# Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

## Proceedings of the $1^{st}$ Workshop on Stochastic Numerics

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submitted: 24th November 1992

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Preprint No. 21
Berlin 1992

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*1991 Mathematics Subject Classification.* 60H10.
*Key words and phrases.* stochastic differential equations, numerical solution, statistical methods, simulation, limit theorems, financial mathematics.

Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin

Fax: + 49 30 2004975
e-Mail (X.400): c=de;a=dbp;p=iaas-berlin;s=preprint
e-Mail (Internet): preprint@iaas-berlin.dbp.de

# PROCEEDINGS

of the $1^{st}$ Workshop on

# STOCHASTIC NUMERICS

September 7 – 12, 1992 in Gosen (near Berlin)
organized by the Institute of Applied Analysis and Stochastics Berlin
together with the Humboldt University Berlin

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# Forword

One aim of the 1$^{st}$ Workshop on Stochastic Numerics was to bring together for the first time leading experts on this field from the eastern countries with those from the west. The support by Deutscher Akademischer Austauschdienst and Deutsche Forschungsgemeinschaft allowed the participation of many colleagues between Torun and Novosibirsk.
Another goal of this workshop was achieved by confronting specialists from stochastic analysis, stochastic numerics, statistics of stochastic processes and different fields of application with new results and trends in the other fields. Thus applied and interdisciplinary aspects played an important role. Modelling of challenging applications, stochastic numerical methods and statistical inference were understood in the framework of their interrelations. Several young mathematicians took the opportunity to consult specialists to improve their profile.

The response to the conference was so overwhelming that many participants expressed their hope that it would be regarded as a kick-off meeting to a series of workshops which enable mathematicians working at universities and in industry to discuss their work, exchange ideas, or even join forces to approach some of the more difficult and pressing problems. Announced as the first workshop on stochastic numerics, the conference provided the ideal setting for identifying such problems and stimulating research activities. Subsequent workshops could possibly involve a section, where progress is discussed and solutions are presented.

Areas that appear to be of particular interest are mathematical finance and quantum physics. We therefore decided to choose problems encountered in these fields to apply our techniques and put the theory to the test. Many papers delivered at the conference dealt with particular stochastic differential equations. Whereas in physics, where relations are often governed by mechanistic laws, problems may well be described by such systems, in finance or quantum physics the dynamic processes are usually not fully understood until now and need to be identified. Hence, the objective is simply to adequately model observable phenomena and discover as much structure as possible. Some of the extended and revised abstracts e.g. by Breckling & Dal Dosso, Rebolledo and Chiarella list a few of the problems encountered which may be tackled using stochastic differential equations, stochastic numerics and statistics of processes. Those with an interest in the kind of problems are encouraged to contact the authors directly for a more detailed discussion. Deutsche Bank Research for instance may also be in a position to provide the data necessary for analysing specific problems in mathematical finance.

These proceedings contain all abstracts submitted to the workshop including several extended and revised after the conference. We included also a list of addresses and e-mail addresses to provide a basis for further communication.

The organisers of the conference would like to thank all those who contributed to this workshop and announce that attempts will be made to have a $2^{nd}$ Workshop on Stochastic Numerics near Berlin from 5 until 10 September 1994.

Berlin, September 1992

U. Küchler E. Platen M. Teuchert

## Contents

First Workshop on Stochastic Numerics
Berlin, September 7-12, 1992

# Stochastic[1] Dynamical Systems

by
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## 1 Basic concepts and results

### 1.1 Dynamical systems

The categories of metric, topological and smooth dynamical systems and their overlaps, time $T = \mathbb{R}$

### 1.2 The concept of a random dynamical system

Cocycles on $T = \mathbb{R}$ and consequences, skew-product flows, example: random differential equations

### 1.3 Stochastic flows through stochastic differential equations

Two-sided stochastic calculus, two-parameter filtration $\mathcal{F}_s^t$, forward and backward integral, results of Kunita

### 1.4 Stochastic dynamical systems through stochastic differential equations

White noise as a metric dynamical system, cocycle property, cocycles with independent increments

[1]The term *stochastic* pertains to the white noise case, while the term *random* pertains to the general case

# 2 Smooth stochastic dynamical systems: The multiplicative ergodic theorem and consequences

## 2.1 Invariant measures for random dynamical systems

Disintegration, Krylov-Bogolyubov method, Fokker-Planck equation $L_{\pm}^{*}\rho^{\pm} = 0$ and pull-back $\lim_{t\to\mp\infty} \varphi(t,\omega)^{-1}\rho^{\pm} = \mu_{\omega}^{\pm}$

## 2.2 The multiplicative ergodic theorem

For linearized random dynamical systems on the tangent bundle

## 2.3 Local smooth theory based on the multiplicative ergodic theorem

Invariant manifolds (in particular stochastic stability), normal forms (conflict between ergodic theory and stochastic analysis), stochastic bifurcation (physicist's versus dynamical concept and interplay)

# 3 A program for numerics and visualization

## 3.1 Generation of stochastic flow

Generate numerically the stochastic flow $\varphi_{s,t}(\omega)$ as a family of mappings for *fixed* $\omega$

## 3.2 Pull-back

Implement and visualize $\varphi(t,\omega)^{-1}x$ for $t \to \pm\infty$

## 3.3 Invariant measures

Solve the forward and backward Fokker-Planck equation $L_{\pm}^{*}\rho^{\pm} = 0$ numerically (via histogram?) and study the parameter dependence of solution, compute and visualize $\lim_{t\to\mp\infty} \varphi(t,\omega)^{-1}\rho^{\pm} = \mu_{\omega}^{\pm}$ and determine character of $\mu_{\omega}^{\pm}$ (Dirac, atomic, random limit cycle, fractal or smooth support)

## 3.4 Invariant manifolds

Compute and visualize center manifold at bifurcation point and use symbolic manipulation to compute the normal form on the center manifold, compute and visualize stable and unstable manifolds for hyperbolic $\mu_{\omega}^{\pm}$ (in particular domain of attraction if $\lambda < 0$)

### 3.5 Bifurcation theory

Compute and draw bifurcation diagram (i.e. new invariant measure bifurcating from reference measure at critical parameter value

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# CERTAIN ASPECTS OF APPLICATION OF NUMERICAL METHODS FOR SOLVING SDE SYSTEMS

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The method is said to be *asymptotically unbiased with the stepsize* h, if while applying it with this stepsise to the scalar linear SDE

$$y(t) = y_0 - \alpha \int_0^t y(s)ds + \sigma w(t), \tag{1}$$

where $\alpha > 0, \sigma \neq 0$ are real coefficients, the distribution of the numerical solution $y_n$ converges as $n \to \infty$ to the normal distribution with zero mean and variance $\sigma^2/(2\alpha)$. The interval $(x, 0)$ is said to be *the interval of asymptotic unbiasedness* of the method, if the latter is asymptotically unbiased with any stepsize $h > 0$, for which $-\alpha h \in (x, 0)$. It is easy to see, that the method for solving SDE systems in the sense of Itô of the form

$$y_{n+1} = y_n + [I - \frac{h}{2}\frac{df}{dy}(y_n)]^{-1}[hf(y_n) + \sqrt{h}\sigma(y_n)\zeta_n] \tag{2}$$

is asymptotically unbiased with the interval of asymptotic unbiasedness $(-\infty, 0)$.

The method is said to be *asymptotically p-stable with stepsise* h, if while applying it with this stepsise to the asymptotically p-stable SDE [1], the equality

$$\lim_{n\to\infty} < |y_n|^p >= 0$$

is satisfied.

Applying method (2) to asymptotically stable in the meansquare scalar SDE in the sense of Itô of the form

$$y(t) = y_0 - \alpha \int_0^t y(s)ds + \sigma \int_0^t y(s)dw(s), \tag{3}$$

where $\alpha > 0, \sigma \neq 0$ are real coefficients, one can obtain, that method (2) is asymptotically stable in the meansquare with any stepsise $h > 0$.

In practical application of numerical methods for simulation of the SDE systems solution, there arises a problem of automatic choosing the integration stepsise and accuracy of simulation of the values of solution at the grid nodes, corresponding to this stepsise. When constructing variable step algorithms for solving SDE systems, one may generalise well-known variable step algorithms for ODE systems, based on the Runge-Kutta or Rosenbrock type methods. For instance, the well-known 6-stage Runge-Kutta method of order 5 from the algorithm RKF45 [2] can be generalised for solving SDE systems in the sense of Itô in the following way:

$$y_{n+1} = y_n + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 +$$

$$+\sqrt{h}(\frac{119}{135}\sigma(y_n) - \frac{6656}{12825}\sigma(y_{n+1}^{(2)}) - \frac{28561}{56430}\sigma(y_{n+1}^{(3)}) +$$
$$+\frac{9}{50}\sigma(y_{n+1}^{(4)}) - \frac{2}{55}\sigma(y_{n+1}^{(5)}))\zeta_n,$$
$$y_{n+1}^{(0)} = y_n,$$
$$y_{n+1}^{(1)} = y_n + \frac{1}{4}k_1,$$
$$y_{n+1}^{(2)} = y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2,$$
$$y_{n+1}^{(3)} = y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3,$$
$$y_{n+1}^{(4)} = y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4,$$
$$y_{n+1}^{(5)} = y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5,$$
$$k_j = hf(y_{n+1}^{(j-1)}) + \sqrt{h}\sigma(y_{n+1}^{(j-1)})\zeta_n, \quad j = 1, ..., 6. \tag{4}$$

The difference $\delta_{n+1} = y_{n+1}^* - z_{n+1}$, where

$$y_{n+1}^* = y_n + \frac{16}{135}k_1^* + \frac{6656}{12825}k_3^* + \frac{28561}{56430}k_4^* - \frac{9}{50}k_5^* + \frac{2}{55}k_6^*,$$
$$z_{n+1} = y_n + \frac{25}{216}k_1^* + \frac{1408}{2565}k_3^* + \frac{2197}{4104}k_4^* - \frac{1}{5}k_5^*,$$
$$k_j^* = hf(y_{n+1}^{(j-1)}), \quad j = 1, ..., 6$$

can be used as an estimate of the error of method (4).

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## APPROXIMATION FOR THE SOLUTIONS OF S.D.E.

**Vlad Bally**

We study the convergence of the flows $X_n(t,x)$, n N, to the flow X(t,x), where X(t,x) solves the S.D.E.

d X(t,x)=f(X(t,x))dt+g(X(t,x))dW(t), X(0,x)=x,

and $X_n(t,x)$ solves the integral equation obtained by replacing the Brownian motion W by a finite variation approximation process $W_n$ ,and the Ito integral is replaced by a Stieldjer's one.

Two types of convergence are considered: convergence in law, on the space $C(R_+ R^n, R^n)$, and in $L^p$ for $\sup_{t,x} X_n(t,x)- X(t,x)$ ( the so called strong convergence ).Actually W is allowed to be a difussion process in the case of the weak convergence and a general martingale in the case of the $L^p$ convergence.

# A STOCHASTIC PARTICLE METHOD FOR SOME ONE-DIMENSIONAL NONLINEAR P.D.E.

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We consider the one-dimensional nonlinear P.D.E. in the weak sense :

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} U - \frac{\partial}{\partial x}\left[\, U \cdot \left(\int b(.,y) U_t(dy)\right)\right] \;,\; 0 < t < T \\ U_{t=0} &= U_0 \end{aligned}$$

When the initial condition is a probability on $\mathbb{R}$, the solution $U_t$ is the distribution of the random variable $X_t$ where $(X_t)$ is a nonlinear stochastic process in the sense of McKean, solution of

$$\begin{cases} dX_t = \int b(X_t, y) U_t(dy)\, dt + \sigma dw_t \\ \qquad\qquad U_t(dy) \text{ is the law of } X_t \\ \\ X_{t=0} = X_0 \quad (\text{with law } U_0). \end{cases}$$

Our purpose is to study a stochastic particle algorithm for the computation of the cumulative distribution function of $U_t$. This method is based upon the moving of particles according to the law of a Markov chain approximating $(X_t)$, and the approximation of $(\mathbb{E}b(x, X_t), t \leq T)$ by means of empirical distributions.

For a bounded function $b$, having bounded derivatives up to the second order, we prove the convergence of the methode when $\Delta t = O\left(\left(\frac{1}{N}\right)^{\frac{2}{3}}\right)$, where $N$ is the number of particles and $\Delta t$ is the time step. In this case the rate of convergence is of ordor $\frac{c}{N^{\frac{1}{3}}}(log(N))^{\frac{1}{2}}$.

## The Analysis of Term Structures

by

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Term structures of interest rates or zero coupon curves as they are also known are one of the most fundamental concepts in finance. They provide the basis for pricing interest rate dependent securities, give rise to bond market indices and can be used for hedging purposes. Further, knowledge of how interest rates develop would also be desirable for effective interest rate management. Analysing term structure dynamics can therefore be regarded as one of the most pressing problems in practice.

The approach sketched below clearly differentiates between cross-sectional and time series aspects. Given a set of bond prices, the first objective is purely cross-sectional, that is, to estimate the term structure of interest rates. It is demonstrated how a bond index system can be based on the concept of a term structure. One problem that needs to be addressed in this context is that of index replication. The second set of objectives relates to the time series aspect and involves an anlysis of the term structure dynamics for forecasting purposes.

### A non-parametric approach to term structure estimation

The problem is that for a bond only single prices are observed, although they may consist of several cash flows. Usually, however, several bond prices are observed and they are related. Such relations, together with the observed data, then enable determination of the term structure. Hence, the task is to come up with a satisfactory method of estimating zero coupon curves from non-zero coupon bonds or, more generally, with an appropriate deconvolution technique.

Breckling & Dal Dosso (1992a) present a new approach to term structure estimation that overcomes this problem and that has a number of advantages over current practice:

- it does not depend on any parametric model, and therefore allows for arbitrary shapes of zero coupon curves and is widely applicable;
- constraints can easily be incorporated;

- only small sets of data are required;
- it is extremely robust against outliers; and
- with respect to the common trade-off between accuracy and smoothness it is optimal.

The technique involves the following optimisation problem

$$\min_{\mathbf{r}} \{ error(\mathbf{r}) + \gamma \; roughness(\mathbf{r}) \}$$

where $\mathbf{r}$ is a vector representing the term structure, $error(\mathbf{r})$ is the error with which bond prices are described, $roughness(\mathbf{r})$ is a statistic that is inversely related to the roughness of the zero coupon curve and $\gamma$ is a smoothness factor that reflects the weight put on smoothness relative to accuracy.

The proposed technique is closely related to kernel estimation with $error(\mathbf{r})$ taking the role of *bias*, that penalises oversmoothing, and $roughness(\mathbf{r})$ that of *variance*, that penalises undersmoothing. In contrast to other techniques the trade-off between accuracy and smoothness has been made explicit, thereby shedding additional light on the process of term structure estimation.

## Proposal of a bond index system

Market indices are an equally important concept in finance. They serve as benchmarks for portfolio managers and give rise to the definition of derivative securities that can be used for hedging purposes or be of outright interest to investors. Generally, two categories of bond market indices can be distinguished: first, portfolio indices that consist of few selected bonds and second, synthetic indices that are based on a model representing the bond market.

Breckling & Dal Dosso (1992b) propose an index system that belongs to the second category. Each market index is based on two concepts, that of a cash flow density describing the market and that of a term structure to evaluate that cash flow. The drawback of these indices is that they are not exactly reproducible, and that they are dependant on the underlying term structure model. However, the advantage of synthetic indices is that they give good market representation, and that they are simple and easy to understand. Furthermore, the concept can easily be adapted to other markets and is internally consistent.

Price and performance indices are defined, and their properties described. A natural decomposition of the term structure translates into a decomposition of the performance index into effects that are associated with price changes, interest accrual and moving up or down the term structure. A decomposition of the cash flow density gives rise to the formulation of subindices.

The proposed index bears great similarity to that published by JP Morgan. However, being a synthetic index it does not suffer from the disadvantages typical of portfolio indices. In particular, the stability and robustness properties that are inherent in the term structure estimation procedure carry over to the index.

### Index replication

Suppose a derivative instrument like the index described above is created and sold by a bank to one of its clients. In order to hedge the risk associated with this position, a portfolio of other tradable securities needs to be purchased, which exhibits a performance that matches that of the index as well as possible. Usually only few securities can be selected for the hedge portfolio, and the task is to find the best and most practical one. All algorithms available to perform this task are cumbersome and compute-intensive, thus urging for better alternatives.

### Term structure dynamics

To manage an interest rate risk effectively, it would be desirable to have an idea of how the term structure changes over time. The approach suggested by Breckling & Dal Dosso (1992c) is based on an eigenvalue decomposition.

Let $\mathbf{r}_t \in \mathcal{R}^n$, $n \in \mathcal{N}$, be the discretised term structure at time t and let $s > 0$ be the forecast horizon. Further, let $\mathbf{x}_{t+s} = \mathbf{r}_{t+s} - \mathbf{r}_t$ denote the change of the term structure between times $t$ and $t + s$. The objective thus is to analyse the structure of the time series $(\mathbf{x}_t)$ and to forecast $\mathbf{x}_{t+s}$.

Assume the random variable $\mathbf{X}_{t+s}$ has expectation $\mathcal{E}(\mathbf{X}_{t+s}) = \mu_{t+s}$, for example $\mu_{t+s} = \mathbf{0}$ if $(\mathbf{x}_t)$ exhibits no trend or $\mu_{t+s} = \mathbf{r}_t - \mathbf{r}_{t-s}$ if the latest trend is expected to continue. As long as a distinction between different points in time is required, all time indices are dropped for ease of notation.

Put $\Psi = \mathcal{E}(\mathbf{X}\ \mathbf{X}')$. That is, $\mathrm{cov}(\mathbf{X}) = \Psi - \mu\ \mu'$. In order to detect a pattern in the term structure changes, a principal component decomposition of the matrix $\Psi$ is carried out. Hence,

$$\Psi = \mathbf{P}\ \mathbf{\Lambda}\ \mathbf{P}'$$

where $\mathbf{P} = (\mathbf{p}_1, ..., \mathbf{p}_n) = (p_{jl})_{jl}$ is the matrix of eigenvectors of $\Psi$ and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is the diagonal matrix of corresponding eigenvalues. This enables us to write

$$\mathbf{X} = \mathbf{P}\ \mathbf{\Lambda}^{1/2}\ \mathbf{U} = \sum_{l=1}^{n} \lambda_l^{1/2}\ \mathbf{p}_l\ U_l$$

where $\mathbf{U} = (U_1, ..., U_n)'$ is a random variable with $\mathcal{E}(U_l) = \nu_l$, $\text{var}(U_l) = 1$ and $\text{cov}(U_j, U_l) = 0$ if $j \neq l$. Writing

$$\nu = \mathcal{E}(\mathbf{U}) = \mathbf{\Lambda}^{-1/2}\ \mathbf{P}'\ \mathcal{E}(\mathbf{X}) = \mathbf{\Lambda}^{-1/2}\ \mathbf{P}'\ \mu \quad ,$$

and setting $\mathbf{Z} = \mathbf{U} - \nu$, yields

$$\mathbf{X} - \mu = \mathbf{P}\ \mathbf{\Lambda}^{1/2}\mathbf{Z} \quad .$$

Note that $\mathbf{P}$ and $\mathbf{\Lambda}$ are fairly stable over time with $\lambda_l \approx 0$ for $\lambda \geq 4$. The approach taken here thus serves to decrease the dimensionality of the problem. Instead of dealing with $(\mathbf{x}_t)$ it is sufficient to analyse the 3-dimensional time series $(z_{1t}, ..., z_{3t})$. Furthermore, there is some evidence that $z_{1t}$ and $z_{2t}$ are amenable to economic interpretation.

Hence, the task is to forecast $(z_{1,t+s}, ..., z_{3,t+s})$ with a forecast error that is as small as possible. Finally, returning to the problem of predicting the term structure at time $t + s$, we get

$$\hat{\mathbf{r}}_{t+s} = \mathbf{r}_t + \hat{\mu}_{\mathbf{t+s}} + \sum_{l=1}^{3} \hat{\lambda}_{t+s,l}^{1/2}\ \hat{\mathbf{p}}_{t+s,l}\ \hat{z}_{t+s,l} \quad .$$

Here, $\hat{\mu}_{\mathbf{t+s}}$, $\hat{\lambda}_{t+s,l}$ and $\hat{\mathbf{p}}_{t+s,l}$ are estimates of the mean of $\mathbf{X}_{t+s}$, of the eigenvalues of $\text{cov}(\mathbf{X}_{t+s})$ and of the corresponding eigenvectors. The values of $\hat{z}_{t+s,l}$ can be determined by fitting a dynamic model that possibly involves economic information.

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# A STOCHASTIC SIMULATION FOR SOLVING SCALAR REACTION-DIFFUSION EQUATIONS

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We want to show here how to use a stochastic branching process to simulate the solution u(t,x) of a nonlinear p.d.e. of the type

$$(1) \qquad \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} + f(u),$$

where f(u) is a function of the type f(u) = u(1-u).

A natural way is to use the McKean connection ([4]) : u(t,x) can be view as the distribution function of the rightmost particle in a branching brownian motion whose the reproduction law is directly related to function f. Hence simulating this process by branching random walks the empirical distribution of the rightmost particle gives an approximation of the solution of (1).

Several other stochastic methods may be proposed to avoid the dependence upon the diffusion coefficient K wich occurs in a classical finite differences method. In Chorin's model ([3]), the diffusion part of equation (1) is approached by a random walk and the reaction part by deterministic jumps.

In the method introduced by Sherman and Peskin ([5]), both steps are stochastic : particles perform gaussian steps and simultaneously die or split into two particles with probabilities depending upon the present state of all living particles. These particles are of course related to the nonlinear term f(u) ; more precisely if at step t, n particles are located at positions $x_i$, i=1,...,n, then compute the quantity

$$u_i = \frac{1}{n} \# \{ j,\ x_j < x_i \} .$$

With probability $1 - \Delta t \frac{1}{n} \sum_i |f'(u_i)|$, nothing happens.

With probability $\Delta t \frac{1}{n} \sum_i |f'(u_i)|$, something happens : it concerns particle i at position $x_i$ with probability $|f'(u_i)| \; / \; \sum_k |f'(u_k)|$ . This particle dies if $f'(u_i) \leq 0$. It is replaced by two particles if $f'(u_i) > 0$.

Sherman and Peskin have shown that this method was efficient in a particular case where the explicit solution of equation (1) is known. For Alain Rouault and myself, it was a challenge to prove the convergence of this simulation model in the general case, with our probabilistic tools.

The natural setting is to consider the approximation model as a spatial branching process with interaction. Since the above probabilities depend only upon the empirical measure of positions, the interaction is of mean-field type.

The branching process is described as follows : n initial particles of mass 1/n generate branching brownian motions (brownian motion is prefered to a discrete process only for simplicity). Living particles define a measure valued process

$$\mu_t^n = \frac{1}{n} \sum_{i=1}^{N_t} \delta_{x_i(t)}$$

where $x_i(t)$ are the positions and $N_t$ is the number of particles alive at time t. The rate of death $\alpha$ and the offspring law $(p_k, \; k \in \mathbb{N})$ of a particle depends on its position and on the present state of the process. It is assumed that $\mu_o^n$ converges in distribution to $\mu_o$, a deterministic measure.

