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An optimal order collocation method for first kind boundary integral equations on polygons

J. Elschner[1] and I.G. Graham[2]

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[1] Institut für Angewandte Analysis
und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany

[2] School of Mathematical Sciences
University of Bath
Bath BA2 7AY
United Kingdom

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Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39
D – 10117 Berlin

Fax: + 49 30 2004975
e-Mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e-Mail (Internet): preprint@iaas-berlin.d400.de

Abstract

This paper discusses the convergence of the collocation method using splines of any order k for first kind integral equations with logarithmic kernels on closed polygonal boundaries in $\mathbb{R}^2$. Before discretization the equation is transformed to an equivalent equation over $[-\pi, \pi]$ using a nonlinear parametrization of the polygon which varies more slowly than arc–length near each corner. This has the effect of producing a transformed equation with a solution which is smooth on $[-\pi, \pi]$. This latter integral equation is shown to be well–posed in appropriate Sobolev spaces. The structure of the integral operator is described in detail, and can be written in terms of certain non–standard Mellin convolution operators. Using this information we are able to show that the collocation method using splines of order k (degree $k-1$) converges with optimal order $O(h^k)$. (The collocation points are the mid–points of subintervals when k is odd and the break–points when k is even, and stability is shown under the assumption that the method may be modified slightly.) Using the numerical solutions to the transformed equation we construct numerical solutions of the original equation which converge optimally in a certain weighted norm. Finally the method is shown to produce superconvergent approximations to interior potentials such as those used to solve harmonic boundary value problems by the boundary integral method. The convergence results are illustrated with some numerical examples.

1 Introduction

In this paper we consider the collocation method using splines of any order k for the integral equation

$$-\frac{1}{\pi}\int_{\Gamma} \log|\boldsymbol{x}-\boldsymbol{\xi}|\, u(\boldsymbol{\xi}) d\Gamma(\boldsymbol{\xi}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \ , \tag{1.1}$$

where $f : \Gamma \rightarrow \mathbb{R}$ is given, $u : \Gamma \rightarrow \mathbb{R}$ is to be found and Γ is a closed polygon in $\mathbb{R}^2$ enclosing a bounded region Ω. This is one of the simplest linear boundary integral equations arising in two dimensional potential theory. It has attracted the attention of many numerical analysts (see the extensive references in Sloan (1992) for example), but despite this a proper explanation of the convergence of the collocation method is still to be given.

The chief difficulty in the analysis of the collocation method for (1.1) is to prove stability, i.e. to demonstrate that the discretization of the integral operator has a bounded inverse in some appropriate sense. This question is completely answered for the case when Γ is smooth (see Arnold & Wendland (1985), Saranen & Wendland (1985), Saranen (1988), Prössdorf & Silbermann (1991) and many later papers cited in Sloan (1992)), but the methods of proof used there do not apply to polygonal Γ. For such Γ the operator on the left hand side of (1.1) can no longer be written as a compact perturbation of a certain well studied isometric isomorphism between H^0 and H^1. Moreover the corners in Γ induce singularities in u and some form of mesh grading is then usually used to restore optimal convergence. A proof of stability of collocation methods for polygonal Γ when graded meshes are being used has so far eluded researchers.

There are some partial results in Costabel & Stephan (1987), Yan (1989), Yan (1990b) and Graham & Yan (1991). In Yan (1989), Yan (1990b) and Graham & Yan (1991) the piecewise constant mid-point collocation method was considered. In Yan (1990b) and in Yan (1989) it was shown that this method is stable and convergent in H^0, provided the angles subtended by Γ at each of its corners are not too close to 0 or 2π. In Graham & Yan (1991) a slightly modified collocation method was considered, allowing these angle restrictions to be removed. However in all of this work there is quite a strong assumption on the uniformity of the meshes which may be used. Hence only suboptimal convergence rates are obtained.

In Costabel & Stephan (1987) the piecewise linear break-point collocation method is considered. Using graded meshes, optimal convergence rates are proved with respect to a weighted $H^{1/2}$ norm. So far these results have not been generalised to higher order splines.

A completely different approach is taken in Chandler (1991) where piecewise constant mid-point collocation for (1.1) is shown to converge in the case of quite general meshes. These results are proved by looking directly at the matrix which arises from the numerical method and use coercivity arguments to show that it is nonsingular. This leads to energy estimates for the error and superconvergence arguments show convergence in Sobolev norms. However at present these results are proved only when Γ is smooth.

In this paper we take a different approach again in which we first reformulate (1.1) in terms of a new unknown with better regularity properties than u. We do this by introducing a nonlinear parametrization $\gamma : [-\pi, \pi] \to \Gamma$ which varies more slowly than arc-length parametrization in the vicinity of each corner of Γ. The equation (1.1) transforms to

$$-\frac{1}{\pi}\int_{-\pi}^{\pi} \log|\gamma(s) - \gamma(\sigma)|\, w(\sigma) d\sigma = g(s) \ , \ s \in [-\pi, \pi] \ , \tag{1.2}$$

where

$$w(\sigma) = |\gamma\prime(\sigma)|\, u(\gamma(\sigma)) \ , \qquad g(s) = f(\gamma(s)) \ . \tag{1.3}$$

By appropriate choice of γ, w can be made smooth local to each corner (provided f is smooth), and hence w can be optimally approximated using splines of any order k on a *uniform* grid. The transformed equation (1.2) is analysed in Section 3. There three main new results – Theorems 1, 2 and 6 – are proved. In particular, in Theorem 2 a careful analysis using Mellin transform techniques shows that the operator

$$Kv(s) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \log|\gamma(s) - \gamma(\sigma)| \ v(\sigma) d\sigma$$

appearing on the left hand side of (1.2) is boundedly invertible from H^0 onto H^1. The proof of Theorem 2 is obtained by first studying the operator $A^{-1}K$ where

$$Av(s) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \log|2\sin(s-\sigma)/2|\, v(\sigma) d\sigma + \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\sigma) d\sigma. \tag{1.4}$$

In Theorem 1 it is shown that in fact

$$A^{-1}K = I + B + E \ , \tag{1.5}$$

where B is a (noncompact) Mellin convolution operator local to each corner, and E is compact. (Thus Theorem 1 generalises earlier results of Yan & Sloan (1988) which are valid only when γ is proportional to arc–length.)

An important consequence of Theorem 2 is Corollary 5 which shows that if γ is properly chosen, then the solution w of (1.2) is smooth, and hence can be well approximated by smooth splines on a uniform grid on $[-\pi, \pi]$. These theoretical results provide the basic prerequisites for a numerical attack on (1.2) using the collocation method. The basis functions used are periodic smooth splines of any order k (degree $k-1$), with collocation points taken to be mid-points of subintervals when k is odd and break-points when k is even. It is well known (see Arnold & Wendland (1985), Saranen & Wendland (1985), Prössdorf & Silbermann (1991)) that this collocation method is stable for the equation $Aw = g$ (which arises when Γ is a circle). In Theorem 9 of §4 we prove the stability and optimal convergence of this collocation method applied to (1.2). As is to be expected when Mellin convolution operators are discretized, the question of stability is rather delicate. Here we follow the spirit of the results in Chandler & Graham (1988), Elschner (1988), Graham & Chandler (1988) and Graham & Yan (1991) and only prove stability of a (possibly) slightly modified collocation method. This modification can be thought of as the discretization of a certain finite section approximation of K (see §4). The modification seems not to be needed in practice, but in any case its use does not affect the convergence of the method. In fact we are able to prove in Theorem 9 that, for h sufficiently small, there exists a unique (possibly modified) collocation solution w_h to (1.2) satisfying the optimal error estimate

$$\| w - w_h \|_0 \leq C h^k \ . \tag{1.6}$$

Here k is the order of the spline basis – e.g. $k = 1$ when piecewise constants are being used.

An important further prerequisite for the stability theory in §4 is the proof that the finite section approximation referred to in the previous paragraph should itself be a stable approximation of K. This result – the third main result in §3 – is established in Theorem 6. The proof proceeds by showing that in fact the operator $A^{-1}K$ is strongly elliptic on H^0.

Since w is defined by (1.3), the estimate (1.6) leads to approximations of u which converge in some weighted H^0 norm. Superconvergence arguments can be used (see Corollary 10) to show that

$$\| w - w_h \|_{-1} \leq C h^{k+\beta} \ , \tag{1.7}$$

where $\beta = 1$ if the collocation method is not modified and $\beta = 1/2$ otherwise. Hence when (1.1) is used to solve Laplace's equation in the usual way, at least $O(h^{k+\frac{1}{2}})$ accurate solutions of the PDE at interior points of the domain are obtained by solving an integral equation using splines of order k.

In §5 experiments with the piecewise constant collocation method ($k = 1$) are reported. These show convergence rates satisfying the estimate (1.6). As for (1.7), when γ is appropriately chosen, we observe in fact that smooth linear functionals of w are approximated by the unmodified collocation method to within $O(h^3)$ (instead of $O(h^2)$ predicted by (1.7)), for the moderate ranges of h which we have used. This is the same rate as has been proved in the case of smooth boundaries by Saranen (1988). It remains an open problem to prove it in the case of the present methods. Nevertheless the present paper presents the first theoretical results on stability and convergence of arbitrarily high order methods for (1.1).

To end this introduction we remark that the idea of resolving singularities in solutions to integral equations by introducing an appropriate change of variable has been proposed at various points in the literature. It is very natural in the case of an integral equation on an open arc, where the use of the cosine transformation (see Yan (1990a)) reduces the unknown to an infinitely smooth function if the data is smooth. This is the starting point for the (global) Chebyshev collocation method for (1.1) on an open arc – see Levesley (1991), Levesley *et al.* (1993), Sloan & Stephan (1993). However as is observed in Levesley *et al.* (1993) the extension of the analysis of this method to the case of a polygon is far from trivial. Change of variable techniques for other sorts of integral equations are considered, for example, in Kress (1990) and Rathsfeld (1988).

The overall plan of the present paper is as follows. The analysis of (1.2) is described in §3. The stability and convergence of the collocation method is given in §4. Numerical experiments are given in §5. The following section contains some necessary preparations.

2 Preliminaries

Let $\mathbb{N}$ denote the positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We assume that the transfinite diameter of Γ is not equal to 1. We assume that Γ has corners $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_r = \boldsymbol{x}_0$ and that for each j, the exterior angle between the sides $\boldsymbol{x}_j - \boldsymbol{x}_{j-1}$ and $\boldsymbol{x}_{j+1} - \boldsymbol{x}_j$ is $(1+\chi_j)\pi$. Let $\hat{\boldsymbol{y}}_j$ denote the unit vector in the direction $\boldsymbol{x}_j - \boldsymbol{x}_{j-1}$. The side joining $\boldsymbol{x}_{j-1}$ to $\boldsymbol{x}_j$ is denoted Γ_j and $|\Gamma_j| = |\boldsymbol{x}_j - \boldsymbol{x}_{j-1}|$ denotes its length. $|\Gamma|$ is the length of Γ.

Our parametrization will be a map $\boldsymbol{\gamma} : [-1,1] \to \Gamma$ defined as follows. First introduce $r+1$ points in $[-\pi, \pi]$:

$$-\pi = S_0 < S_1 < \ldots < S_r = \pi \quad .$$

These will be the preimages of the corner points $\boldsymbol{x}_j$ under the parametrization. Choose a vector $\boldsymbol{q} = (q_1, \ldots, q_r)^T$ of *grading exponents* with $\boldsymbol{q} \geq \mathbf{1} = (1, 1, \ldots, 1)^T$. For notational convenience we extend $\boldsymbol{x}_j$, q_j and S_j to $j \in \mathbb{Z}$ by requiring $\boldsymbol{x}_j$ and q_j to be r–periodic and by defining

$$S_{rk+j} = S_j + 2k\pi \quad , \quad k \in \mathbb{Z} \quad , \quad j = 0, \ldots, r \quad .$$

In addition choose any ε in the range

$$0 < \varepsilon < \frac{1}{2} \min\{S_j - S_{j-1} : j = 1, \ldots, r\} \quad .$$

and let $a > 0$ be a parameter which will be chosen below.

