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K. Fleischmann[1] and I. Kaj[2]

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[1] Institut für Angewandte Analysis
und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin
Germany

[2] Department of Mathematics
Uppsala University
Box 480
S – 75106 Uppsala
Sweden

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LARGE DEVIATION PROBABILITIES FOR SOME RESCALED SUPERPROCESSES

by Klaus Fleischmann and Ingemar Kaj [1)]

Institute of Applied Analysis and Stochastics, Berlin

Department of Mathematics, Uppsala University

Abstract. We consider a class of rescaled superprocesses and derive a full large deviation principle with a good convex rate functional defined on the measure state space. A relatively complete picture of the related non-linear reaction-diffusion equation is accomplished although the rate functional is only partly expressed in terms of solutions of the equation.

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1. Introduction

1.1 Motivation

Since the pioneering paper of Liemant (1969), much has been done in the field of *spatially distributed branching models of infinite populations:* equilibrium theory, convergence theorems, scaling properties, hydrodynamics, sample path properties, random media effects - to mention only some main topics. However, to our knowledge there are only a few papers dealing with *large deviation aspects.* (Generally speaking, large deviation probabilities are of particular interest in statistical physics, in models in random media, and in other respect; the relatively simple branching models may serve as a certain test case only.)

Cox and Griffeath (1985) considered the critical binary branching Brownian motion starting with a homogeneous Poisson particle system of density one and studied in dimensions $d \geq 3$ the asymptotics of the (logarithmic) large deviation probabilities

$$\log \mathcal{P}r\left(t^{-1} \int_0^t ds \; N_s(B) > (1+\varepsilon)\ell(B)\right)$$

as $t \to \infty$ where $N_s(B)$ counts the number of particles at time s in the bounded Borel set $B \subset \mathbb{R}^d$ of volume $\ell(B)$, and $\varepsilon > 0$ has to be *sufficiently small.* This last condition has its origin in the method they use based on cumulants: it guarantees the convergence of some power series expansions. Also, in recent manuscripts of Lee (1992) and Iscoe and Lee (1992) similar restrictions enter into some large deviation probabilities for closely related occupation time processes; the only exception is a dimension $d=3$ result, where a steepness argument could be used.

To remove such "disturbing" conditions was our primary motivation to look for large deviation properties in infinite branching models. A striking technical fact that makes the subject interesting is that in such branching models exponential moments are *infinite* as a rule.

In the present note we are concerned with large deviation probabilities $\log \mathcal{P}r(X^K(t) \in A)$ as $K \to \infty$, where X^K refers to a branching process appropriately *scaled* in time, space and mass, t is a fixed macroscopic time point and A is any open or closed set in the state space of the scaled processes. We restrict our main attention to *supercritical* dimensions d, i.e. to those dimensions where the unscaled process has steady states, and under a critical rescaling we prove a *full* large deviation result.

From the variety of possible choices we decided to work with a measure-valued branching model (*Dawson-Watanabe process, superprocess*), which reduces the number of relevant approximations forced by the scaling and which simplifies the use of some analytical tools.

In the remainder of this Introduction we will briefly describe the model, formulate the main result and provide some heuristic background leading to a dimensionally independent reformulation of the problem.

1.2. Model

Let $\mathrm{X} = [\mathrm{X}, \mathbb{P}^{\kappa,\rho}_{s,\mu}; s\in\mathbb{R}_+, \mu\in\mathcal{M}_a]$ denote the critical *superstable motion* on $\mathbb{R}^d$ with *motion index* $\alpha\in(0,2]$, *"diffusion" constant* $\kappa\geq 0$, and constant *branching rate* $\rho\geq 0$, related via its Laplace transition functionals

(1.2.1) $$\mathbb{E}^{\kappa,\rho}_{s,\mu} \exp(\mathrm{X}(t),\varphi) = \exp(\mu, u_\varphi(t-s)), \qquad 0\leq s\leq t,\ \mu\in\mathcal{M}_a,\ \varphi\in\Phi_-$$

to the solutions $u=u_\varphi$ of the non-linear differential equation

(1.2.2) $$\frac{\partial}{\partial t}u(t,y) = \kappa\Delta_\alpha u(t,y) + \rho u^2(t,y), \qquad t>0,\ y\in\mathbb{R}^d,$$
$$u(0+,\cdot) = \varphi \in \Phi_-.$$

Here $\mathcal{M}_a$ is a Polish space of locally finite (non-negative) measures defined on $\mathbb{R}^d$ with at most potential growth at infinity, determined by some constant a. Moreover, Φ is a related Banach space of continuous test functions on $\mathbb{R}^d$, and Φ_- the subset of its non-positive members. Integrals $\int m(\mathrm{d}x)\, f(x)$ are written as (m,f), and $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian. (For more technical details, see Sections 2 and 3 below.)

In other words, the states of the *time-homogeneous Markov process* X are a-tempered measures, and given the state $\mathrm{X}(s)=\mu$ at an initial time s, the Laplace functional of the random measure $\mathrm{X}(t)$, $t>s$, is described with the help of the solutions u of the non-linear equation (1.2.2).

Roughly speaking, the stochastic evolution of populations $X(s)$ is determined by two components: the population mass is smeared out by the stable flow with generator $\kappa\Delta_\alpha$ (the heat flow in the case $\alpha=2$), and all "differentially small" portions of mass $X(s,dy)$ independently (concerning the space points y) fluctuate in time according to the stochastic equation

$$\text{(1.2.3)} \qquad d\zeta_t = \sqrt{2\rho\zeta_t}\, dW_t, \quad t\geq s,$$

(starting in $\zeta_s=X(s,dy)$ for y "fixed" and with W a standard Wiener process in $\mathbb{R}$) describing the simplest critical continuous state Galton-Watson process (Lamperti process) with "branching rate" ρ.

We mention that such superprocesses serve as *diffusion approximation* for high density branching particle models, where the particles have small mass, move independently according to symmetric α-stable motions and split critically with a large rate and finite variance. (For a recent survey on superprocesses we refer to Dawson (1991).)

For constants $\gamma>0$ and $K>0$ we define the *scaled processes* X^K:

$$\text{(1.2.4)} \qquad (X^K(t),\varphi) := (X(K^\gamma t),\varphi^K), \quad t\geq 0, \quad \varphi^K := K^{-d}\varphi(\cdot/K), \quad \varphi\in\Phi_-;$$

i.e. we speed up the time by a factor K^γ, contract the space and rescale the mass both by the factor K^{-d}.

1.3. Basic Scaling Properties

Before we will formulate our main result, we review the *"basic ergodic theory"* on the scaled processes X^K as $K\to\infty$. To this end we have to distinguish between several parameter constellations.

First consider the situation of a *critical scaling* with which we mean that $\gamma = \alpha\wedge d$ holds:

(∗) In the case of a *subcritical dimension,* i.e. if $d<\alpha$, or more explicitly, $\gamma=d=1<\alpha$, the scaled processes X^K converge in distribution to X but the latter defined with diffusion constant $\kappa=0$ (i.e. the motion component disappears), provided that the initial measures $X^K(0)$ converge in law to some $X(0)$. If $\rho>0$, this means, that the scaling will catch clumps, which in the limit are located in Poissonian points and the sizes of the clumps are independent, fluctuate according to (1.2.3), and for a fixed macroscopic time point t, are exponentially distributed. In other words, this limit can be viewed as a collection of independent copies of processes fluctuating according to the stochastic equation (1.2.3), with initial states ζ_0 according to the limiting initial measure $X(0,dy)$. For details concerning this *time-space-mass scaling limit theorem* we refer to Dawson and Fleischmann (1988).

(**) In the situation of a *critical dimension*, i.e. if $d=\alpha$, or more explicitly, if $\gamma=d=\alpha=1$ or 2, the superprocess is *self-similar*, i.e. that X^K coincides in distribution with X, provided that the initial states $X^K(0)$ and $X(0)$ coincide in law (for instance, if $X(0) = \ell$, the Lebesgue measure; see the Lemma 4.7.1 below).

(***) For *supercritical dimensions* $d > \alpha$ $(= \gamma)$ a *law of large numbers* (LLN) is true: For fixed $t\geq 0$,

$$X^K(t) \xrightarrow[K\to\infty]{\mathcal{P}r} \mathcal{T}^K_t\mu \quad \text{if} \quad X^K(0) \xrightarrow[K\to\infty]{\mathcal{P}r} \mu, \quad \mu\in\mathcal{M}_a,$$

where $\mathcal{T}^K_t\mu$ is the measure which results if the α-stable flow with "diffusion" constant $\kappa\geq 0$ acts on μ over a time period of length t; see the Lemma 4.5.2 below. In this case (if $\rho>0$) also the *Gaussian fluctuations* around the α-stable flow $\mathcal{T}^K\mu$ can be computed, leading to Ornstein-Uhlenbeck processes; see e.g. Dawson, Fleischmann, and Gorostiza (1989) (specialized to a constant medium and to branching with finite variance).

So far we discussed the situation under the critical scaling $\gamma = \alpha\wedge d$. In the case of a *subcritical scaling* $\gamma < \alpha\wedge d$ (i.e. if the microscopic time grows only "moderately"), always a *law of large numbers* holds; see Remark 4.7.5 below. On the other hand, for a *supercritical scaling* $\gamma > \alpha\wedge d$, under reasonable initial conditions one expects a *local extinction* $X^K(t) \xrightarrow[K\to\infty]{\mathcal{P}r} 0$, $t>0$, provided that $d\leq\alpha$, whereas in supercritical dimensions $d>\alpha$ again a law of large numbers should hold.

1.4. Main Result

In this note we fix our attention to *large deviations* related to the law of large numbers (***) above, i.e. with the most interesting LLN since in this case the scaling is critical.

For convenience, similarly to (1.2.4), we introduce a notation μ^K for a scaling of measures μ by

(1.4.1) $$(\mu^K,\varphi) := (\mu,\varphi^K), \quad \mu\in\mathcal{M}_a,\ K>0,\ \varphi\in\Phi_-$$

(with φ^K defined in (1.2.4)).

Theorem 1.4.2 **(large deviation principle).** *Assume that* $d>\alpha=\gamma$. *Fix* $\kappa,\rho\geq 0$, *a measure* $\mu\in\mathcal{M}_a$, $\mu\neq 0$, *and a (macroscopic) time point* $t>0$. *For* $K>0$, *let* μ_K *denote the measure in* $\mathcal{M}_a$ *which satisfies* $(\mu_K)^K = \mu$ *(for instance* $\mu=\ell\equiv\mu_K$*). Then the following large deviation principle holds: There is a lower semi-continuous* **convex** *functional* $\mathbf{S}_{\mu,t}: \mathcal{M}_a \mapsto [0,+\infty]$ *with* $\mathbf{S}_{\mu,t}(\mathcal{T}_t^K\mu) = 0$ *such that,*

(i) *for each open subset* G *of* $\mathcal{M}_a$,

$$\liminf_{K\to\infty} K^{-(d-\alpha)} \log \mathbb{P}^{\kappa,\rho}_{0,\mu_K}\left(\mathbf{X}^K(t) \in G\right) \geq -\inf_{\nu\in G} \mathbf{S}_{\mu,t}(\nu),$$

(ii) *for each closed subset* F *of* $\mathcal{M}_a$,

$$\limsup_{K\to\infty} K^{-(d-\alpha)} \log \mathbb{P}^{\kappa,\rho}_{0,\mu_K}\left(\mathbf{X}^K(t) \in F\right) \leq -\inf_{\nu\in F} \mathbf{S}_{\mu,t}(\nu).$$

(iii) $\mathbf{S}_{\mu,t}$ *is a* **good rate functional:** *all sets* $\{\nu\in\mathcal{M}_a;\ \mathbf{S}_{\mu,t}(\nu)\leq N\}$, $N>0$, *are compact.*

That is, roughly speaking,

$$\mathcal{P}r(\mathbf{X}^K(t) = d\nu) \approx \exp[-K^{d-\alpha}\mathbf{S}_{\mu,t}(\nu)], \quad \text{as } K\to\infty,$$

in the sense of logarithmic equivalence.

The point is that for *(i)* we do not need any smallness condition, i.e. a restriction to small (open) neighborhoods G of $\mathcal{T}_t^K\mu$.

Of course, it is desirable to learn more on the rate functional $\mathbf{S}_{\mu,t}$. In particular, one would like to know the relation to log-Laplace functionals (exponential moments). In fact, if we set

(1.4.3) $$\Lambda_{\mu,t}(\varphi) := \log \mathbb{E}^{\kappa,\rho}_{0,\mu} \exp(\mathbf{X}(t),\varphi) \in (-\infty,+\infty], \qquad \varphi\in\Phi,$$

then under the assumptions in the theorem,

(1.4.4) $$K^{-(d-\alpha)} \log \mathbb{E}^{K,\rho}_{0,\mu_K} \exp(K^{d-\alpha} X^K(t),\varphi) \equiv \Lambda_{\mu,t}(\varphi).$$

Thinking in terms of other large deviation results this immediately raises the *question* whether

(1.4.5) $$S_{\mu,t}(\nu) = \sup_{\varphi\in\Phi} \{(\nu,\varphi) - \Lambda_{\mu,t}(\varphi)\} =: \Lambda^*_{\mu,t}(\nu), \quad \nu\in\mathcal{M}_a.$$

This would be advantageous since, under a boundedness condition, $\Lambda_{\mu,t}(\varphi)$ can be described by means of the equation (1.2.2); see Proposition 3.3.1 and Corollary 3.3.4 below.

From the general theory follows that $S_{\mu,t} \geq \Lambda^*_{\mu,t}$ (see, for instance, [9], Exercise 2.2.23 (ii)). Regarding the converse inequality, we are able to show the relation

(1.4.6) $$\Lambda_{\mu,t}(\varphi) = \sup_{\nu\in\mathcal{M}_a} \{(\nu,\varphi) - S_{\mu,t}(\nu)\}, \quad \varphi\in\Phi_{\mu,t},$$

where $\Phi_{\mu,t}$ denotes the largest *open* set off all those functions $\varphi\in\Phi$ such that $\Lambda_{\mu,t}(\varphi) < +\infty$ (see Subsection 4.6 below). This leaves open whether the relation (1.4.5) holds in full generality.