The main theorem provides the convergence of $\mu^n$ to $\mu$ solution of the equation

$$\frac{\partial \mu_t}{\partial t} = \frac{1}{2} \Delta \mu_t + \gamma(\mu_t)\mu_t$$

where

$$\gamma(x,\mu) = \alpha(x,\mu) \; (\sum_k k p_k(x,\mu) - 1).$$

The convergence is proved by hilbertian techniques, embedding the space of measures in a dual Sobolev space $W_{-3}$. The Sherman-Peskin case corresponds to the choice of $\gamma$ :

$$\gamma(x,\mu) = f'(\mu(]-\infty, \; x])).$$

The fluctuation process

$$Y_t^n = \sqrt{n}\ (\mu_t^n - \mu_t)$$

is shown to converge to a generalized Ornstein-Uhlenbeck process solution of the Langevin equation

$$dY_t = \left( \frac{1}{2} \Delta\mu_t + \tilde{\gamma}'(\mu_t) Y_t \right) dt + dN_t$$

where $\tilde{\gamma}(\mu) = \gamma(\mu)\mu$ and $N_t$ is a continuous gaussian process. For this last result, differentiability assumptions on $\gamma$ are needed.

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# STOCHASTIC NUMERIC METHODS AND APPLICATIONS IN FINANCE AND ECONOMICS

By

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In this brief talk I would like to survey certain areas in finance and economics where I see potential applications of stochastic numerics. These applications should give the flavour of how stochastic numerics, combined with the widespread availability of very powerful, user friendly computer systems, can become of great use to the economic theorist.

I will start with an area where the economic theorist faces the problem of calculating the "fair" value of certain financial instruments, namely derivative securities, which are traded in the world's major financial centres. I would then like to go on and survey two other areas where the need of the economic theorist is not so much the calculation of a specific number, but rather to understand the qualitative behaviour of models of economic behaviour which are inherently stochastic, dynamic and quite often highly nonlinear. In particular I shall discuss some models of asset price dynamics which seek to give a better understanding of the price generating process in finance. I would then like to discuss some models of exchange rate dynamics which are useful for analysis of policy issues in macroeconomics. If there is time I will briefly discuss possible application of stochastic numerics to optimal consumption-investment decisions under uncertainty, and dynamic stochastic economic games.

First Workshop on Stochastic Numerics
Berlin
September 7–12, 1992

# Stochastic versus Deterministic Numerical ODE Integration

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**Abstract.** The talk aims at sampling certain concepts that, on one hand, are useful in the numerical integration of deterministic ODE's and, on the other hand, might be open for extension and consideration towards stochastic ODE's. At the same time, deterministic models (and their recent numerical integration) of stochastic processes are discussed.

## 1. Order and Stepsize Control in Extrapolation Methods

Extrapolation is a convenient means to construct discretization methods of variable up to high order on the basis of rather elementary discretizations of low order [7]. This technique starts from an asymptotic expansion of the discretization error and eliminates successively (and even recursively) the arising expansion coefficients. Examples in non–stiff integration are the explicit Euler discretization (so–called $h$–expansion) and the explicit mid–point rule ($h^2$–expansion). By means of the "stochastic Taylor" expansions of Wagner and Platen (cf. [8]) an extension of extrapolation methods seems to be possible — with at least one additional (additive) term in the Aitken–Neville recursion. The simultaneous selection of order and stepsize can be done as in the deterministic case —

minimization of work per unit step

$$\frac{W_{k_{opt}}}{H_{k_{opt}}} = \min_k \frac{W_k}{H_k} .$$

Upon applying Shannon's information theory (cf. [1]) also in the stochastic context, questions like the maximum optimal order as a function of the required accuracy can be studied. The conjecture is that in numerical SDE integration only low orders pay off due to the additional work from the random number generators — to be verified or falsified! Beyond this aspect, Shannon's information theoretic approach may have an additional attractivity in the context of SDE's: after all, it is used in [1] to characterize a statistical ensemble of problems — which is a rather natural view for SDE's.

## 2. Semi–implicit Stiff Integrators and Newton–Type Uniqueness Theorems

In numerical stiff ODE integration only few concepts carry over to more general problem classes (cf. [7]). Among them is the rather trivially looking idea of "subtracting the stiffness effect" on both sides in the form

$$y' - Ay = f(y) - Ay$$

and discretizing the left–hand differential operator as a whole — see e.g. [2] and references therein. This idea has been used by the author and his co–workers to construct highly efficient stiff extrapolation integrators — see e.g. [2] for a survey. Only recently, the author interpreted this idea in the frame of a simplified Newton iteration in function space — which then immediately supplied associated uniqueness theorems [4].

In PDE's, the linear homogeneous term may characterize an unbounded operator. In SDE's, this term may describe a stochastic perturbation. It is crucial to observe that such extensions necessarily will involve approximation errors of a "rough" operator — which, in turn, means that only Newton–like methods in function space can be implemented. The associated uniqueness theorems will be direct extensions of those in [4]. A satisfactory feature of this theoretical approach is that here nonlinear contractivity arises naturally just as a pole of some scalar function.

## 3. Adaptive Scaling and Global Error Propagation

In deterministic ODE models, the numerical integrators typically only control the local discretization error. The global accumulation of errors will depend on

stability properties of the dynamical system as well. In [3], p. 128 ff., a special adaptive scaling technique has been derived and discussed, which establishes an equidistribution of local errors in the sense

$$\epsilon_{\text{global}} = m \cdot \varepsilon_{\text{local}}$$

with $m$ the number of automatically chosen integration points (an a–posteriori information). The adaptive scaling involves a numerical estimation of local one–sided Lipschitz constants (also: local Ljapunov exponents). Such characterizing quantities also play an important role in the theory of SDE's. An example is given, which surprisingly numerically models a total perturbation of one solution component, which, however, dies out in the progress of computation. Thus, in some cases, deterministic models already include some modelling of stochastic effects in the form of global error propagation properties, wherein discretization errors take the role of statistical perturbations.

## 4. Adaptive Weighted Discrete Galerkin Methods for Macromolecular Processes

Macromolecular processes may stand for a wider class of stochastic processes that play a role in applications such as polymer chemistry or smog reactions. One of the popular computational approaches uses the analytical derivation of statistical moment ODE's (countably many!) and the numerical integration of a finite number of these equations. However, in realistic applications the truncation index of these systems cannot be found in a robust and reliable way — after all, Stieltjes' theorem only states the equivalence of knowing the infinite number of all monomial moments (bounded!) and knowing the probability density function. An alternative is the rather recent approach based on adaptive discrete Galerkin methods — see [5]. In this approach, a discrete Hilbert space $H$ (a sequence space) is constructed from an input of a probability distribution such as geometric or Poisson. An adaptation of free parameters is performed to yield a so–called "moving weight function" as a counterpart of moving nodes in a method of lines treatment for PDE's. Instead of statistical (monomial) moments other generalized moments are computed, which are expansion coefficients associated with the Hilbert space basis (special functions of discrete variables) and, by completeness of $H$, decrease asymptotically — which means that truncation is easier to control. This deterministic approach to modelling a stochastic process leads to tremendous computational speed–up factors (of $10^3$ up to $10^4$) for realistic industrial polyreactions. It might be possible that such an approach may also be useful for rather general SDE's.

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# A MINIMUM-DISTANCE ESTIMATOR FOR DIFFUSION PROCESSES WITH ERGODIC PROPERTIES

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Suppose one observes one path of a stochastic process which solves a stochastic differential equation of the form

$$dX_t = a(\theta, X_t)dt + dW_t, \quad t \geq 0,$$

with a given initial condition $X_0$, where $\theta \in R^d$ is some parameter the true value $\theta_0$ of which is unknown. In order to estimate this parameter we propose the following estimator:

$$\hat{\theta}_T = \arg\min_\theta \int_0^T (X_t - X(\theta)_t)^2 \, dt,$$

where

$$X(\theta)_t = X_0 + \int_0^t a(\theta, X_t)dt.$$

Under the assumption that the observed process has an ergodic property it is shown that an estimator of the above type exists and is strongly consistent and moreover - in case d=1 - asymptotically normal.

# Langevin Methods for Simulating Lattice Field Theory

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September 23, 1992

**Abstract**

We review the application of Langevin algorithms to the simulation of lattice field theories. The problem of simulating lattice field theory can be thought of as the evaluation of certain (very high dimensional) integrals over the configuration space of the field theory system. In practice these integrals may only be evaluated by some statistical sampling method. A well known approach is that of Monte Carlo evaluation. An alternative method which is both effective and interesting in itself is the Langevin scheme first proposed by Parisi and Wu in the context of field theory. We show how to use higher order approximations to the relevant stochastic differential equations in implementing this scheme when it is applied to both linear and non-linear spin models.

## Field Theory on the Lattice

The problem of continuum field theory for a quantum field $\phi(x)$ as viewed by physicists is that of evaluating a FUNCTIONAL INTEGRAL of the following form:

$$\langle\phi(x_1)\cdots\phi(x_N)\rangle = \frac{1}{Z}\int[d\phi]e^{S[\phi]}\phi(x_1)\cdots\phi(x_N), \tag{1}$$

where $x$'s denote points in space-time and $S[\phi]$ is the ACTION given by

$$S[\phi] = -\int d^Dx\left\{\frac{1}{2}(\nabla_x\phi(x))^2 + \frac{1}{2}m^2\phi(x)^2 + \frac{1}{24}\lambda\phi(x)^4\right\}, \tag{2}$$

and the PARTITION FUNCTION $Z$ is

$$Z = \int[d\phi]e^{S[\phi]}. \tag{3}$$

These functional integrals are idealised generalizations of the Wiener-Feynman-Kac integrals for quantum mechanics. Because they make use of the flat measure $[d\phi]$ on $\phi$-function space they require careful definition. Much of what physicists do is an attempt by numerical simulation to realise this definition in practice.

The practical approach frequently adopted is to approximate D-dimensional space time by (let us say) a D-dimensional hyper-cubical lattice with periodic boundary conditions at the edges [1]. That is we allow $x$ to take only values of the form $x =$

$n_1 a\mathbf{e}_1 + \cdots + n_D a\mathbf{e}_D$ where $\{\mathbf{e}_1, \ldots, \mathbf{e}_D\}$ are the basic unit vectors of the hyper-cubical lattice and require that the field obey the condition $\phi(x + La\mathbf{e}_m) = \phi(x), m = 1, \ldots, D$. $a$ being the size of the basic lattice link and $L$ being the number of links in each direction on the lattice. The total number of sites on the lattice is then $L^D$ . In an actual large simulation $L$ might be 32 and $D$ might be 4 so the number of sites in the lattice is $10^6$ . Even larger simulations are envisaged.

On the lattice we will replace $\phi(x)$ with a field variable $\phi_x$ defined at each site of the lattice. The measure $[d\phi]$ is replaced by $\prod_x d\phi_x$ where the product is over all lattice sites $x$ . The continuous derivative $\nabla\phi(x)$ is replaced by a lattice version

$$\nabla_m \phi_x = \frac{1}{a}(\phi_{x+a\mathbf{e}_m} - \phi_x) \tag{4}$$

and the derivative term in the action by $(\nabla\phi)_x^2 = \sum_m (\nabla_m \phi_x)^2$ . Finally the version of the action appropriate to the lattice formulation is

$$S[\phi] = -\sum_x a^D \left(\frac{1}{2}(\nabla\phi)_x^2 + \frac{1}{2}m^2\phi_x^2 + \frac{1}{24}\lambda\phi^4\right) \quad . \tag{5}$$

The information in the theory is contained in the lattice correlation functions

$$\langle \phi_{x_1} \cdots \phi_{x_N} \rangle = \frac{1}{Z}\int \prod_x d\phi_x e^{S[\phi]} \phi_{x_1} \cdots \phi_{x_N} \quad . \tag{6}$$

Now the partition function is

$$Z = \int \prod_x d\phi_x e^{S[\phi]} \quad . \tag{7}$$

The limit back to the continuum is achieved by letting $a \to 0$ and $L \to \infty$ in such a way that $La$ remains constant. In order to achieve a successful result it is also essential that certain more subtle properties of the correlation functions hold. It is not always obvious that the continuum limit is available. Hence the need for numerical simulation as well as deeper analytical investigations.

## Simulation Methods

The field theory problem when formulated on the lattice is reduced to the evaluation of a family very high dimensional integrals. Some kind of sampling method is necessary for their numerical evaluation on a computer. A commonly used method is the MONTE CARLO method [1]. An other method which can have advanteges is the LANGEVIN METHOD. This is of particular interest to this meeting since it makes use of the theory of stochastic differential equations. The method was introduced to field theory calculations by Parisi and Wu [2]. See also the book edited by Damgaard and Hüffel [3] .

The Langevin method is based on the result the probability distribution $P[\phi] \propto e^{P[\phi]}$ can be obtained as the limit as $\tau \to \infty$ of the distribution $P[\phi, \tau]$ where

$$\frac{\partial}{\partial\tau}P[\phi,\tau] = \sum_x \frac{\partial}{\partial\phi_x}\left(\frac{\partial}{\partial\phi_x} - \frac{\partial S}{\partial\phi_x}\right) P[\phi,\tau] \quad . \tag{8}$$

The stochastic differential equation which generates a distribution evolving according to the above partial differential equation is

$$\dot{\phi}_x = \frac{\partial S}{\partial \phi_x} + w_x(\tau) \ . \tag{9}$$

The white noise has the correlator $\langle w_x(\tau) w_{x'}(\tau') = \delta_{xx'}\delta(\tau - \tau')$ . However a practical simulation requires an approximate method of integrating the SDE. The lowest order algorithm is

$$\Delta\phi_x = \frac{\partial S}{\partial \phi_x}\Delta\tau + \sqrt{2\Delta\tau}\eta_x \ . \tag{10}$$

Here each $\eta_x$ for each time step is an independent gaussian random variable with zero mean and unit variance. If we sample the system periodically after letting it equilibriate it will yield an effective distribution $P[\phi]$ that has a systematic error that is $O(\Delta\tau)$. This is the error in the sense of weak convergence. To control systematic errors a higher order algorithm is required. A Runge-Kutta type algorithm with an intermediate step $\phi_x^{(i)}$ which does this has the form

$$\phi_x^{(i)} = \phi_x^{(\text{old})} + \frac{1}{2}\frac{\partial S}{\partial \phi_x}\Delta\tau + \sqrt{\Delta\tau}\eta_x^{(1)} \ . \tag{11}$$

We then move to the final step position

$$\phi_x^{(\text{new})} = \phi_x^{(\text{old})} + \left(\frac{\partial S}{\partial \phi_x}\right)^{(i)}\Delta\tau + \sqrt{\Delta\tau}(\eta_x^{(1)} + \eta_x^{(2)}) \ . \tag{12}$$

Where the superfix indicates that the quantity has been evaluated at the intermediate point and $\eta_x^{(1)}$ and $\eta_x(2)$ are independent gaussian random variables at each step with zero mean and unit variance. The errors in the sense of weak convergence in the induced distribution $P[\phi]$ are now $O(\Delta\tau^2)$ . Yet higher algorithms can be constructed [4, 5].

## Spin Models

It is of interest to consider other lattice models such as spin models since they involve a curved manifold for the field variables. A typical model is one in which each site $x$ of the lattice has associated with it an element $M_x$ of the group $SU(N)$ or $O(N)$. A standard action for this model is

$$S[M] = \frac{\beta}{2}\sum_{x,m}\text{Tr}\left(M^\dagger_{x+\text{e}_m}M_x\right) + \text{c.c.} \quad , \tag{13}$$

and the associated probability distribution is $P[M] = \exp S[M]$ . Expectation values are calculated using the group ($SU(N)$ or $O(N)$) invariant measure $dM_x$ at each site $x$ .

$$\langle f[M]\rangle = \int \prod_x dM_x P[M] f[M] \ . \tag{14}$$

All results for the linear $\phi$-field model can be generalized to these non-linear spin models. We require the concept of COVARIANT DERIVATIVE appropriate to the

group in question [4]. Choose a basis of generators $\{\Lambda_a\}$ for the group in the defining representation. They satisfy the Lie bracket relations

$$[\Lambda_a, \Lambda_b] = C_{abc}\Lambda_c \quad , \tag{15}$$

where the $C_{abc}$ are the structure constants of the Lie algebra. The covariant derivative for the group in question is $D_x^a$ defined so that

$$D_x^a f[M_x] = \frac{\partial}{\partial \epsilon_x^a} f[e^{\epsilon_x \cdot \Lambda} M_x]\Big|_{\epsilon=0} \quad . \tag{16}$$

The Fokker-Planck equation with the stationary solution $P[M] \propto \exp S[M]$ is

$$\frac{\partial}{\partial \tau} P[M,\tau] = \sum_{x,a} D_x^a \left(D_x^a - U_x^a\right) P[M,\tau] \quad , \tag{17}$$

where $U_x^a = D_x^a S[M]$ . The corresponding first order algorithm for the simulation updating procedure is

$$M_x^{(\text{old})} \rightarrow M_x^{(\text{new})} = e^{\epsilon_x \cdot \Lambda} M_x^{(\text{old})} \quad , \tag{18}$$

where $\epsilon_x^a = U_x^a \Delta\tau + \sqrt{2\Delta\tau}\eta_x^a$ , the $\{\eta_x^a\}$ being idependent gaussian random variables as before. The second order algorithm uses an intermediate step $M_x^{(1)}$ constructed in the same way as $M_x^{(\text{old})}$ above with a step $\epsilon_x^{(1)a} = \frac{1}{2} A U_x^a \Delta\tau + \sqrt{B\Delta\tau}\eta(1)a_x$ . Then

$$M_x^{(\text{new})} = e\epsilon_x^{(2)} . \Lambda M_x^{(\text{old})} \quad , \tag{19}$$

where $\epsilon_x^{(2)a} = A U_x^{(1)a} \Delta\tau + \sqrt{B\Delta\tau}\left(\eta_x^{(1)a} + \eta_x^{(2)a}\right)$ . Here $U_x^{(1)a}$ is evaluated at $M(1)_x$ and $A = 1 + \frac{1}{12}C$ and $B = 1 - \frac{1}{12}C$ where $C$ is the Casimir for the adjoint representation of the group i.e. $C_{abc}C_{abd} = C\delta cd$ .

Numerical results for these cases can be found in ref [6] which also sets out the algorithm for a modified Fokker-Planck equation with a *diffusivity tensor.* This modification referred to as Fourier acceleration, is of great use in investigating models near critical points of the parameters.

## Other Applications

We have also considered applications to other problems. A brief list of topics is:

Non-relativistic Quantum Mechanics. [7]

Flow and Diffusion in Random Media [8, 9]

Turbulent Diffusion. [10]

Numerical Integration of Stochastic Differential Equations with state dependent noise. [11]

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# A TEST OF TYPE KOLMOGOROV-SMIRNOV FOR ERGODIC DIFFUSION PROCESSES

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We present a test of type Kolmogorov-Smirnov for ergodic diffusion processes. It is based on an estimate of large deviations probabilities for the supremum of the limit process of the empirical process. We extend the results to the case when some parameters in the drift are estimated during the same observation and we obtain a test of parametric diffusion model. We give an asymptotic speed for the rejection of the model when the drift is miss-specified (see [2] for proofs). We give examples and we show the behaviour of the test on numerical simulations (see D.Talay [7]) for the family of Ornstein-Uhlenbeck processes. The mains references of this paper are [1], [5], [6].

This study was motivated by financial applications, mainly the identification and testing of interest rate evolution models : Vasicheck, Cox-Ingersoll-Ross (see E.Fournié, D.Talay [4], Eric Fournié [3]). These interest rate models behave approximately like Ornstein-Uhlenbeck processes and we have seen we cannot accept or reject this parametric model in an acceptable time, unless the observation belongs to a very different model. But changes in economy are frequent and lead to changes in model which may be considerable and thus impose an observation time too short to allow us to decide. Consequently, because we know only roughly how far the interest rate model is from the observation, it would be very interesting to test the robustness of pricing formula (Black-Scholes,...) to the variations of underlying interest rate model.

**keywords** : *ergodic diffusions, empirical process, supremum of a Gaussian process, two-sided test, parametrical estimation, goodness of fit test, miss-specified drift, asymptotic rejection speed.*

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# A Discretization of a Controlled Stochastic Differential Inclusion

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We consider the following controlled stochastic differential inclusion of Ito-type

$$X(t) \in X^0 + \int_0^t F(s, X(s), U(s))ds + \int_0^t B(s, X(s), U(s))dW(s), \quad t \in [0,T], \tag{1}$$

in the complete probability space $(\Omega, \Im, \mathbf{P})$. $W(t)$ is a q-dimensional Wiener-process. Let $F : \Omega \times [0,T] \times \mathbf{R^n} \times \mathbf{R^m} \longmapsto \mathbf{Comp(R^n)}$ be a $\Im_t$-measurable, uniformly bounded and Lipschitz-continuous multifunction. $B : \Omega \times [0,T] \times \mathbf{R^n} \times \mathbf{R^m} \longmapsto \mathbf{R^{n,q}}$ satisfies the well-known assumptions.

Now the existence of a solution $X(t) \in \mathbf{L_2}(\Omega \times [0,T])$ which is continuous with probability 1 can be shown. The trajectories of the controlled differential inclusion depend on the controls $U(.)$ continuously.

We consider an objective functional of Bolza-type:

$$\bar{J}(U) = \mathbf{E}\{l_0(X(T)) + \int_0^T l_1(t, X(t), U(t))dt\} = Min! \tag{2}$$

Lipschitz-continuity and boundedness of $\bar{J}(.)$ are required. Under corresponding assumptions it is possible to derive a necessary optimality condition as a maximum principle for the set-valued control problem (1)+(2). Like the vector-valued problems the practical computation of an optimal solution of (1)+(2) is very difficult in the general case. For this reason finding so-called $\epsilon$-optimal solutions for the control problem has a great importance. Therefore we introduce a discretized controlled Ito-differential inclusion

$$X_{k+1} \in X_k + \int_{t_k}^{t_{k+1}} F(s, X_\Delta(s), U_\Delta(s))ds + \int_{t_k}^{t_{k+1}} B(s, X_\Delta(s), U_\Delta(s))dW(s), \quad X_0 = X^0, \quad k = 0, 1, \ldots, N-1 \tag{3}$$

with

$$X_\Delta(t) = X_k * \chi_{[t_k, t_{k+1})}, \quad U_\Delta(t) = U_{k+1} * \chi_{(t_k, t_{k+1}]}.$$

By a growing refinement of the decomposition of $[0,T]$ we get a piecewise constant approximation of the solution of (1), that means the solution of (3) converges in the quadratic mean to that of (1). For the corresponding objective functional

$$\bar{J}_\Delta(U_\Delta) = \mathrm{E}\{l_0(X_N) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} l_1(t, X_k, U_{k+1})dt\} = Min! \tag{4}$$

can be shown that now the optimal control of the problem (3)+(4) is an $\epsilon$–optimal control of (1)+(2).