Then, for $j = 1, \ldots, r$, set

$$\gamma(s) = \begin{cases} \boldsymbol{x}_j - a(S_j - s)^{q_j} \hat{\boldsymbol{y}}_j \ , & s \in [S_j - \varepsilon,\ S_j] \ , \\ \boldsymbol{x}_j + a(s - S_j)^{q_j} \hat{\boldsymbol{y}}_{j+1} \ , & s \in [S_j,\ S_j + \varepsilon] \ . \end{cases} \tag{2.1}$$

This leaves $\boldsymbol{\gamma}$ undefined on each $[S_{j-1} + \varepsilon,\ S_j - \varepsilon]$. However we can fill these gaps with smooth connections. This is a standard technique in theory. But, since we are interested in practical algorithms we shall take a little time to give an explicit construction of such connections. Observe that (2.1) implies that for each j,

$$\gamma(s) = \begin{cases} \boldsymbol{x}_{j-1} + \varrho_j^1(s) \hat{\boldsymbol{y}}_j \ , & s \in [S_{j-1},\ S_{j-1} + \varepsilon] \\ \boldsymbol{x}_{j-1} + \varrho_j^2(s) \hat{\boldsymbol{y}}_j \ , & s \in [S_j - \varepsilon,\ S_j] \ , \end{cases}$$

with $\varrho_j^1(s) = a(s - S_{j-1})^{q_{j-1}}$, and $\varrho_j^2(s) = |\Gamma_j| - a(S_j - s)^{q_j}$. It is easy to see by drawing a graph that by choosing $0 < a \le \min_j\{|\Gamma_j|/(S_j - S_{j-1})^{q_j},\ |\Gamma_j|/(S_j - S_{j-1})^{q_{j-1}}\}$ we ensure that $\varrho_j^2 > \varrho_j^1$ on $(S_{j-1},\ S_j)$ for each j. Hence we can introduce a "connecting function" $\lambda(x)$, $x \in \mathbb{R}$ such that λ has $k+1$ continuous derivatives and

$$\begin{aligned} &\lambda(s) = 1,\ s \le S_{j-1} + \varepsilon \ ; \\ &\lambda(s) = 0,\ s \ge S_j - \varepsilon \ ; \\ &\lambda \text{ monotonic decreasing on } [S_{j-1} + \varepsilon,\ S_j - \varepsilon] \ ; \\ &\lambda^{(l)}(S_{j-1} + \varepsilon) = 0 = \lambda^{(l)}(S_j - \varepsilon),\ k + 1 \ge l \ge 1 \ . \end{aligned}$$

(Such a λ is constructed from a spline in Schumaker (1981), p. 141.) It is then easily verified that the function $\varrho_j := \lambda \varrho_j^1 + (1 - \lambda)\varrho_j^2$ agrees with ϱ_j^1 on $[S_{j-1},\ S_{j-1} + \varepsilon]$ and with ϱ_j^2 on $[S_j - \varepsilon,\ S_j]$ and is increasing on $[S_{j-1},\ S_j]$ with $k+1$ continuous derivatives. Defining

$$\gamma(s) = \boldsymbol{x}_{j-1} + \varrho_j(s)(\boldsymbol{x}_j - \boldsymbol{x}_{j-1})$$

(and analogously on each (S_{j-1}, S_j)) then provides a parametrization $\boldsymbol{\gamma}$ of Γ with

$$\gamma'(s) \in C^k(S_{j-1},\ S_j), \quad |\gamma'(s)| \ne 0, \quad s \in (S_{j-1}, S_j)$$

for each j.

With this parametrization (1.1) transforms to (1.2). From now on we abbreviate (1.2) by

$$Kw = g \ . \tag{2.2}$$

The points $\{S_j\}$ must also be chosen so that

$$(S_j - S_{j-1})/2\pi \quad \text{is rational} \tag{2.3}$$

for each j. This ensures that a sequence of uniform meshes on $[-\pi, \pi]$ can be chosen which include $\{S_j\}$ as the mesh points. In addition it would be natural in practice to choose the $\{S_j\}$ so that $(S_j - S_{j-1})/2\pi = |\Gamma_j|/|\Gamma|$. This would mean that the relative lengths of each side of Γ correspond to their relative lengths in parameter space. However there is no such formal requirement in the present theory. Moreover such a choice of $\{S_j\}$ will not be compatible with (2.3) if Γ has sides of irrational length. (In that case $(S_j - S_{j-1})/2\pi$ could typically be chosen as some rational approximation to $|\Gamma_j|/|\Gamma|$ (with arbitrarily high accuracy).

In view of (2.3) then, we can choose $P \in \mathbb{N}$, and $m_j \in \mathbb{N}$ such that

$$P(S_j - S_{j-1}) = 2\pi m_j, \quad j = 1, \ldots, r \ . \tag{2.4}$$

Then, for any $N \in \mathbb{N}$ we can define $n = PN$, $h = 2\pi/n$ and introduce the uniform mesh on $[-\pi, \pi]$:

$$s_i = -\pi + ih, \quad i = 0, \ldots, n \ . \tag{2.5}$$

Observe that when $i = (m_1 + m_2 + \ldots + m_j)N$, we have $s_i = S_j$, so the image of (2.5) under (2.1) includes the corner points $\boldsymbol{x}_j$ and is graded towards them (increasingly sharply) as $\boldsymbol{q}$ increases. Introduce the mid points of subintervals

$$t_i = (s_{i-1} + s_i)/2 \ , \ i = 1, \ldots, n \ .$$

For $k \geq 1$ let V_h^k denote the (smoothest) splines of order k subordinate to the mesh $\{s_i\}$. That is $v \in V_h^k$ if and only if v is 2π–periodic, v reduces to a polynomial of degree $k-1$ on each $[s_{i-1}, s_i]$ and v has $k-2$ continuous derivatives on $[-\pi, \pi]$. (When $k = 1$ the splines are the piecewise constant functions on $[-\pi, \pi]$.) For any $v : [-\pi, \pi] \to \mathbb{R}$ which is well defined at the interpolation points, define the interpolant $Q_h v \in V_h^k$ by requiring

(a) when k is odd $Q_h v(t_i) = v(t_i)$, $i = 1, \ldots, n$;

(b) when k is even $Q_h v(s_i) = v(s_i)$, $i = 0, \ldots, n-1$.

Then the collocation method for (2.2) seeks $w_h \in V_h^k$ such that

$$Q_h K w_h = Q_h g \ . \tag{2.6}$$

We examine the convergence of this solution in §4, after having first proved some analytic properties of K in the following Section.

3 Analytic properties of the transformed equation

Our concern in this section is to prove some analytical results about the operator K and the related operator $A^{-1}K$ which are needed in the stability and convergence analysis of the collocation method.

Let $H^t, t \in \mathbb{R}$, denote the periodic Sobolev space of order t on $[-\pi, \pi]$. Note that $H^0 = L^2[-\pi, \pi]$. Denote the usual norm on H^t by $\|\cdot\|_t$. For any subinterval $J \subset [-\pi, \pi]$, the notation $H^t(J)$ refers to the usual (non-periodic) Sobolev space of order t on J with norm $\|\cdot\|_{t,J}$. Recall that the operator A defined in (1.4) is an isomorphism of H^0 onto H^1, and its inverse is given by

$$A^{-1}v(s) = -HDv(s) + \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\sigma)d\sigma \ , \tag{3.1}$$

where D is the periodic differentiation operator and H denotes the Hilbert singular integral operator

$$Hv(s) = -\frac{1}{2\pi}p.v. \int_{-\pi}^{\pi} \cot\left(\frac{s-\sigma}{2}\right) v(\sigma)d\sigma \ ;$$

see Yan & Sloan (1988). In view of (3.1), we have

$$A^{-1}K = I + B + E \ , \ \text{where } B := -HD(K-A) \ . \tag{3.2}$$

(Here and from now on E denotes a generic compact operator whose value may change from line to line.) It follows from Yan & Sloan (1988) and Graham & Yan (1991) that, for $\boldsymbol{q} = \mathbf{1}$, the operator $D(K-A)$ can be written as the sum of certain Mellin convolution operators and a compact operator. The next theorem shows that a similar decomposition result is true for the operator B in the general case $\boldsymbol{q} \geq \mathbf{1}$.

Theorem 1 . *$B = \sum_{j=1}^{r} B_j + E$, where*

$$B_j v(s) = \begin{cases} \displaystyle\int_{S_j-\varepsilon}^{S_j} b_j^{--}\left(\frac{S_j - s}{S_j-\sigma}\right)\frac{v(\sigma)d\sigma}{(S_j-\sigma)} + \int_{S_j}^{S_j+\varepsilon} b_j^{-+}\left(\frac{S_j - s}{\sigma - S_j}\right)\frac{v(\sigma)d\sigma}{(\sigma-S_j)} , \\ \qquad\qquad s \in [S_j-\varepsilon, \ S_j] \ , \\ \displaystyle\int_{S_j-\varepsilon}^{S_j} b_j^{+-}\left(\frac{s-S_j}{S_j-\sigma}\right)\frac{v(\sigma)d\sigma}{(S_j-\sigma)} + \int_{S_j}^{S_j+\varepsilon} b_j^{++}\left(\frac{s-S_j}{\sigma - S_j}\right)\frac{v(\sigma)d\sigma}{(\sigma-S_j)} , \\ \qquad\qquad s \in [S_j, \ S_j+\varepsilon] \end{cases}$$

and $b_j^{\pm\pm}$ are kernel functions satisfying the estimates

$$\sup_{\mathbb{R}^+}|x^{k+\rho}D^k b_j^{\pm\pm}(x)| < \infty, \quad k \in \mathbb{N}_0, \quad \rho \in (0,1) \ .$$

This result will be shown by applying a localization procedure to the operator B. The proof is then obtained by studying a (2–by–2 matrix–) Mellin convolution operator $\mathcal{B}_j$ on the half–axis which models the behaviour of B local to the corner $\boldsymbol{x}_j$; cf. Elschner (1987) and Prössdorf & Silbermann (1991), Chapter 11 for more general versions of this. The local operator $\mathcal{B}_j$ is constructed by localizing H and $D(K-A)$ to the portion of Γ local to $\boldsymbol{x}_j$ and considering the composition of these two operators. Computing the symbol of $\mathcal{B}_j$ via the Mellin transform, we then obtain the desired estimates for the kernel functions $b_j^{\pm\pm}$ occurring in the decomposition of B.

From this decomposition we are able to prove a theorem on the invertibility of K. To do this we take a closer look at the symbols of the operators $I+\mathcal{B}_j$. After some calculation we find that each of these operators is invertible on $L^2(\mathbb{R}^+)\times L^2(\mathbb{R}^+)$. By gluing the local inverses $(I+\mathcal{B}_j)^{-1}$ together in an appropriate way, we are able to prove that $I+B$ is a Fredholm operator on H^0. Then a homotopy argument is used to show that the index of $I+B$ is zero. Together with the uniqueness of solutions to equation (1.2), this implies

Theorem 2 . *The operator $K: H^0 \to H^1$ has a bounded inverse.*

The proofs of these theorems are given below. To prepare for these proofs, we start by introducing our localization procedure. Choose ε as in §2 and let ψ_j, ψ_j' be 2π–periodic non–negative $\mathcal{C}^\infty$ cut–off functions such that $\psi_j \equiv 1$ in some neighbourhood of S_j, $\psi_j' \equiv 1$ in some neighbourhood of supp ψ_j and supp $\psi_j' \subset [S_j-\varepsilon,\ S_j+\varepsilon]$. Then we shall show below that

$$B = \sum_{j=1}^{r} \psi_j H \psi_j' D(A-K)\psi_j + E \ . \tag{3.3}$$

To verify (3.3), recall that (cf. Yan & Sloan (1988), Section 5)

$$\begin{aligned} D(K-A)v(s) &= -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial s}\log\left| e^{1/2}\,\frac{\gamma(s)-\gamma(\sigma)}{2\sin(s-\sigma)/2}\right| v(\sigma)d\sigma \\ &= \int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\cot\left(\frac{s-\sigma}{2}\right)+c(s,\sigma)\right\}v(\sigma)d\sigma \ , \end{aligned} \tag{3.4}$$

where, for $s \neq S_j$, $j=0,\ldots,r$,

$$c(s,\sigma) := -\frac{1}{\pi}\frac{\gamma_1'(s)(\gamma_1(s)-\gamma_1(\sigma))+\gamma_2'(s)(\gamma_2(s)-\gamma_2(\sigma))}{(\gamma_1(s)-\gamma_1(\sigma))^2+(\gamma_2(s)-\gamma_2(\sigma))^2} \ .$$

Here γ_1, γ_2 are the components of the parametrization γ defined in (2.1). Since γ is a $\mathcal{C}^2$ parametrization on each interval $(S_{j-1},\ S_j)$, the kernel of $D(K-A)$ is uniformly bounded in s and σ as long as the distance of the point (s,σ) to the set $\{(-\pi,\pi),\ (\pi,-\pi)\}\cup\{(S_j,S_j): j=0,\ldots,r\}$ is bounded from below by some positive constant; cf. also Yan & Sloan (1989), Section 5, for the case of the arc–length parametrization. Therefore,

$$D(K-A) = \sum_{j=1}^{r} D(K-A)\psi_j + E = \sum_{j=1}^{r}\psi_j D(K-A)\psi_j + E \ .$$

So from (3.2) we have

$$B = -\sum_{j=1}^{r} H\psi_j D(K-A)\psi_j + E \ .$$

Now, using the identity $\psi_j = \psi_j \psi_j'$ and the fact that $\psi_j H - H\psi_j$ is an integral operator with an infinitely smooth kernel, (3.3) follows.

