NONAUTONOMOUS CONLEY INDEX THEORY AND APPLICATIONS

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Preface

A *nonautonomous dynamical system* describes the evolution of a state under a law which may change with time. For many applications, it appears that nonautonomous dynamical systems provide a suitable abstraction [**16**], spawning the development and refinement of a corresponding theory [**1**, **17**].
In many cases, ordinary differential equations and partial differential equations induce dynamical systems which are governed by the spectrum of a linear operator. Replacing the spectrum by the concept of exponential dichotomies, many results concerning autonomous dynamical systems can likewise be proved in the nonautonomous setting.
Another useful tool in the theory of autonomous dynamical systems is known as *Conley index theory*, a growing collection of various theories sharing a common idea. Originally developed by Charles Conley more than 30 years ago, it has been extended and generalised by numerous mathematicians.
Conley index theory can be defined for discrete-time and continuous-time dynamical systems. A good starting point for learning about Conley index theory in the case of semiflows on not (necessarily) locally compact spaces are Rybakowski's *The Homotopy Index and Partial Differential Equations* [**22**] and a survey article [**5**] by Carbinatto and Rybakowski in conjunction with two articles [**9**, **10**] by Franzosa and Mischaikow, covering attractor-repeller decompositions. It should also be mentioned that a Conley index has been defined for a class of random dynamical systems [**18**].
The present work constructs a nonautonomous extension of the Conley index theory for semiflows, discussing ordinary differential equations and semilinear parabolic equations as (nontrivial) examples. The examples include genuinely nonautonomous equations as well as small perturbations of autonomous equations.
The construction of a nonautonomous Conley index starts with a rather technical definition of nonautonomous index pairs. In a second step, groups of index pairs are associated with an evolution operator (or process) and an invariant set, allowing for the definition of an index which depends solely on the evolution operator and the invariant set. These groups of index pairs are not as obvious as they used to be in the autonomous case. Consequently, the reader will encounter three slightly different notions of index pairs in chapters 2 and 3.
Additionally, Chapter 2 examines the index with respect to an evolution operator of interest, while Chapter 3 shows that evolution operators can also be approximated by exploiting the asymptotic behaviour of the index with respect to the time variable.

In contrast to the difficulties in defining the index, it is fairly easy to prove the existence of a solution defined for large times[1], once a nonzero index pair has been obtained.

Moreover, many results and techniques known from the theory of autonomous dynamical systems exhibit a natural extension to a reasonable nonautonomous setting. These nonautonomous extensions are given primarily at the end of Chapters 2 and 3 of the present treatise. The subsequent chapters deal with a generalisation of hyperbolic equilibria in an asymptotically autonomous setting.

The reader might have noticed the careful choice of the above wording *solutions defined for large times* above. Generally speaking, the index only contains information about the asymptotic behaviour of a dynamical system. The distinction between all times and large times becomes negligible if sufficiently strong recurrence assumptions are made. In the absence of recurrence, full (or entire) solutions[2] can be obtained by approximating the evolution operator by another appropriate evolution operator. Not allowing for a more direct approach is an apparent weakness of the index, but the index admits uniformity properties concerning the existence of solutions, compensating for the disadvantage.

This treatise is organised in six chapters which form two invisible parts. The first part consists of three chapters in which the nonautonomous Conley index theory is developed. In the first chapter, several preliminary definitions and results are gathered, while in the following two chapters, most of the abstract theory is developed. The second chapter focusses on recurrent equations in a broad sense. In the third chapter, the construction of the index is refined. The technical sections, in particular the direct limit formulation of the index, are required in order to apply the theory to nonautomous perturbations of (autonomous) semiflows in a way that yields meaningful results on the persistence of Morse decompositions and connecting orbits.

The second part is again divided into three chapters discussing a class of asymptotically autonomous equations of semilinear parabolic type. These equations preserve the basic (generic) structure of their autonomous counterparts. Still there are Morse sets, but this time they are composed of (possibly multiple) connections between equilibrium solutions of the respective limit equation. In Chapter 4, the asymptotically autonomous problem is approximated by a suitable evolution operator. Subsequently, a generic Morse set is shown to have an index similar to that of a generic equilibrium. By utilising the results of Chapter 5, we can now obtain an arbitrarily small perturbation of the original problem to which the results of Chapter 4 can be applied. The synthesis of these results finally gives rise to the theorems in Chapter 6.

[1]This property is sometimes referrred to as Ważewski property.

[2]A solution $u(t)$ defined for all $t \in \mathbb{R}$ is called a full (or entire) solution.

Contents

CHAPTER 1

Preliminaries

It is the purpose of this chapter to introduce commonly used notation. Additionally, a couple of more or less elementary concepts will be recalled.

1.1. Quotient Spaces

DEFINITION 1.1. Let X be a topological space, and $A, B \subset X$. Denote

$$A/B := A/R \cup \{A \cap B\},$$

where A/R is the set of equivalence classes with respect to the relation R on A which is defined by $x R y$ iff $x = y$ or $x, y \in B$.
We consider A/B as a topological space endowed with the quotient topology with respect to the canonical projection $q : A \to A/B$, that is, a set $U \subset A/B$ is open if and only if

$$q^{-1}(U) = \bigcup_{x \in U} x$$

is open in A.

Recall that the quotient topology is the final topology with respect to the projection q.

REMARK 1.1. *The above definition is compatible with the definition used in [**5**] or [**22**]. The only difference occurs in the case $A \cap B = \emptyset$, where we add $\emptyset$, which is never an equivalence class, instead of an arbitrary point.*

LEMMA 1.2. *Let X be a topological space, $B \subset A \subset X$ be closed subsets.*
We consider X/B and A/B as topological spaces equipped with the respective quotient topology. Then, A/B (equipped with quotient topology) is a closed subspace of X/B.

PROOF. Firstly, A/B is a closed subset of X/B because $(X/B) \setminus (A/B) = X \setminus A$ is open.
Now let τ_Q denote the quotient topology on A/B and τ_0 the subspace topology. The inclusion $(A/B, \tau_Q) \subset X/B$ is continuous, and τ_0 is the coarsest topology with this property, so $k : (A/B, \tau_Q) \to (A/B, \tau_0)$ is continuous.
Moreover, the inclusion $j : (A/B, \tau_Q) \to X/B$ is closed, which implies that k is closed. $\square$

1.2. Homology

Assume that we are given a homology theory (H_*, ∂) characterized by the axioms formulated in [**28**] and introduced by Eilenberg and Steenrod.
In particular, H is a covariant functor from the category of topological pairs to the category of graded abelian groups and homomorphisms of degree 0. We write

$$\mathrm{H}_*(X, A) = (\mathrm{H}_q(X, A))_{q \in \mathbb{Z}}$$

to denote the homology of the pair (X,A).
We also assume that (H_*, ∂) satisfies the *axiom of compact supports* as defined in [**28**]:

> For every pair (X, A) of topological spaces and every $z \in \mathrm{H}_q(X, A)$, there is a pair (X', A') of compact subspaces such that $(X', A') \subset (X, A)$ and z is in the image of the homomorphism induced by this inclusion of subspaces.

As it frequently happens that the homology of quotient spaces is considered, the following notation is introduced.

DEFINITION 1.3. Let X be a topological space and $A \subset X$. We set

$$\mathrm{H}_q[X, A] := \mathrm{H}_q(X/A, A/A) \quad q \in \mathbb{Z},$$

where H_q denotes a homology functor.

1.3. Evolution Operators and Semiflows

Assume we are given a set $\mathfrak{U}$ such that every set mentioned in this work is a subset of $\mathfrak{U}$ unless explicitly noted otherwise. The most notable exception is a new symbol $\diamond \notin \mathfrak{U}$. A function which yields $\diamond$ for a given argument is interpreted as not being defined for that argument.
Let X be a metric space[1]. Define $\overline{A} := A \dot{\cup} \{\diamond\}$ whenever A is a set with $\diamond \notin A$. Note that $\overline{A}$ is merely a set; the notation does not contain any implicit assumption on the topology.

DEFINITION 1.4. Let $\Delta := \{(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+ : \ t \geq t_0\}$. A mapping $\Phi : \ \Delta \times \overline{X} \to \overline{X}$ is called an *evolution operator* if

(1) $\mathscr{D}(\Phi) := \{(t, t_0, x) \in \Delta \times X : \Phi(t, t_0, x) \neq \diamond\}$ is open in $\mathbb{R}^+ \times \mathbb{R}^+ \times X$;
(2) Φ is continuous on $\mathscr{D}(\Phi)$;
(3) $\Phi(t_0, t_0, x) = x$ for all $(t_0, x) \in \mathbb{R}^+ \times X$;
(4) $\Phi(t_2, t_0, x) = \Phi(t_2, t_1, \Phi(t_1, t_0, x))$ for all $t_0 \leq t_1 \leq t_2$ in $\mathbb{R}^+$ and $x \in X$;
(5) $\Phi(t, t_0, \diamond) = \diamond$ for all $t \geq t_0$ in $\mathbb{R}^+$.

A mapping $\pi : \ \mathbb{R}^+ \times \overline{X} \to \overline{X}$ is called *semiflow* if $\tilde{\Phi}(t + t_0, t_0, x) := \pi(t, x)$ defines an evolution operator. An explicit characterization of semiflows will be given below.
To every evolution operator Φ, there is an associated semiflow π on an extended phase space $\mathbb{R}^+ \times X$, defined by:

$$(t_0, x)\pi t := \begin{cases} (t_0 + t, \Phi(t + t_0, t_0, x)) & \Phi(t + t_0, t_0, x) \neq \diamond \\ \diamond & \text{otherwise} \end{cases}$$

[1] It is tacitly assumed that $X \subset \mathfrak{U}$ as well as $\mathbb{R} \times X \subset \mathfrak{U}$.

A function $u: I \to X$ defined on a subinterval I of $\mathbb{R}$ is called a *solution* [2] *of* Φ if $u(t_1) = \Phi(t_1, t_0, u(t_0))$ for all $[t_0, t_1] \subset I$.

A mapping $\pi: \mathbb{R}^+ \times \overline{X} \to \overline{X}$ is a semiflow as defined above iff the following holds:

(1) $\mathscr{D}(\pi) := \{(t,x) \in \mathbb{R}^+ \times X : \pi(t,x) \neq \diamond\}$ is open in $\mathbb{R}^+ \times X$;
(2) π is continuous on $\mathscr{D}(\pi)$;
(3) $\pi(0,x) = x$ for all $(0,x) \in \mathbb{R}^+ \times X$;
(4) $\pi(t_1 + t_2, x) = \pi(t_2, \pi(t_1, x))$ for all $t_1, t_2 \in \mathbb{R}^+$ and $x \in X$;
(5) $\pi(t, \diamond) = \diamond$ for all $t \in \mathbb{R}^+$.

We usually write $x\pi t := \pi(t,x)$.
Strictly speaking, a semiflow in the sense of the above Definition is not a semiflow in the sense of e.g. [**5**]. However, this can be overcome by restricting π to the set of all (t,x) for which π is defined, that is, $\pi(t,x) \neq \diamond$. The advantage of introducing the symbol $\diamond$ will become apparent below: the definition of $\mathrm{Inv}^+(N)$, for instance, can be kept short without introducing any ambiguity.

DEFINITION 1.5. Let X be a metric space, $N \subset X$ and π a semiflow on X. The set

$$\mathrm{Inv}^-_\pi(N) := \{x \in N : \text{ there is a solution } u: \mathbb{R}^- \to N \text{ with } u(0) = x\}$$

is called the *largest negatively invariant subset of* N.
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is called the *largest invariant subset of* N.

In the sequel, it is assumed that X and Y are metric spaces. A suitable abstraction of many nonautonomous problems is given by the concept of skew-product semiflows introduced below.

DEFINITION 1.6. Given a global[3] semiflow τ on Y and a mapping $\Phi: \mathbb{R}^+ \times \overline{Y \times X} \to \overline{X}$, we define $\pi := \pi(\tau, \Phi)$ by:

$$\pi(t,y,x) := \begin{cases} (\tau(t,y), \Phi(t,y,x)) & \Phi(t,y,x) \neq \diamond \\ \diamond & \text{otherwise} \end{cases}$$

[2] If an evolution operator (or process) is defined as the "solution operator" of e.g. a differential equation, a mapping u is a solution of the evolution operator if and only if it is a solution of the differential equation. From that point of view, this is a generalized notion of a solution. If the evolution operator is in fact a semiflow, this definition coincides with the one in [**22**]. In [**17**], entire (full) solutions of a process are defined analogously.
However, often the evolution operator itself is considered to be "the" solution. One could thus be tempted to replace the term "solution" by "motion" but at least in [**25**] or [**3**] this notion is only used for (two-sided) flows. Another possible term would be "continuation", which has a completely different meaning in this context. It should also noted that, at least concerning partial differential equations, it is not uncommon that the term solution requires disambiguation.
[3] Defined for all $t \in \mathbb{R}^+$

If π is a semiflow, then it is called the *skew-product* semiflow associated with (τ,Φ).

The mapping Φ is sometimes referred to as the cocycle mapping (see e.g. [**17**]).
We will usually consider a fixed global semiflow τ on Y, which is denoted by $.^t$ (resp. $y^t := y\tau t$). This semiflow is called t-translation, which is motivated by the prototypical example below.

EXAMPLE 1.1. *Let Z be a metric space, and let $Y := C(\mathbb{R}^+, Z)$ be a metric space such that a sequence of functions converges if and only if it converges uniformly on bounded sets. The translation can now be defined canonically by $(y\tau t)(s) := y(t+s)$ for $s, t \in \mathbb{R}^+$.*

DEFINITION 1.7. For $y \in Y$ let

$$\Sigma^+(y) := \mathrm{cl}_Y\{y^t : t \in \mathbb{R}^+\}$$

denote the positive hull of y. Let Y_c denote the set of all $y \in Y$ for which $\Sigma^+(y)$ is compact.

DEFINITION 1.8. Let $y_0 \in Y$ and $N \subset \Sigma^+(y_0) \times X$ be a closed subset. N is called an *isolating neighborhood* (for K in $\Sigma^+(y_0)\times X$) if $\mathrm{Inv}N \subset \mathrm{int}_{\Sigma^+(y_0)\times X} N$ (and $K = \mathrm{Inv}N$).

The following definition is a consequence of the slightly modified notion of a semiflow (Definition 1.4) but not a semantical change compared to [**5**], for instance.

DEFINITION 1.9. We say that π explodes in $N \subset Y \times X$ if $x\pi[0,t[\subset N$ and $x\pi t = \diamond$.

DEFINITION 1.10. Let $\pi = \pi(.^t, \Phi)$ be a skew-product semiflow and $y \in Y$. Define

$$\Phi_y(t+t_0, t_0, x) := \Phi(t, y^{t_0}, x).$$

It is easily proved that Φ_y is an evolution operator in the sense of Definition 1.4.

1.4. Conley Index Theory

In addition to the most import elementary notions of Conley index theory, this section contains two lemmas which will be used used in the following chapters. Both lemmas are are mainly of technical nature and deal with specific constructions of (FM-)index pairs[4].
Let π be a (local) semiflow on a metric space X. A *solution* u of π is a (continuous[5]) mapping $u : I \to X$, where $I \subset \mathbb{R}$ is an interval such that $u(t_0)\pi(t-t_0)$ is defined and $u(t) = u(t_0)\pi(t-t_0)$ for all $[t_0, t] \subset I$. If u is defined for all $t \in \mathbb{R}$, then u is called a *full solution*.
A closed subset $N \subset X$ is called an *isolating neighborhood* if there does not exist a solution $u : \mathbb{R} \to N$ with $u(0) \in \partial N$. N is called *admissible* if the following holds:

- For every sequence x_n in N and $t_n \to \infty$ in $\mathbb{R}^+$ with $x_n\pi[0, t_n] \subset N$, there is a subsequence $(x'_n, t'_n)_n$ of $(x_n, t_n)_n$ such that $x'_n\pi t'_n$ is convergent.

An admissible set N is *strongly admissible* if:

[4] There are several variants of Conley indices, and consequently the exact meaning of the term index pair varies. In order to disambiguate between different kinds of index pairs, Rybakowski introduced the notion FM-index pair in [**5**].
[5] The semiflow is continuous.

- If $x \in N$ and $x\pi t$ is not defined for some $t \in \mathbb{R}^+$, then there is an $s \leq t$ such that $x\pi s$ is defined and $x\pi s \notin N$.

The term *FM-index pair* has been introduced in [**5**, Definition 2.4] in order to differentiate between index pairs in the sense of [**22**] and index pairs as used in [**10**].

DEFINITION 1.11. A pair (N_1, N_2) is called an *FM-index pair* for (π, S) if:

(a) N_1 and N_2 are closed subsets of X with $N_2 \subset N_1$ and N_2 is N_1-positively invariant;
(b) N_2 is an exit ramp for N_1;
(c) S is closed, $S \subset \operatorname{int}_X(N_1 \setminus N_2)$ and S is the largest invariant set in $\operatorname{cl}_X(N_1 \setminus N_2)$.

Roughly speaking, the positive invariance (PI) of N_2 in the above definition means that a solution starting in N_2 does not leave N_2 without leaving N_1. N_2 being an exit ramp (ER) for N_1 means that a solution starting in N_1 crosses N_2 before leaving N_1. For the sake of completeness, we will also give more formalized definitions of (PI) and (ER):

(PI) If $u : [0, a] \to N_1$ is a solution with $u(0) \in N_2$, then also $u(a) \in N_2$.
(ER) If $u : [0, a] \to X$ is a solution with $u(0) \in N_1$ and $u(a) \notin N_1$, then there exists a $t \in [0, a]$ with $u(s) \in N_1$ for all $s \in [0, t]$ and $u(t) \in N_2$.

In view of Definition 2.6, note that assumption (ER') below is equivalent to (ER).

(ER') If $u : [0, a] \to X$ is a solution with $u(0) \in N_1$ and $u(a) \notin N_1$, then there exists a $t \in [0, a]$ with $u(t) \in N_2$.

It is clear that (ER) implies (ER'). Now suppose that (ER') holds, and set

$$T := \sup\{s \in [0, a] :\ u([0, s]) \subset N_1\}.$$

Since N_1 is assumed to be closed, one must have $T < a$. By (ER') and the choice of T, there is a sequence $s_n \to T+$ in $[0, a]$ such that $u(s_n) \in N_2$ for all $n \in \mathbb{N}$. Due to N_2 being closed, one must have $u(T) \in N_2$, that is, (ER) holds true.

DEFINITION 1.12. An FM-index pair (Definition 1.11) (N_1, N_2) is called *strongly admissible* if $\operatorname{cl}_X(N_1 \setminus N_2)$ is strongly admissible.

The following lemma can be summarized as follows: Consider two semiflows and a continuous mapping which commutes with these semiflows. Then, (continuous) preimages of FM-index pairs are again FM-index pairs.

LEMMA 1.13. *Let π (resp. χ) be a semiflow on a metric space X (resp. Y). Let $f : X \to Y$ be continuous, and assume that*

(1) *(N_1, N_2) is an FM-index pair for $(\chi, \tilde{S})$;*
(2) *if $x\pi t$ is defined, then so is $f(x)\chi t$ and $f(x)\chi t = f(x\pi t)$;*
(3) *π does not explode in $f^{-1}(N_1)$ i.e., if $f(x) \in N_1$ and $x\pi t$ is not defined for all $t \in \mathbb{R}^+$, then $f(x\pi t) \in X \setminus N_1$ for some $t \in \mathbb{R}^+$. Note that $f(x\pi t) \in X \setminus N_1$ includes the implicit assumption that $x\pi t$ is defined.*

Then $(M_1, M_2) := (f^{-1}(N_1), f^{-1}(N_2))$ is an FM-index pair for $(\pi, \operatorname{Inv}_\pi(f^{-1}(\tilde{S})))$. Moreover, if (M_1, M_2) is strongly admissible and f is surjective, then $f(\operatorname{Inv}_\pi(f^{-1}(\tilde{S}))) = \tilde{S}$.

PROOF. Set $(M_1, M_2) := (f^{-1}(N_1), f^{-1}(N_2))$.

(a) It is clear that M_1, M_2 are closed subsets of X and $M_2 \subset M_1$. Let $x \in M_2$ and $t > 0$ such that $x\pi t \notin M_2$. It follows that $f(x) \in N_2$ but $f(x)\chi t \notin N_2$. N_2 is N_1-positively invariant, so there is an $0 \le s \le t$ with $f(x)\chi s \notin N_1$. Hence $x\pi s \notin M_1$, implying that M_2 is M_1-positively invariant.

(b) Let $x \in M_1$ and $x\pi s \notin M_1$ for some $s \ge 0$. Then $f(x) \in N_1$ and $f(x)\chi s \notin N_1$, so there is an $r \in [0, s]$ with $f(x)\chi[0, r] \subset N_1$ and $f(x)\chi r \in N_2$. Hence, $x\pi[0, r] \subset M_1$ and $x\pi r \in M_2$, which proves that M_2 is an exit ramp for M_1.

(c) Firstly, we need to show that $\mathrm{cl}_X(M_1 \setminus M_2) \subset f^{-1}(\mathrm{cl}_Y(N_1 \setminus N_2))$ is an isolating neighborhood. Let $u : \mathbb{R} \to \mathrm{cl}_X(M_1 \setminus M_2)$ be a solution. We have $f \circ u(t) \in \mathrm{cl}_Y(N_1 \setminus N_2)$ for all $t \in \mathbb{R}$, so $f \circ u(0) \in \mathrm{int}_Y(N_1 \setminus N_2)$, which means that $u(0) \in \mathrm{int}_X(M_1 \setminus M_2)$. Therefore, $\mathrm{cl}_X(M_1 \setminus M_2)$ is an isolating neighborhood.

Let $S := \mathrm{Inv}_\pi \mathrm{cl}_X(M_1 \setminus M_2)$. It follows as above that $f(S) \subset \tilde{S}$, so $S \subset f^{-1}(\tilde{S})$. Furthermore, if f is surjective, then for every $y \in \tilde{S}$, there is an $x \in X$ with $f(x) = y$. For every $t \ge 0$, there is a $\tilde{y} \in \tilde{S}$ with $\tilde{y}\chi t = y$ because $y \in \tilde{S}$, which is an invariant set. Thus, there is also an $\tilde{x}$ with $\tilde{x}\pi[0, t] \subset M_1 \setminus M_2$ and $f(\tilde{x}\pi t) = y$. Using the admissibility assumption, one obtains an $x' \in S$ with $f(x') = y$, so $f(S) = \tilde{S}$.

□

LEMMA 1.14. *Let (X, d) be a metric space, and let π be a (local) semiflow on X.*
Let $N \subset X$ be a closed and strongly admissible isolating neighborhood with $\mathrm{Inv}^-(N) = \mathrm{Inv}(N)$. *Then*

$$N^+ := \{x \in N : \ x\pi t \in N \quad \forall t \ge 0\}$$

is a closed, strongly admissible, and positively invariant isolating neighborhood for $\mathrm{Inv} N$.
Moreover, $(N^+, \emptyset)$ is an FM-index pair for $(\pi, \mathrm{Inv} N)$.

PROOF. N^+ is a closed subset of N and therefore again strongly admissible. If $x \in N^+$, then for all $t \ge 0$ also $x\pi t \in N^+$. Hence, N^+ is positively invariant.
In order to show that N^+ is an isolating neighborhood for $\mathrm{Inv} N$, suppose that $x \in \mathrm{Inv} N \cap \partial N^+$. Since N is an isolating neighborhood (for $\mathrm{Inv} N$), one has $x\pi t \in \mathrm{int}_X N$ for all $t \in \mathbb{R}^+$. Since x is also in the boundary of N^+, there is a sequence $x_n \to x$ with $x_n \in N \setminus N^+$ for all $n \in \mathbb{N}$.
For every x_n, one has $r_n := \sup\{s \ge 0 : \ x_n\pi[0, s] \subset N\} < \infty$. If $(r_n)_n$ is bounded, then we can choose a convergent subsequence $r_{n(k)} \to r_0$. One has $x_{n(k)}\pi r_{n(k)} \in \partial N$ for all $k \in \mathbb{N}$, so $x\pi r_0 \in \partial N$. However, $x\pi r_0 \in \mathrm{Inv} N \subset \mathrm{int} N$, which is a contradiction.
Hence $r_n \to \infty$, and, using admissibility, it follows that $x_n\pi r_n$ has a convergent subsequence, that is $x_{n(k)}\pi r_{n(k)} \to x_0 \in \mathrm{Inv}^-(N) \cap \partial N$, in contradiction to $\mathrm{Inv}^-(N) = \mathrm{Inv}(N) \subset \mathrm{int} N$. □

1.5. Recurrent Solutions

Section 2.5 contains several results concerning the existence of recurrent and (positive) Poisson stable solutions. For the sake of completeness, this section contains the required definitions and some auxiliary results used there.
Throughout this section, let Γ be a complete metric space. We consider $C(\mathbb{R}, \Gamma)$, the set of all continuous mappings $\mathbb{R} \to \Gamma$, as a metric space, equipped with a metric d which induces

the compact-open topology. Notice the similarity between the following definition and the notion of an almost periodic function.

DEFINITION 1.15. For $x \in C(\mathbb{R},\Gamma)$, let $x \mapsto x^t$ be defined by $x^t(s) := x(t+s)$, $s, t \in \mathbb{R}$.
$x \in C(\mathbb{R},\Gamma)$ is called *recurrent* if for every $\varepsilon > 0$, there is an $l = l(\varepsilon) > 0$ such that for every $t \in \mathbb{R}$ and every interval $I \subset \mathbb{R}$ of length $\geq l$, there is an $s \in I$ with $d(x^t, x^s) \leq \varepsilon$.

Roughly speaking, the concept of the above definition is equivalent to recurrence à la Birkhoff (see [**3**, Section 2.9]). More precisely, with respect to the flow $(t,x) \mapsto x^t$ on $C(\mathbb{R},\Gamma)$, Definition 1.15 is a special case of [**3**, Definition 2.9.4].
Therefore, u is recurrent if [**3**, Theorem 2.9.7] and only if [**3**, Corollary 2.9.10] $\mathrm{cl}_{C(\mathbb{R},\Gamma)}\{u^t : t \in \mathbb{R}\}$ is a compact minimal set[6].
If one considers, instead of $C(\mathbb{R},\Gamma)$, only solutions of a given semiflow χ on Γ, then the t-translation $.^t$ defines the so-called *lifting flow* [**27**] of χ. If χ is a flow, χ and its lifting flow are topologically conjugate.

LEMMA 1.16. *Let χ be a semiflow on Γ and $K \subset \Gamma$ a non-empty compact invariant set.*
Then there is a compact minimal set $\emptyset \neq K_0 \subset K$.

The proof is straightforward[7] (see also [**3**, Theorem 2.9.1]). Compact minimal sets are important since every solution of a flow lying entirely in a compact minimal set is recurrent. Theorem 1.17 is an adaptation of this result to skew-product semiflows.

THEOREM 1.17. *Let X and Y be metric spaces, and let $\pi = \pi(.^t, \Phi)$ be a skew-product semiflow on $Y \times X$.*
Let $y_0 \in Y$ such that $\Sigma^+(y_0)$ is a compact minimal set. Further let $\emptyset \neq K \subset \Sigma^+(y_0) \times X$ be compact and invariant.
Then there exists a recurrent solution $(v_0, u_0) : \mathbb{R} \to K$ of π with $v_0(0) = y_0$.

Note that $\Sigma^+(y_0) = \omega(y_0)$ since the latter is a non-empty compact invariant subset of the former. In particular, there is a solution $v : \mathbb{R} \to Y$ of $.^t$ with $v(0) = y_0$.

PROOF. We consider the set

$$K' := \{u \in C(\mathbb{R}, K) : \ u \text{ is a solution of } \pi\}.$$

It is easy to see (Lemma 2.28) that K' is compact with respect to the compact-open topology. Hence, the translation $.^t$ extends to a flow on K'.
We may thus choose a compact minimal subset $K'_0 \subset K'$. Let $(v,u) \in K'_0$. (v,u) is a solution of π with $(v(t), u(t)) \in K$ for all $t \in \mathbb{R}$. The set $\omega(v(0)) = \mathrm{cl}_Y\{v(t) : t \in \mathbb{R}^+\}$ is a non-empty compact invariant subset of the compact minimal set $\Sigma^+(y_0)$, so $y_0 \in \omega(v(0))$. Hence, there exists a sequence $(t_n)_n$ in $\mathbb{R}$ with $v(t_n) \to y_0$. By Lemma 2.28, there are a subsequence $\tilde{t}_n$ and a solution (v_0, u_0) of π such that $(v(s+\tilde{t}_n), u(s+\tilde{t}_n)) \to (v_0(s), u_0(s))$ for every $s \in \mathbb{R}$. We thus have $v_0(0) = y_0$ as claimed.
Finally, recall that the compact invariant set K'_0 is a minimal set if and only if (v,u) is recurrent for every $(v,u) \in K'_0$. Therefore, (v_0, u_0) is a recurrent solution. □

[6] i.e., the only compact invariant subset is the empty set
[7] By Zorn's lemma, there is a positively invariant minimal subset, which is then shown to be invariant.

We will also require a variant of Lemma 1.16, which is intended to obtain Poisson stable solutions instead of recurrent solutions.

LEMMA 1.18. *Let χ be a semiflow on Γ, $K \subset \Gamma$ a non-empty compact invariant set and $y_0 \in Y$ such that $y_0 \in \omega(y_0)$.*
Then, given a compact invariant subset $K \subset \Gamma$ with

$$\{(y_0, x) : (y_0, x) \in K \text{ for some } x \in X\} \neq \emptyset,$$

there exists a compact invariant set $K_0 \subset K$ such that $(y_0, x') \in K_0$ for some $(y_0, x') \in K$ and $K_0 = \omega(y_0, x')$ for all $(y_0, x') \in K_0$.

In particular, in Lemma 1.18 we claim that K contains a Poisson stable solution, that is, there is an $(y_0, x') \in K$ with $(y_0, x') \in \omega(y_0, x')$.

PROOF. Consider a set G consisting of all subsets L of K with the following properties:

(1) $\{(y_0, x) : (y_0, x) \in L \text{ for some } x \in X\} \neq \emptyset$
(2) L is compact
(3) L is positively invariant

A partial order on G is given by the inclusion of sets. Let $\{L_i : i \in I\}$ be a totally ordered subset of G, and set $L_0 := \bigcap_{i \in I} L_i$. We need to show that $L_0 \in G$.
Each of the sets

$$L'_i := \{(y_0, x) : (y_0, x) \in L_i \text{ for some } x \in X\}$$

is closed and hence compact. Also, $L_i \supset L_k$ implies $L'_i \supset L'_k$. Therefore,

$$L'_0 := \{(y_0, x) : (y_0, x) \in L_0 \text{ for some } x \in X\} = \bigcap_{i \in I} L'_i$$

is non-empty. It is also easy to see that L_0 is closed, hence compact, and positively invariant. By Zorn's lemma, there exists a maximal element $K_0 \in G$. For every $(y_0, x') \in K_0$ it is clear that $\omega(y_0, x')$ is non-empty, compact and invariant. Moreover, $\{(y_0, x) : (y_0, x) \in \omega(y_0, x')$ for some $x \in X\}$ is non-empty since $y_0 \in \omega(y_0)$. Hence, $\omega(y_0, x') \in G$. K_0 is a maximal element of G, so $K_0 = \omega(y_0, x')$. It is well known that $\omega(y_0, x')$ is invariant as claimed. □

CHAPTER 2

Nonautonomous Homotopy Index

In this chapter, we are going to construct a first, elementary variant of an nonautonomous Conley index, which will be refined later. We rely on many results obtained in the past for the autonomous setting.

We consider genuinely nonautonomous dynamical systems. These can be recast as a skew-product semiflow on an appropriate space. Hence, assuming sufficient compactness, there is a naive approach of extending the index to nonautonomous problems: one considers the index of invariant sets relative to the related skew-product semiflow (see for instance [**21**]). However, this approach has certain disadvantages. In particular, one cannot expect a continuation property. Roughly speaking, continuation means that the index remains unchanged under small perturbations of the dynamical system. In order to illustrate the problem, suppose we are given a dynamical system which depends on a function $f(t,x)$, where t is the time variable, and $x \in X$ represents the state of the system. There is a natural (semi)flow, which acts on f by translation. Let $\Sigma^+(f) := \mathrm{cl}\{f(t+.,.) : \ t \in \mathbb{R}^+\}$ denote the positive hull of f under translation. We are interested in invariant subsets of $\Sigma^+(f) \times X$. This fact already suggests that the index depends on $\Sigma^+(f)$, which might be altered completely by an arbitrarily small change of f.

The naive approach of applying the (autonomous) Conley index to the skew product semiflow on $\Sigma^+(f) \times X$ has another shortcoming: an invariant set $K \subset \Sigma^+(f) \times X$ with non-zero index cannot be empty. However, in order to find a solution which belongs to a given parameter f or a specific $g \in \omega(f)$, additional assumptions on f such as periodicity for instance, are required. In contrast, consider an invariant subset K whose nonautonomous index is non-zero. Then, there exists a solution belonging to the parameter f which is defined for sufficiently large times t. Moreover, to every $g \in \omega(f)$, there is a solution associated with g.

Our construction is based on index pairs for the skew-product semiflow but erases the traces of $\Sigma^+(f)$ in the index. These index pairs are an appropriate modification of previously known concepts of index pairs. We are able to prove that the newly defined index agrees with the Conley index in the autonomous case – which justifies the name Conley index. Additionally, for an important class of linear dynamical systems, it is proved that the index is solely determined by the dimension of an unstable subbundle. This behaviour is related to the existence of exponentional dichotomies [**23**] and is well known from the autonomous case.

This chapter is structured as follows. Section 2.1 introduces the notions of isolating neighborhoods and index pairs for evolution operators. In Section 2.2, we consider skew-product semiflows and examine how (FM-)index pairs for isolated invariant sets relative to the skew-product semiflows induce index pairs for evolution operators. These index pairs will be used to define the index. In Section 2.3, we state an index continuation result based on an asymptotic compactness property and the persistence of isolating neighborhoods.

The remaining parts of the chapter deal with applications. First of all, we compute the index for linear evolution operators in Section 2.4. This is based on a thorough understanding of linear skew product semiflows, which relies on [**23**] although the section is self-contained. In section 2.5, we examine how the theory applies to ordinary differential equations as well as semilinear parabolic equations. Subsequently, we investigate a semilinear parabolic equation with a genuinely nonautonomous non-linearity which is asymptotically linear. We prove that, under reasonable assumptions, there exists a non-trivial bounded solution of this equation defined for large times. This is a generalization of previous results concerning autonomous problems. Furthermore, we investigate the impact of additional recurrence hypotheses, which imply the existence of recurrent solutions and Poisson stable points. There are also two short appendices, putting together important facts and auxiliary results about Conley index theory and recurrent (in a broader sense) solutions.

As an example, let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded domain with smooth boundary and consider the following problem

$$\begin{aligned} \partial_t u - \Delta u &= f(t,x,u(t,x),\nabla u(t,x)) && \\ u(t,x) &= 0 && x \in \partial\Omega \\ u(t,x) &= u_0(x), && \end{aligned} \tag{2.1}$$

where $f = f(t,x,u,v)$ is continuously differentiable in u and v. Moreover, assume that for all $(u,v) \in \mathbb{R} \times \mathbb{R}^N$ with $|u|, |v| \leq k$ and all $(t,x) \in \mathbb{R} \times \Omega$, one has

$$\begin{aligned} \left|f(t,x,u,v)\right| &\leq C_1 \\ \left|f(t,x,u,v) - f(t',x,u',v')\right| &\leq C_2\left(\left|t-t'\right|^\delta + \left|x-x'\right|^\delta + \left|u-u'\right| + \left|v-v'\right|\right), \end{aligned}$$

where $C_1 = C_1(k)$, $C_2 = C_2(k)$ and $\delta = \delta(k)$ are positive constants. Note that the Hölder-continuity in t is required as a compactness condition (cf. Lemma 2.52) yet not for regularity reasons.

Let $p > N$, and choose $X := L^p(\Omega)$. Define an operator

$$\begin{aligned} A: &\quad W^{2,p}(\Omega) \cap W_0^1(\Omega) \to L^p(\Omega) \\ Au := &\quad -\Delta u. \end{aligned}$$

A is a positive sectorial operator. Choose $0 < \alpha < 1$ large enough that there is a continuous embedding $X^\alpha \subset C^1(\bar{\Omega})$ (cf. Lemma 2.50), and let $X^\alpha := A^{-\alpha}(X)$ denote the α-th fractional power space equipped with the norm $\|x\|_\alpha := \|A^\alpha x\|$. (2.1) gives rise to the abstract equation

$$u_t + Au = \hat{f}(t,u), \tag{2.2}$$

where $\hat{f}(t,u)(x) = f(t,x,u(t,x),\nabla u(t,x))$.

Let $\lambda \in \mathbb{R} \setminus \sigma(A)$, and, for the sake of simplicity, assume that $(f(t,x,u,v) - \lambda u)/(|u|+|v|) \to 0$ uniformly in x as $t, |u|+|v| \to \infty$, that is, f is asymptotically linear.

In order to formulate the recurrence assumptions on f, we consider the metric

$$d(g,g') := \sum_{n\in\mathbb{N}} 2^{-n} \frac{\delta_n(g-g')}{1+\delta_n(g-g')},$$

where we set

$$\delta_n(g) := \sup\{\left|g(t,x,u,v)\right| : (t,x,u,v) \in \mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^n \quad |t|,|u|,|v| \le n\}.$$

Convergence with respect to d means convergence uniformly on compact subsets, that is, d induces the compact-open topology on a suitable space.
Corollary 2.59 now implies the following theorem.

THEOREM 2.1.

(1) *There are* $t_0 > 0$ *and a solution* $u_0 : [t_0, \infty[\to X^\alpha$ *of* (2.2) *with* $\sup_{t\ge t_0} \|u_0(t)\|_\alpha < \infty$.

(2) f *is called* Poisson stable *if there is a sequence* $t_n \to \infty$ *such that* $f(t_n + t, x, u, v) \to f(t, x, u, v)$ *with respect to the metric* d*. In this case, there is a Poisson stable bounded solution* $u_0 : \mathbb{R} \to X^\alpha$ *of* (2.2)*, that is,* $u_0(0) \in \omega(u_0(0))$.

(3) f *is called recurrent if for every* $\varepsilon > 0$*, there is an* $l = l(\varepsilon)$ *such that every interval* I *of length* l *contains a* t *with* $d(f, f(t + ., ., ., .)) \le \varepsilon$*. If* f *is recurrent, for instance periodic or Bohr almost periodic in* t*, then there is a bounded recurrent solution* $u_0 : \mathbb{R} \to X^\alpha$.

Note that the recurrency of a solution u_0 means that for every $\varepsilon > 0$, there is a $T = T(\varepsilon)$ such that for every $t \in \mathbb{R}$ the whole solution u_0 is contained in an ε-neighborhood of $u_0([t, t + T]) \subset X^\alpha$.
Further results can be obtained if a solution is already known. The operator A is assumed to be positive. Hence, in the following theorem, the implicit assumption that this solution is stable by linearization is made.

THEOREM 2.2. *Suppose that* $\hat{f}(.,0) \equiv 0$*,* $D\hat{f}(.,0) \equiv 0$*, and* $A - \lambda$ *has at least one negative eigenvalue.*
Then there are constants $\eta_1, \eta_2 > 0$ *such that:*

(1) *There are* $t_0 > 0$ *and a mild solution* $u : [t_0, \infty[\to X^\alpha$ *of* (2.2) *such that* $\eta_1 \le \|u(t)\|_\alpha \le \eta_2$ *for all* $t \in [t_0, \infty[$.

(2) *If* f *is Poisson stable, then there is a Poisson stable solution* $u_0 : \mathbb{R} \to X^\alpha$ *of* (2.2) *with* $\eta_1 \le \|u(t)\|_\alpha \le \eta_2$ *for all* $t \in \mathbb{R}$.

(3) *If* f *is recurrent, then there is a recurrent solution* $u_0 : \mathbb{R} \to X^\alpha$ *of* (2.2) *with* $\eta_1 \le \|u(t)\|_\alpha \le \eta_2$ *for all* $t \in \mathbb{R}$.

The theorems in this section are examples. They follow immediately from Corollary 2.59 and Theorem 2.60.

2.1. Index Pairs for Evolution Operators

In this section, we give a rather technical definition of an isolating neighborhood. Since evolution operators are defined only for positive initial times [1], there is no invariant set obviously

[1]The index as constructed here depends on the behavior of the evolution operator for large initial times. The restriction to positive initial times is not an artificial one but reflects this property.

corresponding to an isolating neighborhood but instead an inner set, which is not unique. Nevertheless, our notion of an isolating neighborhood gives rise to an appropriate variant of index pairs.

FIGURE 2.1. An autonomous index pair

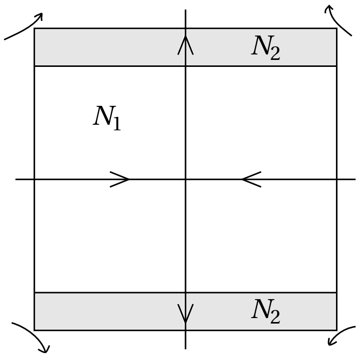
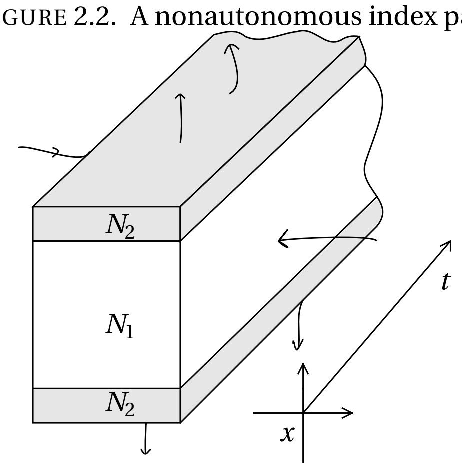


FIGURE 2.2. A nonautonomous index pair



At first glance, it might look strange to start with index pairs rather than invariant sets. The reader who is familiar with Conley index theory knows that index pairs are, roughly speaking, characterized by two properties (and two closed sets): a positive invariant set N_1 and inside this positive invariant an exit set N_2. There is a well known example: a hyperbolic saddle. An autonomous index pair roughly looks like Figure 2.1. An equivalent nonautonomous index pair is sketched in Figure 2.2.

The main result of this section is Theorem 2.7. Roughly speaking, that theorem states that two index pairs which belong to the same inner set and can be ordered by inclusion define the same index. The results of the following section, in particular Lemma 2.16, are based on Theorem 2.7.

The choice of the notion of an isolating neighborhood[2] has yet another, less obvious consequence. Suppose we are given two index pairs $(N_1, N_2) \subset (M_1, M_2)$ for the same inner set that give rise to an index for this inner set. Then the indices obtained from each of the index pairs

[2]As weak as possible, as strong as necessary.

must agree. Hence, if the indices defined by $(N_1, N_2) \subset (M_1, M_2)$ do not agree, then (N_1, N_2) and (M_1, M_2) cannot be index pairs for the same inner set. In this case, Lemma 2.4, which is a direct consequence of the definition of an isolating neighborhood, leads to Theorem 2.19 and its corollaries.

Note that one cannot easily omit the assumption $(N_1, N_2) \subset (M_1, M_2)$. The assumption in Theorem 2.2 that the solution $u \equiv 0$ should be stable is a consequence of this additional (compared to autonomous Conley index arguments) restriction.

2.1.1. Isolating Neighborhoods.

DEFINITION 2.3. Let X be a metric space, Φ be an evolution operator on X and π the associated semiflow. $N \subset \mathbb{R}^+ \times X$ is called an *isolating neighborhood*[3] for (the inner set) $W \subset N$ if

$$\begin{aligned} &\forall t \in \mathbb{R}^+ \; \exists h = h(t) \in \mathbb{R}^+ \; \forall x \in X \\ &((t,x)\pi[0,h] \subset N \implies \exists s \in [0,h] \; (t,x)\pi s \in W). \end{aligned} \tag{2.3}$$

It is trivial to show

LEMMA 2.4. *If $N \subset \mathbb{R}^+ \times X$ is not an isolating neighborhood for $W \subset N$, then*

$$\exists t \in \mathbb{R}^+ \; \forall h \in \mathbb{R}^+ \; \exists x \in X : (t,x)\pi[0,h] \subset N \setminus W.$$

LEMMA 2.5. *Suppose that $N \subset \mathbb{R}^+ \times X$ is an isolating neighborhood for $W \subset N$. Then there is a continuous monotone increasing function $\alpha_0 : \mathbb{R}^+ \to \mathbb{R}^+$ such that* (2.3) *is satisfied for $h(t) = \alpha_0(t)$.*

PROOF. Choose $\alpha_1 : \mathbb{Z}^+ \to \mathbb{R}^+$ such that (2.3) holds with $h(k) = \alpha_1(k)$. Setting

$$\alpha_2(\underbrace{k}_{\in \mathbb{Z}^+} + \underbrace{\xi}_{\in [0,1[}) := \sup_{l \in \{1,2,\dots,k+1\}} \alpha_1(l) + 1,$$

α_2 is monotone increasing, and one has $\alpha_2(k+\xi) \geq \alpha_1(k+1)+1$ for all $(k,\xi) \in \mathbb{Z}^+ \times [0,1[$. Further, let

$$\alpha_0(\underbrace{k}_{\in \mathbb{Z}^+} + \underbrace{\xi}_{\in [0,1[}) := \alpha_2(k) + \xi(\alpha_2(k+1) - \alpha_2(k)) \geq \alpha_2(k) \geq \alpha_1(k+1)+1.$$

Suppose that $(k+\xi, x)\pi[0,s] \subset N$ for some $s \in \mathbb{R}^+$ with $s \geq \alpha_0(k+\xi)$. It follows that $s-(1-\xi) \geq s-1 \geq \alpha_1(k+1)$, so by the choice of α_1, $(k+\xi,x)\pi\tilde{s} = ((k+\xi,x)\pi(1-\xi))\pi(\tilde{s}-(1-\xi)) \in W$ for some $\tilde{s} \in [1-\xi, s]$. Hence, using $h(t) = \alpha_0(t)$, (2.3) is satisfied. □

2.1.2. Index Pairs and Isomorphisms. As before, let X be a metric space, Φ an evolution operator on X and π the associated skew-product semiflow on $\mathbb{R}^+ \times X$.

DEFINITION 2.6. A pair (N_1, N_2) is called an index pair for $W \subset \mathbb{R}^+ \times X$ (resp. (Φ, W)) if

(IP1) $N_2 \subset N_1 \subset \mathbb{R}^+ \times X$, N_1 and N_2 are closed in $\mathbb{R}^+ \times X$;
(IP2) $N_1 \setminus N_2$ is an isolating neighborhood for $W \subset N_1 \setminus N_2$;
(IP3) if $x \in N_1$ and $x\pi t \notin N_1$ for some $t \in \mathbb{R}^+$, then $x\pi s \in N_2$ for some $s \in [0,t]$;
(IP4) if $x \in N_2$ and $x\pi t \notin N_2$ for some $t \in \mathbb{R}^+$, then $x\pi s \in (\mathbb{R}^+ \times X) \setminus N_1$ for some $s \in [0,t]$.

[3] Perhaps, the term *isolating superset* would be more appropriate.

We say that (N_1, N_2) is an index pair (relative to π respectively Φ) if (IP1), (IP3) and (IP4) hold.

In this chapter, the question whether index pairs exist is not treated exhaustingly. In many situations, it is possible to take FM-index pairs[4] which are obtained from the skew-product formulation of the nonautonomous equation in order to derive index pairs in the sense of sense of Definition 2.6.
Given an index pair (N_1, N_2), we consider topological space N_1/N_2, where the "exit set" N_2 is collapsed to a single point. In this chapter, the homotopy type of the pointed space $(N_1/N_2, N_2)$ is called the index. We will show that index pairs give the same index if (2.4) holds or (in Section 2.2) if they are derived from FM-index pairs which belong to the same invariant set in the extended phase space (Lemma 2.16).
Although the majority of the index pairs in this chapter will arise from FM-index pairs in a skew-product setting, this is not mandatory. An alternative concept of binding an index pair to an invariant set will be presented in the next chapter. One reason for this are technical difficulties inherent to the construction of FM-index pairs.
The rest of this section is devoted to the proof of the following theorem. We consider a fixed but arbitrary evolution operator Φ on X and the associated semiflow π on $\mathbb{R}^+ \times X$.

THEOREM 2.7. *Let (N_1, N_2) and (M_1, M_2) be index pairs for (Φ, W) with*

$$(N_1, N_2) \subset (M_1, M_2). \tag{2.4}$$

Inclusion (2.4) *induces an isomorphism*

$$(N_1/N_2, N_2) \simeq (M_1/M_2, M_2)$$

in the homotopy category of pointed spaces.

LEMMA 2.8. *Under the assumptions of Theorem 2.7, there is a continuous monotone increasing function $\alpha_0 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(1) *for every $(t, x) \in M_1$ either $(t, x)\pi s \in N_1$ or $(t, x)\pi s \in M_2$ for some $s \in [0, \alpha_0(t)]$;*
(2) *for every $(t, x) \cap N_1 \cap M_2$ one has $(t, x)\pi s \in N_2$ for some $s \in [0, \alpha_0(t)]$.*

Lemma 2.8 is the only lemma in this section using (IP2).

PROOF. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that (2.3) holds with respect to the index pair (M_1, M_2). In view of Lemma 2.5, we may assume without loss of generality that $\alpha_0 := h$ is continuous and monotone increasing.

(1) Assume that $(t, x)\pi s \notin M_2$ for all $s \in [0, h(t)]$, so by (IP3) $(t, x)\pi s \in M_1 \setminus M_2$ for all $s \in [0, h(t)]$. Hence, by the choice of h, $(t, x)\pi \tilde{s} \in W \subset N_1 \setminus N_2$ for some $\tilde{s} \in [0, h(t)]$.
(2) If $(t, x)\pi s \notin N_2$ for all $s \in [0, h(t)]$, then by (IP3) $(t, x)\pi[0, h(t)] \subset N_1 \setminus N_2$. Consequently, it holds by (IP4) that $(t, x)\pi[0, h(t)] \subset M_2$.

    By the choice of h, we finally obtain that $(t, x)\pi \tilde{s} \in W \subset M_1 \setminus M_2$ for some $\tilde{s} \in [0, h(t)]$, which is a contradiction.

□

[4]Franzosa-Mischaikow-index pair (see [**10**], [**5**])

LEMMA 2.9. *Let* (N_1, N_2) *be an index pair, and define* $f : \mathbb{R}^+ \times N_1 \to N_1/N_2$ *by*

$$f(s,(t,x)) := \begin{cases} (t,x)\pi s & (t,x)\pi[0,s] \subset N_1 \setminus N_2 \\ N_2 & \textit{otherwise.} \end{cases}$$

Then, f is continuous.

PROOF. It is sufficient to prove that f is continuous in every point $(s,(t,x)) \in \mathbb{R}^+ \times N_1$. Suppose that $(s_n,(t_n,x_n)) \in \mathbb{R}^+ \times N_1$ is a sequence with $(s_n,(t_n,x_n)) \to (s,(t,x))$ as $n \to \infty$. If f is not continuous in $(s,(t,x))$, then there are a subsequence $(s'_n,(t'_n,x'_n))$ and a neighborhood U of $f(s,(t,x))$ such that $f(s'_n,(t'_n,x'_n)) \notin U$ for all $n \in \mathbb{N}$. Therefore, there does not exist a subsequence of $(s'_n,(t'_n,x'_n))$ (denoted by the same symbols) such that $f(s'_n,(t'_n,x'_n)) \to f(s,(t,x))$. We consider two cases. Firstly, if $(t,x)\pi[0,s] \subset N_1 \setminus N_2$, then $(t'_n,x'_n)\pi[0,s_n] \subset N_1 \setminus N_2$ provided that n is sufficiently large. It follows that $f(s'_n,(t'_n,x'_n)) = \left[(t'_n,x'_n)\pi s_n\right] \to [(t,x)\pi s] = f(s,(t,x))$. Secondly, assume $(t,x)\pi[0,s'] \subset N_1$ and $(t,x)\pi s' \in N_2$ for some $s' \le s$. Taking subsequences it is sufficient to consider two additional cases. Either for all $n \in \mathbb{N}$ sufficiently large, one has $(t'_n,x'_n)\pi\tilde{s}'_n \in N_2$ for some $\tilde{s}'_n \le s_n$, which implies $f(s'_n,(t'_n,x'_n)) \equiv N_2$, or for all $n \in \mathbb{N}$, $(t'_n,x'_n)\pi\left[0,s'_n\right] \subset N_1 \setminus N_2$, so $(t,x)\pi[0,s] \subset N_1$. Consequently, one has $f(s'_n,(t'_n,x'_n)) = \left[(t'_n,x'_n)\pi s'_n\right] \to [(t,x)\pi s] = N_2$. □

LEMMA 2.10. *Let* (N_1, N_2) *and* (M_1, M_2) *be index pairs with* $(N_1, N_2) \subset (M_1, M_2)$. *Then,*

(a) $(N_1, M_2 \cap N_1)$ *and*
(b) $(N_1 \cup M_2, M_2)$

are index pairs.

PROOF. We need to check the assumptions (IP1), (IP3) and (IP4) of Definition 2.6. It is easy to see that (IP1) holds in both cases.