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# Random Generation of Stochastic Area Integrals

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We describe a method of random generation of the integrals

$$A_{1,2}(t,t+h) = \int_t^{t+h}\int_t^s dw_1(r)dw_2(s) - \int_t^{t+h}\int_t^s dw_2(r)dw_1(s)$$

together with the increments $\Delta w_1(t,t+h)$ and $\Delta w_2(t,t+h)$ of a two-dimensional Brownian path $(w_1(t), w_2(t))$. The motivation is the need for a numerical method for strong solution of general multi-dimensional stochastic differential equations with an order of convergence $O(h)$, where $h$ is the step size. It is known (see eg [4]) that one way to obtain this order is to simulate the so called area integrals $A_{i,j}(t,t+h)$.

The joint density function of $a = A_{1,2}(0,1)$, $b = \Delta w_1(0,1)$ and $c = \Delta w_2(0,1)$ is

$$f(a,b,c) = \frac{1}{2\pi^2}\int_0^\infty \frac{x}{sinh(x)} exp\left(\frac{(b^2+c^2)x}{2tanh(x)}\right) cos(ax)dx$$

However it is possible to reduce the problem essentially to two dimensions by setting

$$r^2 = \Delta w_1(0,1)^2 + \Delta w_2(0,1)^2$$

The integral in the above expression can only be calculated numerically, so there is no 'quick and easy' method of generation available. The method chosen is based on Marsaglia's 'rectangle-wedge-tail' method, generalised to higher dimensions.

In one dimension Marsaglia's method involves dividing an area in $\mathbf{R^2}$ into equal rectangles and setting up tables with an entry for each rectangle. However, when we are dividing a region in $\mathbf{R^3}$ into equal pieces, the number of table entries needed becomes prohibitive. We have therefore been forced to a slightly

more sophisticated analysis of the method to reduce storage requirements while retaining benefits of speed. The final implementation enables one to generate the vector $(a, b, c)$ in about 4.6 times the time it takes to generate a vector of 3 independent numbers from a normal distribution.

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# APPROXIMATIONS WRT. TIME AND CHANCE FOR ITO SDE'S WITH RATES

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We consider an Itô sde of the form:

$$x(t) - x_0 = \int_{t_0}^{t} b(s,x(s))ds + \int_{t_0}^{t} \sigma(s,x(s))dw(s) \ , \ t \in [\, t_0,T] \ , \ x_0 \in \mathbb{R}^d ,$$

where $b \in C([\, t_0,T] \times \mathbb{R}^d;\mathbb{R}^d)$ , $\sigma \in C([\, t_0,T] \times \mathbb{R}^d;L(\mathbb{R}^q;\mathbb{R}^d))$ and w is a q-dimensional standard Wiener process.

Under the assumption that b is of linear growth and $\sigma$ is bounded approximate solutions are constructed which are based on the stochastic Euler method combined with the simultaneous approximation of the Wiener process.

With additional smoothness assumptions this construction is carried out using Milshtein's method instead of Euler's method.

These constructions use a coarse grid for the time discretization and a finer grid for the chance approximation. This has the advantage that time-consuming evaluations of the coefficients b and $\sigma$ are only necessary in the points of the coarse grid.

For both methods the main theorem gives a recipe how to tune up the accuracy of time discretizations and of the approximation of the Wiener process such that both have the same "speed". The resulting convergence rates are given with respect to the $L^p$ norm ($2 \le p < \infty$) of some $C([\, t_0,T] ;\mathbb{R}^d)$-valued random variables having the laws of the exact solution and the approximate solutions, respectively. Thus, these rates apply to the $L^p$ Wasserstein metrics between the aforementioned laws.

# AN APPLICATION OF STOCHASTIC NUMERICAL METHODS IN FINANCING

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Our main interest are numerical investigations for an example in option pricing as the following (see [2]) which is related to the models discussed in [3], [4] and [5].

We consider the process $X = (X^0, X^1, X^2, X^3)^* = (B, S, \sigma, \zeta)^*$ satisfying the Ito stochastic differential equation:

$$\begin{aligned} dB_t &= r(t, X_t)\, B_t\, dt \\ dS_t &= r(t, X_t)\, S_t\, dt + \sigma_t S_t\, dW_t^1 \\ d\sigma_t &= -q(\sigma_t - \zeta_t)\, dt + p\, \sigma_t\, dW_t^2 \\ d\zeta_t &= \frac{1}{\alpha}(\sigma_t - \zeta_t)\, dt \end{aligned} \tag{1}$$

with $p > 0$, $q > 0$, $\alpha > 0$, where $W^1$, $W^2$ are independent Brownian motions under a probability measure $Q$. The first equation describes the bond price $B$, where $r$ is a Markovian instantaneous interest rate. The stock price $S$ follows a generalized geometric Brownian motion since drift and volatility $\sigma$ are not constant. Taking the drift to be $r(t, X_t)$ means that the discounted stock price process $S/B$ is a martingale under the measure $Q$ which we use for pricing.

The processes $\sigma$ and $\zeta$ should be interpreted as the instantaneous and the weighted average volatility of the stock, respectively. The equation for $\sigma$ shows that the instantaneous volatility $\sigma_t$ is disturbed by some external noise (with an intensity parameter $p$) and at the same time continuously pulled back towards the average volatility $\zeta_t$. The parameter $q$ measures the strength of this restoring force or speed of adjustment.

By taking the interest rate $r(t, x)$ to be identically 0 we simplify the model (1). Now we carry out some simulations for the resulting three-dimensional model.

The contingent claim to be considered will be that of an European call option, i.e.,

$$g(X_T) = (S_T - K)^+ \tag{2}$$

with strike price $K = 1.0$. First of all, we simulate (see [2] and [6]) a trajectory of the three dimensional process $X = (X_t)_{0 \le t \le 1} = ((S_t, \sigma_t, \zeta_t))_{0 \le t \le 1}$. Here we are interested in the simulated path itself. Thus these simulations require approximations in the strong sense. We say that an approximate sample path $Y^\Delta$ of the solution $X$ of an Ito stochastic differential equation converges with strong order $\gamma > 0$ if there exist constants $K$ and $\delta_0 < T$ not depending on the step size $\Delta$ such that for all $\Delta \in (0, \delta_0)$ we have the estimate

$$E\left[|X_T - Y^\Delta(T)|\right] \le K\, \Delta^\gamma$$

where $T$ is a given time instant.

Furthermore we want to compute the values $(V_t)$ of the hedging portfolio along the trajectory of $S$, starting with the option price $V_0$. $V_0$ (see [1]) is given by the expectation

$$V_0 = E\left[(S_T - K)^+\right]$$

therefore (see [6]) we only need a weak approximation in order to compute $V_0$. We say that an approximation $Y^\Delta$ converges with weak order $\beta > 0$ as the time step size $\Delta$ tends to 0 if there exist constants $K > 0$ and $\delta_0 < T$ for every function $g : \Re^{m+1} \to \Re$ from a given class $C_p$ of test functions such that for all $\Delta \in (0, \delta_0)$ we have

$$|E\left[g(X_T)\right] - E\left[g(Y^\Delta(T))\right]| \leq K\,\Delta^\beta.$$

As class $C_p$ of test functions it is convenient to use the class of smooth functions which together with all their derivatives have at most polynomial growth.

Finally we compare the values $(V_t)$ of the hedging portfolio with our pay - off function (2) and we see as expected that $(V_t)$ approaches approximatively $\max(0, S_T - K)$ for $t = T$ also in the considered example with past dependent and stochastic volatility.

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# RATE OF CONVERGENCE FOR SOME FUNCTIONALS OF BROWNIAN SEMIMARTINGA

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We consider a d-dimensional "Brownian semimartingale" of the form $dX_t = a_t dW_t + b_t dt$, where $a$ and $b$ are predictable, locally bounded, and $a$ is continuous, and $W$ is a standard Brownian motion. For each $n$ we have an increasing sequence $T(n,i)$ of stopping times, and a sequence of positive $\mathcal{F}_{T(n,i)}$-measurable variables $\Delta(n,i)$ such that $S(n,i) := T(n,i)+\Delta(n,i) \le T(n,i+1)$. We are interested in the limiting behaviour of the processes $U^n_t(g) = \sqrt{\delta_n} \sum_{i:S(n,i)\le t}[g(T(n,i),\xi^n_i) - \alpha^n_i(g)]$ where $g$ is a predictable function of $(\omega,t,x)$ and $\xi^n_i = \Delta(n,i)^{-1/2}[X_{S(n,i)}-X_{T(n,i)}]$ and $\alpha^n_i(g)$ is a suitable centering term, and finally $\delta_n$ is a normalizing sequence tending to 0 and related to the $S(n,i)$'s in such a way that the "empirical measures" $\mu_n = \delta_n \sum_{i\ge 0} \varepsilon_{S(n,i)}$ weakly converge in probability to some limiting (possibly random) measure $\mu$.

We show that, under rather weak assumptions, these processes converge in law in the finite-dimensional sense, and even in the "functional" sense when the limmiting measure $\mu$ has a.s. no atom, to a process of the form $U(g)_t = \int_{[0,t]\times\mathbb{R}^d} g(s,x)B(ds,dx)$: here, $B$ is a martingale measure which may be constructed on an extension of the original space, and which conditionally on $W$ is a Gaussian random measure.

This type of results is related to previous works of Rootzen or Kurtz and Protter about approximation of stochastic integrals, for which it gives rates of convergence for the time-discretized integrals under somewhat weaker assumptions on the regularity of the integrands. The main motivation, however, is in statistics of diffusion processes: it provides a tool allowing to prove LAN or LAMN property for estimation of the diffusion coefficient by discretization of time.

# AN EDUCTION OF THE LANGEVIN EQUATION

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The simplest Langevin equation

$$\frac{d^2X}{dt^2} + a\frac{dX}{dt} = \sigma\frac{dB}{dt} \tag{1}$$

describes the movement of a free particle with coordinate X, submitted to "infinitely many uncorrelated small shocks" represented by "the increments at time t" of the Brownian motion $B$ (see e.g. [3]).

The coefficient $a$ depends on "viscosity" and measures the resistance forces of the medium. The point is that this is a *macroscopic* interpretation, while on the *microscopic level* the nature of interactions can be quite different (the ideal gas!). However the equation (1) is considered to be valid for both cases, with the same meaning of coefficients (see e.g. [4]).

Consider the following system of "microscopic" stochastic differential equations:

$$\begin{aligned} X_t - X_0 &= \int_0^t V_s\,ds \\ V_t - V_0 &= CU_t - C\int_0^t V_{s-}\,dN_s, \end{aligned} \tag{2}$$

where $C = C_{M,m} = 2m/(M+m)$, $N_t = N_t^\lambda = \sum_{k=1}^\infty I(\tau_k \le t)$ is a Poisson point process on $\mathbb{R}^+$ with intensity $\lambda \cdot \ell$ and $U_t = U_t^{m,\lambda} = \sum_{\tau_k \le t} U_k$ with $U_1, U_2, \ldots$ independent and distributed according to $\mathcal{N}(0, kT/m)$ (here $k$ is the Boltzmann constant and $T$ is the absolute temperature).

It seems that the system (2) satisfies all assumptions physicists usually attribute to (1). On the other hand, if $\lambda \to +\infty$ and $m \to 0$ in such a way that $2\lambda m \longrightarrow b$, $0 < b < +\infty$, then by [1] (or [5], [2]) the solutions of (2) converge in law to the solution of the system with the second equation in (2) replaced with

$$M(V_t - V_0) + b\int_0^t V_s\,ds = \sqrt{2bkT}B_t, \tag{3}$$

i.e., to the solution of the Langevin equation (1) with $a = b/M$ and $\sigma = \sqrt{2bkT}/M$.

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# Correction Formulae for Maximum-Likelihood Estimator of Parameters of Stochastic Differential Equations.

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ABSTRACT

The use of the maximum likelihood estimator (MLE) is examined in the situation, where the data come from a "smooth" dynamical system approximated by an Itô equation. It is well known that in such cases correction terms may appear already in the equation. Independently of this fact, in most cases (in particular also when the correction term for the drift coefficient vanishes), additional correction of formulas for estimators of parameters of equation are necessary. The importance of the correction under consideration has been noticed and repetedly indicated in several publications, but in most papers devoted to the estimation of parameters of Itô differential models it was not taken into account. The form of the correction terms seems to be known merely for linear systems.

In this paper a general approach, based upon a known result of Mc Shane's, is employed to prove an appropriate convergence theorem concerning two types of integrals appearing in the formulas of MLE and the explicit form of correction terms is derived for nonlinear systems provided that the parameters enter the drift term linearly. The necessity of using corrections is underlined by a simple example in which the neglection of correction terms leads to an infinite relative error of estimation. This theoretically established fact is also observed on the basis of numerical simulation. The second example, connected with a nonlinear stochastic oscillator illustrates the effectiveness of estimation in presence of corrections in practically important case (it also enlights some of earlier numerical experiences described in the literature).

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# ON THE RATE OF CONVERGENCE OF THE CERTAIN DIFFUSION APPROXIMATIONS

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Let $X^n$ be a sequence of semimartingales and let

$$X_t = X_0 + \int_0^t a(s, X)\,ds + M_t, \qquad t \geqslant 0, \tag{1}$$

where M is a local martingale. Suppose $h$ is a Hölder function on Skorokhod's space $D_{[0,T]}(\mathbf{R}^m)$. We are interested in estimation of the Lévy–Prokhorov distance between distributions of $h(X^n)$ and $h(X)$. In some cases this allows us to examine the rate of convergence of semimartingales with reflection to the diffusion process with reflection.

Let $a(\cdot,\cdot)$ be such that for any $t \in [0,T]$ and $x, y \in D$

$$|a(t,x) - a(t,y)| \leqslant k_t \sup_{s \leqslant t} |x_s - y_s|, \tag{2}$$

$$|a(t,x)| \leqslant k_t\Big(1 + \sup_{s \leqslant t} |x_s|\Big), \tag{3}$$

$$\int_0^t k_s\,ds < \infty.$$

THEOREM 1. *Let $X^n$ be special semimartingales with locally square integrable martingale parts $M^n$ and let $X$ be as in* (1) *with continuous Gaussian local martingale part $M$. Suppose the function $a$ satisfies the conditions* (2), (3), *and*

$$\langle M^{(i)n}, M^{(j)n}\rangle_t = \int_0^t \sigma_s^{(i,j)n}\,ds, \qquad \langle M^{(i)}, M^{(j)}\rangle_t = \int_0^t \sigma_s^{(i,j)}\,ds.$$

*Then, for any $\delta \in (0,1]$, and for any Hölder function $h$ of order $\beta$, $\beta \in (0,1]$, with norm $\|h\|_\beta$, there exists a constant $C = C(T, k, \widehat{\sigma}_T, \|h\|_\beta, \beta)$ such that*

$$\pi_T\big(h(X^n), h(X)\big) \leqslant C\Bigg\{\pi^{1/2}(X_0^n, X_0) + \mathbf{E}^{1/2}\left[\left(\sup_{t \leqslant T}\left|A_t^n - \int_0^t a(s, X^n)\,ds\right|\right) \wedge 1\right]$$
$$+ \mathbf{E}^{1/3}\int_0^T \|\sigma_t^n - \sigma_t\|\,dt + \mathbf{E}^{1/4}\sum_{t \leqslant T} |\Delta M_t^n|^{2+\delta}\Bigg\}^\beta,$$

*where* $\pi_T(\cdot,\cdot)$ *is Lévy - Prokhorov distance,* $\widehat{\sigma}_T = \max_{1\leqslant i\leqslant m}\sup_{t\leqslant T}\sigma_t^{(i,i)}$, *and* $\|\sigma_t\| = \sum_{i,j=1}^m |\sigma_t^{(i,j)}|$.

COROLLARY. *Let the assumptions of Theorem* 1 *be satisfied. Moreover, let*

$$\sup_{t\leqslant T}|\Delta M_t^n| \leqslant \gamma_n,$$

*where* $\gamma_n \downarrow 0$ *for* $n\to\infty$. *Then there exists a constant* $C = C(T,k,\widehat{\sigma}_T,\|h\|_\beta,\beta)$, *such that*

$$\pi_T\big(h(X^n),h(X)\big) \leqslant C\Bigg\{\pi^{1/2}(X_0^n,X_0) + \mathbf{E}^{1/2}\left[\left(\sup_{t\leqslant T}\left|A_t^n - \int_0^t a(s,X^n)\,ds\right|\right)\wedge 1\right]$$
$$+ \mathbf{E}^{1/3}\int_0^T \|\sigma_t^n - \sigma_t\|\,dt + \gamma_n^{1/4}\Bigg\}^\beta.$$

The obtained general results we use for the sequence of normalized vector queuing processes, i.e., for $Y_t^n = n^{-1/2}Q_{nt}^n$, where

$$Q_t^n = Q_0^n + A_t^n + \sum_{k=1}^m \int_0^t \chi_s^n\,dD_s^{nk} - \int_0^t \chi_s^n\,dB_s^n,$$

$A^{(i)n}$, $D^{(i)nk}$, and $B^{(i)n}$ are point processes, $1\leqslant i\leqslant m$, and $\chi_s^n$ is a diagonal matrix with diagonal elements $\chi_s^{(i)n} = \mathbf{1}_{\{Q_{s-}^{(i)n}>0\}}$, $\mathbf{1}_{\{\cdot\}}$ is an indicator of a set $\{\cdot\}$. Denote by $\widetilde{A}^n$, $\widetilde{D}^{nk}$, and $\widetilde{B}^n$ the compensators of point processes $A^n$, $D^{nk}$, and $B^n$ and let

$$\widetilde{A}_t^n = \int_0^t \lambda(n^{-1}Q_s^n)\,ds, \qquad \widetilde{D}_t^{nk} = \int_0^t p_{k\cdot}\,\mu^{(k)}(n^{-1}Q_s^n)\,ds,$$

$$\widetilde{B}_t^n = \int_0^t \mu(n^{-1}Q_s^n)\,ds,$$

where $\lambda^{(i)}(x)$ and $\mu^{(i)}(x)$, $1\leqslant i\leqslant m$, are non-negative functions, $P = (p_{ij})_1^m$ and $p_{k\cdot} = (p_{k1},\dots,p_{km})$.

Let $O = \{x\in\mathbb{R}^m\colon x^{(i)} > 0,\ 1\leqslant i\leqslant m\}$ and $\partial_i O = \{x\in\mathbb{R}^m\colon x^{(i)} = 0\}$, where $\mathbb{R}^m$ has norm $|x| = \max_{1\leqslant i\leqslant m}|x^{(i)}|$. Let $P$ be a $m\times m$ symmetric and non-negative defined matrix.

DEFINITION. For $x\in D$ the pair of the functions $(z,\varphi)$ is called a solution of the Skorokhod's reflection problem $(x;O;I-P)$ if

1) $z = x + (I-P)\varphi$;

2) $z^{(i)}(t) \geqslant 0$, $1 \leqslant i \leqslant m$;

3) $\varphi^{(i)}$ is nondecreasing function with $\varphi^{(i)}(0) = 0$ and satisfying

$$\varphi^{(i)}(t) = \int_0^t \mathbf{1}_{\{x(s) \in \partial, O\}} d\varphi^{(i)}(s), \qquad 1 \leqslant i \leqslant m,$$

where $I$ is a unit matrix.

THEOREM 2. *Let functions $\lambda^{(i)}(x)$ and $\mu^{(i)}(x)$, $1 \leqslant i \leqslant m$, be twice differentiable, their first two derivatives are bounded by constant $L$ and*

$$\lambda(0) + (P^* - I)\mu(0) = 0,$$

*where $P^*$ is transposed matrix, $\|P\| < 1$. If $\sup_n \mathbf{E}|Y_0^n| < \infty$ then there exists a constant $C = C(T, m\|P\|, L)$ such that*

$$\pi_T(Y^n, Y) \leqslant C\{\pi^{1/2}(Y_0^n, Y_0) + n^{-1/8}\},$$

*where $Y$ is the diffusion process with oblique reflection from the boundary $\{x \in \mathbb{R}^m\colon x^{(i)} = 0\}$, $1 \leqslant i \leqslant m$, which satisfies stochastic differential equation*

$$dY_t = bY_t\,dt + \sqrt{c}\,dW_t + (I - P^*)\,d\Phi_t.$$

*Here $(Y, \Phi)$ is the solution of the stochastic Skorokhod's problem $(X; O; I - P^*)$ and $X$ is a solution of stochastic differential equation*

$$dX_t = bh_t(X)\,dt + \sqrt{c}\,dW_t,$$

*where $W$ is a Wiener process and*

$$b^{(i,j)} = \lambda_{ij}(0) + \sum_{k=1}^m p_{ki}\mu_{kj}(0) - \mu_{ij}(0),$$

$$c^{(i,j)} = \begin{cases} -(\mu^{(i)}(0)p_{ij} + \mu^{(j)}(0)p_{ij}), & i \neq j, \\ \lambda^{(i)}(0) + \sum_{k=1}^m p_{ki}\mu^{(k)}(0) + (1 - 2p_{ii})\mu^{(i)}(0), & 1 \leqslant i \leqslant m. \end{cases}$$

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# First-passage time problem for simulated diffusion. Application in neural modeling.

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The original Stein's neuronal model has the following properties: The voltage difference between the membrane potential and the resting potential at the trigger zone of the neuron is described by one-dimensional stochastic process $X = \{X(t),\ t \geq 0\}$ given by stochastic differential equation,

$$\mathrm{d}X = -\tfrac{1}{\tau}X\mathrm{d}t + a\mathrm{d}N^{+}(t) + i\mathrm{d}N^{-}(t)\ ,\ X(0) = x_0, \tag{1}$$

where $\tau > 0$, $a > 0$, $i < 0$ are constants; $N^{+} = \{N^{+}(t),\ t \geq 0\}$, $N^{-} = \{N^{-}(t),\ t \geq 0\}$ are two independent homogeneous Poisson processes with $N^{+}(0) = N^{-}(0) = 0$ and intensities $\lambda$, resp $\omega$. Stein's model with reversal potentials can be given by the stochastic differential equation

$$\mathrm{d}X = -\tfrac{1}{\tau}X\mathrm{d}t + a(V_E - X)\mathrm{d}N^{+}(t) + i(X - V_I)\mathrm{d}N^{-}(t), \quad X(0) = x_0, \tag{2}$$

with the same notation as in (1); $V_I < x_0$ is a constant representing the inhibitory reversal potential and $V_E > x_0$ stands for another constant, the excitatory reversal potential.