Now let us look more closely at the jth term in the sum (3.3). This represents the principal part of B local to the jth corner. Without loss of generality we can assume that this is situated at $\boldsymbol{x}_j = \mathbf{0}$ and that $S_j = 0$. Then the parametrization (2.1) (possibly after rotation) takes the form

$$\gamma(s) = \begin{cases} a(-s)^q(-\cos\chi\pi\,,\ \sin\chi\pi) & ,\ s \in [-\varepsilon, 0] \\ a(s^q, 0) & ,\ s \in [0, \varepsilon]\ , \end{cases} \tag{3.5}$$

where for notational convenience we write χ for χ_j and q for q_j. By construction of the cut–off functions we have

$$\text{supp}\ \psi_j \subset \text{supp}\ \psi_j' \subset [S_j - \varepsilon,\ S_j + \varepsilon] = [-\varepsilon, \varepsilon]\ .$$

We omit the lower index j whenever possible, so that ψ_j is denoted ψ, etc. Note that

$$\frac{1}{2\pi}\cot\frac{s-\sigma}{2} - \frac{1}{\pi}\frac{1}{s-\sigma}$$

is a smooth function on $(-\pi,\pi)\times(-\pi,\pi)$. Therefore, using (3.4) we have

$$D(A-K)\psi = (\mathfrak{H} - \mathfrak{C})\psi + E\ ,$$

where E is compact, where

$$\mathfrak{H}v(s) = -\frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{v(\sigma)d\sigma}{s-\sigma}$$

is the Cauchy singular operator on $\mathbb{R}$ and

$$\mathfrak{C}v(s) = \int_{-\infty}^{\infty} c(s,\sigma)v(\sigma)d\sigma\ ,$$

with $c(s,\sigma)$ defined on $\mathbb{R}\times\mathbb{R}$ using the obvious extension of (3.5) to all $s \in \mathbb{R}$. Thus

$$\psi H\psi' D(A-K)\psi = \psi\mathfrak{H}\psi'(\mathfrak{H} - \mathfrak{C})\psi + E\ . \tag{3.6}$$

In view of (3.4) and (3.5), we have (after some calculations)

$$c(s,\sigma) = \begin{cases} -\operatorname{sign}(s)\,\dfrac{q}{\pi}\,\dfrac{|s|^{q-1}}{|s|^q - |\sigma|^q}\,, & s\sigma > 0\,; \\[2ex] -\dfrac{q}{\pi}\,s^{q-1}\dfrac{s^q + (-\sigma)^q \cos\chi\pi}{s^{2q} + \sigma^{2q} + 2s^q(-\sigma)^q\cos\chi\pi}\,, & s > 0,\quad \sigma < 0\,; \\[2ex] \dfrac{q}{\pi}\,(-s)^{q-1}\dfrac{(-s)^q + \sigma^q\cos\chi\pi}{s^{2q} + \sigma^{2q} + 2(-s)^q\sigma^q\cos\chi\pi}\,, & s < 0,\quad \sigma > 0\,. \end{cases} \tag{3.7}$$

To study the operators $\mathfrak{H}$ and $\mathfrak{C}$ on $L^2(\mathbb{R})$, we identify them with 2-by-2-matrices of Mellin convolution operators $\mathcal{H}$ and $\mathcal{C}$ acting on $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$. We do this by introducing the isometry $\Pi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ defined by

$$\Pi v(s) = (v(s), v(-s))^T, \quad s \in \mathbb{R}^+\,.$$

Then, instead of considering $\mathfrak{H}$ and $\mathfrak{C}$, we consider the equivalent operators

$$\mathcal{H} = \Pi\mathfrak{H}\Pi^{-1}\,, \qquad \mathcal{C} = \Pi\mathfrak{C}\Pi^{-1} \tag{3.8}$$

on $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$.

Lemma 3 . *$\mathcal{H}$ and $\mathcal{C}$ are Mellin convolution operators in the sense described in §(vi) of the Appendix with the symbols*

$$\sigma(\mathcal{H})(z) = \begin{pmatrix} \cot\pi z & -1/\sin\pi z \\ 1/\sin\pi z & -\cot\pi z \end{pmatrix}\,,$$

$$\sigma(\mathcal{C})(z) = \begin{pmatrix} \cot\pi\frac{z-1}{q} & \cos\chi\pi\frac{z-1}{q}/\sin\pi\frac{z-1}{q} \\ -\cos\chi\pi\frac{z-1}{q}/\sin\pi\frac{z-1}{q} & -\cot\pi\frac{z-1}{q} \end{pmatrix}\,.$$

Proof. To verify the first formula, we observe that the operator $\mathcal{H}$ on $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ may be written as a matrix of Mellin convolution operators with kernel:

$$\frac{1}{\pi}\begin{pmatrix} (1-x)^{-1} & -(1+x)^{-1} \\ (1+x)^{-1} & -(1-x)^{-1} \end{pmatrix}.$$

The function $\frac{1}{\pi}(1-x)^{-1}$ is the kernel of the Cauchy singular operator on $\mathbb{R}^+$ and has Mellin transform $\cot\pi z$ (see Appendix §(i), (iii)). From standard transform tables the Mellin transform of $\frac{1}{\pi}(1+x)^{-1}$ is $1/\sin\pi z$, and the first part of the lemma follows. To obtain the second part, we similarly write $\mathcal{C}$ as a matrix-valued Mellin convolution operator with kernel

$$c(x) = \begin{pmatrix} c_{++}(x) & c_{+-}(x) \\ c_{-+}(x) & c_{--}(x) \end{pmatrix}$$

where, by virtue of (3.7), we obtain

$$\begin{aligned} c_{--}(x) &= -c_{++}(x) = \frac{q}{\pi}\frac{x^{q-1}}{x^q-1}, \\ c_{+-}(x) &= -c_{-+}(x) = -\frac{q}{\pi}\frac{x^{2q-1}+x^{q-1}\cos\chi\pi}{1+x^{2q}+2x^q\cos\chi\pi}. \end{aligned}$$

To find the Mellin transforms of these functions, first observe the identity

$$\tilde{f}(z) = \int_0^\infty f(x)x^{z-1}dx = \int_0^\infty f(x^q)x^{qz-q}qx^{q-1}dx = \tilde{g}(qz-q+1)$$

with $g(x) := qx^{q-1}f(x^q)$, or equivalently,

$$\tilde{g}(z) = \tilde{f}\left(\frac{z}{q}+1-\frac{1}{q}\right).$$

Using this, we now obtain

$$\tilde{c}_{++}(z) = -\tilde{c}_{--}(z) = \frac{1}{\pi}\left(\widetilde{\frac{1}{1-x}}\right)\left(\frac{z}{q}+1-\frac{1}{q}\right) = \cot\left(\pi\,\frac{z-1}{q}\right).$$

Further, since

$$\frac{1}{\pi}\left(\widetilde{\frac{x+\cos\chi\pi}{1+x^2+2x\cos\chi\pi}}\right)(z) = \frac{\cos(\chi\pi(z-1))}{\sin\pi z},$$

we have

$$\begin{aligned} \tilde{c}_{+-}(z) &= -\cos\left(\chi\pi\frac{z-1}{q}\right)/\sin\left(\pi\left(\frac{z}{q}+1-\frac{1}{q}\right)\right) \\ &= \cos\left(\chi\pi\,\frac{z-1}{q}\right)/\sin\left(\pi\,\frac{z-1}{q}\right) = -\tilde{c}_{-+}(z). \end{aligned}$$

This completes the proof. □

Proof of Theorem 1. It suffices to check the assertion in the neighbourhood $[-\varepsilon,\varepsilon]$ of the corner point $S_j = 0$ (with parametrization given by (3.5) above). We obtain from (3.6) and (3.8) that

$$\begin{aligned} \psi H\psi' D(A-K)\psi &= \psi\Pi^{-1}\mathcal{H}\Pi\psi'\Pi^{-1}(\mathcal{H}-\mathcal{C})\Pi\psi + E \\ &= \psi\Pi^{-1}\mathcal{H}(\mathcal{H}-\mathcal{C})\Pi\psi + E. \end{aligned} \tag{3.9}$$

(In the last step we used the facts that $\psi'\Pi^{-1}(\mathcal{H}-\mathcal{C})\Pi - \Pi^{-1}(\mathcal{H}-\mathcal{C})\Pi\psi'$ is compact – see Appendix §(iv) – and $\psi'\psi = \psi$). Using Lemma 3 and equation (A.6) of the Appendix, it is easy to check that the symbol of $\mathcal{H}-\mathcal{C}$ is of class $\sum^{-\infty}$; see Appendix (ii) for definition. Then since $\sigma(\mathcal{H})(z)$ is bounded on each strip $\delta < \text{Re } z < 1-\delta$, $\delta \in (0,\frac{1}{2})$; cf. (A.6) again, we have

$$\sigma(\mathcal{H}(\mathcal{H}-\mathcal{C})) = \sigma(\mathcal{H})\sigma(\mathcal{H}-\mathcal{C}) \in \Sigma^{-\infty}.$$

Thus the convolution kernel of $\mathcal{H}(\mathcal{H}-\mathcal{C})$ satisfies the estimates (A.4). Thus evaluating (3.9) at $s \in [-\varepsilon, 0]$ and $s \in [0, \varepsilon]$ we get the local expansion of $B_j v(s)$ in the required form. For a general corner at $S_j \neq 0$ the procedure is the same except that instead of Π, the more general isometry

$$\Pi_j v(s) = (v(s-S_j),\ v(S_j - s)), \quad s \in \mathbb{R}^+$$

is used. □

To prepare for the proof of Theorem 2, we now consider the operator $I + \mathcal{B}$ with

$$\mathcal{B} = \mathcal{H}(\mathcal{H}-\mathcal{C}). \tag{3.10}$$

By (3.8), (3.9) $\mathcal{B}$ can be considered as the local representative of the noncompact part of the operator B near the corner point $\boldsymbol{x}_j = \mathbf{0}$.

Lemma 4 . *There exists a constant c such that* $|\det(I+\sigma(\mathcal{B})(z))| \geq c > 0$, $\operatorname{Re} z = 1/2$.