(a)
- (IP3) Let $(t,x) \in N_1$ and $(t,x)\pi s \notin N_1$ for some $s \ge 0$. Since (N_1,N_2) is an index pair, one obtains immediately that $(t,x)\pi s' \in N_2 \subset N_1 \cap M_2$ for some $s' \in [0,s]$.
- (IP4) Let $(t,x) \in N_1 \cap M_2$ and $(t,x)\pi s \notin N_1 \cap M_2$ for some $s \ge 0$. Firstly, suppose that $(t,x)\pi s \notin M_2$. (M_1,M_2) is an index pair, so $(t,x)\pi s' \in (\mathbb{R}^+ \times X) \setminus M_1 \subset (\mathbb{R}^+ \times X) \setminus N_1$ for some $s' \in [0,s]$. Secondly, if $(t,x)\pi s \notin N_1$, then $(t,x)\pi s' \in N_2$ for some $s' \in [0,s]$ because (N_1,N_2) is an index pair. Hence, there exists an $s'' \in [s',s]$ with $(t,x)\pi s'' \in (\mathbb{R}^+ \times X) \setminus N_1$.

(b)
- (IP3) Let $(t,x) \in N_1 \cup M_2$ and $(t,x)\pi s \notin N_1 \cup M_2$ for some $s \ge 0$. We may assume without loss of generality[5] that $(t,x) \notin M_2$, so the same argument as in (a) applies.
- (IP4) Let $(t,x) \in M_2$ and $(t,x)\pi s \notin M_2$ for some $s \ge 0$. (M_1,M_2) is an index pair and $(\mathbb{R}^+ \times X) \setminus M_1 \subset (\mathbb{R}^+ \times X) \setminus (N_1 \cup M_2)$, so $(t,x)\pi s' \in (\mathbb{R}^+ \times X) \setminus (N_1 \cup M_2)$ for some $s' \in [0,s]$.

□

[5] Otherwise, there is nothing to prove.

LEMMA 2.11. *Let (N_1, N_2) and (M_1, M_2) be index pairs with $N_1 \subset M_1$ and $N_2 = M_2$, and let $\alpha_0 : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that for every $\lambda \in [0,1]$:*

(*) *for every $(t,x) \in M_1$ either $(t,x)\pi s \in N_1$ or $(t,x)\pi s \in M_2$ for some $s \in [0, \alpha_0(\lambda, t)]$;*

Then:

(a) *The mapping $H : [0,1] \times M_1 \to N_1/N_2$ defined by*

$$H(\lambda, t, x) := \begin{cases} [(t,x)\pi\alpha_0(\lambda,t)] & (t,x)\pi[0,\alpha_0(\lambda,t)] \subset M_1 \setminus M_2 \\ N_2 & \textit{otherwise}. \end{cases}$$

is well defined and continuous.

(b) *H gives rise to a continous mapping $\hat{H} := \hat{H}_{M,N} : [0,1] \times M_1/M_2 \to N_1/N_2$, where we set $\hat{H}(\lambda, [t,x]) := H(\lambda, t, x)$.*

(c) *The inclusion $(N_1, N_2) \subset (M_1, N_2)$ induces a homotopy equivalence $i : (N_1/N_2, N_2) \to (M_1/N_2, N_2)$ (provided that at least one function α_0 satisfying (*) exists).*

PROOF. (a) We can understand N_1/N_2 as a subspace of M_1/N_2, so it follows from Lemma 2.9 that H is continuous.

In order to prove that H is well defined, one needs to check if $H(\lambda, t, x) \in N_1/N_2$ for all (λ, t, x). Let (λ, t, x) be given such that $(t,x)\pi[0, \alpha_0(\lambda, t)] \subset M_1 \setminus M_2$. It follows from assumption (*) on α_0 that $(t,x)\pi s' \in N_1$ for some $s' \in [0, \alpha_0(\lambda, t)]$. Consequently, $(t,x)\pi\alpha_0(\lambda,t) \in N_1$ since $N_2 = M_2$ is an exit ramp by (IP3).

(b) This follows immediately because $H([0,1] \times N_2) \subset \{N_2\}$.

(c) Denote $r := \hat{H}_{M,N}(1,.)$. We have

$$r \circ i([t,x]) = \begin{cases} [(t,x)\pi\alpha_0(\lambda,t)] & (t,x)\pi[0,\alpha_0(\lambda,t)] \subset N_1 \setminus N_2 \\ N_2 & \text{otherwise} \end{cases}$$

since $N_2 = M_2$ is an exit ramp for M_1, so $(t,x)\pi[0,s] \subset M_1 \setminus M_2$ implies $(t,x)\pi[0,s] \subset N_1 \setminus N_2$. Using (b) and $\alpha_0'(\lambda, t) = \lambda\alpha_0(1,t)$, it follows that $r \circ i = 1$ in the homotopy category of pointed spaces.

Furthermore $i \circ r = \hat{H}_{M,M}(1,.)$, so again by (b), $i \circ r = 1$ in the homotopy category of pointed spaces.

□

LEMMA 2.12. *Let (N_1, N_2) and (M_1, M_2) be index pairs with $N_2 \subset M_2$ and $N_1 = M_1$, and let $\alpha_0 : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that for every $\lambda \in [0,1]$:*

(**) *for every $(t,x) \cap N_1 \cap M_2$ one has $(t,x)\pi s \in N_2$ for some $s \in [0, \alpha_0(\lambda, t)]$.*

Then:

(a) *The mapping $H : [0,1] \times M_1 \to N_1/N_2$ defined by*

$$H(\lambda, t, x) := \begin{cases} [(t,x)\pi\alpha_0(\lambda,t)] & (t,x)\pi[0,\alpha_0(\lambda,t)] \subset N_1 \setminus N_2 \\ N_2 & \textit{otherwise}. \end{cases}$$

is continuous[6].

[6]In contrast to Lemma 2.11, H is obviously well defined

(b) *H gives rise to a continous mapping $\hat{H} := \hat{H}_{M,N} : [0,1] \times M_1/M_2 \to N_1/N_2$, where we set $\hat{H}(\lambda, [t,x]) := H(\lambda, t, x)$.*

(c) *The inclusion $(N_1, N_2) \subset (N_1, M_2)$ induces a homotopy equivalence $i : (N_1/N_2, N_2) \to (N_1/M_2, M_2)$.*

PROOF. The proof is similar to the proof of Lemma 2.11, so we will omit most of the details.

(b) We need to prove that $H(\lambda, t, x) = N_2$ for all (λ, t, x) with $(t, x) \in M_2$, which is an immediate consequence of assumption (**).

(c) Denote $r := \hat{H}_{M,N}(1, .)$. We have

$$r \circ i([t,x]) = \begin{cases} [(t,x)\pi\alpha_0(\lambda, t)] & (t,x)\pi[0, \alpha_0(\lambda, t)] \subset N_1 \setminus N_2 \\ N_2 & \text{otherwise} \end{cases}$$

Hence, it follows using (b) that $r \circ i = 1$ in the homotopy category of pointed spaces.
Furthermore, by using (IP4) one obtains

$$i \circ r([t,x]) = \begin{cases} [(t,x)\pi\alpha_0(\lambda, t)] & (t,x)\pi[0, \alpha_0(\lambda, t)] \subset N_1 \setminus M_2 \\ M_2 & \text{otherwise,} \end{cases}$$

implying that $i \circ r = 1$.

□

PROOF OF THEOREM 2.7. Using Lemma 2.10, one obtains a chain of index pairs, which is ordered by inclusion:

$$(N_1, N_2) \subset (N_1, M_2 \cap N_1) \subset (N_1 \cup M_2, M_2) \subset (M_1, M_2)$$

Let α_0 be given by Lemma 2.8. The parameter λ in the assumptions of Lemma 2.12 and Lemma 2.11 is not required, that is, it is only assumed that a function α_0 with the respective property exists. Note that this is the only step in this proof that depends on the assumption that (N_1, N_2) and (M_1, M_2) are index pairs for the same inner set W.
One has $(N_1/N_2, N_2) \simeq (N_1/(M_2 \cap N_1), M_2 \cap N_1)$ by Lemma 2.12, and $((N_1 \cup M_2)/M_2, M_2) \simeq (M_1/M_2, M_2)$ by Lemma 2.11. Moreover, N_1 and M_2 are closed, and $N_1 \setminus (N_1 \cap M_2) = (N_1 \cup M_2) \setminus M_2$, so inclusion induces a homeomorphism $(N_1/(M_2 \cap N_1), M_2 \cap N_1) \simeq ((N_1 \cup M_2)/M_2, M_2)$. □

2.2. An Index for Skew Product Semiflows

In this section, we consider a larger class skew-product semiflows, where evolution operators are determined by a parameter $y_0 \in Y$. Under reasonable compactness asumptions, it is possible (and meaningful) to consider isolated invariant subsets of $\omega(y_0) \times X$ respectively $\Sigma^+(y_0) \times X$. Given such an isolated invariant subset K, there are FM-index pairs (see [**5**]) for K in the space $\Sigma^+(y_0) \times X$.
These FM-index pairs in $\Sigma^+(y_0) \times X$ give rise to index pairs in $\mathbb{R}^+ \times X$ in the sense of Definition 2.6. We prove (Lemma 2.16) that for every index pair (N_1, N_2) obtained this way, the pointed space $(N_1/N_2, N_2)$ has the same homotopy type. This homotopy type is the nonautonomous (homotopy) Conley index (Definition 2.13). Subsequently, we will establish basic properties of this index.

Let X and Y be metric spaces. Unless otherwise stated, we will work with one fixed skew-product semiflow $\pi = \pi(.^t, \Phi)$ on $Y \times X$. The notable exception are the continuation lemmas in Section 2.3.2. The canonical semiflow on $\mathbb{R}^+ \times X$ used in the previous section is now denoted by χ, that is, $\chi = \chi(y_0)$ is a skew-product semiflow on $\mathbb{R}^+ \times X$ and $(t, x)\chi s = (t+s, \Phi(s, y_0^t, x))$.

2.2.1. Definition of the Index. The section is devoted to the proof that the nonautonomous Conley-index is well defined. We are going to show that the assumptions of Theorem 2.7 are satisfied and combine this theorem with results concerning the structure of FM-index pairs, which can be found in [**5**]. The notion of a (strongly admissible) isolating neighborhood for the semiflow (and thus for subsets of $Y \times X$) follows [**5**]. This should not be confused with isolating neighborhoods for evolution operators in the sense of Defition 2.3, which are subsets of $\mathbb{R}^+ \times X$.

DEFINITION 2.13. Let $y_0 \in Y$, and let $K \subset \Sigma^+(y_0) \times X$ be an invariant set. Let π_{y_0} denote the restriction of π to $\Sigma^+(y_0) \times X$, that is, $(y, x)\pi_{y_0} t := (y, x)\pi t$.
Let (N_1, N_2) be a strongly admissible FM-index pair for (π_{y_0}, K). Define $r : \mathbb{R}^+ \times X \to \Sigma^+(y_0) \times X$ by $r(t, x) := (y_0^t, x)$.
The homotopy index is

$$\mathrm{h}(\pi, y_0, K) := \mathrm{h}(y_0, K) := \mathrm{h}(r^{-1}(N_1)/r^{-1}(N_2), r^{-1}(N_2)).$$

LEMMA 2.14. *Let y_0, K, r be given by Definition 2.13, and let $N \subset \Sigma^+(y_0) \times X$ be a strongly admissible isolating neighborhood for K.*
Then $M := r^{-1}(N)$ is an isolating neighborhood in the sense of Definition 2.3 for $W := r^{-1}(U)$ whenever U is a neighborhood of K in N.

PROOF. Suppose M is not an isolating neighborhood for W. Then, using Lemma 2.4, there are sequences (t_n, x_n) in M and $h_n \to \infty$ such that $(t_n, x_n)\chi s \in M \setminus W$ for all $s \in [0, h_n]$ and all $n \in \mathbb{N}$. Thus, $(y^{t_n}, x_n)\pi s \in N \setminus U$ for all $s \in [0, h_n]$ and all $n \in \mathbb{N}$. Since N is strongly admissible, one obtains a full solution $u : \mathbb{R} \to N \setminus \mathrm{int}_{\Sigma^+(y_0)\times X}\, U \subset N \setminus K$ of π, in contradiction to the assumption that N is an isolating neighborhood for K. □

LEMMA 2.15. *In addition to the hypotheses of Lemma 2.14, suppose that (N_1, N_2) is an FM-index pair for K with $N_1 \subset N$, and U is a neighborhood of K in $N_1 \setminus N_2$.*
Then $(M_1, M_2) := (r^{-1}(N_1), r^{-1}(N_2))$ is an index pair for $W := r^{-1}(U)$ in the sense of Definition 2.6.

PROOF.
- (IP2) Lemma 2.14 implies that $M := r^{-1}(\mathrm{cl}_X(N_1 \setminus N_2))$ is an isolating neighborhood for $W \subset M_1 \setminus M_2$. Consequently, $M_1 \setminus M_2 \subset M$ is an isolating neighborhood for W (relative to Φ_{y_0}).
- (IP3) Let $(\tau, x) \in M_1$ and $t \in \mathbb{R}^+$ such that $(\tau, x)\chi t \notin M_1$. It follows that $(y^\tau, x) \in N_1$ and $(y^\tau, x)\pi t \notin N_1$, so there is an $s \in [0, t]$ with $(y^\tau, x)\pi s \in N_2$. This in turn implies that $(\tau, x)\chi s \in M_2$.
- (IP4) Let $(\tau, x) \in M_2$ and $t \in \mathbb{R}^+$ such that $(\tau, x)\chi t \notin M_2$. We have $(y^\tau, x) \in N_2$, but $(y^\tau, x)\pi t \notin N_2$. Since π does not explode in N, it does not explode in $N_1 \subset N$,

too. Hence, there is an $s \in [0,t]$ with $(y^\tau, x)\pi s \in (\Sigma^+(y_0)\times X)\setminus N_1$. It follows that $(\tau,x)\chi s \in (\mathbb{R}^+\times X)\setminus M_1$.

□

LEMMA 2.16. *For some $y_0 \in Y$, let $K \subset \Sigma^+(y_0)\times X$ be an invariant set, and let (N_1,N_2) (resp. $(\tilde N_1,\tilde N_2)$) be a strongly admissible FM-index pair for (π,K).*
Then, there is an isomorphism

$$(r^{-1}(N_1)/r^{-1}(N_2), r^{-1}(N_2)) \simeq (r^{-1}(\tilde N_1)/r^{-1}(\tilde N_2), r^{-1}(\tilde N_2))$$

in the homotopy category of pointed spaces.

In other words, Lemma 2.16 says that the homotopy index of Definition 2.13 is independent of the choice of an FM-index pair (N_1,N_2) (in $\Sigma^+(y_0)\times X$). We do not claim that two index pairs (in $\mathbb{R}^+\times X$) which belong to the same inner set define the same homotopy index. This has been proved in Theorem 2.7 under an additional (inclusion) hypothesis.

PROOF. There exist [**5**, Lemma 4.8] an $s\in\mathbb{R}^+$ and a strongly admissible FM-index pair (L_1,L_2) for (π_{y_0},K) such that

$$(N_1,N_2)\subset(N_1,N_2^{-s})\supset(L_1,L_2)\subset(\tilde N_1,\tilde N_2^{-s})\supset(\tilde N_1,\tilde N_2). \tag{2.5}$$

Here, $x\in N_2^{-s}$ (resp. $\tilde N_2^{-s}$) if and only if $x\in N_1$ (resp. $\tilde N_1$) and $x\pi r\in N_2$ (resp. $\tilde N_2$) for some $r\in[0,s]$.
Applying r^{-1} to (2.5) yields

$$(r^{-1}(L_1),r^{-1}(L_2))\subset(r^{-1}(N_1),r^{-1}(N_2^{-s}))\supset(r^{-1}(N_1),r^{-1}(N_2))$$
$$(r^{-1}(L_1),r^{-1}(L_2))\subset(r^{-1}(\tilde N_1),r^{-1}(\tilde N_2^{-s}))\supset(r^{-1}(\tilde N_1),r^{-1}(\tilde N_2)).$$

Suppose that $U\subset\Sigma^+(y_0)\times X$ is a sufficiently small neighborhood of K. It follows from Lemma 2.15 that each of these couples is an index pair for $(\Phi_{y_0}, r^{-1}(U))$. Hence, by Theorem 2.7, each of the above inclusion induces an isomorphism in the homotopy category of pointed spaces. In particular, the pointed spaces $(r^{-1}(N_1/N_2), r^{-1}(N_2))$ and $(r^{-1}(\tilde N_1/\tilde N_2), r^{-1}(\tilde N_2))$ are homotopy equivalent. □

2.2.2. Basic Properties of the Index. Firstly, we claim that the index of an empty invariant set is $\bar 0$, which is the homotopy type of $(\emptyset/\emptyset,\emptyset/\emptyset)$. The lemma below is important despite being a rather weak result, which is useful only in a limited number of cases. Later on, we will develop stronger variants namely Theorem 2.19 and its corollaries.

LEMMA 2.17.

$$\mathrm{h}(y_0,\emptyset)=\bar 0$$

for every $y_0\in Y$.

PROOF. $(\emptyset,\emptyset)$ is a strongly admissible FM-index pair for $\emptyset\subset\Sigma^+(y_0)\times X$. It gives rise to an index pair $(\emptyset,\emptyset)=(r^{-1}(\emptyset),r^{-1}(\emptyset))$ for the evolution operator Φ_{y_0}, so

$$\mathrm{h}(y_0,\emptyset)=(r^{-1}(\emptyset)/r^{-1}(\emptyset),r^{-1}(\emptyset))=(\{\emptyset\},\emptyset)=\bar 0.$$

□

The following theorem justifies the term Conley index. Its meaning is that $\mathrm{h}(y_0, K)$ is an extension of the (autonomous) Conley index theory on metric spaces.

THEOREM 2.18. *Let $y_0 \in Y$ be autonomous, that is, $y_0^t = y_0$ for all $t \in \mathbb{R}^+$. Let $K \subset X$ be an isolated invariant set admitting a strongly admissible neighborhood relative to the semiflow $x\chi t := \Phi(t, y_0, x)$.*
*Then, $\mathrm{h}(y_0, \{y_0\} \times K) = \mathrm{h}(\chi, K)$, where the right-hand side denotes the homotopy index for semiflows as defined in [**22**].*

PROOF. Let (N_1, N_2) be a strongly admissible FM-index pair for (χ, K). It follows that $(\{y_0\} \times N_1, \{y_0\} \times N_2)$ is a strongly admissible FM-index pair for $(\pi, \{y_0\} \times K)$. Thus, using the notation of Definition 2.13, we have

$$r^{-1}(N_i) = \mathbb{R}^+ \times N_i \quad i \in \{1, 2\}.$$

The pointed space $((\mathbb{R}^+ \times N_1)/(\mathbb{R}^+ \times N_2), \mathbb{R}^+ \times N_2)$ is homotopy equivalent to $((\{0\} \times N_1 \cup \mathbb{R}^+ \times N_2)/(\mathbb{R}^+ \times N_2), \mathbb{R}^+ \times N_2)$, which is homeomorphic to $((\{0\} \times N_1)/(\{0\} \times N_2), \{0\} \times N_2)$. $\square$

Primarily, we are not looking for solutions in the ω-limes set of a skew-product semiflow (at least not in this chapter) but of an evolution operator Φ_{y_0} itself. Invariant sets are used to detect these solutions, but we are not interested per se in information about these invariant sets. The following theorem and its corollaries assume that $A \subset K$ is an attractor [7] in K such that the indices of A and K do not agree. We will point out consequences for the evolution operator defined by y_0.
The theorem below has several corollaries can be applied to a number of situations in order to detect solutions. Either no full solution is known, or there is a known solution, and one wants to obtain an additional solution. Prototypical examples are Corollary 2.59 and Theorem 2.60. The seemingly complicated assumptions are caused by the nonautonomous setting. If one was given an autonomous equation or an equation in a restricted class of nonautonomous equations, one could argue as follows: The known solution belongs to an invariant set the index of which does not agree with the index of a maximal compact invariant set. Therefore, the invariant sets do not agree that is, there must exists another (full) solution. Although the argument is still valid even in the nonautonomous cases considered here, its implications are substantially weaker. We obtain a solution of at least one of the equations (evolution operators) given by a parameter $y \in \omega(y_0)$ but not necessarily of the particular equation determined by y_0.

THEOREM 2.19. *Let $y_0 \in Y$ and $K \subset \Sigma^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood N.*
Let $A \subset K$ be an attractor in K, and $\mathrm{h}(y_0, A) \neq \mathrm{h}(y_0, K)$. Then there is a neighborhood N_0 of A such that $r^{-1}(N)$ is not an isolating neighborhood for $r^{-1}(N_0)$.

REMARK 2.1. *If $r^{-1}(N)$ is not an isolating neighborhood for $r^{-1}(N_0)$, then, in view of Lemma 2.4, there is a $t_0 \in \mathbb{R}^+$ and for every $T \in \mathbb{R}^+$ a solution $u : [t_0, t_0 + T] \to r^{-1}(N) \setminus r^{-1}(N_0)$ of Φ_{y_0}.*

[7] a rather strong assumption but presumably necessary

PROOF OF THEOREM 2.19. Let (N_1, N_2, N_3) be an FM-index triple for (π_{y_0}, K, A, R) with $N_1 \subset N$ (see e.g. [**5**] and the references therein).
By Lemma 2.15, the couple $(r^{-1}(N_1), r^{-1}(N_3))$ is an index pair for $(\Phi_{y_0}, r^{-1}(N_1 \setminus N_3))$, and so is $(r^{-1}(N_2), r^{-1}(N_3))$ for (Φ_{y_0}, W), where we set $W := r^{-1}(N_2 \setminus N_3)$. Suppose that $r^{-1}(N)$ is an isolating neighborhood for (Φ_{y_0}, W). Then $r^{-1}(N_1 \setminus N_3) \subset r^{-1}(N)$ is also an isolating neighborhood for (Φ_{y_0}, W). Thus, by Theorem 2.7, there is an inclusion induced isomorphism

$$(r^{-1}(N_2)/r^{-1}(N_3), r^{-1}(N_3)) \simeq (r^{-1}(N_1)/r^{-1}(N_3), r^{-1}(N_3))$$

in the homotopy category of pointed spaces. This means that $\mathrm{h}(y_0, A) = \mathrm{h}(y_0, K)$, in contradiction to our assumptions. We have shown that the conclusions of the theorem hold for $N_0 := N_2 \setminus N_3$. □

COROLLARY 2.20. *Let $y_0 \in Y$ and $K \subset \Sigma^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood N. Assume that* $\mathrm{h}(y_0, K) \neq \bar{0}$.
Then, there is a $t_0 \in \mathbb{R}^+$ such that for every $T \in \mathbb{R}^+$, there is a solution u of Φ_{y_0} which is defined and satisfies $(t, u(t)) \in r^{-1}(N)$ for all $t \in [t_0, t_0 + T]$.

PROOF. Take $A := \emptyset$, and apply Theorem 2.19. One obtains a neighborhood N_0 of A such that such that $r^{-1}(N)$ is not an isolating neighborhood for $r^{-1}(N_0)$.
Whatever N_0 might be, the claim follows directly from Lemma 2.4. □

It is also possible to give a more elementary, straightforward proof for Corollary 2.20. Namely, if its conclusion does not hold, N_2/N_2 is a strong deformation retract of N_1/N_2. The deformation is given by the evolution operator. The crucial point is that for every $t \in \mathbb{R}^+$, there exists a finite time maximum $T = T(t)$ such every point (t, x) reaches the exit set N_2 in a time no longer than T. Otherwise, $N_1 \setminus N_2$ would not be an isolating neighborhood for the empty set, which follows from Lemma 2.15.
Furthermore, using the concept of regular index pairs, we are able to prove Lemma 3.52, a stronger version of Corollary 2.20. It might be possible to generalize Theorem 2.19 as well.

COROLLARY 2.21. *In addition to the assumptions of Theorem 2.19, suppose that $z \in \omega(y_0)$, that is, there is a sequence $t_n \to \infty$ such that $y_0^{t_n} \to z$ as $n \to \infty$.*
Assume that $\mathrm{h}(y_0, A) \neq \mathrm{h}(y_0, K)$, *and let N_0 be given by Theorem 2.19. Then there is a solution $(v, u)\colon \mathbb{R} \to \Sigma^+(y_0) \times X$ such that $v(0) = z$ and*

$$(v(t), u(t)) \in \mathrm{cl}_{Y \times X}(N \setminus N_0) \text{ for all } t \in \mathbb{R}. \tag{2.6}$$

PROOF. Let χ denote the skew-product semiflow on $\mathbb{R}^+ \times X$ which is associated with Φ_{y_0}. It follows from Theorem 2.19 that $r^{-1}(N)$ is not an isolating neighborhood for $W_0 := r^{-1}(N_0)$. Hence, by Lemma 2.4 there must exist a sequence $(x_n)_n$ in X and a $t \in \mathbb{R}^+$ such that $(t, x_n)\chi[0, 2t_n] \subset r^{-1}(N) \setminus W_0$ for all $n \in \mathbb{N}$. Using the admissibility of N and choosing subsequences, we may assume w.l.o.g. that there is a solution $(v, u) : \mathbb{R} \to Y \times X$ such that $(y^{t_n+s}, \Phi(t_n + s - t, t, x_n)) \to (v(s), u(s))$ in $\mathrm{cl}_{Y\times X}(N \setminus N_0)$ for all $s \in \mathbb{R}$. By the choice of the sequence $(t_n)_n$, one has $v(0) = z$. □

Roughly speaking, an invariant set with a non-zero index implies the existence of a whole family of solutions – at least one solution for every parameter $y \in \omega(y_0)$. It is worth noting that the same uniformity principle holds for the existence of connecting orbits of attractor-repeller decompositions (Theorem 3.58).

COROLLARY 2.22. *Let $y_0 \in Y$ and $z \in \omega(y_0)$. Further let $K \subset \Sigma^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood N.*
If $\mathrm{h}(y_0, K) \neq \bar{0}$, then there exists a solution $(v, u) : \mathbb{R} \to N$ with $v(0) = z$.

PROOF. $\emptyset$ is an attractor in K. We have $\mathrm{h}(y_0, \emptyset) = \bar{0}$ by Lemma 2.17. Hence, our claim is a special case (and thus a consequence) of Corollary 2.21. □

2.3. Skew-admissible Isolating Neighborhoods and Continuation

We will consider an asymptotic compactness condition, which resembles Rybakowski's notion of admissibility (see, for example, [**5**, **22**]). The modified condition is therefore called *skew-admissibility*. Based on skew-admissible isolating neighborhoods, we will prove an index continuation theorem. Roughly speaking, index continuation means that the index is preserved under continuous changes of an evolution operator and an associated isolated invariant set.
Throughout this section, we will assume the hypotheses at the beginning of the previous section.

2.3.1. Skew-admissible Isolating Neighborhoods.

DEFINITION 2.23. A subset $N \subset Y \times X$ is called *skew-admissible* provided that the following holds: Whenever $(y_n, x_n)_n$ in N and $(t_n)_n$ in $\mathbb{R}^+$ are sequences such that $t_n \to \infty$, $y_n^{t_n} \to y_0$ in Y and $(y_n, x_n)\pi[0, t_n] \subset N$, the sequence $(\Phi(t_n, y_n, x_n))_n$ has a convergent subsequence $\Phi(t'_n, y'_n, x'_n) \to x_0 \in N$.
N is called *strongly skew-admissible* if it is skew-admissible and π does not explode[8] in N.

DEFINITION 2.24. Let $y_0 \in Y$ and $K \subset \Sigma^+(y_0) \times X$ be an invariant set.
A closed set $N \subset Y \times X$ is called an isolating neighborhood for (y_0, K) provided that:

(1) $K \subset \Sigma^+(y_0) \times X$
(2) $K \subset \mathrm{int}_{Y \times X} N$
(3) K is the largest invariant subset of $N \cap (\Sigma^+(y_0) \times X)$.

We say that N is an isolating neighborhood relative to y_0 if N is an isolating neighborhood relative to $(y_0, (\mathrm{Inv} N) \cap (\Sigma^+(y_0) \times X))$.

There are now two competing notions of an isolating neighborhood with respect to the skew-product semiflow. Definition 1.8 refers to the space $\Sigma^+(y_0) \times X$, whereas Definition 2.24 refers to the space $Y \times X$. Fortunately, both notions of isolation essentially agree.

REMARK 2.2. *Let $y_0 \in Y_c$, and suppose that $K \subset \Sigma^+(y_0) \times X$ is an invariant set admitting a (strongly) skew-admissible isolating neighborhood N. Then $N \cap (\Sigma^+(y_0) \times X)$ is a (strongly) admissible isolating neighborhood in $\Sigma^+(y_0) \times X$.*

[8] $(y, x)\pi\mathbb{R}^+ \subset N \cup \{\diamond\}$ implies $(y, x)\pi\mathbb{R}^+ \subset N$ for all $(y, x) \in N$

As a consequence, we obtain

LEMMA 2.25. *Let $y_0 \in Y_c$, i.e., $\Sigma^+(y_0) \subset Y$ is compact, and suppose that $K \subset \Sigma^+(y_0) \times X$ is an invariant set admitting a strongly skew-admissible isolating neighborhood.*
Then (Definition 2.13) $\mathrm{h}(y_0, K)$ *is defined.*

A converse of Remark 2.2 can be proved as well.

LEMMA 2.26. *Let $y_0 \in Y_c$ and $N \subset \Sigma^+(y_0) \times X$ be an isolating neighborhood for K. Then for $\varepsilon > 0$ small enough*

$$N' := \{(y,x) : \exists x \in X \text{ with } (y',x) \in N \text{ and } d_Y(y,y') \le \varepsilon\}$$

is an isolating neighborhood for (y_0, K).

PROOF. It follows from the compactness of $\Sigma^+(y_0)$ that N' is closed. One still needs to prove that $K \subset \mathrm{int}_{Y\times X} N'$. Let $(y,x) \in K$ be arbitrary. Since $K \subset \mathrm{int}_{\Sigma^+(y_0)\times X} N$, there is a real $\delta' > 0$ such that $U := B_{\delta'}((y,x), \Sigma^+(y_0) \times X) \subset N$. One has $\{(y',x') : d_Y(y',y'') \le \varepsilon$ for some $(y,x') \in U\} \subset N'$, which is a neighborhood of (y,x) in $Y \times X$. Therefore, $K \subset \mathrm{int}_{Y\times X} N'$.
For small $\varepsilon > 0$, K is the largest invariant subset of $N' \cap (\Sigma^+(y_0) \times X)$, which completes the proof[9] □

For the rest of this section, we will be concerned with the proof of the following theorem:

THEOREM 2.27. *Suppose that $(y_n)_n$ is a sequence in Y, $y_0 \in Y_c$ and*[10]

$$d(y_n^t, \Sigma^+(y_0)) \to 0 \text{ as } t, n \to \infty. \tag{2.7}$$

Let N be a strongly skew-admissible isolating neighborhood for (y_0, K).
Then for all $n \in \mathbb{N}$ sufficiently large, there is an invariant subset K_n such that N is an isolating neigborhood for (y_n, K_n).

LEMMA 2.28. *Suppose that $(y_n)_n$ is a sequence in Y, $y_0 \in Y_c$ and*

$$\sup_{t\in\mathbb{R}^+} d(y_n^t, \Sigma^+(y_0)) \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Let $N \subset Y \times X$ be a closed set such that π does not explode in N.
Let $\alpha_n \to \alpha_0$ and $\beta_n \to \beta_0$ be sequences of real numbers. For every $n \in \mathbb{N}$, let $u_n :]\alpha_n, \beta_n[\to N \cap (\Sigma^+(y_n) \times X)$ be a solution[11]*. Assume that the sequence $(u_n(s))_{n\in\mathbb{N}}$ is relatively compact in X for all $s \in]\alpha_0, \beta_0[$.*
Then there are a subsequence $(\tilde{u}_n)_n$ of $(u_n)_n$ and a solution $u :]\alpha_0, \beta_0[\to N \cap (\Sigma^+(y_0) \times X)$ such that $\tilde{u}_n(s) \to u(s)$ uniformly on compact subsets of $\mathbb{R}$.

REMARK 2.3. *Let $(z_n)_n$ be a sequence in Y such that $d(z_n^t, \Sigma^+(y_0)) \to 0$ or $d(z_n^t, y_0^t) \to 0$ as $t, n \to \infty$, then* (2.8) *holds for every sequence $(y_n)_n$ with $y_n \in \omega(z_n)$.*

[9]The details of this last step are omitted. One could use Lemma 2.28 below, which does not rely on the present lemma (in fact, not on a notion of isolation at all).
[10]By an abuse of notation, we write $d(y, \Sigma(y_0)) := \inf_{\tilde{y}\in\Sigma(y_0)} d(y, \tilde{y})$.
[11]of our omnipresent semiflow π

PROOF OF LEMMA 2.28. Let $a_m \to \alpha_0 + 0$ be a sequence converging from above. Using a diagonal sequence argument, (2.8) and the relative compactness of the sequence $(u_n(a_m))_n$, we may choose a subsequence $(\tilde{u}_n)_n$ of $(u_n)_n$ such that for every $m \in \mathbb{N}$ one has $\tilde{u}_n(a_m) \to (z_m, x_m)$ as $n \to \infty$ in $Y \times X$. Finally, define $u(a_m + s) := (z_m, x_m)\pi s$ for all $m \in \mathbb{N}$ and $s \in \mathbb{R}^+$ with $a_m + s < \beta_0$.
If $(z_m, x_m)\pi s$ is defined for $s \in [0, \beta_0 - a_m[$, it follows from the properties of a semiflow that $\tilde{u}_n(a_m + s) \to u(a_m + s)$. Suppose that $(z_m, x_m)\pi s = \diamond$ for some $s \in [0, \beta_0 - a_m[$. π does not explode in N, so there must exist an $s' \in [0, s[$ with $(z_m, x_m)\pi s \in (Y \times X) \setminus N$ which is impossible as $\tilde{u}_n(a_m + s') \to u(a_m + s')$. $\square$

REMARK 2.4. *Let $u : \mathbb{R} \to \Sigma^+(y_0)$ be a solution of the translation semiflow $.^t$, so in particular $u(-n)^n = u(0)$.*
We have $u(-n) = y_0^{t'_n}$ for some $t'_n \in \mathbb{R}^+$, or there is a sequence $t_k \to \infty$ of positive real numbers such that $y_0^{t_k} \to u(-n)$ as $k \to \infty$. In the first case, we can choose $t_k := t'_k + (k-n)$ for $k \geq n$, so in both cases, it follows that $d(y_0^{t_k+n}, u(0)) < 1/n$ for k sufficiently large, implying that $u(0) \in \omega(y_0)$.
The solution u can be chosen arbitrarily, implying that $\mathrm{Inv}\Sigma^+(y_0) = \omega(y_0)$.

PROOF OF THEOREM 2.27. Suppose that the theorem does not hold. Then, by the remark above, there is a subsequence $(\tilde{y}_n)_n$ of $(y_n)_n$ such that for every $n \in \mathbb{N}$, there is a solution $(v_n, u_n) \colon \mathbb{R} \to N \cap (\omega(\tilde{y}_n) \times X)$ with $(v_n(0), u_n(0)) \in \partial N$.
By using Remark 2.3 as well as the compactness of $\Sigma^+(y_0)$, one obtains that $(v_n(s))_{n\in\mathbb{N}}$ is relatively compact. Hence, it follows from the skew-admissibility of N that $((v_n(s), u_n(s)))_{n\in\mathbb{N}}$ is relatively compact for all $s \in \mathbb{R}$.
Remark 2.3 also implies that Lemma 2.28 can be applied, whence one deduces that there is a solution $(v, u) : \mathbb{R} \to N \cap (\Sigma^+(y_0) \times X)$ with $(v(0), u(0)) \in \partial N$, so in contradiction to our assumptions, N is not an isolating neighborhood for (y_0, K). $\square$

2.3.2. Continuation. Lemma 2.33 and Corollary 2.34 rely on the following additional linearity assumptions.

(L1) Y is a linear space[12], and the metric d on Y is invariant, that is, $d(y_1, y_2) = d(y_1 - y_2, 0)$ for all $y_1, y_2 \in Y$.

(L2) The translation $y \mapsto y^t$ is assumed to be linear, that is, $(y_1 + y_2)^t = y_1^t + y_2^t$ and $(\lambda y)^t = \lambda y^t$ for $y, y_1, y_2 \in Y$, $\lambda \in \mathbb{R}$ and $t \in \mathbb{R}^+$.

Before proving Theorem 2.31, which is the main result of this section, we will lift a continuation property for semiflows to evolution operators. The resulting Lemma 2.30 is mainly of technical interest.

DEFINITION 2.29. Suppose that the following holds for all $n \in \mathbb{N} \cup \{0\}$:

(1) $\pi_n = \pi(.^t, \Phi_n)$ is a skew-product semiflow on $Y \times X$;
(2) $y_0 \in Y$ and $K_n \subset \Sigma^+(y_0) \times X$ for all $n \in \mathbb{N}$;
(3) $N \subset Y \times X$ is a strongly π_n-admissible isolating neighborhood for (y_0, K_n).

[12] Addition and scalar multiplication in Y are assumed to be continuous.

Moreover, assume that $\pi_n \to \pi_0$ and N is strongly $(\tilde{\pi}_n)_n$-admissible for every subsequence $(\tilde{\pi}_n)_n$ of $(\pi_n)_n$ (see also [**22**]).
Under these assumptions, we write $(\pi_n, K_n) \to (\pi_0, K_0)$.

LEMMA 2.30. *Suppose that $y_0 \in Y_c$ and $(\pi_n, K_n) \to (\pi_0, K_0)$.*
Then, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, one has

$$\mathrm{h}(\pi_n, y_0, K_n) = \mathrm{h}(\pi_0, y_0, K_0).$$

PROOF. By [**22**, Theorem 12.3], there exist strongly admissible FM-index pairs $(N_{1,n}, N_{2,n})$ and $(\tilde{N}_{1,n}, \tilde{N}_{2,n})$ for K_n, $n \in \mathbb{N} \cup \{0\}$, (in $\Sigma^+(y_0) \times X$) and an $n_0 \in \mathbb{N}$ such that

$$(\tilde{N}_{1,n}, \tilde{N}_{2,n}) \subset (\tilde{N}_{1,0}, \tilde{N}_{2,0}) \subset (N_{1,n}, N_{2,n}) \subset (N_{1,0}, N_{2,0})$$

for all $n \geq n_0$.
Let $n \in \{n \geq n_0\} \cup \{0\}$ be arbitrary, r be given by Definition 2.13[13], and

$$(M_{1,n}, M_{2,n}) := (r^{-1}(N_{1,n}), r^{-1}(N_{2,n}))$$
$$(\tilde{M}_{1,n}, \tilde{M}_{2,n}) := (r^{-1}(\tilde{N}_{1,n}), r^{-1}(\tilde{N}_{2,n})).$$

By Lemma 2.15, $(M_{1,n}, M_{2,n})$ and $(\tilde{M}_{1,n}, \tilde{M}_{2,n})$ are index pairs for $r^{-1}(U)$ whenever $U \subset (N_{1,n} \setminus N_{2,n}) \cap (\tilde{N}_{1,n} \setminus \tilde{N}_{2,n})$ is a neighborhood of K_n in $\Sigma^+(y_0) \times X$.
Consider the following row

$$(\tilde{M}_{1,n}/\tilde{M}_{2,n}, \tilde{M}_{2,n}) \xrightarrow{i} (\tilde{M}_{1,0}/\tilde{M}_{2,0}, \tilde{M}_{2,0}) \xrightarrow{k}$$
$$(M_{1,n}/M_{2,n}, M_{2,n}) \xrightarrow{l} (M_{1,0}/M_{2,0}, M_{2,0})$$

of inclusion induced morphisms. By Theorem 2.7, $k \circ i$ and $l \circ k$ are isomorphisms in the homotopy category of pointed spaces. It follows easily [**22**, Lemma 13.4] that i, k and l are isomorphisms, completing the proof. □

For the rest of this section, we will assume (L1) and (L2) as well as the following hypotheses: Let Γ be a metric space, for instance $\Gamma = [0,1]$, and let $f : \Gamma \to Y_c \times 2^{Y_c \times X}$ be a mapping such that for every $\gamma_0 \in \Gamma$, it holds that $f(\gamma_0) = (y_0, K_0) =: (f_y(\gamma_0), f_K(\gamma_0))$, where:

(C1) There are a neighborhood U of γ_0 in Γ, a strongly skew-admissible set $N \subset Y \times X$ which is, for all $\gamma \in U$, an isolating neighborhood for $f(\gamma)$.

(C2) Whenever $\gamma_n \to \gamma_0$ in Γ, it holds that

$$d(f_y(\gamma_n)^t_n, y^t_0) \to 0 \text{ (in } Y\text{) as } t, n \to \infty.$$

THEOREM 2.31. *The homotopy index* $\mathrm{h} \circ f$ *is constant on connected subsets of* Γ.

REMARK 2.5. *Theorem 2.31 can easily be generalized to the case where Y' is a manifold over Y. However, it is not clear whether the linearity assumptions (L1) and (L2) can be omitted.*

LEMMA 2.32. *Let $\gamma_0 \in \Gamma$, $y_0 := f_y(\gamma_0)$, and let $N \subset Y \times X$ be a strongly skew-admissible set which is an isolating neighborhood for $f(\gamma_0)$.*

[13] Here, it is used that y_0 and thus r are independent of n.

Then there is a neighborhood U of γ_0 in Γ such that for all $\lambda \in [0,1]$ and for all $(y,K) \in f(U)$, N is an isolating neighborhood relative to $y_0 + \lambda(y - y_0)$.

PROOF. If the the lemma does not hold, then there are sequences $\gamma_n \to \gamma_0$ in Γ and λ_n in $[0,1]$, such that for all $n \in \mathbb{N}$, N is not an isolating neighborhood relative to $y_0 + \lambda_n(y_n - y_0)$. For every $t \in \mathbb{R}^+$, the invariance of the metric implies that

$$d(y_0^t + \lambda_n(y_n^t - y_0^t), y_0^t) = d(\lambda_n(y_n^t - y_0^t), 0) \to 0 \text{ as } t, n \to \infty.$$

It follows that (2.7) holds, so by Theorem 2.27, N is an isolating neighborhood relative to $y_0 + \lambda_n(y_n - y_0)$ for all but finitely many n, which is a contradiction. □

LEMMA 2.33. *Let $\gamma_0 \in \Gamma$, $y_0 := f_y(\gamma_0)$, and let $N \subset Y \times X$ be a strongly skew-admissible set which is an isolating neighborhood for $f(\gamma_0)$.*
For $h \in Y_c$, set

$$\tilde{Y}_h := \Sigma^+(y_0) \times \Sigma^+(h)$$

and

$$((y,h),x)\tilde{\pi}_{h,\lambda} t := (\underbrace{(y^t, h^t)}_{=:(y,h)^t}, \Phi(t, y + \lambda h, x)).$$

Then:

(1) *There is a neighborhood U of γ_0 in Γ such that for all $\gamma \in U$ and all $\lambda \in [0,1]$,*

$$N_{h,\lambda} := \{(y,h,x) \in \tilde{Y}_h \times X : (y + \lambda h, x) \in N\}$$

a strongly admissible isolating neighborhood for $((y_0,h), K_{h,\lambda})$ relative to $\tilde{\pi}_{h,\lambda}$, where $h = f_y(\gamma) - y_0$ and

$$K_{h,\lambda} := (\mathrm{Inv}_{\tilde{\pi}_{h,\lambda}} N_{h,\lambda}) \cap (\Sigma^+(y_0,h) \times X).$$

(2) *One has*

$$\mathrm{h}(\tilde{\pi}_{h,\lambda}, (y_0,h), K_{h,\lambda}) = \mathrm{h}(\tilde{\pi}_{h,0}, (y_0,h), K_{h,\lambda}) \tag{2.9}$$

for all $h \in U$ and all $\lambda \in [0,1]$.

PROOF. (1) Let U be given by Lemma 2.32. $\Sigma^+(y_0,h)$ is compact, and N is strongly skew-admissible, so for all $\gamma \in U$ and all $\lambda \in [0,1]$, N is a strongly admissible isolating neighborhood relative to $y_0 + \lambda(y - y_0)$.

We have

$$\mathrm{Inv}_{\tilde{\pi}_{h,\lambda}} N_{h,\lambda} \subset \{(y,h,x) : (y + \lambda h, x) \in \mathrm{Inv}_\pi N\} \cap \big(\Sigma^+(y_0 + \lambda h) \times X\big).$$

Hence, $N_{h,\lambda}$ is an isolating neighborhood relative to (y_0,h) and[14] $\tilde{\pi}_{h,\lambda}$ whenever $\gamma \in U$ and $\lambda \in [0,1]$.

(2) In order to prove our second claim, let $h := f_y(\gamma)$ for some $\gamma \in U$. Consider the mapping $\chi : [0,1] \to \{0,1\}$, where we set $\chi_\lambda = 1$ if and only if (2.9) holds true. It is sufficient to show that χ is locally constant. Arguing by contradiction, we can assume

[14] The definition of an isolating neighborhood depends on a skew-product semiflow, which is usually fixed. Here, this skew-product semiflow $\tilde{\pi}_{h,\lambda}$ is considered to be variable and thus mentioned explicitly.

that there is a sequence $\lambda_n \to \lambda_0$ in $[0,1]$ with

$$\mathrm{h}(\tilde{\pi}_{h,\lambda_n},(y_0,h),K_{h,\lambda_n}) \neq \mathrm{h}(\tilde{\pi}_{h,\lambda_0},(y_0,h),K_{h,\lambda_0})$$

in contradiction to Lemma 2.30.

□

COROLLARY 2.34. *$\mathrm{h}\circ f$ is locally constant, that is, for every $\gamma_0 \in \Gamma$, there is a neighborhood U of γ_0 in Γ such that $\mathrm{h}(\pi, f(\gamma_0)) = \mathrm{h}(\pi, f(\gamma))$ for all $\gamma \in U$.*

PROOF. By assumption (C1), there is a strongly skew-admissible set $N \subset Y \times X$ which is an isolating neighborhood for $f(\gamma_0)$. Furthermore, N is an isolating neighborhood for $f(\gamma)$ for all γ in a sufficiently small neighborhood U_0 of γ_0.
Let U be given by Lemma 2.33, $\gamma \in U \cap U_0$ and $h = f_y(\gamma) - f_y(\gamma_0)$. In view of Lemma 2.33, it is sufficient to prove that for $\lambda \in \{0,1\}$

$$\mathrm{h}(\pi, \underbrace{f_y(\gamma_0)}_{=:y_0} + \lambda h, \underbrace{\mathrm{Inv}_\pi(N) \cap \left(\Sigma^+(y_0+\lambda h)\times X\right)}_{=K'_{y_0+\lambda h}}) = \mathrm{h}(\tilde{\pi}_{\lambda,h},(y_0,h), \underbrace{\mathrm{Inv}_{\tilde{\pi}_{\lambda,h},(y_0,h)} N_{h,\lambda}}_{=K_{h,\lambda}}). \tag{2.10}$$

Note that the choice of U_0 implies that $K'_{y_0} = f_K(\gamma_0)$ and $K'_{y_0+h} = f_K(\gamma)$.
Let (N_1,N_2) be an FM-index pair for $(\pi, y_0+\lambda h)$ with $N_1 \subset N$. By Lemma 1.13,

$$\tilde{N}_i := \{(y,h,x) \in \Sigma^+(y_0,h) \times N : (y+\lambda h, x) \in N_i\} \quad i \in \{1,2\}$$

defines an FM-index pair $(\tilde{N}_1, \tilde{N}_2)$ for $(\tilde{\pi}_{\lambda,h}, \tilde{K}_{\lambda,h})$.
Hence, using solely Definition 2.13,

$$M_i := \{(t,x) \in \mathbb{R}^+ \times N : (y^t, x) \in N_i\} \quad i \in \{1,2\}$$

defines an index pair (M_1, M_2) for $\mathrm{h}(\pi, y_0+\lambda h, K_{h,\lambda})$ and $\mathrm{h}(\tilde{\pi}_{h,\lambda},(y_0,h),\tilde{K}_{h,\lambda})$. Therefore, both indices agree, i.e., (2.10) holds. □

PROOF OF THEOREM 2.31. This is an immediate consequence of Corollary 2.34, stating that the homotopy index $\mathrm{h}\circ f$ is locally constant on Γ, which is connected. □

2.4. Linear Evolution Operators

The index of a linear evolution operator is the homotopy type of a pointed n-sphere, where n is the codimension of the stable bundle determined by the evolution operator. Roughly speaking, this is the main claim of this section and stated in Theorem 2.36.
We begin with several lemmas concerning the structure of the linear evolution operator, respectively the structure of its solutions, that is, stable and unstable subbundles and similar concepts. These initial results resemble those in [**23**]. Nevertheless, full proofs are given[15], for the sake of completeness and readability but also to overcome technical difficulties. Subsequently, using the structural results, we construct a explicit index pair in order to compute the index.
In addition to the hypotheses at the beginning of Section 2.2, assume that X is a normed space. An evolution operator Φ_y, $y \in Y$, is called linear if $\Phi_y(t,t_0,.) : X \to X$ is a continuous

[15]The proofs here are original, but they probably do not contain new ideas.

linear operator for all $t \geq t_0 \in \mathbb{R}^+$. As a consequence, Φ_z is linear for all $z \in \Sigma^+(y)$. We say that $y \in Y$ is linear if Φ_y is linear.
In analogy to [**23**], we call y_0 *weakly hyperbolic* if $\Sigma^+(y_0) \times B$ is a strongly skew-admissible[16] isolating neighborhood for $\omega(y_0) \times \{0\}$ whenever B is a closed bounded neighborhood of 0 in X. Let Y_{cl} denote the set of all linear $y \in Y_c$ for which $\Sigma^+(y) \times B$ is strongly skew-admissible whenever $B \subset X$ is bounded.

LEMMA 2.35. *Let* $\eta_0 > 0$ *and* $y_0 \in Y_{cl}$. y_0 *is weakly hyperbolic if and only if* $\operatorname{Inv}(\Sigma^+(y_0) \times B_{\eta_0}[0]) = \omega(y_0) \times \{0\}$.

PROOF. The set $B_{\eta_0}[0] := \{x \in X : \|x\| \leq \eta_0\}$ is bounded, so the weak hyperbolicity implies that $\operatorname{Inv}(\Sigma^+(y_0) \times B_{\eta_0}[0]) = \omega(y_0) \times \{0\}$.
Conversely, if $B \subset X$ is bounded, then $\varepsilon B \subset B_{\eta_0}[0]$ for small $\varepsilon > 0$. $\Sigma^+(y_0) \times \varepsilon B$ is closed, so $\Sigma^+(y_0) \times \varepsilon B$ is a strongly skew-admissible neighborhood of $\omega(y_0) \times \{0\}$ and $\operatorname{Inv}(\Sigma^+(y_0) \times \varepsilon B) \subset \operatorname{Inv}(\Sigma^+(y_0) \times B_{\eta_0}[0]) = \omega(y_0) \times \{0\}$. It follows that $\operatorname{Inv}(\Sigma^+(y_0) \times B) = \omega(y_0) \times \{0\}$, which completes the proof that y_0 is weakly hyperbolic. $\square$

For $y_0 \in Y$, denote

$$\mathscr{S} := \mathscr{S}_{y_0} := \{(y, x) \in \Sigma^+(y_0) \times X : \sup_{t \in \mathbb{R}^+} \left\|\Phi(t, y, x)\right\| < \infty\}$$
$$\mathscr{S}(y) := \mathscr{S}_{y_0}(y) := \{x : (y, x) \in \mathscr{S}\}.$$

Analogously, we define $\mathscr{U} := \mathscr{U}_{y_0}$ as the set of all (y, x) for which there is a solution $(u, v) : \mathbb{R}^- \to \Sigma^+(y_0) \times X$ with $(v, u)(0) = (y, x)$ and $\sup_{t \in \mathbb{R}^-} \|u(t)\| < \infty$. Then, again

$$\mathscr{U}(y) := \mathscr{U}_{y_0}(y) := \{x : (y, x) \in \mathscr{U}\}.$$

THEOREM 2.36. *Let* $y_0 \in Y_{cl}$ *be weakly hyperbolic, and assume that there is a* $k \in \mathbb{N}$ *with* $k := \operatorname{codim} \mathscr{S}(y_0^t)$ *for all* $t \in \mathbb{R}^+$ *sufficiently large.*
Then,

$$\mathrm{h}(y_0, \omega(y_0) \times \{0\}) = \Sigma^k,$$

where Σ^k *denotes the homotopy type of a pointed* k*-sphere.*

If the integer k in Theorem 2.36 does not exist, then it follows from Lemma 2.40 (c) that $\operatorname{codim} \mathscr{S}(y_0^t) \to \infty$ as $t \to \infty$ and $\operatorname{codim} \mathscr{S}(z) = \infty$ for all $z \in \omega(y_0)$.

LEMMA 2.37. *Let* $y \in Y_{cl}$ *be weakly hyperbolic.*
Assume that $t_n \to \infty$ *and* $(v_n, u_n) : [-t_n, 0] \to \Sigma^+(y) \times X$ *is a sequence of solutions. Let*

$$\alpha_n := \sup_{t \in [-t_n, 0]} \|u_n(t)\|.$$

(a) *If* $\sup_{n \in \mathbb{N}} \alpha_n < \infty$, *then there are a subsequence* $(\tilde{v}_n, \tilde{u}_n)_n$ *of* $(v_n, u_n)_n$ *and a solution* (v, u) *such that* $(\tilde{v}_n(t), \tilde{u}_n(t)) \to (v(t), u(t))$ *for all* $t \in \mathbb{R}^-$.
(b) *Let* $(\beta_n)_n$ *be a sequence with* $\beta_n \in [-t_n, 0]$ *and* $\left\|u_n(\beta_n)\right\| \geq \alpha_n/2$. *Then there is a* $\beta_0 \in \mathbb{R}^+$ *such that for all* $n \in \mathbb{N}$, *either* $\left|\beta_n\right| \leq \beta_0$ *or* $\left|t_n - \beta_n\right| \leq \beta_0$.
(c) *If* $\sup_{n \in \mathbb{N}} \|u_n(0)\| < \infty$ *and* $\sup_{n \in \mathbb{N}} \|u_n(-t_n)\| < \infty$, *then* $\sup_{n \in \mathbb{N}} \alpha_n < \infty$.