The hypothesis of frequency coding of information in the nervous system leads to the study of statistical properties of sequences of time intervals between spike generation. A neuron produces a spike when the membrane voltage (1) or (2) exceeds for the first time a voltage threshold. It is usually assumed, in neuronal models, that the threshold depolarization is a constant $S > x_0$ and that after each spiking the membrane potential is reset to the initial level $X(0) = x_0$. Thus the neuronal firing corresponds to the first passage time (FPT) for the associated stochastic process and the theoretical counterpart of the interspike interval is a random variable $T_S$ defined by the relationship,

$$T_S = \inf\{t \geq 0;\ X(t) \geq S,\ X(0) = x_0\}. \tag{3}$$

For the reason of easier mathematical tractability, the stochastic processes (1) and (2) are often substituted by a diffusion stochastic process. This diffusion approximation can be also well advocated by neurophysiological facts. The limiting diffusion for (1) is well known Ornstein-Uhlenbeck process given by

$$dX = (-\frac{1}{\tau}X + \mu + v)dt + \sigma dW, \tag{4}$$

where $\mu$, $\nu$ and $\sigma > 0$ are constants related to $a$, $i$, $\lambda$ and $\omega$; $W$ is a standard Wiener process. The diffusion approximation for generalized version of (2) has different diffusion coefficients [4] and here we consider only the case, which is studied in detail in [3],

$$dX(t) = (-X/\tau + \mu(V_E - X) + \nu(X - V_I))dt + \sigma\sqrt{(V_E - X)(X - V_I)}dW(t) \tag{5}$$

The FPT problem for the models (4) and (5) can be solved analytically and extensive tables of the FPT distribution and its moments are available, [1], for the model (4). Nevertheless, there is a lack of analytical results when the parameters of these models are time dependent or when a time-dependent threshold is employed, and thus some other methods have to be applied. It holds also for other diffusion models than (5).

To solve the FPT problem for the discontinuous models (1) and (2) is a complicated task and only approximations and numerical results are available [5]. Another method for this purpose is the simulation of the trajectories for (1) and (2) and using these trajectories the properties of FPT can be deduced. The simulation of the trajectories for (4) is a simple task, [2], and for (5) a reflecting boundary condition at $V_I$ has to be imposed. However, the FPT is overestimated in both cases when based on crossings of the trajectories through the threshold $S$. This overestimation has the same origin as the overestimation of FPT which comes due to the sampling performed on the realization of the process. In the present contribution, the properties of the FPT as derived from the simulated trajectories are related to the simulation step for (4) and (5) and compared with analytical and tabulated results.

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# A SURVEY ON PSEUDORANDOM NUMBER GENERATORS

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The classical method in random number generation is the linear congruential method. A linear congruential sequence $(x_n)_{n\geq 0}$ of integers $0 \leq x_n \leq m-1$ is defined by

$$x_{n+1} \equiv ax_n + b \pmod{m}$$

where $m$ is a large integer, and $a, b \in \boldsymbol{Z}$. A sequence of linear congruential pseudorandom numbers in $[0,1]$ is obtained by the normalization $y_n = \frac{x_n}{m}, n \geq 0$. Many theoretical results on the linear congruential pseudorandom numbers are obtained. (An extensive list of references is given in [6]).

Nonlinear congruential methods are designed and analyzed recently as alternatives to the linear congruential method. A general class of nonlinear congruential generators is introduced in [1]. The so called inversive congruential sequence $(x_n)_{n\geq 0}$ of elements of $\boldsymbol{Z}_p$, where $p$ is a prime number, is obtained by

$$x_{n+1} \equiv a\bar{x}_n + b \pmod{p}$$

where $a, b, x_0 \in \boldsymbol{Z}_p$, $\bar{0} = 0$ and $\bar{x} = x^{-1}$ for $x \in \boldsymbol{Z}_p \backslash \{0\}$ ($x^{-1}$ is the inversive of $x$ in the multiplicative group of the finite field of order $p$). Inversive generators with a power of two modulus or an odd prime power modulus are also considered. An excellent survey of inversive generators is given in [3].

Theoretical analysis of the pseudorandom number sequences generated by the inversive method has shown that they behave more randomly than the linear congruential sequences especially in the case of a prime modulus in the sense for example that the $k$–dimensional discrepancy is in accordance with the law of the iterated logarithm (see [4], [5]).

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# Time Discretization of Reflected Stochastic Differential Equations

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In 1983, A.Gerardi, F.Marchetti and A.M.Rosa [1] succeeded in approximating diffusions with boundary conditions by a space discretization of the associated generator. They proved the weak convergence of the approximating pure jump type Markov processes to the diffusion as the jump size vanishes. At the same time, Kinkladze [2] in a more traditional way gave an $L^2$-approximation of a stochastic differential equation in a half-space with instantaneous normal reflection at the boundary; he used a time discretization but the rate of convergence in his scheme was slightly slower than in the usual Euler-Maruyama scheme for non reflected stochastic differential equations. Here we propose a simple improvement of his scheme which yields the same rate of convergence as the Euler-Maruyama scheme. The key point is to use the joint distribution of a Brownian motion with drift and its maximum process, which has been computed by Shepp [4]. Extending a result by Seshadri [3], we get the following theorem:

**Theorem 1** *Let $a = (a_1, \ldots, a_d)$ and let $c$ be a real number. Assume $(B_t)$ is a $d$-dimensional Brownian motion and let*

$$S_t = \sup_{s \leq t} (a.B_s + cs) \ .$$

*Let $U = (U_1, \ldots, U_d)$ be a centered Gaussian random vector with covariance matrix $tI$, and let $V$ be an exponential random variable with parameter $(2t)^{-1}$ independent of $U$. Put*

$$Y = 1/2(a.U + ct + (|a|^2 V + (a.U + ct)^2)^{1/2}) \ .$$

*Then the vectors $(B_t, S_t)$ and $(U, Y)$ have same distribution.*

Now we turn to the simulation of the solution $X$ of a stochastic differential equation with Lipschitz continuous coefficients $b$ and $\sigma$ and normal reflection on the boundary of the halfspace $x_m > 0$. Assume $(W_t, 0 \leq t \leq T)$ is a $d$-dimensional Brownian process. For $n \geq 1$ let $h = T/n$. For $\rho = 0, \ldots, n-1$

and $\rho h < s \leq (\rho+1)h$ we put

$$\begin{array}{lcll} \overline{Z}_s^k & = & \overline{Z}_{\rho h}^k + b^k(\overline{X}_{\rho h})(s-\rho h) + \sigma^k(\overline{X}_{\rho h})(W_s - W_{\rho h}) & k = 1,\ldots,m \\ \overline{X}_s^k & = & \overline{Z}_s^k & k = 1,\ldots,m-1 \\ \overline{X}_s^m & = & \overline{Z}_s^m + \sup_{r\leq s}(\overline{Z}_r^m)^- \ . & \end{array}$$

**Theorem 2** *There exists $C > 0$ such that*

$$E(\sup_{t\leq T} |X_t - \overline{X}_t|^2) \leq Ch \ .$$

From Theorem 1 it follows that each new step in the computation of the successive values of $\overline{X}_{\rho h}$ requires the easy simulation of $d$ new independent standard Gaussian variables and a new independent exponential variable.

A.s. and $L^p$-results may also be stated. If the coefficients $b$ and $\sigma$ depend on $t$ under a Hölder condition, there is no new difficulty. As for the Milstein scheme, it is theoritically valid but there is no simple way to simulate the related joint distribution.

We have dealt with a very particular boundary condition. There are however some other cases where the above method still works: normal reflection in an orthant, or oblique reflection on a hyperplane when the direction of reflection is constant.

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# The effect of noise on solutions of some ODEs

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Model ODEs for physical systems should be able to accommodate the addition of a small amount of noise. If the ODEs have chaotic trajectories with sensitive dependence on initial conditions, it might be expected that the addition of noise would remove all practical predictability. We show that this is not true in a class of systems whose trajectories spend most of their time close to an invariant line with occasional excursions. The dynamics then reduces to a one-dimensional map. It is possible to find expressions for this map in the noiseless case by solving approximations to the ODEs valid close to and away from the invariant line [1] [2]. Here we show how to obtain the noise-affected map by solving an SDE in the region near the invariant line where low-level noise is important. The dynamics is simpler with added noise: we find a noisily periodic orbit for a wide range of parameter values, including those where the corresponding noiseless trajectories are chaotic.

We consider systems of ODEs which reduce to

$$\begin{aligned} \dot{x} &= \mu f(x) - y^2, \\ \dot{y} &= y g(x) \end{aligned} \tag{1}$$

near the line $y = 0$ where $\mu \ll 1$, $f(x) > 0$ and $g(x) = x - \alpha$ where $\alpha = \mathcal{O}(1)$ with any remaining variables being 'slaved' to $x$. Starting from $t = t_0$ with $\dot{x} = 0$ for some $x_0 < \alpha$, $y$ decreases rapidly so that $x$ evolves independently and

$$y = y_0 G(t, t_0)$$

where

$$G(t, t_0) = e^{\int_{t_0}^{t} \tilde{g}(t')dt'} \qquad \tilde{g}(t) = g(x(t)). \tag{2}$$

Noise is introduced by replacing the ODE for $y$ in (1) by the SDE

$$dy_t = y_t(x(t) - \alpha) + \epsilon dW(t) \qquad \epsilon \ll 1 \tag{3}$$

which is a time-dependent Ornstein-Uhlenbeck process with solution [3]

$$y_t = G(t, t_0)\left(y_0 + \epsilon \int_{t_0}^{t} \frac{1}{G(s, t_0)} dW(s)\right), \tag{4}$$

$$\langle y_t \rangle = G(t, t_0) y_0, \tag{5}$$

$$\left\langle y_t^2 \right\rangle = \langle y_t \rangle^2 + \epsilon^2 G^2(t, t_0) \int_{t_0}^{t} \frac{1}{G^2(s, t_0)} ds. \tag{6}$$

Let $x = \alpha$ at $t = t_\alpha$. Near $t = t_\alpha$ we write

$$G(t, t_0) = G(t, t_\alpha) G(t_\alpha, t_0) \simeq e^{\frac{\mu}{2} f(\alpha)(t - t_\alpha)^2} G(t_\alpha, t_0) \tag{7}$$

and note that $G(t_\alpha, t_0) = \mathcal{O}(e^{\frac{-1}{\mu f(\alpha)}})$ when $x_0 - \alpha = \mathcal{O}(1)$.

For $x - \alpha > \mathcal{O}(\sqrt{\mu})$ therefore,

$$\langle y_t \rangle \sim \sqrt{\mu f(x_0)} e^{\frac{-1}{\mu f(\alpha)}} G(t, t_\alpha), \tag{8}$$

$$\langle y_t^2 \rangle - \langle y_t \rangle^2 \simeq \epsilon^2 \sqrt{\frac{\pi}{\mu f(\alpha)}} G^2(t, t_\alpha) \tag{9}$$

If $\langle y_t^2 \rangle - \langle y_t \rangle^2 \ll \langle y_t \rangle^2$ we can treat the noise as a small perturbation to $\langle y_t \rangle$. If however

$$\mu |\ln \epsilon| \geq \mathcal{O}(1) \tag{10}$$

then for $x > \alpha$ we have $\langle y_t^2 \rangle \gg \langle y_t \rangle^2$. The end of the slow phase (at $x = x_1$ when $y_t^2 = \mu f(x)$ so that $\dot{x} = 0$) is thus hastened by the noise, which controls the map $x_0 \to x_1$ . This map is independent of $x_0$ (dependence on initial conditions is lost) as long as $x_0$ is such that $\langle y_t^2 \rangle \gg \langle y_t \rangle^2$ for $x > \alpha$.

When (10) holds, we find a noisily periodic orbit with mean amplitude and period determined by the noise level. The distribution of the amplitude about its mean is, however, $\mathcal{O}(\mu)$ regardless of the noise level. This independence of $\epsilon$ arises because the probability distribution of $y$ for $x - \alpha > \mathcal{O}(\sqrt{\mu})$ is determined by that of $x_c$. $x_c$, defined as the value of $x > \alpha$ for which $|y_t| > \frac{\epsilon}{g(x)}$, is independent of $\epsilon$ because $|y_t| \propto \epsilon$ for $x > \alpha$. The probability distribution of $x_c$ is carried forward by the flow of (1) and becomes the probability distribution of $x_1$. If $\bar{x}_1$ is the mean value of $x_1$ then the standard deviation of $x_1$ is given by:

$$\sigma_{x_1} \simeq \mu \frac{f(\bar{x}_1)}{g(\bar{x}_1)}. \tag{11}$$

The behaviour described in the first paragraph is found in ODEs describing systems such as the resonant interaction of three wave modes [1] and the shear instability of convection [2]. It should be noted that, even apart from questions of physics, trajectories obtained when attempting to solve (1) using single or double precision arithmetic will in fact resemble those of (3) with noise level determined by the computer roundoff error.

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# SECOND ORDER WEAK RUNGE-KUTTA APPROXIMATIONS FOR STRATONOVICH SDE'S

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Milshtein [2] has proposed a numerical integration method of Itô stochastic differential equation

$$X_t = x + \int_0^t f(X_s)\,ds + \int_0^t g(X_s)\,dB_s, \quad t \in [0,T], \tag{1}$$

by discrete time approximations $\bar{X}_t, t \in I_h = \{0, h, 2h, \ldots, T\}$, based on values of Brownian motion $B_t, t \in I_h$, and having the second order accuracy in the following sense:

$$|\mathbf{E}\varphi(\bar{X}_T) - \mathbf{E}\varphi(X_T)| \le Ch^2,$$

for a sufficiently wide class of functions $\varphi: R \to R$ (C does not depend on $h$). Following Kloeden and Platen[1] we call such approximations *the second order weak approximations* (in contrast to *strong approximations* with the convergence of $\mathbf{E}|\bar{X}_T - X_T|$). One can find a wide bibliography on the subject in [1].

Analogously to ordinary differential equations, it is worthwile to have weak approximations with a reduced use of derivatives of coefficients. Such approximations are often called Runge-Kutta type, though the terminology seems to be far from being standardized. Here we define "Runge-Kutta type approximations" as a class of approximations having the following structure:

$$\bar{X}_0 = x, \quad \bar{X}_{t_{k+1}} = A(\bar{X}_{t_k}, h, \Delta B_k), \quad \Delta B_k = B_{(k+1)h} - B_{kh}, t_k = kh,$$

$$A(x,h,b) = x + \sum_{i=0}^{m} q_i F_i h + \sum_{i=0}^{m} r_i G_i b,$$

where

$$\begin{aligned}
&F_0 = f(x), && G_0 = g(x + \alpha_{00} F_0 h),\\
&F_1 = f\big(x + \hat{\alpha}_{10} F_0 h + \hat{\beta}_{10} G_0 b\big), && G_1 = g\big(x + (\alpha_{10} F_0 + \alpha_{11} F_1)\,h + \beta_{10} G_0 b\big),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&F_m = f\left(x + \sum_{i=0}^{m-1} \hat{\alpha}_{mi} F_i h + \sum_{i=0}^{m-1} \hat{\beta}_{mi} G_i b\right), && G_m = g\left(x + \sum_{i=0}^{m} \alpha_{mi} F_i h + \sum_{i=0}^{m-1} \beta_{mi} G_i b\right),
\end{aligned}$$

and

$$\sum_{i=0}^{m} q_i = \sum_{i=0}^{m} r_i = 1.$$

Such approximations we shall call $(m+1)$-stage Runge-Kutta (RK) approximations. A little bit less general approximations (with $\hat{\alpha}_{ij} = \alpha_{ij}$, $\hat{\beta}_{ij} = \beta_{ij}$, $\alpha_{ii} = 0$) were considered

by Rümelin[4] who established their convergence in *quadratic mean sense* to the solution of

$$X_t = x + \int_0^t (f + \lambda gg')(X_s)\, ds + \int_0^t g(X_s)\, dB_s$$

with $\lambda = \sum_{i=1}^m r_i \sum_{j=0}^{i-1} \beta_{ij}$ (thus including both Itô and Stratonovich equations).

In general, there *do not exist* second order weak RK approximations for Itô equations (a derivative-free approximation proposed by Platen[3],[1] is not of the considered form). On the other hand, RK type approximations appear to be suitable for Stratonovich equations. Here is an example of 5-stage second order weak RK approximation for (1) understood in Stratonovich sense :

$$F_0 = f(x), G_0 = g(x+3F_0h),\ F_1 = f\left(x - \frac{1}{4}F_0h + \frac{1}{3}G_0b\right), G_1 = g\left(x - \frac{1}{4}F_0h + \frac{1}{3}G_0b\right),$$

$$F_2 = f\left(x - 3F_0h + \frac{1}{6}(G_0 + G_1)b\right),\ G_2 = g\left(x - 3F_0h + \frac{1}{6}(G_0 + G_1)b\right),$$

$$F_3 = f\left(x + (F_0 - F_1)h + \frac{1}{8}(G_0 + 3G_2)b\right),\ G_3 = g\left(x + (F_0 - F_1)h + \frac{1}{8}(G_0 + 3G_2)b\right),$$

$$G_4 = g\left(x + (\frac{1}{2}G_0 - \frac{3}{2}G_2 + 2G_3)b\right),$$

$$A(x,h,b) = x + \left(\frac{3}{2}F_0 + F_1 - \frac{11}{2}F_2 + 4F_3\right)h + \left(\frac{1}{6}G_0 + \frac{2}{3}G_3 + \frac{1}{6}G_4\right)b.$$

Leaving a detailed discussion for the talk, we shall only note the following:

1. The coefficients $\beta_{ij}$ and $r_i$ in the example are taken from a 5-stage 4th order RK method for ordinary DE. This is not a coincidence. For any such coefficients one can find the remaining ones to obtain the 2nd order weak RK approximation. Unfortunately, this is not true for "classical" 4-stage RK methods (well investigated in the deterministic case) unless the drift $f = 0$. On the other hand, in *one-dimensional case* there *do exist* 4-stage 2nd order weak RK approximations, but the author did not succeed to find a "nice" one.

2. For computational convenience it would be nice to have the coinciding $\hat{\alpha}_{ij} = \alpha_{ij}, \hat{\beta}_{ij} = \beta_{ij}, q_i = r_i$. Unfortunately, as in the example, it is possible only to some extent.

3. No serious problems arise in the case of time-dependent coefficients.

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# ON THE NUMERICAL SOLUTION OF STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

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Stochastic Volterra integro-differential (SVIDEs) are considered , of the form ,

$$(1)\quad y'(t) = f(t,y(t)) + \int_0^t K(t, s, y(s))\, ds + \int_0^t P(t, s , y(s))dW(s) ,$$
$$y(0) = y_0 \quad , t\in I = [0,T] ,$$

and also SVIDEs of the form ,

$$(2)\quad y'(t) = W(t) + f(t,y(t)) + \int_0^t K(t,s,y(s))\, ds ,$$
$$y(0) = y_0 \quad , t\in I = [0,T] ,$$

where $W(t)$ , $t\in I$ is the standard Wiener process . Equations of the form (1) ,(2) have important applications in Population Biology and in Physics (Kannan [4], Makroglou [8] , [9]) and the references therein).

In this presentation is given a description of the application of collocation methods for the numerical solution of (1), (2) ( see Makroglou [8],[9] ) as extended from their application to deterministic VIDEs ( see for example Brunner [2] , Makroglou [7]) combined with simulation techniques developed for the numerical solution of stochastic differential equations (Rao , Borwankar and Ramkrishna [10], Liske and Platen [6] , see also Kloeden and Platen [5] , Gard [3] ) . An outline of the application of the same methods to SVIDEs of second order will also be given extending the work of Aguilar and Brunner [1] .

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# Gnans –
# A Program for Stochastic and Deterministic Dynamical Systems

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Gnans is a program and language for the numerical study of deterministic and stochastic dynamical systems. It is aimed to be a high-level problem solving tool, allowing the user to concentrate on the (mathematical) problem, instead on programming details.

Central for Gnans is the concept of the *system declaration*: a declaration of a dynamical system in a special, equation oriented language. The description consists of declarations of states, parameters etc, together with equations describing the dynamics of the system. As an example, in Figure 1 the code for the van der Pol-Duffing oscillator is shown. Arbitrary C++-code may also be contained in the system description, allowing, among other things, the modeling of systems which are not naturally described by smooth evolution equations. A translator sorts the equations and translates the system to C++ code, which is subsequently compiled and dynamically linked into the executing program. The system equations can now be solved numerically with the speed of a *compiled* program. Several numerical integrators, also for stochastic (Itô-) differential equations, are provided.

In this way, the program can be considered "an initial value problem (IVP) engine". Using a simple script language, this IVP-engine can be programmed. In this way, problems involving loops over parameters and/or initial conditions may be solved.

Gnans has an intuitive graphical user interface, making it possible con-

```
CONTINUOUS TIME SYSTEM vdpol_duffing;

STATE x = 0;
STATE y = 1;
PARAMETER alpha = -1;
PARAMETER beta = 1;
PARAMETER gamma = 0;
PARAMETER delta = -1;
WIENER PROCESS xi1 = 1;
WIENER PROCESS xi2 = 997;
WIENER PROCESS xi3 = 171713;
PARAMETER sigma1 = 1;
PARAMETER sigma2 = 0;
PARAMETER sigma3 = 0;
TIME t;

AT TIME t:
d(x) = y;
d(y) = (alpha*x + beta*y + gamma*cube(x) + delta*sqr(x)*y)*d(t)
     + sigma1*x*d(xi1)
     + sigma2*y*d(xi2)
     + sigma3*d(xi3);
```

Figure 1: Complete Gnans code for the van der Pol-Duffing system.

trol the program and to change all relevant parameters using an intuitive point-and-click interface. Interactive plotting program can be run as child processes, with the possibility to define commands to be sent by the press of a button to the interactive plotting program.

Gnans is designed to run in an environment consisting of networked Unix workstations using the X Window System. It has presently been ported to the following platforms: Sun3, Sun4 (M.I.T. X11R5 and OpenWindows 3), Silicon Graphics Iris, and IBM RS6000. Version 1.0 requires an ANSI C compiler, later versions will require a C++ compiler.

Gnans is copyright © Bengt Mårtensson and Institute for Dynamical Systems, but available free of charge by anonymous ftp from ftp.mathematik.uni-Bremen.de (134.102.232.101). Everyone is invited to use, redistribute, and improve the system. Future users are strongly encouraged to get in touch with the author.

# METHODS FOR PARAMETER ESTIMATION IN STOCHASTIC DIFFERENTIAL EQUATIONS

(Preliminary version)

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**Abstract**

In this paper a method for Maximum Likelihood estimation of parameters in sde's, based on observations in discrete time, is presented. The class of sde's considered is written in the form:

$$dx_t = m(x_t, u_t, \theta, t)dt + E(\theta, t)dw_t, \tag{1}$$

where $x_t$ is the state vector, $u_t$ is a vector of exogenous variables representing controlled inputs or observable disturbances. $\theta$ is a vector of the (unknown) parameters of the model and $w_t$ is a standard Wiener process. The observations are given as

$$y_t = g(x_t, u_t, \theta, t) + F(\theta, t)e_t, \tag{2}$$

where $y_t$ is the dimensional observation vector at time $t$, $e_t$ is the measurement error, assumed to be normally distributed white noise with zero mean and unit covariance.

The Maximum Likelihood estimate is found as the parameters which maximizes the joint probability density of all the observations given the parameters, i.e.

$$\hat{\theta}_{ml} = \arg\max_{\theta} p(y_N, y_{N-1}, y_{N-2}, \cdots | \theta) \tag{3}$$

In order to evaluate the Likelihood function for discrete time data it is necessary to calculate the one step prediction error

$$\epsilon_{t|t-1} = y_t - E[y_t | y_{t-1}, y_{t-2}, \cdots, \theta] \tag{4}$$

If further the prediction errors are assumed to be Gaussian with zero mean and covariance, $R_{t|t-1}$, these are the only quantities needed for determination of the Likelihood function.

