s} < +\infty .$$

Proof. Let $\delta > 0$, $\alpha := \alpha^-(\mathbf{i})$. Then there is some $n_0 \in \mathbf{N}$ such that for $n \geq n_0$ we have $p_{\mathbf{i}(n)} \leq t_{\mathbf{i}(n)}^{\alpha-\delta}$. We use again Falconer's Lemma 3.1., [Fa] to conclude that for $\delta < \alpha - s$

$$\int_{B_K}\int_{\mathcal{J}_\infty} \frac{d\mathbf{a}\ d\nu(\mathbf{j})}{|\pi_{\mathbf{a}}(\mathbf{i}) - \pi_{\mathbf{a}}(\mathbf{j})|^s} \leq c_1 \int_{\mathcal{J}_\infty} \frac{d\nu(\mathbf{j})}{t_{\mathbf{i}\wedge\mathbf{j}}^s} \leq c_1 \sum_{n=0}^{\infty} t_{\mathbf{i}(n)}^{-s} \nu(D_{\mathbf{i}(n)})$$

$$= c_1 \sum_{n=0}^{\infty} t_{\mathbf{i}(n)}^{-s} \cdot p_{\mathbf{i}(n)} \leq c(\mathbf{i}) c_1 \sum_{n=0}^{\infty} t_{\mathbf{i}(n)}^{\alpha-\delta-s} < +\infty \ .$$

■

Proposition 7 *Assume that $t_i < p_i$ for $i = 1, 2, \ldots, k$. Let η be any measure on $\mathcal{J}_\infty$ with finite mass. Then for almost all parameters $\mathbf{a} \in \mathbf{R}^{kd}$ we have*

$$\varliminf_{\varepsilon \to 0} \frac{\log\ \mu(B_\varepsilon(\pi(\mathbf{i})))}{\log\ \varepsilon} \geq \varliminf_{n \to \infty} \frac{\log\ p_{\mathbf{i}(n)}}{\log\ t_{\mathbf{i}(n)}}$$

for η–a.e. $\mathbf{i} \in \mathcal{J}_\infty$.

Proof. By Proposition 6, if $\mathbf{i} \in \mathcal{J}_\infty$ and $s < \alpha^-(\mathbf{i})$, the relation

$$\int_{\mathcal{J}_\infty} \frac{d\nu(\mathbf{j})}{|\pi_{\mathbf{a}}(\mathbf{i}) - \pi_{\mathbf{a}}(\mathbf{j})|^s} < +\infty$$

holds for almost all $\mathbf{a}$. So the set of $(\mathbf{i}, \mathbf{a})$ such that this integral is finite has full $\eta \times m^{dk}$–measure, where m^{dk} denotes the dk–dimensional Lebesgue measure. Hence the set of $\mathbf{i}$, with this integral being finite, has full η–measure for almost all $\mathbf{a}$.

So assume that for $(\mathbf{i}, \mathbf{a})$ the integral is finite. Assume in addition that

$$(*) \qquad \varliminf_{\varepsilon \to 0} \frac{\log\ \mu(B_\varepsilon(\pi(\mathbf{i})))}{\log\ \varepsilon} < \alpha^-(\mathbf{i}) \ .$$

Then there is some sequence $\{\varepsilon_r\}$ with $\varepsilon_r \downarrow 0$ and some $\delta > 0$ such that

$$\frac{\log\ \mu(B_{\varepsilon_r}(\pi(\mathbf{i})))}{\log\ \varepsilon_r} \leq \alpha^-(\mathbf{i}) - \delta \ .$$

Note that the finiteness of the integral considered above implies that $\mu(B_{\varepsilon_r}(\pi(\mathbf{i}))) \downarrow 0$ as $r \to \infty$. So we may assume without any loss of the generality that the following condition is fulfilled for each $r \in \mathbf{N}$:

$$\mu(B_{\varepsilon_r}(\pi(\mathbf{i}))) \geq 2\mu(B_{\varepsilon_{r+1}}(\pi(\mathbf{i}))) .$$

This leads to the following estimate

$$\int_{\mathcal{J}_\infty} \frac{d\nu(\mathbf{j})}{|\pi(\mathbf{i}) - \pi(\mathbf{j})|^s} \geq \sum_{r=1}^{\infty} \int_{\{\mathbf{j} \in \mathcal{J}_\infty : \varepsilon_{r+1} < |\pi(\mathbf{i}) - \pi(\mathbf{j})| \leq \varepsilon_r\}} \frac{d\nu(\mathbf{j})}{|\pi(\mathbf{i}) - \pi(\mathbf{j})|^s}$$

$$\geq \ 1/2 \sum_{r=1}^{\infty} \varepsilon_r^{-s} \mu(B_{\varepsilon_r}(\pi(\mathbf{i}))) \geq 1/2 \sum_{r=1}^{\infty} \varepsilon_r^{-s+\alpha^-(\mathbf{i})-\delta} .$$

This sum is infinite for s sufficiently close to $\alpha^-(\mathbf{i})$ in contradiction to our assumption concerning the integral. So $(*)$ cannot be valid.

■

Proposition 8 *Assume* $\Delta = f(0) = \max \underline{f}(\alpha) \leq 1$ *and* $a < 1/3$. *Moreover, assume that* $t_i < p_i$ *for* $i = 1, 2, \ldots, k$. *Let* $\alpha \geq \alpha^-(1)$. *Then we have*

$$\dim_H(K_\alpha^-) \leq \underline{f}(\alpha)$$

and consequently for $q \leq 1$ *we have*

$$\dim_H(K^-(q)) \leq f^+(q) ,$$

both relations being valid for almost all parameters $\mathbf{a}$.