[16] or strongly admissible, which is equivalent in this case

PROOF. (a) Let $\alpha := \sup_{n\in\mathbb{N}} \alpha_n < \infty$. By our assumptions, the set $\Sigma^+(y) \times B_\alpha[0]$ is strongly skew-admissible. It follows from Lemma 2.28 that there exists a bounded solution $(v,u)\colon \mathbb{R} \to \omega(y) \times X$ with $\|u(0)\| = \alpha$, but y is assumed to be weakly hyperbolic, which is a contradiction.

(b) Arguing by contradiction, we can assume without loss of generality that $\beta_n \to -\infty$ and $\left|t_n - \beta_n\right| \to \infty$. For $n \in \mathbb{N}$, let $(v_n', u_n')(t) := (v_n(t+\beta_n), \alpha_n^{-1} u_n(t+\beta_n))$, which is again a solution.

As in (a), it follows from Lemma 2.28 that there exist a subsequence $(\tilde{v}_n', \tilde{u}_n')_n$ and a solution $(v_0, u_0)\colon \mathbb{R} \to X$ such that

$$(\tilde{v}_n'(t), \tilde{u}_n'(t)) \to (v_0(t), u_0(t)) \text{ for all } t \in \mathbb{R}. \tag{2.11}$$

One has $\|u_0(t)\| \leq 1$ for all $t \in \mathbb{R}$, so $u_0 \equiv 0$ by the weak hyperbolicity. However, $\|u_0(0)\| \geq 1/2$ by the choice of the sequence $(\beta_n)_n$ and since $\tilde{u}_n'(0) \to u_0(0)$.

(c) Suppose to the contrary that $\alpha_n \to \infty$. Assume that each u_n attains its maximal norm at β_n, that is, $\left\|u_n(\beta_n)\right\| = \alpha_n$. We aim for a contradiction, so, in view of (b), we may assume w.l.o.g. that either $t_n - \left|\beta_n\right| \to \beta_0$ or $\left|\beta_n\right| \to \beta_0$.

$t_n - \left|\beta_n\right| \to \beta_0$: Taking subsequences, we can assume w.l.o.g. that $v_n(-t_n) \to z_0 \in \Sigma^+(y)$. We now have

$$1 = \left\|\Phi(t_n - \left|\beta_n\right|, v_n(-t_n), \alpha_n^{-1} u_n(-t_n))\right\| \to \left\|\Phi(\beta_0, z_0, 0)\right\| = 0,$$

which is a contradiction.

$\left|\beta_n\right| \to \beta_0$: Using (a), we may assume w.l.o.g. that there is a solution $(v_0, u_0)\colon \mathbb{R} \to \Sigma^+(y) \times X$ with

$$(v_n(t - \left|\beta_n\right|), \alpha_n^{-1} u_n(t - \left|\beta_n\right|)) \to (v_0(t), u_0(t)) \text{ for all } t \in \left]-\infty, \beta_0\right]. \tag{2.12}$$

It follows that $\|u_0(0)\| = 1$, $u_0(\beta_0) = 0$ and $\sup_{t\in\mathbb{R}^-} \|u_0(t)\| = 1$. Hence, (v_0, u_0) is a bounded solution defined for all $t \in \mathbb{R}$. However, y is weakly hyperbolic, so $u_0 \equiv 0$, which is again a contradiction.

□

LEMMA 2.38. *Let $y \in Y_{cl}$ be weakly hyperbolic.*
Suppose that for all $n \in \mathbb{N}$, $(v_n, u_n)\colon \mathbb{R}^- \to \Sigma^+(y) \times X$ is a solution with $\alpha_n := \sup_{t\in\mathbb{R}^-} \|u_n(t)\| < \infty$ and $\|u_n(0)\| \equiv 1$.
Then, $\sup_{n\in\mathbb{N}} \alpha_n < \infty$, and there are a subsequence $(\tilde{v}_n, \tilde{u}_n)_n$ of $(v_n, u_n)_n$ and a solution $(v_0, u_0)\colon \mathbb{R}^- \to \Sigma^+(y) \times X$ with $(\tilde{v}_n(t), \tilde{u}_n(t)) \to (v_0(t), u_0(t))$ for all $t \in \mathbb{R}^-$.

PROOF. The existence of a convergent subsequence and a limit solution (v_0, u_0) follows from Lemma 2.37 (a) provided that $\sup_{n\in\mathbb{N}} \alpha_n < \infty$.
Arguing by contradicition, we can assume w.l.o.g. that $\alpha_n \to \infty$ as $n \to \infty$. Thus, we have $\alpha_n^{-1} u_n(0) \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, we may choose a $\beta_n \in \mathbb{R}^+$ such that $\left\|u_n(-\beta_n)\right\| \geq \alpha_n/2$. Choosing subsequences, we can assume w.l.o.g. that $\beta_n \to \beta_0 \in [0, \infty]$. It follows from Lemma 2.37 (a) that there is a solution $(v_0', u_0')\colon \left]-\infty, \beta_0\right[\to \Sigma^+(y) \times X$ with

$$(v_n'(-\beta_n + t), u_n'(-\beta_n + t)) \to (v_0'(t), u_0'(t)) \tag{2.13}$$

for an appropriate subsequence $(v_n', u_n')_n$ of $(v_n, \alpha_n^{-1} u_n)_n$ and all $t \in \left]-\infty, \beta_0\right[$. Taking subsequences it is sufficient to consider the following two cases.
$\beta_n \to \infty$: In this case, $(v_n'(t), u_n'(t))$ is defined for all $t \in \mathbb{R}$. Since y is weakly hyperbolic, it follows that $u_0' \equiv 0$, but $\left\|u_0'(0)\right\| \geq 1/2$ due to the pointwise convergence.
$\beta_n \to \beta_0$: We have $u_0(t) \to 0$ as $t \to \beta_0 - 0$, so (v_0', u_0') extends to a solution $(\tilde{v}_0, \tilde{u}_0): \mathbb{R} \to \Sigma^+(y) \times X$ with $\|\tilde{u}_0(t)\| \leq 1$ for all $t \in \mathbb{R}$ and $\tilde{u}_0(0) \neq 0$, in contradiction to our assumption that y is weakly hyperbolic. □

LEMMA 2.39. *Let $y \in Y_{cl}$ be weakly hyperbolic, and let $y_n \to y_0$ be a sequence in $\Sigma^+(y)$. Suppose that for all $n \in \mathbb{N}$, $u_n : \mathbb{R}^+ \to X$ is a solution of Φ_{y_n} with $\alpha_n := \sup_{t \in \mathbb{R}^+} \|u_n(t)\| < \infty$ and $\|u_n(0)\| \equiv 1$.
Then, $\sup_{n \in \mathbb{N}} \alpha_n < \infty$.*

PROOF. Aiming for a contradicition, we can assume w.l.o.g. that $\alpha_n \to \infty$ as $n \to \infty$. Thus, we have $\alpha_n^{-1} u_n(0) \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, we may choose a $\beta_n \in \mathbb{R}^+$ such that $\left\|u_n(\beta_n)\right\| \geq \alpha_n/2$. It is sufficient to consider two cases.
$\beta_n \to \beta_0$: We have $\alpha_n^{-1} u_n(\beta_n) \to \Phi(\beta_0, y_0, 0) = 0$ as $n \to \infty$, in contradiction to $\left\|u_n(\beta_n)\right\| \geq \alpha_n/2$ for all $n \in \mathbb{N}$.
$\beta_n \to \infty$: One can assume w.l.o.g. that $y_n^{\beta_n} \to z \in \Sigma^+(y)$ as $n \to \infty$, so it follows from Lemma 2.37 (a) that there exist a subsequence $(\alpha_n^{-1} u_{n(k)})_k$ of $(\alpha_n^{-1} u_n)_n$ and a solution $(v_0, u_0): \mathbb{R} \to \Sigma^+(y) \times X$ with $(y_{n(k)}^{\beta_{n(k)}+t}, \alpha_n^{-1} u_{n(k)}(\beta_{n(k)} + t)) \to (v_0(t), u_0(t))$ as $k \to \infty$ for all $t \in \mathbb{R}$. Hence, $\sup_{t \in \mathbb{R}} \|u_0(t)\| < \infty$ and $\|u_0(0)\| \geq 1/2$, in contradiction to our assumption that y is weakly hyperbolic. □

LEMMA 2.40. *Let $y_0 \in Y_{cl}$ be weakly hyperbolic. The following statements hold true.*

(a) *Suppose that C is a finite-dimensional linear subspace of X with $\mathcal{S}(y_0) \cap C = \{0\}$. Then, for every $M > 0$ there is a $t_0 = t_0(M) \geq 0$ such that $\left\|\Phi(t, y_0, x)\right\| \geq M\|x\|$ for all $x \in C$ and $t \geq t_0$. Moreover, there is an $\varepsilon > 0$ such that $\left\|\Phi(t, y_0, x)\right\| \geq \varepsilon\|x\|$ for all $t \leq t_0$.*
(b) *Set $C(t) := \Phi(t, y_0, C)$, and let $t_n \to \infty$ and $(c_n)_n$ be sequences with $y_0^{t_n} \to z \in \Sigma^+(y_0)$, $c_n \in C(t_n)$ and $\|c_n\| \leq 1$. Then there is a subsequence $(\tilde{c}_n)_n$ with $\tilde{c}_n \to c_0 \in \mathcal{U}(z)$.*
(c) *$C(t) \cap \mathcal{S}(y_0^t) = \{0\}$ for all $t \in \mathbb{R}^+$. Moreover, $\operatorname{codim} \mathcal{S}(y_0^t) \geq \operatorname{codim} \mathcal{S}(y_0)$ for all $t \in \mathbb{R}^+$ and $\operatorname{codim} \mathcal{S}(z) \geq \operatorname{codim} \mathcal{S}(y_0)$ for all $z \in \omega(y_0)$*
(d) *Suppose that $k = \operatorname{codim} \mathcal{S}(y_0^t) < \infty$ for all $t \in \mathbb{R}^+$. Then $X = \mathcal{U}(z) \oplus \mathcal{S}(z)$ and $\dim \mathcal{U}(z) = \operatorname{codim} \mathcal{S}(y_0) = k$ for all $z \in \omega(y_0)$.*
(e) *Let $y_n \to y$ in $\omega(y_0)$, and let $(x_n)_n$ be a sequence with $x_n \in \mathcal{U}(y_n)$ and $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Then there is a subsequence $(\tilde{x}_n)_n$ of $(x_n)_n$ with $\tilde{x}_n \to x_0$, and $x_0 \in \mathcal{U}(y)$.*

PROOF. (a) Otherwise, there are an $M > 0$ and sequences $t_n \to \infty$ and $x_n \to x_0$ in C with $\|x_n\| \equiv 1$ such that $\left\|\Phi(t_n, y_0, x_n)\right\| \leq M$ for all $n \in \mathbb{N}$. Let

$$\alpha_n := \sup_{s \in [0, t_n]} \left\|\Phi(s, y_0, x_n)\right\|.$$

It follows from Lemma 2.37 (c) that $\sup_{n \in \mathbb{N}} \alpha_n < \infty$. As a consequence, we obtain that $0 \neq x_0 \in C \cap \mathcal{S}(y_0)$, in contradiction to the choice of C.

Now let $t_0 = t_0(1)$, and suppose that ε does not exist. Then there are sequences $t_n \to \tilde{t} \in [0, t_0]$ and $x_n \to x_0$ in C with $\|x_n\| \equiv 1$ such that $\Phi(t_n, y_0, x_n) \to 0$. We have $\Phi(\tilde{t}, y_0, x_0) = 0$, so $0 \neq x_0 \in C \cap \mathscr{S}(y_0)$, which is again a contradiction.

(b) For every $n \in \mathbb{N}$, let $u_n : [0, t_n] \to X$ be a solution of Φ_{y_0} with $u_n(t_n) = c_n$. Using (a), we can assume w.l.o.g. that $u_n(0) \to x_0$ in C. Let $\alpha_n := \sup_{s\in[0,t_n]} \|u_n(s)\|$. We have $\sup\{\alpha_n : n \in \mathbb{N}\} < \infty$ by Lemma 2.37 (c). Now, our claim follows from Lemma 2.37 (a).

(c) Suppose that $X = C \oplus \mathscr{S}(y_0)$, and let $t \in \mathbb{R}^+$. The mapping $\Phi(t, y_0, .)$ is injective on C, that is, $\Phi(t, y_0, x) = 0$ for some $x \in C$ implies $x = 0$. Because otherwise, one has $0 \neq x \in \mathscr{S}(y_0)$. Therefore, $C(t) := \Phi(t, y_0, C)$ is a $\operatorname{codim} \mathscr{S}(y_0)$-dimensional subspace of X with $C(t) \cap \mathscr{S}(y_0^t) = \emptyset$.

Let $z \in \omega(y_0)$ and $\tilde{C} \subset X$ a linear subspace with $X = \tilde{C} \oplus \mathscr{S}(z)$. If $\dim \tilde{C} < \dim C$, then there are sequences $t_n \to \infty$ in $\mathbb{R}^+$ and $x_n \in C(t_n) \cap \mathscr{S}(z)$ with $\|x_n\| \equiv 1$. Using (b), we can assume w.l.o.g. that $x_n \to x_0 \in \mathscr{U}(z)$, where $x_0 \neq 0$. Furthermore, it follows from Lemma 2.39 that $x_0 \in \mathscr{S}(z)$. However, $\mathscr{U}(z) \cap \mathscr{S}(z) = \{0\}$.

(d) Let $X = C \oplus \mathscr{S}(y_0)$, define $C(t)$ as in (b), and let $x \in X$ be arbitrary. We have $X = C(t) \oplus \mathscr{S}(y_0^t)$ for all $t \in \mathbb{R}^+$. Let $z \in \omega(y_0)$ be arbitrary, and choose a sequence $t_n \to \infty$ such that $y^{t_n} \to z$. In view of (c), there is a unique decomposition $x = c_n \oplus s_n$ with $c_n \in C(t_n)$ and $s_n \in \mathscr{S}(y_0^{t_n})$. Set $\alpha_n := \|c_n\|$. Using (b), we can assume w.l.o.g. that $\alpha_n^{-1} c_n \to c_0' \in \mathscr{U}(z)$.

Suppose that $\sup_{n\in\mathbb{N}} \alpha_n = \infty$. Choosing subsequences, we can assume w.l.o.g. that $\alpha_n \to \infty$. One has $\alpha_n^{-1} x \to 0$ and $\alpha_n^{-1} s_n = \alpha_n^{-1} x - \alpha_n^{-1} c_n \to 0 - c_0'$. It follows from Lemma 2.39 that $-c_0' \in \mathscr{S}(z)$, in contradiction to y_0 being weakly hyperbolic.

Therefore, we may assume w.l.o.g. that $\alpha_n \to \alpha_0$, so $c_n \to \alpha_0 c_0' =: c_0 \in \mathscr{U}(z)$ by Lemma 2.37 and $s_0 := x - c_0 \in \mathscr{S}(z)$ by Lemma 2.39. This proves that $X = \mathscr{U}(z) + \mathscr{S}(z)$. Furthermore, y_0 is weakly hyperbolic, so $\mathscr{U}(z) \cap \mathscr{S}(z) = \{0\}$.

Let $x \in \mathscr{U}(z)$ be arbitrary. As before, there are decompositions

$$x = c_n + s_n \in C(t_n) \oplus \mathscr{S}(y^{t_n}). \tag{2.14}$$

Passing (2.14) to the limit, we obtain as above that $x = \lim_{n\to\infty} c_n$. Define $P_n : \mathscr{U}(z) \to C(t_n)$ by $P_n(x) := c_n$, and let $P : X \to \mathscr{U}(z)$ denote the canonical projection along (with kernel) $\mathscr{S}(z)$. We have $P \circ P_n \to \mathrm{id}_{\mathscr{U}(z)}$ by (2.14) and the following remark, so $k = \dim C(t_n) \geq \dim \mathscr{U}(z)$ for sufficiently large $n \in \mathbb{N}$. We further have $\dim C = \dim C(t_n)$ for all $n \in \mathbb{N}$ as proved in (c) and $\dim C = \operatorname{codim} \mathscr{S}(y_0)$ by construction.

(e) This is a direct consequence of Lemma 2.38.

□

Suppose that the assumptions of Theorem 2.36 hold. Since $\mathrm{h}(y_0^t, \omega(y_0^t) \times \{0\}) = \mathrm{h}(y_0, \omega(y_0) \times \{0\})$ for all $t \in \mathbb{R}^+$ and by using Lemma 2.40 (c), one can assume without loss of generality that $\operatorname{codim} \mathscr{S}(y_0^t) = k$ for all $t \in \mathbb{R}^+$.

Let $X = C \oplus \mathscr{S}(y_0)$, and let $C(t)$ be defined as in Lemma 2.40. It is easy to see that the following definition might not be unambiguous, namely if $y_0^t \in \omega(y_0)$ for some $t \in \mathbb{R}^+$.

$$\bar{\mathscr{U}}'(y) := \begin{cases} C(t) & y = y_0^t \\ \mathscr{U}(y) & y \in \omega(y_0) \end{cases}$$

For this reason, we consider an extended parameter space

$$Y' := Y \times [0,1]$$

equipped with a canonical semiflow $(y,\lambda)^t := (y^t, e^{-t}\lambda)$.
Using this extended phase space, an extended unstable bundle $(\bar{\mathscr{U}}(y,\lambda))_{(y,\lambda)\in\Sigma^+(y_0,1)}$ can be defined.

$$\bar{\mathscr{U}}(y,\lambda) := \begin{cases} C(t) & \lambda = e^{-t} \\ \mathscr{U}(y) & \lambda = 0 \end{cases}$$

So far, $\bar{\mathscr{U}}$ is a family of linear subspaces of X. By Lemma 2.40, there is a direct sum decomposition

$$X = \bar{\mathscr{U}}(y,\lambda) \oplus \mathscr{S}(y) \tag{2.15}$$

for every $(y,\lambda) \in \Sigma^+(y_0,1)$. Associated with (2.15), there is a family of projections $P(y,\lambda) \in \mathscr{L}(X, \bar{\mathscr{U}}(y,\lambda))$ with $\ker P(y,\lambda) = \mathscr{S}(y)$.

LEMMA 2.41. *Let $((y_n,\lambda_n,x_n))_n$ be a sequence in $\bar{\mathscr{U}}$ with $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Then there is a subsequence $((y_{n(k)},\lambda_{n(k)},x_{n(k)}))_k$ with $(y_{n(k)},\lambda_{n(k)},x_{n(k)}) \to (y,\lambda,x) \in \bar{\mathscr{U}}$ as $k \to \infty$.*

PROOF. It is clear that $\Sigma^+(y_0,1)$ is compact, so we can assume w.l.o.g. that $(y_n,\lambda_n) \to (y,\lambda)$. Taking subsequences if necessary, it is sufficient to consider three cases.

(1) $\lambda_n, \lambda > 0$: Let $\lambda_n, \lambda = e^{-t_n}, e^{-t}$ and $x_n = \Phi(t_n, y_0, c_n)$ for some $c_n \in C$. By Lemma 2.40 (a) and since C is finite-dimensional, there are subsequences denoted by $n(k)$ such that $c_{n(k)} \to c \in C$ as $k \to \infty$. Hence, $x_{n(k)} \to \Phi(t, y_0, c)$ as $k \to \infty$.
(2) $\lambda_n \to 0$: We have $x_n \in C(t_n)$ for all $n \in \mathbb{N}$. By Lemma 2.40 (b), there are subsequences denoted by $n(k)$ such that $x_{n(k)} \to u \in \mathscr{U}(y)$ as $k \to \infty$.
(3) $\lambda_n \equiv 0$: It follows immediately from Lemma 2.40 (e) that there exists a subsequence $x_{n(k)} \to x_0 \in \mathscr{U}(y)$.

□

LEMMA 2.42. *Let $(y_n,\lambda_n,x_n) \to (y,\lambda,x)$ in $\Sigma^+(y_0,1) \times X$.*
Then $P(y_n,\lambda_n)x_n \to P(y,\lambda)x$.

PROOF. Let $x_n = u_n + s_n \in \bar{\mathscr{U}}(y_n,\lambda_n) + \mathscr{S}(y_n)$ and $x = u + s \in \bar{\mathscr{U}}(y,\lambda) + \mathscr{S}(y)$. We begin by showing that our claim is true under the additional assumption that $(\|u_n\|)_n$ is bounded. Suppose to the contrary that $u_n \not\to u$. We may assume without loss of generality that

$$\|u_n - u\| \geq \eta > 0. \tag{2.16}$$

By Lemma 2.41, there is a subsequence denoted by $n(k)$ such that $u_{n(k)} \to u' \in \bar{\mathscr{U}}(y,\lambda)$. Thus, we have $s_n = x_{n(k)} - u_{n(k)} \to x - u' =: s'$. It follows from Lemma 2.39 that $s' \in \mathscr{S}(y,\lambda)$. This

shows that $x = u + s = u' + s'$, but the decomposition is unique, so $u' = u$, in contradiction to (2.16).
Now suppose that $\alpha_n := \|u_n\| \to \infty$. It follows that $\alpha_n^{-1} x_n = \alpha_n^{-1} u_n + \alpha_n^{-1} s_n \to 0$ as $n \to \infty$. Using the first part of this proof, we obtain that $\alpha_n^{-1} u_n \to 0$ as $n \to \infty$, but $\left\|\alpha_n^{-1} u_n\right\| \equiv 1$. □

LEMMA 2.43. *The sets*

$$\bar{\mathscr{U}} := \{(y, \lambda, x) \in \Sigma^+(y_0, 1) \times X : \; x \in \bar{\mathscr{U}}(y, \lambda)\}$$
$$\bar{\mathscr{S}} := \{(y, \lambda, x) \in \Sigma^+(y_0, 1) \times X : \; x \in \mathscr{S}(y)\}$$

are closed in $Y' \times X$.

PROOF. Let (y_n, λ_n, u_n) be a sequence in $\bar{\mathscr{U}}$ with $(y_n, \lambda_n, u_n) \to (y, \lambda, x)$ in X. It follows from Lemma 2.42 that $0 = P(y_n, \lambda_n)u_n - u_n \to P(y, \lambda)x - x$, so $P(y, \lambda)x = x$, that is, $(y, \lambda, x) \in \bar{\mathscr{U}}$.
Now let (y_n, λ_n, s_n) be a sequence in $\bar{\mathscr{S}}$ with $(y_n, \lambda_n, s_n) \to (y, \lambda, x)$ in X. We have $0 = P(y_n, \lambda_n)s_n \to P(y, \lambda, x) = 0$, so $x \in \mathscr{S}(y)$. □

LEMMA 2.44. *For every* $M \geq 0$, *there is a* $t_0 \in \mathbb{R}^+$ *such that* $\left\|\Phi(t, y, x)\right\| \geq M\|x\|$ *for all* $t \geq t_0$ *and all* $x \in \bar{\mathscr{U}}(y, \lambda)$.

PROOF. Suppose to the contrary that there are sequences $(y_n, \lambda_n) \to (y, \lambda)$, $t_n \to \infty$, and $x_n \in \bar{\mathscr{U}}(y_n, \lambda_n)$ with $\|x_n\| \equiv 1$ such that $\left\|\Phi(t_n, y_n, x_n)\right\| \leq M$ for all $n \in \mathbb{N}$. In view of Lemma 2.41, we may assume without loss of generality that $(y_n, \lambda_n, x_n) \to (y, \lambda, x) \in \bar{\mathscr{U}}$.
Denote $\alpha_n := \sup_{s \in [0, t_n]} \left\|\Phi(s, y_n, x_n)\right\|$. It follows from Lemma 2.37 (c) that $\sup_{n \in \mathbb{N}} \alpha_n < \infty$, so $x \in \mathscr{S}(y)$. We have $x \in \mathscr{S}(y) \cap \bar{\mathscr{U}}(y, \lambda) = \{0\}$ due to the weak hyperbolicity of y_0. However, $\|x\| = \lim_{n \to \infty} \|x_n\| = 1$. □

LEMMA 2.45.

(a) $T(t)\colon C \to C(t)$, $T(t)x := \Phi(t, y_0, x)$ *is an isomorphism for all* $t \in \mathbb{R}^+$
(b) $T'(t)\colon C \to C(t)$, $T'(t)x := T(t)x \cdot \|x\| / \|T(t)x\|$ *is a homeomorphism for all* $t \in \mathbb{R}^+$
(c) *Let* $\bar{C} := \{(t, x) \in \mathbb{R}^+ \times X : \; x \in C(t)\}$ *and define* $\hat{T} : \mathbb{R}^+ \times C \to \bar{C}$ *by* $\hat{T}(t, x) := (t, T'(t)x)$. $\hat{T}$ *is a homeomorphism.*

PROOF. (a) Since $\dim C(t) = k < \infty$, it suffices to show that $T(t)$ is injective for every $t \in \mathbb{R}^+$. This follows from the choice of C because $\Phi(t, y_0, x) = 0$ implies that $x \in \mathscr{S}(y_0)$ but $C \cap \mathscr{S}(y_0) = \{0\}$.
(b) It follows immediately from (a) that there are constants $0 < C_1 = C_1(t) \in \mathbb{R}^+$ and $C_2 = C_2(t) \in \mathbb{R}^+$ such that $C_1 \leq \|x\| / \|T(t)x\| \leq C_2$ for all $0 \neq x \in C$. Hence, $T'(t)$ is well-defined and continuous. For $x' \in C(t)$, set $x := (T(t))^{-1} x'$. One has $(T'(t))^{-1}(x') = x \frac{\|x'\|}{\|x\|}$, which is continuous, too.
(c) This follows from the fact that $T'(t)$ and its inverse depend continuously on t.

□

PROOF OF THEOREM 2.36. Let the semiflow π' on $Y' \times X$ be defined by $(y, \lambda, x)\pi' t = (y^t, \lambda e^{-t}, \Phi(t, y, x))$, and let π_S denote the restriction of π' to $\bar{\mathscr{S}}$. The set

$$N_s := \{(y, \lambda, x) \in \bar{\mathscr{S}} : \; \left\|y\right\| \leq 1\}$$

is a strongly admissible isolating neighborhood for $\omega(y_0,1)\times\{0\}$ relative to π_S. Furthermore, one has $\mathrm{Inv}^-(N_s)=\omega(y_0,1)\times\{0\}$, so it follows from Lemma 1.14 that

$$(N_s^+,\emptyset):=(\{(y,\lambda,x)\in\bar{\mathcal{S}}:\sup_{t\geq 0}\left\|\Phi(t,y,x)\right\|\leq 1\},\emptyset)$$

is a (strongly admissible) FM-index pair for $(\pi_S,\omega(y_0,1)\times\{0\})$.
Let π_U denote the restriction of π' to $\bar{\mathcal{U}}$. By Lemma 2.44, there exists a $\tau>0$ such that $\left\|\Phi(t,y,x)\right\|\geq 2\|x\|$ for all $t\geq\tau$ and all $(y,\lambda,x)\in\bar{\mathcal{U}}(y,\lambda)$. Define

$$N_{u,1}:=\{(y,\lambda,x)\in\bar{\mathcal{U}}:\|x\|\leq 1\}$$
$$N_{u,2}:=\{(y,\lambda,x)\in N_{u,1}:\exists t\leq\tau:\left\|\Phi(t,y,x)\right\|=1\}.$$

It is clear that $N_{u,1}$ and $N_{u,2}$ are closed subsets. Moreover, $N_{u,1}$ is strongly admissible since the X-norm is bounded and $y_0\in Y_{cl}$. It is also clear that $N_{u,2}$ is an exit ramp for $N_{u,1}$. We need to show that $N_{u,2}$ is $N_{u,1}$-positively invariant. Let $(y,\lambda,x)\in N_{u,2}$ and choose $t\geq 0$ maximal with $(y,\lambda,x)\pi'[0,t]\subset N_{u,1}$. By the definition of $N_{u,2}$, there is a $t_1\leq\tau$ with $\left\|\Phi(t_1,y,x)\right\|=1$ and $(y,\lambda,x)\pi'[0,t_1]\subset N_{u,2}$. Due to the choice of τ, there is a $t_2\in[t_1,t_1+\tau]$ with $\left\|\Phi(t_2,y,x)\right\|=2$, so $t\leq t_2$. Hence, $(y,\lambda,x)\pi'[t_1,t]\subset N_{u,2}$. We have shown that $(N_{u,1},N_{u,2})$ is an FM-index pair for $(\pi_u,\omega(y_0,1)\times\{0\})$.
Let $x\in X$ with $\|x\|=1$ and $(y,\mu_0x)\in N_{u,2}$ for some $\mu_0\in\,]0,1]$. Then there is a $t\leq\tau$ such that $\left\|\Phi(t,y,\mu_0x)\right\|=1$, so $\left\|\Phi(t,y,\mu x)\right\|=\mu/\mu_0$. Hence, there is a $\tilde{t}\leq t$ with $\left\|\Phi(\tilde{t},y,\mu x)\right\|=1$. It follows that

$$x\in N_{u,2}\text{ only if }\mu x\in N_{u,2}\text{ for all }\mu\in\left[1,\|x\|^{-1}\right]. \tag{2.17}$$

The next step is to calculate $\mathrm{h}(\pi,y_0,\omega(y_0)\times\{0\})$. By our assumptions, $K_0:=\omega(y_0)\times\{0\}$ (resp. $K_0':=\omega(y_0,1)\times\{0\}$) is an isolated invariant set admitting a strongly admissible isolating neighborhood in $\Sigma^+(y_0)\times X$ relative to π (resp. in $\Sigma^+(y_0,1)\times X$ relative to π'), so $\mathrm{h}(\pi,y_0,K_0)$ and $\mathrm{h}(\pi',(y_0,1),K_0')$ are well-defined. Let (N_1',N_2') be a strongly admissible FM-index pair for (π',K_0'). Then $N_i:=\{(y,x):(y,0,x)\in N_i'\}$ is a strongly admissible FM-index pair by Lemma 1.13 and the assumption that $y_0\in Y_{cl}$. Now, both FM-index pairs (N_1',N_2') and (N_1,N_2) induce the same index pair in $\mathbb{R}^+\times X$, that is, the indices defined by π' and π agree.
We have already constructed an FM-index pair for $(\pi',\omega(y_0,1)\times\{0\})$, namely $(N_{u,1}\oplus N_s^+,N_{u,2}\oplus N_s^+)$, where we set

$$N_{u,1}\oplus N_s^+:=\{(y,\lambda,u+s):(y,\lambda,u)\in N_{u,1}\quad(y,\lambda,s)\in N_s^+\}$$
$$N_{u,2}\oplus N_s^+:=\{(y,\lambda,u+s):(y,\lambda,u)\in N_{u,2}\quad(y,\lambda,s)\in N_s^+\}.$$

According to Definition 2.13, one has $\mathrm{h}(\pi',(y_0,1),K_0')=\mathrm{h}(M_1/M_2,M_2)$, where

$$M_1:=\{(t,x)\in\mathbb{R}^+\times X:(y_0^t,e^{-t},x)\in N_{u,1}\oplus N_s^+\}$$
$$M_2:=\{(t,x)\in\mathbb{R}^+\times X:(y_0^t,e^{-t},u\oplus s)\in N_{u,2}\oplus N_s^+\}.$$

Further define

$$M_2':=\{(t,u\oplus s)\in M_2:\|u\|=1\}.$$

Using (2.17), one can prove that $\mathrm{h}(M_1/M_2,M_2)=\mathrm{h}(M_1/M_2',M_2')$.

Let

$$M_{u,1} := \{(t,x)\in M_1 : \ x \in C(t)\} = \{(t,x) : (y_0^t, e^{-t}, x) \in N_{u,1}\}$$
$$M_{u,2} := M_2' \cap M_{u,1} = \{(t,x) \in M_{u,1} : \ \|x\| = 1\}$$

and consider the inclusion induced morphism $i : (M_{u,1}/M_{u,2}, M_{u,2}) \to (M_1/M_2', M_2')$. We construct a mapping
$H : [0,1] \times (M_1, M_2') \to (M_1/M_2', \{M_2'\})$ by setting

$$H(\mu, (t, u \oplus s)) := (t, u \oplus (\mu \cdot s)).$$

H is well defined because $(y,\lambda,x) \in N_s^+$ implies that $(y,\lambda,\mu\cdot x) \in N_s^+$ for all μ with $|\mu| \le 1$. Now, H gives rise to a continuous mapping $\hat{H} : (M_1/M_2', M_2') \to (M_1/M_2', M_2')$ defined by $\hat{H}(t,[x]) = H(t,x)$. It follows that the homotopy types of $(M_{u,1}/M_{u,2}, M_{u,2})$ and $(M_1/M_2', M_2')$ coincide.
Let $B := \{x \in C : \ \|x\| \le 1\}$ denote the unit ball in $C = C(0)$ and set $(\hat{M}_1, \hat{M}_2) := (\mathbb{R}^+ \times B, \mathbb{R}^+ \times \partial B)$. It follows using Lemma 2.45 (c) that $(M_{u,1}/M_{u,2}, M_{u,2})$ and $(\hat{M}_1/\hat{M}_2, \hat{M}_2)$ are isomorphic. Finally, it is easy to see that $(\hat{M}_1/\hat{M}_2, \hat{M}_2)$ and $(B/\partial B, \partial B)$ are isomorphic in the homotopy category of pointed spaces. Summing it up, we have shown that

$$\mathrm{h}(\pi, y_0, K_0) = \mathrm{h}(B/\partial B, \partial B) = \Sigma^k,$$

where $k = \dim C = \operatorname{codim} \mathscr{S}(y_0)$. □

2.5. Application to Differential Equations

We consider ordinary differential equations as well as semilinear parabolic equations and give examples how the abstract theory developed in the previous sections can be applied to these classes of equations.
We restrict our attention to the simplest possible setting: asymptotically linear equations (without resonance). We are able to generalize existence results, which are well known for autonomous or periodic equations, to large classes of nonautonomous equations (Corollary 2.59 and Theorem 2.60).

2.5.1. Abstract Semilinear Parabolic Equations. Let X be a Banach space and $A : X^1 \subset X^0 \to X^0$ a positive sectorial[17] operator. Here, X^α is the α-th fractional power space defined by A with norm $\|.\|_\alpha := \|A^\alpha.\|_X$.

DEFINITION 2.46. Let $(V, \|.\|_V)$ and $(W, \|.\|_W)$ be arbitrary normed spaces. By $C_b(V,W)$, we denote the space of all continuous mappings $f : V \to W$, which map bounded sets into bounded sets, that is, if $B \subset V$ is bounded with respect to $\|.\|_V$ then so is $f(B)$ with respect to $\|.\|_W$.
C_b is endowed with a metric which induces the bounded-open topology, that is, a sequence of functions in C_b converges iff it converges uniformly on every bounded subset of V.

Fix some $0 \le \alpha < 1$, and consider the parameter space $\mathscr{Y}$ which consists of all functions $y : \mathbb{R} \times X^\alpha \to X^0$ satisfying the following condition: for every bounded (in $\mathbb{R} \times X^\alpha$) set $B \subset \mathbb{R} \times X^\alpha$,

[17] As defined in [**26**, Section 3.6].

there are constants $C = C(y, B) \geq 0$ and $\delta \geq 0$ such that for all $(t, x), (t', x') \in B$:

$$\left\| y(t,x) \right\| \leq C$$
$$\left\| y(t,x) - y(t',x') \right\| \leq C(\left| t - t' \right|^{\delta} + \left\| x - x' \right\|_{\alpha})$$

For $y, y' \in \mathscr{Y}$, define

$$\delta^n(y, y') := \sup\{\left\| y(t,x) - y'(t,x) \right\|_0 : |t| \leq n, \ \|x\|_\alpha \leq n\}.$$

The family $(\delta^n)_{n\in\mathbb{N}}$ defines a metric $d = d_{(\delta^n)_{n\in\mathbb{N}}}$ on $\mathscr{Y}$, where

$$d_{(\gamma_n)_{n\in\mathbb{N}}}(y, y') := \sum_{n\in\mathbb{N}} 2^{-n} \frac{\gamma_n(y, y')}{1 + \gamma_n(y, y')}. \tag{2.18}$$

Note that d induces the bounded-open topology on $\mathscr{Y}$. A t-translation can be defined canonically by $y^t(s) := y(t+s)$. Now it is easy to see that $(\mathscr{Y}, d)$ satisfies the assumptions (L1) and (L2) at the beginning of Section 2.3.2.

REMARK 2.6. *The dynamical system defined by* $(t, y) \mapsto y^t$ *on* $\mathscr{Y}$ *is sometimes referred to as* Bebutov dynamical system *[**25**, p. 32].*

LEMMA 2.47. $(t, y) \mapsto y^t$ *defines a global semiflow on* $(\mathscr{Y}, d)$.

PROOF. We will only prove the continuity with respect to d. Let $y_n \to y_0$ in $\mathscr{Y}$, $t_n \to t_0$ in $\mathbb{R}^+$, and consider an arbitrary bounded set $B \subset \mathbb{R} \times X$. There are constants $C, \delta > 0$ depending on $B' := \{(t + t_n, x) : (t, x) \in B \text{ and } n \in \mathbb{N} \cup \{0\}\}$ such that

$$\begin{aligned} &\left\| y_n(t+t_n, x) - y_0(t+t_0, x) \right\| \\ &\quad \leq \left\| y_n(t+t_n, x) - y_0(t+t_n, x) \right\| + \left\| y_0(t+t_n, x) - y_0(t+t_0, x) \right\| \\ &\quad \leq \left\| y_n(t+t_n, x) - y_0(t+t_n, x) \right\| + C|t_n - t_0|^{\delta} \to 0 \end{aligned}$$

uniformly on B. □

Now, let $\mathscr{X} := X^\alpha$, and define a semiflow π of $\mathscr{Y} \times \mathscr{X}$, where $(v(t), u(t))$ is a solution of π if and only if $v(t) = v(0)^t$ and u is a mild solution of

$$\dot{u}(t) + Au(t) = v(t)(0, u(t))$$

It is well known [**26**, Theorem 47.5] that π is continuous.

REMARK 2.7. *Just like the autonomous Conley index, the index considered here relies on an asymptotic compactness assumption, called skew-admissibility. Skew-admissibility is a straightforward generalization of admissibility.*
For instance, a subset $N \subset \mathscr{Y} \times \mathscr{X}$ *is strongly skew-admissible (Definition 2.23) if* $N_X := \{x : (y, x) \in N : \text{ for some } y \in \mathscr{Y}\}$ *is bounded in* X^α *and the resolvent mapping* $(A + k)^{-1}$ *is compact (see [**22**, Theorem 4.3]), which implies that the inclusion* $X^\alpha \subset X^\beta$ *is compact for* $0 \leq \beta < \alpha \leq 1$.

2.5.2. Ordinary Differential Equations. This section is intended to illustrate how the general theory applies to ordinary differential equations. We do not strive for maximum generality.

Setting $A :=$ id and $X := \mathbb{R}^N$, $N \in \mathbb{N}$, the results of the previous section can be applied. In this particular case, one has $X^\alpha = X$ for all $\alpha \in [0,1]$. Let Y denote the set of all continuous functions $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ with the following additional property: for every bounded set $B \subset \mathbb{R}^N$, there exist constants $C_1 = C_1(B), C_2 = C_2(B), \delta = \delta(B) \in \mathbb{R}^+$ such that

$$\left|f(t,x)\right| \le C_1 \tag{2.19}$$

$$\left|f(t,x) - f(t',x')\right| \le C_2\left(\left|t-t'\right|^\delta + \left|x-x'\right|\right) \tag{2.20}$$

for all $(t,x),(t',x') \in \mathbb{R} \times B$. The set Y is a subset[18] of the set $\mathscr{Y}$ defined in the previous section. We consider Y as a metric (sub)space equipped with the metric d of uniform convergence on bounded subsets of $\mathbb{R} \times \mathbb{R}^N$ as defined for $\mathscr{Y}$ in the previous section.

LEMMA 2.48. *$\Sigma^+(y)$ is compact for every $y \in Y$, that is, $Y = Y_c$.*

A characterization of those functions which are *positively compact* i.e., having a compact positive hull, is [**25**, Theorem III.7]. Namely, a continuous function $y : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ (respectively the motion $t \mapsto y^t$) is positively compact if and only if it is bounded and uniformly continuous on every set $\mathbb{R}^+ \times B$, where $B \subset \mathbb{R}^N$ is bounded. The aforementioned book is also a valuable source for other possible choices of (Y,d).

PROOF. Let $y \in Y$ and $t_n \to \infty$ in $\mathbb{R}^+$. In view of the remark above, one can assume without loss of generality that $y^{t_n} \to y_0 \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ uniformly on bounded (compact) subsets. One needs to prove that $y_0 \in Y$.
Let $B \subset \mathbb{R}^N$ be a bounded subset. There are constants $C_1, C_2, \delta > 0$ depending on y and B such that

$$\begin{aligned}\left|y_0(t,x)\right| &\le \left|y(t_n+t,x) - y_0(t,x)\right| + C_1\\ \left|y_0(t,x) - y_0(t',x')\right| &\le \left|y(t_n+t,x) - y_0(t,x)\right| + \left|y(t_n+t',x') - y_0(t',x')\right|\\ &\quad + C_2\left(\left|t-t'\right|^\delta + \left|x-x'\right|\right)\end{aligned}$$

for all $(t,x),(t',x') \in \mathbb{R} \times B$. Passing to the pointwise limit $n \to \infty$, we obtain for arbitrary $(t,x),(t',x') \in \mathbb{R} \times B$

$$\begin{aligned}\left|y_0(t,x)\right| &\le C_1\\ \left|y_0(t,x) - y_0(t',x')\right| &\le C_2\left(\left|t-t'\right|^\delta + \left|x-x'\right|\right),\end{aligned}$$

where the constants C and δ depend only on $y \in Y$ and the bounded set B. □

2.5.3. Quasilinear Parabolic Equations. Let $\Omega \subset \mathbb{R}^m$, $m \ge 1$ be a bounded domain with smooth boundary, and let $A := A(x,D)$ be given by

$$A(x)u := -\sum_{k,l=1}^m \frac{\partial}{\partial x_k}\left(a_{k,l}(x)\frac{\partial u}{\partial x_l}\right) + c(x)u. \tag{2.21}$$

Assume that $a_{k,l}(x) = a_{l,k}(x)$ and $c(x)$ are real-valued and continuously differentiable in $\bar{\Omega}$.

[18] In contrast to the previous section, it is required that δ can be chosen uniformly in t, which is a slightly stronger assumption.

Suppose that $A(x)$ is strongly elliptic that is, there is a constant $C > 0$ such that

$$\sum_{k,l=1}^{m} a_{k,l}(x)\xi_k\xi_l \geq C\sum_{k=1}^{m}\xi_k^2 \quad \forall(\xi_1,\dots,\xi_m)\in\mathbb{R}^m.$$

The crucial point here is not so much the underlying space X or the elliptic operator but the non-linearity of the abstract equation which arises as Nemytskii operator defined by a sufficiently regular function f. In contrast to the previous section concerning ordinary differential equations, the spaces Y and $\mathscr{Y}$ will differ.
Consider the following equation

$$\begin{aligned} \partial_t u + A(x)u &= f(t,x,u(t,x),\nabla u(t,x)) & \\ u(t,x) &= 0 & x\in\partial\Omega \\ u(t,x) &= u_0(x) & x\in\Omega. \end{aligned}$$

Let $1 < p < \infty$, $X^0 := L^p(\Omega)$, and define an operator A_p with

$$\mathscr{D}(A_p) := W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega),$$

where we set $A_p(u) := A(x)u$. The derivatives are understood in the weak sense.

THEOREM 2.49. *([**20**, Theorem 3.6 in Chapter 7]) Let* $1 < p < \infty$ *and* $c(x)\equiv 0$.
The operator $-A_p$ *is the infinitesimal generator of an analytic semigroup of contractions on* $L^p(\Omega)$.

In particular, A_p is a sectorial operator, and we can assume without loss of generality that $c(x) \geq c_0 > 0$ for all $x\in\Omega$, so A_p is positive. Fix some $p > m$, let $A := A_p$ and let A^α the α-th fractional power of α as defined in [**20**]. The corresponding space is

$$X^\alpha := \{x\in X : \left\|A^\alpha x\right\|_{0,p} < \infty\}$$

equipped with the norm $\|x\|_\alpha := \|A^\alpha x\|_{0,p}$.
It is well-known (see e.g. [**8**, Theorem 5.6.5]) that the inclusion $W^{1,p}(\Omega)\subset L^p(\Omega)$ is compact (completely continuous). Consequently, A has compact resolvent, and any of the inclusions $X^\beta\subset X^\alpha$, $\alpha<\beta$ is compact.
Furthermore, there is a constant $C > 0$ such that

$$\|x\|_{2,p} \leq C\|Au\|_{0,p}. \tag{2.22}$$

LEMMA 2.50. *There is a* $1 > \theta \geq 0$ *such that for all* $\alpha > \theta$ *there is a continuous embedding*[19] $X^\alpha\subset C^1(\bar\Omega)$.

In the following proof[20], this claim is reduced to (2.22) and the Gagliardo-Nirenberg inequality as stated in [**11**, Theorem 10.1]. In view of Lemma 2.50, we can fix an $\alpha\in[0,1[$ such that the inclusion $X^\alpha\subset C^1(\bar\Omega)$ is continuous.

[19]The embedding is defined as usual: $X^\alpha \ni [x]\mapsto x\in C^1(\bar\Omega)$.
[20] The situation in the literature is difficult. There are proofs of this claim which appear to be wrong. Lemma 37.8 in [**26**] is useful, but the details are given only for the Sobolev case (which is similar).

PROOF. There is a $\theta \in [0,1[$ such that

$$-\frac{1}{m} = \theta\left(\frac{1}{p} - \frac{2}{m}\right) + (1-\theta)\frac{1}{p}.$$

It follows from [**11**, Theorem 10.1] that $W^{2,p}(\Omega) \subset C^1(\bar{\Omega})$ and

$$\|x\|_{1,\infty} \le C_1 \|x\|_{2,p}^{\theta} \|x\|_{0,p}^{1-\theta} \quad x \in W^{2,p}(\Omega).$$

We will prove that the lemma holds for $\alpha_0 := \theta$, that is, there is a continuous embedding $X^\alpha \subset C^1(\bar{\Omega})$ provided that $\alpha > \theta$.

By using Young's inequality, one concludes that for all $\varepsilon > 0$

$$\|x\|_{1,\infty} \le C_1 \left(\theta \varepsilon^{1-\theta} \|x\|_{2,p} + (1-\theta)\varepsilon^{-\theta} \|x\|_{0,p}\right)$$

and using (2.22) one finds a constant $C_2 = C_2(\theta)$ such that

$$\|x\|_{1,\infty} \le C_2 \left(\varepsilon^{1-\theta} \|Ax\|_{0,p} + \varepsilon^{-\theta} \|x\|_{0,p}\right) \tag{2.23}$$

From equation (2.6.9) in [**20**] it follows that

$$x = \int_0^\infty t^{\alpha-1} e^{-At} A^\alpha x \,\mathrm{d}t \quad x \in X^\alpha.$$

Suppose [**20**, Theorem 6.13] that $\left\|e^{-At} A^\beta x\right\|_0 \le M t^{-\beta} e^{-\delta t} \|x\|_0$ for all $\beta \in [0,1]$. Subsequently, by using (2.23) with $\varepsilon = t$, we obtain

$$\begin{aligned}
\|x\|_{1,\infty} &\le \int_0^\infty t^{\alpha-1} \left\|e^{-At} A^\alpha x\right\|_{1,\infty} \mathrm{d}t \\
&\le C_2 \int_0^\infty t^{\alpha-1} \left(t^{1-\theta} \left\|e^{-At} A^{1+\alpha} x\right\|_{0,p} + t^{-\theta} \left\|e^{-At} A^\alpha x\right\|_{0,p}\right) \mathrm{d}t \\
&\le C_2 M \int_0^\infty t^{\alpha-\theta-1} e^{-\delta t} \left\|A^\alpha x\right\|_{0,p} + t^{\alpha-\theta-1} e^{-\delta t} \left\|A^\alpha x\right\|_{0,p} \mathrm{d}t \\
&\le 2C_2 M \int_0^\infty t^{\alpha-\theta-1} e^{-\delta t} \,\mathrm{d}t \|x\|_\alpha.
\end{aligned}$$

It is easy to see that $\int_0^\infty t^{\alpha-\theta-1} e^{-\delta t} \,\mathrm{d}t < \infty$ provided that $\alpha > \theta$.

$W^{2,p}(\Omega)$ is dense in $L^p(\Omega)$, so the estimate $\|x\|_{1,\infty} \le C_3 \|x\|_\alpha$ holds for all $x \in X^\alpha$. □

In analogy to the previous section, let Y denote the set of all continuous functions $f : \mathbb{R} \times \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ with the following property: for every bounded set $B \subset \mathbb{R} \times \mathbb{R}^m$, there exist constants $C_1 = C_1(B), C_2 = C_2(B), \delta = \delta(B) \in \mathbb{R}^+$ such that

$$\begin{array}{lcl}
\left|f(t,\omega,u,v)\right| & \le & C_1 \\
\left|f(t,\omega,u,v) - f(t',\omega',u',v')\right| & \le & C_2\left(\left|t-t'\right|^\delta + \left|\omega-\omega'\right|^\delta + \left|u-u'\right| + \left|v-v'\right|\right)
\end{array} \tag{2.24}$$

for all $(t,\omega,u,v),(t',\omega',u',v')\in\mathbb{R}\times\bar{\Omega}\times B$.
We consider a metric $d=d_{(\delta^n)_n}$ on Y, which is defined by (2.18) and a family $(\delta^n)_n$

$$\delta^n(f,f'):=\sup_{x\in\Omega\ \max\{|t|,|u|,|v|\}\le n}\left|f(t,\omega,u,v)-f'(t,\omega,u,v)\right|$$

– in analogy to the metric on $\mathscr{Y}$.
To each $f\in Y$, there is an associated Nemytskii operator $\hat{f}$ defined by

$$\hat{f}(t,x)(\omega):=f(t,\omega,x(\omega),\nabla x(\omega))\quad x\in C^1(\bar{\Omega}).$$

LEMMA 2.51. *The mapping $f\mapsto\hat{f}$, $(Y,d)\to(\mathscr{Y},d)$ is continuous.*

PROOF. The inclusion $X^\alpha\subset C^1(\bar{\Omega})$ is continuous by Lemma 2.50 and the choice of α, so we may assume that $\|x\|_{C^1(\bar{\Omega})}\le C\|x\|_\alpha$ for some $C>0$. Let $x\in X^\alpha$ with $\|x\|_\alpha\le n/C$ and $t\in\mathbb{R}$ with $|t|\le n$. We have $\left|f(t,\omega,x(\omega),\nabla x(\omega))\right|\le\delta^n(f)$ for all $\omega\in\Omega$, so

$$\begin{aligned}\left\|\hat{f}(t,x)\right\|_0&=\left(\int_\Omega\left|f(t,\omega,x(\omega),\nabla x(\omega))\right|^p\,\mathrm{d}\omega\right)^{1/p}\\&\le C\left(\int_\Omega\left(\delta^n(f)\right)^p\,\mathrm{d}\omega\right)^{1/p}\|x\|_\alpha.\end{aligned}\tag{2.25}$$

$f_n\to 0$ in (Y,d) now means $\delta^n(f)\to 0$ for every $n\in\mathbb{N}$. By (2.25), this implies $\left\|\hat{f}(t,x)\right\|_0\to 0$ uniformly on bounded sets. □

In view of Lemma 2.51, it is sufficient for our applications to prove the following lemma (instead of a more involved $\mathscr{Y}$-variant thereof). The statement is analogous to Lemma 2.48, so its proof is omitted.

LEMMA 2.52. *$\Sigma^+(y)$ is compact for every $y\in Y$, i.e., $Y=Y_c$.*

In the following section (Lemma 2.57), we will deal with abstract asymptotically linear equations. In order to check the assumption of asymtotical linearity, the following lemma can be used.

LEMMA 2.53. *Suppose that $f\in Y$, and $\sup_{\omega\in\Omega}f(t,\omega,u,v)/(\max\{|u|,|v|\})\to 0$ as $t\to\infty$, $\max\{|u|,|v|\}\to\infty$. Then, $\|x\|_\alpha^{-1}\left\|\hat{f}(t,x)\right\|_0\to 0$ as $t\to\infty$ and $\|x\|_\alpha\to\infty$.*

PROOF. Let $\varepsilon>0$. By our assumptions, there are constants $r,M\in\mathbb{R}^+$ such that

$$\left|f(t,\omega,u,v)\right|\le\begin{cases}\varepsilon\max\{|u|,|v|\} & \max\{|u|,|v|\}\ge r\\ M & \text{otherwise}\end{cases}$$

whenever $t\ge r$. Recall that the inclusion $X^\alpha\subset C^1(\bar{\Omega})$ is continuous, so there is a constant $C_1>0$ such that $\max\{|x(\omega)|,|\nabla x(\omega)|\}\le C\|x\|_\alpha$ for all $x\in X^\alpha$ and all $\omega\in\Omega$.