Proof. Note that

$$\begin{aligned}\det(I+\sigma(\mathcal{B})) &= \det(I+\sigma(\mathcal{H}(\mathcal{H}-\mathcal{C}))) \\ &= \det(I+\sigma(\mathcal{H})\sigma(\mathcal{H}) - \sigma(\mathcal{H})\sigma(\mathcal{C})) \\ &= \det(-\sigma(\mathcal{H})\sigma(\mathcal{C})) = \det(\sigma(\mathcal{C}))\ ,\end{aligned}$$

since $\sigma(\mathcal{H})\sigma(\mathcal{H}) = -I$ and $\det\sigma(\mathcal{H}) \equiv 1$. Thus it suffices to check that

$$|\det(\sigma(\mathcal{C})(z))| \geq c > 0\ ,\ \operatorname{Re} z = 1/2\ .$$

To obtain this note that from Lemma 3, we have for $z = 1/2 + i\xi$, $\xi \in \mathbb{R}$

$$\det(\sigma(\mathcal{C})(1/2+i\xi)) = \frac{1}{\sin^2(\pi\frac{i\xi-1/2}{q})}\left\{\cos^2\left(\chi\pi\frac{i\xi-1/2}{q}\right) - \cos^2\left(\pi\frac{i\xi-1/2}{q}\right)\right\}.$$

It is readily shown that, since $|\chi| < 1$,

$$\lim_{\xi\to\pm\infty}\det(\sigma(\mathcal{C})(1/2+i\xi)) = 1.$$

Therefore it is sufficient to verify that

$$\Delta(\xi) := \cos^2\left(\chi\pi\frac{i\xi-1/2}{q}\right) - \cos^2\left(\pi\frac{i\xi-1/2}{q}\right) \neq 0\ ,\ \xi \in \mathbb{R}\ . \tag{3.11}$$

To prove (3.11), observe that

$$\Delta(\xi) = \left(\cosh\frac{\chi\pi\xi}{q}\cos\frac{\chi\pi}{2q} + i\sinh\frac{\chi\pi\xi}{q}\sin\frac{\chi\pi}{2q}\right)^2 - \left(\cosh\frac{\pi\xi}{q}\cos\frac{\pi}{2q} + i\sinh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\right)^2,$$

which is the product of the factors

$$I_\pm := \left\{\cosh\frac{\chi\pi\xi}{q}\cos\frac{\chi\pi}{2q} \pm \cosh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\right\} + i\left\{\sinh\frac{\chi\pi\xi}{q}\sin\frac{\chi\pi}{2q} \pm \sinh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\right\} .$$

Then note that since $q \geq 1$ and $|\chi| < 1$ we have Re $I_+ > 0$, $\xi \in \mathbb{R}$. Also, since

$$\text{Im } I_- = \text{sign}(\xi)\left\{\sinh\frac{|\chi|\pi|\xi|}{q}\sin\frac{|\chi|\pi}{2q} - \sinh\frac{\pi|\xi|}{q}\sin\frac{\pi}{2q}\right\} ,$$

it follows that Im $I_- \neq 0$ unless $\xi = 0$. But Re $I_- \neq 0$ when $\xi = 0$, and (3.11) follows. □

Proof of Theorem 2. Recall the expression (3.2) for $A^{-1}K$.

Step 1. We shall first verify that $I + B$ is a Fredholm operator on H^0. By virtue of (3.3), (3.9) and the construction of ψ_j, this operator takes the form

$$I + B = I + \sum_{j=1}^{r} \psi_j \Pi_j^{-1} \mathcal{B}_j \Pi_j \psi_j + E , \tag{3.12}$$

where the matrix operator $\mathcal{B}_j$ is defined by the procedure which led to (3.10) applied to the general corner point $\boldsymbol{x}_j$. That is $\mathcal{B}_j$ is the local representative of B near S_j, whose symbol is given by Lemma 3 with χ and q replaced by χ_j, and q_j, respectively.

By Lemmas 3 and 4, $(I+\sigma(\mathcal{B}_j)(z))^{-1} = I + a_j(z)$ with some matrix symbol $a_j \in \sum^{-\infty}$. Therefore, the Mellin operator $I + \mathcal{D}_j$ with symbol $I + a_j$ is the inverse of $I + \mathcal{B}_j$ in $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$; see Appendix (vi). Consequently, the operator

$$F := I + \sum_{j=1}^{r} \psi_j \Pi_j^{-1} \mathcal{D}_j \Pi_j \psi_j$$

is a left and right regularizer of $I + B$, i.e.

$$F(I + B) = I + E \ , \ (I + B)F = I + E \ .$$

Here we have used the compactness result of Appendix (iv) (in its matrix form) and the representation (3.12). Thus $I+B$ is a Fredholm operator; see Mikhlin & Prössdorf (1980), Chapter 1.

Step 2. We now show by a homotopy argument that the index of $I + B$ is zero. First observe that the symbol of $\mathcal{C}$ given in Lemma 3 is continuous in $q \geq 1$. This continuity is uniform over Re $z = 1/2$. Hence the symbol of $\mathcal{B}$ defined in (3.10) is similarly continuous in q, uniformly over Re $z = 1/2$ as indeed is the symbol of $\mathcal{B}_j$, the generalisation of $\mathcal{B}$ to any corner. For any $t \in [0, 1]$, consider the operators $\mathcal{B}_{j,t}$ which are defined as $\mathcal{B}_j$, but correspond to the exterior angles $(1+\chi_j)\pi$ and grading exponents $tq_j + (1+t)$, and define $I + B_t$ by replacing $\mathcal{B}_j$ with $\mathcal{B}_{j,t}$ in (3.12). For each j, $\mathcal{B}_{j,t}$ is a homotopy of operators as explained in the Appendix, §(v). Then B_t is also a homotopy of operators (i.e. $t \to B_t$ is continuous). Each B_t is Fredholm by Lemma 4 and Step 1. Furthermore, $I+B_0$ has index 0 (cf. Yan & Sloan (1988)). Since the index is a homotopy invariant (cf. Mikhlin & Prössdorf (1980), Chapter 1, Theorem 3.11), we obtain $\mathrm{ind}(I+B) = \mathrm{ind}(I+B_1) = \mathrm{ind}(I+B_0) = 0$.

Step 3. By (3.2), $K = A(I+B)+E$. Also, as is well known, $A : H^0 \to H^1$ is invertible. Hence, by Atkinson's theorem (Prössdorf & Silbermann (1991), Theorem 3.3) $K : H^0 \to H^1$ is Fredholm with index 0. We now complete the proof by verifying that $K : H^0 \to H^1$ has a trivial kernel. To this end let $v \in H^0$ satisfy

$$Kv(s) = -\int_{-\pi}^{\pi} \log|\gamma(s) - \gamma(\sigma)| v(\sigma) d\sigma = 0 \ . \tag{3.13}$$

Substituting $\sigma = \boldsymbol{\gamma}^{-1}(\boldsymbol{\xi}),\ s = \boldsymbol{\gamma}^{-1}(\boldsymbol{x})$, we obtain

$$\int_\Gamma \log|\boldsymbol{x} - \boldsymbol{\xi}|\ |(\gamma^{-1})'(\boldsymbol{\xi})| v(\gamma^{-1}(\boldsymbol{\xi})) d\Gamma(\boldsymbol{\xi}) = 0\ ,\ \boldsymbol{x} \in \Gamma\ .$$

Now if we can show that

$$u(\boldsymbol{\xi}) := |(\gamma^{-1})'(\boldsymbol{\xi})| v(\gamma^{-1}(\boldsymbol{\xi}))$$

is in $L^p(\Gamma)$ for some $p > 1$ then Lemma 1.1 of Yan & Sloan (1988) will show that $u = 0$ and hence $v = 0$. To do this it is sufficient to consider u local to a corner of Γ. So following the local model (3.5) above, we consider $\boldsymbol{\gamma}(s) = (s^q, 0),\ s \in [0, \varepsilon]$. (The constant a is unimportant.) Then, for $\boldsymbol{\xi} = (x, 0),\ \boldsymbol{\gamma}^{-1}(\boldsymbol{\xi}) = x^{1/q}$ and

$$u(\boldsymbol{\xi}) = (1/q) x^{1/q-1} v(x^{1/q})\ .$$

Hence

$$\int_{\Gamma_\varepsilon} |u(\boldsymbol{\xi})|^p d\Gamma(\boldsymbol{\xi}) = \frac{1}{q^p} \int_0^\varepsilon x^{p(1/q-1)} |v(x^{1/q})|^p dx\ ,$$

where $\Gamma_\varepsilon = \boldsymbol{\gamma}([0,\varepsilon])$. Then using the transformation $x^{1/q} \to x$ and setting $\tilde{\varepsilon} = \varepsilon^{1/q}$ we have

$$\begin{aligned} \int_{\Gamma_\varepsilon} |u(\boldsymbol{\xi})|^p d\Gamma(\boldsymbol{\xi}) &= C \int_0^{\tilde{\varepsilon}} x^{(p-1)(1-q)} |v(x)|^p dx \\ &\leq C \left\{ \int_0^{\tilde{\varepsilon}} x^\delta dx \right\}^{(2-p)/2} \left\{ \int_0^{\tilde{\varepsilon}} |v(x)|^2 dx \right\}^{p/2} \ , \end{aligned}$$

by Hölder's inequality, with $\delta := 2(p-1)(1-q)/(2-p)$. Since $\delta > -1$ for some $p > 1$ sufficiently close to 1, we obtain the result. □

Now by assuming f in (1.1) is sufficiently smooth we can prove the regularity of the solution w of (1.2). This is a direct corollary of the uniqueness of the solution to (1.2) proved in Theorem 2.

Corollary 5 . *Let $k \in \mathbb{N}$ and suppose $f \in H^{k+5/2}(\Gamma)$ (where this denotes the usual Sobolev space on Γ). Then the unique solution of (2.2) satisfies $w \in H^k$, provided $q_j > (k+1/2)(1+|\chi_j|)$ for $j = 1, \ldots, r$.*

Proof. Let u be the unique solution of (1.1). From Maz'ya (1991), Chapter 5 we have

$$u(\boldsymbol{\xi}) = \frac{1}{2}\left\{\frac{\partial U}{\partial n_+}(\boldsymbol{\xi}) - \frac{\partial U}{\partial n_-}(\boldsymbol{\xi})\right\}, \ \boldsymbol{\xi} \in \Gamma$$

where U satisfies $\Delta U = 0$ in $\mathbb{R}^2 \backslash \Gamma$, $U = f$ on Γ and $U(\boldsymbol{x}) = O(\log|\boldsymbol{x}|)$ as $|\boldsymbol{x}| \to \infty$, and where $\partial/\partial n_\pm$ denote the normal derivatives on Γ from inside and outside Γ.

Then by Kondratiev (1967) or Theorem 5.1.3.5 of Grisvard (1985), we have $u \in H^{k+3/2}(\Gamma_j)$ with

$$\frac{d^m}{dt^m}\{u(\boldsymbol{x}_j + t\hat{\boldsymbol{y}}_{j+1})\} \leq c_j t^{\alpha_j - 1 - m}, \ \text{as } t \to 0+,$$

for $m = 0, \ldots, k$, where $\alpha_j = (1 + |\chi_j|)^{-1}$. Hence using (2.1) and an induction argument, we have, for $\sigma \in (S_j, \ S_j + \varepsilon]$,

$$|D^m(u \circ \boldsymbol{\gamma})(\sigma)| \leq C_{k,j}(\sigma - S_j)^{q_j(\alpha_j - 1) - m}.$$

Thus by definition of $\boldsymbol{\gamma}$,

$$|D^m w(\sigma)| \leq C_{k,j,\boldsymbol{q}}(\sigma - S_j)^{q_j\alpha_j - m - 1}. \tag{3.14}$$

An analogous bound holds for $\sigma \in [S_j - \varepsilon, S_j)$. But we also know that $w(\sigma) \in \mathcal{C}^k(S_{j-1}, \ S_j)$. Hence the result follows. □

The analysis of the collocation method in the next section will depend heavily on the stability of a "finite section" approximation to the operator $I + A^{-1}(K - A) = A^{-1}K$. To define such an approximation, introduce, for $\tau < \frac{1}{2}\min\{S_j - S_{j-1} : j = 1, \ldots r\}$, the truncation operator

$$T^\tau v(s) = \begin{cases} 0, & s \in [S_j - \tau, \ S_j + \tau], \quad j = 1, \ldots, r \\ v(s), & \text{otherwise}. \end{cases} \tag{3.15}$$

The finite section approximation is then defined to be $I + A^{-1}(K - A)T^\tau$. Its stability is obtained in the following theorem.

Theorem 6 . *There exist $C > 0$ and $\tau_0 > 0$ such that, for all $\boldsymbol{q} \geq \mathbf{1}$, we have*

$$\|(I + A^{-1}(K - A)T^\tau)v\|_0 \geq C\|v\|_0,$$

for all $v \in H^0$, $0 < \tau \leq \tau^0$.

Remark. Observe that, for a fixed τ, the operator $I + A^{-1}(K - A)T^\tau$ is Fredholm of index zero, and so the above inequality shows this operator is invertible and has an inverse which is uniformly bounded for all $\tau \in (0, \tau^0]$.