Proof. Let $\alpha > \alpha^+(0)$. Then in view of Proposition 5 we have $K_\alpha^- \subseteq F_{(\alpha)}^-$ and the assertion follows from Proposition 2. So in the following we treat the case $\alpha^-(1) \leq \alpha \leq \alpha^+(0)$. Then the set $K_{(\alpha)}^-$ consists of the those points of the set F, the lower local dimensions of which are not greater than α. Since the Lipschitz mapping π may only diminish local dimensions, we infer from Proposition 2 that the assertion of the theorem may be violated only for the following situation: There should be some $\alpha' > \alpha$ such that for each positive $\eta < 1/2(\alpha' - \alpha)$ the following set has a Hausdorff dimension greater $\underline{f}(\alpha)$

$$\mathcal{L} := \{\mathbf{i} \in \mathcal{J}_\infty : (\alpha' - \eta, \alpha' + \eta) \cap [\alpha^-(\mathbf{i}), \alpha^+(\mathbf{i})] \neq \emptyset, \ \pi(\mathbf{i}) \in K_{(\alpha)}^-\}$$

for a set of parameters $\mathbf{a}$ with positive Lebesgue measure.
To say it equivalently, the set of those sequences $\mathbf{i}$, such that

$$\frac{\log p_{\mathbf{i}(n)}}{\log t_{\mathbf{i}(n)}}$$

has an accumulation point belonging to $(\alpha' - \eta, \alpha' + \eta)$, but for which the application of π diminishes the lower local dimension to the interval $[0.\alpha]$, should have a Hausdorff dimension greater than $\underline{f}(\alpha)$ with positive probability with respect to $\mathbf{a} \in B_\theta$. Here θ is the number introduced in the proof of Lemma 2.
The following subsets of $\mathcal{J}$ are coverings of $\mathcal{L}$

$$\mathcal{K}_n := \{\mathbf{i} \in \mathcal{J} : \frac{\log p_{\mathbf{i}}}{\log t_{\mathbf{i}}} \in (\alpha' - \eta, \alpha' + \eta), \mu(B_{\varepsilon(\mathbf{i})}(\pi(\mathbf{i}))) \geq (\varepsilon(\mathbf{i}))^{\alpha+\eta}, |\mathbf{i}| \geq n\} ,$$

where as in the proof of Proposition 5 $\varepsilon(\mathbf{i}) = 2\gamma t_{\mathbf{i}}$ and γ is the number defined in the introduction. Just as in the proof of Lemma 2 we may choose γ independent of $\mathbf{a} \in B_\theta$. Observe that for each $\mathbf{i} \in \mathcal{J}_\infty$ and each n we have the relation

$$S_{\mathbf{i}(n)} F \subseteq B_{\gamma t_{\mathbf{i}(n)}}(\pi(\mathbf{i})) \subseteq B_{2\gamma t_{\mathbf{i}(n)}}(\pi(\mathbf{i}(n))) .$$

Choose an arbitrary $\delta > 0$. Let us write for short $p_{\mathbf{i}} \sim t_{\mathbf{i}}^{\alpha'}$ instead of

$$\frac{\log p_{\mathbf{i}}}{\log t_{\mathbf{i}}} \in (\alpha' - \eta, \alpha' + \eta) .$$

Let us write P for the Lebesgue probability measure on B_θ and $\mathbf{E_a}$ to denote the mathematical expectation, where $\mathbf{a}$ is distributed according to P. We have the following estimate, where $c_1, c_2, \ldots$ are suitable constants

$$\begin{aligned} & \mathbf{E_a}\left(\sum_{\mathbf{i}\in\mathcal{K}_n} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta}\right) \\ = \; & \mathbf{E_a}\left(\sum_{\substack{\mathbf{i}\in\mathcal{J}, |\mathbf{i}|\geq n \\ p_{\mathbf{i}} \sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta} \mathbf{1}_{\{\mathbf{a}\in\mathbf{R}^{dk}:\mu(B_{\varepsilon(\mathbf{i})}(\pi(\mathbf{i})))\geq(\varepsilon(\mathbf{i}))^{\alpha+\eta}\}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta} P(\{\mathbf{a}\in\mathbf{R}^{dk} : \mu(B_{\varepsilon(\mathbf{i})}(\pi(\mathbf{i}))) \geq (\varepsilon(\mathbf{i}))^{\alpha+\eta}\}) \\
&\leq \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta} (\varepsilon(\mathbf{i}))^{-\alpha-\eta} \mathbf{E}_{\mathbf{a}} \mu(B_{\varepsilon(\mathbf{i})}(\pi(\mathbf{i}))) \\
&\leq c_1 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha-\eta} \mathbf{E}_{\mathbf{a}} \left(\sum_{\substack{\mathbf{j}\in\mathcal{J}\\ t_{\mathbf{j}}\sim t_{\mathbf{i}}}} p_{\mathbf{j}} \mathbf{1}_{\{\mathbf{a}\in\mathbf{R}^{dk}:|\pi(\mathbf{i})-\pi(\mathbf{j})|\leq 4\gamma t_{\mathbf{i}}\}} \right) .
\end{aligned}
$$