Let $0 \neq x \in X^\alpha$, $\Omega_1 = \{\omega \in \Omega : \max\{|x(\omega)|, |\nabla x(\omega)|\} \leq r\}$. We have

$$\begin{aligned}
\left\|\hat{f}(t,x)\right\|_0 &= \left(\int_\Omega \left| f(t,\omega,x(t,\omega),\nabla x(t,\omega)) \right|^p \,\mathrm{d}x \right)^{1/p} \\
&\leq \left(\int_{\Omega_1} M^p \,\mathrm{d}x \right)^{1/p} + \left(\int_{\Omega \setminus \Omega_1} (\varepsilon \max\{|x(\omega)|, |\nabla x(\omega)|\})^p \,\mathrm{d}x \right)^{1/p} \\
&\leq (M\|x\|_\alpha^{-1} + C_1\varepsilon) \left(\int_\Omega 1 \,\mathrm{d}x \right)^{1/p} \|x\|_\alpha .
\end{aligned}$$

Hence, for every $\varepsilon > 0$, for all $x \in X^\alpha$ with $\|x\|_\alpha$ sufficiently large and for all $t \in \mathbb{R}$ sufficiently large, one has $\left\|\hat{f}(t,x)\right\|_0 \leq C_2\varepsilon\|x\|_\alpha$, where the constant C_2 does not depend on ε. The claim follows immediately. □

2.5.4. Existence of Solutions. In this section, we treat the settings of 2.5.2 and 2.5.3 simultaneously. We therefore use the abstract setting of Section 2.5.1.
Additionally, we tacitly assume that the inclusion $X^\beta \subset X^\alpha$ is compact for $\beta \in]\alpha, 1[$. This holds trivially if X is finite-dimensional and is well known if A has compact resolvent. Recall that $\mathscr{Y}_c$ is the subset of all those $y \in \mathscr{Y}$ for which $\Sigma^+(y)$ is compact.
Firstly, we will compute the index for parameters which are linearizable at 0 respectively ∞ (asymptotically linear). Next, we will derive several existence results, following more or less directly from the index computation.

LEMMA 2.54. *Let $f_n, L_0 \in \mathscr{Y}_c$ and L_0 weakly hyperbolic and linear, i.e., $L_0(t,.) \in \mathscr{L}(X^\alpha, X)$ for all $t \in \mathbb{R}$. Assume further that*

(LIN0) *for every $\varepsilon > 0$, there are $\delta > 0$ such that $\left\|f_n(t,x) - L_0(t)x\right\| \leq \varepsilon\|x\|_\alpha$ for all $(t,x) \in \mathbb{R}^+ \times X^\alpha$ with $t, n \geq 1/\delta$ and $\|x\|_\alpha \leq \delta$.*

Then, there are $\eta_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $\eta \leq \eta_0$ and all $n \geq n_0$, $N_{n,\eta} := \Sigma^+(f_n) \times B_\eta[0, X^\alpha]$ is an isolating neighborhood.

PROOF. Suppose that the claim is not true. Taking a subsequence of $(f_n)_n$ if necessary, we can assume without loss of generality that there exists a sequence $(\eta_n)_n$ of real numbers with $0 \neq \eta_k \to 0$ in $\mathbb{R}^+$ such that for all $n \in \mathbb{N}$, N_{n,η_n} is not an isolating neighborhood. It follows that for every $n \in \mathbb{N}$, there are a $g_n \in \Sigma^+(f_n)$ and a solution $u_n : \mathbb{R} \to B_{\eta_n}[0]$ of

$$\dot{x} + Ax = g_n(t,x)$$

such that $\|u_n(0)\|_\alpha = \eta_n$. Scaling yields (mild) solutions $u_n'(t) := \eta_n^{-1} u_n(t)$ of

$$\dot{x} + Ax = \eta_n^{-1} g_n(t, \eta_n x)$$

with $\left\|u_n'(t)\right\|_\alpha \leq 1$ for all $t \in \mathbb{R}$ and $\left\|u_n'(0)\right\|_\alpha \equiv 1$.
It follows from (LIN0) that

$$L_0(t)x - \eta_n^{-1} f_n(t, \eta_n x) \to 0 \text{ in } \mathscr{Y} \text{ as } t, n \to \infty.$$

The sequence $(u_n'(t))_n$ is relatively compact by Remark 2.7.
Note that $g_n \in \omega(f)$ by Remark 2.4, so in view of Remark 2.3, Lemma 2.28 implies that there is a solution $(v,u)\colon \mathbb{R} \to \Sigma^+(L_0) \times X^\alpha$ with $\sup_{t\in\mathbb{R}} \|u(t)\|_\alpha \le 1$ for all $t \in \mathbb{R}$ and $\|u(0)\|_\alpha = 1$.
Consequently, L_0 is not weakly hyperbolic, which is a contradiction. $\square$

LEMMA 2.55. *In addition to the assumptions of Lemma 2.54, suppose that* $\operatorname{codim} \mathscr{S}_{L_0}(L_0^t) = 0$ *for large* $t \in \mathbb{R}^+$. *Then there is a neighborhood* U *of* $\omega(f)\times\{0\}$ *in* $\Sigma^+(f)\times\{0\}$ *such that* $\operatorname{Inv}^-(U) = \omega(f)\times\{0\}$.

PROOF. Let $U_\varepsilon := \Sigma^+(f)\times B_\varepsilon[0,X^\alpha]$ and suppose to the contrary that $\operatorname{Inv}^-(U_\varepsilon) \neq \operatorname{Inv}(U_\varepsilon)$ for small $\varepsilon > 0$. We may thus choose a sequence $\varepsilon_n \to 0$ and for every n a solution $u : \mathbb{R}^- \to U_\varepsilon$ with $u(0) \in \partial U_\varepsilon$. Proceeding as in the proof of Lemma 2.54, we can conclude that there are $y \in \omega(L_0)$ and $0 \neq x_0 \in \mathscr{U}_{L_0}(y)$. However, by Lemma 2.40 (c), one has $\dim(\mathscr{U}_{L_0}(y)) = 0$. $\square$

THEOREM 2.56. *Suppose that the premises of Lemma 2.54 hold for* $f_n \equiv f$, *and let* $\operatorname{codim} \mathscr{S}_{L_0}(L_0^t) = k_0 \in \mathbb{N}$ *for large* $t \in \mathbb{R}^+$.
Then, $\mathrm{h}(f,K_0) = \Sigma^{k_0}$, *where* $K_0 = \omega(f)\times\{0\}$.

PROOF. Define $\varphi : [0,1] \to \mathscr{Y}$ by $\varphi(\lambda) := \lambda f + (1-\lambda)L_0$.
Let $\lambda \in [0,1]$ be arbitrary. One needs to verify assumptions (C1) and (C2).
It follows from Lemma 2.54 that there exists an $\eta > 0$ such that $\Sigma^+(\varphi(\lambda'))\times B_\eta[0,X^\alpha]$ is an isolating neighborhood in $\Sigma^+(\varphi(\lambda'))\times X^\alpha$ for all λ' in a small neighborhood of λ. An application of Lemma 2.26 gives rise to an isolating neighborhood $N' \subset Y \times X$, so (C1) holds in λ. (C2) holds directly by the assumptions of Lemma 2.54.
Hence, $\mathrm{h}(\varphi(\lambda), \omega(\varphi(\lambda))\times\{0\})$ is defined for all $\lambda \in [0,1]$, and it follows from Theorem 2.31 that $\mathrm{h}(f,K_0) = \mathrm{h}(L_0, \omega(L_0)\times\{0\})$. Finally, an application of Theorem 2.36 proves that $\mathrm{h}(L_0,\omega(L_0)\times\{0\}) = \Sigma^{k_0}$ as claimed. $\square$

There is an analogue of Lemma 2.54 and Theorem 2.56 at infinity (see also [**22**, Theorem II.5.1]). We omit the proofs because of the similarity.

LEMMA 2.57. *Let* $f_n, L_\infty \in \mathscr{Y}_c$ *and* L_∞ *weakly hyperbolic and linear, i.e.,* $L_\infty(t,.) \in \mathscr{L}(X^\alpha, X)$ *for all* $t \in \mathbb{R}$. *Assume further that*

(LIN∞) *for every* $\varepsilon > 0$, *there is an* $r > 0$ *such that* $\left\|f_n(t,x) - L_\infty(t)x\right\| \le \varepsilon\|x\|_\alpha$ *for all* $(t,x) \in \mathbb{R}^+\times X^\alpha$ *and all* $n \in \mathbb{N}$ *with* $t,n \ge r$ *and* $\|x\|_\alpha \ge r$.

Then there are an $\eta_0 > 0$ *and an* $n_0 \in \mathbb{N}$ *such that for all* $\eta \ge \eta_0$ *and all* $n \ge n_0$, $N_{n,\eta} := \Sigma^+(f_n)\times B_\eta[0,X^\alpha]$ *is an isolating neighborhood.*

REMARK 2.8. *Let* η_0 *be given by the previous lemma, and let* $(v,u)\colon \mathbb{R} \to X^\alpha$ *be a solution with* $\alpha := \sup_{t\in\mathbb{R}} \|u(t)\|_\alpha < \infty$. *It follows that* $\alpha < \eta_0$, *so* $(v(t),u(t)) \in N_{\eta_0}$ *for all* $t \in \mathbb{R}$.
In other words, N_{η_0} *is an isolating neighborhood for the largest bounded invariant subset.*

THEOREM 2.58. *Suppose that the premises of Lemma 2.57 hold for* $f_n \equiv f$, *and let* $\operatorname{codim} \mathscr{S}_{L_\infty}(L_\infty^t) = k_\infty \in \mathbb{N}$ *for large* $t \in \mathbb{R}^+$.
Then, $\mathrm{h}(f,K_\infty) = \Sigma^{k_\infty}$, *where* K_∞ *denotes the largest*[21] *compact invariant subset of* $\Sigma^+(f)\times X$.

[21] with respect to inclusion

The next corollary is probably the most typical application of Conley index theory. The index of an empty invariant set is trivial. Conversely, an invariant set with non-trivial index cannot be empty, yet our result is a bit stronger.

COROLLARY 2.59. *Suppose that the assumptions of Theorem 2.58 hold. Then there is an $\eta > 0$ such that the following holds:*

(a) *There is a $t_0 \in \mathbb{R}^+$ and a (mild) solution $u : [t_0, \infty[\to X^\alpha$ of*

$$\dot{x} + Ax = f(t,x) \quad \|x\|_\alpha \le \eta. \tag{2.26}$$

(b) *For every $g \in \omega(f)$, there is a solution $u : \mathbb{R} \to X^\alpha$ of*

$$\dot{x} + Ax = g(t,x) \quad \|x\|_\alpha \le \eta.$$

defined for all $t \in \mathbb{R}$.

(c) *Suppose that f is (positive) Poisson stable, that is, $f \in \omega(f)$. Then there is a solution $u : \mathbb{R} \to X^\alpha$ with $(f, u(0)) \in \omega(f, u(0))$. In particular, this means that $u(0)$ is (positive) Poisson stable, that is, $u(0) \in \omega(u(0))$.*

(d) *Suppose that $\Sigma^+(f)$ is a compact minimal set. Then, there is a recurrent solution $u : \mathbb{R} \to X^\alpha$ of* (2.26).

PROOF. (a) By Lemma 2.57, there are an $\eta > 0$ and an isolating neighborhood $N_\eta := \Sigma^+(f) \times B_\eta[0, X^\alpha]$ for K_∞. K_∞ denotes the largest invariant set contained in N_η and is obviously (Remark 2.8) independent of η provided that η is sufficiently large. By Theorem 2.58 and Corollary 2.20, there are a $t_0' \in \mathbb{R}^+$ and for every $h \in \mathbb{R}^+$ a solution $u_h : [t_0', t_0' + h] \to X^\alpha$ with $\|u_h(s)\|_\alpha \le \eta$ for all $s \in [t_0', t_0' + h]$. Using the compactness of the inclusion $X^\beta \subset X^\alpha$, we may assume without loss of generality that $u_n(t_0'+1) \to x_0$ as $n \to \infty$. Let $t_0 := t_0' + 1$, and let u_0 denote the unique solution of (2.26) with $u_0(t_0) = x_0$.

(b) This follows from Theorem 2.58 and Corollary 2.22.

(c) This follows from (b) and Lemma 1.18.

(d) This follows from (b) and Theorem 1.17.

□

Now, we assume that $u \equiv 0$ is stable by linearization but the asymptotic equation – respectively the largest invariant set K_∞ – has a different index than $\omega(f) \times \{0\}$. We can conclude that there must be another solution, which is, in particular, not a homoclinic connection from 0 to 0. If the indices at 0 and at ∞ differ but 0 is not stable, then we can easily conclude the existence of additional solutions of

$$\dot{x} + Ax = g(t,x) \text{ for some } g \in \omega(f)$$

but not necessarily of the nonautonomous equation

$$\dot{x} + Ax = f(t,x)$$

itself.

THEOREM 2.60. *Let $f \in \mathscr{Y}_c$, and assume that*

(1) *there is a weakly hyperbolic* $L_0 \in \mathscr{Y}_c$, $L_0 \in \mathscr{L}(X^\alpha, X)$, *and (LIN0) holds;*
(2) *there is a weakly hyperbolic* $L_\infty \in \mathscr{Y}_c$, $L_\infty \in \mathscr{L}(X^\alpha, X)$, *and (LIN$\infty$) holds.*

Further let k_0 *(resp.* k_∞*) be given by Theorem 2.56 (resp. Theorem 2.58). Then the following statements hold true.*

(a) *If* $k_0 = 0$ *and* $0 < k_\infty \in \mathbb{N}$, *then there are* $\eta_1, \eta_2 \in]0, \infty[$, $t_0 \in \mathbb{R}^+$ *and a mild solution* $u : [t_0, \infty[\to X^\alpha$ *of*

$$\dot{x} + Ax = f(t, x) \quad \eta_1 \leq \|u(t)\|_\alpha \leq \eta_2. \tag{2.27}$$

(b) *If the assumptions of (a) hold, and* f *is Poisson stable, i.e.* $f \in \omega(f)$, *then there are* $\eta_1, \eta_2 \in]0, \infty[$ *and a mild solution* $u : \mathbb{R} \to X^\alpha$ *of* (2.27) *with* $(f, u(0)) \in \omega(f, u(0))$. *In particular, this means that* u *is Poisson stable.*
(c) *If, in addition to (b),* f *is recurrent, then there are* $\eta_1, \eta_2 \in]0, \infty[$ *and a recurrent mild solution* $u : \mathbb{R} \to X^\alpha$ *of* (2.27).

PROOF. (a) Since $k_0 = 0$, it follows from Lemma 2.55 that K_0 is an attractor in K_∞. In view of Lemma 2.54 and Lemma 2.57, there are $\eta_2 \geq \eta_1' > 0$ such that $\Sigma^+(f) \times B_{\eta_2}[0]$ (resp. $\Sigma^+(f) \times B_{\eta_1'}[0]$) is an isolating neighborhood for K_∞ (resp. K_0).

Now, Theorem 2.19 implies that there exists an isolating neighborhood N_0 of K_0 such that

$$N' := \{(t, x) : \ t \in \mathbb{R}^+ \text{ and } \|x\|_\alpha \leq \eta_2\}$$

is not an isolating neighborhood for (Φ_f, N_0'), where

$$N_0' := \{(t, x) : (f^t, x) \in N_0\}.$$

The largest invariant set in N_0 is $\omega(f) \times \{0\}$. Hence, there is an $\eta_1 > 0$ such that $\omega(f) \times B_{\eta_1}[0] \subset N_0$, and so $\mathbb{R}^+ \times B_{\eta_1}[0] \subset N_0'$. By Lemma 2.4, the fact that N' is not an isolation neighborhood for N_0' means that there exist a $t_0' \in \mathbb{R}^+$ and for every $T \in \mathbb{R}^+$ a solution $u_T : \left[t_0', t_0' + T\right] \to (B_{\eta_2}[0] \setminus B_{\eta_1}[0])$ (of Φ_f).

Finally, choose $t_0 := t_0' + 1$, and for every $n \in \mathbb{N}$, let $u_n : \left[t_0', t_0' + n\right]$ be a solution of $\Phi_{f^{t_0'}}$. Using the compact inclusion $X^\beta \subset X^\alpha$ and standard results, we can assume w.l.o.g. that $u_n(1) \to x_0$ as $n \to \infty$. Consequently, $u(t) := \Phi_f(t, t_0, x_0)$ is a solution defined for all $t \in [t_0, \infty]$ and satisfies $\eta_1 \leq u(t) \leq \eta_2$ for all $t \geq t_0$.

(b) Let (K_0^*, K_0) be an repeller-attractor decomposition of K_∞. If the repeller K_0^* is empty, then $K_0 = K_\infty$. Thus, it follows from the proof of (a) that $K_0^* \neq \emptyset$ and $\eta_1 \leq \|x\|_\alpha \leq \eta_2$ for all $x \in K_0^*$. By Lemma 1.18, there is an $(f, x') \in K_0^*$ with $(f, x') \in \omega(f, x')$.
(c) This follows from the proof of (b) and Theorem 1.17.

□

CHAPTER 3

Further Indices and Attractor-Repeller Decompositions

In the previous chapter, primarily isolated invariant sets are considered. This chapter is concerned with connecting homomorphisms. Connecting homomorphisms are obtained from Morse-decompositions and could be understood as the index of a connection between two given Morse-sets.

First of all, a slightly modified notion of index pairs is introduced. This change allows us to define the index in terms of nonautonomous index pair alone that is, it is not necessary to consider FM-index pairs living in an extended phase space. Nevertheless, invariant sets still depend on an extended phase space and the skew-product formulation, and so does the notion of attractor-repeller decompositions.

Having established the modified index, a couple of well-known concepts (see [**5**, **9**, **10**]) are adapted to the new situation. While this is mostly straightforward, some concepts are simplified versions. Generally speaking, we try to avoid complexity whenever possible. This leads to the concept of weak index filtrations required for the definition respectively the commutativity of a concept known as homology index braids. It also was the main motivtion to avoid the use of (singular) chain complexes and weakly exact sequences in the definition of attractor-repeller sequences. An important result of this chapter is the continuation of Morse-decompositions and their associated homology index braids under small perturbations.

A second important topic is subsumed under the term *uniformly connected attractor-repeller decompositions*. Recall that isolated invariant sets are usually subsets of a space $\omega(y_0) \times X$. Nevertheless, a non-zero index implies the existence of a full solution of Φ_{y_0}, which is the evolution operator associated with the parameter y_0. Moreover, in case of a non-vanishing index there is a full solution of Φ_y for every $y \in \omega(y_0)$. Connecting homomorphisms obey to the same rules. If there is only one parameter $y \in \omega(y_0)$ for which there does not exist a connection (connecting orbit) between attractor and repeller, the connecting homomorphism must vanish. In order to prove this result, the nonautonomous homology index is rewritten as a direct limit. This direct limit structure appears to be similar to the construction of a discrete Conley index.

Finally, the persistence of attractor-repeller decompositions as well as connecting orbits under rather arbitrary but small perturbations is considered in the last section. Note that these perturbations are only required to be C^0-small which does not allow arguments relying on variants of the Banach fixed point theorem (e.g. [**29**]).

The chapter is in mostly self contained. In particular, the existence of index pairs is proved under reasonable assumptions. There are two notable omissions, where this text relies on previous work by other authors: the existence of index filtration and their continuation.

3.1. Related Index Pairs

In this section we give an alternative definition of a nonautonomous Conley index. Essentially, the index is now purely based on nonautonomous index pairs which are subsets of $\mathbb{R}^+ \times X$, where X is an appropriate metric space. It is also often more convenient to compute the index by using the modified definition of this section. The main results are Theorem 3.8 and its corollary.

We say that two index pairs for which the assumptions and thus also the conclusions of Theorem 3.8 hold are *related*. Roughly speaking, related index pairs define the same index[1].

Throughout this section, it is assumed that X and Y are metric spaces, and $\pi = \pi(.^t, \Phi)$ is a skew-product semiflow on $Y \times X$. By $\chi := \chi_{y_0}$ we denote the canonical semiflow $(t,x)\chi_{y_0} s := (t+s, \Phi_{y_0}(s,t,x))$ on $\mathbb{R}^+ \times X$.

DEFINITION 3.1. Let $y_0 \in Y$ and (N_1, N_2) be an index pair in $\mathbb{R}^+ \times X$ relative to χ_{y_0}. Define $r : \mathbb{R}^+ \times X \to \Sigma^+(y_0) \times X$ by $r(t,x) := (y_0^t, x)$.

Let $K \subset \omega(y_0) \times X$ be an (isolated) invariant set. We say that (N_1, N_2) is a (strongly admissible) index pair[2] for (y_0, K) if:

(1) there is a strongly admissible isolating neighborhood N of K in $\Sigma^+(y_0) \times X$ such that $N_1 \setminus N_2 \subset r^{-1}(N)$;

(2) there is a neighborhood W of K in $\Sigma^+(y_0) \times X$ such that $r^{-1}(W) \subset N_1 \setminus N_2$.

DEFINITION 3.2. We say that (y_0, K) is an *invariant pair* if $y_0 \in Y$ and $K \subset \Sigma^+(y_0) \times X$. An invariant pair (y_0, K) is called a *compact invariant pair* provided that K is compact.

Every FM-index pair relative to the skew-product semiflow induces an index pair.

LEMMA 3.3. *Let $y_0 \in Y$ and let (N_1, N_2) be an FM-index pair for $K \subset \Sigma^+(y_0) \times X$ such that N_1 is strongly admissible. Then $(M_1, M_2) := (r^{-1}(N_1), r^{-1}(N_2))$ is an index pair for (y_0, K).*

PROOF. Lemma 2.15 states that (M_1, M_2) is an index pair. We need to prove that the additional[3] assumptions of Definition 3.1 are satisfied. $N := \mathrm{cl}_{Y\times X}(N_1 \setminus N_2)$ is an isolating neighborhood for K, and $M_1 \setminus M_2 = r^{-1}(N_1) \setminus r^{-1}(N_2) \subset r^{-1}(N)$.

Let $W := \mathrm{int}_{\Sigma^+(y_0)\times X}(N_1 \setminus N_2)$, which is a neighborhood of K. We have $r^{-1}(W) \subset r^{-1}(N_1) \setminus r^{-1}(N_2)$. □

The following lemma is not much more than a restatement of Theorem 2.7.

LEMMA 3.4. *Suppose that $(N_1, N_2) \subset (M_1, M_2)$ are index pairs for (y_0, K). The inclusion induced mapping $i : (N_1/N_2, N_2) \to (M_1/M_2, M_2)$ is a homotopy equivalence.*

PROOF. By Definition 3.1, there is a neighborhood W of K such that $r^{-1}(W) \subset (N_1 \setminus N_2) \cap (M_1 \setminus M_2)$. It follows from Definition 3.1 that the closure $\overline{W} := \mathrm{cl}_{Y\times X} W$ is strongly admissible, so by Lemma 2.14, (N_1, N_2) and (M_1, M_2) are index pairs for $(\Phi_{y_0}, r_{y_0}^{-1}(W))$. The claim is now a consequence of Theorem 2.7. □

[1] This is not necessarily a homotopy index, so the vague language is intended.

[2] Every index pair in the sense of Definition 3.1 is assumed to be strongly admissible.

[3] compared to Definition 2.6

DEFINITION 3.5. Let (N_1, N_2) be an index pair in $\mathbb{R}^+ \times X$ (relative to the semiflow χ on $\mathbb{R}^+ \times X$). For $T \in \mathbb{R}^+$, we set

$$N_2^{-T} := N_2^{-T}(N_1) := \{(t,x) \in N_1 : \exists s \le T \ (t,x)\chi s \in N_2\}.$$

LEMMA 3.6. *Let (N_1, N_2) be an index pair for (y_0, K). Then so is (N_1, N_2^{-T}) for every $T \in \mathbb{R}^+$.*

PROOF. We need to check the assumptions of Definition 2.6 and Definition 3.1.

(IP1) We need to show that N_2^{-T} is closed. Suppose that (s_n, x_n) is a sequence in N_2^{-T} with $(s_n, x_n) \to (s, x)$ in N_1. For every $n \in \mathbb{N}$, there is a $t_n \in [0, T]$ such that $(s_n, x_n)\pi t_n \in N_2$. We can assume without loss of generality that $t_n \to t \le T$, so $(s,x)\pi t \in N_2$, which is closed. Thus it holds that $(s,x) \in N_2^{-T}$.

(IP3) Let $x \in N_2^{-T}$ but $x\pi t \notin N_1$ for some $t \in \mathbb{R}^+$. (N_1, N_2) is an index pair, so $x\pi s \in N_2 \subset N_2^{-T}$ for some $s \in [0, t]$.

(IP4) Suppose that $x \in N_2^{-T}$ and $x\pi t \notin N_2^{-T}$ for some $t \in \mathbb{R}^+$. Letting $t_0 := \sup\{s \in \mathbb{R}^+ : \ x\pi[0,s] \cap N_2 = \emptyset\}$, it follows that $t_0 \le T$ and $x\pi t_0 \in N_2$. Furthermore, one has $x\pi[0, t_0] \subset N_2^{-T}$, so $t > t_0$.

Since (N_1, N_2) is assumed to be an index pair, it follows that $x\pi s \in (\mathbb{R}^+ \times X) \setminus N_1$ for some $s \in [t_0, t]$.

(N_1, N_2) is an index pair for (y_0, K), so there is an isolating neighborhood N of K such that $N_1 \setminus N_2^{-T} \subset N_1 \setminus N_2 \subset r^{-1}(N)$.
Let W be an open neighborhood of K such that $r^{-1}(W) \subset N_1 \setminus N_2$. We consider the set

$$W^T := \{x \in W : \ x\pi[0, T] \subset W\}.$$

If $(t, x) \in r^{-1}(W^T) \cap N_2^{-T}$, then $(t,x)\chi_{y_0} T \in r^{-1}(W) \cap N_2 = \emptyset$, so

$$r^{-1}(W^T) \subset N_1 \setminus N_2^{-T}.$$

We need to show that W^T is a neighborhood of K. Suppose to the contrary that there is[4] a sequence $x_n \to x_0 \in K$ in $N \setminus W^T$. For every $n \in \mathbb{N}$, there is a $t_n \in [0, T]$ with $x_n\pi t_n \in (\Sigma^+(y_0) \times X) \setminus W$. We can assume w.l.o.g. that $t_n \to t_0$, so $x_0\pi t_0 \in (\Sigma^+(y_0) \times X) \setminus W$, which is a closed set. However, $x_0 \pi t_0 \in K \subset W$, a contradiction. □

One frequently needs to prove that a couple (N_1, N_2) is not only an index pair but also that it belongs to a certain couple (y_0, K). For this purpose and in conjunction with Lemma 3.6, the following – simple – "sandwich" lemma is useful.

LEMMA 3.7. *Let $y_0 \in Y$, and let (N_1, N_2), (M_1, M_2) and (N_1', N_2') be index pairs with $N_1 \setminus N_2 \subset M_1 \setminus M_2 \subset N_1' \setminus N_2'$.*
If (N_1, N_2) and (N_1', N_2') are index pairs for (y_0, K), then so is (M_1, M_2).

PROOF. One simply needs to check the assumptions of Definition 3.1.

(1) (N_1', N_2') is an index pair for (y_0, K), so there is a strongly admissible isolating neighborhood N of K in $\Sigma^+(y_0) \times X$ such that $M_1 \setminus M_2 \subset N_1' \setminus N_2' \subset r^{-1}(N)$.

[4]As a consequence of the admissibility assumption, K is compact.

(2) (N_1, N_2) is an index pair for (y_0, K), so there is a neighborhood W of K in $\Sigma^+(y_0) \times X$ such that $r^{-1}(W) \subset N_1 \setminus N_2 \subset M_1 \setminus M_2$.

□

We are now in a position to formulate and prove the main result of this section.

THEOREM 3.8. *Let there be given index pairs (N_1, N_2) and (M_1, M_2) for (y_0, K). Further, let $N \subset \Sigma^+(y_0) \times X$ be a strongly admissible neighborhood of K. Then there are a $t_0 \in \mathbb{R}^+$ and an index pair (L_1, L_2) such that*

$$(L_1, L_2) \subset (r^{-1}(N) \cap N_1 \cap M_1, N_2^{-t_0}(N_1) \cap M_2^{-t_0}(M_1)).$$

An important consequence of the theorem above is that the homotopy index of (y_0, K) can be defined as the pointed homotopy type of $(N_1/N_2, N_2)$, where (N_1, N_2) is an index pair for (y_0, K). It coincides[5] with Definition 2.13, so there is no need to redefine the homotopy index. We have merely extended the class of possible or good index pairs.

COROLLARY 3.9. *Under the assumptions of Theorem 3.8, the pointed homotopy types of $(N_1/N_2, N_2)$ and $(M_1/M_2, M_2)$ agree.*

PROOF. By Theorem 3.8, there are an index pair and a constant $t_0 \in \mathbb{R}^+$ for which the following inclusions hold true.

$$(L_1, L_2) \subset (N_1, N_2^{-t_0}) \supset (N_1, N_2)$$
$$(L_1, L_2) \subset (M_1, M_2^{-t_0}) \supset (M_1, M_2)$$

In view of Lemma 3.4 and Lemma 3.6, this readily implies that $(N_1/N_2, N_2)$ and $(M_1/M_2, M_2)$ are isomorphic in the homotopy category of pointed spaces. □

The rest of this section is devoted to the proof of Theorem 3.8. The proof is similar to the proof of [**5**, Lemma 4.8], but instead of using isolating blocks, we will construct appropriate index pairs. In all subsequent lemmas, we will assume that the hypotheses of Theorem 3.8 hold.

Since N is a neighborhood of K, there is an open (in $\Sigma^+(y_0) \times X$) set U with $K \subset U \subset N$. Define $g^+, g^- : \Sigma^+(y_0) \times X \to \mathbb{R}^+$ by

$$g^+(y, x) := \sup\{t \in \mathbb{R}^+ : (y, x)\pi[0, t] \subset U\}$$
$$g^-(y, x) := \sup\{d((y, x)\pi t, \mathrm{Inv}_\pi^-(N)) : t \in \left[0, g^+(y, x)\right]\}.$$

It is easy to see that both functions g^+ and g^- are continuous and monotone decreasing along solutions in U (resp. N), that is, if $u : [0, a] \to U$ (resp. $u : [0, a] \to N$) is a solution of π, then $t \mapsto g^+(u(t))$ (resp. $t \mapsto g^-(u(t))$) is continuous and monotone decreasing on $[0, a]$.

LEMMA 3.10.

(a) *g^+ is lower-semicontinuous.*
(b) *g^- is lower-semicontinuous.*
(c) *$\{g^+ \le c\} := \{(y, x) \in N : g^+(y, x) \le c\}$ is closed.*

[5]Under the assumptions of Theorem 3.8, it follows from [**22**] that there exists an isolating block for K in $\Sigma^+(y_0) \times X$. This isolating block gives rise to an index pair for (y_0, K) as proved in Lemma 3.3.

(d) $\{g^- \le c\} := \{(y,x) \in N : g^-(y,x) \le c\}$ *is closed.*

(e) *For all* $c_1, c_2 > 0$*, the set* $(\{g^- \le c_1\} \cap \{g^+ > c_2\})$ *is a neighborhood of* $K := \mathrm{Inv}(N)$.

PROOF. (a) Let $\varepsilon > 0$ and $(y,x) \in \Sigma^+(y_0) \times X$. Suppose that $(y_n, x_n) \to (y,x)$ in $\Sigma^+(y_0) \times X$ and $g^+(y_n,x_n) \le g^+(y,x) - \varepsilon$ for all $n \in \mathbb{N}$. We can assume w.l.o.g. that $g^+(y_n,x_n) \to t_0$.

First of all, as N is strongly admissible and $(y_n,x_n)\pi s \to (y,x)\pi s$, it follows that $(y,x)\pi s \in N$ for all $s \in [0,t_0]$. Secondly, one has $(y_n,x_n)\pi g^+(y_n,x_n) \in X \setminus U$, which is closed, so $(y,x)\pi t_0 \in X \setminus U$. However, $t_0 \le g^+(y,x) - \varepsilon$, which is a contradiction.

(b) Let $(y,x) \in \Sigma^+(y_0) \times X$ and suppose that $(y_n,x_n) \to (y,x)$ but $g^-(y_n,x_n) \le g^-(y,x) - \varepsilon$ for some $\varepsilon > 0$.

Let $t \in [0, g^+(y,x)[$ be arbitrary. By the lower-semicontinuity of g^+, one has $g^+(y_n,x_n) \ge t$ provided that n is sufficiently large. Furthermore, one has $d((y,x)\pi t, \mathrm{Inv}^-(N)) \le d((y,x)\pi t, d(y_n,x_n)\pi t) + d((y_n,x_n)\pi t, \mathrm{Inv}^-(N))$, so $d((y,x)\pi t, \mathrm{Inv}^-(N) \le g^-(y,x) - \varepsilon$. The last inequality holds for arbitrary $t \in [0, g^+(y,x)[$. We thus have $g^-(y,x) \le g^-(y,x) - \varepsilon$, which is a contradiction.

(c), (d) This follows immediately from the lower-semicontinuity of the respective function.

(e) Arguing by contradiction, we may assume that there are $(y_n,x_n) \to (y,x) \in K$ such that either $g^+(y_n,x_n) \le c_2$ or $g^-(y_n,x_n) > c_1$ for all $n \in \mathbb{N}$. In the first case, it follows that $g^+(y,x) \le c_2$ in contradiction to $(y,x) \in K$. In the second case, we can choose $t_n \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, $t_n \le g^+(y_n,x_n)$ and

$$d((y_n,x_n)\pi t_n, \mathrm{Inv}^-(N)) \ge c_1 > 0. \tag{3.1}$$

Either $(t_n)_n$ has a convergent subsequence or $t_n \to \infty$. Suppose that $(t_{n(k)})_k$ is a subsequence with $t_{n(k)} \to t_0$ as $k \to \infty$. It follows that $d((y,x)\pi t_0, \mathrm{Inv}^-(N)) \ge c_1$, which is a contradiction to $(y,x) \in K$. Thus, one has $t_n \to \infty$, and using the admissibility of N, there is a subsequence $(y_{n(k)}, x_{n(k)})\pi t_{n(k)}$ which converges to a point $(y', x') \in \mathrm{Inv}^-(N)$, in contradiction to (3.1).

□

LEMMA 3.11. *For* $c_1 > 0$ *and* $c_2 > 0$*, set*

$$L_1^{c_1,c_2} := \{g^- \le c_1\} \cap \mathrm{cl}\{g^+ \ge c_2\}$$
$$L_2^{c_1,c_2} := L_1^{c_1,c_2} \cap \{g^+ \le c_2\}$$

and $\hat{L}_i^{c_1,c_2} := r^{-1}(L_i^{c_1,c_2})$, $i = 1,2$.
Then for c_1 *small and* c_2 *large, one has*

(1) $L_1^{c_1,c_2} \subset U$*, and*

(2) $(L_1, L_2) := (\hat{L}_1, \hat{L}_2) := (\hat{L}_1^{c_1,c_2}, \hat{L}_2^{c_1,c_2})$ *is an index pair for* (y_0, K).

PROOF. (1) If $(y,x) \in \mathrm{cl}\{g^+ \ge c_2\}$, then $(y,x)\pi[0,c_2] \subset N$. Hence, if the claim does not hold, there is a point $(y',x') \in K \cap (N \setminus U) = \emptyset$.

(2)(IP1) It follows from Lemma 3.10 (c) and (d) that $L_1^{c_1,c_2}$ and $L_2^{c_1,c_2}$ are closed, so $\hat{L}_1$ and $\hat{L}_2$ are closed by the continuity of r.

(IP3) Let $x \in L_1^{c_1,c_2}$ and $x\pi t \notin L_1^{c_1,c_2}$ for some $t \geq 0$. The semiflow does not explode in N. Hence, there is a $t' \leq t$ such that $x\pi t' \in (\Sigma^+(y_0) \times X) \setminus L_1^{c_1,c_2}$. Choose a sequence $x_n \to x$ in $L_1^{c_1,c_2}$ with $g^+(x_n) \geq c_2$. We have $x_n\pi t \notin L_1^{c_1,c_2}$ for all n sufficiently large, so $x_n\pi s_n \in L_2^{c_1,c_2}$ for some $s_n \leq t$ and all $n \in \mathbb{N}$. We can assume w.l.o.g. that $s_n \to s_0 \leq t$, so $x\pi s_0 \in L_2^{c_1,c_2}$.

(IP4) Let $x \in L_2^{c_1,c_2}$ and $x\pi[0,t] \subset L_1^{c_1,c_2}$. We have $L_1^{c_1,c_2} \subset U$, so $g^+(x\pi s) \leq g^+(x)$ for all $s \in [0,t]$. Hence, $x\pi[0,t] \subset L_2^{c_1,c_2}$.

Furthermore, one has $N \supset L_1^{c_1,c_2} \setminus L_2^{c_1,c_2} \supset W$, where $W := \{g^- \leq c_1\} \cap \{g^+ > c_2\}$ is a neighborhood of K by Lemma 3.10 (e). Thus, $r^{-1}(N) \supset \hat{L}_1^{c_1,c_2} \setminus \hat{L}_2^{c_1,c_2} \supset r^{-1}(W)$, which shows that $(\hat{L}_1, \hat{L}_2)$ is an index pair for (y_0, K).

□

Until now, our proof is based loosely on the respective proof in [**22**] concerning the existence of isolating blocks. However, our claim is significantly weaker, so the proof is - hopefully - easier to follow.

Since both (N_1, N_2) and (M_1, M_2) are index pairs for (y_0, K), we can assume without loss of generality that $r^{-1}(N) \subset N_1 \cap M_1$. Otherwise, one can simply replace N by a sufficiently small neighborhood N', and thereby obtain a stronger result. In order to complete the proof of Theorem 3.8, we need

LEMMA 3.12. *For every $d > 0$, one has $\hat{L}_2^{c,d} \subset N_2^{-T}$ (resp. $\hat{L}_2^{c,d} \subset M_2^{-T}$) provided that c is sufficiently small and T is sufficiently large.*

PROOF. If the lemma is not true, then there are sequences $((t_n, x_n))_n$, $c_n \to 0$ and $T_n \to \infty$ such that $(t_n, x_n) \in \hat{L}_2^{c_n,d}$ and $(t_n, x_n)\pi s \in N_1 \setminus N_2$ for all $s \leq T_n$ and all $n \in \mathbb{N}$.

Taking subsequences and because $c_n \to 0$, we can assume without loss of generality that $(y_0^{t_n}, x_n) \to (y, x) \in \mathrm{Inv}^-(N)$, which is compact because N is strongly admissible. The choice of the sequences implies that $(y, x) \in \mathrm{Inv}^+(N)$, so $(y, x) \in \mathrm{Inv}(N) = K$.

However, $(y_0^{t_n}, x_n)\pi g^+(y_0^{t_n}, x_n) \in N \setminus U$ for all $n \in \mathbb{N}$. Furthermore, $g^+(y_0^{t_n}, x_n) \leq d$ by the choice of $\hat{L}_2^{c,d}$. One may therefore assume w.l.o.g. that $g^+(y_0^{t_n}, x_n) \to t_0$. Consequently, one obtains $(y, x)\pi t_0 \in (N \setminus U) \cap K = \emptyset$, which is an obvious contradiction. □

By using Lemma 3.11, one obtains an index pair $(L_1, L_2) := (\hat{L}_1^{c,d}, \hat{L}_2^{c,d})$ for (y_0, K) provided that c is small and d is large. In view of Lemma 3.12, one can find a possibly even smaller parameter $c > 0$ such that the conclusions of Theorem 3.8 hold for large t_0. The proof of Theorem 3.8 is complete. □

3.2. Categorial Conley Index

A connected simple system is a small category with the following property: if A and B are objects, then there is exactly one morphism $A \to B$.

Understanding the Conley index as a connected simple system is perhaps the most elegant variant of the index. There is no loss of information, and other invariants such as a homotopy or (co)homology index can be derived by applying an appropriate functor. We will show in this

section, that the nonautonomous exentsion of the Conley index defines a connected simple system as well.
Throughout this section, we will assume the hypotheses[6] at the beginning of the previous section.

DEFINITION 3.13. Let $y_0 \in Y$, and let $K \subset \Sigma^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood. The categorial (nonautonomous) Conley index $\mathscr{C}(y_0, K)$ of (y_0, K) is the smallest subcategory of the homotopy category of pointed spaces with the following properties:

(1) Objects of $\mathscr{C}(y_0, K)$ are pairs $(N_1/N_2, N_2)$, where (N_1, N_2) is an index pair for (y_0, K).
(2) If (N_1, N_2) and (M_1, M_2) are index pairs for (y_0, K) with $(N_1, N_2) \subset (M_1, M_2)$, then the inclusion induced morphism $i : (N_1/N_2, N_2) \to (M_1/M_2, M_2)$ in the homotopy category of pointed spaces is a morphism of $\mathscr{C}(y_0, K)$.

For brevity, we also write $[N_1, N_2] := (N_1/N_2, N_2)$.

THEOREM 3.14. *$\mathscr{C}(y_0, K)$ is (well-defined and) a connected simple system.*

The proof below can be sketched as follows: Given two arbitrary index pairs (N_1, N_2) and (M_1, M_2), one constructs a morphism $f : [N_1, N_2] \to [M_1, M_2]$ in $\mathscr{C}(y_0, K)$. This morphism f is a composition of inclusion induced morphisms or their inverse morphisms and therefore necessarily a morphism of $\mathscr{C}(y_0, K)$. These morphisms are then shown to be unique, that is, f depends only on (N_1, N_2) and (M_1, M_2), and invariant with respect to composition. In other words, the proof is nothing but an explicit construction.

PROOF. Let (N_1, N_2) and (M_1, M_2) be arbitrary index pairs for (y_0, K). By Theorem 3.8, there is an index pair (L_1, L_2) for (y_0, K) and a $T \in \mathbb{R}^+$ such that

$$(L_1, L_2) \subset (N_1 \cap M_1, N_2^{-T} \cap M_2^{-T}).$$

Each inclusion of index pairs gives rise to a morphism. We obtain the following diagram, the arrows of which denote isomorphisms (Lemma 3.4) (respectively the inverse morphim) of $\mathscr{C}(y_0, K)$.

$$[N_1, N_2] \longrightarrow \left[N_1, N_2^{-T}\right] \longleftarrow [L_1, L_2] \longrightarrow \left[M_1, M_2^{-T}\right] \longleftarrow [M_1, M_2] \tag{3.2}$$

It follows that there is a morphism in $[N_1, N_2] \to [M_1, M_2]$ in $\mathscr{C}(y_0, K)$, namely the composition of the morphisms in the row above.
Next, we will show that the morphism obtained using this procedure is unique. Firstly, let $T_1 \geq T_2$ be positive real numbers. The following ladder with inclusion induced arrows is commutative.

$$\begin{tikzcd}
{[N_1, N_2]} \arrow[r] & \left[N_1, N_2^{-T_1}\right] & {[L_1, L_2]} \arrow[l] \arrow[r] & \left[M_1, M_2^{-T_1}\right] & {[M_1, M_2]} \arrow[l] \\
{[N_1, N_2]} \arrow[r] \arrow[u] & \left[N_1, N_2^{-T_2}\right] \arrow[u] & {[L_1, L_2]} \arrow[l] \arrow[r] \arrow[u] & \left[M_1, M_2^{-T_2}\right] \arrow[u] & {[M_1, M_2]} \arrow[l] \arrow[u]
\end{tikzcd}$$

[6] i.e. the spaces X, Y and the semiflow π

Hence, the morphism $[N_1,N_2]\to[M_1,M_2]$ defined by (3.2) is independent of T. Secondly, one needs to consider the index pair (L_1,L_2). Suppose (L_1',L_2') is another index pair for (y_0,K) with $(L_1',L_2')\subset(N_1\cap M_1,N_2^{-T}\cap M_2^{-T})$. It follows again from Theorem 3.8 that there exist an index pair (L_1'',L_2'') for (y_0,K) and a constant $T>0$ such that $(L_1'',L_2'')\subset(L_1\cap L_1',L_2^{-T}\cap(L')_2^{-T})$.
We obtain a commutative diagram below, where each arrow denotes an inclusion induced (iso)morphism.

$$\begin{tikzcd}
 & & {[L_1,L_2]} \arrow[d] \arrow[ddl] \arrow[ddr] & & \\
 & & {\left[L_1,L_2^{-T}\right]} \arrow[dl] \arrow[dr] & & \\
{[N_1,N_2]} \arrow[r] & {\left[N_1,N_2^{-2T}\right]} & {\left[L_1'',L_2''\right]} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & {\left[M_1,M_2^{-2T}\right]} & {[M_1,M_2]} \arrow[l] \\
 & & {\left[L_1',(L_2')^{-T}\right]} \arrow[ul] \arrow[ur] & & \\
 & & {\left[L_1',L_2'\right]} \arrow[u] \arrow[uul] \arrow[uur] & &
\end{tikzcd}$$

The morphisms defined by (L_1,L_2) and (L_1',L_2') agree since each arrow in the above diagram denotes an isomorphism (Lemma 3.4).
Finally, we will show that the composition of two morphisms obtained from the above prodecure can be written as in (3.2). Suppose, we are given index pairs (N_1,N_2), (M_1,M_2) and (O_1,O_2) for (y_0,K). By Theorem 3.8, there are an index pair (L_1,L_2) for (y_0,K) and a $T\in\mathbb{R}^+$ such that

$$(L_1,L_2)\subset(N_1\cap M_1\cap O_1,N_2^{-T}\cap M_2^{-T}\cap O_2^{-T}).$$

For every two objects A,B in $\mathscr{C}(y_0,K)$, let $A\to B$ denote the unique morphism defined by (3.2). We also write $B\leftarrow A$ for the inverse (morphism) of $A\to B$. Given morphisms $A\to B$ and $B\to C$, we write $A\to B\to C$ to denote their composition. We need to prove that $A\to B\to C=A\to C$. One has

$$\begin{aligned}
&[N_1,N_2]\to[M_1,M_2]\to[O_1,O_2]\\
=&[N_1,N_2]\to\left[N_1,N_2^{-T}\right]\leftarrow[L_1,L_2]\to\left[M_1,M_2^{-T}\right]\leftarrow[M_1,M_2]\\
&\to\left[M_1,M_2^{-T}\right]\leftarrow[L_1,L_2]\to\left[O_1,O_2^{-T}\right]\leftarrow[O_1,O_2]\\
=&[N_1,N_2]\to\left[N_1,N_2^{-T}\right]\leftarrow[L_1,L_2]\to\left[O_1,O_2^{-T}\right]\leftarrow[O_1,O_2]\\
=&[N_1,N_2]\to[O_1,O_2].
\end{aligned}$$

□

We consider a category CSS($\mathscr{K}$) of connected simple systems in a given category $\mathscr{K}$. Objects of CSS($\mathscr{K}$) are subcategories of $\mathscr{K}$ which are connected simple systems. Let $\mathscr{A}$ and $\mathscr{B}$ be connected simple systems in $\mathscr{K}$. A morphism $\mathscr{A}\to\mathscr{B}$ in CSS($\mathscr{K}$) is a family

$(f_{A,B})_{(A,B)\in\mathrm{Obj}(\mathscr{A})\times\mathrm{Obj}(\mathscr{B})}$, where each $f_{A,B}$ is a morphism $A\to B$ in $\mathscr{K}$ such that

$$\begin{array}{ccc} A & \xrightarrow{f_{A,B}} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f_{A',B'}} & B' \end{array}$$

is commutative. The vertical arrows denote the unique (inner) morphisms in $\mathscr{A}$ respectively $\mathscr{B}$.

If A is an object of $\mathscr{A}$, B is an object of $\mathscr{B}$, and $f: A\to B$ is a morphism, then there is a unique morphism $F\in\mathrm{CSS}(\mathscr{K})$ with $F=F(A,B)=f$. We say that $[f]:=F$ is induced by f.

Now, set $\mathscr{K}=\mathscr{HT}$, the homotopy category of pointed spaces, and given an isolated invariant set $K\subset\Sigma^+(y_0)\times X$ admitting a strongly admissible isolating neighborhood, its index $\mathscr{C}(y_0,K)$ is an object of CSS($\mathscr{HT}$). The morphisms of $\mathscr{C}(y_0,K)$ are called *inner morphisms.*

3.3. Homology Conley Index and Attractor-repeller Sequences

In this section, attractor-repeller decompositions of isolated invariant sets are studied. The main tool are long exact sequences in homology.

3.3.1. Attractor-repeller Decompositions and Index Triples.

DEFINITION 3.15. Let $y_0\in Y$ and $K\subset\Sigma^+(y_0)\times X$ be an isolated invariant set. (A,R) is an *attractor-repeller decomposition* of K if A,R are disjoint isolated invariant subsets of K and for every solution $u:\mathbb{R}\to K$ one of the following alternatives holds true.

(1) $u(\mathbb{R})\subset A$
(2) $u(\mathbb{R})\subset R$
(3) $\alpha(u)\subset R$ and $\omega(u)\subset A$

We also say that (y_0,K,A,R) is an attractor-repeller decomposition.

The α and ω-limes sets are defined as usual.

$$\alpha(u):=\bigcap_{t\in\mathbb{R}^-}\mathrm{cl}_{\Sigma^+(y_0)\times X}\,u(]-\infty,t])$$
$$\omega(u):=\bigcap_{t\in\mathbb{R}^+}\mathrm{cl}_{\Sigma^+(y_0)\times X}\,u([t,\infty[)$$

DEFINITION 3.16. Let $y_0\in Y$ and $K\subset\Sigma^+(y_0)\times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood N. Suppose that (A,R) is an attractor-repeller decomposition of K.

A triple (N_1,N_2,N_3) is called an *index triple* for (y_0,K,A,R) provided that:

(1) $N_3\subset N_2\subset N_1$
(2) (N_1,N_3) is an index pair for (y_0,K)
(3) (N_2,N_3) is an index pair for (y_0,A)

Suppose we are given an isolated invariant set and an attractor-repeller decomposition of the isolated invariant set. Does there exist an index triple?

LEMMA 3.17. *Let $y_0 \in Y$ and $K \subset \Sigma^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood N. Suppose that (A, R) is an attractor-repeller decomposition of K.*
Then there exists an index triple (N_1, N_2, N_3) for (y_0, K, A, R) such that $N_1 \subset r^{-1}(N)$.

PROOF. It is known that there exists an FM-index triple (N_1', N_2', N_3') (see [**5**]) with $N_1 \subset N$. By Lemma 3.3, $(r^-1(N_1'), r^{-1}(N_3'))$ is an index pair for (y_0, K) and $(r^{-1}(N_2'), r^{-1}(N_3'))$ is an index pair for (y_0, A). □

LEMMA 3.18. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Then, (N_1, N_2) is an index pair for (y_0, R).*

PROOF. Firstly, we will show that (N_1, N_2) is an index pair, that is, we need to check Definition 2.6.

(IP3) Let $x \in N_1$ and $t \in \mathbb{R}^+$ such that $x\chi_{y_0} t \notin N_1$. It is known that (N_1, N_3) is an index pair, so $x\chi_{y_0} s \in N_3 \subset N_2$ for some $s \in [0, t]$.

(IP4) Let $x \in N_2$ and $t \in \mathbb{R}^+$ such that $x\chi_{y_0} t \notin N_2$. (N_2, N_3) is an index pair, so $x\chi_{y_0} s \in N_3$ for some $s \in [0, t]$. Since (N_1, N_3) is also an index pair, it follows that $x\chi_{y_0} r \in X \setminus N_1$ for some $r \in [s, t]$.

Recall the mapping r, which can be found in Definition 3.1. Since (N_1, N_3) (resp. (N_2, N_3)) is an index pair for (y_0, K) (resp. (y_0, A)), there is a strongly admissible isolating neighborhood M_K (resp. M_A) such that $N_1 \setminus N_2 \subset r^{-1}(M_K)$ (resp. $N_2 \setminus N_3 \subset r^{-1}(M_A)$). There also exists an open neighborhood W_K (resp. W_A) of K (resp. A) with $r^{-1}(W_K) \subset N_1 \setminus N_3$ (resp. $r^{-1}(W_A) \subset N_2 \setminus N_3$). Recall that $A \cap R = \emptyset$ by the definition of an attractor-repeller decomposition, so there are disjoint open neighborhoods U_A of A and U_R of R. We may assume without loss of generality that $W_A \subset U_A$. Setting $M_R := M_K \setminus W_A$, one has $\mathrm{Inv} M_R \subset R \subset U_R \subset M_R$, which means that M_R is an isolating neighborhood for R.
Moreover, one has

$$N_1 \setminus N_2 = (N_1 \setminus N_3) \setminus (N_2 \setminus N_3) \subset r^{-1}(M_K) \setminus r^{-1}(W_A) = r^{-1}(M_R).$$

Define $N_A' := \mathrm{cl}_{\Sigma^+(y_0)\times X}\, r(N_2 \setminus N_3)$ and $W_R := W_K \setminus N_A'$. One has

$$N_1 \setminus N_2 \supset r^{-1}(W_K) \setminus r^{-1}(N_A') = r^{-1}(W_R).$$

The set $K \cap N_A' \subset M_A$ is positively invariant: Let $x \in K \cap N_A'$ and $x\pi s \in K \setminus N_A'$ for some $s \in \mathbb{R}^+$. There is a sequence (t_n, x_n) in $N_2 \setminus N_3 \subset \mathbb{R}^+ \times X$ such that $r(t_n, x_n) \to x$ as $n \to \infty$. We can assume that $r(t_n, x_n)\pi s \notin N_A'$ for all $n \in \mathbb{N}$, so w.l.o.g. there are reals $s_n \to s_0$ with $(t_n, x_n)\chi_{y_0} s_n \in N_3$ for all $n \in \mathbb{N}$. We have $r(t_n, x_n)\pi s_n \to x\pi s_0$, so $(t_n, x_n)\chi_{y_0} s_n \in r^{-1}(W_K)$ for all but finitely many n, which is a contradiction since $r^{-1}(W_K) \cap N_3 = \emptyset$.
Hence, if $x \in K \cap N_A'$, then $\omega(x) \subset A$, implying that $R \cap N_A' = \emptyset$. Therefore W_R, which is obviously open, is a neighborhood of R. □

LEMMA 3.19. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Then, for every $T \in \mathbb{R}^+$*

$$(N_1, N_2^{-T}, N_3) := (N_1, N_2^{-T}(N_1), N_3)$$

and

$$(N_1, N_2^{-T}, N_3^{-T}) := (N_1, N_2^{-T}(N_1), N_3^{-T}(N_1))$$

are index triples for (y_0, K, A, R).