Of course, one could derive *(i)* with $S_{\mu,t}$ replaced by $\Lambda^*_{\mu,t}$ for open sets G contained in some neighborhood $\mathcal{O}$ of $\mathcal{T}^K_t\mu$ in analogy with the finite-dimensional case (see, for instance, Ellis (1984)), using that $\Lambda_{\mu,t}(\varphi)$ is smooth on $\Phi_{\mu,t}$. Then (1.4.5) would hold at least for $\nu\in\mathcal{O}$.

In the special case of a vanishing "diffusion" constant $\kappa=0$ (i.e. if there is no motion in the model) one has a complete description of $\Lambda_{\mu,t}$, and the Legendre transform $\Lambda^*_{\mu,t}$ can be computed explicitly (see the Appendix). An interesting fact is that as a rule this $\Lambda^*_{\mu,t}$ is *not strongly convex*, hence $\Lambda_{\mu,t}$ is not "steep". This is in contrast with the fact that steepness is often used as a starting point to get the lower bound estimate *(i)* in terms of $\Lambda^*_{\mu,t}$.

1.5. To the Method of Proof: A Heuristic Argument and Reformulation

By scaling properties of the stable semi-group and of the critical continuous-state Galton-Watson process, and by our assumed parameter relations, the time-space-mass scaling of **X** as $K\to\infty$ can be reformulated as a limit in law of X under $\rho\to 0$ (see Lemma 4.7.1 below).

For the sake of a heuristic argument, let us restrict our attention for the moment to the case of a "discrete" branching rate $\rho=1/N$, $N\to\infty$. Then, again by scaling arguments, $\mathbf{X}(t)$ with respect to $\mathbb{P}^{\kappa,1/N}_{0,\mu}$ has the same distribution as $N^{-1}\sum_{i=1}^{N}\mathbf{X}^{i}(t)$, where the $\mathbf{X}^{i}(t)$ are independent and have the law $\mathbb{P}^{\kappa,1}_{0,\mu}$. Now apply an infinite dimensional version of *Cramèr's Theorem*. Here, of course, one has to be careful since the exponential moments of the $(\mathbf{X}^{i}(t),\varphi)$, $\varphi\in\Phi_{+}$, are *infinite* as a rule. But they are finite for all φ in some neighborhood (depending on t) of the origin in the Banach space Φ, and by some additional efforts one can really show that such a version of Cramèr holds in the present case.

To be more precise, our approach is to investigate the large deviation probabilities

$$R^{-1}\log\mathbb{P}^{\kappa,\rho}_{0,R\mu}\Big(R^{-1}\mathbf{X}(t)\in\cdot\Big)\quad\text{as } R\to\infty,$$

which exist *without any dimension restrictions* and may be expressed by means of some rate functional $\mathbf{S}_{\mu,t}$ (see Theorem 4.1.1 below). Then we show as a consequence that Theorem 1.4.2 above holds with the same rate functional $\mathbf{S}_{\mu,t}$.

Concerning technical details, a necessary step in the development is to deal with equation (1.2.2) for initial functions φ which admit also *positive* values. Here one has to take into account that, for given φ and a fixed time interval, solutions u may *not exist* (think of the explosive behavior of the ordinary equation $\frac{d}{dt}u(t) = \rho u^2(t)$, for $\rho>0$ and positive initial values). Perhaps we should add at this place, that (1.2.2) will be handled by transferring it to the corresponding integral equation (*mild* solutions of (1.2.2)). A rather detailed picture is given in the Theorem 2.4.3 below, which in particular covers known results due to Fujita (1966) or Nagasawa and Sirao (1969).

We mention that the methods in this note are useful also for dealing with functional deviations *in time* (and not only in space), for large deviations related to other variants of the law of large numbers (subcritical scaling), and also for large deviations in the case of the occupation time process related to X (as in the model mentioned in the beginning).

1.6. Outline

The relevant tools concerning equation (1.2.2) for φ with possibly changing sign are compiled in Section 2 in a more general set-up than is needed for Theorem 1.4.2 (for the sake of later reference). In Section 3, by analytic continuation methods, the connection to the log-Laplace functionals (exponential moments) is given. The large deviation estimates follow in the final section, where we adopt some methods found in Deuschel and Stroock (1989). An Appendix is devoted to the special case $\kappa=0$.

2. On the Cumulant Equation

2.1. Preliminaries

In this subsection we will introduce the function space Φ^I in which solutions of the equation (1.2.2) will "live".

Fix a dimension $d\geq 1$, a motion index $\alpha\in(0,2]$, a constant a satisfying $d<a\leq d+\alpha$, and introduce the *reference function* $\varphi_a(y) := (1+|y|^2)^{-a/2}$, $y\in\mathbb{R}^d$. Let Φ denote the linear space of all real-valued *continuous* functions φ defined on $\mathbb{R}^d$ with the property that the ratio $\varphi(y)/\varphi_a(y)$ *converges* to a finite limit as $|y|\to\infty$. In Φ we introduce the norm

$$\|\varphi\| := \sup_{y\in\mathbb{R}}|\varphi(y)/\varphi_a(y)|, \qquad \varphi\in\Phi.$$

Then Φ is a separable Banach space. Note that $\mathcal{C}^{comp} \subset \Phi \subset \mathcal{C}_0$ where $\mathcal{C}^{comp}$ and $\mathcal{C}_0=\mathcal{C}_0[\mathbb{R}^d]$ are the spaces of all continuous functions with compact support or vanishing at infinity, respectively, both equipped with the supremum norm $\|\cdot\|_\infty$ of uniform convergence. Moreover, the embedding of Φ into $\mathcal{C}_0$ is continuous, since $\|\varphi\|_\infty\leq\|\varphi\|$, $\varphi\in\Phi$.

Fix a finite closed time interval $I:=[L,T]$, $L\leq T$. Let Φ^I denote the linear space of all *continuous* curves u defined on I and with values in Φ. Equip Φ^I with the supremum norm, denoted by

$$\|u\|_I := \sup\{\|u(t)\|;t\in I\}, \qquad u\in\Phi^I.$$

By setting $u(t,y) := u(t)(y)$, $t\in I$, $y\in\mathbb{R}^d$, we also regard u as a function on $I\times\mathbb{R}^d$, and we get a continuous embedding $\Phi^I \subset \mathcal{C}_0[I\times\mathbb{R}^d]$ since $\|u\|_\infty \leq \|u\|_I$,

$u\in\Phi^I$. Moreover, we immediately obtain:

Lemma 2.1.1. *The spaces* Φ *and* Φ^I *are* **Banach algebras** *with respect to the pointwise product of functions.*

2.2. On the Stable Flow

Recall that $\kappa\geq 0$ is a fixed ("diffusion") constant. If $\kappa>0$, then the *stable semigroup* $\{\mathcal{T}^\kappa_t;t\geq 0\}$ with generator $\kappa\Delta_\alpha = -\kappa(-\Delta)^{\alpha/2}$ possesses continuous transition density functions $p^\kappa(s,t,x,y)=p^\kappa(t-s,y-x)$, $s<t$, $x,y\in\mathbb{R}^d$, with characteristic functions

$$\int dz\ p^\kappa(r,z)\ e^{i\theta\cdot z} = \exp[-\kappa r|\theta|^\alpha], \qquad r>0,\ \theta\in\mathbb{R}^d. \tag{2.2.1}$$

(Note again that with $\alpha=2$ the *heat flow* is included.)

For $\varphi\in\Phi$ we set $\mathcal{T}^{0,I}\varphi := \varphi$ and for $\kappa>0$ define $\mathcal{T}^{\kappa,I}\varphi := \{\mathcal{T}^\kappa_{T-s}\varphi;s\in I\}$ where by definition $\mathcal{T}^\kappa_t\varphi(x) = \int dy\ p^\kappa(t,y-x)\ \varphi(y)$, $t>0$, $x\in\mathbb{R}^d$. The following lemma can be found, for instance, in Dawson and Fleischmann (1988), Lemma 4.1. (Note that $\mathcal{T}^\kappa_t = \mathcal{T}^1_{\kappa t}$.)

Lemma 2.2.2. $[\kappa,\varphi] \mapsto \mathcal{T}^{\kappa,I}\varphi$ *is a* **continuous** *mapping of* $\mathbb{R}_+\times\Phi$ *into* Φ^I.

As a simple consequence we get (see also Dawson and Fleischmann (1992), formula line (3.4)):

Lemma 2.2.3. *The linear operators* $\mathcal{T}^\kappa_t$ *acting in* Φ *are* **uniformly bounded** *for bounded* t *and* κ.

Proof. In fact, for $0 \leq t,\kappa \leq c$,

$$\left\|\mathcal{T}^\kappa_t\varphi\right\| \leq \|\varphi\|\ \left\|\mathcal{T}^1_{\kappa t}\varphi_a\right\| \leq \|\varphi\|\ \left\|\mathcal{T}^{1,J}\varphi_a\right\|_J = \text{const}\ \|\varphi\|,$$

where for the moment we set $J:=[0,c^2]$, and *const* will always denote a finite constant. □

2.3. Another Convolution Map

For $u\in\Phi^I$ we introduce $W^{\kappa,I}u$ by setting

$$(W^{\kappa,I}u)(s) = \int_s^T dr\ \mathcal{T}^\kappa_{r-s}u(r), \qquad s\in I=[L,T].$$

Lemma 2.3.1. $[\kappa,u] \mapsto W^{\kappa,I}u$ *is a* **continuous** *mapping of* $\mathbb{R}_+\times\Phi^I$ *into* Φ^I.

Proof. According to Lemma 2.2.2, $\mathcal{T}^\kappa_{r-s}u(r)$ belongs to Φ, for each pair r,s satisfying $r \ge s$. In view of Lemma 2.2.3,

$$\|\mathcal{T}^{\kappa}_{s}\varphi - \mathcal{T}^{\kappa}_{r}\psi\| \le \|\mathcal{T}^{\kappa}_{s}(\varphi-\psi)\| + \|\mathcal{T}^{\kappa}_{s}\psi - \mathcal{T}^{\kappa}_{r}\psi\|$$

$$\le \text{const}\ \|\varphi-\psi\| + \text{const}\ \|\mathcal{T}^{\kappa}_{|s-r|}\psi - \psi\|, \qquad s,r,\kappa \ge 0,\ \varphi,\psi\in\Phi.$$

Assume $\kappa_n \to \kappa$ and $u_n \to u$ as $n\to\infty$. For $s_n \in I$, by the previous estimates,

$$\left\|W^{\kappa_n,I}u_n(s_n) - W^{\kappa,I}u(s_n)\right\| \le \int_I \mathrm{d}r\ \left\|\mathcal{T}^{1}_{\kappa_n|r-s_n|}u_n(r) - \mathcal{T}^{1}_{\kappa|r-s_n|}u(r)\right\|$$

$$\le \int_I \mathrm{d}r\ \left(\text{const}\ \|u_n(r) - u(r)\| + \text{const}\ \left\|\mathcal{T}^{1}_{|\kappa_n-\kappa|\,|r-s_n|}u(r) - u(r)\right\|\right).$$

In virtue of Lemma 2.2.2, the latter norm expression converges to 0 as $n\to\infty$, for each r. Moreover, by Lemma 2.2.3, it is bounded above by $const\ \|u(r)\| \le const\ \|u\|_I = const$. Hence, by dominated convergence, the integral over the second norm expression converges to 0 as $n\to\infty$. But for the first term we get $\le$ const $\|u_n-u\|_I$ which converges to 0, too. Summarizing,

$$\left\|W^{\kappa_n,I}u_n - W^{\kappa,I}u\right\|_I = \sup_{s\in I}\left\|W^{\kappa_n,I}u_n(s) - W^{\kappa,I}u(s)\right\| \longrightarrow 0 \quad \text{as } n\to\infty,$$

and we are done. □

2.4. Solution via the Implicit Function Theorem

Recall that $I=[L,T]$. Now we have together all ingredients to introduce the functional

(2.4.1) $$\mathbf{F}(\kappa,\rho,\varphi,\psi,u) := u - \mathcal{T}^{\kappa,I}\varphi - W^{\kappa,I}\psi - \rho W^{\kappa,I}(u^2)$$

defined for $[\kappa,\rho,\varphi,\psi,u] \in \mathbb{R}_+\times\mathbb{R}_+\times\Phi\times\Phi^I\times\Phi^I$. We will study the equation

(2.4.2) $$\mathbf{F}(\kappa,\rho,\varphi,\psi,u) = 0$$

which covers (1.2.2). In fact, in more details it can be written as

(2.4.2') $$u(s) = \mathcal{T}^{\kappa}_{T-s}\varphi + \int_s^T \mathrm{d}r\ \mathcal{T}^{\kappa}_{r-s}\psi(r) + \rho\int_s^T \mathrm{d}r\ \mathcal{T}^{\kappa}_{r-s}(u^2(r)), \qquad s\in I,$$

and a formal differentiation to the time variable s yields

(2.4.2") $$-\frac{\partial}{\partial s}u = \kappa\Delta_\alpha u + \psi + \rho u^2, \qquad u\big|_{s=T-} = \varphi.$$

(To rebuild (1.2.2), set $L=0$, $\psi=0$, and reverse the time: $s\mapsto T-t$; later the *backward formulation* is needed to express some functionals of the occupation time process related to X.) Our purpose will be to solve (2.4.2) with the help of the implicit function theorem, for adequate $[\kappa,\rho,\varphi,\psi]$.

Theorem 2.4.3 **(cumulant equation).** *Recall that $1\le d<a\le d+\alpha$, $0<\alpha\le 2$, and $I=[L,T]$ are fixed.*

(i) (uniqueness). *To each* $[\kappa,\rho,\varphi,\psi]\in\mathbb{R}_+^2\times\Phi\times\Phi^I$ *there exists at most one element* $u\in\Phi^I$ *which solves* $\mathbf{F}(\kappa,\rho,\varphi,\psi,u)=0$.

