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The discrete spectrum of the Dirac operators on certain symmetric spaces

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Contents

1 Introduction

In this paper I want to write down some remarks on the existence of harmonic $\mathcal{L}^2$-Spinors on symmetric spaces. It was inspired by an article of U. Bunke [Bun91]. His intention was to give one more explicit calculation of spectral properties of Dirac operators. There are different methods of investigating the spectrum of these operators, but in the case of Riemannian symmetric spaces the techniques symplify enormously because of the possibility of applying methods of harmonic analysis.
For the convenience of the reader I will give a short account of the material from representation theory I shall use later. In the first part I describe the groups of interest by means of roots and certain decompositions of groups and their Lie algebras. In the second section I give an overview over the representations I deal with. Then, after recalling the definition of the Spinor bundle and the Plancherel theorem for the decomposition of $\mathcal{L}^2(G/K)$ I can formulate and prove the main theorem.
I would like to thank U.Semmelmann for bringing this subject to my interest.

2 The groups of interest

In this paper I will deal with semisimple Lie groups of real rank one, more exactly with $Spin(2n,1)_0$, $SU(n,1)$ *and* $Sp(n,1)$. For such groups one has the *Iwasawa decomposition* $G = KAN$, where K is a maximal compact subgroup, A and N are so called vector

groups, for they are diffeomorphic to their Lie algebra via the exponential map, and the dimension of the group A is one.
In Lie groups of real rank one there are only minimal parabolic subgroups . These groups I will denote by P and their *Langlands decomposition* by $P = MAN$, where M is the centralizer in K of A , A and N as above. All these groups are considered as real groups, and their real Lie algebras I will denote by small Gothic letters with subscript 0.

2.1 Satake diagrams

In this subsection I will recall the *Satake diagrams* for the groups I will deal with.
First I give some notation:

- $\bullet$ stands for a simple root α, s.t. $\alpha\big|_{\mathfrak{a}} \equiv 0$, i.e. α is a root of M .
- $\alpha \Longleftarrow \beta$ means that
  - $|\beta|^2 = 2|\alpha|^2$
  - $\sphericalangle(\alpha,\beta) = \frac{3\pi}{4}$
- $\alpha — \beta$
  - $|\beta|^2 = |\alpha|^2$
  - $\sphericalangle(\alpha,\beta) = \frac{2\pi}{3}$

Then one has for the groups mentioned above the following diagrams:

- $Spin(2n,1)_0, \quad n \geq 2$

  $$\begin{array}{ccccccccccc} \circ & — & \bullet & — & \bullet & — \ldots — & \bullet & \Longrightarrow & \bullet & , \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_{n-1} & & \alpha_n & \end{array}$$

  where

  $$\begin{aligned} \alpha_i &= e_i - e_{i+1}, \quad i = 1, \ldots, n \\ \alpha_{n+1} &= e_{n+1} \\ \overline{\alpha} &= e_1 \quad \text{real root} \end{aligned}$$

- $SU(n,1), \quad n \geq 1$

  $$\begin{array}{ccccccccccc} \circ & — & \bullet & — & \bullet & — \ldots — & \bullet & — & \circ & , \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_{n-1} & & \alpha_n & \end{array}$$

  where

  $$\begin{aligned} \alpha_i &= e_i - e_{i+1}, \quad i = 1, \ldots, n-1 \\ \overline{\alpha} &= e_1 - e_{n+1} \quad \text{real root} \end{aligned}$$

- $Sp(n,1)$, $n \geq 2$

$$\underset{\alpha_1}{\bullet} - \underset{\alpha_2}{\circ} - \underset{\alpha_3}{\bullet} - \dots - \underset{\alpha_n}{\bullet} \Longleftarrow \underset{\alpha_{n+1}}{\bullet},$$

where

$$\begin{aligned} \alpha_i &= e_i - e_{i+1}, \quad i = 1, \dots, n \\ \alpha_{n+1} &= 2e_{n+1} \\ \overline{\alpha} &= e_1 + e_2 \quad \text{real root} \end{aligned}$$

Remark 2.1 *For writing the roots there are different conventions, I prefere writing them in orthogonal coordinates, i.e., using the expression*
$(a_1, a_2, \dots, a_n)$ *for* $\sum a_i e_i$, *and* $(e_i, e_j) = \frac{1}{2}\delta_{ij}$, *where* $(\cdot,\cdot)$ *is the scalar product, induced by the Killing form.*
This scaling has been chosen for having base roots α_i *of length 1.*

2.2 Root systems

In the sequel I will denote all root systems connected with the compact Cartan subalgebra $\mathfrak{h}_c$ by Φ (if necessary with sub- or superscripts), and those which correspond to $\mathfrak{h}$ - the split Cartan subalgebra - by Δ.