Proof. For convenience write $M = A^{-1}(K - A)$. To prove the theorem, first observe that the result is true provided we show that

$$\|T^\tau(I+M)T^\tau v\|_0 \geq C\|T^\tau v\|_0 \ , \ v \in H^0 \ , \ 0 < \tau \leq \tau_0 \ . \tag{3.16}$$

To see why this is the case, observe that the space $\ker(I - T^\tau) \times \ker T^\tau$, equipped with the norm

$$\|(u,v)^T\| = \{\|u\|_0^2 + \|v\|_0^2\}^{\frac{1}{2}}$$

is isometrically isomorphic to H^0 under the map

$$v \to (T^\tau v \ , \ (I - T^\tau)v)^T \ .$$

Correspondingly the map $v \to (I + MT^\tau)v$ may be represented by the matrix operator

$$\begin{pmatrix} T^\tau v \\ (I - T^\tau)v \end{pmatrix} \to \begin{pmatrix} T^\tau(I+M)T^\tau & 0 \\ (I-T^\tau)MT^\tau & I \end{pmatrix} \begin{pmatrix} T^\tau v \\ (I - T^\tau)v \end{pmatrix} .$$

If (3.16) is true then for any $\tau \leq \tau_0$ the operator

$$T^\tau(I+M)T^\tau|_{\ker(I-T^\tau)}$$

is invertible on $\ker(I - T^\tau)$, with inverse bounded independently of τ. Thus the map $v \to (I + MT^\tau)v$ is invertible with inverse represented by the map

$$\begin{pmatrix} T^\tau v \\ (I - T^\tau)v \end{pmatrix} \to \begin{pmatrix} (T^\tau(I+M)T^\tau)^{-1} & 0 \\ -(I-T^\tau)MT^\tau[T^\tau(I+M)T^\tau]^{-1} & I \end{pmatrix} \begin{pmatrix} T^\tau v \\ (I - T^\tau)v \end{pmatrix} .$$

This inverse is bounded on $\ker(I - T^\tau) \times \ker T^\tau$ independently of τ and the theorem then follows.

To obtain (3.16) it is sufficient to verify the *strong ellipticity* estimate

$$\mathrm{Re}\langle (I + M + E_0)v, v\rangle \geq C\|v\|_0^2 \ , \ v \in H^0 \ , \tag{3.17}$$

for some particular compact E_0 where $< \cdot \ , \ \cdot >$ denotes the H^0 scalar product. This is because, by the Céa-Polskii Lemma (see e.g. Prössdorf & Silbermann (1991), page 33) (3.17) implies the stability of any Galerkin scheme for $(I + M)v = f$ in H^0. (Recall that $I + M$ is invertible on H^0 by Theorem 2.) Since the Galerkin method in the space $\ker(I - T^\tau)$ consists of finding $v^\tau \in \ker(I - T^\tau)$ such that

$$T^\tau(I+M)v^\tau = T^\tau f$$

it follows that for all $v^\tau \in \ker(I - T^\tau)$,

$$\|T^\tau(I+M)v^\tau\|_0 \geq C\|v^\tau\|_0 \ ,$$

and (3.16) follows.

To prove (3.17) observe that by definition of M and (3.2),

$$I + M = I + A^{-1}(K - A) = A^{-1}K = I + B + E \ ,$$

with E compact. Hence it suffices to prove the estimates

$$\text{Re } \langle (I+\mathcal{B}_j)\boldsymbol{w}, \boldsymbol{w}\rangle \geq C_j \langle \boldsymbol{w} \ , \ \boldsymbol{w}\rangle^2 \ , \ \boldsymbol{w} \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \ , \tag{3.18}$$

for each $j = 1, \ldots, r$, where $\mathcal{B}_j$ is the local representative of B near $\boldsymbol{x}_j$ as defined in (3.12).

Here $< \cdot \ , \ \cdot >$ now denotes the usual scalar product on $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, i.e.

$$\langle \boldsymbol{w} \ , \hat{\boldsymbol{w}}\rangle = \int_0^\infty (\boldsymbol{w}(x), \hat{\boldsymbol{w}}(x))dx$$

with $(\cdot \ , \ \cdot)$ denoting the inner product on $\mathbb{C}^2$.

As before, without loss of generality we prove (3.18) with $\mathcal{B}_j = \mathcal{B}$ given by (3.10). This assumes $\boldsymbol{x}_j = \boldsymbol{0}$ with local parametrization (3.5). To obtain (3.18), recall $\mathcal{H}^2 = -I$, and so $I + \mathcal{B} = -\mathcal{H}\mathcal{C}$. Parseval's equality (see item (i) of Appendix) and Lemma 3 then show that

$$\begin{aligned}\text{Re}\langle (I+\mathcal{B})\boldsymbol{w}, \boldsymbol{w}\rangle &= \frac{1}{2\pi}\int_{Re\ z\ =\ 1/2} \text{Re } (\sigma(\mathcal{H})(z)\sigma(-\mathcal{C})(z)\tilde{\boldsymbol{w}}(z), \tilde{\boldsymbol{w}}(z))|dz| \\ &= \frac{1}{2\pi}\int_{Re\ z\ =\ 1/2} (\text{Re } [\sigma(\mathcal{H})(z)\sigma(-\mathcal{C})(z)]\tilde{\boldsymbol{w}}(z), \tilde{\boldsymbol{w}}(z))|dz|\end{aligned}$$

where for any complex 2×2 matrix D, $\text{Re}[D] := (D + D^*)/2$. The required result is then a consequence of the technical lemma below which examines the eigenvalues of $\text{Re}[\sigma(\mathcal{H})(z)\sigma(-\mathcal{C})(z)]$. □

Lemma 7 . *For any* $q \geq 1$, $\chi \in (-1, 1)$, *there exists a constant* c *such that the eigenvalues* $\lambda_k(z)$ *of the* 2×2 *matrix* $\text{Re } [\sigma(\mathcal{H})(z)\sigma(-\mathcal{C})(z)]$ *satisfy*

$$\lambda_k(z) \geq c > 0 \ , \ k = 1, 2, \quad \text{Re } z = 1/2 \ .$$

Proof. By Lemma 3, we have for $z = 1/2 + i\xi$, and $\xi \in \mathbb{R}$,

$$\sigma(\mathcal{H})(z) = \begin{pmatrix} -ia(\xi) & b(\xi) \\ -b(\xi) & ia(\xi) \end{pmatrix} , \ \sigma(-\mathcal{C})(z) = \begin{pmatrix} -\alpha(\xi) & -\beta(\xi) \\ \beta(\xi) & \alpha(\xi) \end{pmatrix} ,$$

where

$$a(\xi) = \tanh \pi\xi \ , \ b(\xi) = -1/\cosh \pi\xi \ ,$$

$$\alpha(\xi) = \frac{\cosh(\pi\xi/q) \ \cos(\pi/2q) + i \sinh(\pi\xi/q) \ \sin(\pi/2q)}{i \sinh(\pi\xi/q) \ \cos(\pi/2q) - \cosh(\pi\xi/q) \ \sin(\pi/2q)} ,$$

$$\beta(\xi) = \frac{\cosh(\chi\pi\xi/q) \ \cos(\chi\pi/2q) + i \sinh(\chi\pi\xi/q) \ \sin(\chi\pi/2q)}{i \sinh(\pi\xi/q) \ \cos(\pi/2q) - \cosh(\pi\xi/q) \ \sin(\pi/2q)} .$$

Then

$$\text{Re } [\sigma(\mathcal{H})(z)\sigma(-\mathcal{C})(z)] = \begin{pmatrix} A(\xi) & B(\xi) \\ B(\xi) & A(\xi) \end{pmatrix}$$

with

$$A(\xi) = b(\xi)\text{Re}\,\beta(\xi) - a(\xi)\text{Im}\,\alpha(\xi)\ ,$$
$$B(\xi) = b(\xi)\text{Re}\,\alpha(\xi) - a(\xi)\text{Im}\,\beta(\xi)\ .$$

The eigenvalues $\lambda_k(z)$ are

$$\lambda_1(z) = A(\xi) + B(\xi)\ ,\ \lambda_2(z) = A(\xi) - B(\xi)\ .$$

To estimate these eigenvalues observe that (after some calculations) we have

$$\begin{aligned}
A(\xi) &= C(\xi)\left\{\cosh\frac{\chi\pi\xi}{q}\cos\frac{\chi\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q} - \sinh\frac{\chi\pi\xi}{q}\sin\frac{\chi\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\right\}\\
&+ C(\xi)\sinh\pi\xi\left\{\cosh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q} + \cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\sinh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\right\}\ ,\\
B(\xi) &= C(\xi)\left\{\cosh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q} - \sinh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\right\}\\
&+ C(\xi)\ \sinh\pi\xi\left\{\cosh\frac{\chi\pi\xi}{q}\cos\frac{\chi\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q} + \sinh\frac{\chi\pi\xi}{q}\sin\frac{\chi\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\right\}\ ,
\end{aligned}\tag{3.19}$$

with

$$C(\xi) = (\cosh\pi\xi)^{-1}\left\{\cosh^2\frac{\pi\xi}{q}\sin^2\frac{\pi}{2q} + \sinh^2\frac{\pi\xi}{q}\cos^2\frac{\pi}{2q}\right\}^{-1}\ .$$

Then note that when $z = 1/2 + i\xi$, we have

$$\sigma(\mathcal{H})(z)\sigma(-\mathcal{C})(z) \to \begin{pmatrix} \mp i & 0 \\ 0 & \pm i \end{pmatrix}\begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ ,\ \text{as}\ \ \xi \to \pm\infty\ .$$

Hence we must have $\lambda_k(1/2 \pm i\infty) = 1\ ,\ k = 1, 2$. Since the expressions (3.19) are even functions of ξ and χ, and since $C(\xi) > 0$, the lemma follows provided we verify that for all $q \geq 1$,

$$C(\xi)^{-1}(A(\xi) \pm B(\xi)) > 0\ ,\ \xi \geq 0\ ,\ \chi \in [0, 1)\ . \tag{3.20}$$

To prove (3.20) in case of the plus sign, we observe that since $\chi \in [0, 1)$ and $q \geq 1$, we have

$$\cosh\frac{\chi\pi\xi}{q}\cos\frac{\chi\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q} > \sinh\frac{\chi\pi\xi}{q}\sin\frac{\chi\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\ ,$$
$$\cosh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q} > \sinh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\ .$$

To prove (3.20) for the minus sign, we use the estimates

$$
\begin{aligned}
& \left\{\cosh\frac{\chi\pi\xi}{q}\cos\frac{\chi\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}-\sinh\frac{\chi\pi\xi}{q}\sin\frac{\chi\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\right. \\
& \left.+\sinh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}-\cosh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\right\} \\
> \quad & \left\{\cos\frac{\pi}{2q}\sin\frac{\pi}{2q}\left(\cosh\frac{\chi\pi\xi}{q}\cosh\frac{\pi\xi}{q}-\sinh\frac{\chi\pi\xi}{q}\sinh\frac{\pi\xi}{q}\right)\right. \\
& \left.+\cos\frac{\pi}{2q}\sin\frac{\pi}{2q}\left(\sinh^2\frac{\pi\xi}{q}-\cosh^2\frac{\pi\xi}{q}\right)\right\} \\
= \quad & \frac{1}{2}\sin\frac{\pi}{q}\left(\cosh\frac{(1-\chi)}{q}\pi\xi-1\right)\geq 0\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\{\cosh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}+\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\sinh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\right. \\
& \left.-\cosh\frac{\chi\pi\xi}{q}\cos\frac{\chi\pi}{2q}\sinh\frac{\pi\xi}{q}\cos\frac{\pi}{2q}-\sinh\frac{\chi\pi\xi}{q}\sin\frac{\chi\pi}{2q}\cosh\frac{\pi\xi}{q}\sin\frac{\pi}{2q}\right\} \\
& \geq\left\{\cosh\frac{\pi\xi}{q}\sinh\frac{\pi\xi}{q}-\cosh\frac{\pi\xi}{q}\sinh\frac{\pi\xi}{q}\left(\cos\frac{\chi\pi}{2q}\cos\frac{\pi}{2q}+\sin\frac{\chi\pi}{2q}\sin\frac{\pi}{2q}\right)\right\} \\
& =\cosh\frac{\pi\xi}{q}\sinh\frac{\pi\xi}{q}\left(1-\cos\frac{(1-\chi)\pi}{2q}\right)\geq 0\ . \qquad \square
\end{aligned}
$$