(ii) (existence). *There is a* **maximal open convex** *subset* $\mathcal{U}$ *of* $\mathbb{R}_+^2\times\Phi\times\Phi^I$, *such that for each* $[\kappa,\rho,\varphi,\psi]\in\mathcal{U}$ *there exists a solution* $u =: u_{[\kappa,\rho,\varphi,\psi]} \in \Phi^I$ *for which* $\mathbf{F}(\kappa,\rho,\varphi,\psi,u)=0$. *This* $\mathcal{U}$ *includes* $\mathbb{R}_+\times\{0\}\times\Phi\times\Phi^I$ *and* $\mathbb{R}_+^2\times\Phi_-\times\Phi_-^I$, *in particular,* $u_{[\bullet,\bullet,0,0]}=0$.

(iii) (continuity, convexity, and analyticity). *The mapping* $[\kappa,\rho,\varphi,\psi] \mapsto u_{[\kappa,\rho,\varphi,\psi]}\in\Phi^I$ *defined on* $\mathcal{U}$ *is continuous, and for fixed* $[\kappa,\rho]$, *the map* $[\varphi,\psi]\mapsto u_{[\kappa,\rho,\varphi,\psi]}$ *is convex and analytic (with* $[\kappa,\rho,\varphi,\psi]$ *ranging in* $\mathcal{U}$*).*

(iv) (blow-up). *If* $L<T$ *then* $\mathcal{U}$ *is different from* $\mathbb{R}_+^2\times\Phi\times\Phi^I$, *and*

$$\sup\{u_{[\kappa,\rho,\varphi,\psi]}(s,y);\ [s,y]\in I\times\mathbb{R}^d\} \longrightarrow +\infty \quad as \quad [\kappa,\rho,\varphi,\psi] \longrightarrow [\bar\kappa,\bar\rho,\bar\varphi,\bar\psi]\in\partial\mathcal{U},$$

the boundary of $\mathcal{U}$.

(v) (maximum principle). $u_{[\kappa,\rho,\varphi,\psi]}\le 0$ (≥ 0) *provided that* $\varphi,\psi\le 0$ $(\ge 0$, *respectively).*

(vi) (global solutions). *Fix* $[\kappa,\rho]\in\mathbb{R}_+^2$. *If* $[\varphi,\psi]\le 0$, *then even a global solution exists, that is the solution can be extended from* $I=[L,T]$ *to all of* $(-\infty,T]$. *On the other hand, if* $d>\alpha$ *(***supercritical dimension***) and* $[\varphi_+,\psi_+]$ *is sufficiently small in norm, then again a global solution exists.*

(Of course, in our real Banach space setting, *analyticity* at a point means that the power series expansion converges absolutely in a neighborhood of that point; see e.g. Zeidler (1986), Section 8.2.)

To prepare for the proof of the theorem, first note that $\mathbf{F}$ maps $\mathbb{R}_+^2\times\Phi\times\Phi^I\times\Phi^I$ continuously into Φ^I, see the Lemmas 2.2.2, 2.3.1, and 2.1.1. Furthermore, at each point $[\kappa,\rho,\varphi,\psi,u] \in \mathbb{R}_+^2\times\Phi\times\Phi^I\times\Phi^I$, we get the following first *partial (Fréchet) derivative* of $\mathbf{F}$ with respect to u:

(2.4.4) $$D_u^1\mathbf{F}(\kappa,\rho,\varphi,\psi,u)v = v - 2\rho W^{\kappa,I}(uv), \qquad v\in\Phi^I.$$

Consequently, this partial derivative is linear in u and continuous in $[\kappa,\rho,\varphi,\psi,u]$ (again by the Lemmas 2.3.1 and 2.1.1).

Lemma 2.4.5. *For each* $[\kappa,\rho,\varphi,\psi,u] \in \mathbb{R}_+^2\times\Phi\times\Phi^I\times\Phi^I$, *the (bounded linear) operator* $D_u^1\mathrm{F}(\kappa,\rho,\varphi,\psi,u)\colon \Phi^I \mapsto \Phi^I$ *is* **bijective.**

Proof. Suppose $L<T$ (otherwise $W^{\kappa,I}=0$ and $D_u^1\mathrm{F}(\kappa,\rho,\varphi,\psi,u)$ is the identity). Fix $[\kappa,\rho,\varphi,\psi,u]$ and let v belong to Φ^I with $D_u^1\mathrm{F}(\kappa,\rho,\varphi,\psi,u)v=0$. By (2.4.4) and boundedness of the operator $W^{\kappa,I}$ according to Lemma 2.3.1,

$$\|v(s)\| \le \text{const } \|u\|_I \int_s^T dr\ \|v(r)\|, \qquad s\in I.$$

Then Grönwall's Lemma (pass to $\|v(T-s)\|$) implies that $\|v(s)\|\equiv 0$, i.e. $v=0$. Consequently, the first partial derivative under consideration is a *one-to-one* operator.

Let $w\in\Phi^I$. We want to show that there is a $v\in\Phi^I$ with $v-2\rho W^{\kappa,I}(uv)=w$, i.e. that v solves the linear equation

$$v(s) = 2\rho \int_s^T dr\ \mathcal{T}^\kappa_{r-s} u(r)v(r) + w(s), \qquad s\in I. \tag{2.4.6}$$

To this purpose we will decompose the interval I into sufficiently small pieces in order to replace the integral operator in (2.4.6) by an operator with norm strictly smaller than 1, which then will allow us to apply the so-called *main theorem for linear operator equations* in Banach spaces.

Fix $w\in\Phi^I$, let $N>1$ be a natural number (to be specified later), set $\tau := (T-L)/N$, and introduce the intervals $I(i) := [T-(i+1)\tau, T-i\tau]$, $J(i) := [T,T-i\tau]$, $0\le i<N$. Fix i. For $s\in I(i)$, instead of (2.4.6) we get

$$v(s) = 2\rho\ (W^{\kappa,I(i)}(uv))(s) + \mathcal{T}^\kappa_{(T-i\tau)-s}(W^{\kappa,J(i)}(uv))(T-i\tau) + w(s), \tag{2.4.7}$$

Now

$$\left\|W^{\kappa,I(i)}(uv)\right\|_{I(i)} \le C\ \|u\|_I\ \|v\|_{I(i)}\ \tau, \qquad u\in\Phi^I,$$

where the constant C can be chosen independently of i and τ. Fix N so large that $2\rho C\|u\|_I\tau < 1$, in order to ensure that the bounded linear operator $W^{\kappa,I(i)}(u\,\cdot)$ acting in $\Phi^{I(i)}$ has a norm smaller than 1.

First assume that $i=0$. Then the middle expression at the r.h.s. of equation (2.4.7) disappears, and (2.4.7) has a (unique) solution v on $I(0)$; see, for instance, Zeidler (1986), Theorem 1.B.

For a proof by induction on i suppose that v is already constructed on $J(i)$ for some i, $0\le i<N-1$. Then apply the same theorem to extend v continuous-

ly to $I(i) \cup J(i)$. Summarizing, the operator under consideration maps **onto** Φ^I, and the proof is finished. ■

Now we are ready to complete the *Proof of Theorem* 2.4.3.

1° *(uniqueness).* Take $[\kappa,\rho,\varphi,\psi]\in\mathbb{R}_+^2\times\Phi\times\Phi^I$ and assume that $\mathbf{F}(\kappa,\rho,\varphi,\psi,u) = 0 = \mathbf{F}(\kappa,\rho,\varphi,\psi,v)$ for some $u,v\in\Phi^I$. From (2.4.1),

$$\|u(s) - v(s)\| \le \rho \left\| W^{\kappa,I}u^2(s) - W^{\kappa,I}v^2(s) \right\|, \qquad s\in I.$$

Using the Lemmas 2.1.1 and 2.2.3, we can continue with

$$\le \text{const } \|u+v\|_I \int_s^T dr\ \|u(r) - v(r)\|,$$

and again Grönwall's Lemma yields $\|u(s) - v(s)\| \equiv 0$.

2° *(local existence).* Fix a point $[\kappa_0,\rho_0,\varphi_0,\psi_0,u_0]\in\mathbb{R}_+^2\times\Phi\times\Phi^I\times\Phi^I$, and assume that $\mathbf{F}(\kappa_0,\rho_0,\varphi_0,\psi_0,u_0)=0$ (as is the case for $\varphi_0=\psi_0=u_0=0$). Based on (2.4.4), Lemma 2.4.5, and 1°, from the *implicit function theorem* we conclude the existence of an (open) neighborhood $\mathcal{U}_0$ of $[\kappa_0,\rho_0,\varphi_0,\psi_0]$ in $\mathbb{R}_+^2\times\Phi\times\Phi^I\times\Phi^I$ such that there is a unique map $[\kappa,\rho,\varphi,\psi] \mapsto u_{[\kappa,\rho,\varphi,\psi]}$ defined on $\mathcal{U}_0$ with $\mathbf{F}(\kappa,\rho,\varphi,\psi,u_{[\kappa,\rho,\varphi,\psi]})=0$; see for instance, Zeidler (1986), Theorem 4.B. (Here we have to mention that in applying the implicit function theorem we could replace $\mathbb{R}^2$ by $\mathbb{R}_+^2$ where the neighborhoods of points at the half axis $\kappa=0$ or $\rho=0$ are defined in a one-sided way.)

3° *(maximal existence).* Introduce the system $\mathfrak{S}$ of all neighborhoods $\mathcal{V}$ with the properties as $\mathcal{U}_0$ above. If $\mathcal{V}_1$ and $\mathcal{V}_2$ belong to $\mathfrak{S}$ then also $\mathcal{V}_1\cap\mathcal{V}_2$ does, and the corresponding maps $[\kappa,\rho,\varphi,\psi]\mapsto u_{[\kappa,\rho,\varphi,\psi]}$ coincide on $\mathcal{V}_1\cap\mathcal{V}_2$. Set $\mathcal{U} := \bigcup_{\mathcal{V}\in\mathfrak{S}} \mathcal{V}$. We want to show that $\mathcal{U}\in\mathfrak{S}$. Consider a point $[\kappa,\rho,\varphi,\psi]\in\mathcal{U}$. Then it belongs to some $\mathcal{V}\in\mathfrak{S}$, and from the corresponding map we take the solution $u_{[\kappa,\rho,\varphi,\psi]}$. This definition of a map $[\kappa,\rho,\varphi,\psi] \mapsto u_{[\kappa,\rho,\varphi,\psi]}$ on $\mathcal{U}$ makes sense because $\mathfrak{S}$ is closed with respect to the operation of intersection. Thus, $\mathcal{U}$ is the maximal open set of existence, and we will show below in part 7° of the proof that this $\mathcal{U}$ is convex.

4° *(continuity and analyticity).* The continuous dependence of $u_{[\kappa,\rho,\varphi,\psi]}$ on $[\kappa,\rho,\varphi,\psi]$ directly follows by the implicit function theorem from the continuity of $D_u^1\mathbf{F}(\kappa,\rho,\varphi,\psi,u)$ in $[\kappa,\rho,\varphi,\psi]$. For fixed $\kappa,\rho\geq 0$, the first partial derivative of $\mathbf{F}$ with respect to $[\varphi,\psi]$ is given by

$$D^1_{[\varphi,\psi]}\mathbf{F}(\kappa,\rho,\varphi,\psi,u)[\xi,\zeta] = -\mathcal{T}^{\kappa,I}\xi - W^{\kappa,I}\zeta, \qquad [\xi,\zeta]\in\Phi\times\Phi^I, \tag{2.4.8}$$

hence is independent of $[\varphi,\psi]$. Combining this with (2.4.4), we obtain that the first partial derivative $D^1_{[\varphi,\psi,u]}\mathbf{F}(\kappa,\rho,\varphi,\psi,u)$ exists and is even continuous in $[\varphi,\psi,u]$. Next,

$$D_u^2\mathbf{F}(\kappa,\rho,\varphi,\psi,u)vw = -2\rho W^{\kappa,I}(vw), \qquad v,w\in\Phi^I,$$

i.e. $D_u^2\mathbf{F}(\kappa,\rho,\varphi,\psi,u)$ is independent of $[\varphi,\psi,u]$. Consequently, all higher partial derivatives of $\mathbf{F}$ with respect to $[\varphi,\psi,u]$ will disappear (in other words, $\mathbf{F}$ is a polynomial in $[\varphi,\psi,u]$). Therefore $\mathbf{F}(\kappa,\rho,\varphi,\psi,u)$ is *analytic* in $[\varphi,\psi,u]$, for each fixed $[\kappa,\rho]$. Then the analyticity property in the statement **(iii)** follows; see Zeidler (1986), Corollary 4.23.