For illustration let me give the most important root systems for the groups I consider.

Root systems with respect to $\mathfrak{h}_c$

Let Φ^+ be a system of positive compact roots of $\Phi = \Phi(\mathfrak{g}, \mathfrak{h}_c)$ and $\Phi_c = \Phi(\mathfrak{k}, \mathfrak{h}_c)$ be the set of compact roots, i.e. roots with root spaces in $\mathfrak{k}$.
The half sums of positive roots I will denote by δ and δ_c respectively.

1. $\mathfrak{so}(2n,1)$

   - $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j, \varepsilon_i \quad | \quad 1 \leq i < j \leq n\}$
   - $\Phi_c = \{\varepsilon_i \pm \varepsilon_j, \quad | \quad 1 \leq i < j \leq n\}$
   - $$\begin{aligned} \delta &= \left(n - \tfrac{1}{2}, n - \tfrac{3}{2}, \dots, \tfrac{3}{2}, \tfrac{1}{2}\right) \\ \delta_c &= (n-1, n-2, \dots, 1, 0). \end{aligned}$$

2. $\mathfrak{su}(n,1)$

- $\Phi^+ = \{\varepsilon_i - \varepsilon_j, \quad | \quad 1 \le i < j \le n+1\}$
- $\Phi_c = \{\varepsilon_i - \varepsilon_j, \quad | \quad 1 \le i < j \le n\}$
- $$\begin{aligned} \delta &= \left(\tfrac{n}{2}, \tfrac{n-1}{2}, \dots, \tfrac{3}{2}, \tfrac{1}{2},\right) \\ \delta_c &= \left(\tfrac{n-1}{2}, \tfrac{n-2}{2}, \dots, \tfrac{1}{2}, 0\right). \end{aligned}$$

3. $\mathfrak{sp}(n,1)$

- $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_i \quad | \quad 1 \le i < j \le n+1\}$
- $\Phi_c = \{\varepsilon_i \pm \varepsilon_j, \quad 2\varepsilon_i \quad | \quad 1 \le i < j \le n\} \cup \{2\varepsilon_{n+1}\}$
- $$\begin{aligned} \delta &= (n+1, n, \dots, 2, 1) \\ \delta_c &= (n, n-1, \dots, 2, 1, 1) \end{aligned}$$

Non-compact root systems

In this chapter I will introduce the root systems coming from the Iwasawa decomposition, i.e. Δ, Δ_m and Δ_a. Let us start with the definitions.

Definition 2.2 $\Delta_m := \{\alpha \in \Delta \quad | \quad (\alpha, \overline{\alpha}) = 0\}$ *, i.e. it is the root system of* $\mathfrak{t}$ *in* $\mathfrak{m}$*, where* $\mathfrak{t}$ *is the Cartan subalgebra of* $\mathfrak{m}$.

$\Delta_a := \Delta \setminus \Delta_m = \{\alpha \in \Delta \quad | \quad (\alpha, \overline{\alpha}) \neq 0\}$.

Warning : This Δ_a is not the root system of $\mathfrak{a}$ in $\mathfrak{g}$, though it clearly consists of those roots which give a contribution to it.

Now let us write down these systems for our groups. Though it is obvious that the general root system should be the same like in the compact case for convenience I will recall it.

1. $\mathfrak{so}(2n,1)$:
   - $\Delta^+ = \{e_i - e_j, \quad e_i \quad | \quad 1 \le i < j \le n\}$
   - $\Delta_m^+ = \{e_i - e_j, \quad e_i \quad | \quad 2 \le i < j \le n\}$
   - $\Delta_a^+ = \{e_1 - e_j, \quad e_1 \quad | \quad 2 \le j \le n\}$

2. $\mathfrak{su}(n,1)$:

- $\Delta^+ = \{e_i - e_j, \quad | \quad 1 \leq i < j \leq n+1\}$
- $\Delta^+_m = \{e_i - e_j, \quad | \quad 2 \leq i < j \leq n\}$
- $\Delta^+_a = \{e_1 - e_j, \quad e_i - e_{n+1} \quad | \quad 2 \leq j \leq n\}$

3. $\mathfrak{sp}(n,1)$:

- $\Delta^+ = \{e_i - e_j, \quad 2e_i \quad | \quad 1 \leq i < j \leq n+1\}$
- $\Delta^+_m = \{e_1 - e_2\} \cap \{e_i \pm e_j, \quad 2e_i \quad | \quad 3 \leq i < j \leq n+1\}$
- $\Delta^+_a = \{e_1 + e_2, \quad 2e_1, \quad 2e_2\} \cap \{e_1 \pm e_j, \quad e_2 \pm e_j, \quad | \quad 3 \leq j \leq n+1\}$