4 Convergence of the Collocation Method

To analyse (2.6) we write it as a non-standard projection method for a certain second kind equation. To do this first define a projection operator $R_h : H^0 \to V_h^k$ as follows. For $v \in H^0$, let $R_h v \in V_h^k$ be the solution of the collocation equations

$$Q_h A(R_h v) = Q_h A v\ . \tag{4.1}$$

It is known (see Arnold & Wendland (1985), Saranen & Wendland (1985), Prössdorf & Silbermann (1991), pp. 492–493) that this prescription defines $R_h v$ uniquely, and there is a constant R such that for all N,

$$\|R_h\|_0 \leq R\ . \tag{4.2}$$

Using the properties of A discussed at the beginning of §3, it is easily seen that w solves (1.2) if and only if $A(I+M)w = g$, or equivalently

$$(I+M)w = e, \tag{4.3}$$

where $M = A^{-1}(K - A)$ and $e = A^{-1}g$. Correspondingly, the collocation method (2.6) may be written $Q_h A(I + M)w_h = Q_h Ae$. Hence w_h solves (2.6) if and only if $Q_h Aw_h = Q_h A(e - Mw_h)$, and by the above definition of R_h, this is equivalent to $w_h = R_h(e - Mw_h)$. Hence

$$(I + R_h M)w_h = R_h e \ . \tag{4.4}$$

When Γ is smooth, M is compact and elementary arguments applied to (4.3), (4.4) prove the convergence of w_h to w. In the present case M is not compact. An analysis of M in the case when the parametrization of Γ is proportional to arc–length ($\boldsymbol{q} = \mathbf{1}$ in (2.1)) led in Yan (1990b), Yan (1989), Graham & Yan (1991) to suboptional estimates for piecewise constant collocation methods on uniform grids. In §3 of the present paper we have given a detailed analysis of the decomposition of M into

$$M = A^{-1}(K - A) = B + E \ , \tag{4.5}$$

where E is compact on H^0 and B is discussed in Theorem 1. These will lead below to a stability theory for the collocation method and to optimal error estimates (obtained by judicious choice of $\boldsymbol{q}$). However, as discussed in the introduction, the stability theory is proved by allowing the possibility that the method be modified slightly. Thus for any integer $i^* \leq (\min_j\{m_j\})N$, let T^{i^*h} be the truncation operator introduced in (3.15) with $\tau = i^*h$. Define

$$K^{i^*h} = A + (K - A)T^{i^*h},$$

and, instead of considering (2.6), consider instead the slightly more general method

$$Q_h K^{i^*h} w_h = Q_h g \ . \tag{4.6}$$

If $i^* = 0$ then (4.6) is equivalent to (2.6). Otherwise, (4.6) can be got from (2.6) by a slight change to the coefficient matrix of the linear system corresponding to (2.6) (see §4 of Graham & Yan (1991)). By mimicking the derivation of (4.4) from (2.6), it is easily shown that (4.6) is equivalent to

$$(I + R_h M^{i^*h})w_h = R_h e \ , \tag{4.7}$$

where

$$M^{i^*h} = A^{-1}(K - A)T^{i^*h}. \tag{4.8}$$

The stability of (4.6) is obtained with the aid of the following Lemma, which generalises the arguments in Graham & Yan (1991).

Lemma 8 . *For fixed $\boldsymbol{q} \geq \mathbf{1}$ and for each $\epsilon > 0$, there exists $i^* \geq 1$ independent of N such that*

$$\|(I - R_h)M^{i^*h}\|_0 < \epsilon,$$

for all N sufficiently large.

Proof. From (4.8) and (4.5), we have

$$M^{i^*h} = (B + E)T^{i^*h}. \tag{4.9}$$

An expansion of B in terms of Mellin convolution operators local to each corner is given in Theorem 1. Using the periodic C^∞ cut–off functions ψ_j with compact support in $(S_j - \varepsilon,\ S_j + \varepsilon)$ as in the proof of Theorem 1, we have

$$B = \sum_{j=1}^{r} \psi_j B_j \psi_j + E$$

with E compact. The kernel of $\psi_j B_j \psi_j$ is now smooth on $[-\pi,\pi] \times [-\pi,\pi] \backslash \{(S_j, S_j)\}$. Substituting into (4.9) gives

$$M^{i^*h} = \left(\sum_{j=1}^{r} \psi_j B_j \psi_j + E \right) T^{i^*h} \ .$$

Since $\|T^{i^*h}\|_0 = 1$ and since $R_h \to I$ pointwise on H^0, we have $\|(I - R_h)ET^{i^*h}\|_0 \to 0$ as $N \to \infty$. Thus the proof will be complete if we can prove that, for any j, and for any $\epsilon > 0$ there exists $i^* \geq 1$ independent of N such that

$$\|(I - R_h)\psi_j B_j \psi_j T^{i^*h}\|_0 < \epsilon \ , \tag{4.10}$$

for all N sufficiently large.

To obtain (4.10), we shall make use of local spline approximation results by quasiinterpolants as used for example in Schumaker (1981), p. 267.

It follows from the (periodic version) of the results there that if f is any 2π–periodic function on $[-\pi,\pi]$ then there exists $Qf \in V_h^k$ such that

$$\|f - Qf\|_{0,(x_i,x_{i+1})} \leq C\|f\|_{0,(x_{i+1-k},x_{i+k})} \ , \quad f \in H^0 \ . \tag{4.11}$$

Moreover a little further work shows the approximation property

$$\|f - Qf\|_{0,(x_i,x_{i+1})} \leq Ch\|Df\|_{0,(x_{i+1-k},x_{i+k})} \ , \quad f \in H^1 \ . \tag{4.12}$$

This may be derived by the arguments in Elschner (1989) for instance.

Then since $R_h v_h = v_h$ for all $v_h \in V_h^k$, we have on using (4.2)

$$\|(I - R_h)\psi_j B_j \psi_j T^{i^*h} v\|_0 \leq C\|(I - Q)(\psi_j B_j \psi_j T^{i^*h} v)\|_0 \ , \quad v \in H^0 \ . \tag{4.13}$$

Now without loss of generality assume that $S_j = 0$, assume that N is large enough to ensure that $i^*h \leq \varepsilon$, and introduce the notation

$$\begin{aligned} \Lambda &= \{i : x_{i+k} \leq -h \text{ or } x_{i+1-k} \geq h\} \ , \\ \Omega &= \cup\{[x_i, x_{i+1}] : \ i \in \Lambda\} \ . \end{aligned}$$

Then (4.12) implies

$$||(I-Q)(\psi_j B_j \psi_j T^{i^*h} v)||_{0,\Omega} \le Ch||D\psi_j B_j \psi_j T^{i^*h} v||_{0,[-\varepsilon,\varepsilon]\setminus[-h,h]} \ , \tag{4.14}$$

where C depends on k but not on h. Now choose any $\rho \in (0, 1/2)$. A short calculation shows that for $s \in [h, \varepsilon]$,

$$|(D\psi_j B_j \psi_j T^{i^*h} v)(s)|$$

$$\le C \max_{0\le l\le 1} \left\{ \int_{-\varepsilon}^{-i^*h} \left| \left(\frac{d}{ds}\right)^l b_j^{+-} \left(\frac{s}{-\sigma}\right) \right| \frac{|v(\sigma)| d\sigma}{-\sigma} + \int_{i^*h}^{\varepsilon} \left| \left(\frac{d}{ds}\right)^l b_j^{++} \left(\frac{s}{\sigma}\right) \right| \frac{|v(\sigma)| d\sigma}{\sigma} \right\}.$$

Hence on calculating the derivatives on the right hand side and using $||v||_0 \le 1$, we have for $s \in [h, \varepsilon]$,

$$|(D\psi_j B_j \psi_j T^{i^*h} v)(s)|$$

$$\le C \max_{0\le l\le 1} \int_{i^*h}^{\varepsilon} \left\{ \left| D^l b_j^{+-} \left(\frac{s}{\sigma}\right) \right| |v(-\sigma)| + \left| D^l b_j^{++} \left(\frac{s}{\sigma}\right) \right| |v(\sigma)| \right\} \sigma^{-l-1} d\sigma$$

$$= C \max_{0\le l\le 1} s^{-l-\rho} \int_{i^*h}^{\varepsilon} \left(\frac{s}{\sigma}\right)^{l+\rho} \left\{ \left| (D^l b_j^{+-}) \left(\frac{s}{\sigma}\right) \right| |v(-\sigma)| + \left| (D^l b_j^{++}) \left(\frac{s}{\sigma}\right) \right| |v(\sigma)| \right\} \sigma^{\rho-1} d\sigma \tag{4.15}$$

$$\le C s^{-1-\rho} \int_{i^*h}^{\varepsilon} \{|v(-\sigma)| + |v(\sigma)|\} \sigma^{\rho-1} d\sigma \le C s^{-1-\rho} \left\{ (i^*h)^{2\rho-1} \right\}^{\frac{1}{2}}.$$

(The second last step uses Theorem 1 and the last step uses the Cauchy–Schwarz inequality.)
Hence using this and a similar bound for $s \in [-\varepsilon, -h]$, we have

$$\begin{aligned} h||D\psi_j B_j \psi_j T^{i^*h} v||_{0,[-\varepsilon,\varepsilon]\setminus[-h,h]} &\le Ch \left\{ \int_h^{\varepsilon} s^{-2-2\rho} ds \right\}^{\frac{1}{2}} \left\{ (i^*h)^{2\rho-1} \right\}^{\frac{1}{2}} \\ &= C \left\{ (i^*)^{2\rho-1} \right\}^{\frac{1}{2}}. \end{aligned} \tag{4.16}$$

Inserting (4.16) into (4.14) yields

$$||(I-Q)(\psi_j B_j \psi_j T^{i^*h} v)||_{0,\Omega} \le C (i^*)^{\rho-1/2} \ . \tag{4.17}$$

This estimates the norm on the right hand side of (4.13) over $\Omega \subset [-\pi, \pi]$. For the remainder of the norm, use (4.11) to obtain

$$||(I-Q)(\psi_j B_j \psi_j T^{i^*h} v)||_{0,[-\pi,\pi]\setminus\Omega} \le C||\psi_j B_j \psi_j T^{i^*h} v||_{0,[-p_h,\, p_h]} \ , \tag{4.18}$$

with $p_h = (2k-1)h$. Now if $s \in [0, p_h]$ and N is sufficiently large we have

$$
\begin{aligned}
\left|(\psi_j B_j \psi_j T^{i^*h} v)(s)\right| &\leq C\left\{\int_{-\epsilon}^{-i^*h}\left|b_j^{+-}\left(\frac{s}{-\sigma}\right)\right|\frac{|v(\sigma)|d\sigma}{-\sigma}+\int_{i^*h}^{\epsilon}\left|b_j^{++}\left(\frac{s}{\sigma}\right)\right|\frac{|v(\sigma)|d\sigma}{\sigma}\right\} \\
&= C\int_{i^*h}^{\epsilon}\left\{\left|b_j^{+-}\left(\frac{s}{\sigma}\right)\right||v(-\sigma)|+\left|b_j^{++}\left(\frac{s}{\sigma}\right)\right||v(\sigma)|\right\}\frac{d\sigma}{\sigma} \\
&= Cs^{-\rho}\int_{i^*h}^{\epsilon}\left(\frac{s}{\sigma}\right)^{\rho}\left\{\left|b_j^{+-}\left(\frac{s}{\sigma}\right)\right||v(-\sigma)|+\left|b_j^{++}\left(\frac{s}{\sigma}\right)\right||v(\sigma)|\right\}\sigma^{\rho-1}d\sigma \\
&\leq Cs^{-\rho}\int_{i^*h}^{\epsilon}\{|v(-\sigma)|+|v(\sigma)|\}\sigma^{\rho-1}d\sigma
\end{aligned}
$$

for $\rho \in (0,1/2)$, where we have used Theorem 1. Hence, as in the derivation of (4.15), we have

$$
\left|\left(\psi_j B_j \psi_j T^{i^*h} v\right)(s)\right| \leq Cs^{-\rho}\left\{(i^*h)^{2\rho-1}\right\}^{1/2} .
$$

Thus

$$
\begin{aligned}
\|\psi_j B_j \psi_j T^{i^*h} v\|_{0,[0,p_h]} &\leq C\left\{\int_0^{(2k-1)h} s^{-2\rho}ds\right\}^{1/2}(i^*h)^{\rho-1/2} \\
&\leq Ch^{1/2-\rho}(i^*h)^{\rho-1/2} = C(i^*)^{\rho-1/2} .
\end{aligned}
\tag{4.19}
$$

A similar bound holds on $[-p_h, 0]$. Substituting (4.19) into (4.18) and combining with (4.17) proves (4.10). □

Theorem 9 . *Suppose the hypothesis of Corollary 5 holds and suppose for $j = 1,\ldots,r$, we have $q_j > (k+1/2)\,(1+|\chi_j|)$. Then there exists i^* such that (4.6) has a unique solution for all N sufficiently large and*

$$
\|w - w_h\|_0 \leq Ch^k
$$

where C is a constant which depends on w and i^* but is independent of N.