PROOF. Lemma 3.6 implies that (N_1, N_2^{-T}) and (N_1, N_3^{-T}) are index pairs for (y_0, K) for every $T > 0$. Furthermore, assuming that (N_2^{-T}, N_3) is an index pair for (y_0, A), it follows from Lemma 3.6[7] that (N_2^{-T}, N_3^{-T}) is an index pair for (y_0, A).
Hence, we only need to prove that (N_2^{-T}, N_3) is an index pair for (y_0, A).

(IP1) (N_1, N_2^{-T}) is an index pair, so N_2^{-T} is closed.

(IP3) Let $x \in N_2^{-T}$ and $x\chi_{y_0} t \notin N_2^{-T} \supset N_2$. We have $x\chi_{y_0} r \in N_2$ for some $r \le t$. Since (N_2, N_3) is an index pair, we must have $x\chi_{y_0} s \in N_3$ for some $s \in [r, t]$.

(IP4) Let $x \in N_3$ and $x\chi_{y_0} t \notin N_3$. (N_1, N_3) is an index pair, so $x\chi_{y_0} s \in (\mathbb{R}^+ \times X) \setminus N_1 \subset (\mathbb{R}^+ \times X) \setminus N_2^{-T}$ for some $s \in [0, t]$.

(N_1, N_2^{-T}) is an index pair for (y_0, R), so there is an open neighborhood W_R of R such that $r^{-1}(W_R) \subset N_1 \setminus N_2^{-T}$. We may assume that $W_R \cap A = \emptyset$ because $A \cap R = \emptyset$. Let N_K be an isolating neighborhood for K with $N_1 \setminus N_3 \subset r^{-1}(N_K)$. Then $N_A := N_K \setminus W_R$ is an isolating neighborhood for A with

$$N_2^{-T} \setminus N_3 \subset (N_1 \setminus N_3) \setminus (N_1 \setminus N_2^{-T}) \subset r^{-1}(N_A).$$

Since (N_2, N_3) is an index pair for (y_0, A), there is a neighborhood W_A of A with $r^{-1}(W_A) \subset N_2 \setminus N_3$. One has

$$r^{-1}(W_A) \subset N_2 \setminus N_3 \subset N_2^{-T}(N_1) \setminus N_3.$$

□

3.3.2. Long Exact Sequences. The long exact sequence associated with an attractor-repeller sequence is usually defined using the concept of so-called weakly exact sequences (Definition 2.1 in [**9**]). Instead of weakly exact sequences, we use the long exact sequence of triple as a starting point. The advantage is that our definition relies only on an axiomatic characterization of homology yet not necessarily on an underlying chain complex.

LEMMA 3.20. *Let* (N_1, N_2, N_3) *be an index triple for* (y_0, K, A, R). *Then, the projection* $p: N_1/N_3 \to N_1/N_2$ *induces an isomorphism* $\varrho: \mathrm{H}_*(N_1/N_3, N_2/N_3) \to \mathrm{H}_*(N_1/N_2, \{N_2\})$.

The proof will be conducted in three steps, the first two being formulated as separate lemmas.

LEMMA 3.21. *Let* (N_1, N_2) *be an index pair for* (y_0, K) *and define* $f: N_1 \to \mathbb{R}^+$ *by*

$$f(t,x) := \sup\{t_0 \in \mathbb{R}^+ : (t+s, \Phi_{y_0}(t+s, t, x)) \in \mathrm{cl}(N_1 \setminus N_2) \textit{ for all } s \in [0, t_0]\}.$$

Then,

(a) f *is upper semicontinuous and*

(b) *bounded on* N_2.

PROOF. (a) Suppose that f is not upper semicontinuous. Then there is a sequence $(t_n, x_n) \to (t_0, x_0)$ in N_1 such that $f(t_n, x_n) > f(t_0, x_0) + \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$.

[7] $N_3^{-T}(N_1) = N_3^{-T}(N_2^{-T})$

By the definition of f, there is an $s \in [0,\varepsilon[$ with $(t_0 + f(t_0,x_0)+s, \Phi_{y_0}(t_0 + f(t_0,x_0)+s,x_0)) \in (\mathbb{R}^+ \times X)\setminus(\mathrm{cl}(N_1 \setminus N_2))$. It follows that $(t_n + f(t_0,x_0)+s, \Phi_{y_0}(t_n + f(t_0,x_0)+s,t_n,x_n)) \in (\mathbb{R}^+ \times X)\setminus(\mathrm{cl}(N_1 \setminus N_2))$ for all n sufficiently large. Hence, $f(t_n,x_n) < f(t_0,x_0)+\varepsilon$, which is a contradiction.

(b) (N_1,N_2) is an index pair for (y_0,K), so there is a strongly admissible isolating neighborhood $N \subset \Sigma^+(y_0)\times X$ for K such that $N_1 \setminus N_2 \subset r^{-1}(N)$. N is closed, so $\mathrm{cl}(N_1 \setminus N_2) \subset r^{-1}(N)$. Furthermore, there exists an open neighborhood W of K with $r^{-1}(W) \subset N_1 \setminus N_2$. Now, suppose that f is unbounded on N_2. Then there is a sequence (t_n,x_n) in N_2 with $f(t_n,x_n) \to \infty$.

Because $f((t_n,x_n)\chi_{y_0} s) \neq 0$, we must have $(t_n,x_n)\chi_{y_0} s \in N_2 \cap (\mathrm{cl}_{\mathbb{R}^+\times X}(N_1 \setminus N_2))$ for all $s \in \big[0, f(t_n,x_n)\big[$ and all $n \in \mathbb{N}$, so $r(t_n,x_n)\pi s \in N \setminus W$ for all $s \in \big[0, f(t_n,x_n)\big]$.

Since N is strongly admissible, there is a solution $u: \mathbb{R} \to N\setminus W$ of π. However, $u(\mathbb{R}) \subset K$ because N is an isolating neighborhood for K. This is a contradiction since $K \subset W$.

□

LEMMA 3.22. *Let (N_1,N_2) be an index pair for (y_0,K). Then for all $T \in \mathbb{R}^+$ sufficiently large, $N_2^{-T} := N_2^{-T}(N_1)$ is a neighborhood of N_2 in N_1.*

PROOF. By Lemma 3.21 (a), $W^T := f^{-1}([0,T[)$ is open for every $T \in \mathbb{R}^+$. If T is sufficiently large, then $W^T \supset N_2$ by Lemma 3.21 (b), so W^T is a neighborhood of N_2 in N_1. We are going to show that $W^T \subset N_2^{-T}$, which implies that for large $T \in \mathbb{R}^+$, N_2^{-T} is a neighborhood of N_2 as claimed.
In order to prove the inclusion $W^T \subset N_2^{-T}$, let $x \in W^T$ and $\varepsilon > 0$ be arbitrary. We have $x\chi t \notin \mathrm{cl}(N_1 \setminus N_2)$ for some $t \le T+\varepsilon$ solely by the definition of f. Either $x\chi t \in N_1$ and thus $x\chi t \in N_2$ or $x\chi t' \in N_2$ for some $t' \le t$ because (N_1,N_2) is an index pair. Since $\varepsilon > 0$ is arbitrary and N_2 closed, it follows that $x\chi t'' \in N_2$ for some $t'' \le T$, so $x \in N_2^{-T}$. □

PROOF OF LEMMA 3.20. Consider the following sequence of inclusion induced mappings.

$$\mathrm{H}_*(N_1/N_3, N_2/N_3) \xrightarrow{i} \mathrm{H}_*(N_1/N_3, N_2^{-T}/N_3)$$

$$\xrightarrow{\quad k \quad} \mathrm{H}_*(N_1/N_2, N_2^{-T}/N_2) \xrightarrow{l} \mathrm{H}_*(N_1/N_2, N_2/N_2).$$

We will show that i,k,l are isomorphisms.
Firstly, we consider i. Define $\varphi_T : N_1/N_3 \to N_1/N_3$ by

$$\varphi_T([t,x]) := \begin{cases} \big[t+T, \Phi_{y_0}(t+T,t,x)\big] & (t+s, \Phi_{y_0}(t+s,t,x)) \in N_1 \setminus N_3 \text{ for all } s \in [0,T] \\ N_3 & \text{otherwise.} \end{cases}$$

It follows from Lemma 2.9 that φ_T and therefore its restriction to N_2^{-T}/N_3 are continuous. We conclude that N_2/N_3 is a deformation retract of N_2^{-T}/N_3, so i is indeed an isomorphism.
Secondly, choosing T sufficiently large, it follows from Lemma 3.22 that N_2^{-T} is a neighborhood of $N_2 \supset N_3$. Hence, k is an isomorphism by the excision property of homology.

Thirdly, it follows as before that the one-point space N_2/N_2 is a deformation retract of N_2^{-T}/N_2. Hence, k must be an isomorphism as well, which completes the proof. □

In view of Lemma 3.20, we can now make the following definition.

DEFINITION 3.23. Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Let q : $\mathrm{H}_*(N_1/N_3, N_2/N_3) \to \mathrm{H}_*(N_1/N_2, N_2/N_2)$ be inclusion induced and set $\partial = \delta \circ q^{-1}$, where δ is the (natural) connecting homomorphism of the (long exact sequence associated with the) triple $(N_1/N_3, N_2/N_3, N_3/N_3)$.
The long exact sequence associated with (N_1, N_2, N_3) is

$$\longrightarrow \mathrm{H}_*[N_2, N_3] \xrightarrow{i} \mathrm{H}_*[N_1, N_3] \xrightarrow{p} \mathrm{H}_*[N_1, N_2] \xrightarrow{\partial} \mathrm{H}_{*-1}[N_2, N_3] \longrightarrow . \tag{3.3}$$

Here, we denote $\mathrm{H}_*[N_1, N_2] := \mathrm{H}_*(N_1/N_2, \{N_2\})$.

LEMMA 3.24. *Let* (N_1, N_2, N_3) *be an index triple for* (y_0, K, A, R). *The sequence* (3.3) *associated with* (N_1, N_2, N_3) *is exact.*

PROOF. We rewrite (3.3) as follows.

$$\begin{tikzcd} & & & \mathrm{H}_*[N_1, N_2] \arrow[dr, "\partial"] & & \\ \arrow[r] & \mathrm{H}_*[N_2, N_3] \arrow[r, "i"] & \mathrm{H}_*[N_1, N_3] \arrow[ur, "p"] \arrow[r] & \mathrm{H}_*(N_1/N_3, N_2/N_3) \arrow[u, "q"] \arrow[r, "\delta"] & \mathrm{H}_{*-1}[N_2, N_3] \arrow[r] & {} \end{tikzcd}$$

The lower row is the long exact sequence of the triple $(N_1/N_3, N_2/N_3, N_3/N_3)$. The result follows easily because q is an isomorphism by Lemma 3.20. □

LEMMA 3.25. *Let* (N_1, N_2, N_3) *(resp.* (N_1', N_2', N_3')*) be an index triple for* (y_0, K, A, R) *(resp.* (y_0', K', A', R')*).*
The boundary operator ∂ *is natural with respect to continuous mappings* $f : (N_1, N_2, N_3) \to (N_1', N_2', N_3')$, *that is, if* ∂ *and* ∂' *denote the respective boundary operators, then*

$$\begin{tikzcd} \mathrm{H}_*[N_1, N_2] \arrow[r, "\partial"] \arrow[d, "f"] & \mathrm{H}_*[N_2, N_3] \arrow[d, "f"] \\ \mathrm{H}_*\left[N_1', N_2'\right] \arrow[r, "\partial'"] & \mathrm{H}_*\left[N_2', N_3'\right] \end{tikzcd}$$

is commutative.

PROOF. This follows easily from Definition 3.23. The connecting homomorphisms of the long exact sequences associated with a triple are natural, and so are the projections q. □

3.3.3. Homology Conley Index.

DEFINITION 3.26. Let $y_0 \in Y$ and $K \subset \Sigma^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood. Then its categorial Conley index $\mathscr{C}(y_0, K)$ is defined.
The (categorial, nonautonomous) homology Conley index is obtained by applying the homology functor, that is:

(1) If A is an object of $\mathscr{C}(y_0, K)$, then $\mathrm{H}_*(A)$ is an object of $\mathrm{H}_*\mathscr{C}(y_0, K)$.
(2) If f is a morphism of $\mathscr{C}(y_0, K)$, then $\mathrm{H}_*(f)$ is a morphism of $\mathrm{H}_*\mathscr{C}(y_0, K)$.

The above definition immediately leads to the following question: Does a connecting homomorphism ∂ which is defined by a particular index triple give rise to a unique morphism of the homology index?

THEOREM 3.27. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). The connecting homomorphism ∂ that is given by Definition 3.23 gives rise to a unique i.e., independent of (N_1, N_2, N_3), morphism $[\partial]$ in* CSS(gradMod) *and*

$$\longrightarrow \mathrm{H}_*\mathscr{C}(y_0, A) \xrightarrow{[i]} \mathrm{H}_*\mathscr{C}(y_0, K) \xrightarrow{[p]} \mathrm{H}_*\mathscr{C}(y_0, R) \xrightarrow{[\partial]} \mathrm{H}_{*-1}\mathscr{C}(y_0, A) \longrightarrow. \tag{3.4}$$

is a long exact sequence.

(3.4) is called the *(long exact) attractor-repeller sequence* of (y_0, K, A, R). We also say that $[\partial]$ is the connecting homomorphism of (y_0, K, A, R) respectively of the attractor-repeller sequence associated with (y_0, K, A, R).
It is an immediate consequence of Theorem 3.29 below that $[\partial]$ is well-defined. The proof that the morphisms $[i]$ and $[p]$ are well-defined is omitted.

LEMMA 3.28. *Let (N_1, N_2, N_3) and (M_1, M_2, M_3) be index triples for (y_0, K, A, R). Then there is an index triple (L_1, L_2, L_3) such that for some $T > 0$*

$$(L_1, L_2, L_3) \subset (N_1 \cap M_1, N_2^{-T}(N_1) \cap M_2^{-T}(M_1), N_3^{-T}(M_1) \cap M_3^{-T}(M_1)). \tag{3.5}$$

PROOF. By Theorem 3.8, there are index pairs $(\tilde{L}_1, \tilde{L}_3)$ for (y_0, K) and (L_2', L_3') for (y_0, A) which have the required inclusion properties, that is, for some $T' > 0$ it holds that

$$(\tilde{L}_1, \tilde{L}_3) \subset (N_1 \cap M_1, N_3^{-T'}(N_1) \cap M_3^{-T'}(M_1))$$
$$(L_2', L_3') \subset (\tilde{L}_1 \cap N_2 \cap M_2, N_3^{-T'}(N_2) \cap M_3^{-T'}(M_2)).$$

Assume for the moment that there is a constant $T'' > 0$ such that

(1) $L_3' \subset \tilde{L}_3^{-T''} := \tilde{L}_3^{-T''}(\tilde{L}_1)$ and
(2) $(L_2' \cup \tilde{L}_3^{-T''}, \tilde{L}_3^{-T''})$ is an index pair (y_0, A).

By Lemma 3.6, $(\tilde{L}_1, \tilde{L}_3^{-T''})$ is an index pair for (y_0, K), so

$$(L_1, L_2, L_3) := (\tilde{L}_1, L_2' \cup \tilde{L}_3^{-T''}, \tilde{L}_3^{-T''})$$

is an index triple for (y_0, K, A, R). Furthermore, taking $T = T' + T''$, (3.5) is satisfied.
It is therefore sufficient to check the two assumptions above.

(1) Suppose that $(t_n, x_n) \in L_3' \setminus \tilde{L}_3^{-2n}(N_1)$ is a sequence. We have

$$(t_n, x_n)\chi_{y_0}[0, 2n] \subset N_3^{-T'}(N_2) \subset N_3^{-T'}(N_1) \tag{3.6}$$

for all $n \in \mathbb{N}$. $(N_1, N_3^{-T'}(N_1))$ is an index pair for (y_0, K) by virtue of Lemma 3.6, so there is an admissible isolating neighborhood N of K such that $r(t_n, x_n)\pi[0, 2n] \subset N$ for all $n \in \mathbb{N}$.

We may assume without loss of generality that $r(t_n, x_n)\pi n \to (y, x) \in K$, so $(t_n, x_n)\chi_{y_0} n \in N_1 \setminus N_3^{-T'}(N_1)$ provided that N is sufficiently large, in contradiction to (3.6).

(2) We need to check the assumptions of Definition 2.6 and Definition 3.1.
 (IP1) It is clear that L_2 and L_3 are closed sets with $L_2 \subset L_3$.
 (IP3) Let $x \in L_2$ and $x\chi_{y_0} t \notin L_2$ for some $t > 0$. It follows that $x\chi_{y_0} t \notin L_2'$, so $x\chi_{y_0} s \in L_3' \subset L_3$ for some $s \in [0, t]$.
 (IP4) Suppose that $x \in L_3$, but $x\chi_{y_0} t \notin L_3$ for $t > 0$. (L_1, L_3) is an index pair by Lemma 3.6, so $x\chi_{y_0} s \in (\mathbb{R}^+ \times X) \setminus L_1 \subset (\mathbb{R}^+ \times X) \setminus L_2$ for some $s \in [0, t]$.

Finally, (L_2', L_3') is an index pair for (y_0, A). Hence there is an admissible isolating neighborhood $N \subset \Sigma^+(y_0) \times X$ for A with $L_2' \setminus L_3' \subset r^{-1}(N)$. Moreover, there is a neighborhood W of A in $\Sigma^+(y_0)$ such that $r^{-1}(W) \subset L_2' \setminus L_3'$. These inclusions continue to hold for (L_2, L_3): we have $L_2 \setminus L_3 \subset L_2' \setminus L_3' \subset r^{-1}(N)$. Since (L_1, L_3) is an index pair for (y_0, K), there is a neighborhood W_K of K with $r^{-1}(W_K) \subset L_1 \setminus L_3$. The intersection $W_0 := W \cap W_K$ is a neighborhood of A, and $r^{-1}(W_0) \subset L_2 \setminus L_3$.

□

THEOREM 3.29. *Let (N_1, N_2, N_3) and (M_1, M_2, M_3) be index triples for (y_0, K, A, R). Then the following diagram is commutative.*

$$\begin{tikzcd}
\arrow[r] & \mathrm{H}_*[N_1, N_2] \arrow[r, "\partial"] \arrow[d] & \mathrm{H}_*[N_2, N_3] \arrow[r] \arrow[d] & {} \\
\arrow[r] & \mathrm{H}_*[M_1, M_2] \arrow[r, "\partial"] & \mathrm{H}_*[M_2, M_3] \arrow[r] & {}
\end{tikzcd}$$

Its rows represent the long exact sequences associated with the respective index triple, and the vertical arrows denote the respective inner morphism of the categorial Conley index.

PROOF. Assuming that $(N_1, N_2, N_3) \subset (M_1, M_2, M_3)$, the inner morphisms are inclusion induced, so the theorem is merely a reformulation of Lemma 3.25. The general case follows from Lemma 3.28. Let the index triple (L_1, L_2, L_3) be given by that lemma. We have

$$(N_1, N_2, N_3) \subset (N_1, N_2^{-T}, N_3^{-T}) \supset (L_1, L_2, L_3)$$
$$(M_1, M_2, M_3) \subset (M_1, M_2^{-T}, M_3^{-T}) \supset (L_1, L_2, L_3)$$

for some $T > 0$.
By Lemma 3.19, the triples $(N_1, N_2^{-T}, N_3^{-T})$ and $(M_1, M_2^{-T}, M_3^{-T})$ in the middle are index triples. This reduces the general case to the special case covered by Lemma 3.25. □

3.4. Morse Decompositions

3.4.1. Preliminaries. Let $(P, \prec)$ be a strictly partially ordered set, that is, $\prec$ is a relation on P which is irreflexive and transitive.
Using the partial order $\prec$, one defines intervals and attracting intervals. A subset $I \subset P$ is an interval, $I \in \mathscr{I}(P, \prec)$, if $i, j, k \in P$, $i, k \in I$ and $i \prec j \prec k$ implies $j \in I$. An interval $I \in \mathscr{I}(P, \prec)$ is called attracting, $I \in \mathscr{A}(P, \prec)$, if $i, j \in P$, $j \in I$ and $i \prec j$ implies $i \in I$.

DEFINITION 3.30. Let (y_0, K) be a compact invariant pair. A family $(M_p)_{p\in P}$ is called a $\prec$-*ordered Morse-decomposition* (for (y_0, K)) provided that the following holds.

1. (1) The sets M_p, $p \in P$ are closed, invariant and pairwise disjoint.
2. (2) For every solution $u : \mathbb{R} \to K$, either $u(\mathbb{R}) \subset M_p$ for some $p \in P$, or there are $p, q \in P$ such that $p \prec q$, $\omega(u) \subset M_p$ and $\alpha(u) \subset M_q$.

Given an interval $I \in \mathscr{I}(P,\prec)$, let $M(I)$ denote the maximal invariant subset of K such that $(M_p)_{p\in I}$ is a $\prec$-ordered Morse-decomposition. In other words, $M(I)$ contains every Morse-set M_p with $p \in I$ and every connecting orbit[8] between Morse-sets M_p and M_q with $p, q \in I$. The sets $M(I)$ are closed (Corollary 3.33) and hence isolated compact invariant sets.

LEMMA 3.31. *Let (y_0, K) be a compact invariant pair, and let $(M_p)_{p\in P}$ be a $(P,\prec)$-ordered Morse-decomposition.*
Let $I \subset P$ be an interval and $p \in P \setminus I$ be a maximal or minimal element with respect to $(P,\prec)$. Then $(\mathrm{cl}_{Y\times X} M(I)) \cap M_p = \emptyset$.

PROOF. Suppose that the intersection is not empty. We will prove that p can neither be minimal nor maximal.
If $(\mathrm{cl}_{Y\times X} M(I)) \cap M_p \neq \emptyset$, there is a sequence $u_n : \mathbb{R} \to M(I)$ of solutions converging pointwise to a solution $u_0 : \mathbb{R} \to K$ with $u_0(0) \in M_p$. Let

$$
\begin{aligned}
s_n^- &:= \inf\{s \le 0 : u_n([s,0] \subset N_p\} \\
s_n^+ &:= \sup\{s \ge 0 : u_n([0,s] \subset N_p\}
\end{aligned}
$$

where N_p is an isolating neighborhood for M_p in $\Sigma^+(y_0) \times X$.
It is easy to see that $u_n(s_n^-) \in \partial N_p$ for all $n \in \mathbb{N}$. Taking subsequences, we can assume without loss of generality that either $s_n^- \to s_0$ or $s_n^- \to -\infty$. Setting $v_n(t) := u_n(t + s_n)$ and using the compactness of K, we can assume that there is a solution $v : \mathbb{R} \to K$ and $v_n(t) \to v(t)$ pointwise for all $t \in \mathbb{R}$. It follows that $\omega(v) \subset M_p$ as well as $v(0) \in \partial N_p$, so p must not be maximal.
Analogously, using s_n^+, one obtains that p must not be minimal. □

LEMMA 3.32. *Let (y_0, K) be a compact invariant pair, and let $(M_p)_{p\in P}$ be a $(P,\prec)$-ordered Morse-decomposition.*
Let $p \in P$ be a maximal or minimal element with respect to $\prec$. Then $M(P \setminus \{p\})$ is closed (compact).

PROOF. For brevity, we consider only the case that p is maximal. One can argue analogously if p is minimal.
Let $u_n : \mathbb{R} \to M(P \setminus \{p\})$ be a sequence of solutions converging pointwise to a solution $u_0 : \mathbb{R} \to K$ with $u_0(0) \notin M(P \setminus \{p\})$. It follows that $\alpha(u_0) \subset M_p \cap \mathrm{cl}_{Y\times X} M(P \setminus \{p\})$, so $M_p \cap \mathrm{cl}_{Y\times X} M(P \setminus \{p\}) \neq \emptyset$, in contradiction to Lemma 3.31. □

COROLLARY 3.33. *Let $N \subset Y \times X$ be an isolating neighborhood for a compact invariant pair (y_0, K), and let $(M_p)_{p\in P}$ be a $(P,\prec)$-ordered Morse-decomposition.*

[8] If $u : \mathbb{R} \to K$ is a solution with $\alpha(u) \subset M_q$ and $\omega(u) \subset M_p$ for some $p, q \in P$, then $u(\mathbb{R})$ is a connecting orbit between M_p and M_q.

For every $I \in \mathscr{I}(P,\prec)$, the set $M(I)$ is closed (compact).

PROOF. The proof is conducted by induction on the number of elements of $P \setminus I$. If $P \neq I$, there is a maximal or a minimal element p in $P \setminus I$. It follows from Lemma 3.32 that $M(P')$ is compact where we set $P' := P \setminus \{p\}$. Moreover, restricting $\prec$ to P' yields a $(P',\prec)$-ordered Morse-decomposition of $M(P')$. By induction, it follows that $M(I)$ is closed. $\square$

COROLLARY 3.34. *Let $N \subset Y \times X$ be an isolating neighborhood for a compact invariant pair (y_0, K), and let $(M_p)_{p\in P}$ be a $(P,\prec)$-ordered Morse-decomposition.*
For every $I \in \mathscr{I}(P,\prec)$, there is an isolating neighborhood $N(I) \subset Y \times X$ for $(y_0, M(I))$ such that $M_p \cap N(I) = \emptyset$ for all $p \in P \setminus I$.

PROOF. Since $M(I)$ is compact, there exists a closed neighborhood $N(I) \subset N$ of $M(I)$ which is disjoint from M_p for every $p \in P \setminus I$.
Let $u : \mathbb{R} \to N(I)$ be a solution. It follows that $u(\mathbb{R}) \subset K$. As $(M_p)_{p\in P}$ is a Morse-decomposition of K, we must have $u(\mathbb{R}) \subset M(I)$. $\square$

We are now in a position to introduce the notion of *weak*[9] *index filtrations* (for the nonautonomous index).

DEFINITION 3.35. Let (y_0, K) be a compact invariant pair, and let $(M_p)_{p\in P}$ be a Morse-decomposition for (y_0, K).
A *weak index filtration* for $(y_0, K, (M_p)_{p\in P})$ is a family $(N(A))_{A\in\mathscr{A}(P,\prec)}$ of closed subsets of $\mathbb{R}^+ \times X$ such that:

(1) For all $A \in \mathscr{A}(P,\prec)$, $(N(A), N(\emptyset))$ is an index pair for $(y_0, M(A))$.
(2) $A, B \in \mathscr{A}(P,\prec)$ and $A \subset B$ implies that $N(A) \subset N(B)$.

For some $n \in \mathbb{N}$, let $I_1, \dots, I_n$ be intervals. We say that the tuple $(I_1, \dots, I_n)$ is *increasingly ordered* if the order imposed by the indices is compatible with $\prec$, that is, there do not exist $0 \le l < k \le n$ and $(p, q) \in I_l \times I_k$ such that $q \prec p$. If it holds additonally that $I_1, \dots, I_n$ are pairwise disjoint and $I_1 I_2 \dots I_n := I_1 \cup I_2 \cup \dots \cup I_n$ is an interval, we write $(I_1, \dots, I_n) \in \mathscr{I}_n(P,\prec)$.
The following lemma implies in conjunction with Lemma 3.36 that a weak index filtration is sufficient in order to obtain an index triple for every attractor-repeller decomposition $(y_0, M(IJ), M(I), M(J))$ and every $(I, J) \in \mathscr{I}_2(P,\prec)$.

LEMMA 3.36. *Let $(N(A))_{A\in\mathscr{A}(P,\prec)}$ be a weak index filtration, $(I, J, K) \in \mathscr{I}_3(P,\prec)$ and $IJK \in \mathscr{A}(P,\prec)$.*
Then, $(M(J), M(K))$ is an attractor-repeller decomposition of $M(JK)$ and $(N(IJK), N(IJ), N(I))$ is an index triple for $(y_0, M(JK), M(J), M(K))$.

PROOF. Suppose that $u : \mathbb{R} \to M(JK)$ is a solution. Either $u(\mathbb{R}) \subset M_p$ for some $p \in P$, or there are $p \prec q$ such that $\alpha(u) \subset M_q$ and $\omega(u) \subset M_p$. Suppose that neither $u(\mathbb{R}) \subset M(J)$ nor $u(\mathbb{R}) \subset M(K)$ hold. (J, K) is increasingly ordered, so there are $q \in K$ and $p \in J$ such that $\alpha(u) \subset M_q$ and $\omega(u) \subset M_p$. The sets $M(J)$ and $M(K)$ are disjoint by definition and closed

[9]Compare this to the definition of an index filtration given in [**5**].

(hence compact) by Corollary 3.33. We have proved that $(M(J), M(K))$ is an attractor-repeller decomposition of $M(JK)$.
It is easy to see that IJ is an attracting interval. Hence, $(N(IJK), N(IJ), N(\emptyset))$ is an index triple simply because $(N(A))_{A\in\mathscr{A}P,\prec}$ is a weak index filtration. It follows from Lemma 3.18 that $(N(IJK), N(IJ))$ is an index pair for $(y_0, M(K))$. Analogously, one obtains that $(N(IJ), N(I))$ is an index pair for $(y_0, M(J))$, whence it follows immediately (Definition 3.16) that $(N(IJK), N(IJ), N(I))$ is an index triple for $(y_0, M(JK), M(J), (K))$. □

LEMMA 3.37. *Let $J \in \mathscr{I}(P,\prec)$ and set $A := \{p \in P : p \preceq q$ for some $q \in J\}$ and $I := A \setminus J$, where $p \preceq q$ if $p \prec q$ or $p = q$.*
Then, $I \in \mathscr{A}(P,\prec)$, (I, J) is increasingly ordered and $IJ \in \mathscr{A}(P,\prec)$.

PROOF. Let $p, q, r \in P$. If $q \in A$ and $r \prec q$, then $q \preceq q'$ for some $q' \in J$, so $r \prec q'$ and thus $r \in A$, showing that A is an attracting interval.
Suppose that (I, J) is not increasingly ordered, that is, there are $q \in I$ and $p \in J$ such that $p \prec q$. We have $q \prec q'$ for some $q' \in J$, so $q \in J$ since J is an interval. As $q \in I \cap J = \emptyset$ cannot hold, (I, J) must be increasingly ordered.
If I is not an attracting interval, there are $q \in I$ and $r \in P$ such that $r \prec q$ but $r \notin I$. The interval A, however, is attracting, so $r \in J$. Since (I, J) is increasingly ordered, such r and q cannot exist, showing that I is an attracting interval. □

Fix a weak index filtration $(N(A))_{A\in\mathscr{A}(P,\prec)}$. Suppose (I, J) is an increasingly ordered tuple of intervals and IJ is an attracting interval. According to Lemma 3.36, $(N(IJ), N(J))$ is an index pair for $(y_0, M(I))$. Now let (A_0, I, J, K) be increasingly ordered intervals such that A_0IJK is an attracting interval[10], let $(A_0, I, J, K) =: (I_1, I_2, I_3, I_4)$ and set $\mathrm{H}_*(I_k \dots I_l) := \mathrm{H}_*[N(I_1 \dots I_l), N(I_1 \dots I_{k-1})]$ for brevity.
One obtains a commutative diagram (3.7). Except for the connecting homorphism, each morphism is inclusion induced and each row is an attractor-repeller sequence as introduced in Definition 3.23.

FIGURE 3.1.

$$\begin{tikzcd}
\mathrm{H}_*(I) \arrow[r, "i_{I,IJ}"] \arrow[d] & \mathrm{H}_*(IJ) \arrow[r, "p_{IJ,J}"] \arrow[d] & \mathrm{H}_*(J) \arrow[r, "\partial_{J,I}"] \arrow[d] & \mathrm{H}_{*-1}(I) \arrow[d] \\
\mathrm{H}_*(I) \arrow[r, "i_{I,IJK}"] \arrow[d] & \mathrm{H}_*(IJK) \arrow[r, "p_{IJK,JK}"] \arrow[d] & \mathrm{H}_*(JK) \arrow[r, "\partial_{JK,I}"] \arrow[d] & \mathrm{H}_{*-1}(I) \arrow[d] \\
\mathrm{H}_*(IJ) \arrow[r, "i_{IJ,IJK}"] & \mathrm{H}_*(IJK) \arrow[r, "p_{IJK,K}"] & \mathrm{H}_*(K) \arrow[r, "\partial_{K,IJ}"] & \mathrm{H}_{*-1}(IJ) \\
\mathrm{H}_*(J) \arrow[r, "i_{J,JK}"] \arrow[u] & \mathrm{H}_*(JK) \arrow[r, "p_{JK,K}"] \arrow[u] & \mathrm{H}_*(K) \arrow[r, "\partial_{K,J}"] \arrow[u] & \mathrm{H}_{*-1}(J) \arrow[u]
\end{tikzcd} \tag{3.7}$$

[10] A_0 always exists in view of Lemma 3.37.

An aesthetically more pleasing variant of the above diagram is the so-called *homology index braid*, which is depicted in (3.8). Using (3.7), one easily proves its commutativity.

FIGURE 3.2. homology index braid

$$
\begin{tikzcd}
{} \arrow[d] & & {} \arrow[dll] \arrow[drr] & & {} \arrow[d] \\
H_{*-1}(I) \arrow[drr, "i_{I,IJ}"] \arrow[dd, "i_{I,IJK}"] & & & & H_*(K) \arrow[dll, "\partial_{K,IJ}"'] \arrow[dd, "\partial_{K,J}"] \\
& & H_{*-1}(IJ) \arrow[dll, "i_{IJ,IJK}"'] \arrow[drr, "p_{IJ,J}"] & & \\
H_{*-1}(IJK) \arrow[drr, "p_{IJK,JK}"] \arrow[dd, "p_{IJK,K}"] & & & & H_{*-1}(J) \arrow[dll, "i_{J,JK}"'] \arrow[dd, "\partial_{J,I}"] \\
& & H_{*-1}(JK) \arrow[dll, "p_{JK,K}"'] \arrow[drr, "\partial_{JK,I}"] & & \\
H_{*-1}(K) \arrow[drr, "\partial_{K,IJ}"] \arrow[dd, "\partial_{K,J}"] & & & & H_{*-2}(I) \arrow[dll, "i_{I,IJ}"'] \arrow[dd, "i_{I,IJK}"] \\
& & H_{*-2}(IJ) \arrow[dll, "p_{IJ,J}"'] \arrow[drr, "i_{IJ,IJK}"] & & \\
H_{*-2}(J) \arrow[drr] \arrow[d] & & & & H_{*-2}(IJK) \arrow[dll] \arrow[d] \\
{} & & {} & & {}
\end{tikzcd}
\tag{3.8}
$$

3.4.2. Persistence of Attractor-repeller Decompositions. As in the previous section, let P be a finite set equipped with a strict partial order $\prec$.
We will now formulate the main result of this section: the convergence (or persistence) of Morse-decompositions.

DEFINITION 3.38. For all $n \in \mathbb{N}$, let $y_n \in Y_c$ (resp. $y_0 \in Y_c$), $K_n \subset \Sigma^+(y_n) \times X$ (resp. $K \subset \Sigma^+(y_0) \times X$) be an isolated invariant set, and for each $n \in \mathbb{N}$ let $(M_{n,p})_p$ (resp. $(M_p)_p$) be a $(P, \prec)$-ordered Morse-decomposition of K_n (resp. K).
We say that the Morse-decompositions converge, i.e., $(y_n, K_n, (M_{n,p})_p) \to (y_0, K, (M_p)_p)$, provided that the following holds: There is an isolating neighborhood N (resp. N_p) of (y_0, K) (resp. (y_0, M_p)) in $Y \times X$, and there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, it holds that N (resp. N_p) is an isolating neighborhood for (y_n, K_n) (resp. $(y_n, M_{n,p})$).

The following theorem concerning the convergence of Morse-decompositions is the main result of this section.

THEOREM 3.39. *Suppose that $(y_n)_n$ is a sequence in Y, $y_0 \in Y_c$ and*[11]

$$d(y_n^t, \Sigma^+(y_0)) \to 0 \text{ as } t, n \to \infty.$$

Let N (resp. N_p, $p \in P$) be a strongly skew-admissible isolating neighborhood for (y_0, K) (resp. (y_0, M_p), $p \in P$), and let $(M_p)_p$ be a $(P, \prec)$-ordered Morse-decomposition of K.
Then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$:

[11] By an abuse of notation, we write $d(y, \Sigma^+(y_0)) := \inf_{\tilde{y} \in \Sigma^+(y_0)} d(y, \tilde{y})$.

(a) *There is an invariant subset* $K_n \subset \Sigma^+(y_n) \times X$ *(resp.* $M_{n,p} \subset \Sigma^+(y_n) \times X$*) such that* N *(resp.* N_p*) is an isolating neighborhood for* (y_n, K_n) *(resp.* $(y_n, M_{n,p})$*).*

(b) $(M_{n,p})_p$ *is a* $(P, \prec)$*-ordered Morse-decomposition of* K_n*.*

LEMMA 3.40. *Assume the hypotheses of Theorem 3.39, let* $A \in \mathscr{A}(P, \prec)$*, and let* N_A *be a strongly skew-admissible isolating neighborhood for* $(y_0, M(A))$*. Then, for all* n *sufficiently large,*

$$N_A^T := \{(y,x) \in N_A : (y,x)\pi[0,T] \subset N\} \tag{3.9}$$

is an isolating neighborhood for (y_n, K_n') *and* $(y_0, M(A))$*, where* $K_n' := (\mathrm{Inv} N_A^T) \cap (\Sigma^+(y_n) \times X)$*. Moreover,* $N_A^T \cap K_n$ *is positively invariant provided that* T *and* n *are sufficiently large.*

PROOF. First of all, it is easy to see that N_A^T is closed for every $T > 0$. Fix some $T > 0$, and assume that N_A^T is not an (isolating) neighborhood for $(y_0, M(A))$. There must be a sequence $(y_n', x_n) \in N_A \setminus N_A^T$ with $d((y_n', x_n), M(A)) \to 0$ as $n \to \infty$. Consequently, one has $(y_n', x_n)\pi s_n \in \partial N_A$ for some $s_n \le T$. Taking subsequences, we may assume without loss of generality that $(y_n', x_n) \to (y, x) \in M(A)$ and $s_n \to s_0$. We thus have $(y, x)\pi s_0 \in M(A) \cap \partial N_A$, but N_A is an isolating neighborhood for $M(A)$, so (y, x) must not exist. We have proved that N_A^T is an isolating neighborhood for $(y_0, M(A))$. Subsequently, Theorem 2.27 implies that N_A^T is an isolating neighborhood for (y_n, K_n') provided that n is sufficiently large. This proves our first claim.

We still need to prove the positive invariance property for T and n large. Suppose to the contrary that we are given sequences $T_n \to \infty$ and $(y_n', x_n) \in K_n \cap N_A^{T_n}$ such that $s_n := \sup\{s \in \mathbb{R}^+ : (y_n', x_n)\pi[0,s] \subset N_A^{T_n}\} < \infty$ for all $n \in \mathbb{N}$. We must have $(y_n', x_n)\pi(s_n + T_n) \in \partial N_A$ by the choice of s_n.

In view of Lemma 2.28, there is a solution $u : \mathbb{R}^- \to K \cap N_A$ with $(y_n', x_n)\pi(s_n + T_n) \to u(0)$, so $u(0) \in \partial N_A$. Hence, u can be extended to a full solution $u' : \mathbb{R} \to K$ with $\alpha(u) \subset M(A)$. Since A is an attracting interval, we conclude that $u'(\mathbb{R}) \subset M(A)$, which is a contradiction since N_A is an isolating neighborhood for $(y_0, M(A))$. □

LEMMA 3.41. *Assume the hypotheses of Theorem 3.39, let* $A \in \mathscr{A}(P, \prec)$*, and let* N_A *be an isolating neighborhood for* $(y_0, M(A))$*. Define* N_A^T *by* (3.9)*.*

Then, for all $n \in \mathbb{N}$ *and* $T \in \mathbb{R}^+$ *sufficiently large as well as for every solution* $u : \mathbb{R} \to (\Sigma^+(y_n) \times X) \cap N$*, it holds that either*[12] $\alpha(u) \cap \mathrm{int}_{Y \times X} N_A^T = \emptyset$ *or* $u(\mathbb{R}) \subset N_A^T$*.*

PROOF. Choose n and T large enough that the conclusions of Lemma 3.40 hold.

In particular, $K_n \cap N_A^T$ is positively invariant. If $\alpha(u) \cap \mathrm{int}_{Y \times X} N_A^T \neq \emptyset$, then there is a sequence $t_n \to -\infty$ such that $u(t_n) \in N_A^T \cap K_n$. It follows from the positive invariance of $N_A^T \cap K_n$ that $u(t_n + s) \in N_A^T \cap K_n$ for all $s \in \mathbb{R}^+$, and thus $u(t) \in N_A^T \cap K_n$ for all $t \in \mathbb{R}$ because $t_n \to -\infty$. □

PROOF OF THEOREM 3.39. (a) This is merely a restatement of Theorem 2.27.

(b) In view of (a), one can assume without loss of generality that for all $p \in P$, the set $N_p \subset Y \times X$ is an isolating neighborhood for $(y_n, M_{n,p})$ for all $n \in \mathbb{N}$.

---

[12] It is easy to see that the two following conditions are mutually exclusive for large n. Namely, $u(\mathbb{R}) \subset N_A^T$ implies that $\alpha(u) \subset N_A^T$. If the latter set is an isolating neighborhood, one immediately obtains that $\alpha(u) \subset \mathrm{int}_{Y \times X} N_A^T$.

We are going to prove that for $n \in \mathbb{N}$ large, $(M_{n,p})_{p\in P}$ is a $(P,\prec)$-ordered Morse-decomposition of K_n by induction of the cardinality of P. Let $p_0 \in P$ be a maximal element, so $A := P \setminus \{p_0\}$ is an attracting interval. By Corollary 3.34, there exists a strongly skew-admissible isolating neigbhorhood $N_A \subset N$ for $(y_0, M(A))$ such that $N_A \cap M_{p_0} = \emptyset$.

Thus, we can assume by induction that for all $n \geq n_0(A)$, $(M_{n,p})_{p\in A}$ is a Morse-decomposition of $K_n' := (\mathrm{Inv} N_A) \cap (\Sigma^+(y_n) \times X)$. Replacing N_A by N_A^T and choosing T large, we can additionally assume that $N_A \cap K_n$ is positively invariant, and N_A satisfies the conclusions of Lemma 3.41 for all $n \geq n_0 := n_0(P) \geq n_0(A)$. Set $N'_{p_0} := N \setminus \mathrm{int}_{Y\times X} N_A$ and $M_{n,p_0} := (\mathrm{Inv} N_{p_0}) \cap (\Sigma^+(y_n) \times X)$. It is easy to see that N'_{p_0} is another isolating neighborhood for (y_0, M_{p_0}). Hence, there exists an $n_0 = n_0(p_0, A) \geq n_0(A)$ such that for all $n \geq n_0$, N'_{p_0} is an isolating neighborhood for M_{n,p_0}.

Let $n \geq n_0(p_0, A)$, and let $u\colon \mathbb{R} \to N \cap (\Sigma^+(y_n) \times X)$ be a solution. Either $u(\mathbb{R}) \subset N_A^T$, in which case the induction argument applies, or $\alpha(u) \cap \mathrm{int}_{Y\times X} N_A^T = \emptyset$.

In the second case, one has $\alpha(u) \subset N'_{p_0}$, so $\alpha(u) \subset M_{n,p_0}$ by the choice of n_0. Either $u(\mathbb{R}) \subset N'_{p_0}$, implying that $u(\mathbb{R}) \subset M_{n,p_0}$, or $\omega(u) \subset N_A$ since $N_A \cap K_n$ is positively invariant for all $n \geq n_0$.

Hence, for $n \geq n_0(p_0, A)$, $(M_{n,p})_p$ is a $(P,\prec)$-ordered Morse-decomposition of K_n. □

3.4.3. Continuation. Let P be a finite set and $\prec$ a strict partial order on P. Consider an isolated invariant set and a $(P,\prec)$-ordered Morse-decomposition of this invariant set. A continuous change of a dynamical system which preserves the invariant set and its Morse-decomposition preserves the categorial Conley index – and thus also the homotopy index and every other index which can be obtained from it. It also preserves the homology index braid and, in particular, its homomorphisms.
As in the previous chapter, we make the standing assumption that Y is linear. More precisely, it is assumed throughout this section that (L1) and (L2) hold.
We will now, mutatis mutandis, proceed as in the previous chapter. The first step is to find an appropriate replacement for the homotopy index, which is called *continuation class.*

DEFINITION 3.42. Let Ω be a set. Consider the set $\mathscr{P}$ of all tuples (Y, X, π, y_0, K, M), where $X, Y \subset \Omega$ are metric spaces, π is a skew-product semiflow on $Y \times X$, $y_0 \in Y_c$, $N \subset Y \times X$ is a skew-admissible isolating neighborhood for (y_0, K) and $(M_p)_{p\in P}$ a $(P,\prec)$-ordered Morse-decomposition.
Define an equivalence relation on $\mathscr{P}$ as follows: (Y, X, π, y_0, K, M) and $(Y', X', \pi', y_0', K', M')$ are related if there exists a family $(\theta_I)_{I\in\mathscr{I}(P,\prec)}$ such that:

(1) $\theta(I)\colon \mathscr{C}(y_0, M(I)) \to \mathscr{C}(y_0', M'(I))$ is an isomorphism.

(2) For every $(I,J)\in\mathscr{I}_2(P,\prec)$, the following ladder, the rows of which are attractor-repeller sequences, is commutative.

$$\begin{tikzcd}
\arrow[r] & \mathrm{H}_*\mathscr{C}(y_0,M(I)) \arrow[r] \arrow[d, "\mathrm{H}_*\theta(I)"] & \mathrm{H}_*\mathscr{C}(y_0,M(IJ)) \arrow[r] \arrow[d, "\mathrm{H}_*\theta(IJ)"] & \mathrm{H}_*\mathscr{C}(y_0,M(J)) \arrow[r] \arrow[d, "\mathrm{H}_*\theta(J)"] & \mathrm{H}_{*-1}\mathscr{C}(y_0,M(I)) \arrow[r] \arrow[d, "\mathrm{H}_{*-1}\theta(I)"] & {} \\
\arrow[r] & \mathrm{H}_*\mathscr{C}(y_0',M'(I)) \arrow[r] & \mathrm{H}_*\mathscr{C}(y_0',M'(IJ)) \arrow[r] & \mathrm{H}_*\mathscr{C}(y_0',M'(J)) \arrow[r] & \mathrm{H}_{*-1}\mathscr{C}(y_0',M'(I)) \arrow[r] & {}
\end{tikzcd} \tag{3.10}$$

The *continuation class* $\mathrm{ContCl}(y_0,K,M):=\mathrm{ContCl}(\pi,y_0,K,M):=\mathrm{ContCl}(Y,X,\pi,y_0,K,M)$ is the equivalence class of (y_0,K,M) under the above relation. The notation of Y, X and π is omitted whenever possible.

Usually the skew-product semiflow π remains unchanged. As before, there is a notable exception, stated below.

LEMMA 3.43. *Suppose that $y_0\in Y_c$ and*[13] $((\pi_n,y_0),K_n,(M_{n,p})_{p\in P})\to((\pi_0,y_0),K_0,(M_p)_{p\in P})$. *Then, there is an $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, one has*

$$\mathrm{ContCl}(\pi_n,y_0,K_n,(M_{n,p})_{p\in P})=\mathrm{ContCl}(\pi_0,y_0,K_0,(M_p)_{p\in P}).$$

PROOF. This lemma serves primarily as an interface to the results of [**6**]. Theorem 3.4 in [**6**] implies that there are index filtrations $(\hat N_n(I))_{I\in\mathscr{I}(P,\prec)}$ and $(\hat N_n'(I))_{I\in\mathscr{I}(P,\prec)}$ for $(\pi_n,K_n,(M_{n,p})_{p\in P})$ (resp. $(\pi_0,K_0,(M_p)_{p\in P})$) possessing the required nesting property, that is,

$$\hat N_n(I)\subset\hat N_0(I)\subset\hat N_n'(I)\subset\hat N_0'(I)\quad I\in\mathscr{I}(P,\prec). \tag{3.11}$$

By using weak index filtrations, we may limit our attention to attracting intervals.
For every $A\in\mathscr{A}(P,\prec)$, define

$$\begin{aligned}
N(A) &:= \{(t,x):(y_0^t,x)\in\hat N(A)\}\\
N'(A) &:= \{(t,x):(y_0^t,x)\in\hat N'(A)\}
\end{aligned}$$

For every $A\in\mathscr{A}(P,\prec)$, it holds that $(\hat N_n(A),\hat N_n(\emptyset))$ (resp. $(\hat N_n'(A),\hat N_n'(\emptyset))$) is an FM-index pair for $M_n(A)$, so by Lemma 3.3 $(N_n(A),N_n(\emptyset))$ and $(N_n'(A),N_n'(\emptyset))$ are index pairs for $(y_0,M_n(A))$ – with respect to π_n.
Let $(I,J)\in\mathscr{I}_2(P,\prec)$ and $IJ,I\in\mathscr{A}(P,\prec)$. By Lemma 3.36, $(N_n(IJ),N_n(I))$ and $(N_n'(IJ),N_n'(I))$ (resp. $(N(IJ),N(I))$ and $(N'(IJ),N'(I))$) are index pairs for $(y_0,M_n(I)):=(y_0,M(I))$ with respect to the skew-product semiflow π_n (resp. $(y_0,M(I))$ with respect to the skew-product semiflow π_0). From (3.11), we obtain the following inclusion induced morphisms:

$$[N_n(IJ),N_n(I)]\xrightarrow{i}[N(IJ),N(I)]\xrightarrow{j}\big[N_n'(IJ),N_n'(I)\big]\xrightarrow{k}[N'(IJ),N'(I)]$$

Lemma 3.4 implies that each of the morphisms $j\circ i$, $k\circ j$ is a homotopy equivalence. Hence, i,j,k are homotopy equivalences, and so is

$$\theta(J):=[i]:\ \mathscr{C}(\pi_n,y_0,M_n(J))\to\mathscr{C}(\pi_0,y_0,M(J)).$$

[13]Take $Y:=\{\pi_k:\ k\in\mathbb{N}\cup\{0\}\times\Sigma^+(y_0)$ and $d((\pi_n,y),(\pi_m,y')):=\big|\rho(n)-\rho(m)\big|+d(y,y')$, where $\rho(n)=1/n$ if $n>0$ and $\rho(0)=0$. Then Definition 3.38 can be applied.

In view of Lemma 3.37, we can always find an interval I such that the above construction of $\theta(J)$ is possible. The next step is to prove that $\theta(J)$ is well-defined, that is, independent of I. Suppose that $(I', J) \in \mathscr{I}_2(P, \prec)$ and $I'J, I' \in \mathscr{A}(P, \prec)$. Then $I_0 := I' \cap I$ is again an attracting interval and so is $I_0 J = I'J \cap IJ$. It is thus sufficient to prove that the morphisms $\theta(J)$ defined by I_0 and I agree. This follows easily from the commutativity of the diagram below because the vertical (inclusion-induced) morphisms are inner morphisms of the categorial Conley index,

$$\begin{tikzcd}
{[N_n(I_0J), N_n(I_0)]} \arrow[r, "i_0"] \arrow[d] & {[N(I_0J), N(I_0)]} \arrow[d] \\
{[N_n(IJ), N_n(I)]} \arrow[r, "i"] & {[N(IJ), N(I)]}
\end{tikzcd}$$

so $[i] = [i_0]$. The same argument applies to j and k.

Finally, let $(I, J, K) \in \mathscr{I}_3(P, \prec)$ and consider (3.12), where every morphism is inclusion induced except for the connecting homomorphism of the respective attractor-repeller sequence. It is

FIGURE 3.3.

$$\begin{tikzcd}
\mathrm{H}_*[N_n(IJ), N_n(I)] \arrow[r] \arrow[d] & \mathrm{H}_*[N_n(IJK), N_n(I)] \arrow[r] \arrow[d] & \mathrm{H}_*[N_n(IJK, N_n(IJ)] \arrow[r, "\partial_n"] \arrow[d] & \mathrm{H}_{*-1}[N_n(IJ), N_n(I)] \arrow[d] \\
\mathrm{H}_*[N(IJ), N(I)] \arrow[r] & \mathrm{H}_*[N(IJK), N(I)] \arrow[r] & \mathrm{H}_*[N(IJK, N(IJ)] \arrow[r, "\partial"] & \mathrm{H}_{*-1}[N(IJ), N(I)]
\end{tikzcd} \tag{3.12}$$

clear that inclusion induced morphisms commute. From Lemma 3.25, one obtains that the square with the connecting homomorphisms is commutative as well. The commutativity of (3.10) follows. □

For the rest of this section, we will make the following general assumptions. Let P be a finite set and $\prec$ a strict partial order on P, Γ a metric space and $f = (y(\gamma), K(\gamma), (M_p(\gamma))_{p\in P})_{\gamma\in\Gamma}$ a family such that for all $\gamma \in \Gamma$

(1) $y(\gamma) \in Y_c$
(2) there is a strongly-skew-admissible isolating neighborhood for $(y(\gamma), K(\gamma))$
(3) $(M_p(\gamma))_{p\in P}$ is a $(P, \prec)$-ordered Morse-decomposition of $K(\gamma)$

DEFINITION 3.44. We say that f is continuous at $\gamma_0 \in \Gamma$ if whenever $\gamma_n \to \gamma_0$ in Γ:

(C1') $(y(\gamma_n), K(\gamma_n), (M_p(\gamma_n))_{p\in P}) \to (y(\gamma_0), K(\gamma_0), (M_p(\gamma_0))_{p\in P})$.
(C2') $d(y^t_{\gamma_n}, y^t_{\gamma_0}) \to 0$ as $n, t \to \infty$.

f is continuous if it is continuous in every point $\gamma_0 \in \Gamma$.

The rest of this section is devoted to the proof of the following continuation theorem.

