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Feedback stabilization of nonlinear discrete–time systems

Wolfdietrich Müller and Klaus R. Schneider

Abstract. It is the merit of D. Aeyels [4] to have shown a way in which center manifold theory can be used in a constructive manner to find a smooth feedback control for stabilizing an equilibrium of a continuous–time system described by a nonlinear ordinary differential eqution $\dot{x} = f(x, u)$. In this paper we are going to extend Aeyels' approach to nonlinear discrete–time systems described by equations of the type

$$x(k+1) = f(x(k), u(k)), \quad k = 0, 1, 2, \ldots,$$

where we assume that f is sufficiently smooth and satisfies $f(0,0) = 0$. In critical cases, i.e. in situations where the linearization of the system in the neighborhood of the equilibrium includes non–controllable modes, under some non–resonance conditions we derive sufficient conditions for the existence of a smooth nonlinear stabilizing feedback.

Keywords: Discrete–time control system, Smooth feedback stabilization, Center manifold.

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1 Introduction

The problem of local stabilization of equilibria of nonlinear continuous–time control systems by smooth state feedback has a long history [7]. In situations where the linearization of the system in the neighborhood of an equilibrium includes non–controllable modes, this problem can be solved only by a nonlinear feedback. In case that the dimension of the system is not greater than two there is a well–known algorithm to construct such a feedback. It is the merit of D. Aeyels [4] to use center manifold theory [11] to extend this algorithm to higher–dimensional systems. (In its essence this approach corresponds to the use of the Pliss reduction principle [26]). Let the continuous–time system be described by a nonlinear ordinary differential equation

$$\dot{x} = f(x, u) \tag{1.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is supposed to be sufficiently smooth, $f(0,0) = 0$. A vector function $\tilde{u} : \Omega \to \mathbb{R}^m$, where Ω is a neighborhood of the origin in $\mathbb{R}^n$ and $\tilde{u}(0) = 0$, is said to be a (local) *stabilizing feedback control* [7] for system (1.1) if the zero solution of the *closed–loop system* $\dot{x} = f(x, \tilde{u}(x))$ is asymptotically stable. In

Aeyels' paper a constructive approach for finding a smooth stabilizing feedback is presented for systems of the type

$$\dot{x} = f(x) + bu, \quad f(0) = 0 \tag{1.2}$$

where b is an n–vector and u is a scalar control (i.e. $m = 1$). In particular, the case where the matrix $A = (\partial f/\partial x)(0)$ has two purely imaginary eigenvalues, has been extensively discussed.

In this paper we are going to extend Aeyels' approach to nonlinear discrete–time systems described by equations of the type

$$x(k+1) = f(x(k), u(k)), \quad k = 0, 1, 2, \ldots, \tag{1.3}$$

where, as in the continuous–time case, we assume that f is sufficiently smooth and satisfies $f(0,0) = 0$.

Stabilization problems for discrete–time systems have been extensively studied in the last years. For basic results see e.g. the textbooks of J.P. LaSalle [22] and E.D. Sontag [28]. J. Hammer (see [16]) developed in a series of papers a strategy for extending certain results of linear stabilization theory to nonlinear recursion equations. I. Dzesov, G. Leonov and V. Reitmann [12] obtained a frequency criterion for stabilization of discrete–time systems by harmonic external excitation. A.V. Lun'kov [24] investigated the preservation of stabilizability in case of a finite–difference discretization of a continuous–time system. As a powerful tool for obtaining stabilizability results for nonlinear systems, the method of Lyapunov functions was used by many authors, see e.g. papers of P.D. Krut'ko [21], K.K. Lee and A. Araposthatis [23], J. Tsinias [29], J. Tsinias and N. Kalouptsidis [30] and a recent paper of C.I. Byrnes, W. Lin and B.K. Ghosh [10] where sufficient condition for global stabilization of systems of the type $x(k+1) = f(x(k), u(k))$ are obtained. In two papers of Byrnes and Lin [8, 9], stabilizability results based on discrete–time passive systems theory are presented.

In this paper we use an approach based on center manifold theory. After recalling some stability concepts in Section 2 and stabilizability results for linear discrete–time systems in Section 3, we present a general description of this approach in Section 4. The remaining sections are devoted to a detailed study of certain particular critical cases: In Section 5 we discuss the case of a one–dimensional critical subsystem which, as it turns out, in a certain sense covers the situation of a continuous–time system with a two–dimensional critical subsystem investigated by Aeyels [4]; in Section 6 we consider a discrete–time system with a two–dimensional critical subsystem with two simple conjugate complex eigenvalues on the unit circle.

2 Notation

We recall some stability concepts for the uncontrolled discrete–time system

$$x(k+1) = f(x(k)), \quad f : \mathbb{R}^n \to \mathbb{R}^n, \quad f(0) = 0 \tag{2.1}$$

(see e.g. [15], [22]). Let $x(k;x_0)$ denote the solution of (2.1) with initial value $x(0;x_0) = x_0$. The origin of (2.1) is said to be

- *stable* (in the sense of Lyapunov) if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|x_0| < \delta(\varepsilon)$ implies $|x(k;x_0)| < \varepsilon$ for all $k \geq 0$;
- *attractive* if there exists a number $\eta > 0$ such that $|x_0| < \eta$ implies
$$\lim_{k\to\infty} x(k;x_0) = 0;$$
- *asymptotically stable* if it is stable and attractive;
- *exponentially stable* if there exist positive numbers η, γ and q with $0 < q < 1$ such that $|x_0| < \eta$ implies $|x(k;x_0)| < \gamma\, q^k$ for all $k \geq 0$;
- *unstable* if it is not stable.

Consider now the discrete–time control system

$$x(k+1) = f(x(k), u(k)), \quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n. \tag{2.2}$$

From [7] we adapt the following concepts. A point x_1 is said to be *reachable from* x_0 *at step* k_1 if there exists an open–loop control $\{u(0), \ldots, u(k_1 - 1)\}$ such that the corresponding solution of (2.2) starting from x_0 at $k = 0$ reaches the point x_1 at $k = k_1$. Let $R(k_1, x_0)$ denote the set of all points which are reachable from x_0 at step k_1. The set $R(x_0) = \cup_{k_1 \geq 0} R(k_1, x_0)$ is called the *reachable set* of x_0. System (2.2) is said to be *locally controllable at* x_0 if x_0 belongs to the interior of $R(k, x_0)$ for each $k > 0$, it is said to be *completely controllable* if $R(x_0) = \mathbb{R}^n$ for each $x_0 \in \mathbb{R}^n$.