Proof. By the Remark following the statement of Theorem 6, if i^* is fixed then the finite section operator $I + M^{i^*h} = I + A^{-1}(K-A)T^{i^*h}$ is invertible for N sufficiently large, and has an inverse which is bounded independently of N. Then since

$$
I + R_h M^{i^*h} = (I + M^{i^*h}) - (I - R_h)M^{i^*h},
$$

Lemma 8 shows that there exists fixed i^* such that for N sufficiently large, $(I + R_h M^{i^*h})^{-1}$ exists and $\|(I + R_h M^{i^*h})^{-1}\|_0 \leq C$, with C independent of N. Then comparing (4.3) and (4.7), we have

$$
\begin{aligned}
w - w_h &= (I + R_h M^{i^*h})^{-1}\,(w + R_h M^{i^*h} w - R_h e) \\
&= (I + R_h M^{i^*h})^{-1}\left\{(w - R_h w) + R_h(M^{i^*h} - M)w\right\}.
\end{aligned}
$$

Hence taking norms and using (4.2) and Theorem 2,

$$\begin{aligned} \|w - w_h\|_0 &\leq C\left\{\|w - R_h w\|_0 + \|A^{-1}(K-A)\|_0 \, \|(T^{i^*h} - I)w\|_0\right\} \\ &\leq C\left\{\|w - R_h w\|_0 + \|(I - T^{i^*h})w\|_0\right\} \\ &\leq C\left\{\|w - v_h\|_0 + \|(I - T^{i^*h})w\|_0\right\}, \end{aligned} \tag{4.20}$$

for any $v_h \in V_h^k$, where the final inequality uses $R_h v_h = v_h$ and (4.2). Now by choosing v_h to satisfy the usual approximation property for smooth splines (see, e.g., Prössdorf & Silbermann (1991), p. 44), we have

$$\|w - v_h\|_0 \leq C h^k \|w\|_k. \tag{4.21}$$

Also from the last line of the proof of Corollary 5 we have

$$\int_{S_j - i^*h}^{S_j + i^*h} |w(\sigma)|^2 d\sigma \leq C \int_{S_j}^{S_j + i^*h} (\sigma - S_j)^{2\rho_j} d\sigma,$$

where $\rho_j = q_j/(1 + |\chi_j|) - 1$. Now by choice of q_j stated in the hypothesis we have $\rho_j > k - \frac{1}{2}$ and so

$$\int_{S_j - i^*h}^{S_j + i^*h} |w(\sigma)|^2 d\sigma \leq C(i^*h)^{2\rho_j + 1} \leq C(i^*)^{2k} h^{2k} .$$

This inequality is true for all $j = 1, \ldots, r$ and so

$$\|(I - T^{i^*h})w\|_0 \leq C h^k . \tag{4.22}$$

Combining this with (4.21) in (4.20) yields the result. □

Since w_h approximates w, with w defined by (1.3), and since u is the solution of the original boundary integral equation, it is of interest to construct approximations to u from w_h and to examine their accuracy. Since the parametrization $\boldsymbol{\gamma}$ defines a bijection between $[-\pi, \pi]$ and Γ, we can define an approximation u_h to u by setting

$$u_h(\boldsymbol{\gamma}(\sigma)) = |\boldsymbol{\gamma}'(\sigma)|^{-1} w_h(\sigma).$$

Now let $\lambda_{\boldsymbol{q}}$ be the bijective map on $[-\pi, \pi]$ with the property that

$$\boldsymbol{\gamma}(\lambda_{\boldsymbol{q}}(\sigma)) = \boldsymbol{x}_j + \left(\frac{s - S_j}{S_{j+1} - S_j}\right)(\boldsymbol{x}_{j+1} - \boldsymbol{x}_j), \; s \in [S_j, \; S_{j+1}] .$$

(That is $(\boldsymbol{\gamma} \circ \lambda_{\boldsymbol{q}})\,(\sigma)$ is a constant multiple of arc length on each side of Γ.) Then

$$\begin{aligned} &\int_{-\pi}^{\pi} |u((\boldsymbol{\gamma} \circ \lambda_{\boldsymbol{q}})(\sigma)) - u_h((\boldsymbol{\gamma} \circ \lambda_{\boldsymbol{q}})(\sigma))|^2 \; |(\boldsymbol{\gamma}' \circ \lambda_{\boldsymbol{q}})(\sigma)|^2 \; |\lambda_{\boldsymbol{q}}'(\sigma)| d\sigma \\ &\qquad = \int_{-\pi}^{\pi} |w(\sigma) - w_h(\sigma)|^2 d\sigma = O(h^{2k}) . \end{aligned} \tag{4.23}$$

Thus (4.23) shows that $u_h \to u$ with $O(h^k)$ in a certain weighted H^0 norm. It is easily seen that the weight vanishes with $O(|s - S_j|^{1-1/q_j})$ as $s \to S_j$ for each $j = 1, \ldots, r$, which is natural since the solution has a singularity at each of those points.

To conclude this section we state a Corollary of Theorem 9. Its proof is entirely analogous to the proof of Theorem 8 of Graham & Yan (1991).

Corollary 10 . *Under the hypothesis of Theorem 9,*

$$\|w - w_h\|_{-1} \le C h^{k+\beta}$$

where $\beta = 1$ *if* $i^* = 0$, *and* $\beta = \frac{1}{2}$ *if* $i^* \ge 1$.

5 Numerical Experiments

To illustrate the convergence results in Theorem 9 and Corollary 10 we solved equation (1.1) when Γ is the boundary given by

$$\gamma_1(s) = \sin s \ \ (\cos(1-\chi)s \ , \ \ \sin(1-\chi)s)^T \ , \ \ s \in [0, \pi] \ .$$

This is the boundary of a "teardrop-shaped" region which has a single corner at $s = 0$ (or $s = \pi$) and is smooth elsewhere. The exterior angle between the tangents at $s = 0$ is $(1 + \chi)\pi$. (In Fig.1 this boundary is depicted for the special case $\chi = 3/4$.). This contour is most easily described using this parametrization, but clearly a simple scaling would transfer it to $s \in [-\pi, \pi]$ as assumed in the theoretical sections of this paper.

The results of Sections 3 and 4 are given for polygonal boundaries only. (This is the simplest case which is relevant to engineering calculations.) However using perturbation arguments it should be possible to derive the same results for curvilinear polygons (as found in Atkinson & Graham (1992) for second kind equations, for example). So this example presents a simple model problem with one corner which is a reasonable test of the validity of Theorem 9 and Corollary 10. From Grisvard (1985) we expect that

$$u(\gamma_1(\sigma)) = \begin{cases} O(\sigma)^{\alpha-1} \ , & \sigma \to 0+ \ , \\ O(\pi - \sigma)^{\alpha-1} \ , & \sigma \to \pi- \ , \end{cases}$$

where

$$\alpha = (1 + |\chi|)^{-1} \ . \qquad (5.1)$$

To reformulate (1.1) in terms of a solution which is smoother at $\sigma = 0$, $\sigma = \pi$, we choose a grading exponent $q \ge 1$, and set

$$\gamma(s) = \begin{cases} \gamma_1((\pi/2)^{1-q} s^q) \ , & s \in [0, \pi/2] \ , \\ \gamma_1(\pi - (\pi/2)^{1-q}(\pi - s)^q) \ , & s \in [\pi/2, \pi] \ . \end{cases}$$

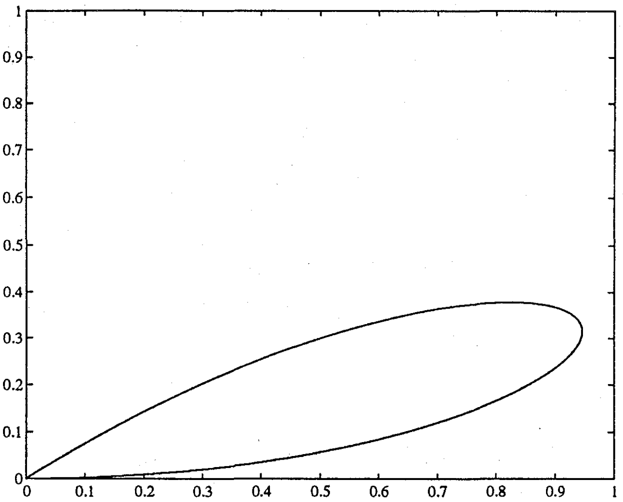

Figure 1: The contour Γ for the case $\chi = 3/4$

It is easily seen that $\gamma(s)$ is $O(s^q)$ as $s \to 0+$, $O(\pi - s)^q$ as $s \to \pi-$, and that γ'' is smooth on $[0, \pi/2]$ and on $[\pi/2, \pi]$, but suffers a jump discontinuity at $\pi/2$. Thus we have $w := |\gamma'| u \circ \gamma \in H^1$, provided

$$q > 3/2\alpha \ , \tag{5.2}$$

with α given by (5.1). So it is expected that the theoretical results proved in Theorem 9 and Corollary 10 will hold when $k = 1$. (A parametrization which is smoother on $(0, \pi)$ would be needed if we wanted to use higher order splines.)

We solved the reformulated equation (2.2) by the (unmodified) piecewise constant mid–point collocation method (i.e. (2.6) with $k = 1$). Then analogously to Theorem 9 we expect

$$\|w - w_h\|_0 = O(h^{\alpha - 1/2}) \tag{5.3}$$

when $q = 1$, but

$$\|w - w_h\|_0 = O(h) \tag{5.4}$$

when $q > 3/2\alpha$. From Corollary 10 we expect

$$|(v, w) - (v, w_h)| = O(h^{\alpha + 1/2}) \ , \tag{5.5}$$

when $q = 1$, and

$$|(v, w) - (v, w_h)| = O(h^2) \tag{5.6}$$

when $q > 3/2\alpha$.

The implementation of the collocation method (2.6) requires calculation of the integrals

$$K_{ij} := \int_{s_{j-1}}^{s_j} \log|\gamma(t_i) - \gamma(\sigma)| d\sigma \ ,$$

for $i, j = 1, \ldots, n$. These are done by the standard singularity subtraction technique, i.e. we write

$$K_{ij} = \int_{s_{j-1}}^{s_j} \{k_1(t_i, \sigma) + k_2(t_i, \sigma)\} d\sigma, \tag{5.7}$$

where

$$k_2(t, \sigma) = \log\{|t - \sigma||2\pi - t + \sigma||2\pi - \sigma + t|\}$$

and

$$k_1(t, \sigma) = \log|\gamma(t) - \gamma(\sigma)| - k_2(t, \sigma)$$

The second integral in (5.7) is done analytically. The first is done by the two–point Gauss rule. If Γ were smooth this rule would be sufficient to guarantee $O(h)$ convergence in H^0 for approximations of w and $O(h^3)$ convergence for approximations to smooth linear functionals of w (see Graham & Atkinson (1993), for example). It is also sufficient to preserve the orders of convergence here, but as yet we have no proof of this fact.

Results for $\chi = 3/4$, $\chi = 9/10$ and $q = 1$, $q = 3$ are given in Tables 1–4. In all cases we used

$$f \equiv 1 \tag{5.8}$$

in (1.1). The exact solution w of (2.2) is unknown in this case. So, to test (5.3), (5.4) we first computed w^*, the approximation to w using $n = 1024$. We used w^* as an "exact" solution with which we compared w_h. The norms $\|w^* - w_h\|_0$ were computed using the mid point rule on the mesh with 1024 subintervals. To test (5.5), (5.6) we chose $v \equiv 1$ and again used w^* instead of w, computing the integrals exactly.