5° *(blow-up)*. Assume that $[\kappa_n,\rho_n,\varphi_n,\psi_n] \longrightarrow [\bar\kappa,\bar\rho,\bar\varphi,\bar\psi]\in\partial\mathcal{U}$ and that the corresponding solutions $u_n := u_{[\kappa_n,\rho_n,\varphi_n,\psi_n]}$ satisfy $\|u_n\|_\infty \leq C$, $n\geq 1$, for some finite constant C. From (2.4.1) and (2.4.2), for $s\in I$,

$$\|u_{n+m}(s) - u_n(s)\| \leq \left\|\mathcal{T}^{\kappa_{n+m}}_{T-s}\varphi_{n+m} - \mathcal{T}^{\kappa_n}_{T-s}\varphi_n\right\|$$

$$+\left\|W^{\kappa_{n+m},I}\psi_{n+m}(s) - W^{\kappa_n,I}\psi_n(s)\right\| + \left\|\rho_{n+m}W^{\kappa_{n+m},I}u^2_{n+m}(s) - \rho_n W^{\kappa_n,I}u_n^2(s)\right\|.$$

From the Lemmas 2.2.2 and 2.3.1, the first two terms on the r.h.s. are of the order $o(1)$ as $n,m\to\infty$, uniformly in s. Since the sequence ρ_n is bounded, the remaining term can be estimated from above by

$$\leq o(1) + \text{const}\,\left(\|u_{n+m}\|_\infty + \|u_n\|_\infty\right)\int_s^T dr\,\|u_{n+m}(r) - u_n(r)\|$$

(the $o(1)$ is again uniform in s). Using the boundedness of the sequence $\|u_n\|_I$ and Grönwall's inequality we get $\|u_{n+m}-u_n\|_I = o(1)$ as $n,m\to\infty$. Hence the u_n form a Cauchy sequence in the Banach space Φ^I. Let $\bar u$ denote its limit. From the Lemmas 2.1.1, 2.2.2, and 2.3.1 we conclude that $\mathbf{F}(\bar\kappa,\bar\rho,\bar\varphi,\bar\psi,\bar u)=0$. However, this contradicts the statement in 2° since by assumption $[\bar\kappa,\bar\rho,\bar\varphi,\bar\psi]$ does not belong to the maximal open set $\mathcal{U}$ of existence. Therefore $\|u_n\|_\infty$ is unbounded. From (2.4.2') as well as the Lemmas 2.2.2 and 2.3.1,

$$u_n \geq \mathcal{T}^{\kappa_n,I}\varphi_n + W^{\kappa_n,I}\psi_n \xrightarrow[n\to\infty]{} \mathcal{T}^{\bar\kappa,I}\bar\varphi + W^{\bar\kappa,I}\bar\psi \in \Phi^I$$

which yields the claimed blow-up property.

6° *(points of non-existence).* Take $[\kappa,\rho,\varphi,\psi]\in\mathbb{R}_+^2\times\Phi\times\Phi^I$ with $\rho>0$, $[\varphi,\psi]\geq 0$, and $\varphi\neq 0$. Let $\theta>0$. From 2° we know that $[\kappa,\rho,\theta\varphi,\theta\psi]$ belongs to $\mathcal{U}$ for θ sufficiently small. Assume that it belongs to $\mathcal{U}$ for all $\theta>0$. Applying the operator $\mathcal{T}^\kappa_{s-L}$ on the solution $u_\theta := u_{[\kappa,\rho,\theta\varphi,\theta\psi]}$ at time $s\in I=[L,T]$, we get

$$\mathcal{T}^\kappa_{s-L}u_\theta(s) = \theta\mathcal{T}^\kappa_{T-L}\varphi + \theta\int_s^T dr\ \mathcal{T}^\kappa_{r-L}\psi(r) + \int_s^T dr\ \mathcal{T}^\kappa_{r-L}(u_\theta^2(r)).$$

Setting $\mathcal{T}^\kappa_{s-L}u_\theta(s)(y) =: f_\theta(s)$, $s\in I$, for a fixed $y\in\mathbb{R}$, from Jensen's inequality we obtain

$$f_\theta(s) \geq f_\theta(T) + \int_s^T dr\ f_\theta^2(r), \qquad s\in I.$$

Therefore f_θ dominates the solution of the equation

$$g(s) = f_\theta(T) + \int_s^T dr\ g^2(r), \qquad s\in I,$$

for all $\theta>0$. But the latter equation is solvable only for $(T-L)f_\theta(T) < 1$ and its solution $g(s) = f_\theta(T)/(1 - (T-s)f_\theta(T))$, $s\in I$, explodes as $(T-L)f_\theta(T) \uparrow 1$. On the other hand, $L<T$ by assumption, and $f_\theta(T) = \theta\mathcal{T}^\kappa_{T-L}\varphi(y)$ ranges continuously from $0+$ to $+\infty$ on $\{\theta>0\}$ for an appropriate y by our assumption on φ. This is certainly a contradiction. Consequently, $[\kappa,\rho,\theta\varphi,\theta\psi]$ does not belong to $\mathcal{U}$ for θ sufficiently large.

7° *(convexity).* At this stage we use the standard iteration scheme, which we recall here without going into any details. (For this technique, see for instance Dawson and Fleischmann (1988), Proposition 4.6, or also Dawson and Fleischmann (1992).) Fix $\kappa,\rho\geq 0$. Let $[\kappa,\rho,\varphi,\psi]$ belong to $\mathcal{U}$. Set

$$u_0:=0, \quad u_{n+1} = \mathcal{T}^{\kappa,I}\varphi + W^{\kappa,I}\psi + \rho W^{\kappa,I}(u_n^2), \qquad n\geq 0. \tag{2.4.9}$$

We may assume that I is sufficiently small (otherwise decompose I as in the proof of Lemma 2.4.5). Then we get $u_n \xrightarrow[n\to\infty]{} u_{[\kappa,\rho,\varphi,\psi]}=:u$ in Φ^I. Take additionally $[\kappa,\rho,\varphi',\psi']\in\mathcal{U}$ and consider the corresponding approximating functions u_n' of the solution $u_{[\kappa,\rho,\varphi',\psi']} = :u'$. For a constant $0<\beta<1$, we want to show that $u_{[\kappa,\rho,\varphi^\beta,\psi^\beta]} =: u^\beta$ exists, where $[\varphi^\beta,\psi^\beta] := \beta[\varphi,\psi] + (1-\beta)[\varphi',\psi']$, and that

$$u^\beta \leq \beta u + (1-\beta)u'. \tag{2.4.10}$$

To this end, by using (2.4.9), show by induction that

$$u_n^\beta \leq \beta u_n + (1-\beta)u_n' \tag{2.4.11}$$

holds. On the other hand,

$$\|u^{\beta}_{n+m}(s) - u^{\beta}_{n}(s)\| \le \text{const} \int_s^T dr\ \|u^{\beta}_{n+m-1}(r) - u^{\beta}_{n-1}(r)\|,$$

because from (2.4.9) and (2.4.11),

$$\|u^{\beta}_{n}\|_I \le (\|u_n\|_I + \|u'_n\|_I) \vee (\|\mathcal{T}^{\kappa,I}\varphi\|_I + \|W^{\kappa,I}\psi\|_I) \le \text{const}.$$

Consequently, u^{β}_{n} converges in Φ^I to the desired solution u^{β} as $n\to\infty$, and the inequality (2.4.10) is obvious. Summarizing, $\mathcal{U}$ and u have the desired convexity properties.

8° *(special cases).* If $\rho=0$ then $u = \mathcal{T}^{\kappa,I}\varphi + W^{\kappa,I}$. If $[\varphi,\psi]\ge 0$, then obviously $u_{[\kappa,\rho,\varphi,\psi]}\ge 0$, (if they exist). On the other hand, if $[\varphi,\psi]\le 0$ then non-positive solutions $u_{[\kappa,\rho,\varphi,\psi]}$ can again be constructed by the iteration scheme. Since I is arbitrary, we can easily extend the solutions to all of $\mathbb{R}_+$. Such global solutions exist also if, for $[\kappa,\rho]\in\mathbb{R}^2_+$ fixed, $[\varphi_+,\psi_+]$ is sufficiently small in norm, provided that we are in supercritical dimensions $d>\alpha$; we refer to Fujita (1966) or Nagasawa and Sirao (1969). This finishes the proof of Theorem 2.4.3. □

3. Log-Laplace Functionals

3.1. Preliminaries: The a-Vague Topology

Recall that $0<\alpha\le 2$ and $d<a\le d+\alpha$. To Φ we introduce the "dual" set $\mathcal{M}_a$ of all (locally finite non-negative) measures μ defined on $\mathbb{R}^d$ such that $(\mu,\varphi_a) < +\infty$, or equivalently, $(\mu,\varphi) < +\infty$ for all $\varphi\in\Phi_+$. For instance, all finite measures and the Lebesgue measure ℓ belong to this set of *a-tempered measures.* We endow $\mathcal{M}_a$ with the *a-vague topology*. By definition, this is the coarsest topology such that all real functions $\mu\mapsto(\mu,\varphi)$, $\varphi \in \mathcal{C}_+^{\text{comp}}\cup\{\varphi_a\}$, are continuous. Hence all the mappings $\mu\mapsto(\mu,\varphi)$, $\varphi\in\Phi$, are continuous.

Let $[\Phi^*,\|\cdot\|_*]$ denote the dual Banach space to $[\Phi,\|\cdot\|]$. Then $\mathcal{M}_a$ can be considered as a *closed* topological subspace of Φ^* equipped with the *weak* topology* (i.e. the a-vague topology in $\mathcal{M}_a$ is nothing else than the topology induced in $\mathcal{M}_a$ by the weak* topology in Φ^*). Note that

(3.1.1) $$|(\mu,\varphi)| \le \|\varphi\|\ (\mu,\varphi_a), \qquad \varphi\in\Phi,\ \mu\in\mathcal{M}_a,$$

from which in particular follows that the "duality" relation $(\cdot,\cdot)$ between $\mathcal{M}_a$ and Φ is *continuous* in both "components", and that

(3.1.2) $$\|\mu\|_* = (\mu,\varphi_a), \quad \mu\in\mathcal{M}_a.$$

There exists a sequence $\{f_n;n\geq 1\}$ of functions in $\mathcal{C}_+^{comp}$ such that

$$\rho_a(\mu,\nu) := \sum_{n=0}^{\infty} 2^{-n-1}\left(1 - \exp[-|(\mu,f_n)-(\nu,f_n)|/\|f_n\|]\right), \quad \mu,\nu \in \mathcal{M}_a,$$

where $f_0 := \varphi_a$, is a *complete translation-invariant metric* on $\mathcal{M}_a$ which generates the a-vague topology; cf. Kallenberg (1983), Appendix A.7. Hence, $\mathcal{M}_a$ is a *Polish* space.

Lemma 3.1.3. *Each open ball* $B(\nu,r) := \{\mu\in\mathcal{M}_a;\ \rho_a(\mu,\nu)<r\}$, $\nu\in\mathcal{M}_a$, $r>0$, *is a* **convex** *subset of* $\mathcal{M}_a$.

Proof. This can be concluded from the inequality

(3.1.4) $$\rho_a(\theta\mu_1+(1-\theta)\mu_2,\nu) \leq \rho_a(\mu_1,\nu) \vee \rho_a(\mu_2,\nu), \quad 0\leq\theta\leq 1, \quad \mu_1,\mu_2,\nu \in \mathcal{M}_a,$$

which follows from the corresponding property of the Euclidean metric entering into the exponents in the definition of ρ_a , combined with the fact that the function $1-e^{-r}$, $r\geq 0$, is monotonously increasing. □

Lemma 3.1.5. *A subset* A *of* $\mathcal{M}_a$ *is* **relatively compact** *if and only if there is a natural number* k *such that* $A \subseteq \{\mu\in\mathcal{M}_a;\|\mu\|_*\leq k\}$ *holds.*

Proof. If for a sequence $\{\mu_n;n\geq 1\} \subseteq A$ we have $\|\mu_n\|_* = (\mu_n,\varphi_a) \xrightarrow[n\to\infty]{} \infty$, then this sequence cannot have a subsequence which a-vaguely converges in $\mathcal{M}_a$, thus A is not a relatively compact subset of $\mathcal{M}_a$. On the other hand,

$$\{\mu\in\mathcal{M}_a;\|\mu\|_*\leq k\} = \mathcal{M}_a \cap \{\varphi^*\in\Phi^*;\ |(\varphi^*,\varphi)|\leq k,\ \forall\varphi\in\Phi \text{ with } \|\varphi\|\leq 1\}, \quad k\geq 1,$$

and the Banach-Alaoglu theorem implies the (weak*) relative compactness of all these sets; see, for instance, Rudin (1973), Theorem 3.15. □

Finally, from the definition of ρ_a we conclude that

(3.1.6) $$\rho_a(\mu,\nu) \leq \|\mu-\nu\|_*, \quad \mu,\nu \in \mathcal{M}_a.$$

3.2. Superstable Motion in $\mathbb{R}^d$

Recall that $0<\alpha\leq 2$, $d<a\leq d+\alpha$, and $\kappa,\rho \geq 0$. A critical *superstable motion* **X** in $\mathbb{R}^d$ with motion index α, "diffusion" constant $\kappa\geq 0$, and (constant) branching

rate $\rho\ge 0$ can be defined as a time-homogeneous Markov process $[X,\mathbb{P}^{\kappa,\rho}_{s,\mu};s\in\mathbb{R},\mu\in\mathcal{M}_a]$ with *continuous* trajectories in $\mathcal{M}_a$ and with Laplace transition functionals

(3.2.1) $$\mathbb{E}^{\kappa,\rho}_{s,\mu}\exp\langle X(t),\varphi\rangle = \exp\Big\langle\mu,u_{[\kappa,\rho,\varphi,0]}(-(t-s))\Big\rangle,\qquad s\le t,\ \mu\in\mathcal{M}_a,\ \varphi\in\Phi_-,$$

where $u_{[\kappa,\rho,\varphi,0]}=u$ solves

(3.2.2) $$u(s) = \mathcal{T}^{\kappa}_{-s}\varphi + \rho\int_s^0 dr\ \mathcal{T}^{\kappa}_{r-s}(u^2(r)),\qquad s\le 0,$$

or as a short-hand, $$-\frac{\partial}{\partial s}u = \kappa\Delta_\alpha u + \rho u^2,\qquad u\big|_{s=0-} = \varphi;$$

that is, $u_{[\kappa,\rho,\varphi,0]}$, $\varphi\in\Phi_-$, is the unique extension from $I\subset\mathbb{R}_-$ to $\mathbb{R}_-$ of the solution according to Theorem 2.4.3. (For the construction of the process, cf. for instance Dawson (1991), in particular Proposition 5.6.4.)