Here we wrote $t_{\mathbf{j}} \sim t_{\mathbf{i}}$ for short, meaning the following property of $\mathbf{j} : t_{\mathbf{j}} \leq t_{\mathbf{i}}$ and $t_{\mathbf{j}(|\mathbf{j}|-1)} > t_{\mathbf{i}}$. That is, we consider all those finite sequences $\mathbf{j}$ which have the property that $t_{\mathbf{j}} \leq t_{\mathbf{i}}$, but the sequence obtained from $\mathbf{j}$ by omitting the last member yields a t which is still greater than $t_{\mathbf{i}}$. Observe in the following that these $\mathbf{j}$ from a covering of $\mathcal{J}_\infty$.
We can continue the above chain of inequalities as follows, using again Falconer's lemma 3.1 and Čebyshev's inequality

$$
\begin{aligned}
&= c_1 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha-\eta} \sum_{\substack{\mathbf{j}\in\mathcal{J}\\ t_{\mathbf{j}}\sim t_{\mathbf{i}}}} p_{\mathbf{j}} P(\{\mathbf{a}\in\mathbf{R}^{dk} : |\pi(\mathbf{i})-\pi(\mathbf{j})| \leq 4\gamma t_{\mathbf{i}}\}) \\
&\leq c_1 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha-\eta} \sum_{\substack{\mathbf{j}\in\mathcal{J}\\ t_{\mathbf{j}}\sim t_{\mathbf{i}}}} p_{\mathbf{j}} P\left(\left\{\mathbf{a}\in\mathbf{R}^{dk} : \frac{|\pi(\mathbf{i})-\pi(\mathbf{j})|^{-1+\eta}}{(4\gamma t_{\mathbf{i}})^{-1+\eta}} \geq 1\right\}\right) \\
&\leq c_2 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha+1-2\eta} \sum_{\substack{\mathbf{j}\in\mathcal{J}\\ t_{\mathbf{j}}\sim t_{\mathbf{i}}}} p_{\mathbf{j}} \mathbf{E}_{\mathbf{a}} |\pi(\mathbf{i})-\pi(\mathbf{j})|^{-1+\eta} \\
&\leq c_3 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha+1-2\eta} \sum_{\substack{\mathbf{j}\in\mathcal{J}\\ t_{\mathbf{j}}\sim t_{\mathbf{i}}}} p_{\mathbf{j}} t_{\mathbf{i}\wedge\mathbf{j}}^{-1+\eta} \\
&= c_3 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha+1-2\eta} \sum_{\substack{\mathbf{j}\in\mathcal{J}\\ \mathbf{j}\leq\mathbf{i}}} p_{\mathbf{j}} t_{\mathbf{j}}^{-1+\eta} .
\end{aligned}
$$

Here in the last step we used that the inner sum is formed over a covering of $\mathcal{J}_\infty$. Now in view of $t_i < p_i$, $i = 1, 2, \ldots, k$, for sufficiently small η we may estimate the last expression by

$$\begin{aligned}
&\leq c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha+1-2\eta}\cdot p_{\mathbf{i}}\cdot t_{\mathbf{i}}^{-1+\eta}\\
&= c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha-\eta}\cdot p_{\mathbf{i}} \leq c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta-\alpha-\eta}\cdot t_{\mathbf{i}}^{\alpha'-\eta}\,.
\end{aligned}$$

Now in view of the assumption concerning α there is a unique q in $[0,1]$ such that $\alpha\in[\alpha^-(q),\alpha^+(q)]$. Then the last expression equals

$$\begin{aligned}
&c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\alpha q+\beta(q)+\alpha'-\alpha+\delta-2\eta} \leq c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\alpha q+\beta(q)+q(\alpha'-\alpha)+\delta-2\eta}\\
=\;& c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{\alpha' q+\beta(q)+\delta-2\eta} \leq c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} (t_{\mathbf{i}})^{(\alpha'+\eta)q+\beta(q)+\delta-(2+q)\eta}\\
\leq\;& c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J},|\mathbf{i}|\geq n\\ p_{\mathbf{i}}\sim t_{\mathbf{i}}^{\alpha'}}} p_{\mathbf{i}}^q t_{\mathbf{i}}^{\beta(q)+\delta-(2+q)\eta} \leq c_4 \sum_{\substack{\mathbf{i}\in\mathcal{J}\\ |\mathbf{i}|\geq n}} p_{\mathbf{i}}^q t_{\mathbf{i}}^{\beta(q)+\delta-(2+q)\eta}\,.
\end{aligned}$$

For η being sufficiently small this last expression tends to zero as $n\to\infty$ by Lemma 6. Hence for almost all $\mathbf{a}$ the sums

$$\sum_{\mathbf{i}\in\mathcal{K}_n} (t_{\mathbf{i}})^{\underline{f}(\alpha)+\delta}$$

over the coverings $\mathcal{K}_n$ tend to zero, which leads to the conclusion that the Hausdorff dimension of $\mathcal{L}$ is not greater than $\underline{f}(\alpha)$. The proof is complete.

■

Now we can prove

Theorem 1 *Assume* $\Delta = f(0) = \max \underline{f}(\alpha) \leq 1$ *and* $a < 1/3$. *Moreover, assume that* $t_i < p_i$ *for* $i = 1, 2, \dots, k$. *For any real* q *we have the relation*

$$f^-(q) \leq \dim_H(K(q))$$

for almost all parameters $\mathbf{a}$. *Consequently, if* β *is smooth at* q, *for* $\alpha = \alpha(q)$ *we get*

$$\underline{f}(\alpha) \leq \dim_H(K_\alpha)$$

almost surely in $\mathbf{a}$.
For $q \leq 1$ *we get*

$$f^-(q) \leq \dim_H(K(q)) \leq f^+(q) ,$$

and consequently, if β *is smooth at* q, *for* $\alpha = \alpha(q)$ *we have*

$$\dim_H(K_\alpha) = \underline{f}(\alpha)$$

for almost all $\mathbf{a}$ *with respect to the* dk*–dimensional Lebesgue measure.*

Proof. 1. It follows from Propositions 5, 7 and 1 that $K(q)$ supports μ^q, so that as in the proof of Proposition 4 we conclude that this set has at least dimension $f^-(q)$.
2. Since $K(q) \subseteq K^-(q)$, the second estimate follows from Proposition 8. ■