Consider the discrete–time control system (2.2) with $f(0,0) = 0$. It is said to be *locally feedback stabilizable* if there exists a function $\tilde{u} : \Omega \to \mathbb{R}^m$, where Ω is a neighborhood of the origin in $\mathbb{R}^n$ and $\tilde{u}(0) = 0$, such that the zero solution of the closed–loop system $x(k+1) = f(x(k), \tilde{u}(x(k)))$ is asymptotically stable. The function $\tilde{u}$ is called a *stabilizing feedback.*

Let A be a real square matrix. By $\sigma(A)$ we denote the spectrum of the matrix A. We shall use the notation

$$\begin{aligned}
\sigma^-(A) &= \{\lambda \in \sigma(A) : \ |\lambda| < 1\}, \\
\sigma^1(A) &= \{\lambda \in \sigma(A) : \ |\lambda| = 1\}, \\
\sigma^+(A) &= \{\lambda \in \sigma(A) : \ |\lambda| > 1\}.
\end{aligned}$$

The matrix A will be called *stable* if $\sigma(A) = \sigma^-(A)$, and *critical* if $\sigma(A) = \sigma^1(A)$.

In what follows, in order to simplify notation we denote variables on the advanced (i.e. $(k+1)$–th) time level by a "hat" and simply omit the argument on the ground level.

3 Linear systems

In this section we consider the linear system

$$\dot{x} = Ax + Bu \tag{3.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$. The zero solution of the uncontrolled system

$$\dot{x} = Ax$$

is

(*i*) *exponentially stable* if A is stable,

(*ii*) (nonasymptotically) *stable* if $\sigma^+(A) = \emptyset$, $\sigma^1(A) \neq \emptyset$ and all $\lambda_j \in \sigma^1(A)$ are simple,

(*iii*) *unstable* if A does not obey (*i*) or (*ii*)

(see e.g. [15]). Consider now the controlled system (3.1). The matrix

$$C = (B\ AB\ \ldots\ A^{n-1}B)$$

is called the controllability matrix of (3.1). If rank $C = n$ then — as is well known (see e.g. [18], [13]) — (A, B) is a controllable pair, i.e. system (3.1) is completely controllable. In this case for an arbitrarily prescribed set of eigenvalues $\sigma^{(0)} = \{\lambda_1^{(0)}, \ldots, \lambda_n^{(0)}\}$ a matrix $K^{(0)}$ can be found such that $\sigma(A + K^{(0)}B) = \sigma^{(0)}$ holds ("arbitrary pole assignment"). Thus by locating $\sigma^{(0)}$ in the interior of the unit ball $|\lambda| < 1$, it is easily seen that in this case system (3.1) is exponentially stabilizable by the (linear) feedback $\tilde{u}(x) = K^{(0)}x$.

Assume now rank $C = d < n$. Then system (3.1) can be transformed into a system

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u \end{aligned} \tag{3.2}$$

with dim $x_1 = n - d$, dim $x_2 = d$, and (A_{22}, B_2) a controllable pair. Depending on the properties of $\sigma(A_{11})$, three cases have to be distinguished:

(i) $\sigma^+(A_{11}) \neq \emptyset$. Then system (3.2) includes unstable uncontrollable modes. Thus, system (3.1) can not be stabilized.

(ii) $\sigma^+(A_{11}) = \emptyset$, $\sigma^1(A_{11}) = \emptyset$, i.e. A_{11} is a stable matrix. Then system (3.1) is stabilizable by linear feedback.

(iii) $\sigma^+(A_{11}) = \emptyset$, $\sigma^1(A_{11}) \neq \emptyset$. Then system (3.2) includes critical uncontrollable modes. Therefore, the linear system (3.1) can not be stabilized.

When passing to nonlinear systems, it will turn out that the situation of case (iii) is the most interesting one, because in this case under certain assumptions a stabilization by nonlinear feedback will be possible. We note that in this case by
— applying a feedback control $u = K\,x_2 + v$ such that $\sigma(A_{22} + B_2K)$ is located in the interior of the unit circle and is separated from $\sigma(A_{11})$,
— performing a block–diagonalizing linear transformation,
— collecting all critical modes in a vector y_1 and all stable modes in a vector y_2,
system (3.2) can be transformed into a system

$$\begin{aligned} \hat{y}_1 &= A_1\,y_1 \\ \hat{y}_2 &= A_2\,y_2 + B\,v \end{aligned} \tag{3.3}$$

where A_1 is critical and A_2 is stable.

4 Nonlinear systems. Linearization and reduction

Our aim is to derive conditions under which the nonlinear discrete–time system

$$\hat{z} = f(z, u). \tag{4.1}$$

with $f(0,0) = 0$ can be stabilized by a smooth feedback $u = \tilde{u}(z)$. In order to get a more transparent representation of our approach we shall restrict our investigations to systems of the type (1.2), i.e.

$$\hat{z} = f(z) + \tilde{B}u \tag{4.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth, $f(0) = 0$, $\tilde{B}$ is an $n \times m$–matrix, but the same procedure is applicable to the fully nonlinear system (4.1). We rewrite (4.2) as

$$\hat{z} = Az + \tilde{B}u + \text{h.o.t.} \tag{4.3}$$

where $A = Df(0)$ (by Df we denote the Jacobian of f), and h.o.t. means higher order terms in the state variable.

As described in the preceding section, after a linear transformation we can represent (4.3) in the form

$$\begin{aligned} \hat{z}_1 &= A_{11}z_1 + f_1(z_1, z_2) \\ \hat{z}_2 &= A_{21}z_1 + A_{22}z_2 + f_2(z_1, z_2) + B_2u \end{aligned} \tag{4.4}$$

where (A_{22}, B_2) is a controllable pair and f_1, f_2 consist of higher order terms. With respect to $\sigma(A_{11})$, again the three cases (i) to (iii) of the classification from the end of Section 3 have to be distinguished. From well–known results on stability in the first approximation [15], [22], it is clear that the nonlinear system (4.4) can not be stabilized in case (i) and can be locally stabilized by linear feedback in case (ii). So, in what follows, we restrict our investigation to case (iii). As indicated at the end of Section 3, we consider the system

$$\begin{aligned} \hat{x} &= A_1\,x + f(x, y) \\ \hat{y} &= A_2\,y + g(x, y) + Bv \end{aligned} \tag{4.5}$$

assuming

(H_1) A_1 is a critical matrix, that is, all eigenvalues of A_1 are located on the unit circle.

(H_2) A_2 is a stable matrix.

(H_3) f and g are C^l–functions near the origin and satisfy $f(0,0) = 0$, $g(0,0) = 0$, $Df(0,0) = 0$, $Dg(0,0) = 0$.