THEOREM 3.45. *Suppose that Γ is connected and f is continuous. Then,* $\mathrm{ContCl} \circ f$ *is constant.*

LEMMA 3.46. *Let $\gamma_0 \in \Gamma$, and let N (resp. N_p) be a strongly skew-admisible isolating neighborhood for $(y(\gamma_0), K(\gamma_0))$ (resp. $(y(\gamma_0), (M_p(\gamma_0))_{p\in P})$). Then there is a neighborhood U of γ_0 in Γ such that for all $\gamma \in U$ and all $\lambda \in [0, 1]$:*

(a) *There is a set* $K_{\gamma,\lambda}$ *(resp.* $M_{\gamma,\lambda,p}$*,* $p\in P$*) such that* N *(resp.* N_p*,* $p\in P$*) is an isolating neighborhood for* $(\underbrace{\lambda y(\gamma_0)+(1-\lambda)y(\gamma)}_{=:y(\gamma,\lambda)}, K_{\gamma,\lambda})$ *(resp.* $(y(\gamma,\lambda), M_{\gamma,\lambda,p})$*,* $p\in P$*).*

(b) $(M_{\gamma,\lambda,p})_{p\in P}$ *is a* $(P,\prec)$*-ordered Morse-decomposition of* $K_{\gamma,\lambda}$

(c) $f(\gamma)=(y(\gamma,0),K_{\gamma,0},(M_{\gamma,0,p})_{p\in P})$.

LEMMA 3.47. *Let* N *and* N' *be strongly admissible isolating neighborhoods for* (y_0,K)*, and let*

$$d(y_n^t,y_0^t)\to 0 \text{ as } t,n\to\infty.$$

Then $K_n:=\operatorname{Inv}N\cap(\Sigma^+(y_n)\times X)=\operatorname{Inv}N'\cap(\Sigma^+(y_n)\times X)=:K_n'$ *for all but finitely many* $n\in\mathbb{N}$.

PROOF. Arguing by contradiction, we can assume without loss of generality that there exists a sequence $x_n\in K_n\setminus K_n'$. By using Theorem 2.27, one obtains that N' is an isolating neighborhood for (y_n,K_n') for all but finitely many $n\in\mathbb{N}$. Hence, the solution through x_n leaves N' at least once.
Therefore, one can choose a sequence $(x_n')_{n\ge n_0}$ with $x_n'\in K_n\setminus N'$ for all $n\ge n_0$. By Lemma 2.28, there is a convergent subsequence $x_n''\to x_0\in\operatorname{Inv}(N)$.
Because N is an isolating neighborhood for K, one has $x_0\in K$. One the other hand, $x_n''\in(Y\times X)\setminus N'$ for all $n\in\mathbb{N}$ implies that $x_0\in(Y\times X)\setminus\operatorname{int}N'$. However, $K\subset\operatorname{int}N'$, which is a contradiction. □

PROOF OF LEMMA 3.46. (a), (b) Otherwise, there are sequences $\gamma_n\to\gamma_0$ and $\lambda_n\in[0,1]$ such that (a) or (b) are not satisfied. However,

$$d(y(\gamma_n,\lambda_n)^t,y(\gamma_0)^t)\to 0 \text{ as } t,n\to\infty,$$

so in particular $d(y(\gamma_n,\lambda_n)^t,\Sigma^+(y(\gamma_0)))\to 0$ as $t,n\to\infty$. This is a contradiction to Theorem 3.39.

(c) First of all, it is clear that $y(\gamma,0)=y(\gamma)$.
By (C1'), there is an isolating neighborhood N' (resp. N_p', $p\in P$) for $(y(\gamma),K(\gamma))$ (resp. $(y(\gamma),M_p(\gamma))$, $p\in P$). It follows from Lemma 3.47 that $K_{\gamma,0}=K(\gamma)$ (resp. $M_{\gamma,0,p}=M_p(\gamma)$, $p\in P$) for all γ in a sufficiently small neighborhood of γ_0.

□

LEMMA 3.48. *Let* $\gamma_0\in\Gamma$*, and let* N *(resp.* N_p*,* $p\in P$*) be a strongly skew-admisible isolating neighborhood for* $(y(\gamma_0),K(\gamma_0))$ *(resp.* $(y(\gamma_0),M_p)$*). Let* $U\subset\Gamma$ *as well as* $K_{\gamma,\lambda}$ *and* $M_{\gamma,\lambda,p}$ *be given by Lemma 3.46.*
Let $\gamma\in U$*,* $y_0:=y(\gamma_0)$*,* $h:=y(\gamma)-y(\gamma_0)$*, and consider the spaces* $Y':=\Sigma^+(y_0)\times\Sigma^+(h)$ *and* $Y'':=[0,1]\times Y'$*. On* $Y'\times X$*, define a family* $(\pi_\lambda)_{\lambda\in[0,1]}$ *of semiflows by*

$$(y,h,x)\pi_\lambda t:=(y^t,h^t,\Phi(t,y+\lambda h,x)).$$

On $Y''\times X$*, one defines* $(\lambda,y,h,x)\pi' t:=(\lambda,(y,h,x)\pi_\lambda t)$*.*
Define $j\colon Y'\times X\to Y\times X$ *by* $j(y_0,h,x):=(y_0+h,x)$*, and fix an arbitrary* $\gamma\in U$*. Then:*

(a) *The sets* $(M_{\lambda,p}')_p:=(j^{-1}(M_{\gamma,\lambda,p}))_p$ *form a Morse-decomposition of* $K_\lambda':=j^{-1}(K_{\gamma,\lambda})$ *relative to* π_λ*.*
Furthermore, $(\lambda_n,K_{\lambda_n}',(M_{\lambda_n,p}')_p)\to(\lambda,K_\lambda',(M_{\lambda,p}')_p)$ *whenever* $\lambda_n\to\lambda$ *in* $[0,1]$*.*

(b) $\mathrm{ContCl}((\lambda, y_0, h), K'_\lambda, (M'_{\lambda,p})_p)$ *is independent of* $\lambda \in [0,1]$.
(c) $\mathrm{ContCl}(f(\gamma)) = \mathrm{ContCl}((1, y_0, h), K'_1, (M'_{1,p})_p) = \mathrm{ContCl}((0, y_0, h), K'_0, (M'_{0,p})_p) = \mathrm{ContCl}(f(\gamma_0))$.

PROOF. By Lemma 3.46 respectively the choice of U, $N \subset Y \times X$ (resp. $N_p \subset Y \times X$) is an isolating neighborhood for $(y(\gamma,\lambda), K_{\gamma,\lambda})$ (resp. $(y(\gamma,\lambda), M_{\gamma,\lambda,p})$) for all $\lambda \in [0,1]$ and all $\gamma \in U$. It is easy to see[14] that $N' := j^{-1}(N)$ (resp. $N'_p := j^{-1}(N_p)$) is an isolating neighborhood for $(\lambda, y_0, h, K'_\lambda)$ (resp. $(\lambda, y_0, h, M'_{\lambda,p})$) and $\lambda \in [0,1]$.

(a) Let $u : \mathbb{R} \to K'_\lambda$ be a solution of π_λ. It follows that $j \circ u$ is a solution of π, so either $j \circ u \subset M_{\gamma,\lambda,p}$ for some $p \in P$ or there are $p \prec q$ in P such that $\alpha(j \circ u) \subset M_{\gamma,\lambda,q}$ and $\omega(j \circ u) \subset M_{\gamma,\lambda,p}$. In the first case, we immediately conclude that $u \subset M'_{\lambda,p}$. In the second case, one has $u(t) \in N'_{\lambda,q}$ for all t sufficiently small and $u(t) \in N'_{\lambda,p}$ for all t sufficiently large. Since N'_q and N'_p are isolating neighborhoods for $(\lambda, y_0, h, M'_{\lambda,q})$ and $(\lambda, y_0, h, M'_{\lambda,p})$, it follows that $\alpha(u) \subset M'_{\lambda,q}$ and $\omega(u) \subset M'_{\lambda,p}$.

In conjunction with the remarks on N' and N'_p, it follows immediately from the definition that $((\lambda_n, y_0, h), K'_{\lambda_n}, (M'_{\lambda_n,p})_p) \to ((\lambda, y_0, h), K'_\lambda, (M'_{\lambda,p})_p)$ whenever $\lambda_n \to \lambda$ as claimed.

(b), (c) Recall the notation of Lemma 3.46. For every interval $I \subset P$, there is an isolated invariant subset $M_{y(\gamma,\lambda)}(I)$. If (N_1, N_2) is an index pair for $(y(\gamma,\lambda), M_{y(\gamma,\lambda)}(I))$, then (N_1, N_2) is also an index pair for $((\lambda, y_0, h), M'_\lambda(I))$, where $M'_\lambda(I)$ is defined with respect to the Morse-decomposition $(M'_{\lambda,p})_{p \in P}$. Consequently,

$$\mathrm{ContCl}(y(\gamma,\lambda), K_{\gamma,\lambda}, M'_{\gamma,\lambda}) = \mathrm{ContCl}((\lambda, y_0, h), K'_\lambda, (M'_{\lambda,p})_p).$$

Suppose that $\chi(\lambda) := \mathrm{ContCl}((\lambda, y_0, h), K_\lambda, (M_{\lambda,p})_p)$ is not constant. In order to make the next step visible, we explicitly note that

$$\mathrm{ContCl}((\lambda, y_0, h), K_\lambda, (M_{\lambda,p})_p) = \mathrm{ContCl}(Y'', X, \pi', (\lambda, y_0, h), K_\lambda, (M_{\lambda,p})_p).$$

Define $j' : Y'' \times X \to Y' \times X$ by $j'((\lambda, y, h), x) := ((y, h), x)$. By the same argument as above, one proves:

$$\mathrm{ContCl}(Y'', X, \pi', (\lambda, y_0, h), K''_\lambda, (M''_{\lambda,p})_p) = \mathrm{ContCl}(Y', X, \pi_\lambda, (y_0, h), K''_\lambda, (M''_{\lambda,p})_p) \tag{3.13}$$

If χ is not constant, there must exist a sequence $\lambda_n \to \lambda_0$ such that $\chi(\lambda_n) \neq \chi(\lambda_0)$, which, in view of (3.13), contradicts Lemma 3.43.

□

PROOF OF THEOREM 3.45. Firstly, we will prove that $\mathrm{ContCl} \circ f$ is locally constant. Let $\gamma_0 \in \Gamma$, and let the isolating neighborhoods N and N_p, $p \in P$ be determined by (C1'). It follows from Lemma 3.48 above, that $\mathrm{ContCl} \circ f$ is constant in a neighborhood U of γ_0.
We have shown that $\mathrm{ContCl} \circ f$ is locally constant. Moreover, Γ is connected, which completes the proof. □

[14] Firstly, the set $j^{-1}(K_\lambda)$ is invariant because $\Sigma^+(y_0, h)$ is compact. Secondly, $\mathrm{Inv} N' \cap (\Sigma^+(y_0, h) \times X) \subset j^{-1}(K_\lambda)$. One can argue analogously for $M_{\lambda,p}$, $p \in P$.

3.5. The Homology Conley Index as a Direct Limit

Let (N_1, N_2) be an index pair. Another index pair – representing the same index – is $(N_1[t,\infty[, N_2[t,\infty[)$, where $t > 0$ is arbitrary and $N_i[t,\infty[= \{(s,x) \in N_i :\ s \geq t\}$ for $i \in \{1,2\}$. Apparently, only the behavior at large times is relevant. In the present section, this limit behavior will be examined. Throughout this section we consider the homology index. Finite sections of an index pair (N_1, N_2), that is, sets of the form $N_i\big[\alpha,\beta\big] = N_i \cap (\big[\alpha,\beta\big] \times X)$, in conjunction with appropriate morphisms form a direct system. The index $\mathrm{H}_*(N_1/N_2, N_2/N_2)$ is then proved to be isomorphic to a direct limit obtained from these sections.
It is interesting to note that this result (in particular Lemma 3.56) resembles techniques employed in [**19**]. In this paper, however, we will focus on the use of the direct limit representation of the index as a tool. The results in this section are crucial for the following section.
For the rest of this section, let Λ be a set and $\leq$ a partial order on Λ. Recall [**28**] that a *direct system of sets* is a family $(A_\alpha)_{\alpha\in\Lambda}$ of sets and a family of functions $(f_{\alpha,\beta})$, where $\alpha,\beta \in \Lambda$, $\alpha \leq \beta$ and $f_{\alpha,\beta} : A_\alpha \to A_\beta$.
The *direct limit* $\operatorname{dirlim}(A_\alpha, f_{\alpha,\beta})$ of $(A_\alpha, f_{\alpha,\beta})$ is the set of equivalence classes in $\bigcup_{\alpha\in\Lambda}\{\alpha\} \times A_\alpha$ under the relation $\sim$, which is defined as follows: Let $\alpha,\beta \in \Lambda$ and $(a,b) \in A^\alpha \times A^\beta$. $(\alpha,a) \sim (\beta,b)$ if and only if there is a $\gamma \in \Lambda$ such that $\alpha,\beta \leq \gamma$ and $f_{\alpha,\gamma}(a) = f_{\beta,\gamma}(b)$.
Let (X,d) be a complete metric space, and $V \subset \mathbb{R}^+ \times X$. We set

$$V(t) := \{x : (t,x) \in V\}$$
$$V([a,b]) := V[a,b] := \{(t,x) \in V :\ t \in [a,b]\}.$$

DEFINITION 3.49. An index pair (N_1, N_2) is called *regular* (with respect to y_0) if the (inner) exit time $T_i : N_1 \to [0,\infty]$, $T_i(x) := \sup\{t \in \mathbb{R}^+ :\ x\chi_{y_0}[0,t] \subset N_1 \setminus N_2\}$ is continuous.

The main motivation for regular index pairs are Lemma 3.50 below and Lemma 3.56 at the end of this section. As stated subsequently in Lemma 3.51, it is easily possible to obtain regular index pairs by modifying (enlarging) the exit set appropriately.

LEMMA 3.50. *Let (N_1, N_2) be a regular index pair in $\mathbb{R}^+ \times X$.*
Consider the direct system $(A_\alpha, f_{\alpha,\beta})$ for $\alpha,\beta \in \Lambda$, where Λ denotes the set of all non-empty compact subintervals of $\mathbb{R}^+$ ordered by inclusion, and $A_\alpha := H_*[N_1(\alpha), N_2(\alpha)]$. For $\alpha \subset \beta$, let $i_{\alpha,\beta} : (N_1(\alpha), N_2(\alpha)) \to (N_1(\beta), N_2(\beta))$ denote the respective inclusion and set $f_{\alpha,\beta} := \mathrm{H}_*(i_{\alpha,\beta}) : A_\alpha \to A_\beta$.
Then, the inclusions $i_\alpha : (N_1(\alpha), N_2(\alpha)) \to (N_1, N_2)$ induce an isomorphism

$$j : \operatorname{dirlim}(\mathrm{H}_*(N_1(\alpha), N_2(\alpha)), f_{\alpha,\beta}) \to \mathrm{H}_*[N_1, N_2], \quad [(\alpha,x)] \mapsto \mathrm{H}_*(p \circ i_\alpha)(x),$$

where $p : N_1 \to N_1/N_2$ denotes the canonical projection.

LEMMA 3.51. *Let (N_1, N_2) be an index pair for (y_0, K). Then there are a constant $\tau \in \mathbb{R}^+$ and a set $N_2' \subset N_1$ such that:*

(1) $N_2 \subset N_2' \subset N_2^{-\tau}$
(2) *(N_1, N_2') is a regular index pair for (y_0, K).*

PROOF. By Lemma 3.22, N_2^{-T} is a neighborhood of N_2 in N_1 provided that T is sufficiently large. It follows that $N_2 \cap \mathrm{cl}(N_1 \setminus N_2^{-T}) = \emptyset$ as well. By Urysohn's lemma, there exists a continuous function $f : N_1 \to [0,1]$ such that $f(x) = 0$ on N_2 and $f(x) = 1$ on $\mathrm{cl}(N_1 \setminus N_2^{-T})$.
Set

$$\lambda(x) := \int_0^{T(x)} f(x\chi_{y_0} s)\,\mathrm{d}s,$$

where $T(x) := \sup\{t \in \mathbb{R}^+ : \; x\chi_{y_0}[0,t] \subset N_1 \setminus N_2\}$, in order to guarantee that the integrand is defined.
It is easy to see that $\lambda(x) = 0$ on N_2 and $\lambda(x) \le T(x)$ for all $x \in N_1$. Next, we are going to prove that

$$T(x) - T \le \lambda(x). \tag{3.14}$$

One has $\lambda(x) \ge 0$ for all $x \in N_1$, so let $x \in N_1$ with $T(x) > T$. It follows that $f(x\chi_{y_0} s) = 1$ for all $s \in [0, T(x) - T]$, so

$$T(x) - T = \int_0^{T(x)-T} f(x\chi_{y_0} s)\,ds \le \lambda(x).$$

We need to show that λ is continuous. Suppose that $x_n \to x_0$ is a sequence and $\lambda(x_0) < \infty$. Initially, assume that $T(x_n)$ is unbounded, so it is possible extract a subsequence x_n' with $T(x_n') \to \infty$. We have $x_n'\chi_{y_0} s \in N_1 \setminus N_2^{-T}$ for all $s < T(x_n') - T$ and all $n \in \mathbb{N}$, so $x_0\chi_{y_0} s \in \mathrm{cl}(N_1 \setminus N_2^{-T}) \subset N_1 \setminus N_2$ for all $s \in \mathbb{R}^+$, which in turn implies that $T(x_0) = \infty$. However, $\lambda(x_0) \ge T(x_0) - T = \infty$, which is a contradiction. Consequently, the sequence $(T(x_n))_n$ must be bounded.
We further have $x_n\chi_{y_0}[0, T(x_n)] \subset N_1$ and $x_n\chi_{y_0} T(x_n) \in N_2$ for all $n \in \mathbb{N}$. $T(x_n)$ is bounded, so we may choose a subsequence x_n' with $T(x_n') \to t_0 < \infty$. It follows that

$$\lambda(x_n') \to \int_0^{t_0} f(x\chi_{y_0} s)\,ds = \lambda(x_0),$$

where the last equality stems from the facts that N_2 is positively invariant and $f(x) = 0$ on N_2. This readily implies that $\lambda(x_n) \to \lambda(x_0)$.
Finally if $\lambda(x_0) = \infty$, then $x_0\chi_{y_0} s \in N_1 \setminus N_2^{-T}$ for all $s \in \mathbb{R}^+$. Arguing by contradiction, assume that there exists a subsequence x_n' with $\lambda(x_n') \le t_0$ for all $n \in \mathbb{N}$. From (3.14), one obtains that $x_n'\chi_{y_0} t_n \in N_2^{-T}$ for some $t_n \in [0, t_0]$. Taking subsequences, we may assume w.l.o.g. that $t_n \to t_0' \le t_0$, so $x_0\chi_{y_0} t_0' \in N_2^{-T}$, implying that $\lambda(x_0) \le T(x_0) \le t_0' + T$. This is a contradiction and completes the proof that λ is continuous.
It is easy to see that $N_2' := \lambda^{-1}([0, T+1])$ is a closed neighborhood of N_2^{-T} in N_1. Also, λ is monotone decreasing along the semiflow, so (N_1, N_2') is an index pair.
By (3.14), it holds that $N_2' \subset N_2^{-\tau}$, where $\tau := 2T + 1$. It follows from Lemma 3.7 in conjunction with Lemma 3.6 that (N_1, N_2') is an index pair for (y_0, K).
Let $x \in N_1 \setminus N_2'$ and recall the definition $T_i(x) := \sup\{t \in \mathbb{R}^+ : \; x\chi_{y_0}[0,t] \subset N_1 \setminus N_2'\}$ of the inner exit time. We have $\lambda(x\chi_{y_0} T_i(x)) = T + 1$ and $f(x) = 1$ on $N_1 \setminus N_2'$, so $\lambda(x) = T_i(x) + T + 1$. λ is continuous as already proved, so (N_1, N_2') is a regular index pair as claimed. □

Using regular index pairs, it is easy to prove the following stronger version of Corollary 2.20.

LEMMA 3.52. *Let $y_0 \in Y$ and $K \subset \Sigma^+(y_0) \times X$ an isolated invariant set admitting a strongly admissible isolating neighborhood.*
If (N_1, N_2) is an index pair for (y_0, K), and $\mathrm{h}(N_1/N_2, N_2) \neq \bar{0}$, then there are $t_0 \in \mathbb{R}^+$ and a solution $u : [t_0, \infty[\to N_1 \setminus N_2$ of Φ_{y_0}.

PROOF. In view of Lemma 3.51, one may assume without loss of generality that (N_1, N_2) is a regular index pair. Suppose that (N_1, N_2) is such that for every $t_0 \in \mathbb{R}^+$ there does not exist a solution $u : [t_0, \infty[\to N_1 \setminus N_2$ of Φ_{y_0}. Then the (continuous) exit time T_i satisfies

$$T_i(x) = \sup_{t \in \mathbb{R}^+} \{x\chi_{y_0}[0,t] \subset N_1 \setminus N_2\} < \infty \text{ for all } x \in N_1.$$

It is easy to see that for each $x \in N_1$, $x\chi_{y_0}[0, T_i(x)] \subset N_1$ and $x\chi_{y_0}T_i(x) \in N_2$. One can define $H : [0,1] \times N_1 \to N_1$ by

$$H(\lambda, x) := x\chi_{y_0}(\lambda T_i(x)).$$

H is continuous, and $H(\lambda, x) = x$ for all $(\lambda, x) \in [0,1] \times N_2$. Consequently, $(N_1/N_2, N_2)$ and $(N_2/N_2, N_2)$ are homotopy equivalent, completing the proof because $\mathrm{h}(N_2/N_2, N_2) = \bar{0}$. □

LEMMA 3.53. *Let (N_1, N_2) be a regular index pair. Then the projection $p : N_1 \to N_1/N_2$ induces an isomorphism $p_* : \mathrm{H}_*(N_1, N_2) \to \mathrm{H}_*(N_1/N_2, N_2/N_2)$.*

PROOF. The (inner) exit time $T := \sup\{t \in \mathbb{R}^+ : x\chi_{y_0}[0,t] \subset N_1 \setminus N_2\}$ is continuous. Therefore, $N_2' := N_2^{-1} = \{x \in N_1 : x\pi s \in N_2 \text{ for some } s \in [0,1]\}$ is a neighborhood of N_2 in N_1. Define $H : [0,1] \times N_1 \to N_1$ by $H(\lambda, x) := x\chi_{y_0}(\lambda \min\{T(x), 1\})$. Using H, we conclude that there are inclusion induced isomorphisms

$$\mathrm{H}_*(N_1, N_2) \to \mathrm{H}_*(N_1, N_2')$$
$$\mathrm{H}_*(N_1/N_2, N_2) \to \mathrm{H}_*(N_1/N_2, N_2'/N_2).$$

Using the excision property of homology, it follows that p induces an isomorphism $\mathrm{H}_*(N_1, N_2') \to \mathrm{H}_*(N_1/N_2, N_2'/N_2)$. □

PROOF OF LEMMA 3.50. In view of Lemma 3.53, it is sufficient to consider the inclusion induced mapping

$$j' : \mathrm{dirlim}(\mathrm{H}_*(N_1(\alpha), N_2(\alpha)), \mathrm{H}_*(i_{\alpha,\beta})) \to \mathrm{H}_*(N_1, N_2).$$

j' is an isomorphism since H is assumed to be a homology theory with compact supports. □

LEMMA 3.54. *Let the direct system $(A_\alpha, f_{\alpha,\beta})$ be defined as in Lemma 3.50, $a < c$, and $\alpha := [a, c] \subset [b, c] =: \beta$.*
Then, $f_{\alpha,\beta}$ is an isomorphism.

PROOF. Let $h > 0$ and $\gamma := [d - h, d] \subset \mathbb{R}^+$ be an otherwise arbitrary interval. Since (N_1, N_2) is assumed to be a regular index pair, the inner exit time $T(x) := \sup\{t \in \mathbb{R}^+ : x\chi_{y_0}[0,t] \subset N_1 \setminus N_2\}$ is continuous.

We can define a continuous mapping $H: [0,1]\times N_1(\gamma)\to N_1(\gamma)$ by

$$H(\lambda,(d-t,x)) := \begin{cases}(d-t,x)\chi_{y_0}(\lambda t) & \lambda t\le T(x)\\ (d-t,x)\chi_{y_0}T(x) & \lambda t > T(x).\end{cases}$$

It follows that $(N_1'(\gamma),N_2(\gamma))$ is a strong deformation retract of $(N_1(\gamma),N_2(\gamma))$, where we set

$$N_1' := N_1(d)\cup N_2.$$

Therefore, the inclusion $(N_1'(\gamma),N_2(\gamma))\subset(N_1(\gamma),N_2(\gamma))$ defines an isomorphism

$$\mathrm{H}_*(N_1'(\gamma),N_2(\gamma))\to\mathrm{H}_*(N_1(\gamma),N_2(\gamma)).$$

Moreover, the inclusion $(N_1'(\alpha),N_2(\alpha))\to(N_1'(\beta),N_2(\beta))$ induces an isomorphism in homology by excision[15].

Summing up, there is a commutative diagram, where every arrow denotes an isomorphism induced by the inclusion of the respective subspaces:

$$\begin{tikzcd}
\mathrm{H}_*(N_1(\alpha),N_2(\alpha)) \arrow[rrr, bend left] & \mathrm{H}_*(N_1'(\alpha),N_2(\alpha)) \arrow[l] \arrow[r] & \mathrm{H}_*(N_1'(\beta),N_2(\beta)) \arrow[r] & \mathrm{H}_*(N_1(\beta),N_2(\beta))
\end{tikzcd}$$

□

LEMMA 3.55. *Let $y_0\in Y$, (N_1,N_2) be a regular index pair for Φ_{y_0} and $\varepsilon>0$ be arbitrary (not necessarily small). Let Γ denote the set of all non-empty compact subintervals of $\mathbb{R}^+$ ordered by inclusion. Let $\varepsilon>0$, $\alpha,\beta\in\Gamma$, $N_2^{-\varepsilon}:=N_2^{-\varepsilon}(N_1)$ and $A_\alpha^\varepsilon:=\mathrm{H}_*(N_1(\alpha),N_2^{-\varepsilon}(\alpha))$. Finally, let $f_{\alpha,\beta}^\varepsilon: A_\alpha^\varepsilon\to A_\beta^\varepsilon$ be inclusion induced.*
Then for every[16] pair $[a,c]\subset[b,c]$ of subintervals of $\mathbb{R}^+$, it holds that $f_{\alpha,\beta}^\varepsilon$ is an isomorphism.

PROOF. As in the proof of Lemma 3.54, it can be shown that $(N_1',N_2'):=(N_1(c)\cup N_2([a,c]),N_2^{-\varepsilon}(c)\cup N_2([a,c])$ is a strong deformation retract of $(N_1[a,c],N_2^{-\varepsilon}([a,c]))$.
We have $N_2^{-\varepsilon}=T^{-1}([0,\varepsilon])$, where $T: N_1\to[0,\infty]$ denotes the inner exit time $T(x):=\sup\{t\in\mathbb{R}^+ : x\chi_{y_0}[0,t]\subset N_1\setminus N_2\}$. Due to the continuity of T, $N_2^{-\varepsilon/2}(N_1)\cap N_1'$ is a neighborhood of N_2 in N_1'. Hence, $(N_1(c),N_2^{-\varepsilon}(c))\subset(N_1',N_2')$ induces an isomorphism in homology by excision. Further details are omitted. □

LEMMA 3.56. *Let (N_1,N_2) be a regular index pair, $\varepsilon>0$, and let the direct sytem $(A_\alpha^\varepsilon,f_{\alpha,\beta})$ be defined as in Lemma 3.55.*
Suppose we are given a strictly monotone increasing sequence $(a_n)_n$ in $\mathbb{R}^+$ with $a_n\to\infty$. Define a direct system $(B_k^\varepsilon,g_{k,l})$, where $k,l\in\mathbb{N}$, $k\le l$ and $B_k^\varepsilon:=A_{\{a_k\}}^\varepsilon$. Moreover, for $k\le l$ and in view of Lemma 3.55, we can define

$$g_{k,l} := f_{\{a_l\},[a_k,a_l]}^{-1}\circ f_{\{a_k\},[a_k,a_l]}.$$

Then, the inclusions $i_k: (N_1(\{a_k\}),N_2^{-\varepsilon}(\{a_k\})\to(N_1,N_2^{-\varepsilon})$ induce an isomorphism

$$g:\ \mathrm{dirlim}(B_k^\varepsilon,g_{k,l})\to\mathrm{H}_*[N_1,N_2^{-\varepsilon}],\quad [k,x]\mapsto\mathrm{H}_*(p\circ i_k)(x),$$

where p is given by Lemma 3.53.

[15]Here, the assumption $a\neq c$ is used.
[16]In contrast to Lemma 3.54, the case $\alpha=\{c\}$ is included.

PROOF. First of all, we need to prove that g is well defined. Let there be given two representations $[k,x]=[l,x']$ of the same element in $\operatorname{dirlim}(B_k^\varepsilon, g_{k,l})$, that is, $g_{k,l}(x)=x'$. The following diagram with inclusion induced morphisms is commutative.

$$\begin{tikzcd}
B_k^\varepsilon \arrow[r] \arrow[rr, bend left, "g_{k,l}"] \arrow[dr, "\mathrm{H}_*(i_k)"'] & \mathrm{H}_*(N_1([a_k,a_l]), N_2^{-\varepsilon}([a_k,a_l])) \arrow[d] & B_l^\varepsilon \arrow[l] \arrow[dl, "\mathrm{H}_*(i_l)"] \\
& \mathrm{H}_*(N_1,N_2) &
\end{tikzcd}$$

Consequently, g is well defined.

Let the isomorphism $j:\ \operatorname{dirlim}(A_\alpha^\varepsilon, f_{\alpha,\beta}) \to \mathrm{H}_*[N_1, N_2^{-\varepsilon}]$ be given by Lemma 3.50 with (N_1,N_2) replaced by $(N_1,N_2^{-\varepsilon})$. It is clear that $j([a_k,x]) = g([k,x])$ for all $[k,x]\in \operatorname{dirlim}(B_k^\varepsilon, g_{k,l})$. It follows that g is an epimorphism because $B_k^\varepsilon = A_{\{a_k\}}^\varepsilon$ by definition.

Assume that $g([k,x])=0$. Since j is an isomorphism, it follows that $[\{a_k\},x]=0$, so there exists a compact interval $[a,b]$ with $a\le a_k\le b$ such that $f_{\{a_k\},[a,b]}(x)=0$. We even have $f_{\{a_k\},[a,a_l]}(x)=0$ provided that $b\le a_l$. By Lemma 3.55, one has $g_{k,l}(x)=0$, so $[k,x]=\big[l,g_{k,l}(x)\big]=[l,0]=0$. We have proved that g is a monomorphism. $\square$

3.6. Uniformly Connected Attractor-repeller Decompositions

In analogy to the previous section, let $V\subset Y\times X$ and define[17]

$$\begin{aligned} V(y) &:= \{x:\ (y,x)\in V\} \\ V(U) &:= V\cap(U\times X) \qquad\qquad \text{where } U\subset Y. \end{aligned}$$

DEFINITION 3.57. Let (y_0,K,A,R) be an attractor-repeller decomposition. We say that A and R are *not uniformly connected* (in K) if there exists an $y\in\omega(y_0)$ such that $K(y)=A(y)\cup R(y)$. Otherwise, (y_0,K,A,R) is called *uniformly connected.*

The following theorem is the main result of this section. The rest of the section is devoted to its proof. The strategy is to exploit Lemma 3.56 together with the assumption that A and R are not uniformly connected.

THEOREM 3.58. *Let (y_0,K,A,R) be an attractor-repeller decomposition, and let there exist a strongly admissible isolating neighborhood $N\subset\Sigma^+(y_0)\times X$ for K.*
The connecting homomorphism of the associated attractor-repeller sequence is trivial if (y_0,K,A,R) is not uniformly connected.

LEMMA 3.59. *Let (y_0,K,A,R) be an attractor-repeller decomposition such that A and R are not uniformly connected. Suppose that K is compact.*[18]
Then there is a $y'\in\omega(y_0)$ and a neighborhood U of y' in $\Sigma^+(y_0)$ such that $K(y)=A(y)\cup R(y)$ for all $y\in U$.

[17]for arbitrary spaces Y and X, in particular also $Y=\mathbb{R}^+$

[18]This follows from the assumptions of Theorem 3.58

PROOF. Since A and R are not uniformly connected, there must exist a $y' \in \omega(y_0)$ such that $K(y') = A(y') \cup R(y')$. Since K and thus also A and R are compact, there exist isolating neighborhoods N_A of A and N_R of R which are disjoint, that is, $N_A \cap N_R = \emptyset$.
Suppose that the lemma does not hold. Then there is a sequence $(y_n, x_n) \in K$ such that $y_n \to y'$ and $x_n \in K(y_n) \setminus (N_A \cup N_R)$. Due to the compactness of K, we may assume without loss of generality that $(y_n, x_n) \to (y', x_0) \in K$. Thus, $(y', x_0) \in K \setminus (A \cup R)$, which is a contradiction. □

Let $y' \in \omega(y_0)$ and $U \subset \Sigma^+(y_0)$ a closed neighborhood of y' for which the conclusions of Lemma 3.59 hold. There is a sequence $t_n \to \infty$ in $\mathbb{R}^+$ such that $a_n := y_0^{t_n} \in U$ for all $n \in \mathbb{N}$. By the choice of U, one has $A(a_n) \cap R(a_n) = \emptyset$ for all $n \in \mathbb{N}$. For the rest of this section and unless otherwise stated, let $\varepsilon > 0$ be an arbitrary but fixed parameter.

LEMMA 3.60. *Let $N_A \subset \Sigma^+(y_0) \times X$ (resp. $N_R \subset \Sigma^+(y_0) \times X$) be an isolating neighborhood for A (resp. R).*
Then, there is an isolating neigbhorhood $N' \subset \Sigma^+(y_0) \times X$ for K such that $N'(U) \subset N_A(U) \cup N_R(U)$.

PROOF. Let $N'_\varepsilon := \mathrm{cl}_{\Sigma^+(y_0)\times X} \bigcup_{(y,x)\in K} B_\varepsilon(y, x)$. We claim that for all $\varepsilon > 0$ sufficiently small, $N'_\varepsilon(U) \subset N_R \cup N_A$. Otherwise, one obtains[19] using the compactness of K that there is a point $(y, x) \in K(U) \setminus \mathrm{int}_{\Sigma^+(y_0)\times X}(N_R \cup N_A)$. Since $N_R \cup N_A$ is an isolating neighborhood for $A \cup R$ and $K(U) = A(U) \cup R(U)$, one has $(y, x) \in \emptyset$, which is an obvious contradiction. □

As before, let π_{y_0} denote the restriction of π to $\Sigma^+(y_0) \times X$. Let (M_1, M_2, M_3) be an FM-index triple for (π_{y_0}, K, A, R) such that M_1 is strongly π_{y_0}-admissible, that is, (M_1, M_2, M_3) is a triple of closed subsets of $\Sigma^+(y_0) \times X$. Since $\mathrm{cl}_{Y\times X}(M_1 \setminus M_2)$ is an isolating neighborhood for R, it is possible to choose an isolating neighborhood $N_R \subset N$ for R with $N_R \cap M_2 = \emptyset$.
Furthermore, by Lemma 3.60 and because there exists an isolating neighborhood $N'' \subset M_1 \setminus M_3$ for K, there is also an isolating neighborhood $N' \subset M_1 \setminus M_3$ with

$$N'(U) \subset N_R \dot\cup \underbrace{\mathrm{cl}_{\Sigma^+(y_0)\times X}(M_2 \setminus M_3)}_{=:N_A}.$$

The isolating neighborhoods N_A and N_R are disjoint by the choice of N_R.
Recall that $r : \mathbb{R}^+ \times X \to \Sigma^+(y_0) \times X$ is defined by $r(t, x) := (y_0^t, x)$. By Lemma 3.17, there exists an index triple (N_1, N_2, N_3) with $N_1 \subset r^{-1}(N')$. Consequently, one has $N_1(\tilde U) \subset r^{-1}(N_A) \dot\cup r^{-1}(N_R)$ and $N_1 \subset r^{-1}(M_1) \setminus r^{-1}(M_3)$, where

$$\tilde U := \{t \in \mathbb{R}^+ : \ y_0^t \in U\}.$$

LEMMA 3.61. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Then there is a set $N'_3 \supset N_3$ such that (N_1, N'_3) is a regular index pair for (y_0, K), $N'_3 \subset N_3^{-\tau}(N_1)$ for some $\tau \geq 0$ and $(N_2^{-\tau}(N_1), N'_3)$ is a regular index pair for (y_0, A).*

PROOF. It follows from Lemma 3.51 that there exist a set $N'_3 \supset N_3$ and a constant $\tau \in \mathbb{R}^+$ such that (N_1, N'_3) is a regular index pair for (y_0, K) and $N_3 \subset N'_3 \subset N_3^{-\tau}(N_1)$. This means in

[19] Recall that U is closed.

particular that the (inner) exit time $T: N_1 \to [0,\infty]$, $T(x) := \sup\{t \in \mathbb{R}^+ : x\chi_{y_0}[0,t] \subset N_1 \setminus N_3'\}$ is continuous.
One needs to prove that $(N_2^{-\tau}(N_1), N_3')$ is an index pair.

(IP3) Let $x \in N_2^{-\tau}(N_1)$ and $x\chi_{y_0} t \notin N_2^{-\tau}(N_1)$ for some $t \geq 0$. One cannot have $x\chi_{y_0}[0,t] \subset N_1$, so $x\chi_{y_0} s \in N_3'$ for some $s \leq t$.

(IP4) Let $x \in N_3'$ and $x\chi_{y_0} t \notin N_3'$. It follows that $x\chi_{y_0} s \in (\mathbb{R}^+ \times X) \setminus N_1 \subset (\mathbb{R}^+ \times X) \setminus N_2^{-\tau}(N_1)$ for some $s \in]0,t]$ as (N_1, N_3') is an index pair.

By Lemma 3.19, $(N_2^{-\tau}(N_1), N_3)$ and $(N_2^{-\tau}(N_1), N_3^{-\tau}(N_1))$ are index pairs for (y_0, A). By using Lemma 3.6 and the sandwich lemma 3.7, one concludes that $(N_2^{-\tau}(N_1), N_3')$ is an index pair for (y_0, A).
Finally, the exit time with respect to the index pair $(N_2^{-\tau}(N_1), N_3')$ is the restriction of T to $N_2^{-\tau}(N_1)$ and therefore continuous that is, the index pair is regular. □

Having proved Lemma 3.61, we can assume without loss of generality that (N_1, N_2, N_3) is an index triple and (N_1, N_3) as well as (N_2, N_3) are regular index pairs.

LEMMA 3.62. *Denote* $\hat{M}_2 := r^{-1}(M_2)$, $N_2^{-\varepsilon} := N_2^{-\varepsilon}(N_1)$ *and* $N_3^{-\varepsilon} := N_3^{-\varepsilon}(N_1)$. *Then* $(N_2^{-\varepsilon} \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2)$ *and* $(N_1 \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2)$ *are both index pairs for* (y_0, A).

The proof is simple: By construction, one has $N_1 \subset r^{-1}(M_1 \setminus M_3)$ and M_2 is M_1-positively invariant. For the convenience of the reader and for the sake of completeness, a more detailed version is also provided.

PROOF. Recall that $N_2^{-\varepsilon} \subset N_1$ and $N_1 \cap \hat{M}_3 = \emptyset$ by construction. Hence, $N_1 \cap \hat{M}_2$ (resp. $N_2^{-\varepsilon} \cap \hat{M}_2$) is N_1-positively invariant (resp. $N_2^{-\varepsilon}$-positively invariant), that is, $x\chi_{y_0}[0,t] \subset N_1$ (resp. $x\chi_{y_0}[0,t] \subset N_2^{-\varepsilon}$) and $x \in \hat{M}_2$ implies $x\chi_{y_0}[0,t] \subset \hat{M}_2$.
We need to prove that $(N_1 \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2)$ (resp. $(N_2^{-\varepsilon} \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2)$) is an index pair.

(IP1) $N_1 \cap \hat{M}_2$, $N_2^{-\varepsilon} \cap \hat{M}_2$ and $N_3^{-\varepsilon} \cap \hat{M}_2$ are closed.

(IP3) Let $x \in N_1 \cap \hat{M}_2$ (resp. $N_2^{-\varepsilon} \cap \hat{M}_2$) and $t_0 := \sup\{s \in \mathbb{R}^+ : x\chi_{y_0}[0,s] \subset N_i$ resp. $N_2^{-\varepsilon}\}$. $(N_i^{-\varepsilon}, N_3^{-\varepsilon})$ is an index pair (Lemma 3.19), so $x\chi_{y_0} t_0 \in N_3^{-\varepsilon}$. It follows that $x\chi_{y_0}[0,t_0[\subset N_1 \cap \hat{M}_2$ (resp. $N_2^{-\varepsilon} \cap \hat{M}_2$), so $x\chi_{y_0} t_0 \in \hat{M}_2$ because M_2 and thus $\hat{M}_2$ are closed. Therefore, $x\chi_{y_0} t_0 \in N_3^{-\varepsilon} \cap \hat{M}_2$.

(IP4) Let $i = 1,2$, $x \in N_3^{-\varepsilon} \cap \hat{M}_2$ and $x\chi_{y_0} t \notin N_3^{-\varepsilon} \cap \hat{M}_2$ for some $t \in \mathbb{R}^+$. Since $(N_1, N_3^{-\varepsilon})$ is an index pair and $\hat{M}_2$ is N_1-positively invariant, one has $x\chi_{y_0} s \in (\mathbb{R}^+ \times X) \setminus N_1$ for some $s \in [0,t]$. Consequently, $x \in (\mathbb{R}^+ \times X) \setminus (N_2^{-\varepsilon} \cap \hat{M}_2) \supset (\mathbb{R}^+ \times X) \setminus N_1$.

One needs to prove that both pairs are index pairs for (y_0, A). $N' := \mathrm{cl}_{\Sigma^+(y_0) \times X}(M_2 \setminus M_3)$ is a strongly admissible isolating neighborhood of A in $\Sigma^+(y_0) \times X$ and

$$(N_2^{-\varepsilon} \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2) \subset (N_1 \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2) \subset r^{-1}(N').$$

Moreover, there exists a neighborhood W of A such that $r^{-1}(W) \subset N_2^{-\varepsilon} \setminus N_3^{-\varepsilon}$. Setting $W_0 := M_2 \cap W$, one has

$$r^{-1}(W_0) = \hat{M}_2 \cap r^{-1}(W) \subset (N_2^{-\varepsilon} \cap \hat{M}_2) \setminus (N_3^{-\varepsilon} \cap \hat{M}_2) \subset (N_1 \cap \hat{M}_2) \setminus (N_3^{-\varepsilon} \cap \hat{M}_2).$$

We have proved that $(N_2^{-\varepsilon} \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2)$ and $(N_1 \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2)$ are index pairs for (y_0, A). □

LEMMA 3.63. *Let* $\hat{M}_i := r^{-1}(M_i)$, $i = 2,3$. *Then* $N_3^{-\varepsilon}(N_1) \cap \hat{M}_2 = (N_3 \cap \hat{M}_2)^{-\varepsilon}(N_1 \cap \hat{M}_2)$.

PROOF. Let $x \in N_3^{-\varepsilon}(N_1) \cap \hat{M}_2$. There is a real number $0 \le t \le \varepsilon$ with $x\chi_{y_0} t \in N_3$. Since $N_1 \cap \hat{M}_3 = \emptyset$, one must have $x\chi_{y_0} t \in \hat{M}_2$, so $x \in (N_3 \cap \hat{M}_2)^{-\varepsilon}(N_1 \cap \hat{M}_2)$.
The remainder of the proof is almost trivial and thus omitted. □

We will now complete the proof of Theorem 3.58. Consider the long exact sequence

$$\mathrm{H}_*[N_1, N_3^{-\varepsilon}] \longrightarrow \mathrm{H}_*[N_1, N_2^{-\varepsilon}] \xrightarrow{\delta} \mathrm{H}_{*-1}[N_2^{-\varepsilon}, N_3^{-\varepsilon}] \xrightarrow{\mathrm{H}_*(i)} \mathrm{H}_{*-1}[N_1, N_3^{-\varepsilon}]$$

associated with the index triple $(N_1, N_2^{-\varepsilon}, N_3^{-\varepsilon})$. Since this is an exact sequence, it is sufficient to prove that $\mathrm{H}_*(i)$ is a monomorphism, where $i:\ N_2^{-\varepsilon}/N_3^{-\varepsilon} \to N_1/N_3^{-\varepsilon}$ is inclusion induced. The following diagram is commutative because each homomorphism is inclusion induced.

$$\begin{tikzcd}
\mathrm{H}_*[N_2^{-\varepsilon}, N_3^{-\varepsilon}] \arrow[r, "\mathrm{H}_*(i)"] & \mathrm{H}_*[N_1, N_3^{-\varepsilon}] \\
\mathrm{H}_*[N_2^{-\varepsilon} \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2] \arrow[u, "\mathrm{H}_*(k)"] \arrow[r, "\mathrm{H}_*(l)"] & \mathrm{H}_*[N_1 \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2] \arrow[u, "\mathrm{H}_*(m)"']
\end{tikzcd}$$

It follows from Lemma 3.62 and Lemma 3.4 that $\mathrm{H}_*(l)$ and $\mathrm{H}_*(k)$ are isomorphisms. In order to prove that $\mathrm{H}_*(i)$ is a monomorphism, we only need to show that $\mathrm{H}_*(m)$ is mono.
Recall that $N_1(\tilde{U}) = (N_1(\tilde{U}) \cap N_A) \cup (N_1(\tilde{U}) \cap N_R)$, so $N_1(\tilde{U}) \cap \hat{M}_2 = N_1(\tilde{U}) \cap N_A$. Therefore, the inclusion

$$m_n:\ (N_1(\{a_n\}) \cap \hat{M}_2, N_3^{-\varepsilon}(\{a_n\}) \cap \hat{M}_2) \to (N_1(\{a_n\}), N_3^{-\varepsilon}(\{a_n\}))$$

induces a monomorphism $\mathrm{H}_*(m_n)$.
Set $B_k^\varepsilon := \mathrm{H}_*(N_1(\{a_n\}) \cap \hat{M}_2, N_3^{-\varepsilon}(\{a_n\}) \cap \hat{M}_2)$ (resp. $(B')_k^\varepsilon := \mathrm{H}_*(N_1(\{a_n\}), N_3^{-\varepsilon}(\{a_n\}))$, and let $g_{k,l}$ (resp. $g'_{k,l}$) and i_k (resp. i'_k) be defined by Lemma 3.56 accordingly. Setting

$$m'([k,x]) := [k, \mathrm{H}_*(m_k)(x)],$$

one obtains a commutative diagram:

$$\begin{tikzcd}
\mathrm{dirlim}(B_k^\varepsilon, g_{k,l}) \arrow[r, "\simeq"] \arrow[d, "m'"] & \mathrm{H}_*[N_1 \cap \hat{M}_2, N_3^{-\varepsilon} \cap \hat{M}_2] \arrow[d, "\mathrm{H}_*(m)"] \\
\mathrm{dirlim}((B')_k^\varepsilon, g'_{k,l}) \arrow[r, "\simeq"] & \mathrm{H}_*[N_1, N_3^{-\varepsilon}]
\end{tikzcd}$$

The horizontal arrows denote the canonical isomorphisms given by Lemma 3.56. Additionally, Lemma 3.63 has been used.
Therefore, it is sufficient to prove that m' is a monomorphism. Suppose that $m'([k,x]) = 0$ for some $[k,x] \in \mathrm{dirlim}(B_k^\varepsilon, g_{k,l})$, that is, $g'_{k,l}(\mathrm{H}_*(m_k)(x)) = 0$ for some $l \ge k$.

The following diagram commutes because each of the homomorphisms is induced by inclusions.

$$\begin{tikzcd}
B_k^\varepsilon \arrow[r] \arrow[d, "\mathrm{H}_*(m_k)"] \arrow[rr, bend left, "g_{k,l}"] & \mathrm{H}_*(N_1[a_k,a_l]\cap \hat{M}_2, (N_3')^{-\varepsilon}[a_k,a_l]\cap \hat{M}_2) \arrow[d] & B_l^\varepsilon \arrow[l] \arrow[d, "\mathrm{H}_*(m_l)"] \\
(B')_k^\varepsilon \arrow[r] \arrow[rr, bend right, "g'_{k,l}"'] & \mathrm{H}_*(N_1[a_k,a_l], (N_3')^{-\varepsilon}[a_k,a_l]) & (B')_l^\varepsilon \arrow[l]
\end{tikzcd}$$

One has $g'_{k,l}(\mathrm{H}_*(m_k)(x)) = \mathrm{H}_*(m_l)(g_{k,l}(x)) = 0$, so $g_{k,l}(x) = 0$ by the injectivity of $\mathrm{H}_*(m_l)$. This in turn implies that $[k,x] = \left[l, g_{k,l}(x)\right] = 0$. We have shown that m' is a monomorphism, which completes the proof of Theorem 3.58.

3.7. Nonautonomous C^0-small Perturbations of (Autonomous) Semilinear Parabolic Equations

Let X be a Banach space and A a sectorial operator defined on a dense subset $\mathscr{D}(A) \subset X$. We are interested in mild solutions of

$$u_t + Au = \hat{f}(t,u), \tag{3.15}$$

which happen to be strong solutions due to regularity assumptions. Let us further assume that A has compact resolvent.

As often, the operator A is assumed to be positive, so there is a family of fractional power spaces X^α defined by A. The respective norm is given by $\|x\|_\alpha := \|A^\alpha x\|_X$.

Let Y denote another metric space. With $f \in Y$ there is an associated mapping $\hat{f}$, which serves a a parameter for the evolution operator defined by (3.15). A typical example for $\hat{f}$ is assigning the Nemitskii operator associated with a function f.

The situation treated below is prototypical. Its description is not meant to be exhaustive. The reader who is interested in a more detailed exposition is referred to Section 2.5 in Chapter 2.

For the rest of this section, we will consider a specific choice of Y. Our focus lies on C^0-small nonautonomous perturbations of autonomous equations, the main result being the persistence of Morse-decompositions and certain solutions: Morse-sets with a non-zero index as well as connecting orbits with a non-vanishing connecting homomorphism. A typical Morse-set with non-zero index might be a hyperbolic equilibrium and a typical Morse-set with a non-vanishing connecting homomorphism might be a transversal heteroclinic solution (see [**14**]).

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and let Y denote the set of all continuous functions $f : \mathbb{R} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ which are subject of the following restriction:

> For some $\delta > 0$ and every $C_1 > 0$, there are constants $C_2 = C_2(C_1)$ and $C_3 = C_3(C_1)$ such that for all $(t,x,u) \in \mathbb{R} \times \Omega \times \mathbb{R}$ with $|u| \le C_1$
>
> $$\left|f(t,x,u)\right| \le C_2$$
>
> and for all $(t_1,x_1,u_1),(t_2,x_2,u_2) \in \mathbb{R} \times \Omega \times \mathbb{R}$ with $|u_1|,|u_1| \le C_1$
>
> $$\left|f(t_1,x_1,u_1) - f(t_2,x_2,u_2)\right| \le C_3\left(|t_1-t_2|^\delta + |x_1-x_2|^\delta + |u_1-u_2|\right)$$

Defining addition and scalar multiplication pointwise as usual, Y becomes a linear space. As before, we consider a family $(\delta_n)_{n\in\mathbb{N}}$ of seminorms

$$\delta_n(f) := \sup\{\left|f(t,x,u)\right| : (t,x,u)\in\mathbb{R}\times\Omega\times\mathbb{R} \text{ with } |t|,|u|\leq n\}.$$

These seminorms give rise to an invariant metric d on Y:

$$d(f_1,f_2) := \sum_{n=1}^{\infty} 2^{-n}\frac{\delta_n(f_1-f_2)}{1+\delta_n(f_1-f_2)}.$$

The metric d induces the compact-open topology on Y.
Denote

$$h:\mathbb{R}\to\mathbb{R}\quad h(t) := \begin{cases}(t+1)\sin\ln(t+1) & t>0\\ 0 & t\leq 0\end{cases}$$

and $t_n := e^{2\pi n}-1$, $n\in\mathbb{N}$. Then, $\ln(t_n+s)-2\pi n\to 0$ as $n\to\infty$ uniformly for s lying in bounded subsets of $\mathbb{R}$. Hence, one has

$$h(t_n+s) = h(t_n) + \int_{t_n}^{t_n+s} \sin\ln(t+1)+\cos\ln(t+1)\,\mathrm{d}t$$

so that $h(t_n+s)\to s$ as $n\to\infty$ uniformly on bounded sets. A refined variant of this construction will be studied in Chapter 4.
We are interested in full solutions of a perturbed equation. Suppose that $f\in Y$ is the parameter associated with the perturbed equations. Computing the index with respect to f would imply the loss of all the information contained in f for negative times. The index would be determined by the equation's behaviour at large times. This restriction can be overcome by using the auxialiary function h defined above. It allows to embed f into the ω-limes set of a related parameter, namely:

$$f.h := ((t,x,u)\mapsto f(h(t),x,u)).$$

It is easy to see that $f\in Y$ implies $f.h\in Y$ and, from the calculations above, it follows that $(f.h)^{t_n}\to f$ in Y, that is, uniformly on bounded subsets.
Combining this approach with the abstract results of the whole chapter, one obtains the following theorem.