Remark. The estimate of the $f(\alpha)$–spectrum of the self–affine fractal given by the preceding theorem is not completely satisfactory for two reasons. At the one hand, if the function β is not smooth at q, then the two values $f^-(q)$ and $f^+(q)$ are different. Our conjecture is that β is indeed everywhere smooth in our situation under quite general conditions.
The second gap is that we are not able to prove the estimate from above for all α. We give some comment concerning this problem.
Figure 1 shows the spectral function $\underline{f}(\alpha)$ in a typical situation. As mentioned above, we have to cope with the problem that under the mapping π the local dimension might decrease so that the part of the fractal with some local dimension α could get a higher Hausdorff dimension than $\underline{f}(\alpha)$, theoretically even the Hausdorff dimension $\dim_H(\mathcal{J}^{>}_{(\alpha)})$ of the part $\mathcal{J}^{>}_{(\alpha)}$ of the symbolic

fractal $\mathcal{J}_\infty$, where the local dimension is greater α. This might happen only for values of α such that $\dim_H(\mathcal{J}^{>}_{(\alpha)}) > \dim_H(\mathcal{J}_{(\alpha)})$, i.e. for $\alpha < \alpha^-(0)$, see figure 1.

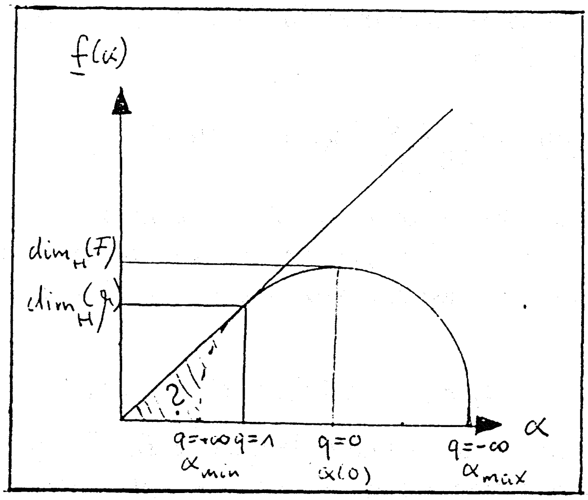

Fig. 1

Analyzing this problem more carefully we observe that π may not diminish the local dimension of a part of $\mathcal{J}_\infty$ without diminishing the Hausdorff dimension at least for the same quantity. This leads to the conclusion that this "bad" effect could only happen at that part of the function $\underline{f}$, where the slope exeeds one, which corresponds to $q > 1$ resp. $\alpha < \alpha^-(1)$. This is essentially what we prove in Proposition 8. Performing some of the estimates in that proof with greater exertion we can slightly improve this result to values of q beyond one. In fact, we find that if the slope of $\underline{f}$ at α does not exceed α^{-1}, then the Hausdorff dimension remains unchanged. To give a proof for *all* q resp. α would require to show that under π it is impossible that, given $\eta > 0$, the number of $\mathbf{i} \in \mathcal{J}$ with $t_\mathbf{i} \sim \varepsilon$ and such that $\pi(\mathbf{i}) \in B_\varepsilon(\mathbf{x})$, is finally less than $\varepsilon^{-\eta}$ as $\varepsilon \to 0$. To say it in other words, we would have to show that it is almost impossible (in $\mathbf{a}$) that exponentially many $\mathbf{i}$ (compared to the lenght of $\mathbf{i}$) are mapped to almost the same position under π, where this means that the distance is of the order $t_\mathbf{i}$. It seems very unlikely that such a coincidence may happen, but we have no proof that it happens only with Lebesgue measure zero (in $\mathbf{a}$). So we confine ourselves to $q \le 1$, since the slight improvement mentioned above is no real progress compared with the final goal to show the result for all q.

4 The Renyi Dimensions

In analogy to the introduction of the measures $\nu^{q,s}$ – which are in fact nothing but the Renyi measures for the symbolic fractal equipped with the measure ν – we introduce a two–parameter family $\mu^{q,s}$, $q, s \in \mathbf{R}$, of outer measures carried by $F(\mathbf{a})$ by the set–up.

$$\mu^{q,s}(E) := \lim_{\varepsilon\to 0} \inf_{\mathcal{B}\in C_\varepsilon(E)} \left\{ \sum_{B\in\mathcal{B}} (\mu(B))^q \cdot (\mathrm{diam} B)^s \right\} , \qquad E \subseteq F(\mathbf{a}) .$$

Here $C_\varepsilon(E)$ denotes the set of all countable or finite coverings of E by balls of diameter less than ε and of positive μ–measure. It is quite easy to check that the $\mu^{q,s}$ are outer measures. Now to each $q \in \mathbf{R}$ and $E \subseteq F(\mathbf{a})$ we define a number $d_q(E)$ by

$$d_q(E) := \inf\{s|\mu^{q,s}(E) < +\infty\} .$$

(The Renyi dimensions are usually defined as $D_q(E) := (1-q)^{-1} d_q(E)$ for $q \neq 1$ and as the negative derivative –supposed it exists– of $d_q(E)$ at $q = 1$.) Observe that $\mu^{q,s}(E)$ is a convex function of q, s for any subset E of F. Consequently $d_q(E)$ is a convex function of q.
As it was to be expected, the Renyi dimensions are closely connected with the $f(\alpha)$–spectrum.