Note that by the continuity properties of solutions and by (3.1.1) the Laplace functional expression (3.2.1) is *continuous in all its variables* s,t,μ,φ as described. Note also that if $\rho=0$ then **X** reduces to the stable flow $\{\mathcal{T}^{\kappa}_t\mu;t\ge 0\}$ in $\mathcal{M}_a$ defined by $\langle\mathcal{T}^{\kappa}_t\mu,\varphi\rangle := \langle\mu,\mathcal{T}^{\kappa}_t\varphi\rangle$, $\varphi\in\Phi_+$.

3.3. Exponential Moments

The (weighted) *occupation time process* **Y** related to **X** is defined by $Y(t) := \int_0^t ds\ X(s)$, $t>0$. Now we want to describe the exponential moments of $[X(t),Y(t)]$, $t\ge 0$, with the help of solutions to the equation (2.4.2').

Proposition 3.3.1 **(log-Laplace functional).** *Fix* $I=[L,T]$, $L<T$, $\kappa,\rho\ge 0$, *and let* $\mathfrak{B} := \mathfrak{B}[\kappa,\rho]$ *denote the set of all those* $[\varphi,\psi]\in\Phi\times\Phi^I$ *such that* $V[\varphi,\psi] := v$ *defined by*

(3.3.2) $$v(s,y) := \log\mathbb{E}^{\kappa,\rho}_{s,\delta_y}\exp\Big[\langle X(T),\varphi\rangle + \int_s^T dr\ \langle X(r),\psi(r)\rangle\Big],\qquad [s,y]\in I\times\mathbb{R}^d,$$

satisfies $\sup\{v_{[\kappa,\rho,\varphi,\psi]}(s,y);\ [s,y]\in I\times\mathbb{R}^d\} < +\infty$. *Then* $\mathfrak{B}$ *is an* **open convex** *set which covers* $\Phi_-\times\Phi_-^I$, *and* $[\varphi,\psi]\in\mathfrak{B}[\kappa,\rho]$ *if and only if* $[\kappa,\rho,\varphi,\psi]\in\mathcal{U}$ *with* $\mathcal{U}$ *defined in Theorem* 2.4.3 **(ii)**. *In this case* $V[\varphi,\psi] = u_{[\kappa,\rho,\varphi,\psi]}$, *the (unique) solution to* (2.4.2').

Note that this proposition provides a *probabilistic representation* of the solutions to (2.4.2').

Proof of Proposition 3.3.1. 1°. Fix $[\kappa,\rho,\varphi,\psi]\in\mathbb{R}_+^2\times\Phi\times\Phi^I$, and assume for the moment that φ,ψ are non-negative. Then, for all $\theta\le 0$, the functions $V[\theta\varphi,\theta\psi]$ defined in (3.3.2) belong to Φ^I and solve (2.4.2'), hence coincide with $u_\theta :=$ $u_{[\kappa,\rho,\theta\varphi,\theta\psi]} \in \Phi^I$. In fact, if $\psi=0$, then this is a version of the Laplace functional (3.2.1), and the formula can be extended to (3.3.2) by approximating ψ by appropriate step functions and using that **X** is a Markov process; see Iscoe (1986).

2°. Now drop the additional assumption $\varphi,\psi\ge 0$. Let $\varphi=\varphi_+-\varphi_-$, $\psi=\psi_+-\psi_-$ denote the minimal decomposition with $\varphi_+,\psi_+\ge 0$, $\varphi_-,\psi_->0$. Then from 1° we know that $v^{\tilde\theta} :=$ $V[\theta_1\varphi_++\theta_2\varphi_-,\theta_3\psi_++\theta_4\psi_-]$ belongs to Φ^I and satisfies (2.4.2') with $[\theta\varphi,\theta\psi]$ replaced by $A(\tilde\theta) := [\theta_1\varphi_++\theta_2\varphi_-,\theta_3\psi_++\theta_4\psi_-]$, for all $\tilde\theta := [\theta_1,\dots,\theta_4] \le 0$, that is, $v^{\tilde\theta} = u_{[\kappa,\rho,\theta_1\varphi_++\theta_2\varphi_-,\theta_3\psi_++\theta_4\psi_-]} =: u_{\tilde\theta}$ for non-positive $\tilde\theta$.

3°. Keeping the notations from the previous step of proof, set

(3.3.3) $\qquad \tilde\Theta := \{\tilde\theta\in\mathbb{R}^4;\ A(\tilde\theta)\in\mathfrak{B}\},\quad \Theta := \{\tilde\theta\in\mathbb{R}^4;\ [\kappa,\rho,\theta_1\varphi_++\theta_2\varphi_-,\theta_3\psi_++\theta_4\psi_-]\in\mathcal{U}\}$

with $\mathcal{U}$ defined in Theorem 2.4.3 **(ii)**. Note that $\mathbb{R}_-^4 \subseteq \tilde\Theta\cap\Theta$. By Hölder's and the triangular inequality, $\tilde\Theta$ is a convex subset of $\mathbb{R}^4$. On the other hand, the properties of $\mathcal{U}$ yield that Θ is an open convex subset of $\mathbb{R}^4$. Fix for the moment $[s,y]\in I\times\mathbb{R}^d$. Well-known properties of bilateral Laplace functions imply that $\tilde\theta\mapsto v^{\tilde\theta}(s,y)$ is an *analytic* function on the interior $\tilde\Theta^\circ$ of $\tilde\Theta$. On the other hand, $\tilde\theta\mapsto u_{\tilde\theta}(s,y)$ is an analytic function on Θ. But by 2° both coincide on $\{\tilde\theta;\tilde\theta\le 0\}$, and by uniqueness of analytic continuation we conclude that $v^{\tilde\theta}(s,y)$ $= u_{\tilde\theta}(s,y)$ on $\tilde\Theta^\circ\cap\Theta$, and that both $v^{\tilde\theta}(s,y)$ and $u_{\tilde\theta}(s,y)$ are branches of a unique analytic function defined on $\tilde\Theta^\circ\cup\Theta$. Since $[s,y]$ is arbitrary, the $(I\times\mathbb{R}^d)^{\mathbb{R}}$-valued mappings $v^{\tilde\theta}$ and $u_{\tilde\theta}$ coincide on $\tilde\Theta^\circ\cap\Theta$.

4°. By definition, $v^{\tilde\theta}$ has an infinite supremum outside of $\tilde\Theta$. Therefore $\Theta\subseteq\tilde\Theta$. On the other hand, the supremum of $u_{\tilde\theta}$ blows up if $\tilde\theta$ approaches the boundary $\partial\Theta$ of Θ. Moreover, again by Hölder's inequality, $v^{\tilde\theta}$ is convex in $\tilde\theta\in\tilde\Theta$. Hence, Θ cannot be strictly included in $\tilde\Theta$, that is $v=u$ on $\Theta=\tilde\Theta^\circ=\tilde\Theta$. Passing to $\theta_1=\theta_3=\theta$ and $\theta_2=\theta_4=-\theta$, we get that $[\theta\varphi,\theta\psi]\in\mathfrak{B}$ if and only if $[\kappa,\rho,\theta\varphi,\theta\psi]\in\mathcal{U}$, and in this case $V[\theta\varphi,\theta\psi] = u_{[\kappa,\rho,\theta\varphi,\theta\psi]}$. Specialize to $\theta=1$ to finish the proof. □

From δ-initial measures we may pass to any initial measure:

Corollary 3.3.4 **(exponential moments).** *Fix $I=[L,T]$, $L<T$, and $\mu\in\mathcal{M}_a$. Then*

$$\mathbb{E}^{\kappa,\rho}_{L,\mu} \exp\left[(X(T),\varphi) + \int_L^T dr\ (X(r),\psi(r))\right] = \exp(\mu,u_{[\kappa,\rho,\varphi,\psi]}(L))$$

if $[\kappa,\rho,\varphi,\psi]\in\mathcal{U}$, *where* $u_{[\kappa,\rho,\varphi,\psi]}$ *solves the equation* (2.4.2') *according to Theorem* 2.4.3.

Proof. This follows from Proposition 3.3.1 if we approximate μ by discrete measures with a finite set of atoms and use the branching property and obvious continuities. □

Once a solution passes the blow-up boundary, it should stay at infinity (compare for Baras and Cohen (1987)):

Conjecture 3.3.5 **(complete blow-up).** Fix $I=[L,T]$, $L<T$. If $\kappa>0$ and $[\kappa,\rho,\varphi,\psi] \notin \mathcal{U}\cup\partial\mathcal{U}$ then

$$\mathbb{E}^{\kappa,\rho}_{L,\mu} \exp\left[(X(T),\varphi) + \int_L^T dr\ (X(r),\psi(r))\right] = +\infty, \qquad \mu\in\mathcal{M}_a.$$

4. Large Deviation Estimates

4.1. Reformulation of the Large Deviation Principle Theorem 1.4.2

As announced in Subsection 1.5, we will derive the following general result. Note that here only our basic parameter assumptions $1\le d<a\le d+\alpha$, $0<\alpha\le 2$, $\kappa,\rho\ge 0$ are enforced.

Theorem 4.1.1. **(version of the large deviation principle).** *Fix $t>0$ and $\mu\in\mathcal{M}_a$, $\mu\ne 0$. There exists a lower semi-continuous* **convex** *good rate functional* $\mathbf{S}_{\mu,t}$: $\mathcal{M}_a \mapsto [0,+\infty]$ *with* $\mathbf{S}_{\mu,t}(\mathcal{T}^{\kappa}_t\mu) = 0$ *such that,*

(i) *for each open subset G of $\mathcal{M}_a$,*

$$\liminf_{R\to\infty} R^{-1} \log \mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}X(t) \in G\right) \ge - \inf_{\nu\in G} \mathbf{S}_{\mu,t}(\nu),$$

(ii) *for each closed subset F of $\mathcal{M}_a$,*

$$\limsup_{R\to\infty} R^{-1} \log \mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}X(t) \in F\right) \le - \inf_{\nu\in F} \mathbf{S}_{\mu,t}(\nu).$$

The proof, to which we devote the next four subsections, is based on a general methodology for large deviation probabilities as presented in Chapters II and III of Deuschel and Stroock (1989), in conjunction with the results on superprocess log-Laplace functionals developed in Sections 2 and 3.

4.2. Supermultiplicativity

As an immediate preparation for the proof of the previous theorem, we formulate the following simple lemma.

Lemma 4.2.1. *Fix $t>0$, $\mu\in\mathcal{M}_a$ and a convex Borel subset A of $\mathcal{M}_a$. Then the function*

(4.2.2) $$f(R) := \mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}\mathbf{X}(t)\in A\right), \qquad R>0,$$

is supermultiplicative: $f(R+S) \geq f(R)\, f(S),\ R,S>0.$

Proof. Fix $R,S > 0$. Let $[\mathbf{X}',\mathbf{X}'']$ be distributed according to the product measure $\mathbb{P}^{\kappa,\rho}_{0,R\mu} \times \mathbb{P}^{\kappa,\rho}_{0,S\mu}$. Then

$$f(R)f(S) \leq \mathbb{P}^{\kappa,\rho}_{0,R\mu}\times\mathbb{P}^{\kappa,\rho}_{0,S\mu}\left(R^{-1}\mathbf{X}'(t)\in A,\ S^{-1}\mathbf{X}''(t)\in A\right).$$

However, if both $R^{-1}\mathbf{X}'(t)$ and $S^{-1}\mathbf{X}''(t)$ belong to A then also its convex combination $(R+S)^{-1}(\mathbf{X}'(t)+\mathbf{X}''(t))$ is in A. But by the *branching property*, which follows directly from the form of the Laplace functional (3.2.1), $\mathbf{X}'(t)+\mathbf{X}''(t)$ has the law $\mathbb{P}^{\kappa,\rho}_{0,(R+S)\mu}$. Summarizing,

$$f(R)f(S) \leq \mathbb{P}^{\kappa,\rho}_{0,(R+S)\mu}\left((R+S)^{-1}\mathbf{X}(t)\in A\right) = f(R+S). \qquad \square$$

Lemma 4.2.3. *In addition to the assumptions on A and f in the previous lemma, suppose that $A\subseteq\mathcal{M}_a$ is open. If now $f(R)>0$ for some $R>0$ then f is bounded away from 0 on some non-empty open interval.*

Proof. Assume that $f(R)>0$ for a fixed $R>0$. Since $\mathcal{M}_a$ is Polish, by the regularity of finite measures we find a compact set $C\subset A$ such that even $f(R) \geq \mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}\mathbf{X}(t)\in C\right) > 0$; see e.g. Bauer (1981), Satz 41.3. The convex hull $\hat{C}\subseteq A$ of the totally bounded subset C of the locally convex space Φ^* (concerning the weak* topology) is totally bounded; see, for instance Rudin (1973), Theorem 3.24. Moreover, it is a closed subset of $\mathcal{M}_a$. (In fact, if $\nu_n\in\hat{C}$ converges (a-vaguely) to $\nu\in\mathcal{M}_a$ as $n\to\infty$, then by compactness take such a subsequence that all terms in the representations $\nu_n = \theta_n\alpha_n + (1-\theta_n)\beta_n$, $0\leq\theta_n\leq 1$, $\alpha_n,\beta_n\in C$, converge, which implies that $\lim_{n\to\infty} \nu_n$ belongs to $\hat{C}$.) Consequently, $\hat{C}$ is an ($a^{\perp}$-vaguely) compact subset of A, and by the relative compactness criterion Lemma 3.1.5 we get

(4.2.4) $$\sup\{\|\nu\|_*;\nu\in\hat{C}\} =: K < \infty.$$

So far we mainly proved that there exists a compact convex set $\hat{C}\subset A$ such that $f(R) \geq g(R) := \mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}X(t)\in\hat{C}\right) > 0$. We may choose a δ such that $0 < 2\delta < \rho_a(\hat{C},M_a\setminus A)$, and a natural number $s_0 > K/\delta$. Recall (3.1.6). Write $G := \{\nu\in M_a;\ \|\nu-\hat{C}\|_*<\delta\} \subset A$. For $0\leq r<R$ and a natural number s, let $[X',X'']$ be distributed according to $\mathbb{P}^{\kappa,\rho}_{0,sR\mu} \times \mathbb{P}^{\kappa,\rho}_{0,r\mu}$. Then by the branching property,

$$f(S) \geq \mathbb{P}^{\kappa,\rho}_{0,sR\mu}\times\mathbb{P}^{\kappa,\rho}_{0,r\mu}\left(S^{-1}(X'(t)+X''(t))\in A\right) \quad \text{with} \quad S=sR+r.$$