THEOREM 3.64. *Suppose that $f\in Y$ is autonomous, and let $K\subset X^\alpha$ be a compact invariant set with respect to the evolution operator (semiflow) defined by* (3.15). *Let $N\subset X^\alpha$ be a strongly admissible (e.g. bounded) isolated neighborhood of K.*
Let (A,R) be an attractor-repeller decomposition of K, and assume that the associated connecting homomorphism $\partial:\ \mathrm{H}_*\mathscr{C}(f,A)\to\mathrm{H}_*\mathscr{C}(f,R)$ defined by the homology attractor-repeller sequence does not vanish.
Let $N_A\subset X^\alpha$ (resp. N_R) be a isolating neighborhood for A (resp R), and suppose that $N_A\cap N_R=\emptyset$. Then, there exists an $\varepsilon>0$ such that the following holds true for all $f'\in Y$ with $d(f,f')<\varepsilon$:

(a) *If $u:\mathbb{R}\to N\subset X^\alpha$ is a solution of*

$$u_t + Au = \hat{f}'(t,u), \tag{3.16}$$

then either $u(\mathbb{R}) \subset N_A \cup N_R$ *or* $\alpha(u) \subset N_R$ *and* $\omega(u) \subset N_A$.

(b) *There is a solution* $u : \mathbb{R} \to N_R$ *of* (3.16).

(c) *There is a solution* $u : \mathbb{R} \to N_A$ *of* (3.16).

(d) *There is a solution* $u : \mathbb{R} \to N$ *of* (3.16) *such that* $\alpha(u) \subset N_R$ *and* $\omega(u) \subset N_A$.

Using the same arguments as below the theorem can be generalized to partially ordered Morse decompositions. Moreover, Morse-decomposition are still preserved under small perturbations if the connecting homomorphism is zero, but one can no longer deduce the existence of solutions[20].

PROOF. First of all, note that $f.h = f$ as f is autonomous. Furthermore, $Y \times N$, $Y \times N_A$ and $Y \times N_R$ are isolating neighborhoods for $(f.h, K)$, $(f.h, A)$ and $(f.h, R)$ respectively. Therefore, (a) follows from Theorem 3.39. We will now consider (a segment of) the attractor-repeller sequence:

$$\longrightarrow \mathrm{H}_* \mathscr{C}(f,R) \xrightarrow{\partial} \mathrm{H}_{*-1} \mathscr{C}(f,A) \longrightarrow$$

Since, $\partial \neq 0$, one necessarily has $\mathrm{H}_* \mathscr{C}(f,R) \neq 0$ and $\mathrm{H}_* \mathscr{C}(f,A) \neq 0$. By using Theorem 3.45, one proves that for all f' in a neighborhood of f given by (a), the attractor-repeller sequence above extends to a commutative ladder:

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}_* \mathscr{C}(f,R) \arrow[r, "\partial"] \arrow[d, "\simeq"] & \mathrm{H}_{*-1} \mathscr{C}(f,A) \arrow[r] \arrow[d, "\simeq"] & {} \\
{} \arrow[r] & \mathrm{H}_* \mathscr{C}(f',R') \arrow[r, "\partial'"] & \mathrm{H}_{*-1} \mathscr{C}(f',A') \arrow[r] & {}
\end{tikzcd}$$

Here, we set $R' := (\mathrm{Inv}\, Y \times N_R) \cap (\Sigma^+(f') \times X)$ and $A' := (\mathrm{Inv}\, Y \times N_A) \cap (\Sigma^+(f') \times X)$. Consequently, in view of Corollary 2.22 and because $f' \in \omega(f'.h)$, (b) and (c) must hold. Finally, claim (d) is a consequence of Theorem 3.58, which claims that $K' = (\mathrm{Inv}\, Y \times N) \cap (\Sigma^+(f') \times X^\alpha)$ is uniformly connected. $\square$

[20]At least not with these arguments.

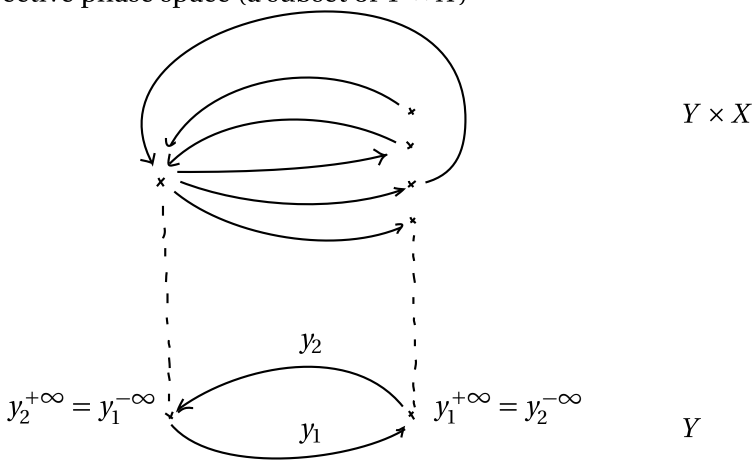
CHAPTER 4

Cycles of Asymptotically Autonomous Equations

In this chapter, we consider semilinear parabolic[1] equations – respectively evolution operators or skew product semiflows defined by equations of this type. We are interested in equations which are asymptotically autonomous in time.
Such an equation can be understood as a transition between two autonomous equations. If these autonomous equations are nice enough, they might be gradient-like. And the nonautonomous equation might exhibit invariant sets which consist of these of these equilibria – all having the same Morse-index – as well as isolated connections which are solutions of the transitional equations.
Additionally, we do not consider a single asymptotically autonomous equation but a finite number of them which form a cycle [2]. It is invariant sets with this strucure which are the subject of the present chapter. More precisely, we construct an evolution operator the ω-limes set of which is the asymptotically autonomous cycle, and prove subsequently that the homology index is zero in all dimensions except for the critical dimension determined by the Morse index of the equilibria.
The main result is Theorem 4.12, which resembles Theorem 2.36. In the context of asymptotically autonomous equations, the former theorem is a weak generalization of the latter.

[1]including ordinary differential equations on finite-dimensional spaces
[2]The reader who is curious about applications will find examples in Chapter 6.

FIGURE 4.1. An asymptotically autonomous cycle (in Y) and an invariant subset of the respective phase space (a subset of $Y \times X$)



4.1. The Parameter Space $\mathscr{Y}$

While the pattern behind the proofs seems to apply to more general situations, the author has not found an appropriate general form. Apparently, every additional abstraction only hides a relatively simple idea. Therefore, we limit our attention to semilinear equations of the form (4.1) with a specific right-hand side.

Let X^0 be a Banach space, and let $\|.\|_0$ denote its norm. As before, one considers a positive, sectorial operator A defined on a dense subspace. The fractional power spaces X^α are defined with respect to $\|.\|_\alpha := \|A^\alpha.\|_0$. Throughout this chapter, a fixed α with $0<\alpha<1$ is considered. In contrast to the usual notation, we write $X := X^\alpha$ and $\|.\| := \|.\|_X := \|.\|_\alpha$.

Suppose that A has compact resolvent, which means that the inclusion $X^\beta \subset X^\alpha$ for $\beta > \alpha$ is compact (completely continuous). Let $\mathscr{Y}$ denote the space of all continous mappings $y : \mathbb{R}\times X^\alpha \to X^0$ defined as follows:

(Y1) For every bounded set $B \subset \mathbb{R}\times X^\alpha$, there are constants $C, \delta > 0$ such that:

$$\left\|y(t,x)-y(t',x')\right\|_0 \le C\left(\left|t-t'\right|^\delta + \left\|x-x'\right\|_\alpha\right)$$

(Y2) $\mathscr{Y}$ is a subspace of $C_b^0(\mathbb{R}\times X, X^0)$.

(Y3) Let $\mathscr{Y}^1$ denote the subspace of all $y\in\mathscr{Y}$ for which $y(t,.)$ is differentiable and denote its derivative by $\mathrm{D}_x\, y$. Suppose for every bounded set $B\subset\mathbb{R}\times X^\alpha$, there are constants $C,\delta>0$ such that:

$$\left\|\mathrm{D}_x\, y(t,x)-\mathrm{D}_x\, y(t',x')\right\|_{\mathscr{L}(X^\alpha,X^0)} \le C\left(\left|t-t'\right|^\delta + \left\|x-x'\right\|_\alpha\right)$$

(Y4) Denote

$$\begin{aligned}\delta_n(y,y') \;&:=\; \sup\{\left\|\mathrm{D}_x(y-y')(t,x)\right\|_{\mathscr{L}(X^\alpha,X^0)} :\ t\le n \text{ and } \|x\|_\alpha \le n\}\\ &\quad +\sup\{\left\|(y-y')(t,x)\right\|_0 :\ t\le n \text{ and } \|x\|_\alpha\le n\}.\end{aligned}$$

and let $d_1 := d_{(\delta_n)_n}$ be given by (2.18).

For an arbitrary $y\in\mathscr{Y}$, $\Sigma^+(y)$ need not be compact. Therefore, in previous sections, more restricted parameter spaces were considered. However, in this chapter, the compactness is obtained by the construction of y_0 (Lemma 4.11).

DEFINITION 4.1. Let $u:\ \mathbb{R}\to X^0$ be a Hölder-continuous function, and define $f\ominus u\in\mathscr{Y}$ by $(f\ominus u)(t,x) := f(t,x+u(t)) - f(t,u(t))$.

LEMMA 4.2. *Suppose that $f\in\mathscr{Y}_c$ (resp. $f\in\mathscr{Y}_c^1$) and $u_0:\ \mathbb{R}\to X$ is globally bounded and Hölder-continuous that is, there are constants $C_1, C_2, \delta>0$ such that*

$$\|u(t)\|_0 \le C_1 \textit{ for all } t\in\mathbb{R}$$

$$\left\|u(t)-u(t')\right\|_0 \le C_2\left|t-t'\right|^\delta \textit{ for all } t,t'\in\mathbb{R}.$$

Then $f\ominus u_0\in\mathscr{Y}_c$ (resp. $\mathscr{Y}_c^1$).

PROOF. Let $t_n\to\infty$ be a sequence in $\mathbb{R}^+$. The theorem of Arzelà-Ascoli and the compact inclusion $X^\alpha\subset X^0$ imply that there exists a subsequence $(t_n')_n$ and a continuous function $u_0:\mathbb{R}\to X^0$ such that $u^{t_n'}\to u_0$ uniformly on compact (bounded) subsets of $\mathbb{R}$. Furthermore, there are constants $C_1, C_2, \delta > 0$ such that

$$\|u_0(t)\|_0 \le C_1 \text{ for all } t \in \mathbb{R}$$
$$\left\|u_0(t) - u_0(t')\right\|_0 \le C_2 \left|t - t'\right|^{\delta} \text{ for all } t, t' \in \mathbb{R}.$$

Since $y_0 \in \mathscr{Y}_c$ (resp. $\mathscr{Y}_c^1$), one can choose a subsequence $(t''_n)_n$ of $(t'_n)_n$ such that $f^{t''_n} \to f_0$. Consequently, $(f \ominus u)^{t''_n} \to f_0 \ominus u_0 \in \mathscr{Y}$, (resp. $\mathscr{Y}_c^1$ since $\mathrm{D}_x(f \ominus u)(t, x) = \mathrm{D}_x f(t, x + u(t))$) proving that $\Sigma^+(f_0 \ominus u_0)$ is compact. □

Let the evolution operator Φ_y on X^α be given by (mild) solutions of

$$u_t + Au = y(t, u) \tag{4.1}$$

The skew-product semiflow π is defined as usual on the space $\mathscr{Y} \times X = \mathscr{Y} \times X^\alpha$.
One can prove the following lemma, which is the main motivation for Definition 4.1.

LEMMA 4.3. *Let $y_0 \in \mathscr{Y}_c$ and u_0 a solution of Φ_{y_0}.*
Then $y \ominus u_0$ is defined, and $u_0 + v$ is a solution of Φ_{y_0} whenever v is a solution of $\Phi_{y_0 \ominus u_0}$.

4.2. Weakly Hyperbolic Solutions and Structural Assumptions

DEFINITION 4.4. Suppose that (Y, d) is a metric space.
Let $u : \mathbb{R} \to X^0$ be a locally Hölder-continuous function, and let $L \in Y_l$ be a linear operator.
We say that L is the *derivative of y at u* provided that the following holds:
For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left\|(y \ominus u)(t, x) - L(t, x)\right\|_0 \le \varepsilon \|x\|_\alpha$$

for all $(t, x) \in \mathbb{R} \times X^\alpha$ with $\|x\|_\alpha \le \delta$.
In this case, y is also called *differentiable at u*, and we write $\mathrm{D}\, y(u) := L$ to denote the derivative.

In other words, the derivative is the parameter corresponding to the linear variational equation.
We are now in a position to give a precise definition of weak[3] hyperbolicity:

DEFINITION 4.5. A solution $u : \mathbb{R} \to X$ of Φ_y, $y \in Y$ is called *weakly hyperbolic* if:

(a) $\sup_{t \in \mathbb{R}} \|u(t)\| < \infty$
(b) y is differentiable at u
(c) If $w : \mathbb{R} \to X$ is a bounded solution of $\Phi_{\mathrm{D}\, y(u)}$, then $w \equiv 0$.

An invariant subset $K \subset Y \times X$ is said to be *weakly hyperbolic* if for every solution $(v, u) : \mathbb{R} \to K$, u is a weakly hyperbolic solution of $\Phi_{v(0)}$.

DEFINITION 4.6. Let (Y, d) be a metric space[4] and $(t, y) \mapsto y^t$ a global flow on Y.
Recall that a parameter $y \in Y$ is called *autonomous* if $y^t = y$ for all $t \in \mathbb{R}$.

[3] A common notion [**17**] of hyperbolicity in a nonautonomous setting is the existence of an exponential dichotomy, which is, generally speaking, a stronger claim.
[4] No additional assumptions on Y are made.

$y \in Y$ is called *asymptotically autonomous* if there are autonomous $y^{\pm\infty} \in Y$ such that $d(y^{\pm t}, y^{\pm\infty}) \to 0$ as $t \to \infty$.
The set of all asymptotically autonomous $y \in Y$ is denoted by Y_{aa}.

DEFINITION 4.7. For $y_1, y_2 \in Y_{aa}$, we set $y_1 \preceq y_2$ if $d(y_1^t, y_2^{-t}) \to 0$ as $t \to \infty$.
A tuple $(y_1, y_2, \dots, y_n) \in Y^n$ is said to be an *asymptotically autonomous cycle (in (Y,d))* if $y_1 \preceq y_2 \preceq \dots \preceq y_n$ and $y_n \preceq y_1$.

LEMMA 4.8. *Assume that $y_n \to y$ in $\mathscr{Y}^1$ and $y_n(t,0) = y(t,0) = 0$ for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$. Then:*

$$\varepsilon^{-1} y_n(t, \varepsilon x) \to \mathrm{D}_x\, y(t,0)x \text{ in } \mathscr{Y} \text{ as } n, \varepsilon \to \infty, 0$$

PROOF. One has

$$\varepsilon^{-1} y_n(t,\varepsilon x) = \varepsilon^{-1}(y_n(t,\varepsilon x) - y(t,\varepsilon x)) + \varepsilon^{-1} y(t, \varepsilon x).$$

Furthermore,

$$\varepsilon^{-1}(y_n - y)(t, \varepsilon x) = \int_0^1 \mathrm{D}_x(y_n - y)(t, \varepsilon s x) s x \,\mathrm{d}s \to 0 \text{ in } Y \text{ as } n \to 0,$$

that is uniformly for bounded x and ε.
Similarly, one can prove that $\varepsilon^{-1} y(t, \varepsilon x) \to \mathrm{D}_x\, y(t,0)$ in Y as $\varepsilon \to 0$. □

LEMMA 4.9. *Let $y \in \mathscr{Y}^1$ be asymptotically autonomous in $\mathscr{Y}^1$ that is, there are $y^{\pm\infty} \in \mathscr{Y}^1$ such that $d_1(y^t, y^{\pm\infty}) \to 0$ as $t \to \pm\infty$. Assume further that for all $z \in \Sigma(y)$, $u \equiv 0$ is a weakly hyperbolic solution of Φ_z*
Then, for small $\varepsilon > 0$, there does not exist a solution $u : \mathbb{R} \to X$ with $\sup_{t\in\mathbb{R}} \|u(t)\| \le \varepsilon$ and $u \not\equiv 0$.

PROOF. First of all, note that by standard results π does not explode in $\Sigma(y_0) \times B_\delta[0, X]$. Suppose to the contrary that there are sequences $\varepsilon_n \to 0$ of positive real numbers and $u_n \to X$ of solutions of Φ_y with $\sup_{t\in\mathbb{R}} \|u_n(t)\| = \varepsilon_n$. Let $(t_n)_n$ be a sequence such that for each $n \in \mathbb{N}$ $\|u_n(t_n)\| \ge \varepsilon/2$. $u_n'(t) := \varepsilon_n^{-1} u_n(t_n + t)$ is a solution of Φ_{y_n}, where $y_n(t,x) := \varepsilon_n^{-1} y(t + t_n, \varepsilon_n x)$. Taking subsequences, we can assume without loss of generality that $y^{t_n} \to z \in \Sigma(y)$. It follows from Lemma 4.8 that $y_n(t,x) \to \mathrm{D}_x\, z(t,x)$ in $\mathscr{Y}$. $\Sigma(y) \times B_1[0]$ is strongly admissible, so by Lemma 2.28 there exists a subsequence $(u_n')_n$ of $(u_n)_n$ converging pointwise to a solution $u' : \mathbb{R} \to B_1[0]$ of $\Phi_{\mathrm{D}_x z}$ with $\left\|u'(0)\right\| \ge 1/2$. This solution cannot exist since the solution $u \equiv 0$ of Φ_y is assumed to be weakly hyperbolic. □

The following lemma is the main result of this section.

LEMMA 4.10. *Assume:*

- (H1) *$(y_1, \dots, y_N)$ is an asymptotically autonomous cycle in $(\mathscr{Y}^1, d_1)$.*
- (H2) *$y_0 \in \mathscr{Y}_c$ and $\omega(y_0) = \bigcup_{i=1,\dots,N} \Sigma(y_i)$, where $\Sigma(y) := \mathrm{cl}_{\mathscr{Y}}\{y^t : t \in \mathbb{R}\}$.*
- (H3) *$y_0 \in \mathscr{Y}_c$, $K \subset \Sigma^+(y_0) \times X$ is a weakly hyperbolic invariant set, and there is a strongly skew-admissible isolating neighborhood for (y_0, K).*

Then:

(a) *For each y_i, $i \in \{1,\dots,N\}$, there are only finitely many solutions $(v,u)\colon \mathbb{R} \to K$ with $v(0) = y_i^{\infty}$. Each of these solutions is a fixed point solution that is, $u \equiv e$ for some $e \in X$.*

(b) *For each y_i, $i \in \{1,\dots,N\}$, there are finitely many solutions $(v,u)\colon \mathbb{R} \to K$ with $v(0) = y_n$. Each of these solution is a heteroclinic connection.*

PROOF. (a) We consider an autonomous equation given by a parameter y_i^{∞}. In this case, the evolution operator $\phi := \Phi_{y_i^{\infty}}$ defines a semiflow on X. Let $u : \mathbb{R} \to X$ be a bounded solution of ϕ. Since the evolution operator ϕ is autonomous, $u^t(s) := u(t+s)$ with $t > 0$ real are solutions as well.

Since y_i is asymptotically autonomous and u bounded in X, one has

$$\sup_{t\in\mathbb{R}} \left\| y_i(t,u(t)) \right\|_0 < \infty.$$

Hence, it follows using standard estimates that

$$\left\| u(s) - u^t(s) \right\| \le M e^{-\delta t} + C \int_0^t r^{-\alpha} e^{-\delta r} \, \mathrm{d}r \to 0 \text{ as } t \to 0, \tag{4.2}$$

where the constants $M, C, \delta > 0$ are independent of $s \in \mathbb{R}$.

Subsequently, $u' \equiv 0$ and $u'_t := u^t - u$ are solutions of $y_i^{\infty} \ominus u$. One has $u'_t \to 0$ uniformly on $\mathbb{R}$ as $t \to 0$ by (4.2). Moreover, $u' \equiv 0$ is weakly hyperbolic, so, in view of Lemma 4.9, one must have $u'_t \equiv 0$ for sufficiently small t, implying that u is constant.

It can be proved analogously that there exist only finitely many points of the form $(y_i^{\infty}, e) \in K$.

(b) First of all, let $i \in \{1,\dots,N\}$, and let $(v,u) : \mathbb{R} \to K$ be a solution. The ω-limes set (resp. α-limes set) is a non-empty connected subset of $K \cap (\{y_i^{\infty}\} \times X)$ (resp. $K \cap (\{y_i^{-\infty}\} \times X)$). In view of (a), $\omega((v,u)) = \{(y_i^{\infty}, e^+)\}$ and $\alpha((v,u)) = \{(y_i^{-\infty}, e^-)\}$, where e^+ and e^- are fixed points.

Let $i \in \{1,\dots,N\}$, and let e^- (resp. e^+) be an arbitrary fixed point of $y_i^{-\infty}$ (resp. y_i^{∞}). It follows from Corollary 3.33 that the set

$$\begin{aligned} K_{(i,e^-),(i,e^+)} := \; & \{(v(t),u(t))\colon t \in \mathbb{R},\, v(0) = y_i \text{ and} \\ & (v,u)\colon \mathbb{R} \to K \text{ is a solution with } u(t) \to e^{\pm} \text{ as } t \to \pm\infty\} \\ & \cup \{(y_i^{-\infty}, e^-), (y_i^{\infty}, e^+)\} \end{aligned}$$

is compact.

Let $(v,u)\colon \mathbb{R} \to K_{(i,e^-),(i,e^+)}$ be a solution. By using Lemma 4.9, one can prove that $\mathrm{cl}_{\mathscr{Y}^1 \times X}\{(v(t),u(t))\colon t \in \mathbb{R}\}$ is an isolated invariant subset of $\Sigma^+(y_0) \times X$.

Thus, $K_{(i,e^-),(i,e^+)}$ is composed of finitely many orbits [**15**, Lemma 5.3]. The claim follows because, in view of (a), there are only finitely many sets $K_{(i,e^-),(i,e^+)}$ which need to be considered.

$\square$

4.3. Construction of an Initial Element y_0

In this section, we consider $(Y,d) = (\mathscr{Y},d)$ or $(Y,d) = (\mathscr{Y}^1,d_1)$ simultaneously. In order to apply the nonautonomous Conley index to an asymptotically autonomous cycle, we require an element $y_0 \in Y$ such that

$$\omega(y_0) = \bigcup_{i=1}^{N} \Sigma(y_i), \tag{4.3}$$

where $(y_1,\dots,y_N)$ is a given asymptotically autonomous cycle and $\Sigma(y) = \mathrm{cl}_{\mathscr{Y}^1}\{y^t :\ t \in \mathbb{R}\}$. Extend y_k periodically by $y_k := y_l$ if $k-l \in N\cdot\mathbb{N}$ and set

$$y_0(s,u) := y_0((y_k)_k)(s,u) := \sum_{k=1}^{\infty} y_k'(s,u) \tag{4.4}$$

where

$$y_k'(s,u) := \begin{cases} 0 & s<0 \\ 0 & t \le -\frac{\pi}{2} \\ (1-\frac{2|t|}{\pi})y_k(s\sin t,u) & -\frac{\pi}{2} \le t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le t \end{cases}$$

and $k\pi/2 + t = \ln(s+1)$, $k \in \mathbb{Z}$.

LEMMA 4.11. *Assume that either $(Y,d) = (\mathscr{Y},d)$ or $(Y,d) = (\mathscr{Y}_1,d_1)$.*
Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in Y, every y_k being asymptotically autonomous in Y. Suppose that the set $\{y_k :\ k \in \mathbb{N}\}$ is finite and for all $k \in \mathbb{N}$, $d(y_k^t, y_{k+1}^{-t}) \to 0$ as $t \to 0$. Let y_0 be defined by (4.4).
Then:

(a) $y_0 \in Y$;
(b) $\omega(y_0) = \bigcup_{i=1}^{N}\Sigma(y_i')$ *with* $\{y_i' :\ i = 1,\dots,N\} = \bigcap_{n\in\mathbb{N}}\{y_k :\ n \le k \in \mathbb{N}\}$. *In particular,* $\Sigma^+(y_0)$ *is compact in* Y.

PROOF. (a) We will show that $y_k' \in Y$ for all $k \in \mathbb{Z}$. Subsequently, since sum in (4.4) is finite on compact subintervals of the time-variable, so the formal series converges in (Y,d) to the pointwise finite sum.

Suppose that $(Y,d) = (\mathscr{Y},d)$.

For every bounded set $B \subset \mathbb{R}\times X$, there are constants $C_1 > 0$ and $0 < \delta \le 1$ such that

$$\left\| y_k(s,u) - y_k(s',u')\right\|_0 \le C_1\left(\left|s-s'\right|^{\delta} + \left\|u-u'\right\|\right)$$

for all $k \in \mathbb{Z}$ and all $(s,u),(s',u') \in B$.

Suppose that $k\pi/2 + t = \ln(s+1)$ and $k\pi/2 + t' = \ln(s'+1)$. By differentiation, one obtains that $\left|t-t'\right| = \left|\ln(s+1) - \ln(s'+1)\right| \le \left|s-s'\right|$ provided that $s,s' \ge 0$, so

$$\varrho_k(s) := \begin{cases} 0 & s<0 \\ 0 & t \le -\pi/2 \\ (1-\frac{2|t|}{\pi}) & -\pi/2 \le t \le \pi/2 \\ 0 & \pi/2 \le t \end{cases}$$

is Lipschitz-continuous, i.e. $\left|\varrho_k(s)-\varrho_k(s')\right| \le C_2\left|s-s'\right|$.

Again by diffentiation, one can show that

$$\left|s\sin t - s'\sin t'\right| \le 2\left|s-s'\right|.$$

Consequently,

$$\begin{aligned}\left\|y_k'(s,u)-y_k'(s',u')\right\|_0 &\le \left(\varrho_k(s)-\varrho_k(s')\right)\left\|y_k(s\sin t,u)\right\|_0\\ &\quad+\varrho_k(s')\left\|y_k(s\sin t,u)-y_k(s'\sin t',u')\right\|_0\\ &\le C_2\left|s-s'\right|+C_1\left(2^\delta\left|s-s'\right|^\delta+\left\|u-u'\right\|\right),\end{aligned}$$

implying that $y_k' \in \mathscr{Y}$ as claimed. Mutatis mutandis, the same calculation (applied to the derivatives D_x) proves that $y_k \in \mathscr{Y}^1$ implies $y_k' \in \mathscr{Y}^1$.

(b) First of all, we are going to prove that $\Sigma(y_k) \subset \omega(y_0)$ for every $k \in \{1,\dots,N\}$. Let $(n_m')_m$ be a sequence in $\mathbb{N}$ such that $n_m' \to \infty$ and $y_{n_m'} \equiv y_k$. Define $(s_m)_{m\in\mathbb{N}}$ by

$$s_m := e^{n_m'\pi/2}-1.$$

It is easy to see (cf. Section 3.7) that

$$t_m(h) := \ln(s_m+h+1)-n_m'\pi/2 \to 0 \text{ as } m\to\infty$$

and

$$(s_m+h)\sin t_m(h) \to h \text{ as } m\to\infty$$

uniformly for h in bounded subsets of $\mathbb{R}$.

It follows that $y_k^{s_m\sin t_m(0)} \to y_k$ in Y and consequently that $y_0^{s_m} \to y_k$ as $m\to\infty$.

Secondly, suppose that $s_m \to \infty$ is an arbitrary sequence. We will show that there exists a subsequence, denoted by s_m', such that $y_0^{s_m'} \to y \in \Sigma(y_k)$ for some $k \in \{1,\dots,N\}$. This result readily implies that $\omega(y_0) \subset \bigcup_{k=1}^N \Sigma(y_k)$ as claimed.

As before, we write[5]

$$\ln(s_m+1) = k_m\pi/2+t_m \quad k_m\in\mathbb{N} \quad t_m\in[-\pi/4,\pi/4].$$

We can now choose subsequences - denoted by the same symbols - such that $y_{k_m} \equiv y$ and $y_{k_m+1} \equiv z$ are constant, $t_m \to t_0$, and the following alternative holds:

(I) $s_m\sin t_m \to s_0$: We necessarily have $\sin t_m \to 0$, so $t_0 = 0$. Hence, $(s_m+h)\sin t_m(h) \to s_0+h$ uniformly on bounded sets, which implies that $y_0^{s_m} \to y_{k_0}^{s_0}$ as $m\to\infty$.

(II) $s_m\sin t_m \to \infty$: We have $t_0 \ge 0$, so we can assume w.l.o.g. that $t_m \ge 0$ for all $m\in\mathbb{N}$. One has

$$\begin{aligned}y^{s_m}(h,.) &= (1-\frac{2|t_m(h)|}{\pi})y(s_m(h)\sin t_m(h),.)\\ &\quad+\frac{2|t_m(h)|}{\pi}z(s_m(h)\sin(t_m(h)-\pi/2),.)\end{aligned}$$

[5]It is not required that the numbers k_m and t_m are unique.

where $(s_m + h)\sin t_m(h) \to \infty$ uniformly on bounded sets. Since $y_{k_m} \equiv y$ and $y_{k_m+1} \equiv z$, we know by assumption that $\omega(y) = \alpha(z) = \{y'\}$, and so $y_0^{s_m} \to y' \in \Sigma(y_{k_0})$.

(III) The case $s_m \sin t_m \to -\infty$ leads to $t_0 \le 0$ and can be treated analogously to (II).

□

4.4. The Main Theorem

The following theorem is the main result of this chapter. We fix $(Y, d) = (\mathscr{Y}^1, d_1)$ as the underlying space, i.e. the semiflow π is defined on $\mathscr{Y}^1 \times X$.

THEOREM 4.12. *Assume (H1), (H2) and (H3). Moreover, let y_0 be defined by* (4.4)*, and let q_0 denote the Morse-index of an arbitrary equilibrium solution (cf. Lemma 4.10) of K.*
Then for all $q \neq q_0$:

$$\mathrm{H}_q\, \mathscr{C}(y_0, K) \simeq 0.$$

The rest of this section is devoted to the proof of Theorem 4.12. First of all, we are going to state and prove a couple of auxiliary lemmas.
According to the assumptions (H1)–(H3) respectively Lemma 4.10 relying solely on these assumptions, the invariant subset K consists of finitely many connections between finitely many equilibria. Due to the cyclic nature of the underlying nonautonomous equations, K can be seen, geometrically, as a union of finitely many circles.
Suppose that we walk through these circles (Figure 4.1 on p. 81). Each equilibrium is a junction, where one decides which route to follow. Now assume that the theorem does not hold. Then, exploiting the direct limit formulation of the previous chapter, there is an element of dimension $q \neq q_0$ which, along a certain route in these circles, never becomes zero. One can assume without loss of generality that this non-zero path our element takes corresponds to a solution that is constantly zero.
Therefore, the index associated with this zero solution must have a non-trivial q-th homology, in contradiction to Theorem 2.36.
For technical reasons, we introduce an additional hypothesis, which could be understood as a non-degeneracy condition:

(H4) $N \ge 2$, and for all $k, l \subset \{1, \dots, N\}$, either $k = l$ or $\omega(y_k) \neq \omega(y_l \ominus u_0)$ whenever $u_0 : \mathbb{R} \to X$ is globally Hölder-continuous and converges as $t \to \pm\infty$.

Suppose that Theorem 4.12 holds under the additional hypothesis (H4). If $N = 1$, we can replace the asymptotically autonomous cycle (y_1) by (y_1, y_1) without affecting the definition of y_0, thus leaving the index unchanged.
We can now replace X by $X' := \mathbb{R} \times X$ and y_k by an appropriate function y'_k $(k = 1, \dots, N)$ such that

$$y'_k(t, (s, x)) := (\mu_k(t) \cdot s, y_k(t, x))$$

where $\mathbb{R} \ni \mu_k(t) < 0$ and $\lim_{t\to\infty} \mu_k(t) = \lim_{t\to\infty} \mu_l(t)$ implies $[k]_{\mathbb{R}/N\mathbb{Z}} = [l]_{\mathbb{R}/N\mathbb{Z}}$. Note that (Y1)–(Y4), adapted to the extended space X', are still satisfied.
It follows that (H4) holds with respect to $(y'_1, \dots, y'_k)$, (H1), (H2) and (H3) continue to hold, and the Morse-indices remain unchanged. Therefore, $\mathrm{H}_q\, \mathscr{C}(y'_0, \{0\} \times K) \simeq 0$ for $q \neq q_0$.

Finally, if (N_1, N_2) is an index pair for (y_0, K), then so is $(N_1', N_2') := ([-1,1] \times N_1, [-1,1] \times N_2)$ for $(y_0', \{0\} \times K)$. It follows that $\mathrm{H}_q\, \mathscr{C}(y_0, K) \simeq \mathrm{H}_q\, \mathscr{C}(y_0', \{0\} \times K)$, so (H4) is not required for the conclusions of Theorem 4.12 to be valid.

DEFINITION 4.13. Define $\mathbb{R}/N\mathbb{Z} := \{x + N\mathbb{Z} : x \in \mathbb{R}\}$ as usual. Let $x, y \in [0, N[$. A metric d on $\mathbb{R}/N\mathbb{Z}$ is given by

$$d([x],[y]) := \begin{cases} |x-y| & |x-y| \le N/2 \\ N - |x-y| & |x-y| > N/2. \end{cases}$$

We also write

$$[a,b]_{\mathbb{R}/N\mathbb{Z}} := \{[s]_{\mathbb{R}/N\mathbb{Z}} : s \in [a,b]\}$$

DEFINITION 4.14. Let $I \subset \mathbb{R}^+$ be an interval. We say that a continuous mapping $q : I \to \mathbb{R}/N\mathbb{Z}$ is non-decreasing if there exist a continuous and monotone non-decreasing function $q' : I \to \mathbb{R}$ such that $q(t) = [q'(t)]$ for all $t \in I$.

Non-decreasing in the sense of the above definition describes – understanding $\mathbb{R}/N\mathbb{Z}$ as a 1-sphere – a positive rotation of the point described by q on the sphere. The term *non-decreasing* is (obviously) motivated by technical reasons.

LEMMA 4.15. *In addition to the hypotheses of Theorem 4.12, let us assume that (H4) holds. Then there exists a continuous function $q : \Sigma^+(y_0) \to \mathbb{R}/(N\mathbb{Z})$ such that:*

(1) *q is non-decreasing with respect to translation, that is $t \mapsto q(y^t)$ is non-decreasing for all $y \in \{y_0^t : t \in \mathbb{R}^+\}$.*
(2)
$$q(y_k^t) = \left[k + \frac{1}{\pi}\arctan t\right]_{\mathbb{R}/N\mathbb{Z}}. \tag{4.5}$$
In particular, the restriction $q : \omega(y_0) \to \mathbb{R}/N\mathbb{Z}$ is a homeomorphism.

In the sequel, it is tacitly assumed that $q : \Sigma^+(y_0) \to \mathbb{R}/(N\mathbb{Z})$ is a mapping for which the conclusions of Lemma 4.15 hold.

PROOF. Using the concrete definition of y_0, the proof is rather straightforward. Let $s \in \mathbb{R}^+$ such that $[-\pi/8, \pi/8] \ni t := \ln(s+1) - k\pi/2$ with $k \in \mathbb{Z}$, and define:

$$q'(s) := k + \frac{1}{\pi}\arctan s \sin t$$

One has $q'(e^{(k+1/4)\pi/2} - 1) < k + 1/2 < q'(e^{(k+3/4)\pi/2} - 1)$, so it is possible to extend q' to a continuous and monotone increasing function $\mathbb{R}^+ \to \mathbb{R}^+$.
The mapping $h : \mathbb{R} \to \{y_0^t : t \in \mathbb{R}\}$, $h(t) := y_0^t$ is continuous and bijective, for if it was not bijective, y_0 would have to be periodic, which is, in view of $\omega(y_0)$, not the case. We can thus define $q(y_0^s) := [q'(s)]_{\mathbb{R}/N\mathbb{Z}}$. In order to prove that h is a homeomorphism, implying that q is continuous, suppose to the contrary that $h(s_n) \to h(s)$ but $s_n \not\to s$. Taking a subsequence, we can assume without loss of generality that $s_n \to \infty$. One has $h(s) = y_0^s \in \omega(y_0)$, so y_0 must be an asymptotically autonomous parameter for which (H2) holds. This is only possible if $N = 1$, but $N \ge 2$ is assumed in (H4). This is a contradiction, proving that h is a homeomorphism.

Subsequently, one needs to show that q can be continuously extended to $\Sigma^+(y_0)$. Suppose we are given a sequence $s_n \to \infty$ of positive real numbers. We can write $t_n = \ln(s_n+1) - k_n\pi/2$, where $t_n \in [-\pi/4, \pi/4[$ and $k_n \in \mathbb{Z}$. There are subsequences $t'_n = \ln(s'_n+1) - k'_n\pi$ such that $\left[k'_n\right] \to [k_0]$ (in $\mathbb{R}/N\mathbb{Z}$), $t'_n \to t_0$ and $s'_n \sin t'_n \to s_0 \in \mathbb{R} \cup \{\pm\infty\}$.
In the proof of Lemma 4.11, we saw that

$$y_0^{s'_n} \to \begin{cases} z \in \alpha(y_{k_0}) \cap \omega(y_{k_0-1}) & s_0 = -\infty \\ y_{k_0}^{s_0} & s_0 \in \mathbb{R} \\ z \in \omega(y_{k_0}) \cap \alpha(y_{k_0+1}) & s_0 = \infty \end{cases}$$

It is clear that $s_0 \in \mathbb{R}$ implies $t'_n \to 0$ as $n \to \infty$. Using (H4), it follows that[6] $y_0^{s'_n} \to y_{k_0}^{s_0} \in \omega(y_0)$ only if $\left[k'_n\right] \to [k_0]$ and $s'_n \sin t'_n \to s_0$, so

$$q(y_0^{s_n}) \to \begin{cases} \left[k_0 + \frac{1}{\pi} \arctan s_0\right]_{\mathbb{R}/N\mathbb{Z}} & y_0^{s_n} \to y_{k_0}^{s_0} \\ \left[k_0 + \frac{1}{2}\right]_{\mathbb{R}/N\mathbb{Z}} & y_0^{s_n} \to z \in \omega\{y_{k_0}\}. \end{cases} \tag{4.6}$$

Lastly, using (4.6), it is easy to see that q is non-decreasing on $\omega(y_0)$, too. □

LEMMA 4.16. *In addition to the hypotheses of Theorem 4.12 assume (H4). Define* $\dot{q} : \Sigma^+(y_0) \times X \to \mathbb{R}/N\mathbb{Z}$ *by* $\dot{q}(y,x) := q(y)$.
For a compact subset $K_0 \subset K$, *set*

$$U_\varepsilon(K_0) := \{(y,x) \in \Sigma^+(y_0) \times X : \ d((y,x), K_0) \le \varepsilon\}.$$

Let $[a,b]_{\mathbb{R}/N\mathbb{Z}} \cap (1/2 + \mathbb{Z}) = \{k_0 + 1/2\}$ *with* $k_0 \in \mathbb{Z}$, *let* $E := E_{k_0+1/2} \subset K$ *denote the set of all fixed points* (y,x) *with* $\dot{q}(x) = k_0 + 1/2$.
Lastly, let V *be a neighborhood of* $K \cap \dot{q}^{-1}([a,b])$.
There is an $\varepsilon_0 > 0$ *such that the following holds for all* $0 < \varepsilon \le \varepsilon_0$*:*

(a) $U_\varepsilon(K)$ *is an isolating neighborhood for* (y_0, K).
(b)
$$U_\varepsilon(K) \cap \dot{q}^{-1}([a,b]_{\mathbb{R}/N\mathbb{Z}}) \subset V$$

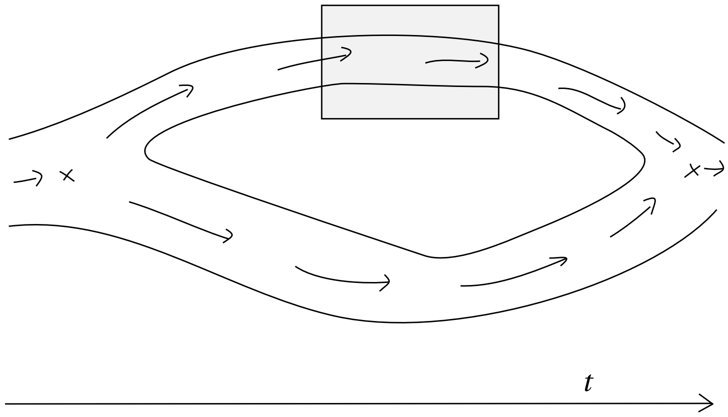
PROOF. (a) There exists an isolating neighborhood N for (y_0, K) by assumption (H3). $U_\varepsilon(K)$ is a neighborhood of K for every $\varepsilon > 0$. It is easy to see that for small $\varepsilon > 0$, one has $U_\varepsilon(K) \subset N$, so $U_\varepsilon(K)$ is an isolating neighborhood for (y_0, K) as claimed.
(b) Suppose to the contrary that there are sequences $\varepsilon_n \to 0$, $t_n \to \infty$ and $\left((y_n, x_n) \in U_{\varepsilon_n}(K) \cap \dot{q}^{-1}([a,b]_{\mathbb{R}/N\mathbb{Z}})\right) \setminus V$. Due to the compactness of K, there are subsequences, denoted by the same symbols, such that $(y_n, x_n) \to (y,x) \in K \setminus \dot{q}^{-1}([a,b]_{\mathbb{R}/N\mathbb{Z}})$ but $\dot{q}(y,x) \in [a,b]_{\mathbb{R}/N\mathbb{Z}}$ by continuity, which is a contradiction.

□

LEMMA 4.17. *In addition to the hypotheses of Theorem 4.12, suppose that (H4) holds. Let* (N_1, N_2) *be an index pair for* (y_0, K).

[6] Recall that $\omega(y_k) = \{y_k^{\infty}\}$ and $\alpha(y_k) = \{y_k^{-\infty}\}$.

FIGURE 4.2. An index pair, following an invariant set with two branches. One of the branches can be interrupted (Lemma 4.17) by removing the grey area. The arrow below indicates the evolution of time.



Let $((a_n, b_n, c_n))_n$ *be a sequence of non-negative real numbers such that* $c_n < a_{n+1} < b_{n+1} < c_{n+1}$ *for all* $n \in \mathbb{N}$. *Let* $N_1 = N_{1,1} \cup N_{1,2}$ *be a union of closed subsets, and suppose that the sets* $N_{1,1}([a_n, c_n])$ *and* $N_{1,2}([a_n, c_n])$ *are disjoint.*
Lastly, define

$$M_1 := N_1 \setminus \{(t,x) \in N_{1,2} : \ t \in]b_n, c_n[\text{ for some } n \in \mathbb{N}\}$$

$$M_2 := (N_2 \cap M_1) \cup \{(t,x) \in N_{1,2} : \ t = b_n \text{ for some } n \in \mathbb{N}\}$$

Then (M_1, M_2) *is an index pair. Moreover, if* (N_1, N_2) *is regular, then so is* (M_1, M_2).

PROOF. Recall that the semiflow $\chi := \chi_{y_0}$ on $\mathbb{R}^+ \times X$ is defined by $(t,x)\chi s := (t+s, \Phi_{y_0}(t,0,x))$.

(IP1) It is easy to see that M_1 and M_2 are closed.

(IP3) Let $(t,x) \in M_1$ such that $(t,x)\chi[0,s_0] \not\subset M_1$. Firstly, suppose that $(t,x)\chi[0,s_0] \not\subset N_1$. It follows that $(t,x)\chi s \in N_2$ for some $s \in [0,s_0]$. Secondly, if $(t,x)\chi[0,s_0] \subset N_1$, then there are $s \in [0,s_0]$ and $n \in \mathbb{N}$ such that $b_n < t+s < c_n$ and $(t,x)\chi(t+s) \in N_{1,2}$. It follows that $(t,x)\chi s' \in M_2$ for some $s' \in [0,s]$.

Let $\tau := \sup\{s \geq 0 : \ (t,x)\chi[0,s] \subset M_1\} < \infty$. By the arguments above, we obtain a sequence $\tau_n \to \tau + 0$ such that $(t,x)\chi\tau_n \in N_2 \cup M_2$ for all $n \in \mathbb{N}$. Since $N_2 \cap M_2$ is closed, it follows that $(t,x)\chi\tau \in (N_2 \cup M_2) \cap M_1 = M_2$.

(IP4) Let $(t,x) \in M_2$ and $(t,x)\chi s_0 \notin M_2$. If $(t,x)\chi[0,s_0] \not\subset N_1$, then $(t,x)\chi s \in (\mathbb{R}^+ \times X) \setminus N_1$ for some $s \in [0,s_0]$ since (N_1, N_2) is an index pair.

Otherwise, $(t,x)\chi[0,s_0] \subset N_1$ implies immediately that $(t,x)\chi s_0 \in N_1 \setminus M_1 \subset (\mathbb{R}^+ \times X) \setminus M_1$.

We need to prove that (M_1, M_2) inherits the regularity of (N_1, N_2). Recall that the inner exit time with respect to the index pair (N_1, N_2) is given by

$$T^+(x) := \sup\{t \in \mathbb{R}^+ : \ x\chi[0,t] \subset N_1 \setminus N_2\}.$$

We need to show that the exit time T_M^+ with respect to (M_1, M_2) is continuous as well.

Let $(t,x) \in M_1$ be arbitrary. We consider a couple of cases:

(1) $t + T_M^+(t,x) = b_n$ for some $n \in \mathbb{N}$:
   (a) $(t,x)\chi T_M^+(t,x) \in N_{1,2}$: One has $(t',x')\chi T^+(t',x') \in N_{1,2}$ for all (t',x') in a small neighborhood of (t,x). Thus, $T_M^+(t',x') = \min\{b_n - t', T^+(t,x)\}$ for all those (t',x'), implying that T_M^+ is continuous in (t,x).
   (b) $(t,x)\chi T_M^+(t,x) \in N_{1,1}$: $T_M^+(t',x') = T^+(t',x')$ for all (t',x') in a small neighborhood of (t,x). Therefore, T_M^+ is continuous in (t,x).

(2) $t + T_M^+(t,x) \neq b_n$: One has $(t,x)\chi T_M^+(t,x) \in N_2$, so $T_M^+(t,x) = T^+(t,x)$. M_2 is closed, so $T_M^+(t,x)$ is lower semicontinuous. We further have $T_M^+(t',x') \leq T^+(t',x') \to T^+(t,x) = T_M^+(t,x)$ as $(t',x') \to (t,x)$, implying that T_M^+ is upper semicontinuous and therefore continuous in (t,x).

□

LEMMA 4.18. *Assume the hypotheses of Theorem 4.12.*
For every $k \in \mathbb{N}$, let $(v_k, u_k) : \mathbb{R} \to K$ be a heteroclinic solution with $v_k(0) = y_k$ for all $k \in \mathbb{N}$, and assume that $d((v_k(t), u_k(t)), (v_{k+1}(-t), u_{k+1}(-t))) \to 0$ as $t \to \infty$.
In analogy to (4.4) *define*

$$u_0(t) := u_0((u_k)_k)(t) := \sum_{k=1}^{\infty} u_k'(t),$$

where

$$u_k'(s) := \begin{cases} 0 & s \leq 0 \\ 0 & t \leq -\pi/2 \\ (1 - \frac{2|t|}{\pi})u_k(s \sin t) & -\pi/2 \leq t \leq \pi/2 \\ 0 & \pi/2 \leq t \end{cases}$$

and $k\pi/2 + t = \ln(s+1)$.
Then:

(a) *There are reals $C_1, C_2, \delta > 0$ such that $\|u_0(t)\| \leq C_1$ for all $t \in \mathbb{R}$ and $\left\|u_0(t) - u_0(t')\right\| \leq C_2\left|t - t'\right|^{\delta}$ for all $t, t' \in \mathbb{R}$.*

(b) *Every $z \in \omega(y_0 \ominus u_0)$ can be written as $z = y \ominus u$, where $y \in \omega(y_0)$ and $u \in \omega(u_0)$ is a solution of Φ_y. Assuming (H4), this decomposition is unique.*
   The mapping $p : \Sigma^+(y_0 \ominus u_0) \to \Sigma^+(y_0) \times \Sigma^+(u_0)$, $p(y \ominus u) := (y,u)$ is a semi-conjugacy[7].

(c) *$\omega(y_0 \ominus u_0) \times \{0\}$ is an isolated invariant subset of $\omega(y_0 \ominus u_0) \times X$.*

(d) *Recall that q_0 is given by Theorem 4.12.*

$$\mathrm{H}_q\, \mathscr{C}(y_0 \ominus u_0, \omega(y_0 \ominus u_0) \times \{0\}) \simeq \begin{cases} \mathbb{Z} & q = q_0 \\ 0 & q \neq q_0. \end{cases} \tag{4.7}$$

PROOF. (a) K is compact, hence bounded. Consequently, there are $C_3, \delta > 0$ such that

$$\left\|u(t) - u(t')\right\| \leq C_3\left|t - t'\right|^{\delta} \qquad t, t' \in \mathbb{R}$$

[7] surjective, continuous and commutes with the canonical semiflow

for every solution $u : \mathbb{R} \to K$. The crucial point is that the bound C_3 depends only on K but not on the specific solution u.

$s \mapsto s\sin(\ln(s+1) - k\pi/2)$ is uniformly Lipschitz-continuous, and thus for all $k \in \mathbb{N}$

$$\left\|u_k'(t) - u_k'(t')\right\| \le C_4 \left|t - t'\right|^{\delta} \quad t, t' \in \mathbb{R}.$$

Finally, observe that for every $t \in \mathbb{R}$, there are at most two summands $u_k'(t)$ which do not vanish. It follows that u_0 is bounded and Hölder-continuous, where both constants depend solely on K.

(b) Let there be given sequences $(k_n)_n$ in $\mathbb{N}$ and $(t_n)_n$ in $[-\pi/2, \pi/2]$ such that $k_n \to \infty$, $[k_n]_{\mathbb{Z}/N\mathbb{Z}} \to [k_0]_{\mathbb{Z}/N\mathbb{Z}}$, and set $s_n := \mathrm{e}^{k_n\pi/2 + t_n} - 1$. Let us further assume that $u_{k_n}^{s_n} \to u'$ uniformly on compact intervals, and $s_n \sin t_n \to s_0$. As in the proofs of Lemma 4.11 and Lemma 4.15, it follows that $y_0^{s_n} \to y_{k_0}^{s_0}$ as $n \to \infty$.

Since $u_{k_n}^{s_n}$ is a solution of $y_{k_n}^{s_n}$, u' must be a solution of $y_{k_0}^{s_0}$, and $y_0^{s_n} \ominus u_0^{s_n} \to y_{k_0}^{s_0} \ominus u'$.

Every $z \in \omega(y_0 \ominus u_0)$ is the limit of a sequence $(y_0 \ominus u_0)^{s_n}$. Taking subsequences, we obtain a sequence $(s_n')_n$ such that $u_0^{s_n'} \to u'$, $y_0^{s_n'} \to y'$ and u' is a solution of $\Phi_{y'}$. Thus $z = y' \ominus u'$.

In order to prove uniqueness, suppose that $z = y' \ominus u' = y'' \ominus u''$. It follows that $y' = y'' \ominus (u'' - u')$. Due to (H4), we must have $y' = y''$.

Finally, assume that p is not continuous. Then there is a sequence of the form $(y_n \ominus u_n)_n$ such that $y_n \ominus u_n \to y_0 \ominus u_0$ but $d(y_n, y_0) \ge \varepsilon > 0$ or $d(u_n, u_0) \ge \varepsilon > 0$ for all $n \in \mathbb{N}$. We can assume without loss of generality that $y_n \to y'$ and $u_n \to u'$ as $n \to \infty$. Clearly, $y_0 \ominus u_0 = y' \ominus u'$. From the uniqueness of the decomposition, one obtains $y_0 = y'$ and $u_0 = u'$, which is a contradiction.

(c) It follows from (H3) that $D_x(y_0 \ominus u_0)$ is weakly hyperbolic. Hence, setting $f_n \equiv y_0 \ominus u_0$ and $L_0 = D_x(y_0 \ominus u_0)$, (LIN0) is satisfied. It follows from Lemma 2.54 that $\omega(y_0 \ominus u_0) \times \{0\}$ is an isolated invariant set.

(d) This follows from Theorem 2.56. More precisely, it follows that there exists a q_0 such that (4.7) holds. By using Lemma 2.40, one concludes that q_0 must agree with the Morse index of the equilibrium solution $u \equiv 0$ with respect to y_k^{∞} for arbitrary $k \in \{1, \ldots, N\}$.

□

PROOF OF THEOREM 4.12. In view of the remarks following the statement of the theorem, it is sufficient to conduct the proof under the additional hypothesis (H4).

Throughout this proof, $\varepsilon > 0$ is a fixed number. There are various conditions on ε that will be formulated later although it would be possible to collect all of these assumptions at the beginning of the proof.