Proposition 9 *Under the assumptions of Theorem 1, we have for almost all* $\mathbf{a}$ *(with to the Lebesgue measure on* $\mathbf{R}^{\mathbf{dk}}$*)*

$$d_q(F(\mathbf{a})) \leq \beta(q) \qquad q \leq 1 ,$$
$$d_q(F(\mathbf{a})) = \beta(q) \qquad q \leq 0 .$$

Proof. 1. First assume $0 \leq q \leq 1$. Fix some small $\delta > 0$. We show that we can cover the interval $[0,1]$ by a finite number of intervals I such that the Renyi dimension (with parameter q) of the set of points, the lower local dimension of which belongs to I, is not greater than $\beta(q) + \delta$ for almost all $\mathbf{a} \in \mathbf{R}^d$:

$$d_q(K(I)) \leq \beta(q) + \delta \quad \text{for } K(I) := \left\{ \mathbf{x} \in F(\mathbf{a}) : \varliminf_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(\mathbf{x}))}{\log \varepsilon} \in I \right\} . \quad (1)$$

Observe that there is no point in $F(\mathbf{a})$ with local dimension 1 or greater. First let $0 \le \alpha \le \alpha^+(1) - \delta/4$. Then by Lemma 1 for each $\xi > 0$ we find some covering $\mathcal{B}$ of $K(I)$ $I = [\alpha, \alpha + \delta/4]$, consisting of balls with diameter less than ξ and such that

$$\sum_{B \in \mathcal{B}} (\mathrm{diam} B)^{\alpha + \delta/2} < \xi \ . \tag{2}$$

The proof of Lemma 1 show that we can choose the centers of the $B \in \mathcal{B}$ in $K(I)$ and, moreover, we may choose these balls so small that

$$\mu(B) \le (\mathrm{diam} B)^{\alpha - \delta/2} \ ,$$

since the local dimension of the center of each such B is in I. So we get

$$\sum_{B \in \mathcal{B}} (\mu(B))^q \cdot (\mathrm{diam} B)^{\beta(q)+\delta} \le \sum_{B \in \mathcal{B}} (\mathrm{diam} B)^{q\alpha - q\delta/2 + \beta(q) + \delta}$$
$$\le \sum_{B \in \mathcal{B}} (\mathrm{diam} B)^{q\alpha + \beta(q) + \delta/2} \ .$$

Now we have $q\alpha + \beta(q) - \beta(q) + \alpha(q-1) + \alpha > \beta(q) + \alpha^+(1)(q-1) + \alpha \ge \alpha$, the latter relation being valid since β is convex. So by (2)

$$\sum_{B \in \mathcal{B}} (\mu(B))^q \cdot (\mathrm{diam} B)^{\beta(q)+\delta} \le \xi \ .$$

i.e. we proved (1) in that case. Since $[0, \alpha^+(1)]$ can be covered by finitely many of such I, we get

$$d_q(K([0, \alpha^+(1))) \le \beta(q) + \delta \ .$$

We consider the set $[\alpha^+(1), 1]$. Obviously this set can be covered by a finite number of intervals $I = [\alpha, \alpha + \delta/4]$. We learn from the proof of Proposition 8 that for almost all $\mathbf{a}$ with respect to the dk–dimensional Lebesgue measure we may cover $K(I)$ by balls with maximum radius ξ such that

$$\mu(B) \le (\mathrm{diam} B)^{\alpha - \delta/2}$$

and

$$\sum_{B \in \mathcal{B}} (\mathrm{diam} B)^{\underline{f}(\alpha) + \delta/2} < \xi \ .$$

Consequently,

$$\sum_{B\in\mathcal{B}} (\mu(B))^q \cdot (\mathrm{diam} B)^{\beta(q)+\delta} \le \sum_{B\in\mathcal{B}} (\mathrm{diam} B)^{q\alpha+\beta(q)+\delta/2}$$
$$\le \sum_{B\in\mathcal{B}} (\mathrm{diam} B)^{\underline{f}(\alpha)+\delta/2} < \xi .$$

The last estimate follows from the definition of $\underline{f}$. So for $0 \le q \le 1$ we proved that $d_q(F) \le \beta(q)$. For $q < 0$ this relation is trivially fulfilled, since by the definition of β we find coverings $\mathcal{K}$ of arbitrary small diameter of $\mathcal{J}_\infty$ such that

$$\sum_{\mathbf{i}\in\mathcal{K}} p_{\mathbf{i}}^1 \cdot t_{\mathbf{i}}^{\beta(q)+\delta} < \xi ,$$

and that yields in view of the definition of γ, which we may choose independently of $\mathbf{a}$ for bounded $\mathbf{a}$,

$$\sum_{\mathbf{i}\in\mathcal{K}} (\mu(B_{2\gamma t_{\mathbf{i}}}(\pi(\mathbf{i}))))^q \cdot (2\gamma t_{\mathbf{i}})^{\beta(q)+\delta} < (2\gamma)^{\beta(q)+\delta} \cdot \xi .$$

So for any $q \le 1$ we have $d_q(F) \le \beta(q)$ for almost all $\mathbf{a}$. Consequently for an arbitrary dense countable subset Q of $(-\infty, 1]$ we get for almost all $\mathbf{a}$ the relation

$$d_q(F) \le \beta(q), \qquad q \in Q .$$

Since $\beta(q)$ is a finite convex function and $d_q(F)$ is convex, we conclude that both functions are continuous and this implies the first part of the assertion of the Proposition.