Our goal is to find a nonlinear feedback

$$v = V(x,y), \quad V(0,0) = 0, \tag{4.6}$$

such that the zero solution of the closed–loop system (4.5), (4.6) is stable. In order to tackle this problem we use the center manifold theorem for maps (see e.g. [25], [17], [19], [6]). With respect to the finite–dimensional system

$$\begin{aligned} \hat{x} &= A_1 x + f(x,y) \\ \hat{y} &= A_2 y + g(x,y) \end{aligned} \tag{4.7}$$

it reads as follows:

Theorem 4.1 (Center manifold theorem). *Consider the mapping $\Psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, given by (4.7). Assume hypotheses (H_1) – (H_3) to be valid. Then there exist an $\varepsilon > 0$ and a C^l–mapping s from $S_\varepsilon := \{x \in \mathbb{R}^{n_1} : \ |x| < \varepsilon\}$ into $\mathbb{R}^{n_2}$, satisfying $s(0) = 0$, $Ds(0) = 0$ such that*

a) the manifold $\Gamma_s = \{(x,y) : \ y = s(x), \ |x| < \varepsilon\}$, is invariant with respect to the mapping Ψ, that is, if for $|x| < \varepsilon$ and $|\hat{x}| < \varepsilon$, $\Psi(x, s(x)) = (\hat{x}, \hat{y})$, then $\hat{y} = s(\hat{x})$,

b) the manifold Γ_s is locally attracting for Ψ, that is, there is a δ such that if $|x| < \varepsilon$, $|y| < \delta$, and if $(x_k, y_k) = \Psi^k(x,y)$ are such that $|x_k| < \varepsilon$, $|y_k| < \delta$ for all $k > 0$ then

$$\lim_{k\to\infty} |y_k - s(x_k)| = 0.$$

The dynamics on the center manifold Γ_s is given by

$$\hat{x} = A_1 x + f(x, s(x)). \tag{4.8}$$

Under the hypotheses of Theorem 4.1 the asymptotic behavior of small solutions of (4.8) determines the behavior of the full system (4.7) near the origin. Concerning the stability of the origin this so–called reduction principle can be formulated as follows

Theorem 4.2 *If the origin of (4.8) is locally asymptotically stable, then the origin of (4.7) is also locally asymptotically stable.*

From the property that Γ_s is an invariant manifold with respect to (4.7), that is, from the validity of the relation

$$\Psi(x, s(x)) = (\hat{x}, s(\hat{x}))$$

we obtain that $s(x)$ has to satisfy the recurrence relation

$$A_2\, s(x) + g(x, s(x)) = s\left(A_1 x + f(x, s(x))\right) \tag{4.9}$$

and the conditions

$$s(0) = 0, \quad Ds(0) = 0. \tag{4.10}$$

This can be used to approximate $s(x)$ by a Taylor expansion. Concerning an approximation of $s(x)$ by a function $h(x)$ the following result holds true.

Theorem 4.3 *If a function $h(x)$ exists with $h(0) = 0$, $Dh(0) = 0$ which, in a certain neighborhood of $x = 0$, satisfies* (4.9) *modulo terms of p-th degree and higher ($p \leq l$), then*

$$s(x) = h(x) + O(|x|^p). \tag{4.11}$$

Our approach to apply Theorem 4.1 – Theorem 4.3 for deriving conditions which guarantee the existence and constructive determination of a smooth stabilizing feedback can be characterized by the following steps.

1. We make a polynomial Ansatz for the feedback $V(x, y)$ as well as for the center manifold $s(x)$.

2. We put our Ansatz polynomials into the recurrence relation (4.9) for the center manifold and equate corresponding coefficients.

3. We derive conditions guaranteeing that free coefficients of the feedback polynomial $V(x, y)$ can be chosen in such a way that the origin of the reduced system (4.8) is asymptotically stable.

We illustrate our approach by means of control systems satisfying

(H_4) dim y = dim u, B is invertible.

Under this assumption we introduce a new control function $w(x)$ by

$$v = B^{-1}(-A_2 y - g(x, y) + w(x)).$$

Substituting this relation into (4.5) we get

$$\begin{array}{rcl} \hat{x} & = & A_1 x + f(x, y) \\ \hat{y} & = & w(x) \end{array} \tag{4.12}$$

(with A_1 critical), and we try to determine $w(x)$ in such a way that the origin of (4.12) is asymptotically stable.

In Section 5 we discuss the simple case where dim $x = 1$, in Section 6 we consider the case where dim $x = 2$ and A_1 has two conjugate complex eigenvalues on the unit circle.

5 One–dimensional critical subsystem

In this section we discuss the case when $\dim x = 1$. Then the notion "A_1 critical" means $A_1 = 1$ or $A_1 = -1$. Assume first $A_1 = 1$, i.e. consider the system

$$\begin{aligned} \hat{x} &= x + f(x,y) \\ \hat{y} &= w(x). \end{aligned} \tag{5.1}$$

We represent $f(x,y)$ in the form

$$f(x,y) = f_{11}\,x^2 + x f_{12} y + (f_{22} y, y) + f_{111}\,x^3 + \text{h.o.t.} \tag{5.2}$$

where f_{12} is an n-vector and f_{22} an $n \times n$-matrix, f_{11} and f_{111} are scalars. Our aim is to derive conditions on the coefficients f_{ij} such that the origin can be stabilized by a smooth feedback. To this end we choose w to be quadratic in x, i.e.

$$w(x) = \alpha\, x^2, \tag{5.3}$$

where α is an n-vector to be determined appropriately. Let the center manifold of (5.1) be represented in the form

$$y = s(x) = a\, x^2 + \text{h.o.t.} \tag{5.4}$$

It is clear that the vector a depends on the feedback w. The recurrence relation (4.9) for system (5.1) reads

$$w(x) = s\,(x + f(x, s(x)))\,,$$

i.e.

$$\alpha\, x^2 = a\,(x + \text{h.o.t.})^2 + \text{h.o.t.} = a\, x^2 + \text{h.o.t.} \tag{5.5}$$

So, equating the coefficients multiplying x^2, we obtain

$$a = \alpha, \tag{5.6}$$

that is, we can assign the lowest order terms of the center manifold arbitrary values by appropriately adjusting the vector α in the quadratic feedback law. For the reduced system (4.8) we obtain from (5.1), (5.2), (5.4), (5.6)

$$\hat{x} = x + f_{11}\, x^2 + (f_{111} + \alpha f_{12})\, x^3 + \text{h.o.t.} \tag{5.7}$$

By investigating the stability properties of (5.7) we get via Theorem 4.2 the following results:

Theorem 5.1 *Consider system* (5.1) *with* $f(x,y)$ *given by* (5.2).

(i) If $f_{11} \neq 0$ *then system* (5.1) *is unstable and can not be stabilized.*

(ii) If $f_{11} = 0,\ f_{111} < 0$ *then the uncontrolled system* $(\alpha = 0)$ *is already asymptotically stable.*

(iii) If $f_{11} = 0,\ f_{111} > 0,\ f_{12} = 0$ *then system* (5.1) *is unstable and can not be stabilized.*

(iv) If $f_{11} = 0,\ f_{111} \geq 0,\ f_{12} \neq 0$ *then system* (5.1) *can be stabilized by a quadratic feedback* (5.3) *satisfying* $\alpha f_{12} + f_{111} < 0$.