Singularity and Integrality

Definition 2.3 *A root α is called* singular *if $(\alpha,\gamma) = 0$, for some $\gamma \in \Delta^+$.*
A root α is called integral *(or* algebraically integral*) if $\frac{2(\alpha,\gamma)}{|\gamma|^2} \in \mathbb{Z}$ for all $\gamma \in \Delta^+$.*
This leads to an easy criterion of regularity/integrality :

1. $\mathfrak{so}(2n,1)$
   regular roots : $(a_1, a_2, \ldots, a_n)$ s.t. $a_i \neq a_j \quad \forall i \neq j$
   integral roots : $(a_1, a_2, \ldots, a_n)$ s.t. $a_i - a_j \in \mathbb{Z}, 2a_j \in \mathbb{Z} \forall i \neq j$

2. $\mathfrak{su}(n,1)$
   regular roots : $(a_1, a_2, \ldots, a_{n+1})$ s.t. $a_i \neq a_j \quad \forall i \neq j$
   integral roots : $(a_1, a_2, \ldots, a_{n+1})$ s.t. $a_i - a_j \in \mathbb{Z} \forall i \neq j$

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3 Representations

In this section I give a short overview of those representations which will be of interest below. So I will describe the representations of the complementary series with some properties and the representations of the discrete series. I dont want to give a full account of the representation theory, this can be found, e.g., in [Wil91], [Kna86] .
For a better understanding of the structure of representations of semi-simple Lie groups it is often enough to analyse the corresponding Harish–Chandra–module . Let us recall the definition [Wal73].

Definition 3.1 *A Harish–Chandra module is a module V over the universal enveloping algebra $U(\mathfrak{g})$, together with an action of K, s.t.*

1. *the representation of $\mathfrak{k}$ is an algebraic direct sum of irreducible finite dimensional continuous representations of $\mathfrak{k}$, each isomorphism class occuring with finite multiplicity, (admissibility);*
2. *the representation of $\mathfrak{k}$, as a subalgebra of $\mathfrak{g}$, conicides with the differential of the K–action, (compatibility)*
3. *V has finite length as a $U(\mathfrak{g})$–module.*

For our purpose this category is large enough: Let (π, V_π) be an irreducible representation of G (or even an induced representation), and take the module
$V := \{v \in V_\pi | \dim(\operatorname{span}\pi(K)v) < \infty\}$, then this is a Harish–Chandra module.

Remark 3.2 *In the sequel I will be a bit careless about notations and often write representation instead of Harish–Chandra module or regard characters of Harish–Chandra module. This can be justified by the following theorem of S. Ju. Prischepionok [Pri76]:*

Theorem 3.3 *If V is a Harish–Chandra module for G, then V may be realized as the space of K–finite vectors in some admissible finite length representation π of G, on a Hilbert space V_π.*

3.1 The induced representation

In this part I would like to recall the definition and some well known properties of the induced representations (cf. [Kna86]).

Definition 3.4 *Let $Q \subset G$ be a subgroup, (W, γ) a representation of Q. Then one can define the bundle $G \underset{Q}{\times} W \to G/Q$. The space of sections ($\mathcal{L}^2, C^\infty$ etc.) carries a left G–action. This construction is denoted by $Ind_Q^G(W)$.*

Now let $P = MAN$ be "the" minimal parabolic subgroup and $G = KAN$. Then I can decompose

$g = kan = k(g)a(g)n(g), \; p = m'a'n'$ (with the obvious notations).

Moreover, one can define the following projections

$$\begin{aligned} g \mapsto [g]_p &:= g \cdot P = \{g \cdot m'a'n' | m'a'n' \in P\} \\ k \mapsto [k]_m &:= k \cdot M = \{k \cdot m' | m' \in M\} \end{aligned}$$

and one has a map

$$\begin{array}{lrcl} \varphi: & G/P & \to & K/M \\ & [g]_P & \mapsto & [k(g)]_M \\ & kan\; m'a'n' = km'aa'n'' = (km')aa'n'' & \mapsto & (km'). \end{array}$$

So one can see that φ is well defined.

We can define an inverse map

$$\begin{array}{rcl} \varphi^{-1}: K/_M & \to & G/_P \\ [k]_M & \mapsto & [k]_P \\ km' & \mapsto & km' \end{array}$$

This shows that φ is a bijection.