$\epsilon_{t|t-1}$ and $R_{t|t-1}$ may be calculated by using a kalman filter in the linear case and approximate filters in the general case.

An example is given where the method is used to identify the thermal characteristics of a building. In this example the dynamics is described by a system of linear stochastic differential equations, hence an exact explicit solution exist, which is utilized in the calculations.

# 1 Introduction

A number of authors have been dealing with the numerical solutions of sde's in general, and specifically numerical approximations of the solution of the Ito type stochastic differential equation,

$$d\boldsymbol{x}_t = \boldsymbol{m}(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{\theta}, t)dt + \boldsymbol{\sigma}(\boldsymbol{x}_t, \boldsymbol{\theta}, t)d\boldsymbol{w}_t, \tag{5}$$

where $\boldsymbol{x}_t$ is the state vector, $\boldsymbol{u}_t$ is a vector of exogenous variables representing controlled inputs or observable disturbances. $\boldsymbol{\theta}$ is a vector of the (unknown) parameters of the model and $\boldsymbol{w}_t$ is a standard Wiener process, see (Gard, 1988) and (Kloeden, Platen, & Schurz, 1991). When the step further is taken, to estimate the parameters in models of the kind of Eq. 1 based on discrete observations, the literature is more spare. Graebe (1990) has done some work in this area, but mainly on the simpler model without the $\boldsymbol{x}_t$ dependency in the last term of Eq. 1. The equation for the discrete observations is given by:

$$\boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{\theta}, t) + \boldsymbol{F}(\boldsymbol{\theta}, t)\boldsymbol{e}_t, \tag{6}$$

where $\boldsymbol{y}_t$ is the dimensional observation vector at time $t$, $\boldsymbol{e}_t$ is the measurement error, assumed to be normally distributed white noise with zero mean and unit covariance.

For the purpose of parameter estimation, it may be advantageous to consider the linearized version of the above model, Eq. 1 (without the $\boldsymbol{x}_t$ dependency in $\boldsymbol{\sigma}$) and Eq. 2 about some reference signal $(\boldsymbol{x}^*, \boldsymbol{u}^*)$. This yields

$$d\boldsymbol{x}_t = \boldsymbol{A}(\boldsymbol{\theta}, t)\boldsymbol{x}_t dt + \boldsymbol{B}(\boldsymbol{\theta}, t)\boldsymbol{u}_t dt + \boldsymbol{E}(\boldsymbol{\theta}, t)d\boldsymbol{w}_t \tag{7}$$

$$\boldsymbol{y}_t = \boldsymbol{C}(\boldsymbol{\theta}, t)\boldsymbol{x}_t + \boldsymbol{D}(\boldsymbol{\theta}, t)\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{\theta}, t)\boldsymbol{e}_t \tag{8}$$

where the matrices are calculated by $\boldsymbol{A}(\boldsymbol{\theta}, t) = \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{x}}\Big|_{\substack{\boldsymbol{u}=\boldsymbol{u}^* \\ \boldsymbol{x}=\boldsymbol{x}^*}}$, $\boldsymbol{B}(\boldsymbol{\theta}, t) = \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\Big|_{\substack{\boldsymbol{u}=\boldsymbol{u}^* \\ \boldsymbol{x}=\boldsymbol{x}^*}}$ etc.

Other models may lead directly to a formulation as Eq. 3 and Eq. 4, without linearizing. This is the case, if the underlying model is basically linear or close to linear. In these cases an exact solution exist, and it may be advantageous computationally and numerically to use this fact even if the model is only close to linear.

In the following sections methods for Maximum Likelihood estimation of parameters in sde's, based on observations in discrete time, is developed. First the requirements for estimation is

discussed, then an exact and approximative solutions are presented. Some of the numerical and computational details are also considered. Finally an application is given, where the exact solution has been used for estimation of the heat dynamics of a building, based on measurements.

## 2 The Maximum Likelihood Method

In the following it is assumed, that a finite set of observations are obtained at equally spaced time intervals, with unit sampling time. Hence the time index $t \in \{0, 1, 2, \cdots, N\}$, where $N$ is the number of observations. In order to derive the likelihood function, the following set of observations is introduced,

$$y^t = [y_t, y_{t-1}, \cdots, y_1, y_0] \tag{9}$$

i.e. $y^t$ is a matrix containing all the observations up to and including time $t$. Finally, let $\theta$ denote a vector of all the unknown parameters.

The likelihood function is the joint probability density of all the observations assuming that the parameters are known, i.e.

$$\begin{aligned} \mathcal{L}(\theta, y^N) &= p(y^N|\theta) \\ &= p(y_N|y^{N-1}, \theta)p(y^{N-1}|\theta) \\ &= \Big(\prod_{t=1}^{N} p(y_t|y^{t-1}, \theta)\Big)p(y_0|\theta) \end{aligned} \tag{10}$$

where successive applications of the rule $p(a\,,\,b) = p(a\,|\,b)\,p(b)$ is used to express the likelihood function as a product of conditional densities. Furthermore, it is convenient to introduce the one-step prediction error (or innovation),

$$\epsilon_{t|t-1} = y_t - E(y_t|y^{t-1}, \theta) \tag{11}$$

The prediction errors are now assumed to be Gaussian with zero mean and covariance,

$$R_{t|t-1} = V(y_t|y^{t-1}, \theta) \tag{12}$$

If the criterion to minimize is specified as the negative log likelihood, it is given by

$$L(\theta, y^N) = -\log \mathcal{L}(\theta, y^N) \tag{13}$$

$$= \frac{1}{2}\sum_{t=1}^{N}\Big(\epsilon'_{t|t-1}R^{-1}_{t|t-1}\epsilon_{t|t-1} + \log\,\det R_{t|t-1} + n_y \log 2\pi\Big) \tag{14}$$

where $n_y$ is the dimension of $y$. The simple expression for $\mathcal{L}$ is derived from the fact the the normal distribution is characterized solely by the mean and covariance.

The Maximum Likelihood estimator has some attractive characteristics, which motivates the choice of it. Under mild conditions the maximum likelihood estimates are asymptotically normally distributed (Ljung, 1987), with parameters

$$\sqrt{N}(\hat{\theta}_N - \theta) \overset{asympt.}{\sim} N(0, M_F^{-1}) \tag{15}$$

where $M_F = E(\partial^2 L(\theta, y^N)/\partial\theta^2)$ is the Fisher information matrix. When minimizing the criterion Eq. 14 using a quasi-Newton method, an iteratively updated Hessian matrix evaluated at the minimizing parameter value is returned. This matrix may be used as an estimate of $M_F$. The inverse of Fishers information matrix is used as an estimate of the covariance of the parameters. The variances serves as a basis for calculating t-test statistics for testing the hypothesis that the parameter is equal to zero. The correlation between the estimates is readily found based on the variance matrix.

# 3 The Predictor

Under the assumptions stated in the previous section it is necessary to calculate the prediction error, $\epsilon_{t|t-1}$, and the covariance of the prediction error, $R_{t|t-1}$, in order to calculate the likelihood function. This is under the assumption of Gaussian innovations.

The evaluation of $\epsilon_{t|t-1}$ involves the calculation of $\hat{y}_{t|t-1} = E(y_t|y^{t-1}, \theta)$, where the caret, denotes the expectation operation,

$$\hat{y}_{t|t-1} = \int_{-\infty}^{\infty} g(\xi, u_t, \theta, t) p(\xi|y^{t-1}) d\xi \tag{16}$$

$$R_{t|t-1} = \int_{-\infty}^{\infty} g(\xi, u_t, \theta, t) g(\xi, u_t, \theta, t)' p(\xi|y^{t-1}) d\xi + F(\theta, t) F(\theta, t)' \tag{17}$$

The exact calculation of these numbers requires the entire conditional density of the state vector, conditioned on earlier observations $p(x_t|y^{t-1})$ for all $t$. This density is not in the general case available, it is therefore necessary to employ approximations. One such possible approximation is known as a statistically linearized filter. While this technique typically assume the density of $x$ to be Gaussian, it is not based on a direct linearization of the system dynamics. In principle it therefore does not require $m$ and $E$ to be linearizable, allowing saturations and the like. This filter however have great computational requirements.

For the class of linear models of Eq. 7 and Eq. 8 there exist an exact solution. The kalman filter provides the exact solution in this case. The following equations are obtained for updating the state estimate:

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t \epsilon_t \tag{18}$$

$$P_{t|t} = P_{t|t-1} - K_t R_{t|t-1} K_t' \tag{19}$$

$$K_t = P_{t|t-1} C' R_{t|t-1}^{-1} \tag{20}$$

The formulas for the prediction of the state and observation are given by,

$$\frac{\partial}{\partial\tau}\hat{\boldsymbol{x}}_{\tau|t} = \boldsymbol{A}\hat{\boldsymbol{x}}_{\tau|t} + \boldsymbol{B}\boldsymbol{u}_\tau\,, \quad t\leq\tau<t+1 \tag{21}$$

$$\frac{\partial}{\partial\tau}\boldsymbol{P}_{\tau|t} = \boldsymbol{A}\boldsymbol{P}_{\tau|t} + \boldsymbol{P}_{\tau|t}\boldsymbol{A}' + \boldsymbol{E}\boldsymbol{E}'\,, \quad t\leq\tau<t+1 \tag{22}$$

$$\hat{\boldsymbol{y}}_{t+1|t} = \boldsymbol{C}\hat{\boldsymbol{x}}_{t+1|t} + \boldsymbol{D}\boldsymbol{u}_{t+1} \tag{23}$$

$$\boldsymbol{R}_{t+1|t} = \boldsymbol{C}\boldsymbol{P}_{t+1|t}\boldsymbol{C}' + \boldsymbol{F}\boldsymbol{F}' \tag{24}$$

The initial conditions are $\hat{\boldsymbol{x}}_{1|0} = \boldsymbol{\mu}_0$ and $\boldsymbol{P}_{1|0} = \boldsymbol{V}_0$. The time dependencies of the matrices have been suppressed for clarity in the kalman filter equations. This implementation of the kalman filter thus involves the solution of a set of ordinary differential equations for each sampling instant. If on the other hand the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are time invariant it is possible to calculate an explicit solution for Eq. 21 and Eq. 22,

$$\hat{\boldsymbol{x}}_{t+1|t} = \boldsymbol{\Phi}\hat{\boldsymbol{x}}_{t|t} + \boldsymbol{\Gamma}\boldsymbol{u}_t \tag{25}$$

$$\boldsymbol{P}_{t+1|t} = \boldsymbol{\Phi}\boldsymbol{P}_{t|t}\boldsymbol{\Phi}' + \boldsymbol{R}_1 \tag{26}$$

where the matrices $\boldsymbol{\Phi}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{R}_1$ are calculated as,

$$\boldsymbol{\Phi}(\tau) = e^{\boldsymbol{A}\tau}\,; \quad \boldsymbol{\Gamma}(\tau) = \int_0^\tau e^{\boldsymbol{A}s}\boldsymbol{B}\,ds\,; \quad \boldsymbol{R}_1(\tau) = \int_0^\tau \boldsymbol{\Phi}(s)\boldsymbol{E}\boldsymbol{E}'\boldsymbol{\Phi}(s)'ds \tag{27}$$

and $\tau$ is the sampling time. This implementation of the kalman filter thus involves the calculation of the exponential of a matrix. This calculation may be done once for a given set of parameters if the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are time invariant. If the time dependence is slow compared to the dominating eigenvalues of the system, this implementation may be used for time varying systems, by evaluating Eq. 27 for each sampling instant, assuming constant $\boldsymbol{A}$ and $\boldsymbol{B}$ within a sampling time. This solution requires less computations and is more robust.

# 4 Numerical Stability

All practitioners, have realized that one thing is the theoretical development of an algorithm, but another thing is the implementation of the algorithm on a computer. The problems arises due to the finite arithmetic in the computer, which introduces a rounding error for every calculation with real numbers. In certain cases these rounding errors will accumulate so the solution could be misleading.

For the algorithms outlined in the previous sections, specific care for the numerics should be faced on the kalman filter, the calculation of the exponential of a matrix and on the optimization routine.

## 4.1 A stable kalman filter

It is well known that the kalman filter in some situations is numerical unstable. The problems arise when some of the variances, because of rounding errors, become non-positive definite.

Therefore careful handling of the equations for the variances Eq. 19, 20, 24 and 26 is needed in order to stabilize the kalman filter. Since all variances should be symmetric and positive definite, it is desirable to use their Cholesky factorization. The $\boldsymbol{LDL'}$-factorization have been chosen, also called square root free Cholesky decomposition, where $\boldsymbol{L}$ is unit lower matrix and $\boldsymbol{D}$ is diagonal. This method have accuracies comparable with the usual method (with no factorization) using twice the numerical precision.

An equation for updating a factorized matrix is

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{G}\boldsymbol{D}_g\boldsymbol{G}' \tag{28}$$

where $\tilde{\boldsymbol{A}}$ is known from other considerations to be positive definite, and $\boldsymbol{D}_g$ is a diagonal matrix. Thus it is necessary to compute a unit lower triangular matrix $\tilde{\boldsymbol{L}}$ and a diagonal matrix $\tilde{\boldsymbol{D}}$ with $\tilde{d}_i > 0$ such that

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{L}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{L}}' \tag{29}$$

Algorithms to solve this problem is found in (Bierman, 1977) and (Madsen & Melgaard, 1991). It is obvious that Eq. 24 and Eq. 26 easy are brought into this form. Eq. 19 can be rewritten as

$$\begin{aligned}
\boldsymbol{P}_{t|t} &= \boldsymbol{P}_{t|t-1} - \boldsymbol{K}_t\boldsymbol{R}_{t|t-1}\boldsymbol{K}_t' \qquad \Leftrightarrow \\
\boldsymbol{P}_{t|t} &= \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{C}'\boldsymbol{R}_{t|t-1}^{-1}\boldsymbol{C}\boldsymbol{P}_{t|t-1}' \qquad \Leftrightarrow \\
\tilde{\boldsymbol{L}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{L}}' &= \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}' - \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}'\boldsymbol{C}'[\boldsymbol{L}_r\boldsymbol{D}_r\boldsymbol{L}_r']^{-1}\boldsymbol{C}[\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}']' \qquad \Leftrightarrow \\
\tilde{\boldsymbol{L}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{L}}' &= \boldsymbol{L}[\boldsymbol{D} - \boldsymbol{G}\boldsymbol{D}_r^{-1}\boldsymbol{G}']\boldsymbol{L}' \qquad \text{where} \qquad \boldsymbol{G} = \boldsymbol{D}\boldsymbol{L}'\boldsymbol{C}'\boldsymbol{L}_r'^{-1}
\end{aligned} \tag{30}$$

The expression $[\boldsymbol{D} - \boldsymbol{G}\boldsymbol{D}_r^{-1}\boldsymbol{G}']$ in Eq. 30 is in the form Eq. 28, and can thus be solved for the factors $\bar{\boldsymbol{L}}\bar{\boldsymbol{D}}\bar{\boldsymbol{L}}'$, and we have $\tilde{\boldsymbol{L}} = \boldsymbol{L}\bar{\boldsymbol{L}}$ and $\tilde{\boldsymbol{D}} = \bar{\boldsymbol{D}}$.

This implementation of the kalman filter is able to handle the multiple-input, multiple-output case with a high grade of accuracy and stability, see e.g. (Bierman, 1977).

## 4.2 Calculation of exp(A)

An approach for calculating the exponential of a matrix, which in general is very robust against ill conditioned matrices, is the so-called *Padé approximation* to $exp(\boldsymbol{A})$, which is defined by

$$\boldsymbol{R}_q(\boldsymbol{A}) = \boldsymbol{D}_q(-\boldsymbol{A})^{-1}\boldsymbol{D}_q(\boldsymbol{A}) \tag{31}$$

where

$$\boldsymbol{D}_q = \sum_{j=0}^{q} c_j\boldsymbol{A}^j$$

$$c_j = \frac{(2q-j)!q!}{(2q)!j!(q-j)!}$$

In general, the Padé approximation is very useful, and it is suggested in Moler & Loan (1978) as the most adequate in most cases.

However, the roundoff difficulties and the computing costs increases as the spread of eigenvalues or $\|A\|$ increases. These difficulties can be reduced by combining the above method with the method of *scaling and squaring*. This method utilize the following fundamental property of the exponential function

$$e^{A} = (e^{A/m})^m \tag{32}$$

The idea is to choose m as a power of two for which $e^{A/m}$ can be reliably and efficiently computed by the Padé approximation, and afterwards to finally compute the exponential of A by repeated squaring. Usually m is chosen as the smallest power of two for which $\|A\|_2/m$ is less than one. According to Moler & Loan (1978) the resulting algorithm is one of the most efficient.

## 4.3 The optimization algorithm

A crucial point in any statistical analysis based on the maximum likelihood method is concerned with the actual maximization of the likelihood function. In our case an explicit method is not available, and the maximization has to be performed using a numerical method of iterative character.

The problem may be formulated as to find the minimizing point of a nonlinear function, i.e. $\arg\min f(x)$, where $f : \mathcal{R}^n \rightarrow \mathcal{R}$.

Among optimization methods, Newton-Raphson's method has shown to be exceptionally effective. A minimum of $f$ is found where $g = \partial f/\partial x = 0$. The method is based on a Taylor expansion of $g$ to first order

$$g(x_n + h) = g(x_n) + h\frac{\partial g}{\partial x}(x_n) + o(h) \tag{33}$$

Putting $g(x_n + h) = 0$ and neglecting $o(h)$ the algorithm takes the form

$$h_n = -H(x_n)^{-1}g(x_n) \tag{34}$$

$$x_{n+1} = x_n + h_n \tag{35}$$

where the Hessian $H = \partial^2 f/\partial x^2$ in the regular case is positive definite. It can be shown that the iterations converge towards the minimizing point $x_*$, where $g(x_*) = 0$. Since we are not able to provide the optimization procedure with the first and second partial derivatives of $f(x)$, these will have to be approximated by the algorithm. Finite-difference approximation

is used for the first derivative $g$, and a secant approximation $\boldsymbol{B}_n$ is used for the Hessian. The secant approximation is more effective and robust than a finite-difference Hessian in the optimization. This class of secant methods are called *quasi-Newton*, and the most successful seems to be the BFGS method for iterative Hessian approximation combined with soft line search, (Dennis & Schnabel, 1983).

The gradient is found by a forward difference approximation, switching to a central difference when nearing the optimum. The central difference approximation is more accurate, but also more computer intense.

Since the Hessian is positive definite it is represented by its $\boldsymbol{LDL}'$ factorization, and the iterative updating of the Hessian is performed on the factors, hence increasing the numerical stability of the algorithm.

# 5 An example: Thermal dynamics of a building

The application considered in this paper is related to the Commission of the European Communities' (CEC) research project called PASSYS. The aim of this project is to establish a common basis within the European Community for determining the energy dynamics of building components, especially components related to passive solar energy. Passive solar design has been recognized as an important potential for energy conservation. Many new components and systems have been developed in the late 1970s and 1980s. However, very little is known about their actual thermal and solar dynamic characteristics. A further uncertainty is the unknown performance when the components are applied to buildings and exposed to variations in climate.

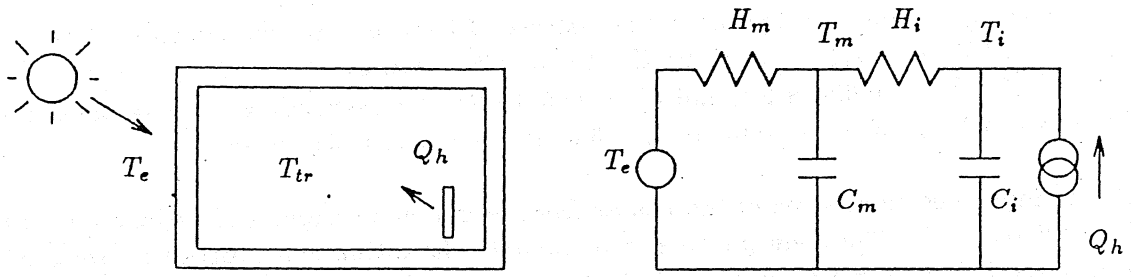
Within the PASSYS project, a test procedure for building components, using short-term performance data from a test cell, is developed and defined. The south wall and, for some test cells, the roof are removable. The test cell is calibrated using a highly insulated opaque south wall. Different south wall components could then be inserted in place of the calibration wall.

The test cell has a test room of 13.8 $m^2$ ground surface and 38 $m^3$ air volume with an adjoining service room to the north, accommodating measuring and air-conditioning equipment. The U-value of the envelope is less than 0.1 $W/m^2K$. The south wall, which is the actual passive solar system, is fixed in an insulated frame. Any kind of wall can be incorporated in this frame. A further description of the test cell is found in Wouters & Vandaele (1990).

The thermal characteristic of buildings is frequently approximated by a simple network with resistors and capacitances – see, for instance Madsen & Holst (1992). In this section, such a lumped parameter model for the dynamics of the test cell is presented.

The dominating heat capacity of the test cell is located in the outer wall. For such buildings, the model with two time constants shown in Figure 1 is frequently found adequate. The states of the model are given by the temperature, $T_i$, of the indoor air and possibly inner

Figure 1: *A model with two time constants of the test building and the equivalent electrical network.*




part of the walls with heat capacity $C_i$, and by the temperature, $T_m$, of the heat accumulating medium, with the heat capacity $C_m$. $H_i$ is the transmittance of heat transfer between the room air and the walls, while $H_m$ is the heat transmittance between the inner part of the walls and the external surface of the walls. The input to the system is the heat supply, $Q_h$, and the outdoor surface temperature, $T_e$. By considering the outdoor surface temperature instead of the outdoor air temperature, the effect of solar radiation is taken into account.

In state space form the model is written,

$$\begin{bmatrix} dT_i \\ dT_m \end{bmatrix} = \begin{bmatrix} -H_i/C_i & H_i/C_i \\ H_i/C_m & -(H_i+H_m)/C_m \end{bmatrix} \begin{bmatrix} T_i \\ T_m \end{bmatrix} dt + \begin{bmatrix} 0 & 1/C_i \\ H_m/C_m & 0 \end{bmatrix} \begin{bmatrix} T_e \\ Q_h \end{bmatrix} dt + \begin{bmatrix} dw_i(t) \\ dw_m(t) \end{bmatrix}. \tag{36}$$

An additive noise term is introduced to describe deviations between the model and the true system. Hence, the model of the heat dynamics is given by the (matrix) stochastic differential equation

$$d\boldsymbol{T} = \boldsymbol{A}\boldsymbol{T}dt + \boldsymbol{B}\boldsymbol{U}dt + d\boldsymbol{w}(t) \tag{37}$$

where $w(t)$ is assumed to be a *wiener process* with incremental covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{1,i}^2 & 0 \\ 0 & \sigma_{1,m}^2 \end{bmatrix}. \tag{38}$$

The measured air temperature is naturally encumbered with some measurement errors, and hence the measurement equation is written

$$T_{tr}(t) = (1\ 0) \begin{bmatrix} T_i \\ T_m \end{bmatrix} + e(t) \tag{39}$$

where $e(t)$ is the measurement error, assumed to be normally distributed with zero mean and variance $\sigma_2^2$.