Proof Criteria for stability or instability of the zero solution of the scalar equation (5.7) are easily obtained by using Lyapunov functions (cf. [15]). Note that in the discrete–time case the "time derivative" of a Lyapunov function $V(x)$ is $DV(x) = V(\hat{x}) - V(x)$. The zero solution of an autonomous system $\hat{x} = F(x),\ F(0) = 0$, is asymptotically stable if a function V can be found such that $V(x)$ is positive definite and $DV(x)$ is negative definite in some domain $|x| < h$; it is unstable if a function V can be found which has a "region $V < 0$" including points arbitrarily close to the origin and such that $DV(x) < 0$ in the interior of that region. Thus, in case (i) take $V(x) = -f_{11}\,x$ (< 0 if $\operatorname{sgn} x = \operatorname{sgn} f_{11}$), $DV(x) = -f_{11}^2\,x^2 + O(x^3)$, in case (iii) take $V(x) = -x^2,\ DV(x) = 2f_{111}\,x^4 + O(x^6)$; in both cases instability can be concluded independently of the value of α. In case (ii) take $V(x) = x^2,\ DV(x) = 2f_{111}\,x^4 + O(x^6)$; the equilibrium is stable independent of the value of α. In case (iv) take $V(x) = x^2,\ DV(x) = 2\,(f_{111} + \alpha f_{12})\,x^4 + O(x^6)$. Under our assumption we can choose w such that the coefficient $f_{111} + \alpha f_{12}$ is negative and therefore the equilibrium is stable. □

Consider now the case where $A_1 = -1$ holds, i.e. where the critical eigenvalue is $\lambda = -1$. We use the same approach as before, i.e. we consider the system

$$\begin{aligned} \hat{x} &= -x + f(x,y) \\ \hat{y} &= w(x) \end{aligned} \tag{5.8}$$

with (5.2), (5.3), (5.4). Instead of (5.5) we have now $\alpha\,x^2 = a\,(-x + \text{h.o.t.})^2 + \text{h.o.t.} = a\,x^2 + \text{h.o.t.}$, so (5.6) remains unchanged, and investigating the stability properties of the reduced system

$$\hat{x} = -x + f_{11}\,x^2 + (f_{111} + \alpha f_{12})\,x^3 + \text{h.o.t.} \tag{5.9}$$

we obtain

Theorem 5.2 *Consider system* (5.8) *with* $f(x,y)$ *given by* (5.2).

(i) If $f_{11}^2 + f_{111} > 0$ *then the uncontrolled system* ($\alpha = 0$) *is asymptotically stable.*

(ii) If $f_{11}^2 + f_{111} < 0,\ f_{12} = 0$ *then system* (5.8) *is unstable and can not be stabilized.*

(iii) If $f_{11}^2 + f_{111} \leq 0,\ f_{12} \neq 0$ *then system* (5.8) *can be stabilized by a quadratic feedback* (5.3) *satisfying* $f_{11}^2 + f_{111} + \alpha f_{12} > 0$.

Proof We use the Lyapunov function $V(x) = x^2 - f_{11}\,x^3$. This function is positive in a neighborhood of $x = 0$, its "time derivative" with respect to (5.9) is $DV(x) = -2\,(f_{11}^2 + f_{111} + \alpha f_{12})\,x^4 + O(x^5)$. In case (i) $DV(x)$ is negative definite in a neighborhood of $x = 0$, in case (iii) it can be made negative definite by choosing

w such that $f_{11}^2 + f_{111} + \alpha f_{12}$ is positive, therefore in both cases the equilibrium $x = 0$ of (5.9) is stable. To prove instability in case (ii) we remark that in this case the function $-V(x)$ defines a "region $V < 0$" since now $-DV(x)$ is negative definite in a neighborhood of $x = 0$. $\square$

Remark 5.1 Note that an iterated application of the mapping (5.9) leads to

$$\hat{\hat{x}} = x - 2(f_{11}^2 + f_{111} + \alpha f_{12})x^3 + \text{h.o.t.},$$

that is, to a mapping of the type (5.7) where the coefficient of the quadratic term automatically vanishes. Thus, the stability results of Theorem 5.2 could have been obtained as a corollary from Theorem 5.1.

Remark 5.2 An interesting result closely related to Theorem 5.2 is presented in a recent paper of Abed, Wang and Chen [3]. In previous papers of Abed and Fu [1], [2], for continuous–time systems a method of computing stabilizing feedback controls in critical cases had been developed which was connected with a strategy of controlling the bifurcations which may occur in the equilibrium point if the system depends on an additional scalar parameter. In [3] this strategy has been extended to discrete–time systems $\hat{x} = f_\mu(x, u)$ in the specific situation of a period doubling bifurcation. This type of bifurcation occurs in the uncontrolled system $\hat{x} = f_\mu(x, 0)$, $f_\mu(0, 0) = 0$ for all μ, if its linearization at $x = 0$ has an eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = -1$, $\lambda_1'(\mu) \neq 0$ and all remaining eigenvalues have magnitude less than unity. For $\lambda = 0$, in the case where the critical mode of the linearized system is uncontrollable, this system is of a similar type as the systems covered by Theorem 5.2. If a certain inequality is satisfied then the equilibrium of the system $\hat{x} = f_0(x, u)$ can be stabilized by a quadratic feedback, and the same feedback can be used to control the direction of the bifurcation and the stability of the bifurcating period–2 orbit in a certain neighborhood of $\mu = 0$.

Remark 5.3 In what follows we show how Theorem 5.1 can be applied to derive sufficient conditions to stabilize a continuous control system with a two–dimensional critical subsystem.

We consider the control system

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x, y) \\ \frac{dy}{dt} &= Cy + g(x, y) + Bu \end{aligned} \tag{5.10}$$

under the assumptions

(A_1) A is a 2×2–matrix with $\sigma(A) = \{\pm i\omega\}$, $\omega > 0$.

(A_2) Let Ω be a neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R}^n$. $f : \Omega \to \mathbb{R}^2$ and $g : \Omega \to \mathbb{R}^n$ are sufficiently smooth and satisfy $(f, g)(0, 0) = (0, 0)$, $(f, g)_{x,y}(0, 0) = (0, 0)$.

(A_3) (C, B) is a controllable pair.

Assumptions (A_1), (A_3) imply that without loss of generality we can assume

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \qquad \mathrm{Re}\,\sigma(C) < 0. \tag{5.11}$$

Our aim is to find a condition guaranteeing the existence of a feedback control $u = \tilde{u}(x)$ stabilizing (5.10). Let $x^T = (x_1, x_2)$. Concerning $u = \tilde{u}(x)$ we make the Ansatz

$$\tilde{u}(x) = \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 + \alpha_{22}x_2^2 \tag{5.12}$$

where α_{ij} are n–vectors to be determined such that the origin of the closed–loop system is asymptotically stable. Substituting (5.12) into (5.10) we get

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x,y) \\ \frac{dy}{dt} &= Cy + g(x,y) + B\tilde{u}(x). \end{aligned} \tag{5.13}$$

Assumptions $(A_1) - (A_3)$ imply the existence of a center manifold $y = h(x)$ to (5.13) which can be represented in the form

$$h(x) = s_{11}x_1^2 + s_{12}x_1x_2 + s_{22}x_2^2 + \text{h.o.t.} \tag{5.14}$$

where s_{ij} are n–vectors. The behavior of (5.13) on the center manifold is described by the system

$$\frac{dx}{dt} = Ax + f(x, h(x)). \tag{5.15}$$

Under our assumptions, the so–called reduction principle [11] is valid, that is, if the origin of the reduced system (5.15) is asymptotically stable then the origin of the full system (5.13) is also asymptotically stable. Via $h(x)$, system (5.15) depends on the feedback $\tilde{u}(x)$. The problem to be solved is: Under which conditions can we determine the coefficients α_{ij} of the feedback $\tilde{u}(x)$ such that the origin of (5.15) is asymptotically stable?