Now recall the definition of the induced representation. Let (γ, W) be a representation of P, for simplicity I will write $p.w$ instead of $\gamma(p)w$.
Then we had defined (e.g.)

$$\mathrm{Ind}_p^G(W) = \mathcal{C}(G) \underset{P}{\times} W = \{f : G \to W | f(gp) = p^{-1} \cdot f(g)\}$$

as a section in the bundle $\mathcal{C}(G) \underset{P}{\times} W \to G/P$; i.e. we have a "twist" with the representation space W and the following diagramm commutes:

$$\begin{array}{ccc} g & \longrightarrow & f(g) \\ \updownarrow & & \updownarrow \\ g \cdot p & \longrightarrow & p^{-1} \cdot f(g) = a'^{-1} m'^{-1} f(g) \;. \end{array}$$

Now I want to describe this bundle as a bundle over K/M: let f be a section of $\mathcal{C}(G) \underset{P}{\times} W$, i.e. $f(gm'a'n') = a'^{-1}m'^{-1}f(g)$, then $f(kan) = a^{-1}f(k)$.

Let us define the correspondence

$f \mapsto \tilde{f} := f\Big|_K$

and, the inverse

$\tilde{f} \mapsto f \qquad$ by $\qquad f(g) := a^{-1}(g)\tilde{f}(k(g))$.

Then it is obvious that

$$\tilde{f} \in \Gamma(\mathcal{C}(K) \underset{M}{\times} W) \overset{1-1}{\longleftrightarrow} f \in \Gamma(\mathcal{C}\, G) \underset{P}{\times} (W).$$

The G actions are given by:

$$\begin{aligned} g\cdot f(g_0) &= f(g^{-1}g_0) \\ g\cdot \tilde{f}(k_0) &= a(gk_0)^{-1}\tilde{f}(k(gk_0)) \end{aligned} \qquad (\#)$$

Infinitesimal character

Now let me recall a fact about the infinitesimal character of an induced representation (cf. [Vog79], [Kna86], [Dix77]). It is well known that there exists an isomorphism γ

$$\gamma : Z(\mathfrak{g}) \to S(\mathfrak{h})^W$$

mapping the center $Z(\mathfrak{g})$ of the universal enveloping algebra onto the Weyl invariants of the symmetric algebra of a Cartan subgroup of $\mathfrak{g}$ $S(\mathfrak{h})^W$, which can be regarded as an algebra of polynomials on $\mathfrak{g}^*$ invariant under the action of the Weyl group .
Now let the center of the universal enveloping algebra act by a scalar (e.g. on an irreducible component). Then there exists a $\lambda \in \mathfrak{g}^*$ s.t.

$$Z.v = \gamma(Z)(\lambda)v := \chi_\lambda(Z)v \qquad \text{for all} \qquad Z \in Z(\mathfrak{g})\ .$$

This map χ_λ is called the infinitesimal character of V; it holds that
$\chi_\lambda = \chi_{\lambda'} \iff \lambda = w\cdot\lambda'$ for some $w \in W$.
Now let (W,σ) be a representation of M with highest weight δ, $\nu \in \mathfrak{a}^*$; let e^ν denote the corresponding A–character.
Then, according to Harish–Chandra the representation $I_{\sigma,\nu}$ has the infinitesimal character $\chi_{\delta+\rho_m,\nu}$, where $(\delta+\rho_m,\nu)$ is regarded as an element of $\mathfrak{t}^*\oplus\mathfrak{a}^* = \mathfrak{h}^*$, $\mathfrak{a}$ the non compact part of $\mathfrak{h}$ (cf. [Vog81]).

Frobenius reciprocity

Let $Q \subset G$ be an arbitrary subgroup, W a Q–module and let $V\big|_Q$ denote the restriction of the G–module V to Q. Then one has

$$\mathrm{Hom}_G(V, \mathrm{Ind}_Q^G(W)) \simeq \mathrm{Hom}_Q(V\big|_Q, W).$$

If we take P as inducing group, we see in # that the action of A is dropped and it holds that

$$\mathrm{Ind}_P^G(W)\Big|_K = \mathrm{Ind}_M^K(W)\ .$$

Corollary 3.5

$$\begin{aligned} Hom_K(U, Ind_P^G(W)\big|_K) &= Hom_K(U, Ind_M^K(W)) = \\ &= Hom_M(U\big|_M, W) \end{aligned}$$

The Langlands classification

Let us regard as the inducing subgroup a minimal parabolic subgroup $P = MAN$. Let $(\gamma, W_\gamma) \in \hat{M}$, $\nu \in \mathfrak{a}^*$, and define a representation $(\gamma \otimes \nu \otimes 1, W \otimes \mathbb{C}_\nu)$ on MAN. Then I will denote by $I_{\gamma,\nu}$ the representation $Ind_P^G(W \otimes \mathbb{C}_{\nu+\rho_a})$. Now I can state the Langlands classification (for sake of simplicity only in the rank one case) ; for reference see, e.g. [CM82], [Lan73].