The following parameter values have been estimated in an earlier experiment on a test cell: $H_i$ = 55.29 W/K , $H_m$ = 13.86 W/K , $C_i$ = 325.0 Wh/K, $C_m$ = 387.8 Wh/K, $\sigma_{1,i}^2$ = 0.00167 K$^2$, $\sigma_{1,m}^2$ = 0.00978 K$^2$, and $\sigma_2^2$ = 0.00019 K$^2$. Corresponding to these parameters, the time constants of the system are $\tau_1$ = 3.03 hours and $\tau_2$ = 54.28 hours.

This example describes some of the results from several simulations and estimations of the PASSYS test cell. The main purpose is to validate the estimation procedure. By considering several simulated sequences this investigation considers both the mean values and the variances of the estimated parameters.

The input and output signals in the model are:

$T_e$ is measured surface temperature, from a Danish test cell.

$Q_h$ the heat supply, is a PRBS (pseudo-random binary sequence) of order $n$ = 6 and $T_{prbs}$ = 8 hours, switching between 0 W and 300 W.

$T_{tr}$ room temperature, simulated with the specified model.

In this example we have chosen the sampling time $T_{sampl}$ = 20 minutes, and we have measurements from 21 days, which equals 1512 observations per simulation series. We have simulated 10, in principal, equal series, but with different starting point for the noise sequences. A contraction of the estimation results from the 10 series is presented in Table 1. The estimates

Table 1: *Results from estimation of 10 series. $x_{simul}$ is the simulated parameters, $\bar{x}$ is the mean of the estimated parameters, $s_x$ is the empirical standard deviation of the estimated parameters and $\bar{s}$ is the mean of the estimated standard deviation.*

| Parameter | $x_{simul}$ | $\bar{x}$ | $s_x$ | $\bar{s}$ |
|---|---|---|---|---|
| $H_i$ | 55.290 W/K | 56.315 W/K | 1.9609 W/K | 1.9765 W/K |
| $H_m$ | 13.860 W/K | 13.565 W/K | 0.53025 W/K | 0.52408 W/K |
| $C_i$ | 325.00 Wh/K | 322.31 Wh/K | 3.7832 Wh/K | 3.2851 Wh/K |
| $C_m$ | 387.78 Wh/K | 375.51 Wh/K | 10.816 Wh/K | 13.896 Wh/K |
| $\sigma_{1,i}$ | 0.040866 K | 0.040947 K | 0.014283 K | 0.014293 K |
| $\sigma_{1,m}$ | 0.098894 K | 0.103083 K | 0.050865 K | 0.063209 K |
| $\sigma_2$ | 0.013784 K | 0.013733 K | 0.006605 K | 0.005718 K |

are Maximum Likelihood obtained from an implementation of the exact explicit method described in the previous sections. It is seen that not only are the estimated parameters close

to the the simulated, but also the uncertainty on the estimates (the standard deviation) are close to the empirical standard deviation. This imply, that this method provides reliable confidence regions for the parameters, and it is possible to make test statistics for model reduction etc.

# 6 Conclusion

Stochastic differential equations are very useful for modelling physical systems, where the a priori knowledge about the system is widely used in the model specification. In the paper it has been shown, which steps to take for the estimation of parameters in stochastic differential equations based on discrete measurements. It has also been rendered, that careful handling of the numerical implementation, of the algorithms, is very important for stable computation.

A practical example from a CEC research project has demonstrated the use and strength of the method presented, for the estimation and validation of parameters in a model of stochastic differential equations.

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# ON MARTINAGALE APPROACH TO STOCHASTIC APPROXIMATION ALGORITHM

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Stochastic approximation algorithsms as the quite natural generalization the well-known Newtons method were originated by classical papers of Robbins - Monro and Kiefer - Wolfowitz. The model of stochastic approximation algorithm under our consideration in the talk is the following (on some stochastic basis)

$$d\Theta_t = - \gamma_t R(\Theta_{t-}) da_t - \gamma_t dM_t \qquad (1)$$

Here R is a lipshitz type reggression function with the unique root $\Theta^*$, M is a local square integrable martingale a is a local increasing predictable process and $\gamma$ is a positive predictable process which satisfy to the following type conditions

$$\int_0^\infty \gamma_s da_s = \infty \text{ and } \int_s^\infty \gamma_s^2 da_s < \infty.$$

The main goal of the talk is to describe the approach to the algorithm (1) and its asymptotic behaviour (a.s. -convergence, mean-square convergence, asymtotic normality) as $t \to \infty$ which based on the some comprehansive asymptotic results for local martigales.

The model (1) contains many particular cases (discrete-time models, diffusion models, etc). Which studied early separatly. In

particular we give full investigation on the model (1) with Gaussian errors M.

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# THE SOLVING OF THE BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATION BY THE NUMERICAL INTEGRATION OF STOCASTIC EQUATIONS

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Let $G$ be a bounded domain in $R^n$, $Q = G \times [T_0, T)$ be a cylinder in $R^{n+1}$, $\Gamma = \bar{Q} \setminus Q$. Let us consider the first boundary value problem for the equation of parabolic type

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(t,x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(t,x)\frac{\partial u}{\partial x^i} + c(t,x)u + g(t,x) = 0, \quad u|_\Gamma = \varphi(t,x).$$

The probabilistic representation of the solution of the problem is connected with the stochastic differential system

$$dX = b(s,X)\,ds + \sigma(s,X)\,dw(s).$$

For example, if $c = 0, g = 0$, we have

$$u(t,x) = E\,\varphi(\tau, X_{t,x}(\tau))$$

where $\tau$ is the entrance time of the process $(s, X_{t,x}(s))$ in the boundary $\Gamma$.

In one of the suggesting algorithms of the weak approximation of $(\tau, X_{t,x}(\tau))$ of the report the general step is the following. Let $(t_k, \bar{X}_k) \in Q$ be an approximate point at $k$-th step. Let $\xi_k$ be an uniformly distributed unit vector on the surface of the open sphere $u \in R^n$ with the center in the origin, $\tau_k$ be a some number. Then $(t_{k+1}, \bar{X}_{k+1})$ is generated via the formulae

$$t_{k+1} = t_k + \frac{\tau_k^2}{n}, \quad \bar{X}_{k+1} = \bar{X}_k + \frac{\tau_k^2}{n} b_k + \tau_k \sigma_k \xi_k, \quad t_0 = t, \quad \bar{X}_0 = x.$$

Let $U(t,x,\tau)$ be an open ellipsoid which is obtained out of the sphere $U$ by virtue the linear operator $\tau\sigma(t,x)$ and the shift $x + \frac{\tau^2}{n} b(t,x)$. For $(t,x) \in Q$ and sufficiently small $\tau$ we have the cylinder $\Pi(t,x,\tau) = U(t,x,\tau) \times [t, t + \frac{\tau^2}{n}) \in Q$. It is clear that the point $(t_{k+1}, \bar{X}_{k+1})$ belongs to the boundary of the upper base of the cylinder $\bar{\Pi}(t,x,\tau)$.

Let $\Gamma_{\tau^2}$ be an intersection of the $\tau^2$ - neighbourhood of the boundary $\Gamma$ with the cylinder $Q$. Now we state the way of generating of a sequence $\tau_k$ which defines an algorithm. Let $\tau$ be a some sufficiently small number. We put $\tau_k = \tau$ if $\Pi(t_k, \bar{X}_k, \tau)$ belongs to $Q$. If $\Pi(t_k, \bar{X}_k, \tau)$ does not belong to $Q$ and the point $(t_k, \bar{X}_k)$ does not belong to $\Gamma_{\tau^2}$ then we find $\tau_k < \tau$ such that $U(t_k, \bar{X}_k, \tau_k)$ is tangent to the boundary of the domain $G$. Moreover, it is assumed that $\tau_k$ is so small that always $\bar{X}_k \in U(t_k, \bar{X}_k, \tau_k)$. Then we

simulate the point $(t_{k+1}, \bar{X}_{k+1})$ by virtue of the formulae in which $\xi_k$ does not depend on $\xi_1, \ldots, \xi_{k-1}$. Such a random walk is ended at the random step $\varkappa$ as $(t_\varkappa, \bar{X}_\varkappa) \in \Gamma_{\tau^2}$. At last we find the point $(\bar{t}, \bar{X})$ on $\Gamma$ which is the nearest to $(t_\varkappa, \bar{X}_\varkappa)$. It is stated that $(\bar{t}, \bar{X})$ is the weak approximation of $(\tau, X_{t,x}(\tau))$ with the error $O(\tau^2)$, i.e.

$$E\,\varphi(\bar{t}, \bar{X}) - u(t,x) = E\left[\varphi(\bar{t}, \bar{X}) - \varphi(\tau, X_{t,x}(\tau))\right] = O(\tau^2).$$

At the same time the average value $E\varkappa = O(1/\tau^2)$.

One can also construct a similar algorithm with the some order of convergence in which $\Pi$ is replaced by the right parallelepiped and at each step the random walk is realized on the vertices of the upper base. Let us note that higher order approximations, in which every point $(t_{k+1}, \bar{X}_{k+1})$ is simulated in the given neighbourhood of $(t_k, \bar{X}_k)$, can be derived. Similar results can be received for equations of elliptic type too.

The material of the report is directly connected with weak approximation of solutions of stochastic differential equations 1-6. Unlike these works in the present report attention has attracted to the boundedness of the simulated increments of the solutions. Such an approach is necessary both for the solution of boundary value problems and in a number of other cases.

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# Variance Reduction for Simulated Diffusions

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**Abstract**

The talk will concern variance reduction techniques for the Monte-Carlo integration of functionals of the solutions of Itô stochastic differential equations (sdes). The Monte Carlo method for sdes offers a means of calculating solutions to certain types of parabolic partial differential equation and so has applications in various fields including stochastic control, particle physics and econometrics; it involves the representation of the required integrals as means of random variables, and the simulation of many outcomes of these random variables. The variables concerned are defined on infinite dimensional Wiener spaces, and cannot therefore be simulated directly—they must at some stage be approximated by variables defined on high, but finite dimensional spaces. The approach taken is to construct variance reduced random variables on the infinite dimensional spaces which can subsequently be approximated by any of a number of known finite difference methods.

The methods of control variates and importance sampling are developed. In both cases a perfect variate (ie. one which is unbiasssed and has zero variance) is first constructed by means of Haussmann's integral representation theorem for functionals of Itô processes. These involve terms which cannot be calculated exactly but which can be approximated to yield unbiassed estimators of the desired integrals with reduced variances. A number of methods of approximating these terms will be described and numerical results will be presented for one of the methods.

# Stochastic Numerical Treatment of Nonlinear Diffusions

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We are concerned with the problems arising in the simulation of nonlinear diffusion processes, namely the diffusion processes whose transition mechanisms depend on their own temporal pdf, $u(t,x)dx = P\{X_t \in [x, x+dx)\}$ :

$$(1) \quad \begin{cases} dX_t(\omega) = a(t, X_t, H(X_t; u(t)))dt + b(t, X_t, H(X_t; u(t)))dW_t(\omega), \quad (0 \le t \le T) \\ X_0(\omega) = \xi(\omega), \\ H(x; u(t)) = \int H(x-y)u(t,y)dy \end{cases}$$

where $a(t,x,y)$, $b(t,x,y)$ and $H(x)$ are some real, smooth functions and $W_t(\omega)$ $(0 \le t)$ is the Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

The mechanism being nonlinear the general scheme for the numerical approximation of the process should be carried through such discretization formula like:

$$(2) \qquad \overline{X}_{t_{k+1}} = F(t_k, h, \{\overline{X}_{t_{k-k_0}}, \cdots, \overline{X}_{t_k}\}, \Delta_k W, \mathrm{M}\overline{u}(t_k))$$

where $h = T/N$, $t_k = k \cdot h$, $\Delta_k W = W(t_{k+1}) - W(t_k)$ and $\mathrm{M}\overline{u}(t)$ is an estimator for the pdf $\overline{u}(t,x)$ of the $\overline{X}_{t_k}$'s, constructed from a certain number of samples, say $\{\overline{X}_{t_k}(\omega^i),\ 1 \le i \le N_0\}$.

The formula (2) shows that in the nonlinear case we have to deal with two kinds of errors,

(e,1) the error caused by the discretization,

(e,2) the error associated to the density estimator.

One of the principal characteristics of such nonlinear problem is that the error of type (e,1) can not be independent of the error of the type (e,2). So much researches have been done on the analysis of each of these errors by many authors (see the references given in Kloeden-Platen [1] and in Silverman [3]), especially in the context of the simulation of linear diffusions but very little (as far as the author knows) for the nonlinear problems. In the recent article ([2]) we studied the efficiency of the Euler scheme applied to our nonlinear problem in the following way:

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$$
(3) \qquad \begin{cases} Y_0^h(\omega) &= \xi(\omega), \\ Y_{k+1}^h(\omega) &= Y_k^h(\omega) + a(t_k, Y_k^h, H(Y_k^h : \mathrm{M}\overline{u}(t_k))) \cdot h \\ & \qquad +b(t_k, Y_k^h, H(Y_k^h : \mathrm{M}\overline{u}(t_k))) \cdot \Delta_k W \\ Y_t^h(\omega) &= Y_k^h(\omega) + \dfrac{t - t_k}{h} \cdot \Delta_k Y^h, \quad for \ \ t \in [t_k, t_{k+1}) \ . \end{cases}
$$

It is shown there that, if the accuracy of the estimator $\mathrm{M}\overline{u}(t,x)$ is such that,

$$
E' \int |\overline{u}(t,x) - \mathrm{M}\overline{u}(t,x)|^{2p} dx < C(1/N_0)^{2p\eta},
$$

then , for $\forall \epsilon > p$ it holds the following estimate,

$$
E(\sup_t |X_t - Y_t^h|^{2p}) \leq C h^{\gamma'} (\log \frac{1}{h})^{\epsilon}, \quad \gamma' = p \cdot (2\alpha \wedge 1 \wedge 2\eta).
$$

where $\alpha$ is the order of Hölder continuity in $t$ of the coefficients $a(t,x,y)$, $b(t,x,y)$.

In this talk, we are going to discuss about the amelioration of this result and show some relevant results.

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# NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS

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The talk gives a survey on results obtained partly together with Peter E. Kloeden and others during the preparation of the monograph Kloeden/Pl. [1] and the book Kloeden/Pl./Schurz [2]. These results are mainly related to papers by Milstein, Talay, Tubaro and Newton.

The stochastic Stratonovich Taylor formula derived in Kloeden/Pl. [3] turnes out to be the appropriate expansion of Ito diffusions for the construction of higher strong order approximations of integer strong order. For the evaluation of the involved multiple stochastic integrals one can use various relations between multiple Ito and Stratonovich integrals, see Kloeden/Pl. [4], and may approximate the appearing basic multiple Stratonovich integrals as described in Kloeden/Pl./Wright [5].

For stiff stochastic differential equations it is crucial to develop a concept of numerical stability which leads to the construction of higher strong order implicit schemes, see Kloeden/Pl. [6].

As an application it was shown in Kloeden/Pl./Schurz [7] that higher strong order explicit and implicit schemes allow to derive efficient and stable filters for noisy observations of hidden Markov chains.

Another application of strong schemes provides a method for testing discretized parametric estimators for diffusion process, see Kloeden/Pl./Schurz/Sørensen [8]. For the study of nonlinear stochastic dynamical systems strong higher order methods are applied in Kloeden/Pl./Schurz [9].

Higher order weak Taylor schemes are described in Mikulevicius/Pl. [10] covering also the case of Ito process with jump component. The weak order of the Euler scheme is investigated in Mikulevicius/Pl. [11] under Hölder continuity of the drift and diffusion coefficients.

Some investigations on extrapolation methods for the weak approximation of Ito diffusions are summarized in Kloeden/Pl./Hofmann [12]. As an application of strong and weak methods solving SDEs we studied in Hofmann/Pl./Schweizer [13] the pricing of options under incompleteness and stochastic volatility.

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# ON INTEGRO-DIFFERENTIAL BELLMAN EQUATION

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Let $L^k$, $k \in \mathbf{N}$ be operators acting on functions $u \in C_0^\infty((0,1) \times \mathbb{R}^d)$ by the formula

$$L^k u(t,x) = \sum_{i,j=1}^{d} a_{ij}^k(t,x) u_{x_i x_j}(t,x) + \sum_{i=1}^{d} b_i^k(t,x) u_{x_i}(t,x) + \int \left[ u(t,x+y) - u(t,x) - \sum_{i=1}^{d} u_{x_i}(t,x) y_i 1_{|y| \leqslant 1} \right] \pi^k(t,x,dy)$$

Here $a^k(t,x)$ is a symmetric non-negative definite $d \times d$-matrix, $b^k(t,x) \in \mathbb{R}^d$ and $\pi^k(t,x,\cdot)$ is a measure on $\mathbf{R}^d \backslash \{0\}$ such that $\int |y|^2 \bigwedge 1 \pi^k(t,x,dy) < \infty$.

Let $D = D([0,1], \mathbf{R}^d)$ be the Skorokhod space, $X_t = X_t(\omega) = \omega_t$ be a canonical process in $D$,

$$D_u^s = \sigma\{X_r, s \leqslant r \leqslant u\}, \; D = \bigvee_{u \in [0,1]} D_u^0, \; \mathbb{D}^s = \{D_{u+}^s, u \in [s,1]\}.$$

A function $\alpha: [0,1] \times D \to \mathbf{N}$ is a step strategy if there exists a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$, $n \in \mathbf{N}$ of the interval $[0,1]$ and a family $\{\beta_i, \; i = 0,1,\ldots,n\}$ of measurable functions $\beta_i: \mathbb{R}^d \to \mathbf{N}$ such that $\alpha_t(\omega) = \beta_i(X_{t_i}(\omega))$ if $t_i \leqslant t < t_{i+1}$. $\mathcal{A}$ denotes the set of all step strategies.

Under appropriate conditions for any $s \in [0,1]$, $x \in \mathbb{R}^d$, $k \in \mathbf{N}$ there is a unique solution $\mathbf{P}_{s,x}^k$ of the $(s,x,L^k)$ — martingale problem and $(X_t, \mathbf{D}^s, \mathbf{P}_{s,x}^k)$ is strongly Markovian. Let $\mathbf{P}_{s,x}^\alpha$, $s \in [0,1]$, $x \in \mathbb{R}^d$, $\alpha \in \mathcal{A}$ be a unique probability measure on $(D, \mathcal{D})$ such that for any $u \in C_0^\infty(\mathbb{R}^d)$ the process

$$u(X_t) - \int_s^t L_r^{\alpha_r} u(X_r) dr$$

is a $(\mathbb{D}^s, \mathbf{P}^\alpha_{s,x})$–martingale and $\mathbf{P}^\alpha_{s,x}(X_r = x,\ 0 \leqslant r \leqslant s) = 1$.

Let $f\colon \mathbf{N} \times [0,1] \times \mathbb{R}^d \to \mathbb{R}^1$ be a bounded measurable function and

$$v(s,x) = \sup_{\alpha \in \mathcal{A}} \mathbf{E}^\alpha_{s,x} \int_s^1 f^{\alpha_t}(t, X_t)dt.$$

We shall discuss the cases when the cost function $v$ is a classical, viscosity or semi-convex solution to the problem

$$\frac{\partial u}{\partial t} + \sup_k (L^k u + f^k) = 0 \qquad \text{in} \quad (0,1) \times \mathbb{R}^d, \quad u(1,\cdot) = 0.$$

The talk will be based on the results presented in [1]–[6].

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# ON THE STOCHASTIC SIMULATION OF QUANTUM SYSTEMS

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ABSTRACT

In this conference we will consider two types of problems in connection with Quantum Models. First, we will show how to use Quantum Stochastic Differential Equations to give a precise formulation of some Master Equations. Secondly, we will discuss the simulation of such equations by means of *classical* stochastic differential equations and processes. In particular, a model of the Laser electromagnetic field will be considered.

Quantum Stochastic Analysis provides theorems on existence and uniqueness of solutions for a wide class of Stochastics Differential Equations. This was done by Hudson and Parthasarathy [5], later extended by Fagnola [3], among other authors (see also Parthasarathy [6]). Following Accardi [7], a Markovian cocycle is a family $(V(t))_{t\geq 0}$ of contractive operators with a time shift covariance and localisation properties. Unitary Markovian cocycles in the Boson–Fock space with strongly continuous reduced semigroup can be obtained solving a quantum stochastic differential equation of the form

$$dV(t) = \sum_{i,j} V(t) L^i_j d\Lambda^j_i,\ V(0) = I, \tag{1}$$

where $\Lambda^j_i$ are the basic integrators of Hudson–Parthasarathy quantum stochastic calculus, and operators $L^i_j$ play the role of the infinitesimal generator: in [5] they are assumed to be bounded; in [3] this hypothesis is weakened. A *state* of a quantum system is represented in that framework by a positive trace class operator $\rho$ with unit trace; its evolution is given by the expression

$$\rho_t = V(-t)\rho V(t),\ (t \geq 0). \tag{2}$$

To simulate equations (1), (2), consists in the construction of a classical probability approach to them. This leads to the construction of a special class of classical diffusion processes.

Indeed, assume we choose our basic spaces in such a form that operators $\rho_t$ could be associated to a probability density $x \mapsto \rho(x,t)$ on $\mathbb{R}^d$. To state the evolution, (1) is replaced by a differential equation satisfied by $\rho(x,t)$. One main point in the simulation is to approach such a differential equation by a Fokker–Planck equation

and $\rho(x,t)$ by a solution of it in the sense of distributions. In general, the stated Fokker–Planck equation will have no unique weak solution..

The problem is then the following. Given a weak solution $\rho(x,t)$ to a Fokker–Planck equation, does there exist a diffusion process $(X_t)_{t\geq 0}$ such that the distribution of the random variable $X_t$ has $\rho(\cdot,t)$ as a probability density with respect to Lebesgue measure in $\mathbb{R}^d$?

This problem was stated by Nelson ([8], [9])in different terms. In those papers, Nelson followed the ideas of Guerra and Morato to introduce the dynamics [4]. This works as follows. Assume the kinematics of a particle's motion is given by a stochastic differential equation of diffusion type. Introduce an action functional of the diffusion process as the integral of a suitable Lagrangian operator. Use the Calculus of Variations to derive Hamilton–Jacobi optimal control equation. If the control field is chosen as the drift of the diffusion (the velocity field), then one gets Schrödinger equation from Hamilton–Jacobi's formalism.

The mathematical problem in the above approach is that the assumed diffusion could not exists! Many authors have been giving different sufficient conditions to construct such a kind of processes ([2], [13], [1]). In [11], Rebolledo has introduced a new class of diffusions which can be constructed under a single assumption on the density $\rho(\cdot,t)$. This assumption is that of finite entropy. The key is the use of an approximation procedure which relies on previous results in [10]. The constructive method so developed is also applied in Quantum Optics in [12].