System (5.15) belongs to the class of dynamical systems

$$\begin{aligned} \dot{z}_1 &= -\omega z_2 + \sum_{2\le i+j\le 3} a_{ij}\, z_1^i z_2^j + \text{h.o.t.}, \\ \dot{z}_2 &= \omega z_1 + \sum_{2\le i+j\le 3} b_{ij}\, z_1^i z_2^j + \text{h.o.t.} \end{aligned} \tag{5.16}$$

By using polar coordinates ($z_1 = r\cos\varphi$, $z_2 = r\sin\varphi$), near the origin system (5.16) is equivalent to the first order equation

$$\frac{dr}{d\varphi} = k_1(\varphi)r + k_2(\varphi)r^2 + k_3(\varphi)r^3 + \text{h.o.t.} \tag{5.17}$$

having $r = 0$ as trivial solution. The behavior of the solutions of (5.17) can be described by means of the *Poincaré map* ψ defined by

$$\psi(\varrho) := R(2\pi; \varrho) \tag{5.18}$$

where $R(\varphi;\varrho)$ denotes the solution of (5.17) satisfying $R(0;\varrho) = \varrho$. In particular, the stability of the origin of system (5.16) can be determined by considering the discrete–time map

$$\hat{\varrho} = \psi(\varrho). \tag{5.19}$$

It is well known [27] that under our conditions the Taylor expansion of ψ at zero reads

$$\psi(\varrho) = \varrho + \alpha_3\,\varrho^3 + \text{h.o.t.} \tag{5.20}$$

Thus, according to Theorem 5.1, the origin of (5.16) is asymptotically stable if the coefficient α_3 is negative. In our situation, α_3 is determined by the coefficients of (5.16) as follows (cf. [27], [5])

$$\begin{aligned}\alpha_3 \;=\; \frac{\pi}{4\omega}\Big\{&\frac{1}{\omega}\left[a_{11}(a_{20}+a_{02}) + 2(a_{02}b_{02} - a_{20}b_{20}) - b_{11}(b_{20}+b_{02})\right]\\ &+ 3(b_{03}+a_{30}) + a_{12} + b_{21}\Big\}.\end{aligned} \tag{5.21}$$

Now we apply this result to the reduced system (5.15). Since α_3 depends only on terms up to order three in the Taylor expansion and since the expansion of the center manifold $h(x)$ starts with terms of order two it is sufficient to represent $f(x,y)$ in the form

$$\begin{aligned} f_1(x,y) &= \sum_{2\le i+j\le 3} \tilde{a}_{ij}\,x_1^i x_2^j + (x_1 a_1 + x_2 a_2)y + \text{h.o.t.}\\ f_2(x,y) &= \sum_{2\le i+j\le 3} \tilde{b}_{ij}\,x_1^i x_2^j + (x_1 b_1 + x_2 b_2)y + \text{h.o.t.}\end{aligned} \tag{5.22}$$

Replacing y by (5.14) and taking into consideration the formula (5.21) for the α_3–coefficient we get that we can control the stability of the origin of (5.15) via the expression

$$\alpha_3 = \tilde{\alpha}_3 + \frac{\pi}{4\omega}\left[3(a_1 s_{11} + b_2 s_{22}) + a_1 s_{22} + a_2 s_{12} + b_1 s_{12} + b_2 s_{11}\right] \tag{5.23}$$

where $\tilde{\alpha}_3$ represents the contributions of the coefficients $\tilde{a}_{ij}$, $\tilde{b}_{ij}$. To investigate the relations between the coefficients s_{ij} of the center manifold and the coefficients α_{ij} of the feedback we use the invariance condition for the center manifold,

$$Ch(x) + g(x,h(x)) + B\tilde{u}(x) = h'(x)(Ax + f(x,h(x))). \tag{5.24}$$

By assumption (A_2) we have

$$g(x,y) = g_{11}x_1^2 + g_{12}x_1x_2 + g_{22}x_2^2 + \text{h.o.t.} \tag{5.25}$$

where g_{ij} are n–vectors and h.o.t. here also include quadratic terms in y and bilinear terms in x, y. Inserting (5.12), (5.14), (5.25) into (5.24) and equating the coefficients of x_1^2, x_1x_2, x_2^2, we obtain the equations

$$\begin{aligned} Cs_{11} + B\alpha_{11} + g_{11} &= \omega\, s_{12}\\ Cs_{12} + B\alpha_{12} + g_{12} &= 2\,\omega\,(s_{22} - s_{11})\\ Cs_{22} + B\alpha_{22} + g_{22} &= -\,\omega\, s_{12}.\end{aligned} \tag{5.26}$$

It is easy to prove that the vectors s_{ij} are unique linear functions L_{ij} of the vector $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_{22})$. Therefore we have:

Lemma 5.3 *If there is an α such that*

$$(3a_1 + b_2)L_{11}(\alpha) + (a_2 + b_1)L_{12}(\alpha) + (a_1 + 3b_2)L_{22}(\alpha) \neq 0 \qquad (5.27)$$

then system (5.10) *with* (5.11) *is stabilizable by a smooth feedback* $\tilde{u}(x)$.

If we consider system (5.10) in the specific situation where $\dim y = 1$, $C = -k$ ($k > 0$), $B = 1$, $g(x, y) = 0$, $\omega = 1$, $f_1(x, y) = 0$, $f_2(x, y)$ given by (5.22) where now b_1, b_2 are some scalar coefficients, then we arrive at the example system

$$\begin{array}{rcl} \dot{x}_1 & = & -x_2 \\ \dot{x}_2 & = & x_1 + f_2(x_1, x_2, y) \\ \dot{y} & = & -ky + \tilde{u}(x_1, x_2) \end{array} \qquad (5.28)$$

discussed in [4]. In this case the equations (5.26) can be written as

$$\begin{pmatrix} k & 1 & 0 \\ -2 & k & 2 \\ 0 & -1 & k \end{pmatrix} \begin{pmatrix} s_{11} \\ s_{12} \\ s_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix}; \qquad (5.29)$$

the coefficient matrix of this system is nonsingular, so s_{11}, s_{12}, s_{22} can be assigned arbitrary values by appropriately adjusting the feedback parameters $\alpha_{11}, \alpha_{12}, \alpha_{22}$. Solving this system we obtain

$$\begin{array}{rcl} s_{11} = L_{11}(\alpha) & = & ((2 + k^2)\alpha_{11} - k\alpha_{12} + 2\alpha_{22})/(k^3 + 4k), \\ s_{12} = L_{12}(\alpha) & = & (2k\alpha_{11} + k^2\alpha_{12} - 2k\alpha_{22})/(k^3 + 4k), \\ s_{22} = L_{22}(\alpha) & = & (2\alpha_{11} + k\alpha_{12} + (2 + k^2)\alpha_{22})/(k^3 + 4k). \end{array}$$