Theorem 3.6 *Let V be an irreducible Harish–Chandra module for G . Then, either V is tempered, or there exists an induced representation $I_{\gamma,\nu}$ s.t. $Re(\nu) > 0$ on $\mathfrak{a}_0^*$ and V may be realized as the unique irreducible quotient of $I_{\gamma,\nu}$.*
Moreover, the data (W_ν, ν) associated to V , is unique (for a fixed choice of a positive root system).

Thus we get two kinds of irreducible Harish–Chandra modules :

1. the Harish–Chandra module, coming from the unique quotients of induced representations and
2. the Harish–Chandra module, coming from the so called discrete series (which exists, by a result of Harish-Chandra, because of the equality of the ranks of G and K ; i.e. G has a compact Cartan subalgebra H_c).

Notation 1 *The irreducible quotiont of the induced representation π I will denote by $\overline{\pi}$.*

Parametrisation

In this section I will describe the parametrisation of the induced representations with real positive A–parameter and regular infinitesimal character. I prefer the parametrisation "by infinitesimal character". Of course the infinitesimal character does not depend on the particular choice of the root system, but I mean the following:
For $(W, \sigma) \in \hat{M}$ with highest weight δ, $\nu \in \mathfrak{a}^*$ we had defined the representation $I_{\delta,\nu}$ and we had seen that this representation has the infinitesimal character $\chi_{\delta+\rho_m,\nu}$, $(\delta + \rho_m \oplus \nu)$

regarded as an element of $\mathfrak{t}^* \oplus \mathfrak{a}^* = \mathfrak{h}^*$ This root $(\delta + \rho_m \oplus \nu)$ I will take as parameter for the representation $I_{\delta,\nu}$. The representation of the roots as elements of the direct sum is not very convenient, so I will write them in the usual way by their orthogonal projectors.
It follows from the properties of Schmids translation functor that it is enough to regard representations with trivial infinitesimal character (for explanation look, e.g., in [Col85]). Now let me give the concrete parametrisation for the groups of interest.

Proposition 3.7 *The non–equivalent induced representations with trivial infinitesimal character are parametrized by*

1. $\mathfrak{so}(2n, 1)$

$$\rho_{0i} = \left(n + \frac{1}{2} - i, n - \frac{1}{2}, n - \frac{3}{2}, \ldots, n + 1\frac{1}{2} - i, n - 1\frac{1}{2} - i, \ldots, \frac{1}{2}\right)$$

for $1 \leq i \leq n$.

2. $\mathfrak{su}(n, 1)$

$$\rho_{ij} = \left(\frac{n}{2} - i, \frac{n}{2}, \ldots, \frac{n}{2} + 1 - i, \frac{n}{2} - 1 - i, \ldots, j - \frac{n}{2}, j - \frac{n}{2} - 2, \ldots, -\frac{n}{2}, j - \frac{n}{2} - 1\right)$$

for $0 \leq i \leq n, \quad 0 \leq j \leq n - i$

3. $\mathfrak{sp}(n, 1)$

$$\begin{aligned} \rho^{\pm}_{ij} &:= (n + 1 - i, \pm(n + 1 - j), n + 1, \ldots, n + 2 - i, n - i, \ldots, 1) \\ \rho^{\pm}_{ji} &:= (\pm(n + 1 - j), -(n + 1 - i), n + 1, \ldots, n + 2 - i, n - i, \ldots, 1) \end{aligned}$$

for $0 \leq i < j \leq n + 1$

Remark 3.8 *The parametrizing for the extreme values of* i, j *should be clear; I prefered the notation from above for better evidence.*

Notation 2 *I will denote by* π_{ij} *the representation with infinitesimal character* ρ_{ij}.

3.2 The complementary series

Unitary induced representations

Definition 3.9 *A representation is called* unitary *if*

1. *it is irreducible and*
2. *it exists a non–degenerate scalar product s.t. the group acts by unitary transformations.*

Induced representations can be classified as follows:

1. non–unitary induced representations
2. complementary series representations
3. principal series representations.

The latter is defined very simply, namely, one has to take the inducing data $(W, \gamma) \in \hat{M}$ and $\nu \in i\mathfrak{a}_0^*$; this choice generates a unitary representation of G.
In general it is hard to determine whether an induced representation is unitary, but in the case of a real A–parameter ($\nu \in \mathfrak{a}_0^*$) there can exist a scalar product on $I_{\gamma,\nu}$ making the representation unitary. The irreducible quotient of this representations then belongs to the complementary series. In real rank one case the description of the unitary dual is given in [BSK80], but I dont want to recall it here, because in our setting only the principal series will be of some interest.