2. Let $q \le 0$, $\delta < 0$. We denote by $K_\tau(q)$ the subset of $K(q)$, consisting of all points $\mathbf{x}$ in $K(q)$ such that

$$\mu(B'_\tau(\mathbf{x})) \le (\tau')^{\alpha^-(q)-\delta} \qquad \text{for all } \tau' \le \tau .$$

Then

$$K(q) = \bigcup_{\tau>0} K_\tau(q) .$$

Hence by Theorem 1 we find some $\tau_0 < 1$ with $\dim_H(K_{\tau_0}(q)) \ge f^-(q) - \delta/2$. Then we have for $s < \beta(q) - (1-q)\delta$ and any covering of F with diameter less than $\tau_0/2$

$$S(\mathcal{B}, q, s) := \sum_{B\in\mathcal{B}} (\mu(B))^q \cdot (\mathrm{diam} B)^s \ge S(\mathcal{B}, q, s) := \sum_{B\in\mathcal{B}'} (\mu(B))^q \cdot (\mathrm{diam} B)^s,$$

where $\mathcal{B}'$ denotes the subset of $\mathcal{B}$ consisting of balls which have a non-empty intersection with $K_{\tau_0}(q)$. So $\mathcal{B}'$ is a covering of $K_{\tau_0}(q)$. Now we may find to each $B \in \mathcal{B}'$ some ball B^* with diameter between diam B and $2 \cdot \text{diam}B$ such that the center of B^* belongs to $K_{\tau_0}(q)$ and $B \subseteq B^*$. This yields, if we denote by $\mathcal{B}^*$ the collection of all those B^*,

$$S(\mathcal{B}^*, q, s) := \sum_{B \in \mathcal{B}^*} (\mu(B))^q \cdot (\text{diam}B)^s \leq 2^s S(\mathcal{B}, q, s) ,$$

so that

$$\begin{aligned} S(\mathcal{B}, q, s) &\geq 2^{-s} \sum_{B \in \mathcal{B}^*} (\text{diam}B)^{s+q(\alpha^-(q)-\delta)} \\ &\geq 2^{-s} \sum_{B \in \mathcal{B}^*} (\text{diam}B)^{\beta(q)+q\alpha^-(q)-\delta} = 2^{-s} \cdot \sum_{B \in \mathcal{B}^*} (\text{diam}B)^{f^+(q)-\delta} . \end{aligned}$$

First let us assume that β is smooth at q. Then $f^+(q) = f^-(q)$. Since the Hausdorff dimension of $K_{\tau_0}(q)$ is at least $f(q) - \delta/2$, the last expression becomes arbitrary large for coverings $\mathcal{B}$ of arbitrary small diameter, this being valid for almost all $\mathbf{a}$. So for those $q \leq 0$, where β is smooth, we obtain $d_q(F) \geq \beta(q)$ for almost all $\mathbf{a}$. Since β is a finite convex function, we find a dense countable subset Q of $(-\infty, 0]$ where it is smooth. So for almost all $\mathbf{a}$ we have $d_q(F) \geq \beta(q)$ for each $q \in Q$, which in view of the fact that $d_q(F)$ is convex yields that for almost all $\mathbf{a}$ we have $d_q(F) \geq \beta(q)$ even for all $q \in \mathbf{R}$. This proves the Proposition.

■

5 Appendix

We prove here the fact that we can find a finite nonzero modification of $\nu^{q,\beta(q)}$. This is a consequence of some standard knowledge about net measures, as it was presented by [Be], [Ro], [Fa1]. Nevertheless we give a detailed exposition of this proof.

1. First note that by Lemma 6 in the case $q < 1$ we have to treat the case of infinite total mass, whereas for $q > 1$ we have cope with the possibility that $\nu^{q,\beta(q)}$ is the zero measure.

To begin with, let us show that in the infinite mass case we find a modification with zero mass. In fact, if $\nu^{q,\beta(q)}(\mathcal{J}_\infty) = +\infty$, then $S(q, \beta(q), \mathcal{J}_n)$ tends to

infinity as $n \to +\infty$ with a sub-exponential speed by Lemma 6, so that $h_q(\mathbf{i}) := (S(q,\beta(q),\mathcal{J}_{|\mathbf{i}|}))^{-2}$ is an admissible choice in the sense of section 1. Obviously this yields as modified measure $\overline{\nu}^{q,\beta(q)}$ the zero measure.
The fact that a modification can turn the zero measure (in the case $q > 1$) into a measure with non-zero total mass seems to be less obvious. We are going to show this in the second step.
2. Let $q > 1$. Choose any strictly increasing sequence $\{s_l\}_{l=1,2,\ldots}$ tending to $\beta(q) < 0$. We learn from the fourth step of the proof of Lemma 6 that for each l there is some number $k_l > 0$, some $n_l \in \mathbf{N}$ and some $\tau_l > l$ such that for any finite non-reducible covering $\mathcal{K}$ of $\mathcal{J}_\infty$ from $S(q,s_l,\mathcal{K}) \leq 1$ and $S(q,s_l,\mathcal{K}(r)) > M$ we would infer $S(q,s_l,\mathcal{K}(r+n_l)) > \tau_l \cdot M - k_l$ for any $r \in \mathbf{N}$. Choose some τ_l' with $1 < \tau_l' < \tau_l$ and choose an increasing sequence $\{M_l\}$ in such a way that $\tau_l \cdot M_l - k_l > \tau_l' \cdot M_l$. Let k_0 be the first index such that $S(q,s_1,\mathcal{J}_{k_0}) > M_1$ and let $k_1 = k_0 + m_1 \cdot n_1$, where $m_1 \in \mathbf{N}$ we choose in such a way that $(\tau_1')^{m_1} \cdot M_1 > M_2$. Then we choose m_2 such that $(\tau_2')^{m_2} \cdot M_2 > M_3$ and we put $k_2 = k_1 + m_2 \cdot n_2$ and so on. We define an expression $\eta(\mathbf{i})$, $\mathbf{i} \in \mathcal{J}$, as follows. For $|\mathbf{i}| \leq k_1$ let $\eta(\mathbf{i}) := (t_\mathbf{i})^{s_1}$. For $k_1 < |\mathbf{i}| \leq k_2$ we put

$$\eta(\mathbf{i}) := (t_{\mathbf{i}(k_1)})^{s_1} \left(t_\mathbf{i}/t_{\mathbf{i}(k_1)}\right)^{s_2} ,$$

and in the general case $k_l < |\mathbf{i}| \leq k_{l+1}$ we put

$$\eta(\mathbf{i}) := \left(t_{\mathbf{i}(k_1)}\right)^{s_1} \cdot \left(t_{\mathbf{i}(k_2)}/t_{\mathbf{i}(k_1)}\right)^{s_2} \cdot \left(t_{\mathbf{i}(k_3)}/t_{\mathbf{i}(k_2)}\right)^{s_3} \cdot \ldots \cdot \left(t_\mathbf{i}/t_{\mathbf{i}(k_l)}\right)^{s_{l+1}} .$$