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# APPROXIMATION OF RANDOM ELEMENTS

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Let $X(\omega)$ be an $H-$ valued random element defined on $(\Omega, S, P)$ and let $A$ be a $P^X$ - measurable linear operator from the Hilbert space $H$ into the Hilbert space $H_1$ . Then the random element $AX(\omega)$ is said to be an **approximation** of $X(\omega)$ (in $H_1$ with respect to the approximation operator $A$). In order to characterize the quality of approximation in $H$ , we introduce a $P^{AX}$ - measurable linear operator $B : H_1 \longrightarrow H$ called **back projector**. The back projector maps the approximation $AX(\omega)$ in an associated $H$ - valued random element $BAX(\omega)$. The **approximation error** is then given by $l(X(\omega), BAX(\omega)$, where $l$ denotes a (non-negative measurable) loss functional defined on the product space $H \times H$. In order to obtain a non-random measure for the approximation $AX(\omega)$ of $X(\omega)$, we choose the mean of $l(X(\omega), BAX(\omega))$ . $r(BA) = El(X(\omega), BAX(\omega)$ is said to be the **approximation risk** with respect to the operator $A$, the loss function $l$ and the back projector $B$ (See [1]).
This talk is concerned with some problems at the approximation of Gaussian random elements with values in separable Hilbert space. Some of the results are also true for non-Gaussian random elements and in more general function spaces. Furthermore, we discuss
- finite element approximations ([2])
- wavelet-approximations and
- kernel approximations.

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# Computer experiments for the noisy Duffing - van der Pol equation

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Keywords : random dynamical systems, stochastic bifurcation, invariant measures.

## Abstract

In the field of random dynamical systems a lot of work has been done to build up a bifurcation theory. Roughly speaking, stochastic bifurcation means the change of qualitative behaviors of the dynamical system. Those behaviors are characterized here by so called invariant measures. In this work we show some numerical results for those measures. We first explain these objects shortly as follows (for details and the general concept of random dynamical systems see e.g. Arnold [1] or Arnold and Crauel [2]).

Take the canonical Wiener probability space $(\Omega,\mathcal{F},P)$ and the shift $\{\vartheta_t\}_{t\in\mathbb{R}}$ on $\Omega$, $\vartheta_t(\omega(\cdot)) := \omega(\cdot + t) - \omega(t)$ for $\omega(\cdot) \in \Omega$. Define the $\sigma$-field $\mathcal{F}_s^t := \sigma\{\omega(u) - \omega(v) \mid s \le u, v \le t\}$, $s,t \in \mathbb{R}$. We have $\mathcal{F} = \mathcal{F}_{-\infty}^{+\infty}$. Let a stochastic differential equation on a smooth manifold $X$ (e.g. $\mathbb{R}^d$) be given. Under some appropriate conditions there exists a stochastic flow, say $\varphi_{s,t}$, $s,t \in \mathbb{R}$ (see Kunita [6]). Particularly, $\varphi_{s,t}(\omega)x$ is the solution of the stochastic differential equation at time $t$ starting from $x$ at time $s$. Now define $\varphi(t-s,\vartheta_s\omega) := \varphi_{s,t}(\omega)$, then $\varphi(t,\omega) : X \to X$ is a random dynamical system over $(\Omega,\mathcal{F},P,\vartheta_t)$ (for the proof see e.g. Arnold [1]).

Denote by $Pr(\mathrm{X})$ the space of all probability measures over X. First we restrict $\varphi(t,\omega)$ to positive times ($t \in \mathbb{R}^+$), i.e. $\{\varphi(t,\omega)\}_{t\ge 0}$ is a random dynamical system over $(\Omega, \mathcal{F}_0^\infty, P|_{\mathcal{F}_0^\infty}, \{\vartheta_t\}_{t\ge 0})$. We say a measure $\rho \in Pr(\mathrm{X})$ is a stationary measure for the one-point motion $\varphi_{0,t}(\omega)x$ if for all $t \ge 0$

$$\int_X P\{\varphi_{0,t}(\omega)x \in A\}\, d\rho(x) = \rho(A), \qquad A \in \mathcal{B}(X),$$

i.e. $\rho$ is an invariant initial distribution for the transition probability $\mathrm{P}(t,x,A) = P\{\varphi_{0,t}(\omega)x \in A\}$ and thus a stationary solution of the corresponding Fokker-Planck equation.

Let us return to the case $t \in \mathbb{R}$. For a given random dynamical system we can define a so-called skew product flow $\Theta_t : X \times \Omega \to X \times \Omega$, by $(x,\omega) \mapsto (\varphi(t,\omega)x, \vartheta_t(\omega))$. A measure $\mu \in Pr(X \times \Omega)$ is called invariant for $\varphi(t,\omega)$ if it has marginal $P$ on $\Omega$ and is invariant under $\Theta_t$ for all $t$. Now we can define in general the disintegration of a given invariant measure $\mu$ on $(X \times \Omega, \mathcal{B} \otimes \mathcal{F})$ as a measurable map $\mu_\cdot : \Omega \to Pr(X)$, $\omega \mapsto \mu_\omega$ which satisfies

$$\mu(B \times C) = \int_C \mu_\omega(B)\, dP(\omega), \qquad \text{for all } B \in \mathcal{B}(X) \text{ and } C \in \mathcal{F}.$$

The disintegration is unique P-a.s. and exists if e.g. X is a Polish space. We know that a measure $\mu$ is invariant for the random dynamical system $\varphi$ if and only if $\varphi(t,\cdot)\mu_\omega = \mu_{\vartheta_t\omega}$ P-a.s. for all $t \in \mathbb{R}$ (see Arnold and Crauel [2]).

In particular, the measure $\rho \times P$ on $(X \times \Omega, \mathcal{B} \otimes \mathcal{F}_0^\infty)$ is invariant for $\{\Theta_t\}_{t\ge 0}$ and $\rho$ can be considered as a disintegration of $\rho \times P$ on $(\Omega, \mathcal{F}_0^\infty)$.

In bifurcation theory we study parameterized families of random dynamical systems. We say a stochastic bifurcation occurs if new invariant measures appear when these parameters pass certain critical values. These bifurcations can be characterized by Lyapunov exponents. Sometimes these bifurcations can be described by the occurrence of new stationary measures $\rho$. This method is often used by physicists and engineers. We will see in the following that for the construction of $\rho$ only the information in $\mathcal{F}_0^\infty$ is needed whereas $\mu_\omega$ needs information from $\mathcal{F}_{-\infty}^\infty$. Consequently, we can expect to get more insight from $\mu_\omega$ then from $\rho$ in the dynamics of a given family of random dynamical systems.
The main idea of our numerical procedure for $\rho$ and $\mu_\omega$ is explained in the following.
Firstly, by the ergodic theorem $\rho$ can be estimated by the occupation measure

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\varphi(t,\omega)x)\,ds = \int_X f(y)\,d\rho(y) \qquad for\ \rho - a.a.\ x,\ f \in L^1(\rho). \tag{1}$$

Secondly, following Baxendale [3] we define a random process with values in $Pr(\mathrm{X})$ by $\{\rho_{-t} := \varphi_{-t,0}\rho\}_{t\geq 0}$ (where $\varphi_{-t,0}\rho(A) := \rho(\varphi_{-t,0}^{-1}(A)), \forall A \in \mathcal{B}(X)$). Then $\{\rho_{-t}\}_{t\geq 0}$ is a $Pr(\mathrm{X})$-valued martingal relative to the filtration $\mathcal{F}_{-t}^0$ in the sense that for all bounded and continuous functions $f : X \to \mathbb{R}$, $\mathbf{E}\big(\int_X f\,d(\rho_{-t-s}) \mid \mathcal{F}_{-t}^0\big) = \int_X f\,d(\rho_{-t}),\ P - a.s.$
Using some characteristics of the flow and of $\{\rho_{-t}\}$ one derives that

$$\lim_{t\to\infty} \varphi_{-t,0}\rho = \mu_\omega \quad P - a.s. \qquad and \qquad \mathbf{E}(\mu_\omega) = \rho, \tag{2}$$

where the convergence is in the weak topology on $Pr(\mathrm{X})$. Now the numerical realization is easily done by solving the forward equation to construct $\rho$ from (1) and by solving the backward equation to construct $\mu_\omega$ from (2). These equations are solved numerically by using the schemes which are described in the book of Kloeden and Platen [5].
As an example we consider the following noisy Duffing - van der Pol equation

$$\ddot{x} = (\alpha + \sigma_1\xi_1)x + (\beta + \sigma_2\xi_2)\dot{x} + \gamma x^3 + \delta x^2\dot{x} + \sigma_3\xi_3, \tag{3}$$

where $\xi_1, \xi_2$ and $\xi_3$ are independent one-dimensional white noise processes and $\gamma = \delta = -1$. For $\sigma_1 = \sigma_2 = \sigma_3 = 0$, one derives the deterministic equation which is well studied in their local and global bifurcation behavior (see Holmes and Rand [4]). Recently, a great number of efforts are made to study the stochastic bifurcations of (3), mainly on the level of $\rho$. Here we present some numerical results of stochastic bifurcation of (3), on the $\rho$ level and especially on the $\mu_\omega$ level. Due to the lack of space we only show two particular cases. The systematical presentation of our numerical results will appear later.

In Figure 1 and Figure 2 the left-hand sides show the stable solution $\rho$ of the Fokker-Planck equations, and the right-hand sides show the support of the corresponding invariant measures $\mu_\omega$ for different values of $\alpha$ and $\beta$. Here we only explain shortly one particular case. So, let us look at Figure 1. Here we take $\sigma_1 = \sigma_3 = 0, \sigma_2 = \frac{1}{10}$, i.e. only $\beta$ is perturbed by noise. Let $\alpha < 0$ be fixed. We see that if $\beta < 0$ the Fokker-Planck equation of (3) has the solution $\rho = \delta_0$ and disintegration $\mu_\omega = \delta_0$. That is clear, because the noise does not effect the deterministic fixed point at zero. If $\beta > 0$ the $\rho$ becomes a measure whose density looks like a crater and $\mu_\omega$ becomes a new invariant measure, whose support is a closed circle. In this case we say that (3) undergoes a stochastic Hopf bifurcation.

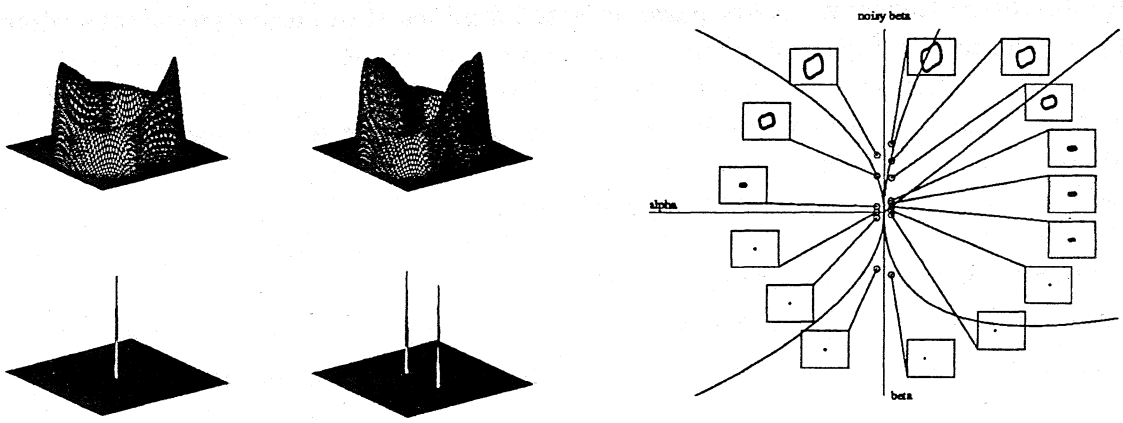


**Figure 1.** Noisy $\beta$ : Solutions of the Fokker-Planck equation (left) and supports of $\mu_\omega$ in the parameter plane ($\alpha$,$\beta$) (right)

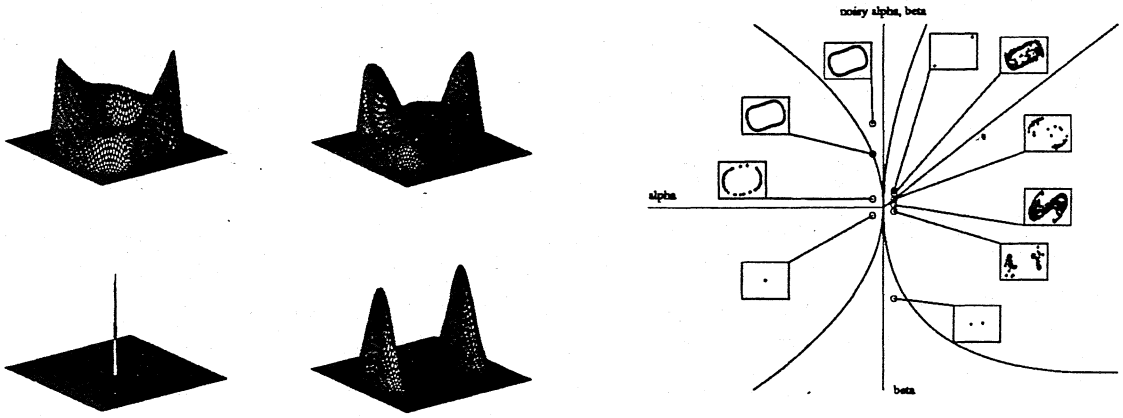


**Figure 2.** Noisy $\alpha$,$\beta$ : Solutions of the Fokker-Planck equation (left) and supports of $\mu_\omega$ in the parameter plane ($\alpha$,$\beta$) (right)

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# Approximate Markov Chain Filtering

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Comprehensive studies on stochastic filtering theory has been made by several authors in the 60'ies and 70'ies. See Wonham [1], Zakai [2], Kallianpur and Striebel [3] and the outstanding treatise of Kallianpur [4] and many others, just to name only few of them. Special interest arises in the nonlinear Markov chain filtering problem having only discrete time observations. Here we are going to construct approximate filters similar to Newton [5]. What is happening with the approximate filters stated below if the time step size used converges to zero in discrete time situations? Does it reflect the exact filter? Can one obtain an order of the approximate filter analogous to the numerical solutions of stochastic differential equations described by Kloeden & Platen [6] ? What are the assumptions we have to make in order to get approximations for the exact Markov chain filter? These and similar questions are answered here.

The problem can be described shortly by the following: Let $t \in [0,T]$. Assume it is given an unobservable continuous time Markov chain $\xi = \{\xi_t, \quad t \in [0,T]\}$ with finite state space $S = \{s_1, \ldots, s_d\}$ on the probability space $(\Omega, A, I\!P)$. This Markov chain is not directly observable, but we do have observations $W = \{W_t : 0 \leq t \leq T\}$ satisfying the equation $W_t = \int_0^t h(\xi_s)ds + W_t^*$ ($m$–dimensional) where $W_t^*$ is a Wiener process under $I\!P$ and $h : S \to I\!R^m$ is assumed to be a quadratic integrable function. Now we are interested in filtering as much information on the Markov chain as we can get under given filtration $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$. Such information objects are called filters and can be described by $\Pi_t(g) = I\!E(g(\xi_t)|\mathcal{F}_t) = \sum_{k=1}^d g(s_k)X_t^k / \sum_{k=1}^d X_t^k$ $(t \in [0,T], s_k \in S)$ for any bounded function $g : S \to I\!R$. The $d$–dimensional process $X_t = \{X_t^1, \ldots, X_t^d\}$ occuring above satisfies the Zakai equation $X_t = p(0) + \int_0^t AX_s\, ds + \int_0^t H\, X_s\, dW_s$ where $p(0)$ is the vector of initial probabilities, $A$ is the intensity matrix of the Markov chain and $H$ denotes the diagonal matrix with $h(s_i)$ in its diagonal. The

corresponding approximate filters $\Pi_t^\delta(g)$ for a given time step size $\delta > 0$ are obtained by the use of the approximation $Y_t^{\delta,k}$ instead of $X_t^k$ in $\Pi_t(g)$. Using the strong approximation techniques developed by Milstein and Newton [5] and recently extended by Kloeden & Platen [6] one gets the following answer on the above questions proved in [7] :

**Theorem:** (Convergence of approximate Markov chain filters)
An approximate Markov chain filter $\Pi_t^\delta(g)$ with time step size $\delta$ converges on the time interval $[0,T]$ with the strong order $\gamma > 0$ to the optimal least squares filter $\Pi_t(g)$ for a given bounded function $g$ if the discrete time approximation $Y_t^\delta$ used in $\Pi_t^\delta(g)$ converges on $[0,T]$ to the solution $X_t$ of the Zakai equation with the same order.

Several numerical examples show higher order approximations are most efficient for highly accurate, approximate Markov chain filtering. In some cases one needs implicit approximations.

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# ARBITRARY ORDER APPROXIMATIONS FOR LINEAR SYSTEMS OF SDE's

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Though the expansion in Taylor series in general form of the solution of SDE was obtained by Platen[1], it is impossible for general SDE to construct numerical methods of order grater then 2 in meansquare sense due to the necessity to compute random values of the form $\int W_i dW_j$, if $i \neq j$ with high order. Therefor the investigation of possibility of constructing schemes having arbitrary order of consistency, as it is in the ordinary differential equations theory, as always attach one's interest. But this problem can be resolved at least for special classes of SDE.

For linear systems of the form

$$X(t) = X(t_0) + A\int_{t_0}^{t} X(s)ds + \sigma\int_{t_0}^{t} dW(t), \qquad t \in [t_0, T]$$

in [2] the schems of the following form were suggested

$$X_{n+1} = R_q(Ah)X_n + \sum_{i=0}^{l} \frac{A^i\sigma}{i!} \cdot \int_0^h \tau^i dW^{t_n}(\tau),$$

where $R_q(Ah)$ - is approximation of matrix exponent of order $q$, polynomial or rational, and having the order of consistency in meansqure

$$E(|X(t_{n+1}) - X(t_n)|^2 / X(t_n) = X_n) = O\left(h^{min(2q+2,2l+3)}\right), \quad while\ h \to 0.$$

Here the normal dependent random vectors $\xi_n^{(i)} = \int_0^h \tau^i dW^{t_n}(\tau)$ has covariance matrix $E(\xi_n^{(i)} \cdot \xi_n^{(j)}) = I \cdot h^{i+j+1}/(i+j+1)$ and can be simulated as

$$\xi_n^{(i)} = \sqrt{h} \cdot diag(1, h, \ldots, h^l) \cdot L \cdot \zeta_n,$$

where elements of low triangular matrix $L$ are computed via the following rules

$$l_{i,1} = 1/i, \text{ for all } i, \qquad l_{i,j} = 0, \text{ if } i < j,$$

$$l_{i,j} = \frac{\sqrt{2j-1}}{i} \cdot \prod_{k=1}^{j-1} \frac{i-k}{i+k}, \qquad \text{if } i \geq j \neq 1,$$

and $\zeta_n$ are normal random vectors with zero mathematical expectations and unique covariance matrix.

For linear systems in Stratonovich sense of the form

$$X(t) = X(t_0) + A\int_{t_0}^{t} X(s)ds + \sum_{j=1}^{m} \sigma^{(j)} \int_{t_0}^{t} X(\tau)dw_j(\tau), \qquad t \in [t_0, T]$$

under the assumptions $\sigma^{(i)} \cdot \sigma^{(j)} = 0$ for any $i, j = 1, \ldots, m$ one can construct schemes of the following form

$$X_{n+1} = \Phi_q(t_{n+1}, t_n) \cdot X_n,$$

where

$$\Phi_q(t_1, t_0) = R_q(A(t_1 - t_0)) + \sum_{k=0}^{q-1} \frac{1}{k!} \sum_{j=1}^{m} \sigma^{(j)} A^k \xi_{kj}^{(1)}$$
$$+ \sum_{k=0}^{q-2} \frac{1}{k!} A \sum_{j=1}^{m} \sigma^{(j)} A^k \xi_{kj}^{(2)} + \cdots$$
$$+ \sum_{k=0}^{1} \frac{1}{k!} A^{q-2} \sum_{j=1}^{m} \sigma^{(j)} A^k \xi_{kj}^{(q-1)} + A^{q-1} \sum_{j=1}^{m} \sigma^{(j)} \xi_{0j}^{(q)},$$
$$\xi_{kj}^{(q)} = \int_{t_0}^{t_1} \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{q-1}} (\tau_q - t_0)^k dw_j(\tau_q) d\tau_{q-1} \cdots d\tau_1.$$

This schemes may have an arbitrary order of concistency if $\sigma^{(i)} \cdot \sigma^{(j)} = 0$ that let us eliminate the terms with random quantities $\int W_i dW_j$, and there remain only easily simulated normal random quantities $\xi_{kj}^{(q)}$ with zero mathematical expectations and mutual covariances

$$E(\xi_{k_1 j}^{(p_1)} \cdot \xi_{k_2 j}^{(p_2)}) = \int_{t_0}^{t_1} \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{p_1-2}} \int_{t_0}^{t_1} \int_{t_0}^{\varphi_1} \cdots \int_{t_0}^{\varphi_{p_2-2}}$$
$$\times \int_{t_0}^{\min(\tau_{p_1-1}, \varphi_{p_2-1})} (\mu - t_0)^{k_1+k_2} d\mu d\varphi_{p_2-1} \cdots d\varphi_1 d\tau_{p_1-1} \cdots d\tau_1.$$

Some concret schemes of different order and numerical results are presented in [3].

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# ON APPROXIMATION OF SOLUTIONS OF SDE'S WITH REFLECTING BOUNDARY CONDITIONS

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We consider a d-dimensional stochastic differential equation on a domain $D$ with reflecting boundary condition

$$X_t = H_t + \int_0^t f(X_{s-})dZ_s + K_t, \quad t \in R, \tag{1}$$

where $Z$ is a semimartingale, $X$ is a reflecting process on $\overline{D} = D \cup \partial D$ and $K$ m is a bounded variation process with variation increasing only, when $X_t \in \partial D$. We assume as in [1] and [2] that $D$ is a general domain satisfying the conditions (A) and (B). Let $f$ be Lipschitz continuous and bounded and let $|\Delta Y| + L|\Delta Z| < r_0$, where $L, r_0 > 0$ and $r_0$ is some constant depending on a region $D$, only ( for example if $D$ is convex then $r_0 = \infty$). Then using discrete approximations of the solution of the SDE (1), which are constructed with the natural analogy to Euler's formula, we can show existence and uniqueness of strong solution of the SDE (1) ( see [2, Theorem 5] ).

We investigate the rate of convergence for such approximations in the case $D = [0, \infty) \times R^{d-1}$, $Z = W$, where $W$ is a standard Wiener process.