3.3 Discrete series

Denote by L_{H_c} the integral forms on $\mathfrak{h}_{c_i}$ where the elements of L_{H_c} are those which lift to characters of H_c.
We say that $\Lambda \in L_{H_c}$ is regular if $(\Lambda, \alpha) \neq 0$ for all $\alpha \in \Phi$. Corresponding to each regular $\Lambda \in L_{H_c}$ Harish–Chandra has constructed certain invariant eigendistributions on G . Two of these coincide precisely when their parameters Λ are related by an element of W_K, the Weyl group of K. Each Θ_Λ is the character of a discrete series representation and conversely ([HC65a], [HC65b], [HC65c]).
The infinitesimal character of this corresponding representation is χ_Λ.
If we fix an infinitesimal character χ_ρ, the non–equivalent discrete series representations are labeled by the set: $\{w\rho \quad | \quad w \in W/W_K\}$. Let me give the parameters ρ explicitly.

Proposition 3.10 *The discrete series are parametrised by*

$\mathfrak{so}(2n,1)$ $\quad \{\delta_0 = (n-\frac{1}{2},\dots,\frac{3}{2},\frac{1}{2}) \quad , \quad \delta_1 = (n-\frac{1}{2},\dots,\frac{3}{2},-\frac{1}{2})\}$

$\mathfrak{su}(n,1)$ $\quad \{\delta_i = (\frac{n}{2},\dots,\frac{n}{2}+1-i,\frac{n}{2}-1-i,\dots,-\frac{n}{2},\frac{n}{2}-i)\}$
$i = 0,\dots,n$

$\mathfrak{sp}(n,1)$ $\quad \{\delta_i = (n+1,\dots,n+2-i,n-i,\dots,1,n+1-i)\}$
$i = 0,\dots,n$

4 The Spinor bundle

In this section I will recall the definition of the Spinor bundle and its properties which will be used in the sequel. For a full account see e.g. [BGV90] and [Par67].
Let V be a vector space of dimension d with a (non–degenerate) scalar product, then one can associate the so called *Clifford algebra* $Cl(V)$ of dimension 2^d:
defined as the algebra over $\mathbb{R}$ generated by V with the relations
$vw + wv = -2(v,w) \quad \forall v,w \in V.$

Definition 4.1 *The graded vector space W is called* Clifford module *, if it carries an action of the algebra $Cl(V)$ which is even with respect to the grading of W.*

Proposition/Definition 4.2 *For V an even dimensional Euklidean oriented vector space there exists a unique $\mathbb{Z}_2$–graded Clifford module $S = S^+ \oplus S^-$, called the* Spinor module *, such that*
$Cl(V)^{\mathbb{C}} \simeq End(S).$

Using the fact that $Spin(V) \subset Cl(V)^+$ (the even elements of Cl(V)), we see that $S^{\pm}$ are representations of $Spin(V)$.
Let us now regard our symmetric space G/K , where G is real semi–simple, K the maximal compact subgroup of the same rank as G and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then these spaces have a Spin structure, i.e. there exists a lift of the principal bundle $G \times_K SO(\mathfrak{p})$ to the bundle $G \times_K Spin(\mathfrak{p})$. To this bundle we can associate the vector bundle

$$\mathcal{S} := G \times_K Spin(\mathfrak{p}) \times_{Spin(\mathfrak{p})} S \simeq G \times_K S.$$

This bundle is called the *Spinor bundle.*

Remark 4.3 *Here in fact it does not matter whether I consider the pair (G,K) or, e.g., the coverings $(\widetilde{G},\widetilde{K})$. This is of some imprtance in the case of the group $SU(n,1)$, for there the maximal compact subgroup K has a non–trivial fundamental group π_1.*

Now let me look a bit closer at this bundle from the representation theoretical point of view. Therefore I recall a very important theorem of Parthasaraty ([Par67]). Before stating his result I need some notations.

The existence of a homogeneous Spin–structure means the existence of a map $\tilde{\alpha}$ which makes the following diagramm commutative:

$$\begin{tikzcd} & \mathrm{Spin}(\mathfrak{p}) \arrow[d, "\Psi"] \\ K \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"] & \mathrm{SO}(\mathfrak{p}) \end{tikzcd}$$

This lift exists iff $\alpha_*(\pi_1(K)) = \Psi_*(\pi_1(Spin(\mathfrak{p}))$. This can by obtained by replacing G by a suitable covering if necessary. This is correct in case of noncompact symmetric spaces because of the existence of a diffeomorphism $\varphi :\ G \underset{\longrightarrow}{\sim} K \exp \mathfrak{p}$ which actually means that all topology of G is in fact topology of K ([Hel78]). Now we chose the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{so}(p)$ in such way that

$\mathfrak{k} \supseteq \mathfrak{h}_c \longrightarrow \alpha_*(\mathfrak{h}_c) \subseteq \mathfrak{t}$.