But a sum belongs to A certainly if the first summand belongs to G and the second summand has a $\|\cdot\|_*$-norm smaller than δ:

(4.2.5) $$f(S) \geq \mathbb{P}^{\kappa,\rho}_{0,sR\mu}\left(S^{-1}X'(t)\in G\right)\ \mathbb{P}^{\kappa,\rho}_{0,r\mu}\left(\|S^{-1}X''(t)\|_*<\delta\right).$$

The first factor on the right hand side can be estimated further in the same way: $(sR+r)^{-1}X'(t)\in G$ is certainly fulfilled if $(sR)^{-1}X'(t)\in\hat{C}$ and if the $\|\cdot\|_*$-norm of the difference of both "vectors" is smaller than δ. But this is actually true under $(sR)^{-1}X'(t) \in \hat{C}$ and $s\geq s_0$:

$$\left\|(sR+r)^{-1}X'(t) - (sR)^{-1}X'(t)\right\|_* = \left\|r(sR+r)^{-1}(sR)^{-1}X'(t)\right\|_* \leq s^{-1}K < \delta.$$

Thus, the first factor at the r.h.s. of the inequality (4.2.5) can be estimated from below by

(4.2.6) $$\geq \mathbb{P}^{\kappa,\rho}_{0,sR\mu}\left((sR)^{-1}X'(t)\in\hat{C}\right) = g(sR) \geq (g(R))^s > 0, \qquad s\geq s_0,$$

where we applied Lemma 4.2.1 to the (compact) convex set $\hat{C}$. Concerning the second factor at the r.h.s. of (4.2.5), pass to the complement and proceed for $\theta>0$ as follows:

$$\mathbb{P}^{\kappa,\rho}_{0,r\mu}\left(\|S^{-1}X''(t)\|_*\geq\delta\right) = \mathbb{P}^{\kappa,\rho}_{0,r\mu}\left((X''(t),\theta\varphi_a)\geq S\delta\theta\right) \leq e^{-S\delta\theta}\ \mathbb{E}^{\kappa,\rho}_{0,r\mu} \exp(X''(t),\theta\varphi_a).$$

By Corollary 3.3.4 with $I=[-t,0]$ and using time-homogeneity, we may continue with

$$\leq e^{-sR\delta\theta}\ \exp\left[(r\mu,u_{[\kappa,\rho,\theta\varphi_a,0]}(-t))\right]$$

which is finite for a $\theta>0$ sufficiently small. But the second exponential expression is bounded in $r\leq R$, whereas the first one converges to 0 as $s\to\infty$. Consequently, the second factor on the r.h.s. of (4.2.5) is bounded away from zero for all sufficiently large $S=sR+r$. Combined with (4.2.6) we conclude that $f(S)$ is bounded away from zero on some non-empty open interval. This finishes the proof. □

4.3. Weak Large Deviation Principle

Let $\mathfrak{A}$ denote the system of all those subsets of $\mathcal{M}_a$ which are non-empty, open, and convex. Fix $\mu\in\mathcal{M}_a$, $t>0$, and, for the moment, $A\in\mathfrak{A}$. In Lemma 4.2.1 go over to $-\log f$ to conclude that the function

$$\sigma(R) := -\log\mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}\mathrm{X}(t) \in A\right) \in [0,+\infty], \qquad R>0,$$

is *subadditive*, i.e. $\sigma(R+S) \le \sigma(R)\sigma(S)$, $R,S>0$. Moreover, Lemma 4.2.3 yields that σ is either bounded on some non-empty open interval, or identically $+\infty$. Hence, the subadditivity of σ implies that all the limits

$$\text{(4.3.1)} \qquad \mathbf{S}_{\mu,t}(A) := -\lim_{R\to\infty} R^{-1} \log \mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}\mathrm{X}(t) \in A\right) \in [0,+\infty], \qquad A\in\mathfrak{A},$$

exist; see, for instance Lemma 4.2.5 in [9]. Recall that by Lemma 3.1.3 all open balls $B(\nu,r)$, $r>0$, $\nu\in\mathcal{M}_a$, belong to $\mathfrak{A}$. By monotonicity, set

$$\text{(4.3.2)} \qquad \mathbf{S}_{\mu,t}(\nu) := \lim_{r\downarrow 0} \mathbf{S}_{\mu,t}(B(\nu,r)) = \sup\{\mathbf{S}_{\mu,t}(A);\ \nu\in A\in\mathfrak{A}\}, \qquad \nu\in\mathcal{M}_a.$$

Obviously, $\mathbf{S}_{\mu,t}:\mathcal{M}_a\mapsto[0,+\infty]$ is a *lower semi-continuous* functional. For *convexity*, it is enough to show that

$$\text{(4.3.3)} \qquad \mathbf{S}_{\mu,t}((\nu_1+\nu_2)/2) \le (\mathbf{S}_{\mu,t}(\nu_1) + \mathbf{S}_{\mu,t}(\nu_2))/2, \qquad \nu_1,\nu_2 \in \mathcal{M}_a.$$

Set $(\nu_1+\nu_2)/2 =: \nu$, take any $A\in\mathfrak{A}$ with $\nu\in A$, and choose $A_i\in\mathfrak{A}$ such that $\nu_i\in A_i$ and $A \supseteq (A_1+A_2)/2$. Then, by (4.3.1) and the branching property,

$$\mathbf{S}_{\mu,t}(A) = -\lim_{R\to\infty} (2R)^{-1} \log \mathbb{P}^{\kappa,\rho}_{0,2R\mu}\left((2R)^{-1}\mathrm{X}(t) \in A\right)$$
$$\le (\mathbf{S}_{\mu,t}(A_1) + \mathbf{S}_{\mu,t}(A_2))/2 \le (\mathbf{S}_{\mu,t}(\nu_1) + \mathbf{S}_{\mu,t}(\nu_2))/2,$$

and (4.3.2) implies (4.3.3).

It is easy to see that from (4.3.1) and (4.3.2) we get

$$\text{(4.3.4)} \qquad \liminf_{R\to\infty} R^{-1}\log\mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}\mathrm{X}(t)\in G\right) \ge -\inf_{\nu\in G} \mathbf{S}_{\mu,t}(\nu), \qquad \text{open } G \subseteq \mathcal{M}_a.$$

On the other hand, if C is a compact subset of $\mathcal{M}_a$ and $i := \inf_{\nu\in C} \mathbf{S}_{\mu,t}(\nu)$ is positive, then for $0<\varepsilon<i$ we find finitely many open balls $B_1,\ldots,B_M$ which cover C and satisfy $\mathbf{S}_{\mu,t}(B_m) \ge i-\varepsilon$, $1\le m\le M$. Then again with (4.3.1) and (4.3.2),

we finally obtain

(4.3.5) $\limsup_{R\to\infty} R^{-1}\log\mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}X(t)\in C\right) \le -\inf_{\nu\in C} \mathbf{S}_{\mu,t}(\nu),$ compact $C \subseteq \mathcal{M}_a$.

Summarizing, with (4.3.4) and (4.3.5) we proved that the family $R^{-1}\log\mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}X(t) \in \cdot\right)$, $R>0$, satisfies a *weak large deviation principle* with the convex rate functional $\mathbf{S}_{\mu,t}:\mathcal{M}_a \mapsto [0,+\infty]$.

4.4. Full Large Deviation Principle

For convenience, we formulate the following lemma. Recall the set $\Phi_{\mu,t}$ introduced after (1.4.6).

Lemma 4.4.1. *Fix* $\mu\in\mathcal{M}_a$ *and* $t>0$. *For all* $\varphi\in\Phi_{\mu,t}$,

$$\lim_{N\to\infty} \lim_{R\to\infty} R^{-1}\log\mathbb{E}^{\kappa,\rho}_{0,R\mu}\left\{\exp(X(t),\varphi);\ (R^{-1}X(t),\varphi) \ge N\right\} = -\infty.$$

Proof. Fix μ,t,φ as in the lemma and set $I=[-t,0]$. Since $\Phi_{\mu,t}$ is open by definition, we find a $\theta>0$ such that also $(1+\theta)\varphi$ belongs to $\Phi_{\mu,t}$. As in the proof of Lemma 4.2.3, we can use an exponential moment inequality to get

(4.4.2) $R^{-1}\log\mathbb{E}^{\kappa,\rho}_{0,R\mu}\left\{\exp(X(t),\varphi);\ (R^{-1}X(t),\varphi) > N\right\} \le -\theta N + R^{-1}\Lambda_{R\mu,t}((1+\theta)\varphi),$

$R,N>0$. But the exponential moments $\Lambda_{\mu,t}(\varphi)$ introduced in (1.4.3) satisfy

(4.4.3) $\Lambda_{R\mu,t}(\varphi) = \int R\mu(dy)\ \log\ \mathbb{E}^{\kappa,\rho}_{0,\delta_y} \exp(X(t),\varphi) = R\Lambda_{\mu,t}(\varphi), \quad R>0.$

Hence, the r.h.s. in (4.4.2) is finite, and letting first $R\to\infty$ and then $N\to\infty$, the claim follows. □

By Lemma 4.4.1 with $\varphi=\theta\varphi_a$ and $\theta>0$ sufficiently small,

$$\lim_{N\to\infty} \lim_{R\to\infty} R^{-1}\log\mathbb{P}^{\kappa,\rho}_{0,R\mu}\left((R^{-1}X(t),\theta\varphi) \ge N\right) = -\infty.$$

From the compactness Lemma 3.1.5 and (3.1.2) we learn that to each $M>0$ we find a compact set $C_M\subseteq\mathcal{M}_a$ such that

$$\limsup_{R\to\infty} R^{-1}\log\mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}X(t) \in C_M\right) \le -M.$$

In other words, we have *exponential tightness.* Together with the results of the previous subsection we get a *full large deviation principle with the convex good rate functional* $\mathbf{S}_{\mu,t}$; see, [9], Lemma 2.1.5.

4.5. Law of Large Numbers

We start with a simple but important *scaling property* of the superprocess:

Lemma 4.5.1. *Fix* $\mu\in\mathcal{M}_a$ *and a constant* $c\geq 0$. *If* $\mathbf{X}$ *is distributed according to* $\mathbb{P}^{\kappa,\rho}_{0,\mu}$ *then* $c\mathbf{X}$ *has the law* $\mathbb{P}^{\kappa,c\rho}_{0,c\mu}$.

Proof. By the Markov property, this directly follows from the identity $u_{[\kappa,\rho,c\varphi,0]} = cu_{[\kappa,c\rho,\varphi,0]}$, $\varphi\in\Phi_-$, via (3.2.1) and (3.2.2). □

As a complement to Theorem 4.1.1 we add here the following

Lemma 4.5.2 **(law of large numbers).** *Fix* $t>0$, $\mu\in\mathcal{M}_a$. *For all neighborhoods* $\mathcal{U}(\mathcal{T}^{\kappa}_t\mu)$ *of* $\mathcal{T}^{\kappa}_t\mu$,

$$\mathbb{P}^{\kappa,\rho}_{0,R\mu}\left(R^{-1}\mathbf{X}(t)\in\mathcal{U}(\mathcal{T}^{\kappa}_t\mu)\right)\xrightarrow[\kappa\to\infty]{} 1,$$

Proof. By Lemma 4.5.1,

$$\mathbb{E}^{\kappa,\rho}_{0,R\mu}\exp(R^{-1}\mathbf{X}(t),\varphi) = \mathbb{E}^{\kappa,\rho/R}_{0,\mu}\exp(\mathbf{X}(t),\varphi) = \exp(\mu,u_{[\kappa,\rho/R,\varphi,0]}(-t)), \qquad \varphi\in\Phi_-.$$

The claim then follows from continuity properties, since

$$(\mu,u_{[\kappa,0,\varphi,0]}(t)) = (\mu,\mathcal{T}^{\kappa}_t\varphi) = (\mathcal{T}^{\kappa}_t\mu,\varphi). \qquad □$$

By the LLN Lemma 4.5.2, Lemma 3.1.3, and (4.3.1) we have $\mathbf{S}_{\mu,t}(B(\mathcal{T}^{\kappa}_t\mu,r)) = 0$ for all $r>0$, and (4.3.2) implies that $\mathbf{S}_{\mu,t}(\mathcal{T}^{\kappa}_t\mu) = 0$. This completes the proof of Theorem 4.1.1.

4.6. On the Relation between the Rate Functional and Exponential Moments

Fix again $t>0$ and $\mu\in\mathcal{M}_a$. By (4.4.3),

$$R^{-1}\log\mathbb{E}^{\kappa,\rho}_{0,R\mu}\exp(\mathbf{X}(t),\varphi) \equiv \Lambda_{\mu,t}(\varphi), \qquad \varphi\in\Phi,$$

and by Lemma 4.4.1 and *Varadhan's Theorem* (see [9], Theorem 2.1.10) we obtain that (1.4.6) holds for all $\varphi\in\Phi_{\mu,t}$. Note that

$$\Lambda_{\mu,t}(\varphi) = (\mu,u_{[\kappa,\rho,\varphi,0]}(-t)), \qquad \varphi\in\Phi_0\subseteq\Phi_{\mu,t},$$

by Corollary 3.3.4, where Φ_0 is the set of all those $\varphi\in\Phi$ such that

$$\sup\{\Lambda_{\delta_y,s}(\varphi);\ 0\leq s\leq t,\ y\in\mathbb{R}^d\} < +\infty.$$

4.7. Proof of Theorem 1.4.2

Here we come back to our scaled processes $\mathbf{X}^{K}$ defined in (1.2.4). The large deviation principle of Theorem 1.4.2 is in fact a consequence of Theorem 4.1.1 combined with some scaling properties. First of all, $\mathbf{X}^{K}$ coincides in law with the original process $\mathbf{X}$ but with other parameters κ,ρ. Recall the notation (1.4.1).