Now consider the net measure $\widetilde{\nu}$ which is obtained by substituting the expression $t_\mathbf{i}^{\beta(q)}$ in the defining relation of $\nu^{q,\beta(q)}$ by $\eta(\mathbf{i})$. We find that $\widetilde{\nu}(\mathcal{J}_\infty) \geq 1$. In fact, assume there would be a finite covering $\mathcal{K} \in C_{k_1}(\mathcal{J}_\infty)$ with the property $\sum_{\mathbf{i}\in\mathcal{K}} p_\mathbf{i}^q \cdot \eta(\mathbf{i}) \leq 1$. Then we see that the k_0-section $\mathcal{K}(k_0)$ of $\mathcal{K}$ (which is simply $\mathcal{J}_{k_0}$) fulfils

$$\sum_{\mathbf{i}\in\mathcal{K}(k_0)} p_\mathbf{i}^q \cdot \eta(\mathbf{i}) = S(q,s_1,\mathcal{J}_{k_0}) > M_1 .$$

We see by induction that

$$\sum_{\mathbf{i}\in\mathcal{K}(k_{l-1})} p_\mathbf{i}^q \cdot \eta(\mathbf{i}) > M_l .$$

In fact, in the same way as in the proof of Lemma 6, step 4., we get

$$\sum_{\mathbf{i}\in\mathcal{K}(k_l)} p_\mathbf{i}^q \cdot \eta(\mathbf{i}) \geq \left(\sum_{\mathbf{i}\in\mathcal{K}(k_{l-1})} p_\mathbf{i}^q \cdot \eta(\mathbf{i})\right) \cdot (\tau_l')^{m_l} > M_l \cdot (\tau_l')^{m_l} > M_{l+1} .$$

So again we have a contradiction to the assumption that $\tilde{\nu}(\mathcal{J}_\infty) < 1$.
Now let $h_q(\mathbf{i}) := \eta(\mathbf{i})/t_{\mathbf{i}}^{\beta(q)}$. We find that this is a modifying function in the sense of section 1. Consequently we found a modification of $\nu^{q,\beta(q)}$ which has non–zero total mass.
3. We are now in a position where we do not need to treat the cases $q < 1$ and $q > 1$ separately: In either case we have modifications of $\nu^{q,\beta(q)}$, which have zero and infinite total mass (or 2. yields already a measure with finite mass). We have to find something between these extrema.
Let us denote the modifying functions by h_q^0 and h_q^∞ (one of them is constantly equal to one), and let us write $\nu_\infty^{q,\beta(q)}$ for the modification of $\nu^{q,\beta(q)}$ with infinite mass. Next we prove that in the case where $\nu_\infty^{q,\beta(q)}$ has no atoms we find a compact set $\mathcal{E} \subseteq \mathcal{J}_\infty$ with $\nu_\infty^{q,\beta(q)}(\mathcal{E}) = 1$. This is a standard result for net measures, we follow closely the considerations in [Fa1], proof of 5.4.
Let us write $\tilde{\nu}$ instead of $\nu_\infty^{q,\beta(q)}$ for short and denote by $\tilde{\nu}_r$ the set function which is defined by

$$\tilde{\nu}_r(\mathcal{I}) := \inf_{\mathcal{K}\in C_r(\mathcal{I})} \sum_{\mathbf{i}\in\mathcal{K}} h_q^\infty(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)}, \qquad \mathcal{I} \subseteq \mathcal{J}_\infty .$$

Since $\tilde{\nu}_r(\mathcal{J}_\infty) \uparrow \tilde{\nu}(\mathcal{J}_\infty) = +\infty$, we find some r_0 with $\tilde{\nu}_{r_0}(\mathcal{J}_\infty) > 1$. There is a compact subset $\mathcal{E}_1$ of $\mathcal{J}_\infty$ with $\tilde{\nu}_{r_0}(\mathcal{E}_1) = 1$. In fact, let us perform the following construction. We take the elements of $\mathcal{J}_{r_0}$ in lexicographic order and remove one by one from $\mathcal{J}_\infty$ those disks $D_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{J}_{r_0}$, which can be removed leaving the $\tilde{\nu}_{r_0}$–value of the remaining compact not less than one. Then we do the same with the new compact and $\mathcal{J}_{r_0+1}$ and so on. We get a descending sequence $\mathcal{E}_{(1)}$, $\mathcal{E}_{(2)}$, ... of compacts the intersection of which is the desired compact $\mathcal{E}_1$. Indeed, assume first that $\tilde{\nu}_{r_0}(\mathcal{E}_1) < 1$. This would mean we find a finite covering $\mathcal{K} \in C_{r_0}(\mathcal{E}_1)$ with

$$\sum_{\mathbf{i}\in\mathcal{K}} h_q^\infty(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)} < 1 .$$

We denote by $\mathcal{E}'$ the compact set $\bigcup_{\mathbf{i}\in\mathcal{K}} D_{\mathbf{i}}$, which covers $\mathcal{E}_1$. Observe that $\mathcal{J}_\infty \setminus \mathcal{E}'$ is compact, too. The intersection of the descending sequence of compact sets $\mathcal{E}_{(i)} \setminus \mathcal{E}'$ is empty. So $\mathcal{K}$ covers already one of the $\mathcal{E}_{(i)}$, which is in contradiction to their construction.
Now assume that $\tilde{\nu}_{r_0}(\mathcal{E}_1) = 1+\delta$, where $\delta > 0$. Since $\nu_\infty^{q,\beta(q)}$ has no atoms, the expression $h_q^\infty(\mathbf{i}(n)) \cdot p_{\mathbf{i}(n)}^q \cdot t_{\mathbf{i}(n)}^{\beta(q)}$ takes arbitrary small values for each $\mathbf{i} \in \mathcal{E}_1$.