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# Exponential Families of Stochastic Processes

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Many important statistical stochastic process models are exponential families in the sense that the likelihood function corresponding to observing the process in the time interval $[0,t]$ has an exponential representation of the form (2). The exponential structure of the likelihood function implies several probabilistic and statistical properties of the model. Several scientists have contributed to the theory of exponential families of stochastic processes. In the lecture a review is given of some of the authors' own work as published in Küchler & Sørensen [2–6]. In these papers references can be found to work by other authors on the subject.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ be a filtered statistical space, where the class of probabilty measures $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subseteq R^k$ is indexed by a k-dimensional parameter, and where the filtration $\{\mathcal{F}_t : t \geq 0\}$ is right-continuous. The events in $\mathcal{F}_t$ are those which can be observed in the experiment with index $t$ in a certain family of experiments. This could, for instance, be observation of a stochastic process in the time interval $[0,t]$. By the definition $\mathcal{F}_t = \mathcal{F}_{[t]}$ also discrete time stochastic processes are covered ($[t]$ denotes the integer part of $t$). Let $P_\theta^t$ denote the restriction of $P_\theta$ to the $\sigma$-field $\mathcal{F}_t$. The class $\mathcal{P}$ is called an exponential family with respect to the filtration $\{\mathcal{F}_t\}$ if there exists a measure $\mu$ such that

$$P_\theta^t << \mu^t, \quad t \geq 0, \ \theta \in \Theta \tag{1}$$

and

$$\frac{dP_\theta^t}{d\mu^t} = \exp\left(\gamma_t(\theta)^* B_t - \Phi_t(\theta)\right), \quad t \geq 0, \ \theta \in \Theta. \tag{2}$$

Here * denotes transposition of the k-dimensional column vector $\gamma$. The k-dimensional stochastic process $B$ is adapted to the filtration $\{\mathcal{F}_t\}$, and $\Phi$ and $\gamma$ are non-random functions of $t$ and the parameter $\theta$. If the filtration is generated by observation of a stochastic process, we speak of an exponential family of stochastic processes, and (2) is the likelihood function corresponding to observation of the process in $[0,t]$.

Next, we list some examples of stochastic process models which for observation in $[0,t]$ are exponential families. A first example is the family of solutions to the stochastic differential equations

$$dX_t = \left[ a(t,X_t) + \sum_{i=1}^{m} \gamma^{(i)}(\theta) b^{(i)}(t,X_t) \right] dt + c(t,X_t) dW_t, \quad \theta \in \Theta \subseteq R^k, \tag{3}$$

where the functions $a, b$ and $c$ are known. Another example is the class of counting processes with intensity

$$\lambda_t = \kappa H_t, \ \kappa > 0, \tag{4}$$

where $H$ is a predictable process. Concrete examples are the Poisson processes ($H_t = 1$), the pure births processes ($H_t = X_{t-}$), and the logistic birth processes ($H_t = X_{t-}(K - X_{t-})$). An example that combines the first two types of models is the family of the diffusions with jumps given by

$$dX_t = \left[ a(t,X_t) + \sum_{i=1}^{m} \gamma^{(i)}(\theta) b^{(i)}(t,X_t) \right] dt + g\left(t, X_{t-}, \Delta Z_t^\theta\right) dZ_t^\theta + c(t,X_{t-}) dW_t, \tag{5}$$

where $Z^\theta$ belongs to a class of compound Poisson processes whose jump distributions form a classical exponential family of distributions. Some concrete examples are given in Sørensen [7]. Further examples are the class of Markov processes with a state space of a fixed finite dimension and subfamilies thereof (continuous time as well as discrete time), discrete time Markov processes with transition densities of the form

$$f_\theta(y|x) = \exp\left( \sum_{i=1}^{m} \gamma^{(i)}(\theta) m^{(i)}(y,x) - \Phi(\theta) \right) \tag{6}$$

(Heyde & Feigin [1]), the Galton-Watson branching processes and the Gaussian autoregressive processes of fixed order $m$.

In the lecture it is proved that most exponential families of stochastic processes are curved exponential families. Results about existence and uniqueness, consistency and asymptotic normality of the maximum likelihood estimator is shown for two important types of exponential families of processes: 1) those with a time-continuous likelihood function (for instance diffusions), 2) those which can be obtained by a stochastic time transformation of a family of Lévy processes. An exponential family of the last-mentioned type has a likelihood function of the form

$$\frac{dP_\theta^t}{dP_0^t} = \exp\left[\theta^* A_t - \kappa(\theta) S_t\right], \ \theta \in \Theta \subseteq R^k, \tag{7}$$

where the process $S$ is 1–dimensional and $\kappa$ is a cumulant transform. Finally, the full exponential family generated by the curved exponential family on $\mathcal{F}_t$ is considered briefly. This family, which plays a role in several modern statistical techniques, is called the envelope family on $\mathcal{F}_t$. Stochastic process interpretations of the envelope families will be discussed.

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# CONVERGENCE AND STABILITY OF IMPLICIT RUNGE–KUTTA METHODS FOR SYSTEMS WITH MULTIPLICATIVE NOISE

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(JULY 7, 1992)

**Abstract.**

A class of *implicit* Runge–Kutta schemes for stochastic differential equations affected by *multiplicative* Gaussian white noise is shown to be optimal with respect to global order of convergence in quadratic mean. A test equation is proposed in order to investigate the stability of discretization methods for systems of this kind. Here *stability* is intended in a truly probabilistic sense, as opposed to the recently introduced extension of A–stability to the stochastic context, given for systems with additive noise. Stability regions for the optimal class are also given.

*1980 AMS Subject Classification:* 65L20 (primary), 60H10, 34F05, 65L07, 93E15

*Keywords and phrases:* Numerical stability, Runge–Kutta methods, implicit methods, stochastic differential equations, stochastic stability.

# Analysis of a stochastic particle method for a non-linear P.D.E.

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In a recent paper [1], E.G. Puckett proposed a stochastic particle method for the non linear P.D.E. in $[0,T] \times \mathbb{R}$ :

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au = \Delta u + f(u) \\ \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

where $1 - u_0$ is the cumulative function, supposed to be continuous, of a probability distribution, and $f$ is a function satisfying properties ensuring, in particular, that the solution $u(t,x)$ takes values in $[0,1]$.

His justification of the method and his analysis of the error were based on a splitting of the operator $A$ ; a rough presentation of the algorithm is the following : after having located $N$ particles on the real axis in an appropriate way at time 0 and given them weights $\frac{1}{N}$, at each time step, first one numerically solves the O.D.E.

$$\frac{\partial v}{\partial t} = f(v)$$

with the approximation of $u$ at the previous step as initial condition, on a time interval of length $\Delta t$ (this operation changes the weights of the particles), second one approximately solves

$$\frac{\partial v}{\partial t} = \Delta v$$

by randomly moving the particles during a time interval equal to $\Delta t$.

The upper bound of the random error on $u(T,x)$ in $L^1(\Omega \times \mathbb{R})$ is shown to be of order $\frac{1}{N^{\frac{1}{4}}}$, provided $\Delta t = \frac{1}{N^{\frac{1}{2}}}$.

In the last Section of the paper, Puckett presents numerical results which obviously show that this estimation is very pessimistic.

We propose a completely different interpretation of the method, and a completely different analysis of the error. This permits us to extend the algorithm to more general situations :

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + f(u) \\ \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

where $L$ is a strongly elliptic second order operator with smooth coefficients, and also to get much better results on the rate of convergence.

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# STATISTICAL INFERENCES ABOUT SEMIMARTINGALES

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Let $(\Omega,\mathcal{F},P_\theta)$, $\theta\in\Theta\subset R^1$, be a family of probability spaces with filtration $F=(\mathcal{F}_t)_{t\geq 0}$. Let $X=(X_t)_{t\geq 0}$ be the semimartingale with respect to $(F,P_\theta)$ and $T^\theta=(B^\theta,C^\theta,\nu^\theta)$ is its triplet. We suppose that there is $\theta_o\in\Theta$ such that the measure P is locally absolutely continuous with respect to the measure $P_{\theta_o}$ for all $\theta\in\Theta$ and $L(\theta)=(L_t(\theta))_{t\geq 0}$ is respective local density. Then $C^\theta=C$ for every $\theta\in\Theta$. Let random measures $\mu^\theta(dt,dx)$ and random functions $Y(\theta)=\{Y(t,x;\theta),t\geq 0,x\in R^1\}$ have the same sense as in [2] and function $\gamma(\theta)=(\gamma_t(\theta))_{t\geq 0}$ is defined by

$$B^\theta_t - B^{\theta_o}_t - x\cdot I(|x|\geq 1)\cdot(Y(\theta)-1)*\nu^{\theta_o}_t = \gamma(\theta)\circ C_t, \qquad t\geq 0.$$

We consider asymtotical statistical inferences about semimartingales which are founded on maximum likelihood method. In particular, we consider estimation problems of the unknown parameter $\theta$ under the observations of the process $\{X_s,\ 0\leq s\leq t\}$ as $t\to\infty$.

We suppose that the functions $\gamma(\theta)$ and $Y(\theta)$ are twice continuous differentiable with respect to the parameter $\theta$ and

$$\partial\gamma(\theta)/\partial\theta=\dot\gamma(\theta), \qquad \partial^2\gamma(\theta)/\partial\theta^2=\ddot\gamma(\theta)$$
$$\partial Y(\theta)/\partial\theta=\dot Y(\theta), \qquad \partial^2 Y(\theta)/\partial\theta^2=\ddot Y(\theta)$$

In addition, we suppose that its possible a change of differentiation and an integration in necessary cases.

Theorem. Let $\theta\in\Theta$ be a true value of the parameter and the following conditions are satisfied with some nonnegative

increasing function $\varphi_t$ ( $\varphi_t \to \infty$ as $t \to \infty$ ):

A. $\varphi_t^{-2} \cdot \sup_{s \le t} \int_{R^1} (\dot{Y}(s,x;\theta)/Y(s,x;\theta))^2 \, \nu^{\theta}(\{s\},dx) \xrightarrow{P_\theta} 0;$

B. there exists some increasing function $K(x;\theta)$ with bounded variation such that

$$\varphi_t^{-2} \cdot \{I(\bar{R}_+)(x) \cdot \dot{\gamma}^2(\theta) \circ C_t + I(Y(\theta) < x \cdot \varphi_t \cdot Y(\theta)) \cdot (\dot{Y}(\theta)/Y(\theta))^2 * \nu_t^{\theta}\} \xrightarrow{P} K(x;\theta)$$

for each x which is a continuity point for K and for $x=+\infty$ ($\bar{R}_+ = R_+ \cup \{+\infty\}$).

Then it exist the maximum likelihood estimator $\hat{\theta}_t$ for parameter $\theta$ which is consistent and random variable $K(+\infty;\theta) \cdot \varphi_t \cdot (\hat{\theta}_t - \theta)$ has infinitely divisible limit distribution with the characteristic function

$$\psi(\lambda) = \exp\left\{ \int (e^{i\lambda x} - 1 - i\lambda x) \cdot x^{-2} dK(x;\theta) \right\}.$$

In particular, if the function K from the condition B is such that $K(x;\theta)=0$ for $x \le 0$ and $K(x;\theta)=\sigma^2(\theta)>0$ for $x>0$, then the random variable $\sigma(\theta) \cdot \varphi_t \cdot (\hat{\theta}_t - \theta)$ are asymptotically normal with the parameters (0,1).

The results of this talk are generalizations of author's one [1] which concern statistics of Markovian processes.

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# APPROXIMATION SCHEMES FOR STOCHASTIC FUNCTIONAL EQUATIONS OF RETARDED TYPE

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We consider strong approximations of discrete type for linear stochastic functional equations of the form

$$(1)\quad \begin{cases} dx(t) = [L(x_t) + f(t)]dt + g(t)dw(t), \quad 0 \le t \le T \\ x(0) = \eta\ ,\ x(s) = \phi(s),\ s \in J := [-r, 0] \end{cases}$$

and of the Lie-Trotter type for nonlinear stochastic functional equations of the form

$$(2)\quad \begin{cases} dx(t) = F(t, x(t-r), x(t))dt + G(t, x(t-r), x(t))dw(t), 0 \le t \le T \\ x(s) = I(s),\ s \in J, \end{cases}$$

where : $L : L^2(J, R^d) \longrightarrow R^d$ is defined by

$$L\phi = \sum_{i=0}^{\nu} A_i \phi(-r_i) + \int_J A(s)\phi(s)ds\ ,$$

$0 = r_0 < r_1 < \ldots < r_\nu = r$, $(A_i)_{1 \le i \le \nu}$ dxd-matrices,

$A \in L^2(J, R^d \otimes R^d)$, $x_t(s) = x(t+s)$, $s \in J$,

$F(t,x,y), G(t,x,y) : [0,T] \times R^{2d} \longrightarrow R^d$ are continuous and satisfy the Lipschitz condition in $x, y$, uniformly in $t$,

$(w(t))_{t \in [0,T]}$ is a real brownian motion,

$(\eta, \phi)$ (resp. $(I(t))_{t \in [0,T]}$) is a random variable with values in the Hilbert space $Z := R^d \times L^2(J, R^d)$ (resp. a continuous $R^d$-valued process) ( initial conditions ).

We shall assume that $\eta, \phi, I$ are independent of $w$.

By using the equivalence between (1) and a stochastic evolution equation (SEE for short) in the Hilbert space $Z$ we introduce an

abstract discrete approximation scheme (DAS for short) for the SEE, which approximates the solution of the SEE by a sequence of systems of discrete stochastic difference equations of succesively higher dimension. Standard schemes, as the Averaging/Finite difference scheme and the Spline/Variational scheme, are covered by the general DAS.

For equation (2) we introduce some approximation schemes suggested by the Lie-Trotter formula( the splitting up method ).

The method consists in a separation of the diffusion and the drift terms and obtaining in this way two simpler equations, one of them is deterministic ( depending on a random parameter ) and the other one is stochastic ( without drift ).

This method has been used by A. Rascanu [2] for standard Ito equations and by A. Bensoussan, R. Glowinski and A. Rascanu [1] for parabolic stochastic equations.

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# A note on Poisson process generation

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keywords: Poisson process generation, compensator
AMS-classification: 60G55, 65U05

## 1 Introduction

Let $\pi$ be the standard Poisson process. Suppose that $M$ is a counting process with a compensator $A$ defined with respect to the filtration $\mathcal{F}$. Put

$$u_t(A,I) = A_s - s,\ u_t^*(A,I) = \sup_{s\le t}|u_s(A,I)| \text{ and } p_t(\Delta A) = \sum_{s\le t}(\Delta A_s)^2,$$

where $I_s = s$ and $\Delta A_s = A_s - \lim_{u\uparrow s} A_u$. Denote by $||\pi_T - M_T||$ the variational distance of integer valued random variables $\pi_T$ and $M_T$ by $\rho_T(\pi, M)$ the Prohorov distance between the laws of the processes $\pi$ and $M$ in the Skorohod space $d_{[0,T]}$. For a nonegative random variable $X$ put $\nu(X) = \inf_{\epsilon>0}\{P(X \ge \epsilon) \le \epsilon\}$. Below $T$ is a fixed time.

**Theorem 1** (Brown (1982), Kabanov(1983)) *For any $T > 0$ we have*

$$||\pi_T - M_T|| \le E|A_T - T| + E\rho_T(A). \tag{1}$$

It is possible to generalize the bound in Theorem 1 to finite dimensional distributions and also to the variational distance $V_T(\pi, M)$ between the laws of the processes $M$ and $\pi$ in the Skorohod space $d_{[0,T]}$. Using the Hellinger process (see [1])

$$h_t(A,I) = \int_0^t \left(\sqrt{dA/d(A+I)} - \sqrt{dI/d(A+I)}\right)^2 + \sum_{s\le t}(1-\Delta A_s)^2$$

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it is even possible to give upper and lower bounds for $V_T(M,\pi)$. However, when the process $M$ jumps only at predictable times, these bounds are useless to obtain any information about the quality of an approximation. If we use the Prohorov distance instead of the variational distance we have the following

**Theorem 2** (Nikunen and Valkeila (1991)) For any $T > 0$ we have

$$\rho_T(\pi, M) \leq \nu(u_T^*(A, I)) + E\left(|A_T - T| + \Delta A_T + \frac{3}{2} p_T(A)\right). \tag{2}$$

One cannot give a lower bound in terms of the compensators for $\rho_T(\pi, M)$.
We shall give an upper bound for $\rho_T(\pi, M)$ in the special case when $M$ is a renewal counting process. We discuss the accurancy of Poisson process generation.

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# EFFICIENT EMPIRICAL ESTIMATORS FOR MARKOV STEP PROCESSES

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The distribution of a Markov step process is determined by the transition distribution and the mean holding time, which may depend on the state. We suppose that both are unknown, introduce a class of functionals which determines the transition distribution and the mean holding time, and construct estimators for the functionals. Assuming that the process is positive Harris recurrent and aperiodic, and that the mean holding time is bounded and bounded away from zero, we show that the estimators are efficient, as the observation time tends to infinity, in the sense of a convolution theorem. This generalizes a recent result for Markov chains (Greenwood and Wefelmeyer, 1992) which, in turn, generalizes the efficiency result for empirical estimators in the case of independent, identically distributed observations (Levit, 1974).

# Simultaneous Normal Form and Center Manifold Reduction with MAPLE

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To study the random dynamical systems, analogously to the deterministic one, we need the stochastic version of normal form theory. For the random dynamical systems perturbed by small noise the stochastic normal form is studied by many physicists and engineers (e.g. Coullet et al. [4], Sri Namachchivaya and Lin [5], Nicolis and Nicolis [6]). In this case a stochastic normal form is only a small perturbation of the deterministic one. The unified approach is recently presented by Arnold and Xu Kedai [1], [2] and [3], using completely different methods which are based on the multiplicative ergodic theorem.

The main ideas of stochastic normal form can be explained by considering the following example: The noise van der Pol-Duffing oscillator

$$\ddot{y} + \dot{y} - (\alpha + \sigma\xi(\theta_t\omega))y + y^3 + y^2\dot{y} = 0, \tag{1}$$

where $\xi(\theta_t\omega)$ is a zero mean stationary stochastic process and $\sigma$ is a strength parameter. By defining $x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and diagonalizing one can rewrite (1) as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (\alpha + \sigma\xi(\theta_t\omega))\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -(x_1 + \frac{1}{\sqrt{2}}x_2)^3 \\ \sqrt{2}(x_1 + \frac{1}{\sqrt{2}}x_2)^3 \end{pmatrix} \\ &=: f(\theta_t\omega, x, \alpha, \sigma). \end{aligned}$$

Take now a random near identity transformation $x(t,\omega) = H(\theta_t\omega, x_c(t,\omega), x_s(t,\omega), \alpha, \sigma) = \begin{pmatrix} x_c(t,\omega) \\ x_s(t,\omega) \end{pmatrix} + h(\theta_t\omega, x_c(t,\omega), x_s(t,\omega), \alpha, \sigma)$, where $h$ is nonlinear part, we obtain

$$\begin{aligned} \begin{pmatrix} \dot{x}_c \\ \dot{x}_s \end{pmatrix} &= (1 + D_{x_c,x_s}h)^{-1}\left(f(\theta_t\omega, h(\theta_t\omega, x_c, x_s, \alpha, \sigma)) - \frac{\partial h}{\partial t}\right) \\ &=: \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_c \\ x_s \end{pmatrix} + g(\theta_t\omega, x_c, x_s, \alpha, \sigma). \end{aligned}$$

**The aim of normal form theory is:** Try to find out such a transformation $H$, so that the nonlinear part $g$ in above equation is as "simple" as possible, preferably linear.

One of main theorems in [3] tells us that there does exist a transformation $x = H(\omega, x_c, x_s, \alpha, \sigma) = \begin{pmatrix} x_c + h_c(\omega, x_c, x_s, \alpha, \sigma) \\ x_s + h_s(\omega, x_c, x_s, \alpha, \sigma) \end{pmatrix}$ such that the equation

$$\left.\begin{aligned} \dot{x}_c &= g_c(\theta_t\omega, x_c, \alpha, \sigma) \\ \dot{x}_s &= -x_s + g_s(\theta_t\omega, x_c, x_s, \alpha, \sigma) \end{aligned}\right\} \tag{2}$$

is in stochastic normal form. A procedure, with which one can calculate the normal form and center manifold simultaneously, is also given. But the computation efforts of this procedure is enormous so that it is impossible to work with it manually. To overcome this problem we have developed a algorithm which is well suited for symbolic computation and tested by computer algebra program MAPLE.

The idea of this algorithm is: For the above equation, by simple computation we have

$$\begin{pmatrix} g_c \\ g_s \end{pmatrix} = f\left(\theta_t\omega, \begin{pmatrix} x_c \\ x_s \end{pmatrix} + h, \alpha, \sigma\right) + \begin{pmatrix} 0 \\ x_s \end{pmatrix} + (D_{x_c,x_s}h)\begin{pmatrix} -g_c \\ x_s - g_s \end{pmatrix} - \frac{\partial h}{\partial t}.$$

We now develop $f, g$ and $h$ into Taylor form in $x_c x_s \alpha\sigma$. By comparing the coefficients of $x_c x_s \alpha\sigma$ in both sides we obtain then step by step all homological equations, which have to be solved to obtain the coefficients $h_{c,s}^{lmnk}$ and $g_{c,s}^{lmnk}$ of $x_c^l x_s^m \alpha^n \sigma^k$ . For example, for $x_c, x_s$ up to third order and $\alpha, \sigma$ up to second order the normal form on the center manifold is

$$\begin{aligned} \dot{x_c} = g_c \;\; = \;\; & (\xi\sigma + g_c^{1002}\sigma^2 + \alpha + g_c^{1011}\alpha\sigma - \alpha^2)x_c \\ & + (-1 + g_c^{3001}\sigma + g_c^{3002}\sigma^2 + 3\alpha + g_c^{3011}\alpha\sigma - 18\alpha^2)x_c^3, \end{aligned}$$

and the approximate stochastic center manifold is

$$\begin{aligned} M_c(\omega) \;\; = \;\; & \left\{ \begin{pmatrix} x_c \\ (h_s^{1001}\sigma + h_s^{1002}\sigma^2 - \sqrt{2}\alpha + h_s^{1011}\alpha\sigma + 2\sqrt{2}\alpha^2)x_c \\ + \left(\sqrt{2} + h_s^{3001}\sigma + h_s^{3002}\sigma^2 - 7\sqrt{2}\alpha + h_s^{3011}\alpha\sigma - 47\sqrt{2}\alpha^2\right)x_c^3 \end{pmatrix} : x_c \in \mathbb{R}, (\alpha, \sigma) \in \mathbb{R}^2 \right\} \end{aligned}$$

It is worth to point out how the normal form can be used to study the stochastic bifurcation. The Lyapunov exponent of above equation corresponding to the trivial invariant measure $\delta_0$ can be read-off to be $\lambda = \alpha - \alpha^2 - c\sigma^2$, where $c := \int_0^\infty e^s B(s)ds > 0$ and $B(t) := \mathbb{E}\xi(\omega)\xi(\theta_t\omega)$ is the correlation function of the mean-square continuous process $\xi(\theta_t\omega)$. We have proved (see Xu Kedai [7]) that in this case the noise van der Pol-Duffing oscillator undergoes stochastic pithfork bifurcation. That is, for small $|\alpha|$ and $\sigma$, if $\alpha \leq \alpha_c := \frac{1-\sqrt{1-4c\sigma^2}}{2} = c\sigma^2 + O(\sigma^4)$, it has only one invariant measure $\delta_0$; if $\alpha > \alpha_c$, it has another two invariant measures more. Note that in deterministic case, i.e. $\sigma = 0$, the bifurcation point $\alpha_c = 0$.

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