Because of $a_1 = a_2 = 0$, (5.27) reads

$$b_1\, L_{12}(\alpha) + b_2\, (L_{11}(\alpha) + 3L_{22}(\alpha)) \neq 0.$$

It is easily seen that this inequality can be satisfied if $b_1 \neq 0$ or $b_2 \neq 0$. Equation (5.23) reads

$$\alpha_3 = \tilde{\alpha}_3 + \frac{\pi}{4}\Big[b_1 s_{12} + b_2(s_{11} + 3s_{22})\Big],$$

so if $b_1 \neq 0$ we choose the parameters α_{ij} such that $s_{11} = s_{22} = 0$, $s_{12} = -qb_1$, if $b_1 = 0$, $b_2 \neq 0$ we choose the α_{ij} such that $s_{12} = s_{22} = 0$, $s_{11} = -qb_1$. In both cases, by taking $q > 0$ sufficiently large we can guarantee that α_3 is negative and system (5.28) is stabilizable by a quadratic feedback.

6 Two–dimensional critical subsystem

In this section we consider system (4.12) with $\dim x = 2$. We restrict our investigation to the case where A_1 has two conjugate complex eigenvalues with nonzero imaginary parts. While in the continuous–time case the critical eigenvalues are located on the imaginary axis, in the the discrete–time case they are to be located on the unit circle $|\lambda| = 1$, i.e. we have to take $\lambda_{1,2} = e^{\pm i\varphi}$, $\varphi \neq 0$. In order to be able to use certain stability results from [17], [14], we assume

$$\lambda^j \neq 1 \quad \text{for} \quad j = 1, 2, 3, 4. \tag{6.1}$$

Situations where this assumption is violated will be the subject of further investigations.

First we consider the case where $\dim y = 1$. So we suppose system (4.12) to have the form

$$\begin{aligned} \hat{x}_1 &= x_1 \cos\varphi - x_2 \sin\varphi + F(x_1, x_2, y) \\ \hat{x}_2 &= x_1 \sin\varphi + x_2 \cos\varphi + G(x_1, x_2, y) \\ \hat{y} &= w(x_1, x_2) \end{aligned} \tag{6.2}$$

where F, G are assumed to have Taylor expansions

$$\begin{aligned} F(x_1, x_2, y) &= F_{11}x_1^2 + F_{12}x_1x_2 + F_{22}x_2^2 \\ &\qquad + F_{13}x_1y + F_{23}x_2y + F_{33}y^2 + \text{h.o.t.} \\ G(x_1, x_2, y) &= G_{11}x_1^2 + G_{12}x_1x_2 + G_{22}x_2^2 \\ &\qquad + G_{13}x_1y + G_{23}x_2y + G_{33}y^2 + \text{h.o.t.} \end{aligned} \tag{6.3}$$

and w is chosen as a quadratic form in x_1, x_2,

$$w(x_1, x_2) = \alpha x_1^2 + 2\beta x_1 x_2 + \gamma x_2^2. \tag{6.4}$$

The procedure of using center manifold theory is a bit more involved than in the one–dimensional case but runs along the same pattern. Assume the center manifold to be described by

$$y = s(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + \text{h.o.t.} \tag{6.5}$$

The recurrence relation (4.9) for system (6.2) reads

$$w(x_1, x_2) = s(x_1 \cos\varphi - x_2 \sin\varphi + F(x_1, x_2, y),\ x_1 \sin\varphi + x_2 \cos\varphi + G(x_1, x_2, y))$$

(where y is to be understood as an abbreviation for $s(x_1, x_2)$), i.e.

$$\begin{aligned} \alpha x_1^2 &+ 2\beta x_1x_2 + \gamma x_2^2 = a\,(x_1\cos\varphi - x_2\sin\varphi + F(x_1,x_2,y))^2 \\ &+ b\,(x_1\cos\varphi - x_2\sin\varphi + F(x_1,x_2,y))\,(x_1\sin\varphi + x_2\cos\varphi + G(x_1,x_2,y)) \\ &+ c\,(x_1\sin\varphi + x_2\cos\varphi + G(x_1,x_2,y))^2 + \ldots \end{aligned}$$

Equating the coefficients multiplying x_1^2, x_1x_2, x_2^2, we obtain the equations

$$\begin{aligned} \alpha &= a\cos^2\varphi + b\sin\varphi\cos\varphi + c\sin^2\varphi \\ 2\beta &= -2a\sin\varphi\cos\varphi + 2b(\cos^2\varphi - \sin^2\varphi) + 2c\sin\varphi\cos\varphi \\ \gamma &= a\sin^2\varphi - b\sin\varphi\cos\varphi + c\cos^2\varphi \end{aligned}$$

which can be written as a system

$$\begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi & \sin^2\varphi \\ -\sin\varphi\cos\varphi & \cos^2\varphi - \sin^2\varphi & \sin\varphi\cos\varphi \\ \sin^2\varphi & -\sin\varphi\cos\varphi & \cos^2\varphi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}. \tag{6.6}$$

Denote the coefficient matrix of this system by M. An easy calculation gives $\det M = 1-\frac{1}{2}(\sin 2\varphi)^2 > 0$. This implies that, as in the one–dimensional case, the lowest order terms in the Taylor expansion of the center manifold can be assigned arbitrary values by appropriately choosing the values of the parameters α, β, γ in the control law (6.4).

For the reduced system (4.8) we obtain

$$\begin{aligned} \hat{x}_1 &= x_1 \cos\varphi - x_2 \sin\varphi + F(x_1, x_2, ax_1^2 + bx_1x_2 + cx_2^2 + \text{h.o.t.}) \\ \hat{x}_2 &= x_1 \sin\varphi + x_2 \cos\varphi + G(x_1, x_2, ax_1^2 + bx_1x_2^2 + cx_2^2 + \text{h.o.t.}) \end{aligned} \tag{6.7}$$

where F, G are given by (6.3). Investigating the stability of this system we get the following result:

Theorem 6.1 *Consider system* (6.2) *with* F, G *given by* (6.3) *and* $\lambda = e^{i\varphi}$ *satisfying* (6.1). *If at least one of the inequalities*

$$\begin{gathered} (3F_{13} + G_{23}) \cos\varphi + (3G_{13} + F_{23}) \sin\varphi \neq 0 \\ (F_{23} + G_{13}) \cos\varphi + (G_{23} + F_{13}) \sin\varphi \neq 0 \\ (F_{13} + 3G_{23}) \cos\varphi + (G_{13} + 3F_{23}) \sin\varphi \neq 0 \end{gathered} \tag{6.8}$$

is satisfied, then the system is stabilizable by quadratic feedback.