Let $\pm\lambda_1, \ldots, \pm\lambda_m$, where $2m = \dim \mathfrak{p}$, be the weights of the standard representation of $\mathfrak{so}(n)$. Furthermore we need the Weyl group $W^1 := \{\sigma \in W \mid \sigma(\Phi^+(\mathfrak{g}, \mathfrak{h}_c)) \supset \Phi_c^+\}$.

Proposition 4.4 *(Parthasaraty) The representation of K on $S^\pm$ has the character $\chi_\pm$ which decomposes as follows:*

$$\chi_\pm = \bigoplus_{\substack{\sigma \in W^1 \\ \det \sigma = \pm 1}} \tau_{\sigma\delta - \delta_c}$$

4.1 The Plancherel formula

In this section I want to recall the results of Atyiah/Schmid concerning the decomposition of $\mathcal{L}^2(G/K, \mathcal{S})$ ([AS77]). In this part I adhere to their notation. So let V_j denote the representation space corresponding to the representation j, and let $d(j)$ be the Plancherel measure on $\widehat{G}$. Let $\widehat{G}_d$ denote the discrete part of the unitary dual of the group G. Then we have

$$\begin{aligned} \mathcal{L}^2(G/K, \mathcal{S}) &= \mathcal{L}^2(G) \underset{K}{\otimes} S = \qquad \text{Plancherel formula} \\ &= \left(\sum_{j \in \widehat{G}_d} V_j^* \otimes V_j d(j) + \int_{j \in \widehat{G} \setminus \widehat{G}_d} V_j^* \otimes V_j d(j) \right) \underset{K}{\otimes} S = \\ &= \sum_{j \in \widehat{G}_d} V_j \otimes Hom_K(V_j, S) d(j) + \int_{j \in \widehat{G} \setminus \widehat{G}_d} V_j \otimes Hom_K(V_j, S) d(j). \end{aligned}$$

We see that it is necessary to analyse the spaces $Hom_K(V_j, S)$, where V_j corresponds to representations from the unitary dual of G. Moreover, a nontrivial contribution to the non–discrete part give only the principal series representations.

4.2 Non–discrete spectrum

In the case of $j \in \widehat{G} \setminus \widehat{G}_d$ we can use the Frobenius reciprocity:

$$Hom_K(V_{\sigma,\nu}; S^{\pm}) \quad = \quad Hom_M(\sigma, S^{\pm}\Big|_M)$$

Now we have to investigate the groups $Hom_M(\sigma, S^{\pm})$. Here we can use the branching rules, proved by Baldoni–Silva in [BS80].

1. $\mathfrak{so}(2n, 1)$
   Let us recall the highest weights of the K–representations contained in the Spinor representation:
   $\delta_{in} = (\frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$.
   Then we have non-trivial homomorphisms with the restricted Spinor representations $S^{\pm}\Big|_M$ with multiplicity one if the heighest weight μ of the inducing M–representation is of the form
   $\mu = (\frac{1}{2}, \ldots, \frac{1}{2}, 0)$.

2. $\mathfrak{su}(n, 1)$
   Here let us do the same procedure:
   $\delta_{in} = (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{n}{2} - i)$.
   In this case we have a novelty: to one weight δ_{in} correspond three possible weights of the inducing representation:

   $\mu = (\delta_{in} = (\frac{n+1-i}{4}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{n+1-i}{4}).$

   $\mu = (\delta_{in} = (\frac{n+1-i}{4}, \frac{1}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{n+1-i}{4}).$

   $\mu = (\delta_{in} = (\frac{n+1-i}{4}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{n+1-i}{4}).$

3. $\mathfrak{sp}(n, 1)$
   $\delta_{in} = (1, 1, \ldots, 1, 0, \ldots, 0, n - i)$
   Here we have a lot of possible homomorphisms even with higher multiplicities, but it would take too much time to write them down here. For reference see [BS80]

Now I can put together the results from 3.2.2 about decomposition of induced representation and these branching rules in order to get the information about the homomorphisms between the K–types of unitary representations (not belonging to the discrete series) and the Spinor module. For illustration I will regard the case of $Spin(2n,1)$. Non–trivial homomrphisms can occur only if the highest weight of the inducing M representation is $\mu = (\frac{1}{2},\ldots,\frac{1}{2},0)$. This leads to the unitary principal series $Ind(\mu, \mathbb{C}_{i\nu})$, where ν is real and we get for the non–discrete part of $\mathcal{L}^2$

$$\mathcal{L}^2_n = \int\limits_{\mathfrak{a}_0} V_{(\overline{\mu},i\nu)} \otimes \mathbb{C}^2 d(\overline{\mu}, i\nu),$$

where $\overline{\mu}$ is the restriction of μ to $\mathfrak{m}$.