Interior potentials are important in boundary integral calculations. For example if $\boldsymbol{x}$ lies inside Γ the potential

$$U(\boldsymbol{x}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \log|\boldsymbol{x} - \gamma(\sigma)| w(\sigma) d\sigma$$

solves $\Delta U = 0$ inside Γ with $U = f$ on Γ. In the case of data (5.8) $U \equiv 1$ and so errors in computed values of U are easily found. With w approximated by w_h and $\boldsymbol{x}$ a fixed randomly chosen point with distance no more than $0.01 \times \sqrt{2}$ from $(\cos(1-\chi)\pi/2, \ \sin(1-\chi)\pi/2)/2$, we computed

$$U_h := \frac{1}{\pi} \int_{-\pi}^{\pi} \log|\boldsymbol{x} - \gamma(\sigma)| w_h(\sigma) d\sigma$$

using the two point Gauss rule on each subinterval of $[-\pi, \pi]$. This should converge to $U(\boldsymbol{x}) = 1$ as $n \to \infty$ and will give a good idea of how the method of the present paper performs when used as a boundary integral solver for Laplace's equation. Again this is a smooth functional of w_h and its convergence should be governed by (5.5), (5.6). Results are given in Tables 1–4. Estimated rates of convergence are in the columns headed "Eoc".

From Tables 2,4 it is clear that when $q = 3 > 3/2\alpha$, (5.4) is satisfied. From Tables 1,3 we see that when $q = 1$ (5.3) is satisfied but appears somewhat pessimistic. A bigger surprise is the rate of convergence of linear functionals $\int_{-\pi}^{\pi} w_h$ and U_h. For $q = 3$ these both converge with $O(h^3)$ which is better than that predicted by Corollary 10 and is the same order which has been proved in the case of smooth Γ (see Saranen (1988)). When $q = 1$, $\int_{-\pi}^{\pi} w_h$ converges with only slightly better than the rate predicted by Corollary 10 but again U_h converges with $O(h^3)$. Whether this is just an artifact of the special case solved here, or whether this good convergence of interior potentials is true in more general situations is a question which merits further investigation.

n	$\|w^* - w_h\|_2$	Eoc	$\int_{-\pi}^{\pi} w^* - \int_{-\pi}^{\pi} w_h$	Eoc	$\|U_h - 1\|$	Eoc
8	2.33 (-1)		6.28 (-2)		4.11 (-3)	
		0.14		1.09		5.41
16	2.12 (-1)		2.94 (-2)		9.70 (-5)	
		0.17		1.13		2.15
32	1.89 (-1)		1.34 (-2)		2.18 (-5)	
		0.19		1.16		3.00
64	1.66 (-1)		5.99 (-3)		2.73 (-6)	
		0.25		1.22		3.02
128	1.40 (-1)		2.58 (-3)		3.36 (-7)	
		0.31		1.32		3.01
256	1.13 (-1)		1.03 (-3)		4.17 (-8)	
						3.01
512					5.19 (-9)	

Table 1
$\chi = 3/4$, $(\alpha = 0.571)$, $q = 1$

n	$\|w^* - w_h\|_0$	Eoc	$\|\int_{-\pi}^{\pi} w^* - \int_{-\pi}^{\pi} w\|$	Eoc	$\|U_h - 1\|$	Eoc
8	7.57 (-2)		4.60 (-3)		8.73 (-3)	
		0.99		2.39		3.55
16	3.82 (-2)		8.81 (-4)		7.45 (-4)	
		1.00		2.82		5.06
32	1.91 (-2)		1.25 (-4)		2.23 (-5)	
		1.00		2.96		3.11
64	9.56 (-3)		1.61 (-5)		2.59 (-6)	
		1.01		2.97		3.00
128	4.76 (-3)		2.05 (-6)		3.23 (-7)	
		1.04		3.00		3.00
256	2.32 (-3)		2.56 (-7)		4.04 (-8)	
						3.00
512					5.04 (-9)	

Table 2

$\chi = 3/4,\ (\alpha = 0.571),\ q = 3$

n	$\|w^* - w_h\|_0$	Eoc	$\int_{-\pi}^{\pi} w^* - \int_{-\pi}^{\pi} w_h$	Eoc	$\|U_h - 1\|$	Eoc
8	2.81 (-1)		6.41 (-2)		9.93 (-4)	
		0.12		0.94		0.78
16	2.59 (-1)		3.35 (-2)		5.78 (-4)	
		0.13		1.01		1.76
32	2.36 (-1)		1.66 (-2)		1.71 (-4)	
		0.17		1.07		5.64
64	2.10 (-1)		7.89 (-3)		3.44 (-6)	
		0.22		1.13		3.70
128	1.81 (-1)		3.60 (-3)		2.65 (-7)	
		0.28		1.25		3.00
256	1.49 (-1)		1.51 (-3)		3.30 (-8)	
						3.01
512					4.11 (-9)	

Table 3

$\chi = 9/10,\ (\alpha = 0.526),\ q = 1$

n	$\|w^* - w_h\|_0$	Eoc	$\int_{-\pi}^{\pi} w^* - \int_{-\pi}^{\pi} w_h$	Eoc	$\|U_h - 1\|$	Eoc
8	6.41 (-2)		2.34 (-2)		4.22 (-2)	
		0.98		2.77		2.50
16	3.24 (-2)		3.43 (-3)		7.47 (-3)	
		0.99		4.09		3.55
32	1.63 (-2)		2.02 (-4)		6.39 (-4)	
		0.98		4.52		5.06
64	8.24 (-3)		8.77 (-6)		1.91 (-5)	
		0.99		3.62		6.19
128	4.14 (-3)		7.15 (-7)		2.62 (-7)	
		1.02		3.66		2.92
256	2.04 (-3)		5.67 (-8)		3.46 (-8)	
						3.00
512					4.33 (-9)	-

Table 4

$\chi = 9/10$, $(\alpha = 0.526)$, $q = 3$

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Appendix: Mellin convolution operators

Here we recall some basic facts about Mellin convolution operators on the half-axis which are needed in the analysis of §3; see e.g. Eskin (1981), Elschner (1988), Elschner (1987).

(i) For a given function $v : \mathbb{R}^+ \to \mathbb{C}$, its *Mellin transform* is $\tilde{v}(z) = \int_{\mathbb{R}^+} s^{z-1} v(s) ds$. The operator $v \longrightarrow \tilde{v}$ is an isometric isomorphism from $L^2(\mathbb{R}^+)$ onto $L^2(\{\text{Re } z = 1/2\})$ (cf. Eskin (1981)§2) and we have *Parseval's equality*

$$\int_{\mathbb{R}^+} v\bar{w} ds = \frac{1}{2\pi} \int_{\text{Re } z = 1/2} \tilde{v}\bar{\tilde{w}} |dz| \; .$$

For a given bounded function $a(z)$ on the line Re $z = 1/2$, the Mellin (pseudodifferential) operator $\mathcal{A}$ with symbol $\sigma(\mathcal{A})(z) := a(z)$ is defined by

$$\mathcal{A}v(s) = \frac{1}{2\pi i} \int_{\text{Re } z = 1/2} s^{-z} a(z) \tilde{v}(z) dz \; , \; v \in L^2(\mathbb{R}^+) \; . \tag{A.1}$$

Thus $\mathcal{A}$ is a continuous operator on $L^2(\mathbb{R}^+)$ with norm bounded by

$$\|\mathcal{A}\| \leq \sup\nolimits_{\text{Re } z = 1/2} |a(z)| \; . \tag{A.2}$$

If $\mathcal{B}$ is another Mellin operators with symbol b, then $\mathcal{BA}$ is obviously again a Mellin operator having the symbol ba. In particular, if the symbol of $\mathcal{A}$ satisfies the estimate $|a(z)| \geq c > 0$, Re $z = 1/2$, then $\mathcal{A}$ is continuously invertible on $L^2(\mathbb{R}^+)$, where its inverse is the Mellin operator with symbol $1/a$.

(ii) The symbol $a(z)$ is said to be of class $\sum^{-\infty}$ if it is analytic in the strip $0 <$ Re $z < 1$ and if, the estimates

$$a(z) = O((1 + |z|)^{-k}) \; , \; |z| \to \infty \; , \; k \in \mathbb{N}_0$$

hold uniformly in each substrip $\delta <$ Re $z < 1 - \delta$, $\delta \in (0, 1/2)$. If $a \in \sum^{-\infty}$, then the kernel function $\kappa(x)$, $x \in \mathbb{R}^+$, defined by

$$\kappa(x) = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} x^{-z} a(z) dz \; , \; 0 < \delta < 1 \; , \tag{A.3}$$

fulfils the estimates

$$\sup_{x \in \mathbb{R}^+} |x^{\rho+k} D^k \kappa(x)| < \infty \; , \text{ for all } k \in \mathbb{N}_0 \; , \; 0 < \rho < 1 \; . \tag{A.4}$$

This is a simple consequence of the analyticity and the decay at infinity of the symbol. The converse statement is also valid; cf. Eskin (1981)Lemma 2.3. Moreover, the operator $\mathcal{A}$ with symbol $a \in \sum^{-\infty}$ can be written as the *Mellin convolution*

$$\mathcal{A}v(s) = \int_0^\infty \kappa\left(\frac{s}{\sigma}\right) v(\sigma)\frac{d\sigma}{\sigma} ,$$

and the symbol is the Mellin transform of the kernel function:

$$a(z) = \int_0^\infty x^{z-1}\kappa(x)dx . \tag{A.5}$$

Note that (A.4) for $k = 0$, implies $|\kappa(x)| \leq Cx^{-\rho}$, $x \in \mathbb{R}^+$. Taking $\rho < 1/2$ for $x \in [0,1]$ and $\rho > 1/2$ for $x \in [1,\infty)$ yields $\kappa \in L^2(\mathbb{R}^+)$. It can also easily be shown that $x^{-1/2}\kappa(x)$ is integrable on $\mathbb{R}^+$ if $a \in \sum^{-\infty}$ so that the right-hand side of (A.5) may be interpreted as a convergent integral.

(iii) The Cauchy singular operator on $\mathbb{R}^+$

$$\mathcal{H}v(s) = \frac{1}{\pi}\, p.v. \int_0^\infty \frac{v(\sigma)d\sigma}{\sigma - s}$$

is a Mellin operator having the symbol $\sigma(\mathcal{H})(z) = \cot \pi z$ which is analytic for $0 < \text{Re } z < 1$ and satisfies the estimates

$$\sigma(\mathcal{H})(z) = \mp i + O((1+|z|)^{-k}), \quad \text{Im } z \to \pm\infty , \quad k \in \mathbb{N}_0 \tag{A.6}$$

uniformly in each substrip $\delta < \text{Re } z < 1-\delta$, $\delta > 0$; see Eskin (1981)§15. Then (A.3) with $a(z) = \cot \pi z$ and $\kappa(x) = (1-x)^{-1}$ holds in the sense of distributions, whereas (A.5) has then to be interpreted as a Cauchy principal value. For additional information on $\mathcal{H}$ see Jörgens (1982) §13.5.

(iv) Let ψ be a smooth function function with compact support on $[0,\infty)$. If $a \in \sum^{-\infty}$ or $a(z) = \cot \pi z$ and $\mathcal{A}$ is the corresponding Mellin operator (A.1), then the commutator $\psi\mathcal{A} - \mathcal{A}\psi$ is compact on $L^2(\mathbb{R}^+)$, see Eskin (1981)§15. For $a \in \sum^{-\infty}$, the operator ψA is compact on $L^2(\mathbb{R}^+)$ if in addition $0 \notin \text{supp } \psi$.

(v) A family $a_t(z)$, $0 \leq t \leq 1$, of bounded function on Re $z = 1/2$ is called a *homotopy of symbols* if

$$\lim_{t' \to t} \sup_{\text{Re } z = 1/2} |a_{t'}(z) - a_t(z)| = 0 , \quad 0 \leq t \leq 1 .$$

Then by (A.2), the corresponding family of Mellin operators $\mathcal{A}_t$ with symbols a_t is a homotopy of bounded operators on $L^2(\mathbb{R}^+)$, i.e. the map $t \longrightarrow \mathcal{A}_t$ is continuous on $[0,1]$ in the operator norm of $L^2(\mathbb{R}^+)$.

(vi) Assertions (i) – (v) extend in an obvious way to the case of matrix operators if we replace symbols and kernels by matrix functions. In particular, if $\mathcal{A}$ is a Mellin operator with a 2-by-2-matrix symbol $a(z)$ satisfying $|\det(a(z))| \geq c > 0$, Re $z = 1/2$, then $\mathcal{A}$ is invertible on $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, where its inverse has symbol $a(z)^{-1}$.

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