Hence we find some $n \geq r_0$ making this expression less than δ. Now it is obvious that we may remove $D_{\mathbf{i}(n)}$ from $\mathcal{E}_1$, and the remaining compact has still a $\widetilde{\nu}_{r_0}$–value not less than one, since any covering of $\mathcal{E}_1 \setminus D_{\mathbf{i}(n)}$ plus $D_{\mathbf{i}(n)}$ yields a covering of $\mathcal{E}_1$. But this contradicts the construction of $\mathcal{E}_1$.
So we have a compact $\mathcal{E}_1$ with the property $\widetilde{\nu}_{r_0}(\mathcal{E}_1) = 1$. We even find compacts $\mathcal{E}_1^0 \supseteq \mathcal{E}_1^1 \supseteq \ldots$ such that $\widetilde{\nu}_{r_0+1}(\mathcal{E}_1^r) = \widetilde{\nu}_{r_0+2}(\mathcal{E}_1^r) = \ldots = \widetilde{\nu}_{r_0+r}(\mathcal{E}_1^r) = 1$. Again we prove this by induction: Choose $\mathcal{E}_1^0 = \mathcal{E}_1$. Let $\mathcal{E}_1^r$ be given. For each $\mathbf{i} \in \mathcal{J}_{r_0+r}$ we consider the set $\mathcal{E}_1^r \bigcap D_{\mathbf{i}}$. If $\widetilde{\nu}_{r_0+r+1}(\mathcal{E}_1^r \bigcap D_{\mathbf{i}}) \leq h_q^{\infty}(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)}$, we leave this part of $\mathcal{E}_1^r$ unchanged. But if this relation is not fulfilled, we substitute $\mathcal{E}_1^r \bigcap D_{\mathbf{i}}$ by a compact subset $\mathcal{F}$ with $\widetilde{\nu}_{r_0+r+1}(\mathcal{F}) = h_q^{\infty}(\mathbf{i}) \cdot p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)}$. This is possible, see the construction of $\mathcal{E}_1$. In that way we do not change the values of $\widetilde{\nu}_{r_0+j}(\mathcal{E}_1^r)$, $0 \leq j \leq r$. Doing this for all $\mathbf{i} \in \mathcal{J}_{r_0+r}$, we get $\mathcal{E}_1^{r+1} \subseteq \mathcal{E}_1^r$ as desired. We denote the compact limit of these sets by $\mathcal{E}$. The same argument as above shows that $\widetilde{\nu}_r(\mathcal{E}) = 1$ for each $r \geq r_0$. So we have $\widetilde{\nu}(\mathcal{E}) = \nu_{\infty}^{q,\beta(q)}(\mathcal{E}) = 1$.
4. Now we are in a position to prove the existence of a modification of $\nu^{q,\beta(q)}$ with total mass one. If $\nu_{\infty}^{q,\beta(q)}$ is non–atomic, by 3. we find a compact $\mathcal{E}$ with $\nu_{\infty}^{q,\beta(q)}(\mathcal{E}) = 1$. Now choose

$$h_q(\mathbf{i}) = \begin{cases} h_q^{\infty}(\mathbf{i}) & for \quad \mathcal{E} \bigcap D_{\mathbf{i}} \neq \emptyset \\ h_q^0(\mathbf{i}) & else \end{cases} \quad , \qquad \mathbf{i} \in \mathcal{J} \, .$$

Let us denote the corresponding modified measure by $\overline{\nu}^q$. It is obvious that $\overline{\nu}^q(\mathcal{E}) = 1$. On the other hand, on each $D_{\mathbf{i}}$ which has no intersection with $\mathcal{E}$ the measure $\overline{\nu}^q$ coincides with the zero measure by definition of h_q^0. So it is the zero measure on the countable union of all these disks, i.e. on $\mathcal{J}_{\infty} \setminus \mathcal{E}$. This is the proof in the non–atomic case. The case where there is an atom $\mathbf{i}_0$ of $\nu_{\infty}^{q,\beta(q)}$ is easy: The expression

$$h_q^{\infty}(\mathbf{i}_0(n)) \cdot p_{\mathbf{i}_0(n)}^q \cdot t_{\mathbf{i}_0(n)}^{\beta(q)}$$

must have a non–zero lower limit and it can only have a sub–exponential growth (since $S(q, \beta(q), \mathcal{J}_r)$ has a sub–exponential growth). So by

$$h_q(\mathbf{i}) = \begin{cases} (p_{\mathbf{i}}^q \cdot t_{\mathbf{i}}^{\beta(q)})^{-1} & for \quad \mathbf{i} < \mathbf{i}_0 \\ h_q^0(\mathbf{i}) & else \end{cases} \quad , \qquad \mathbf{i} \in \mathcal{J}$$

we get a modifying function and obviously we have for the modified measure $\overline{\nu}^q = \delta_{\mathbf{i}_0}$. The proof is finished.

■

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