4.3 Discrete spectrum

The case of the discrete spectrum requires some more work. One way of its investigation was proposed in [Bun91] for the group $Spin(2n,1)$. He describes the discrete series by their K-types as subrepresentations of some induced representation.
I prefer to use the theorem of minimal K-type for showing the non-existence of any discrete spectrum for the spaces of interest (cf. [Kna86], [Vog79]).

Definition 4.5 *Let π be an admissible representation of G . Then the* minimal K-types *are those λ_0 among all occuring K-types λ which minimize the functional $|\lambda + 2\delta_c|^2$.*

Theorem 4.6 *(Vogan) Let π_λ be a representation of the discrete series with Harish-Chandra parameter λ. Then the following holds true:*

1. $\pi_\lambda \Big|_K$ *contains the irreducible K-module with the highest weight $\lambda - \delta_c + \delta_n(\Psi)$ exactly with multiplicity one,*
2. *all other irreducible K-representations have highest weights of the form* $\lambda - \delta_c + \delta_n(\Psi) + A$

where Ψ is the positive system in which λ is dominant, and A is a sum of positive roots in Ψ.

Now I can investigate the discrete spectrum.

Proposition 4.7 *For the groups $Spin(2n,1), SU(n,1), Sp(n,1)$ there are no homomorphisms between the Spinor module and the discrete part of the spectrum.*

Lemma 4.8 *The smallest possible Harish–Chandra parameters for the discrete series for* $\mathfrak{so}(2n,1)$ *and* $\mathfrak{sp}(n,1)$ *are the* δ_i.

proof
For λ being a Harish–Chandra parameter it is necessary that $\Lambda \in L_{H_c}$ is regular, where L_{H_c} are the integral forms on $\mathfrak{h}_{c_i}$. From regularity it follows that no two entries could have the same absolute value and no entry can be equal to 0. Integrality fixes a certain distance between the successive entries. Furthermore it is sufficient to restrict to one positive chamber, for the length does not change under the action of the Weyl group. So one can demand without loss of generality that the entries are not increasing:
$a_1 \geq a_2 \geq \ldots$.
By combining these two conditions one proves the lemma. q.e.d.

Proof of the proposition:
proof
I will prove the result for $\mathfrak{so}(2n,1)$ and $\mathfrak{sp}(n,1)$ and for $\mathfrak{su}(n,1)$ separatly. I start with $\mathfrak{so}(2n,1)$ and $\mathfrak{sp}(n,1)$. From the theorem about the minimal K-type and the lemma I know that all possible K-types are of the form $2\delta_{in} + A$ for A positive with respect to the positive system which contains δ_i.
Now I have to check whether there could be the K-weights δ_{in} among them. Without loss of generality I can restrict myself to the consideration of δ_{nn} . Obviosly this root can not lie in the representation parametrized by δ_n. Assume that there is some j s.t. $\delta_{nn} = 2\delta_{jn} + A$. The roots δ_{jn} are non-positive with respect to $\Phi^+ = \Psi(\delta_n)$, so it is impossible to find such A. In case of $\mathfrak{su}(n,1)$ one has to work on the double covering and therefore there are different integrality conditions, i.e. the difference between two successive entries is at least half integer (I regard again only the case of decreasing entries). Then the theorem on minimal K-types yields to the inequality:
$\lambda + A = \delta_c$.
This is impossible because there is a the zero in the end of δ_c what contradicts to the the assumption that λ should lie in Φ^+. q.e.d.

4.4 Discrete spectrum of the Dirac operator

Now I have all the information I need for stating the main theorem.

Theorem 4.9 *There exists no discrete spectrum for the Dirac operator defined above on the symmetric spaces* G/K, *where* G *one of the groups* $Spin(2n,1), SU(n,1)$ *or* $Sp(n,1)$.

proof
For the square of the Dirac operator on a symmetric space one has the following formula,

due to Parthasaraty [Par67].
$D^2 = -\Omega + c$, where c is some constant and Ω the Casimir operator of G, or, more precisely, its action by derivation of the translation. Now the Casimir operator acts on irreducible representation spaces as a scalar,in particular, it leaves these spaces invariant. So one gets all the $\mathcal{L}^2$ eigenvectors from the discrete part of the Plancherel formula. If there is no discrete spectrum, obviously there can not be any $\mathcal{L}^2$ eigenvectors of D^2 q.e.d.

Remark 4.10 *If one traces through the proof of this theorem it is clear that the same remains true for general non–compact symmetric spaces,which admit a homogeneous Spin structure.*

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