Lemma 4.7.1 **(space-time-mass scaling).** *For* $K \geq 1$, *let* μ_K *belong to* $\mathcal{M}_a$, *and set* $\kappa_K := \kappa K^{\gamma-\alpha}$ *as well as* $\rho_K := \rho K^{\gamma-d}$. *Then*

$$\mathbb{P}^{\kappa,\rho}_{0,\mu_K}\left(X^K(t)\in(\cdot)\right) = \mathbb{P}^{\kappa_K,\rho_K}_{0,(\mu_K)^K}\left(X(t)\in(\cdot)\right), \qquad K\geq 1,\ t>0.$$

Proof. Fix $K\geq 1$. By the *self-similarity* of the stable transition density functions $p^{\kappa}(t) := p^{\kappa}(t,\cdot)$, $t>0$, introduced in Subsection 2.2, we have

$$p^{\kappa}(K^{\gamma}t) = \left(p^{\kappa_K}(t)\right)^K, \qquad t>0, \tag{4.7.2}$$

(which directly follows from (2.2.1)). This implies

$$\mathcal{T}^{\kappa}_{K^{\gamma}t}(\varphi^K) = \left(\mathcal{T}^{\kappa_K}_t\varphi\right)^K, \qquad t\geq 0,\ \varphi\in\Phi_-. \tag{4.7.3}$$

But $K^{-d}(\psi^2)^K = (\psi^K)^2$, $\psi\in\Phi_-$, and the uniqueness of solutions $u=u_{[\kappa,\rho,\varphi,0]}$ to equation (3.2.2) yields

$$u_{[\kappa,\rho,\varphi^K,0]}(-K^{\gamma}t) = \left(u_{[\kappa_K,\rho_K,\varphi,0]}(-t)\right)^K, \qquad t\geq 0,\ \varphi\in\Phi_-.$$

Then from (3.2.1) for $t\geq 0$, $\varphi\in\Phi_-$,

$$\mathbb{E}^{\kappa,\rho}_{0,\mu_K}\exp\langle X^K(t),\varphi\rangle = \mathbb{E}^{\kappa,\rho}_{0,\mu_K}\exp\langle X(K^{\gamma}t),\varphi^K\rangle = \exp\left\langle \mu_K, u_{[\kappa,\rho,\varphi^K,0]}(-K^{\gamma}t)\right\rangle.$$

By the previous identity and again by (3.2.1) we can continue with

$$= \exp\left\langle \mu_K,\left(u_{[\kappa_K,\rho_K,\varphi,0]}(-t)\right)^K\right\rangle = \exp\left\langle (\mu_K)^K, u_{[\kappa_K,\rho_K,\varphi,0]}(-t)\right\rangle = \mathbb{E}^{\kappa_K,\rho_K}_{0,(\mu_K)^K}\exp\langle X(t),\varphi\rangle.$$

This coincidence of Laplace functionals implies the claim. □

The Lemmas 4.7.1 and 4.5.1 are now the essential steps in order to see that Theorem 1.4.2 follows from Theorem 4.1.1. In fact, for $\gamma=\alpha$,

$$\mathbb{P}^{\kappa,\rho}_{0,\mu_K}\left(X^K(t)\in(\cdot)\right) = \mathbb{P}^{\kappa,\rho_K}_{0,\mu}\left(X(t)\in(\cdot)\right) = \mathbb{P}^{\kappa,\rho}_{0,K^{d-\alpha}\mu}\left(K^{-(d-\alpha)}X(t)\in(\cdot)\right), \qquad K\geq 1, \tag{4.7.4}$$

and we have only to set $K^{d-\alpha}=:R$ and take into account that $d>\alpha$, by assumption.

Remark 4.7.5. Under *subcritical scaling*, that is $\gamma < \alpha\wedge d$, the *law of large numbers*

$$X^K(t) \xrightarrow[K\to\infty]{\mathcal{P}r} \mu \quad \text{if} \quad X^K(0) \xrightarrow[K\to\infty]{\mathcal{P}r} \mu,$$

mentioned in the end of Subsection 1.2 above, follows similarly as in the proof of Lemma 4.5.2, since here $\kappa_K\to 0$ in view of Lemma 4.7.1 and $\mathcal{T}^0_t$ equals the identity operator. ■

Appendix: On the Model without Spatial Motion

The purpose of this Appendix is to compute the Legendre transform of the log-Laplace functional $\Lambda_{\mu,t}$ of $X(t)$ in the case $\kappa=0$, as announced in Subsection 1.4.

Fix $\rho>0$ and $\varphi\in\Phi$. Then our equation (2.4.2') (with $\psi=0$) degenerates to the ordinary equation

$$u(s,y) = \varphi(y) + \rho \int_s^0 dr\, u^2(r,y), \qquad s<0, \tag{A.1}$$

which has the (pointwise) solution

$$u(s,y) = \begin{cases} \varphi(y)/(1+\rho s\varphi(y)) & \text{if } \rho s\varphi(y) > -1 \\ +\infty & \text{otherwise.} \end{cases}, \qquad s<0,\ y\in\mathbb{R}^d. \tag{A.2}$$

By analytic continuation as in the proof of Proposition 3.3.1 we conclude that

$$\log \mathbb{E}^{0,\rho}_{0,\delta_y} \exp(X(t),\varphi) = u(-t,y) \in (-\infty,+\infty], \qquad t>0.$$

In addition, fix $t>0$ and $\mu\in\mathcal{M}_a$. Then

$$\Lambda_{\mu,t}(\varphi) = \log \mathbb{E}^{\kappa,\rho}_{0,\mu} \exp(X(t),\varphi) = \int\mu(dy)\ u(-t,y) \in (-\infty,+\infty].$$

with u given in (A.2).

Next we introduce some notation. Each $\nu\in\mathcal{M}_a$ may be uniquely decomposed in $\mathcal{M}_a$ into $\nu = \nu_{ac} + \nu_\partial + \nu_\infty$. Here $\nu_{ac}(dy) =: g_{ac}(y)\mu(dy)$ is absolutely continuous with respect to μ whereas ν_∂ and ν_∞ are singular with respect to μ. By definition, ν_∂ is concentrated on the (uniquely determined) closed support $\mathcal{S}$ of μ and $\nu_\infty(\mathcal{S}) = 0$. Recall that Φ^* is the dual space to the Banach space Φ but equipped with the weak* topology.

Proposition A.3. *The Legendre transform*

$$\Lambda^*_{\mu,t}(\varphi^*) := \sup_{\varphi\in\Phi}\{(\varphi^*,\varphi) - \Lambda_{\mu,t}(\varphi)\}, \qquad \varphi^*\in\Phi^*, \tag{A.4}$$

has the following form: For $\nu\in\mathcal{M}_a$,

$$\Lambda^*_{\mu,t}(\nu) = (\rho t)^{-1} \int\mu(dy)\ (\sqrt{g_{ac}(y)}-1)^2 + \nu_\partial(\mathbb{R}^d)/\rho t + \nu_\infty(\mathbb{R}^d)\cdot(+\infty), \tag{A.5}$$

whereas $\Lambda^*_{\mu,t}(\varphi^*)=+\infty$ *for the remaining* $\varphi^*\in\Phi^*$ *(we use the convention* $0\cdot(+\infty)=0$*).*

Note that $\Lambda^*_{\mu,t}$ is *positively homogeneous along* ν_∂, hence it is *not strongly convex* at $\nu=\nu_\partial\neq0$. Roughly speaking, strong convexity is violated by some measures ν which spatially "deviate" inside the closed support of the starting measure μ. Note also that $\Lambda^*_{\mu,t}$ is *not continuous* in the vague topology. In fact, let $d=1$, $\rho t=1$, let μ be the uniform distribution on the interval $(-1,1)$, and ν_ε be the mean zero Gaussian distribution with variance $\varepsilon>0$, but restricted to $(-1,1)$. Then $\nu_\varepsilon=(\nu_\varepsilon)_{ac}$ converges vaguely to $\delta_0=(\delta_0)_\partial$ as $\varepsilon\to 0$ but $\mathbf{S}_{\mu,t}(\nu_\varepsilon) \xrightarrow[\varepsilon\to 0]{} 2$ whereas $\mathbf{S}_{\mu,t}(\delta_0) = 1$. On the other hand, mention that $\Lambda_{\mu,t}$ is *steep* if μ is concentrated on a finite set of atoms.

Proof of Proposition A.3. 1°. Without loss of generality we may assume that $\rho=1$ (otherwise make a time change). Also, by the special form of $\Lambda_{\mu,t}$, in the definition of $\Lambda^*_{\mu,t}$ the supremum can be restricted to those $\varphi\in\Phi$ such that $t\varphi(y) < 1$ μ-a.e. (since (φ^*,φ) is always finite).

2°. To prove that $\Lambda^*_{\mu,t} = +\infty$ outside $\mathcal{M}_a$, we fix $\varphi^*\in\Phi^*$ and assume that $\sup\{(\varphi^*,\varphi) - \Lambda_{\mu,t}(\varphi)\} < +\infty$ where φ runs through the set just described. Then we have to show that φ^* can be generated by a measure in $\mathcal{M}_a$. To this purpose we want to apply the Daniell-Stone Theorem; see, for instance, Bauer (1974), Satz 39.4. Indeed, Φ is a Stone Vektorverband, and we will show that φ^* is non-negative and that $(\varphi^*,\varphi_n) \longrightarrow 0$ as φ_n pointwise monotonously decreases to 0 (as $n\to\infty$). Assume that there exists a non-negative $\varphi\in\Phi$ such that $(\varphi^*,\varphi) < 0$. Then $\theta\varphi \le 0$ for $\theta<0$, and the supremum in the definition of the Legendre transform can be estimated below by taking into account only $\theta\varphi$:

$$\Lambda^*_{\mu,t}(\varphi^*) \ge \theta(\varphi^*,\varphi) \quad \text{since} \quad -\Lambda_{\mu,t}(\theta\varphi) \ge 0.$$

Letting $\theta \to -\infty$ we get a contradiction to the assumed finiteness. Hence, φ^* is non-negative. Suppose that in Φ there exists a sequence $\varphi_n \downarrow 0$ pointwise as $n\to\infty$ and such that $(\varphi^*,\varphi_n) \ge \varepsilon$, $n\ge 1$, for some $\varepsilon>0$. All φ_n are continuous and will vanish as $|y|\to\infty$. Hence, $\|\varphi_n\|_\infty \longrightarrow 0$ as $n\to\infty$. Thus, for each $\theta>0$, $0 \le \theta\varphi_n \le \|\theta\varphi_n\|_\infty < 1/t$ for all sufficiently large n. Therefore,

$$\Lambda^*_{\mu,t}(\varphi^*) \ge \theta(\varphi^*,\varphi_n) - \Lambda_{\mu,t}(\theta\varphi_n) \ge \theta\varepsilon - \int d\mu\ \theta\varphi_n/(1-t\theta\varphi_n)$$

for sufficiently large n. But even

$$1 - t\theta\varphi_n(y) \ge 1 - t\theta\|\varphi_n\|_\infty \ge 1/2$$

as $n\to\infty$. Hence,

$$\Lambda^*_{\mu,t}(\varphi^*) \ge \theta\varepsilon - 2\int \mu(dy)\ \theta\varphi_n(y).$$

However, the latter integral is finite and tends to 0 by monotone convergence as $n\to\infty$. Thus $\Lambda^*_{\mu,t}(\varphi^*) \ge \theta\varepsilon$, for all $\theta>0$. Letting $\theta\to\infty$ we arrive at the desired contradiction. Summarizing, φ^* is an abstract integral and can then be represented by some measure ν. Here ν is defined on the smallest σ-field making all $\varphi\in\Phi$ measurable, which is nothing else than the usual Borel σ-field on $\mathbb{R}^d$. Of course, ν has the needed finiteness property, i.e. it belongs to $\mathcal{M}_a$. It remains to calculate $\Lambda^*_{\mu,t}$ on $\mathcal{M}_a$.

3^o. By calculus methods one can easily handle the *"zero-dimensional"* case: For $a\geq 0$,

$$\sup_{\theta<1/t}(a\theta - \theta/(1-t\theta)) = t^{-1}(\sqrt{a} - 1)^2$$

where the supremum is uniquely "realized" at $\theta=(1 - 1/\sqrt{a})/t$ (read $\theta=-\infty$ when $a=0$).

4^o. Next we will deal with the case $\nu_\infty\neq 0$. Here we have to show that $\Lambda^*_{\mu,t}(\nu) = +\infty$. Now there is a bounded Borel set $B \subseteq \mathbb{R}^d\setminus\mathcal{S}$ with $\nu_\infty(B) > 0$. By regularity, there is even a compact set $C \subseteq B$ with $\nu_\infty(C) > 0$. Since the closed sets C and $\mathcal{S}$ are apart by a positive (Euclidean) distance, for all sufficiently small $\varepsilon>0$ the open ε-neighborhood $\mathcal{U}_\varepsilon(C) =: \mathcal{U}$ of C is also disjoint to $\mathcal{S}$. For such ε we may choose some $\psi_\varepsilon\in\Phi$ with the property that $\varepsilon^{-1}1_C \leq \psi_\varepsilon \leq \varepsilon^{-1}1_{\mathcal{U}}$. Then $\Lambda_{\mu,t}(\psi_\varepsilon) = 0$, and

$$\Lambda^*_{\mu,t}(\nu) \geq (\nu,\psi_\varepsilon) \geq \varepsilon^{-1}\,\nu_\infty(C) \longrightarrow +\infty \quad \text{as} \quad \varepsilon\to 0.$$

5^o. For the remaining proof we can assume that $\nu_\infty=0$. Then

$$\Lambda^*_{\mu,t}(\nu) = \sup\{(\nu_\partial+\nu_{ac},\varphi) - \Lambda_{\mu,t}(\varphi)\} \tag{A.6}$$

where the supremum is taken over those $\varphi\in\Phi$ such that $t\varphi(y) < 1$ μ-a.e. We can estimate from above as follows (recall that $g_{ac}=:g$ is the density of ν with respect to μ):

$$\Lambda^*_{\mu,t}(\nu) \leq \nu_\partial(\mathbb{R}^d)/t + \int\mu(dy)\,\sup_{\theta<1/t}\,[g(y)\theta - \theta/(1-t\theta)]$$

i.e. we pass to pointwise supremes, using for the first term that $\varphi \leq 1/t$ on $\mathcal{S}$ by the continuity of φ. Together with 3^o we get the desired expression as an upper estimate for $\Lambda^*_{\mu,t}(\nu)$. It remains to deal with estimations from below. Here the *key idea of proof* consists in choosing a $\psi\in\Phi$ such that approximately $\psi(y) \approx 1/t$ for those y where ν_∂ has its mass, whereas $\psi(y) \approx (1 - 1/\sqrt{g(y)})/t$ on the "support" of ν_{ac}. Here of course some technical work has to be done.