Proof In [14], explicit stability conditions are presented for a system

$$\begin{aligned} \hat{x}_1 &= x_1 \cos\varphi - x_2 \sin\varphi + f(x_1, x_2) \\ \hat{x}_2 &= x_1 \sin\varphi + x_2 \cos\varphi + g(x_1, x_2) \end{aligned} \tag{6.9}$$

where $\lambda = e^{i\varphi}$ satisfies (6.1). It is proved that, by a smooth change of coordinates, this system can be transformed into a normal form which, in polar coordinates r, θ, reads

$$\begin{aligned} \hat{r} &= r + c_3 r^3 + \text{h.o.t.} \\ \hat{\theta} &= \theta + c_1 + c_2 r^2 + \text{h.o.t.} \end{aligned} \tag{6.10}$$

and that the equilibrium of (6.9) is asymptotically stable if the number c_3 in (6.10) is negative. This number is given by

$$c_3 = c_3^{(2)} + c_3^{(3)}$$

where $c_3^{(2)}$ is calculated from the second order derivatives of f, g with respect to x_1, x_2 (for details see [14], p. 163) and $c_3^{(3)}$ is calculated from the third order derivatives and is given by

$$c_3^{(3)} = \mathrm{Re}(\overline{\lambda}\,\xi_{21}), \tag{6.11}$$

$$\xi_{21} = \frac{1}{8}(b_1 + ib_2), \tag{6.12}$$

$$\begin{aligned} b_1 &= \tfrac{1}{2}(f_{x_1x_1x_1} + f_{x_1x_2x_2} + g_{x_1x_1x_2} + g_{x_2x_2x_2}), \\ b_2 &= \tfrac{1}{2}(g_{x_1x_1x_1} + g_{x_1x_2x_2} + f_{x_1x_1x_2} + f_{x_2x_2x_2}). \end{aligned} \tag{6.13}$$

When looking for possibilities to control this number c_3 in the specific situation of system (6.7), one easily sees that this only can be done by assigning suitable values to the lowest order coefficients a, b, c of the center manifold (6.5) (which, as we already stated, can be performed by appropriately choosing the coefficients α, β, γ in the control law (6.4)). Looking more thoroughly on the way in which c_3 depends on a, b, c, one sees that $c_3^{(2)}$ is totally independent of a, b, c, while $c_3^{(3)}$ depends on a, b, c only via terms which are linear in y, i.e. (see (6.3)) via the coefficients $F_{13}, F_{23}, G_{13}, G_{23}$. Inserting (6.5) into (6.3) and abbreviating by c_F, c_G all terms that do not depend on a, b, c, we get

$$F = c_F + F_{13}\left(ax_1^3 + bx_1^2x_2 + cx_1x_2^2\right) + F_{23}\left(ax_1^2x_2 + bx_1x_2^2 + cx_2^3\right),$$
$$G = c_G + G_{13}\left(ax_1^3 + bx_1^2x_2 + cx_1x_2^2\right) + G_{23}\left(ax_1^2x_2 + bx_1x_2^2 + cx_2^3\right),$$

and from (6.13) we obtain

$$\begin{aligned} b_1 &= c_{b_1} + (3F_{13} + G_{23})\,a + (F_{23} + G_{13})\,b + (F_{13} + 3G_{23})\,c, \\ b_2 &= c_{b_2} + (3G_{13} + F_{23})\,a + (G_{23} + F_{13})\,b + (G_{13} + 3F_{23})\,c \end{aligned} \tag{6.14}$$

where again c_{b_1}, c_{b_2} denote (possible) additional terms not depending on a, b, c.

Note that $\overline{\lambda} = \cos\varphi - i\sin\varphi$. So from (6.11), (6.12) we get $c_3^{(3)} = \frac{1}{8}(b_1\cos\varphi + b_2\sin\varphi)$. Inserting (6.14) gives

$$\begin{aligned} c_3^{(3)} = c_{c_3} &+ \left[(3F_{13} + G_{23})\cos\varphi + (3G_{13} + F_{23})\sin\varphi\right] a \\ &+ \left[(F_{23} + G_{13})\cos\varphi + (G_{23} + F_{13})\sin\varphi\right] b \\ &+ \left[(F_{13} + 3G_{23})\cos\varphi + (G_{13} + 3F_{23})\sin\varphi\right] c\,; \end{aligned} \tag{6.15}$$

so $c_3^{(3)}$ can be assigned arbitrary values (and, in particular, sufficiently large negative values in order to make the 'stability number' $c_3 = c_3^{(2)} + c_3^{(3)}$ negative) if at least one of the inequalities (6.8) is satisfied. □

The following lemma gives a complete list of the situations in which all three inequalities (6.8) are violated.

Lemma 6.2 *The equations*

$$\begin{aligned} (3F_{13} + G_{23})\cos\varphi + (3G_{13} + F_{23})\sin\varphi = 0 \\ (F_{23} + G_{13})\cos\varphi + (G_{23} + F_{13})\sin\varphi = 0 \\ (F_{13} + 3G_{23})\cos\varphi + (G_{13} + 3F_{23})\sin\varphi = 0 \end{aligned} \tag{6.16}$$

are simultaneously satisfied if and only if one of the following three exceptional situations occurs:

$$\begin{array}{ll} (i) & G_{13} = F_{13},\ G_{23} = F_{23},\ \tan\varphi = -1; \\ (ii) & G_{13} = -F_{13},\ G_{23} = -F_{23},\ \tan\varphi = 1; \\ (iii) & G_{13} = -F_{23},\ G_{23} = -F_{13},\ \tan\varphi = F_{13}/F_{23}. \end{array} \tag{6.17}$$

Proof Take (6.16) as an overdetermined system for $\cos\varphi$, $\sin\varphi$. In order to have a solution, the rank of the coefficient matrix has to be equal to one, i.e. all second order subdeterminants have to vanish:

$$\begin{aligned}(3F_{13}+G_{23})(G_{23}+F_{13})-(F_{23}+G_{13})(3G_{13}+F_{23}) &= 0\\ (3F_{13}+G_{23})(G_{13}+3F_{23})-(F_{13}+3G_{23})(3G_{13}+F_{23}) &= 0\\ (F_{23}+G_{13})(G_{13}+3F_{23})-(F_{13}+3G_{23})(G_{23}+F_{13}) &= 0.\end{aligned} \tag{6.18}$$

Subtracting the third equation of (6.18) from the first one,we get

$$G_{13}^2+G_{23}^2=F_{13}^2+F_{23}^2, \tag{6.19}$$

and from the second equation we get

$$G_{13}G_{23}=F_{13}F_{23}. \tag{6.20}$$

Assume F_{13}, F_{23} to be given. Then all admissible pairs (G_{13},G_{23}) have to satisfy the equations (6.19) and (6.20). It is easily seen that the four pairs

$$\begin{aligned}&\text{(a)}\quad (G_{13},G_{23})=(F_{13},F_{23}),\\ &\text{(b)}\quad (G_{13},G_{23})=(-F_{13},-F_{23}),\\ &\text{(c)}\quad (G_{13},G_{23})=(F_{23},F_{13}),\\ &\text{(d)}\quad (G_{13},G_{23})=(-F_{23},-F_{13})\end{aligned} \tag{6.21}$$

satisfy these equations. On the other hand, a simple geometric consideration shows that there are no further solutions: In the (G_{13},G_{23})–plane, (6.19) describes a circle and (6.20) describes a hyperbola, so there can be no more than four intersection points.