6°. We start with the case $\mu(\{g < 1-\delta\}) = +\infty$ for some $\delta>0$. Then also $\mu(\{(\sqrt{g}-1)^2 > \delta^2\}) \geq \mu(\{\sqrt{g} < 1-\delta\}) = +\infty$ for some (in the following fixed) $\delta>0$, and we have to show that $\Lambda^*_{\mu,t}(\nu) \geq +\infty$. Let $A\subseteq\mathcal{S}$ be a supporting Borel set of μ with the property that $\nu_\partial(A) = 0$. By our assumption, to each $K>0$ there is a compact set $C := C_K \subseteq \{\sqrt{g} < 1-\delta\} \cap A$ with $\mu(C) > K$. By regularity, we find a bounded open neighborhood $\mathcal{U}(C_K) =: \mathcal{U}$ such that $(\nu+\mu)(\mathcal{U}\setminus C) < 1$. Choose $\psi\in\Phi$ with

(A.7) $$- t^{-1}\delta(1-\delta)^{-1}1_{\mathcal{U}} \leq \psi \leq - t^{-1}\delta(1-\delta)^{-1}1_C.$$

Now

$$\Lambda^*_{\mu,t}(\nu) \geq (\nu_\partial+\nu_{ac},\psi) - \Lambda_{\mu,t}(\psi),$$

and using both estimates of (A.7) and since $\nu_\partial(C) = 0$ and $r \mapsto r/(1-r)$ is monotone, we can continue with

$$\geq - (\nu_\partial+\nu_{ac})(\mathcal{U}\setminus C)\ t^{-1}\delta(1-\delta)^{-1} - \int_C \mu(dy)\ [g(y)\ t^{-1}\delta(1-\delta)^{-1} - t^{-1}\delta]$$
$$\geq - t^{-1}\delta(1-\delta)^{-1} + \mu(C)\ t^{-1}\delta^2 \geq - t^{-1}\delta(1-\delta)^{-1} + K\ t^{-1}\delta^2$$

which tends to $+\infty$ as $K\to\infty$.

7°. By the previous step of proof, from now on we can assume that $\mu(\{g<1-\delta\}) < +\infty$ for all $\delta>0$. Let E denote the halfopen unit cube $[0,1)^d$ in $\mathbb{R}^d$, and let z_i, $i=1,2,\ldots$, run through all points of the lattice $\mathbb{Z}^d$. Each Borel set $B\subseteq\mathbb{R}^d$ can be decomposed into disjoint bounded sets by setting $B_i := B \cap (E+z_i)$, $i\geq1$. We will apply this construction (and reserve the index i for it) to the sets $\mathcal{S}\setminus A$ and $A \cap \{g \geq 1-\delta\}$, $\delta>0$, which have possibly infinite mass with respect to ν_∂ and μ. (Although we could also deal separately with the cases $\nu_\partial(\mathcal{S}\setminus A) = +\infty$ and $\mu(\{g\geq1+\delta\}) = +\infty$ similarly as in 6°, since then $\Lambda^*_{\mu,t}(\nu) = +\infty$.)

8°. For the next steps of proof we fix a number $\delta := 2^{-m}$, $m>1$, and set $\varepsilon := \varepsilon_{\delta,n} = \sqrt{\delta 2^{-n}}$, $n\geq1$. For $i\geq1$ choose compact sets $C_{\varepsilon,i} \subseteq (\mathcal{S}\setminus A)_i$ such that

(A.8) $$\nu_\partial((\mathcal{S}\setminus A)_i \setminus C_{\varepsilon,i}) < \varepsilon^2 2^{-i}.$$

For $1 \leq j \leq (1-\delta)/\varepsilon^2$ we introduce the Borel sets

$$B_{\varepsilon,j} := \{(j-1)\varepsilon^2 \leq g < j\varepsilon^2\} \cap A.$$

Select compact sets $K_{\varepsilon,j} \subseteq B_{\varepsilon,j}$ satisfying $\mu(B_{\varepsilon,j}\setminus K_{\varepsilon,j}) < \varepsilon^4$. For $i\geq1$ take compact subsets $L_{\varepsilon,i}$ of $(\{1-\delta \leq g < 1\} \cap A)_i$ with the property that

$$\mu\Big((\{1-\delta\leq g<1\}\cap A)_i \setminus L_{\varepsilon,i}\Big) < \varepsilon^2 2^{-i}.$$

Finally, for $i\geq 1$ and $0 \leq k < (1-\varepsilon^2)/\varepsilon^4$ set

$$B_{\varepsilon,i,k} := (\{1+k\varepsilon^2 \leq g < 1+(k+1)\varepsilon^2\}\cap A)_i$$

and take compact sets $C_{\varepsilon,i,k} \subseteq B_{\varepsilon,i,k}$ such that

$$(\nu_{ac}+\mu)(B_{\varepsilon,i,k} \setminus C_{\varepsilon,i,k}) < \varepsilon^6 2^{-i}. \tag{A.9}$$

Note that all these compact sets $C_{\varepsilon,i}$, $K_{\varepsilon,j}$, $L_{\varepsilon,i}$, and $C_{\varepsilon,i,k}$ (where i,j,k are running as above) have pairwise a positive distance. Now choose $\psi_\varepsilon\in\Phi$ with the property that

$$\psi_\varepsilon(y) = \begin{cases} (1-\varepsilon)/t & \text{on } C_{\varepsilon,i} \\ (1 - 1/\sqrt{j\varepsilon^2}\,)/t & \text{on } K_{\varepsilon,j} \\ 0 & \text{on } L_{\varepsilon,i} \\ (1 - 1/\sqrt{1+(k-1)\varepsilon^2})/t & \text{on } C_{\varepsilon,i,k} \end{cases}$$

where $i\geq 1$, $1 \leq j \leq (1-\delta)/\varepsilon^2$, and $0 \leq k < (1-\varepsilon^2)/\varepsilon^4$. Moreover, we impose

$$-1/\varepsilon t \leq \psi_\varepsilon(y) \leq (1-\varepsilon)/t \tag{A.10}$$

for the remaining y. This choice of ψ_ε is actually possible since ψ_ε has these bounds also on all the compact sets above (for the fixed ε).

9°. Now we are ready to provide the estimates from below. In fact, $\Lambda^*_{\mu,t}(\nu) \geq (\nu_\partial,\psi_\varepsilon) + I_1 + I_2 + I_3$ where the last three terms refer to the integral $\int d\mu\ (g\psi_\varepsilon - \psi_\varepsilon/(1-t\psi_\varepsilon))$ restricted to $\{g<1-\delta\}$, to $\{1-\delta\leq g<1\}$, and to $\{g\geq 1\}$, respectively. First of all,

$$(\nu_\partial,\psi_\varepsilon) \geq \nu_\partial(\textstyle\bigcup_i C_{\varepsilon,i})\ (1-\varepsilon)/t - \nu_\partial((\mathcal{S}\setminus A) \setminus \bigcup_i C_{\varepsilon,i})/\varepsilon t$$

where the first term converges to the desired expression $\nu_\partial(\mathbb{R}^d)/t$ as $\varepsilon\to 0$, whereas, using (A.8), the second term can be estimated further from below by $\geq -\ \varepsilon/t$, converging to zero as $\varepsilon\to 0$.

10°. Turning to I_1 we proceed as follows. On each set $K_{\varepsilon,j}$ for the integrand we have

$$g\psi_\varepsilon - \psi_\varepsilon/(1-t\psi_\varepsilon) \geq (\sqrt{j\varepsilon^2}\,-1)^2/t$$

since $g < j\varepsilon^2$ and noting that ψ_ε is non-positive because of $j\varepsilon^2<1$. Further, on $B_{\varepsilon,j}\setminus K_{\varepsilon,j}$ use $0\leq g\leq 1$ and (A.10) to get $g\psi_\varepsilon - \psi_\varepsilon/(1-t\psi_\varepsilon) \geq -\ 2/t\varepsilon$. Decomposing $\{g < 1-\delta\} = \bigcup_j (K_{\varepsilon,j} \cup (B_{\varepsilon,j}\setminus K_{\varepsilon,j}))$ we obtain

$$I_1 \geq \sum_j \left[\mu(K_{\varepsilon,j})\ (\sqrt{j\varepsilon^2}\,-1)^2/t - \mu(B_{\varepsilon,j}\setminus K_{\varepsilon,j})\ 2/t\varepsilon\right].$$

Since $\mu(B_{\varepsilon,j} \setminus K_{\varepsilon,j}) < \varepsilon^4$ and taking into account that there are at most $1/\varepsilon^2$ indices j, further

$$I_1 \geq \sum_j \mu(K_{\varepsilon,j}) (\sqrt{j\varepsilon^2}-1)^2/t - 2\varepsilon/t.$$

Now set $f_\varepsilon(y) := \sum_j j\varepsilon^2 1_{B_{\varepsilon,j}}(y)$ to get

$$I_1 \geq t^{-1} \int \mu(dy)\, 1_{\bigcup_j K_{\varepsilon,j}}(y) \left(\sqrt{f_\varepsilon(y)} - 1\right)^2 - 2\varepsilon/t.$$

Recall that $\varepsilon^2 = \delta 2^{-n}$, and let $n\to\infty$. Then on $\{g < 1-\delta\} \cap A$ we have $f_\varepsilon \longrightarrow g$ and $1_{K_{\varepsilon,j}} \longrightarrow 1$ pointwise. But f_ε is bounded by 1 and $\mu(\{g < 1-\delta\}) < +\infty$, thus by bounded convergence we get $I_1 \geq t^{-1} \int_{\{g<1-\delta\}} d\mu\, (\sqrt{g}-1)^2$ where the latter expression finally converges to $t^{-1} \int_{\{g<1\}} d\mu\, (\sqrt{g}-1)^2$ as $\delta\to 0$.

11°. Since $\psi_\varepsilon(y) = 0$ in the main term of I_2, its estimation results into the error term

$$I_2 \geq \sum_i \mu\left((\{1-\delta\leq g<1\}\cap A)_i \setminus L_{\varepsilon,i}\right) 2\varepsilon/t \geq -2\varepsilon/t \xrightarrow[\varepsilon\to 0]{} 0.$$

12°. Finally, ψ_ε is non-negative on each $\bigcup_i C_{\varepsilon,i,k}$. Hence, on these sets $g \geq 1 + k\varepsilon^2 =: \eta_k$, and then

$$I_3 \geq \sum_k \Big[\mu(\textstyle\bigcup_i C_{\varepsilon,i,k}) \left(\eta_k (1-1/\sqrt{\eta_k})/t - (\sqrt{\eta_k}-1)/t\right)$$
$$- \nu_{ac}(\textstyle\bigcup_i B_{\varepsilon,i,k} \setminus \bigcup_i C_{\varepsilon,i,k})/\varepsilon t - \mu(\bigcup_i B_{\varepsilon,i,k} \setminus \bigcup_i C_{\varepsilon,i,k})/\varepsilon t\Big].$$

By (A.9) we can continue with

$$\geq \sum_k \Big[\mu(\textstyle\bigcup_i C_{\varepsilon,i,k}) (\sqrt{\eta_k} - 1)^2/t - 2\varepsilon^5/t\Big].$$

Using the notation $h_\varepsilon(y) := \sum_k \eta_k 1_{\bigcup_i B_{\varepsilon,i,k}}(y)$ and taking into account that we have at most $1/\varepsilon^4$ indices k, the latter expressions can be written as and estimated from below by

$$t^{-1} \int d\mu\, 1_{\bigcup_{i,k} C_{\varepsilon,i,k}} (\sqrt{h_\varepsilon}-1)^2 - 2\varepsilon/t.$$

Here we can additionally assume that in 8° the construction of the sets $C_{\varepsilon,i,k}$ had been done in such a way that the union $\bigcup_{i,k} C_{\varepsilon,i,k}$ *monotonously* increases to $\{g\geq 1\}$ as $n\to\infty$ (via $\varepsilon = \sqrt{\delta 2^{-n}}$). But h_ε converges monotonously to $g1_{\{g\geq 1\}}$ and then by monotone convergence as $n\to\infty$ we arrive at the estimate $I_3 \geq t^{-1}\int_{\{g\geq 1\}} d\mu\, (\sqrt{g}-1)^2$.

13°. Combining the estimates in 9°-12°, we get the desired lower bound, and the proof of Proposition A.3 is complete. □

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INSTITUTE OF APPLIED ANALYSIS
AND STOCHASTICS
HAUSVOGTEIPLATZ 5-7
D-O-1086 BERLIN, GERMANY
e-mail: fLeischmann@iaas-berLin.dbp.de

DEPARTMENT OF MATHEMATICS,
UPPSALA UNIVERSITY
BOX 480
S-751 06 UPPSALA, SWEDEN
e-mail: ingemar.kaj@math.uu.se

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