Thus, looking for solutions of (6.18), we can be sure that only pairs from (6.21) are candidates. But it turns out that not every pair from (6.21) is a solution of (6.18). For the pairs (a), (b), (d) one easily confirms that the equations (6.18) identically vanish. But for pairs of type (c) the first and third equations of (6.18) are satisfied only if the additional condition $F_{13}^2=F_{23}^2$ is fulfilled; these pairs are then of type (a) or (b) as well.

Returning to the equations (6.16), for each type of pairs an additional condition on φ is obtained; pairs of type (a), (b), (d) fit into situations (i), (ii), (iii), respectively, of (6.17). □

Using the result of Lemma 6.2, we obtain from Theorem 6.1

Corollary 6.3 *Consider system* (6.2) *with* F, G *given by* (6.3) *and* $\lambda=e^{i\varphi}$ *satisfying* (6.1). *If* φ *and the coefficients* $F_{13},F_{23},G_{13},G_{23}$ *are such that none of the exceptional situations* (i), (ii), (iii) *from* (6.17) *occurs, then system* (6.2) *is stabilizable by quadratic feedback.*

Now we turn to the general case where $\dim y=n$. In this situation almost everything formally looks as before. We consider system (6.2) with F, G given by

$$\begin{aligned}F(x_1,x_2,y) &= F_{11}x_1^2+F_{12}x_1x_2+F_{22}x_2^2\\ &\qquad +x_1F_{13}y+x_2F_{23}y+(F_{33}y,y)+\text{h.o.t.}\\ G(x_1,x_2,y) &= G_{11}x_1^2+G_{12}x_1x_2+G_{22}x_2^2\\ &\qquad +x_1G_{13}y+x_2G_{23}y+(G_{33}y,y)+\text{h.o.t.}\end{aligned} \tag{6.22}$$

where now $F_{13}, F_{23}, G_{13}, G_{23}$ are n–vectors and F_{33}, G_{33} are $n \times n$–matrices, and a quadratic vector function w given by (6.4). Note that the coefficients α, β, γ in (6.4) and a, b, c in (6.5) are n–vectors. The recurrence relation has to be satisfied componentwise, so by equating coefficients of x_1^2, x_1x_2, x_2^2, we obtain n decoupled systems of type (6.6) each connecting certain components of a, b, c with the corresponding components of α, β, γ. Thus, as before, the lowest order terms in the Taylor expansion of the center manifold can be assigned arbitrary values by appropriately choosing the values of the parameters α, β, γ in the control law (6.4). The proof of Theorem 6.1 runs without changes, only terms like $F_{13}a$ etc. are now scalar products. The statement of Theorem 6.1 remains valid; the inequalities (6.8) are to be interpreted as vector inequalities. Applying Lemma 6.2 to each component of these vector inequalities, we obtain the following final result:

Theorem 6.4 *Consider system* (6.2) *with* $\dim y = n$, *with* F, G *given by* (6.22) *and* $\lambda = e^{i\varphi}$ *satisfying* (6.1). *If* φ *and the coefficient vectors* $F_{13}, F_{23}, G_{13}, G_{23}$ *are such that none of the exceptional situations*

$$\begin{array}{ll} (i) & G_{13}^{(i)} = F_{13}^{(i)},\ G_{23}^{(i)} = F_{23}^{(i)}\ (i = 1, \ldots, n),\ \tan\varphi = -1; \\ (ii) & G_{13}^{(i)} = -F_{13}^{(i)},\ G_{23}^{(i)} = -F_{23}^{(i)}\ (i = 1, \ldots, n),\ \tan\varphi = 1; \\ (iii) & G_{13}^{(i)} = -F_{23}^{(i)},\ G_{23}^{(i)} = -F_{13}^{(i)},\ F_{13}^{(i)}/F_{23}^{(i)} = \tan\varphi\ (i = 1, \ldots, n) \end{array}$$

occurs, then system (6.2) *is stabilizable by quadratic feedback.*

As an illustrating example with $n = 1$, we consider the system

$$\begin{aligned} \hat{x}_1 &= \tfrac{1}{2}x_1 - \tfrac{\sqrt{3}}{2}x_2 + (q\,x_1 + x_2)\,y + \tilde{F}(x_1, x_2) \\ \hat{x}_2 &= \tfrac{\sqrt{3}}{2}x_1 + \tfrac{1}{2}x_2 - (x_1 + q\,x_2)\,y + \tilde{G}(x_1, x_2) \\ \hat{y} &= \tfrac{1}{2}y + u \end{aligned} \tag{6.23}$$

where q is a real parameter and $\tilde{F}(x_1, x_2)$, $\tilde{G}(x_1, x_2)$ are arbitrary nonlinear functions consisting of second and higher order terms. The linearization of (6.23) at $x = y = 0$ has two uncontrollable critical modes with eigenvalues $\lambda_{1/2} = e^{\pm i\pi/3}$ and a stable controllable mode with $\lambda_3 = \frac{1}{2}$. Note that system (6.23) fits into situation (*iii*) of Corollary 6.3. So our result states that (6.23) can be stabilized by quadratic feedback if $F_{13}/F_{23} \neq \tan\varphi$, i.e. if $q \neq \sqrt{3}$. One particular way of doing it is the following: Take $u = -\frac{1}{2}y + w$, $w = w(x_1, x_2) = \frac{\delta}{4}(x_1^2 - 2\sqrt{3}\,x_1x_2 + 3\,x_2^2)$ where δ is a real constant to be fixed later. Then the coefficients of the second order terms of the center manifold (6.5) are $a = \delta$, $b = 0$, $c = 0$, and (6.15) has the form

$$c_3^{(3)} = c_{c_3} + (q - \sqrt{3})\,\delta.$$

The 'stability number' $c_3 = c_3^{(2)} + c_3^{(3)}$ can be made negative by taking $\delta < -|c_3^{(2)} + c_{c_3}|/(q - \sqrt{3})$ if $q > \sqrt{3}$ and by taking $\delta > |c_3^{(2)} + c_{c_3}|/(\sqrt{3} - q)$ if $q < \sqrt{3}$.

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