We neither assume nor do we require backwards uniqueness in this proof, yet we need to separate distinct orbits. Suppose there are $k = k(i)$ connecting orbits[8] in K, which are solutions with respect to y_i. Choosing $t > 0$ sufficiently large, it holds for every $i \in \{1, \ldots, N\}$ that $K(y_i^{-t})$ consists of exactly $k(i)$ points, each of them representing exactly one connecting

[8] An orbit is understood here as a subset of the phase space.

orbit. For every $z \in \omega(y_i)$, $K(z)$ is a set of finitely many fixed points of Φ_z. Denote $z_i := y_i^{-2t}$, $i \in \{1,\dots,N\}$.
By (4.5), one has for all $i \in \{1,\dots,N\}$

$$q(z_i) = \left[i + \frac{1}{\pi} \arctan(-2t) \right]_{\mathbb{R}/N\mathbb{Z}}.$$

In view of the above equation, the following definition is useful.

$$q_0 := \frac{1}{\pi} \arctan(-2t).$$

Because q is non-decreasing, there exists a continuous and monotone increasing function $q' : \mathbb{R}^+ \to \mathbb{R}$ such that $\left[q'(a)\right]_{\mathbb{R}/N\mathbb{Z}} := q(y_0^a)$. Suppose that $(a_l)_l$ is an ascending sequence of non-negative real numbers such that $\{a_l : l \in \mathbb{N}\} = (q')^{-1}(\{q_0\})$.
By Lemma 3.51, there exists[9] a regular index pair (N_1, N_2) for (y_0, K) with $N_1 \subset r^{-1}(U_\varepsilon(K))$. Let $\varepsilon' > 0$ be arbitrary but fixed, $\alpha \subset \mathbb{R}^+$ a closed interval, and set

$$A_{\alpha,q} := A^{\varepsilon'}_{\alpha,q} := \mathrm{H}_q(N_1(\alpha), N_2^{-\varepsilon'}(\alpha))$$

as well as

$$B_{k,q} := B^{\varepsilon'}_{k,q} := A^{\varepsilon'}_{\{a_k\},q} \quad (k,q) \in \mathbb{N} \times \mathbb{Z}.$$

Given closed intervals $\alpha \subset \beta \subset \mathbb{R}^+$, we define $f_{\alpha,\beta} : A_\alpha \to A_\beta$ to be the inclusion induced homomorphism. We also define $g_{k,l} : B_{k,q} \to B_{l,q}$, $k \le l$ by

$$g_{k,l} := f^{-1}_{[a_k,a_l],\{a_l\}} \circ f_{\{a_k\},[a_k,a_l]}.$$

It follows from Lemma 3.56 that

$$\operatorname{dirlim}_{k\to\infty}(B_{k,q}, g_{k,l})_{k\le l} \simeq \mathrm{H}_q[N_1, N_2^{-\varepsilon'}] \simeq \mathrm{H}_q\, \mathscr{C}(y_0, K) \quad q \in \mathbb{Z}.$$

In order to prove the main theorem by contradiction, assume that $\mathrm{H}_q\, \mathscr{C}(y_0,K) \ne 0$ for some $q \ne q_0$ that is, there exist $k_0 \in \mathbb{N}$ and $\eta_0 \in B_{k_0}$ such that $g_{k_0,l}(\eta_0) \ne 0$ for all $l \ge k_0$.
Choosing $\delta > 0$ sufficiently small, the families $(B_\delta((y,x)))_{(y,x)\in \dot{q}^{-1}(\{q_0+k\})}$ are disjoint for some $\delta > 0$ and for all $k \in \mathbb{Z}$. By using Lemma 4.16, one can assume that

$$U_\varepsilon(K) \cap q^{-1}(\{\left[q_0+k\right]_{\mathbb{R}/N\mathbb{Z}}\}) \subset \bigcup_{(y,x)\in K\cap q^{-1}(\{\left[q_0+k\right]\})} B_\delta((y,x)).$$

Therefore, setting $U_{k,x} := (U_\varepsilon(K) \cap B_\delta((z_k,x)))(y_0^{a_k})$

$$U_\varepsilon(K)(y_0^{a_k}) = \dot{\bigcup}_{x\in K(z_k)} U_{k,x}$$

is a disjoint union taken over finitely many sets – one representing each orbit. Each $B_{k,q}$ is isomorphic to a direct sum of homologies of subspaces, namely

$$B_k := B_{k,q} \simeq \bigoplus_{x\in K(y_0^{a_k})} \underbrace{\mathrm{H}_q(N_1(a_k)\cap U_{k,x}, N_2^{-\varepsilon'}(a_k)\cap U_{k,x})}_{=:B_{k,x}=:B_{k,x,q}}.$$

Consequently, $g_{k,l} = \bigoplus_{x'\in K(y_0^{a_l})} \sum_{x\in K(y_0^{a_k})} g_{(k,x),(l,x')}$, where

$$g_{(k,x),(l,x')} : B_{k,x} \to B_{l,x'} \quad g_{(k,x),(l,x')}(\eta) := p_{l,x'} \circ g_{k,l}(\eta)$$

[9] In view of (H3), it is clear that there exists an index pair for (y_0, K), which need not be regular, though.

and $p_{l,x'} : B_l \to B_{l,x'}$ is the canonical projection.
The non-zero element η_0 can be written as finite sum of elements in $B_{k_0,x}$, so we may assume without loss of generality that $\eta_0 \in B_{k_0,x_0}$. For all $l \geq k_0$, we have

$$\begin{aligned} g_{k_0,l}(\eta_0) = \sum g_{(l-1,x^{(l-k_0-1)}),(l,x^{(l-k_0)})} \circ g_{(l-2,x^{(l-k_0-2)}),(l-1,x^{(l-k_0-1)})} \\ \circ \cdots \circ g_{(k_0,x_0),(k_0+1,x')}(\eta_0) \neq 0, \end{aligned} \tag{4.8}$$

where the sum taken over all $l-k_0$-tuples $(x', x^{(2)}, x^{(3)}, \dots, x^{(l-k_0)}) \in K(z_{k_0+1}) \times K(z_{k_0+2}) \times \cdots \times K(z_l)$. In other words, the sum is taken over all possible paths starting from (k_0, x_0).

LEMMA 4.19. *Let* $\Gamma := ((k_0+l, x_l))_{l=1,\dots,n}$, *be a tuple in* $(\mathbb{N} \times X)^n$ *such that* $x_l \in K(y_0^{a_{k_0+l}})$ *for all* $l \in \{1, \dots, n\}$.
Define $g_\Gamma := g_{(k_0,x_0),\gamma}$ *if* $\Gamma = \{\gamma\}$ *and*

$$g_\Gamma := g_{(k_0+l-1,x_{l-1}),(k_0+l,x_l)} \circ g_{((k_0+1,x_0+1),\dots,(k_0+l-1,x_{l-1}))}$$

otherwise.
Then, there is a sequence $((k_0+l, x_l))_{l\in\mathbb{N}}$ *such that* $g_{\Gamma_n}(\eta_0) \neq 0$ *for every finite tuple of the form* $\Gamma_n = ((k_0+1, x_1), \dots, (k_0+n, x_n))$.

PROOF. By (4.8), there are tuples Γ of arbitrary length such that $g_\Gamma(\eta_0) \neq 0$. Suppose that $(\Gamma_n)_{n\in\mathbb{N}}$ is a sequence such that for all $n \in \mathbb{N}$, $\Gamma_n = (\gamma_{1,n}, \dots, \gamma_{n,n})$ is an n-tuple and $g_{\Gamma_n}(\eta_0) \neq 0$ for all $n \in \mathbb{N}$.
Since each of the sets $K(y_k^{-t})$ is finite, it is possible to choose a subsequence $(\gamma'_{i,n})_n$ of $(\gamma_{i,n})_n$ for every $i \in \mathbb{N}$ such that $\gamma'_{i,n}$ converges i.e., $\gamma'_{i,n} \equiv \gamma_i$ for large $n \in \mathbb{N}$.
A standard diagonal sequence argument implies that there exists a subsequence $(\Gamma_{n(k)})_k$ such that for all $i \in \mathbb{N}$, $\gamma_{i,n(k)} \equiv \gamma_i$ for all $i \in \mathbb{N}$ and all $k \in \mathbb{N}$ with $i \leq n(k)$. It follows that $g_{(\gamma_1,\dots,\gamma_i)}(\eta_0) \neq 0$ for all $i \in \mathbb{N}$. Hence, one can choose $(k_0+l, x_l) := \gamma_l$ in order to satisfy the conclusions of this lemma. □

By Lemma 4.19, there exists a sequence $((k_0+l, x_l))_l$ such that $x_l \in K(z_{k_0+l})$ and

$$g_{\Gamma_n}(\eta_0) \neq 0 \text{ for all } n \in \mathbb{N}, \tag{4.9}$$

where we set $\Gamma_n = ((k_0+1, x_1), \dots, (k_0+n, x_n))$.
To every pair (l, x_l) with $x_l \in K(z_l)$, there is a unique solution $(v_l, u_l) : \mathbb{R} \to K$ with $(v, u)(0) = (z_l, x_l)$.
For $l \in \mathbb{N}$, define $K_{1,l} \subset \mathbb{R}^+ \times Y \times X$ by

$$K_{1,l}(s) := \begin{cases} \emptyset & t < -\pi/2 \\ \{(v_l, u_l)(s \sin t)\} & -\pi/2 \leq t \leq \pi/2 \\ \emptyset & \pi/2 < t, \end{cases}$$

where $l\pi/2 + t = \ln(s+1)$.
Letting $q'_0 := \frac{1}{\pi} \arctan(-t)$, it holds for every $k \in \mathbb{Z}$ that the set $K_{[q_0,q'_0]} := \{(y,x) \in K : q(y) \in [q_0+k, q'_0+k]_{\mathbb{R}/N\mathbb{Z}}\}$ consists of exactly one component $S_{k,x}$ for each orbit denoted by $x \in K(z_k)$. For small $\varepsilon' > 0$, the sets $V_{k,x} := \mathrm{cl}_{\Sigma^+(y_0)\times X}\, U_{\varepsilon'}(S_{k,x})$ are pairwise disjoint.

By the choice of ε and in view of Lemma 4.16 we can assume that for all $k\in\mathbb{Z}$

$$\{(y,x)\in U_\varepsilon(K):\ q(y)\in\left[q_0+k,q_0'+k\right]_{\mathbb{R}/N\mathbb{Z}}\}\subset\bigcup_{i=1}^{n}V_i. \tag{4.10}$$

Let

$$K_1:=\bigcup_{l\in\mathbb{N}}K_{1,l},$$

and

$$N_{1,1}:=\{(t,x)\in N_1:\ d((y_0^t,x),K_1(t))\le\varepsilon\}$$
$$N_{1,2}:=\{(t,x)\in N_1:\ d((y_0^t,x),K_1(t))\ge\varepsilon\},$$

where $d(x,K_1(t)):=\sup\{\left\|x-y\right\|:\ y\in K_1(t)\}$ denotes the usual distance from a point to a set. It is clear that $N_1=N_{1,1}\cup N_{1,2}$ and also that $N_{1,1}$ as well as $N_{1,2}$ are closed subsets of N_1. Furthermore, because $N_1\subset r^{-1}(U_\varepsilon(K))$ and by (4.10), $N_{1,1}(t)\cap N_{1,2}(t)=\emptyset$ for all $t\in\mathbb{R}^+$ with $q'(t)\in\left[q_0+k,q_0'+k\right]_{\mathbb{R}/N\mathbb{Z}}$.
Let (M_1,M_2) be defined as follows:

$$M_1:=N_1\setminus\{(t,x)\in N_{1,2}:\ q'(t)\in\left]q_0+k,q_0'+k\right[\text{ for some }k\in\mathbb{Z}\}$$
$$M_2:=(N_2\cap M_1)\cup\{(t,x)\in N_{1,2}:\ q'(t)=q_0+k\text{ for some }k\in\mathbb{Z}\}$$

It follows from Lemma 4.17 that (M_1,M_2) is a regular index pair.
Let

$$\tilde{A}_{\alpha,q}:=\mathrm{H}_q(M_1(\alpha),M_2^{-\varepsilon'}(\alpha))$$
$$\tilde{B}_{k,q}:=\tilde{A}_{\alpha,q}(\{a_k\}),$$

let $\tilde{f}_{\alpha,\beta}:\ \tilde{A}_\alpha\to\tilde{A}_\beta$, $\alpha\subset\beta$ be inclusion induced and let $\tilde{g}_{k,l}:\ \tilde{B}_{k,q}\to\tilde{B}_{l,q}$, $k\le l$ be given by the formula

$$\tilde{g}_{k,l}:=\tilde{f}^{-1}_{[a_k,a_l],\{a_l\}}\circ\tilde{f}_{\{a_k\},[a_k,a_l]}.$$

One has $(N_1(\{a_k\}),N_2(\{a_k\}))\subset(M_1(\{a_k\}),M_2(\{a_k\}))$ for all $k\in\mathbb{N}$ and $(N_1(\{a_k\}),N_2^{-\varepsilon'}(\{a_k\}))\subset(M_1(\{a_k\}),M_2^{-\varepsilon'}(\{a_k\}))$ provided that $\varepsilon'>0$ is small. One can prove[10] the following diagram with inclusion induced homomorphisms i_k and i_l $(k_0\le k\le l)$ is commutative:

$$\begin{tikzcd}
B_{k,q} \arrow[r, "g_{k,l}"] \arrow[d, "i_k"] & B_{l,q} \arrow[d, "i_l"] \\
\tilde{B}_{k,q} \arrow[r, "\tilde{g}_{k,l}"] & \tilde{B}_{l,q} \\
B_{k,x_k} \arrow[r, "g_{(k,x_k),(l,x_l)}"] \arrow[u, "\simeq"] & B_{l,x_l} \arrow[u, "\simeq"]
\end{tikzcd} \tag{4.11}$$

For large indices $k\in\mathbb{N}$, i_k is – up to an isomorphism – a projection associated with one component of a direct sum:

$$\begin{aligned}\mathrm{H}_q(N_1(\{a_k\}),N_2^{-\varepsilon'}(\{a_k\}))\ &\simeq\ \mathrm{H}_q(N_1(\{a_k\})\cap N_{1,1},N_2^{-\varepsilon'}(\{a_k\})\cap N_{1,1})\\ &\quad\oplus\mathrm{H}_q(N_1(\{a_k\})\cap N_{1,2},N_2^{-\varepsilon'}(\{a_k\})\cap N_{1,2}).\end{aligned}$$

[10]Intuitively, we have interrupted all connections except for one. The calculation itself is rather tedious.

Because $\mathrm{H}_q[M_1, M_2] \simeq \operatorname{dirlim}(\tilde{B}_{k,l}, \tilde{g}_{k,l})$ by Lemma 3.56, it follows from (4.9) and (4.11) that

$$\mathrm{H}_q[M_1, M_2] \neq 0 \text{ for some } q \neq q_0. \tag{4.12}$$

In order to obtain a contradiction, we consider two cases:

(I) For all $k \geq k_0$, $\|u_k(t) - u_{k+1}(-t)\| \to 0$ as $t \to \infty$, or

(II) there are arbitrarily large $k \in \mathbb{N}$ with $u_k(t) - u_{k+1}(-t) \not\to 0$ as $t \to \infty$.

In other words, either the ω-limes set of u_k (a single fixed point) and the α-limes set of u_{k+1} agree for all but finitely many indices k or they do not. We will show that in either case $\mathrm{H}_q[M_1, M_2] = 0$ ($q \neq q_0$ has been obtained from the assumption that $\mathrm{H}_q\,\mathscr{C}(y_0, K) \neq 0$), which is a contradiction and completes the proof of Theorem 4.12 under the additional assumption (H4).

(I) The assumptions of Lemma 4.18 are satisfied. Let u_0 be given by that lemma, and define $y_0' := y_0 \ominus u_0$. Let N be a strongly admissible isolating neighborhood for (y_0, K), and let $p = (p_1, p_2)\colon\ \Sigma^+(y_0') \to \Sigma^+(y_0) \times \Sigma^+(u_0)$ be defined by Lemma 4.18. It follows that $N' := P^{-1}(N)$ is a strongly admissible isolating neighborhood, where we set $P: \Sigma^+(y_0') \times X \to \Sigma^+(y_0) \times X$, $(y, x) \mapsto (p_1(y), x + p_2(y)(0))$. We may thus denote $K' := \operatorname{Inv} N'$.

Recall that (M_1, M_2) is an index pair for Φ_{y_0}, so

$$M_i' := \{(t, x - u_0(t))\colon (t, x) \in M_i\} \quad i \in \{1, 2\}, \tag{4.13}$$

defines an index pair (M_1', M_2') for $\Phi_{y_0'}$.

We claim that (M_1', M_2') is an index pair for (y_0', K_0'), where

$$K_0' := \underbrace{\big(\omega(y_0 \ominus u_0) \times \{0\}\big)}_{=:A} \cup \underbrace{\{(y \ominus u_0, x) \in K'\colon\ q(y) \cap (\mathbb{Z} + 1/2) \neq \emptyset\}}_{=:R}.$$

R is the set of fixed points (in K') of the autonomous equations.

Let $W_\delta := U_\delta(K_0')$ be a neighborhood of K_0'. We aim to prove that for small $\delta > 0$,

$$r_{y_0'}^{-1}(W_\delta) \subset M_1' \setminus M_2'. \tag{4.14}$$

As (N_1, N_2) is an index pair for (y_0, K), there is a neighborhood W_K of K in $\Sigma^+(y_0) \times X$ such that $r_{y_0}^{-1}(W_K) \subset N_1 \setminus N_2$.

If (4.14) does not hold, there is a sequence $(y_0^{t_n} \ominus u_0^{t_n}, x_n) \to (z, x_0) \in K_0'$ such that for all $n \in \mathbb{N}$ it holds that $(t_n, x_n) \notin M_1' \setminus M_2'$ and $(t_n, x_n + u_0(t_n)) \notin M_1 \setminus M_2$, which is equivalent.

Since $(y_0^{t_n}, x_n + u(t_n)) \to P(z, x_0) \in K$, it follows that $(t_n, x_n + u(t_n)) \in N_1 \setminus N_2$ for all but finitely many $n \in \mathbb{N}$. Altogether, one has $(t_n, x_n + u_n(t_n)) \in (N_1 \setminus N_2) \setminus (M_1 \setminus M_2)$ for large $n \in \mathbb{N}$.

Thus, $q'(t_n) \in \big[q_0 + k, q_0' + k\big]$ for some $k \in \mathbb{Z}$, implying that $x_0 = 0$ by the choice of K_0', whence it follows that $x_n + u_n(t_n) \in N_{1,1} \subset M_1$ for large $n \in \mathbb{N}$, so $x_n + u_n(t_n) \in M_1 \setminus M_2$, which is a contradiction. This proves (4.14).

Trivially, $M_1' \setminus M_2' \subset r_{y_0'}^{-1}(\tilde{N})$, where we set

$$\tilde{N} := W_\delta \cup \operatorname{cl}_{Y \times X} r(M_1' \setminus M_2').$$

We need to prove that $\tilde{N}$ is an isolating neighborhood for K_0'. It is clear that $\tilde{N} \subset N'$ for small $\delta > 0$, so Inv$\tilde{N} \subset$ InvN'. Suppose that $\tilde{N}$ is not an isolating neighborhood for K_0', so there is a point $(\tilde{y}, \tilde{x}) \in (\mathrm{Inv}\tilde{N}) \setminus K_0'$.

In view of Lemma 4.18, one has $\tilde{y} = (y_l \ominus u_l)^{s_0}$ for some uniquely determined $l \in \mathbb{N}$ and $s_0 \in [-\infty, \infty]$. By the choice of K_0', one must have $s_0 \in \mathbb{R}$. We can even assume that $s_0 = -2t$, so $q(y_l^{s_0}) = q_0$. $K'(\tilde{y})$ is finite by assumption, so provided $\delta > 0$ is small enough, there is a sequence $(s_n, x_n) \in M_1' \setminus M_2'$ such that

$$((y_0')^{s_n}, x_n) \to (\tilde{y}, \tilde{x}).$$

It follows from Lemma 4.18 that

$$y_0^{s_n} \to y_l^{s_0}.$$

As in the proof of Lemma 4.15, we write $t_n + k_n\pi/2 = \ln(s_n + 1)$, where $k_n \in \mathbb{N}$ and $-\pi/4 < t_n \le \pi/4$. Taking subsequences, we can assume without loss of generality that $u_{k_n} \equiv u_l$, $s_n \sin t_n \to s_0$ and $[k_n]_{\mathbb{R}/N\mathbb{Z}} = [l]_{\mathbb{R}/N\mathbb{Z}}$.

One has $(s_n, x_n + u_l(-2t)) \in M_1 \setminus M_2$ for all $n \in \mathbb{N}$, so

$$d((y_0^{s_n}, x_n + u_l(-2t)), (v_l, u_l)(-2t)) \le \varepsilon.$$

Taking $n \to \infty$, we obtain

$$d((y_l^{-2t}, \tilde{x} + u_l(-2t)), (y_l^{-2t}, 0 + u_l(-2t))) \le \varepsilon.$$

By (4.10), we must have $\tilde{x} = 0$, so $(\tilde{y}, \tilde{x}) \in K_0'$, which is a contradiction, proving that $\tilde{N}$ is an isolating neighborhood for K_0'.

The proof that (M_1', M_2') is an index pair for (y_0', K_0') is now complete, and thus

$$\mathrm{H}_q\,\mathscr{C}(y_0, K_0') \neq 0 \quad q \neq q_0.$$

By Corollary 2.22, one has

$$\mathrm{H}_*\,\mathscr{C}(y_0 \ominus u_0, R) = 0,$$

so using Theorem 3.27, one deduces that

$$\mathrm{H}_*\,\mathscr{C}(y_0 \ominus u_0, A) \simeq \mathrm{H}_*\,\mathscr{C}(y_0 \ominus u_0, K_0').$$

However, it follows immediately from Lemma 4.18 that

$$\mathrm{H}_q(\mathscr{C}(y_0 \ominus u_0, A)) \simeq 0 \quad q \neq q_0.$$

(II) In this case, there is a sequence $l(n) \in \mathbb{N}$ with $l(n) \to \infty$ such that $\omega(u_{l(n)}) \cap \alpha(u_{l(n)+1}) = \emptyset$. By the choice of $\varepsilon > 0$, that is by the choice of a sufficiently small isolating neighborhood at the beginning of the proof and as $N \ge 2$ by (H4), we can assume that $\omega(u_{l(n)})$ and $\alpha(u_{l(n)+1})$ belong to distinct connected components of

$$U_\varepsilon(K) \cap q^{-1}(\big[q_0 + l(n), 0 + l(n) + 1\big]_{\mathbb{R}/N\mathbb{Z}}) =: W_n.$$

Each connected component of W_n contains exactly one fixed point e of $\Phi_{y^\infty_{l(n)}}$. Taking subsequences, one can assume without loss of generality that $W_n \equiv W$ does not depend on n.

Since $\mathrm{H}_q[M_1, M_2] \neq 0$, it follows from Corollary 2.20 that the evolution operator Φ_{y_0} has solutions of arbitrary length. More precisely, there is a $t_0 \in \mathbb{R}^+$ and for every $T \in \mathbb{R}^+$ a solution $u' : [t_0, t_0 + T] \to X$ of Φ_{y_0} such that $(t, u'(t)) \in M_1 \setminus M_2$ for all $t \in \left[t_0', T\right]$.

By the construction of (M_1, M_2), there exist a solution u' and an $l_0 = l(n_0) \in \mathbb{N}$ such that $a_{l_0} \geq t_0$, $u'(a_{l_0}) \in U_\varepsilon((v_{l_0}, u_{l_0})(-2t))$ and $u'(a_{l_0+1}) \in U_\varepsilon(v_{l_0+1}, u_{l_0+1})(-2t))$.

However, $U_\varepsilon((v_{l_0}, u_{l_0})(-2t)) \cap W$ belongs to the same connected component of W as $\omega(v_{l_0}, u_{l_0})$ and $U_\varepsilon(v_{l_0+1}, u_{l_0+1})(-2t)) \cap W$ belongs to the same connected component of W as $\alpha(v_{l_0+1}, u_{l_0+1})$, which is a contradiction.

$\square$

CHAPTER 5

The Generic Structure of an Asymptotically Autonomous Semilinear Parabolic Equation

Let $\Omega \subset \mathbb{R}^m$, $m \geq 1$ be a bounded domain with smooth boundary. As an illustrative example for the abstract result in the following section, consider the following problem

$$\begin{aligned} &\partial_t u - \Delta u = f(t,x,u(t,x),\nabla u(t,x)) && \\ &u(t,x) = 0 && x \in \partial\Omega \\ &u(t,x) = u_0(x) && x \in \Omega \end{aligned} \tag{5.1}$$

Suppose that f is sufficiently regular and $f(t,x,u,v) \to f^{\pm}(x,u)$ as $t \to \pm\infty$ uniformly on compact subsets. Note that the limit nonlinearities $f^{\pm}$ are independent of the gradient ∇u. The limit problems

$$\begin{aligned} &\partial_t u - \Delta u = f^{\pm}(x,u(t,x)) && \\ &u(t,x) = 0 && x \in \partial\Omega \\ &u(t,x) = u_0(x) && x \in \Omega \end{aligned} \tag{5.2}$$

define local gradient-like semiflows on an appropriate Banach space $\tilde{X}$. It is well-known that for generic $f^{\pm}$, every equilibrium of (5.2) is hyperbolic. Hence, a solution $u : \mathbb{R} \to \tilde{X}$ is either an equilibrium solution or a heteroclinic connection.
It has been proved [**4**] that for a generic f the semiflow induced by

$$\begin{aligned} &\partial_t u - \Delta u = f(x,u(t,x),0) && \\ &u(t,x) = 0 && x \in \partial\Omega \\ &u(t,x) = u_0(x) && x \in \Omega \end{aligned}$$

Morse-Smale. The Morse-Smale property particularly means that:

(1) every bounded subset of $\tilde{X}$ contains only finitely many equilibria;
(2) a connection[1] between e^- and e^+ can only exist if the respective Morse-indices satisfy $m(e^+) < m(e^-)$.

In this chapter, motivated by the results of the previous chapter, an analogue to the Morse-Smale property will be proved for asymptotically autonomous semilinear parabolic equations which are asymptotically autonomous, for example (5.1). Roughly speaking, the general situation is as follows: Equilibria in the autonomous case correspond to connections between two equilibria having the same Morse-index, and every bounded set contains only finitely

[1]non-trivial, that is, except for constant solutions

many such connections. Furthermore, a connection between equilibria e^- and e^+ can only exist if $m(e^+) \leq m(e^-)$.
The proof of our results is similar to the relevant parts of [**4**], applying an abstract transverality theorem to a suitable differential operator. Compared to [**4**], the problem appears to be less involved. It is also possible to remove assumptions on the injectivity (respectively the denseness of the range) of certain linear evolutions operators. Therefore, we are able to formulate the main result, Theorem 5.4 in a rather abstract setting.
We will now apply Theorem 5.4 to the concrete problem (5.1). Let $p > m \geq 1$, $X := L^p(\Omega)$, which is reflexive, and define an operator

$$\begin{aligned} A: & \qquad W^{2,p}(\Omega) \cap W_0^1(\Omega) \to L^p(\Omega) \\ Au := & \qquad -\Delta u. \end{aligned}$$

A is a positive sectorial operator and has compact resolvent. As usual, define the fractional power space X^α as the range of $A^{-\alpha}$ equipped with the norm $\|x\|_\alpha := \|A^\alpha x\|_X$. For $\alpha < 1$ sufficiently large, the space X^α is continuously imbedded in $C^1(\bar{\Omega})$ (Lemma 2.50). Hence, f gives rise to a Nemitskii operator $\hat{f}: \mathbb{R} \times X^\alpha \to X$, where

$$\hat{f}(t,u)(x) := f(t,x,u(x),\nabla u(x)).$$

Suppose that for some $\delta > 0$:

(1) $f(t,.) \to f^\pm$ uniformly on sets of the form $\Omega \times B_\eta(0) \times B_\eta(0) \subset \Omega \times \mathbb{R} \times \mathbb{R}^m$, where $\eta > 0$ and $f^\pm: \Omega \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable in its second variable with $\partial_u f^\pm(x,u)$ being continuous,
(2) every equilibrium of (5.2) is hyperbolic,
(3) $f(t,x,.,.)$ is C^∞, and
(4) each partial derivative of $f(t,x,.,.)$ is continuous in x, bounded and Hölder-continuous in t with Hölder-exponent δ uniformly on sets of the form $\mathbb{R} \times \Omega \times B_\eta[0,\mathbb{R}] \times B_\eta[0,\mathbb{R}^m]$, $\eta > 0$.

Let $C_0^{0,\delta}(\mathbb{R} \times \bar{\Omega})$ denote the set of all in t Hölder-continuous (with exponent $\delta > 0$) functions $g: \mathbb{R} \times \Omega \to \mathbb{R}$ with $g(t,x) \to 0$ as $t \to \pm\infty$ uniformly on Ω. $C_0^{0,\delta}(\mathbb{R} \times \bar{\Omega})$ is endowed with the norm

$$\|g\| := \sup_{(t,x)\in\mathbb{R}\times\Omega} |g(t,x)| + \sup_{(t,x)\neq(t',x)\in\mathbb{R}\times\Omega} \frac{|g(t,x)-g(t',x)|}{|t-t'|^\delta}.$$

THEOREM 5.1. *There is a residual subset $Y \subset C_0^{0,\delta}(\mathbb{R} \times \bar{\Omega})$ such that for all $g \in Y$ the following holds:*
For every solution of $u: \mathbb{R} \to W^{2,p}(\Omega)$ of

$$u_t + Au = \hat{f}(t,u) + \hat{g}(t),$$

it holds that:

(1) *There are equilibria $e^\pm$ of*

$$u_t + Au = \hat{f}^\pm(u)$$

such that $u(t) \to e^\pm$ in $C(\bar{\Omega})$ as $t \to \pm\infty$.

(2) *$m(e^-) \geq m(e^+)$ and $m(e^-) = m(e^+)$ only if*

$$v_t + Av = D\hat{f}(t, u(t))v$$

does not have a non-trivial ($L^p(\Omega)$-) bounded solution.

Since there are continuous imbeddings $X^1 \subset C(\bar{\Omega}, \mathbb{R}) \subset X^0$, the above theorem follows immediately from Corollary 5.5.

5.1. Abstract Formulation of the Result

In this section, we introduce some additional notation. Subsequently, the main result, Theorem 5.4, is formulated.
Let X and Y be normed spaces and $\Omega \subset X$ be open. $\mathscr{L}(X, Y)$ is the space of all continuous linear operators $X \to Y$ endowed with the usual operator norm.
$C_B^k(\Omega, X)$ denotes the space of all k-times continuously differentiable mappings $\Omega \to X$ with bounded derivatives up to order k. The spaces endowed with the usual norm

$$\|y\| := \sup_{x\in\Omega} \max\{\|y(x)\|, \ldots, \|D^k y(x)\|\}$$

The space $C_B^{k,\delta}(\Omega, Y)$ is the subspace of $C_B^k(\Omega, Y)$ which consists of all functions in $C_B^k(\Omega, Y)$ whose k-order derivative is Hölder-continuous with exponent $\delta > 0$. In the case $\delta = 0$, we simply set $C_B^{k,0}(\Omega, Y) := C_B^k(\Omega, Y)$. On $C_B^{k,\delta}(\Omega, Y)$, we consider the norm

$$\|y\| := \|y\|_{C_B^k(\mathbb{R},X)} + \sup_{x,x'\in\Omega\, x\neq x'} \frac{\|D^k y(x) - D^k y(x')\|}{\|x - x'\|^\delta}.$$

$C_{B,0}^{k,\delta}(\mathbb{R}, Y)$ denotes the closed subspace of $C_B^{k,\delta}(\mathbb{R}, Y)$ containing all functions x with $x(t) \to 0$ as $|t| \to \infty$.
Let $\eta > 0$ and $i_\eta : B_\eta(0) \cap \Omega \to \Omega$ the inclusion mapping. Let $C_b^{k,\delta}(\Omega, Y)$ denote the set of all functions $f : \Omega \to Y$ such that $f \circ i_\eta \in C_b^{k,\delta}(\Omega \cap B_\eta(0), Y)$ for all $\eta > 0$. These spaces are equipped with an invariant metric

$$d(f, f') := d(f - f', 0) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|(f - f') \circ i_\eta\|}{1 + \|(f - f') \circ i_\eta\|}.$$

This metric induces the respective topology of uniform convergence on bounded sets, that is, $f_n \to f$ in $C_b^k(\Omega, Y)$ (resp. $C_b^{k,\delta}(\Omega, Y)$) if $f_n \circ i_\varepsilon \to f \circ i_\varepsilon$ in $C_B^{k,\delta}(\Omega \cap B_\varepsilon(0), Y)$ for every $\varepsilon > 0$.

DEFINITION 5.2. We say that an evolution operator $T(t, s)$ on a normed space X admits an *exponential dichotomy* on an interval J if there are constants $\gamma, M > 0$ and a family $(P(t))_{t\in J}$ in $\mathscr{L}(X, X)$ such that:

(1) $T(t, s)P(s) = P(t)T(t, s)$ for $t \geq s$.
(2) The restriction $T(t, s) : \mathscr{R}(P(s)) \to \mathscr{R}(P(t))$ is an isomorphism. Its inverse is denoted by $T(s, t)$, where $s < t$.
(3) $\|T(t, s)(I - P(s))\|_{\mathscr{L}(X,X)} \leq Me^{-\gamma(t-s)}$ for $t \geq s$.
(4) $\|T(t, s)P(s)\|_{\mathscr{L}(X,X)} \leq Me^{\gamma(t-s)}$ for $t < s$.

We also refer to the the family of projections as an exponential dichotomy.

DEFINITION 5.3. Let π be a semiflow on a normed space X. We say that π is *simple gradient-like* if:

(a) Every equilibrium e of π is isolated[2].
(b) For every bounded solution $u: \mathbb{R} \to X$, one has $u(t) \to e^-$ as $t \to -\infty$ and $u(t) \to e^+$ as $t \to \infty$.
(c) There is a partial order $\prec$ on the set E of all equilibria such that $e^+ \prec e^-$ whenever u satisfies (b).
(d) If u is given by (b) and $e^- = e^+$, then $u \equiv e$.

Unless otherwise stated, let X be a reflexive Banach space and A a positive sectorial operator defined on subspace $X^1 \subset X$. $X^\alpha := \mathscr{R}(A^{-\alpha})$ denotes the α-th fractional power space with the norm $\|x\|_\alpha := \|A^\alpha x\|$. We will assume that the operator A has compact resolvent.
Fix some $\delta \in]0,1[$, and let $f \in C_b^{1,\delta}(\mathbb{R} \times X^\alpha, X)$ be asymptotically autonomous, that is, there are $g^\pm \in C_b^{1,\delta}(X^\alpha, X)$ such that $f(t,.) \to g^\pm$ in $C_b^{1,\delta}(X^\alpha, X)$ as $t \to \pm\infty$. We consider solutions of

$$u_t + Au = f(t,u) \tag{5.3}$$

and its limit equations

$$u_t + Au = g^\pm(u). \tag{5.4}$$

The above equations define evolution operators (respectively semiflows in the autonomous case) on X^α.
By an equilibrium e of (5.4), we mean a point $e \in X^\alpha$ such that $u: \mathbb{R} \to X^\alpha$, $t \mapsto e$, solves (5.4). We say that an equilibrium e is *hyperbolic* if the linearized equation

$$u_t + Au = Dg^\pm(e)u$$

admits an exponential dichotomy $(P(t))_{t\in\mathbb{R}}$. The Morse-index of e is $m(e) := \dim \mathscr{R}\, P(t)$, where $t \in \mathbb{R}$ can be chosen arbitrarily.

THEOREM 5.4. *Assume that:*

(a) *Every equilibrium e of* (5.4) *is hyperbolic.*
(b) $f \in C_b(\mathbb{R} \times X^\alpha, X)$
(c) $f(t,.) \to g^\pm$ *in* $C_b^1(X^\alpha, X)$ *as* $t \to \pm\infty$
(d) $f(t,.)$ *is* C^∞ *for each* $t \in \mathbb{R}$, $D_x^k f(t,x)$ *Hölder-continuous in* t *with Hölder-exponent* δ *uniformly on sets of the form* $\mathbb{R} \times B_\eta(0) \subset \mathbb{R} \times X^\alpha$, *and* $\left\|D^k f(t,x)\right\| \le C(k, \|x\|_\alpha)$ *for all* $k \in \mathbb{N} \cup \{0\}$ *and all* $(t,x) \in \mathbb{R} \times X^\alpha$.
(e) *The semiflows induced by* (5.4) *are simple gradient-like.*

Let $\beta \in [0,1]$, *and let* $C_{B,0}^{0,\delta}(\mathbb{R}, X^\beta)$ *denote the complete subspace of all* $x \in C_B^{0,\delta}(\mathbb{R}, X^\beta)$ *with* $\|x(t)\|_\alpha \to 0$ *as* $|t| \to \infty$. *Then, for a generic*[3] $g \in C_{B,0}^{0,\delta}(\mathbb{R}, X^\beta)$, *every bounded solution* $u: \mathbb{R} \to X^\alpha$ *of*

$$u_t + Au = f(t,u) + g(t) \tag{5.5}$$

satisfies:

[2] The term *simple* refers to this hypothesis.
[3] i.e., there is a residual subset of $C_B^{0,\delta}(\mathbb{R}, X^\beta)$ such that all g in this subset have the stated property

(1) *There are equilibria e^-, e^+ of the respective limit equation* (5.4) *such that* $\left\|u(t)-e^-\right\|_\alpha \to 0$ *as* $t \to -\infty$ *and* $\left\|u(t)-e^+\right\|_\alpha \to 0$ *as* $t \to \infty$.

(2) $m(e^+) \le m(e^-)$ *and* $m(e^-) = m(e^+)$ *only if the linear equation*

$$v_t + Av = Df(t, u(t))v \tag{5.6}$$

does not have a non-trivial bounded solution $v : \mathbb{R} \to X^\alpha$.

Note that (2) is equivalent to the existence of an exponential dichotomy for (5.6) (cf. the proof of Lemma 5.15).

PROOF. (1) Since the limit equations (5.4) are simple gradient-like, this is a consequence of Lemma 5.7.

(2) This follows from Theorem 5.10 together with Lemma 5.24.

□

COROLLARY 5.5. *Let E be a normed space such that $X^1 \subset E \subset X^0$, the inclusions being continuous.*
Moreover, assume the hypotheses of Theorem 5.4. Then the conclusions of Theorem 5.4 hold for a generic $g \in C^{0,\delta}_{B,0}(\mathbb{R}, E)$.

PROOF. Let Y denote the set of all $g \in C^{0,\delta}_{B,0}(\mathbb{R}, X)$ such that 0 is a regular value of $\Phi(., g)$. It follows from Theorem 5.10 that $Y = \bigcap_{n\in\mathbb{N}} Y_n$, where each Y_n is open and dense in $C^{0,\delta}_{B,0}(\mathbb{R}, X)$. A second application of Theorem 5.10 proves that $Y \cap C^{0,\delta}_{B,0}(\mathbb{R}, X^1)$ is dense in $C^{0,\delta}_{B,0}(\mathbb{R}, X)$. By the continuity of the inclusions, each of the sets $Y_n \cap C^{0,\delta}_{B,0}(\mathbb{R}, E)$ is open in $C^{0,\delta}_{B,0}(\mathbb{R}, E)$. Moreover, $Y \cap C^{0,\delta}_{B,0}(\mathbb{R}, X^1)$ is a dense subset of each $Y_n \cap C_{B,0}(\mathbb{R}, E)$, which proves that $\bigcap_{n\in\mathbb{N}}(Y_n \cap C_{B,0}(\mathbb{R}, E)) = Y \cap C_{B,0}(\mathbb{R}, E)$ is residual. □

5.2. A Skew Product Semiflow and Convergence of Solutions

Let $Y \subset C_b(\mathbb{R} \times X^\alpha, X)$ denote the subspace, that is, equipped with a metric of convergence uniformly on bounded sets, of all functions $f : \mathbb{R} \times X^\alpha \to X$ such that:

(1) $f(t, .) \in C^1_b(X^\alpha, X)$ for all $t \in \mathbb{R}$
(2) $t \mapsto f(t, .)$ is a Hölder-continuous function $\mathbb{R} \to C^1_b(X^\alpha, X)$

The above assumptions are rather strong, but we do not strive for maximum generality here. It is easy to prove

LEMMA 5.6. *For every $f \in Y$, the translation $t \mapsto f^t(s, x) := f(t + s, x)$, $\mathbb{R} \to Y$ is continuous.*

Let $Y_0 \subset Y$ be a compact subspace of Y which is invariant with respect to translations. We consider solutions of the semilinear parabolic equation

$$\dot{u} + Au = y(t, u). \tag{5.7}$$

These induce a skew-product semiflow $\pi := \pi_{Y_0}$ on $Y_0 \times X^\alpha$, where we set $(y, x)\pi t := (y^t, u(t))$ if there exits a solution $u : [0, t] \to X^\alpha$ of (5.7) with $u(0) = x$. It follows[4] from [**26**, Theorem 47.5] that π is continuous.

[4]e.g.

Now suppose that $y^t \to y^-$ as $t \to -\infty$ and $y^t \to y^+$ as $t \to \infty$, where $y^-, y^+ \in Y$ are autonomous. It is easily seen that the set $Y_0 := \mathrm{cl}_Y\{y^t : t \in \mathbb{R}\} = \{y^t : t \in \mathbb{R}\} \cup \{y^-, y^+\}$ is compact. Moreover for y^+ (resp. y^-), (5.7) defines a semiflow on X^α, which is denoted by χ_{y^+} (resp. χ_{y^-}).
It is easy to see that the two lemmas still hold true in a more general setting, replacing the boundedness in X^α by an asymptotic convergence assumption, admissibility [**22**] for example.

LEMMA 5.7. *Assume that χ_{y^+} (resp. χ_{y^-}) is simple gradient-like*[5]*, and let $u : \mathbb{R} \to X^\alpha$ be a bounded solution of* (5.7)*. Then, $u(t)$ converges to an equilibrium of χ_{y^+} (resp. χ_{y^-}) as $t \to \infty$ (resp. $t \to -\infty$).*

PROOF. We consider only the case $t \to \infty$ because $t \to -\infty$ can be treated analogously. Suppose to the contrary that $N \subset X^\alpha$ is bounded and $u : \mathbb{R} \to Y_0 \times N$ is a solution with $\omega(u) \neq \{(y^+, e_0)\}$, where e_0 denotes a minimal equilibrium in $\{x : (y^+, x) \in \omega(u)\}$. The minimality refers to the partial order $\prec$ introduced in Definition 5.3.
Let $E \subset \{y^+\} \times X^\alpha$ denote set set of all equilibria in $\omega(u)$. Pick an $\varepsilon > 0$ such that $B_\varepsilon[(y^+, e_0)] \cap E = \{(y^+, e_0)\}$ and a sequence $t_n \to \infty$ with $u(t_n) \to (y^+, e_0)$. There are $s_n \geq t_n$ such that $d(u(s_n), (y^+, e_0)) = \varepsilon$ and $u([t_n, s_n]) \subset B_\varepsilon\big[(y^+, e_0)\big]$.
We claim that $|t_n - s_n| \to \infty$ as $n \to \infty$. Otherwise, we may assume without loss of generality that $r_n := |t_n - s_n| \to r_0$. The continuity of the semiflow implies that $\partial B_\varepsilon[(y^+, e_0)] \ni u(s_n) \to (y^+, e_0)\pi r_0$, which is a contradiction. Choosing a subsequence $(s'_n)_n$ of $(s_n)_n$, we can assume that $u(s'_n) \to (y^+, x_0) \in \partial B_\varepsilon[(y^+, e_0)] \cap \mathrm{Inv}^-_\pi(B_\varepsilon[(y^+, e_0)])$. Since χ_{y^+} is simple gradient-like, one has $(y^+, x_0)\pi t \to (y^+, e)$ as $t \to \infty$ for some $e \in E$, so $e \prec e_0$, in contradiction to the minimality of e_0. $\square$

5.3. Surjectivity

We consider the following (Banach) spaces:

$$\mathscr{X} := C_B^{1,\delta}(\mathbb{R}, X) \cap C_B^{0,\delta}(\mathbb{R}, X^1)$$
$$\mathscr{Y} := C_{B,0}^{0,\delta}(\mathbb{R}, X^\beta) := \{y \in C_B^{0,\delta}(\mathbb{R}, X^\beta) : y(t) \to 0 \text{ as } t \to \pm\infty\} \quad 0 \leq \beta \leq 1$$
$$\mathscr{Z} := C_B^{0,\delta}(\mathbb{R}, X).$$

Here, we choose $\|x\|_{\mathscr{X}} := \|x\|_{C_B^{1,\delta}(\mathbb{R},X)} + \|x\|_{C_B^{0,\delta}(\mathbb{R},X^1)}$.
A function $f : \mathbb{R} \times X^\alpha \to X^0$ gives rise to a Nemytskii operator $\hat{f}$ defined by

$$\hat{f}(u)(t) := f(t, u(t)).$$

LEMMA 5.8. *Under the hypotheses (b), (c) and (d) of Theorem 5.4, $\hat{f}$ maps bounded Hölder-continuous functions to bounded Hölder-continuous functions, that is, $\hat{f}(C_B^{0,\delta}(\mathbb{R}, X^\alpha)) \subset C_B^{0,\delta}(\mathbb{R}, X^0)$*

LEMMA 5.9. *Under the hypotheses (b) and (d) of Theorem 5.4, $\hat{f} : C_B^{0,\delta}(\mathbb{R}, X^\alpha) \to C_B^{0,\delta}(\mathbb{R}, X)$ as defined above is C^∞.*

[5]according to Definition 5.3 equilibria are isolated

PROOF. Suppose that $u, u', v \in C_B^{0,\delta}(\mathbb{R}, X^\alpha)$ satisfy $\|u\|, \left\|u'\right\| \le M$, and set $B(t) := \mathrm{D}_x f(t, u(t))$.
For arbitrary $v \in C^{0,\delta}(\mathbb{R}, X^\alpha)$ and $t, s \in \mathbb{R}^+$, one has the estimates

$$\begin{aligned} \|B(t)v(t)\|_0 &\le C_1(M)\|v\|_{C_B(\mathbb{R},X^\alpha)} \\ \|B(t+s)v(t+s) - B(t)v(t)\|_0 &\le C_2(M)s^\delta\|v\|_{C_B(\mathbb{R},X^\alpha)} + C_1(M)s^\delta\|v\|_{C_B^{0,\delta}(\mathbb{R},X^\alpha)}, \end{aligned}$$

where $C_1(M), C_2(M)$ are derived from the assumptions on $\mathrm{D}_x f$.
Now, set $B(t, y) := \mathrm{D}_x f(t, u(t) + y) - \mathrm{D}_x f(t, u(t))$. We have

$$\begin{aligned} B(t,y) &= \int_0^1 \mathrm{D}_x^2 f(t, x+\lambda y)y \, \mathrm{d}\lambda \\ B(t,y_1) - B(t,y_2) &= \big(\mathrm{D}_x f(t, x+y_1) - \mathrm{D}_x f(t,x)\big) - \big(\mathrm{D}_x f(t, x+y_2) - \mathrm{D}_x f(t,x)\big) \\ &= \int_0^1 \mathrm{D}_x^2 f(t, x+y_2+\lambda(y_1-y_2))(y_1-y_2)\, \mathrm{d}\lambda, \\ B(t+s,y) - B(t,y) &= \big(\mathrm{D}_x f(t+s, x+y) - \mathrm{D}_x f(t+s,x)\big) - \big(\mathrm{D}_x f(t, x+y) - \mathrm{D}_x f(t,x)\big) \\ &= \int_0^1 \big[\mathrm{D}_x^2 f(t+s, x+\lambda y) - \mathrm{D}_x^2(t, x+\lambda y)\big]\, y \, \mathrm{d}\lambda, \end{aligned}$$

so

$$\begin{aligned} \left\|B(t,y)z\right\|_0 &\le C_3(M)\left\|y\right\|_\alpha \|z\|_\alpha \\ \left\|B(t+s,y_2)z_2 - B(t,y_1)z_1\right\|_0 &\le C_3(M)\big(\left\|y_2\right\|\|z_2 - z_1\|_\alpha \\ &\quad + \left\|y_2 - y_1\right\|_\alpha \|z_1\|_\alpha + s^\delta \left\|y_1\right\|_\alpha \|z_1\|_\alpha\big) \end{aligned}$$

whence it follows that $[D\hat{f}(u)]v(t) := Df(t, u(t))v(t)$ satisfies

$$\left\|\mathrm{D}\hat{f}(u+u') - \mathrm{D}\hat{f}(u)\right\|_{\mathscr{L}(C_B^{0,\delta}(\mathbb{R},X^\alpha), C_B^{0,\delta}(\mathbb{R},X))} \le C_3(M)\left\|u'\right\|_{C_B^{0,\delta}(\mathbb{R},X^\alpha)}. \tag{5.8}$$

In particular, one has

$$D\hat{f} \in C_b\big(C_B^{0,\delta}(\mathbb{R}, X^\alpha),\ \mathscr{L}(C_B^{0,\delta}(\mathbb{R}, X^\alpha), C_B^{0,\delta}(\mathbb{R}, X))\big).$$

Moreover,

$$f(t, x+y) = f(t,x) + \mathrm{D}f(t,x)y + \int_0^1 \big(\mathrm{D}f(t, x+\lambda y) - \mathrm{D}f(t,x)\big)\, y \ \mathrm{d}\lambda,$$

so

$$\hat{f}(u+u') - \hat{f}(u) - \mathrm{D}\hat{f}(u)u' = \int_0^1 \big(\mathrm{D}\hat{f}(u+\lambda u') - \mathrm{D}\hat{f}(u)\big)\, u' \ \mathrm{d}\lambda,$$

and by (5.8),

$$\left\|\hat{f}(u+u') - \hat{f}(u) - \mathrm{D}\hat{f}(u)u'\right\| \le C_3(M)\left\|u'\right\|^2,$$

which proves that $\hat{f}$ is continuously differentiable and $D\hat{f}$ as defined above is indeed the derivative. The higher derivatives can be treated analogously. □

Define $\Phi := \Phi_f : \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ by

$$\Phi(u,v)(s) := u_t(s) + Au(s) - f(s, u(s)) - v(s).$$

Φ is continuous by the choice of $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$.

Recall that a subset of a topological space is nowhere dense if the interior of its closure is empty. A countable union of nowhere dense sets is called meager and the complement of a meager set residual. The following theorem is the main result of this section.

THEOREM 5.10. *Under the hypotheses of Theorem 5.4, the set of all $y \in \mathscr{Y}$ such that* 0 *is a regular*[6] *value of* $\Phi(.,y)$ *is residual (in* $\mathscr{Y}$*).*

In order to prove Theorem 5.10, we need to check the premises of the following theorem, which is a simplified version of [**4**, Theorem 2.1] (see also [**13**, Theorem 5.4]).

THEOREM 5.11. *Let X, Y, Z be open subsets of Banach spaces, r a positive integer, and $\Phi : X \times Y \to Z$ a C^r map. Assume that the following hypotheses are satisfied:*

(1) *For each $(x,y) \in \Phi^{-1}(\{0\})$, $D_x\Phi(x,y) : X \to Z$ is a Fredholm operator of index less than r.*
(2) *For each $(x,y) \in \Phi^{-1}(\{0\})$ $D\Phi(x,y) : X \times Y \to Z$ is surjective.*
(3) *The projection $p : (x,y) \mapsto y : \Phi^{-1}(\{0\}) \to Y$ is σ-proper, that is, there is a countable system of subsets $V_n \subset \Phi^{-1}(\{0\})$ such that $\bigcup_{n\in\mathbb{N}} V_n = \Phi^{-1}(\{0\})$ and for each $n \in \mathbb{N}$ the restriction $p_n : V_n \cap \Phi^{-1}(\{0\}) \to Y$ of p is proper.*

Then the set of all $y \in Y$ such that 0 *is a regular value of* $\Phi(.,y)$ *is residual in Y.*

Using Lemma 5.9, it is easy to see that Φ is C^∞. In particular, we have

$$D\Phi(u_0, v_0)(u, v) = u_t + Au - D\hat{f}(u_0)u - v.$$

Now, suppose that $\Phi(u_0, v_0) = 0$, that is, u_0 is a (classical) solution of

$$u_t + Au = \hat{f}(u) + v_0.$$

Under the assumptions of Theorem 5.10, it follows from Lemma 5.7 that $u(t)$ converges to a (hyperbolic) equilibrium $e^\pm$ of the respective limit equation as $t \to \pm\infty$.

PROOF OF THEOREM 5.10. Initially, define

$$\mathscr{X}_n := \{x \in \mathscr{X} : \|x(t)\|_\alpha < n \text{ for all } t \in \mathbb{R}\} \quad n \in \mathbb{N}.$$

It is clear that $\mathscr{X} = \bigcup_{n\in\mathbb{N}} \mathscr{X}_n$.

Since each equilibrium of (5.4) is hyperbolic, there are only finitely many equilibria e with $\|e\|_\alpha \le n$. Hence, there is an $m \in \mathbb{N}$ such that $m(e) \le m$ whenever e is an equilibrium of (5.4) with $\|e\|_\alpha \le n$.

Furthermore, there is an $\varepsilon = \varepsilon(n) > 0$ such that $\left\|e - e'\right\|_\alpha > \varepsilon$ for every pair (e, e') of equilibria with $\|e\|_\alpha \le n$ and $\left\|e'\right\|_\alpha \le n$. Define

$$\mathscr{X}_{n,m} := \{x \in \mathscr{X}_n : x(t) \in \bigcup_e B_\varepsilon[e] \text{ for } |t| \ge m\},$$

where the union is taken over all equilibria e with $\|e\|_\alpha \le n$.

Let $(u_0, y_0) \in \mathscr{X}_n \times \mathscr{Y}$ be a solution of $\Phi(u_0, y_0) = 0$. By our assumptions and [**4**, Lemma 4.a.11], assumption (CH) in Lemma 5.13 is satisfied. Hence, it follows from Theorem 5.16 and Lemma

[6] $D_x\Phi(x_0, y) : \mathscr{X} \to \mathscr{Z}$ is surjective whenever $\Phi(x_0, y) = 0$

5.18 that for every solution $u_0 \in \mathscr{X}_n$, $D_x\Phi(u_0, y_0): \mathscr{X} \to \mathscr{Z}$ is a Fredholm operator and its (Fredholm) index is bounded by m. Furthermore,

$$L(u,v) := u_t + Au - D_u f(t,u_0)u + v \tag{5.9}$$

defines a surjective operator $\mathscr{X} \times W \to \mathscr{Z}$, where $W = \text{span}\{w_1,\dots,w_m\}$ and $w_1,\dots,w_m \in \mathscr{Y}$ have compact support.

In order to apply Theorem 5.11, we need to show that the map $(x,y) \mapsto y : \Phi^{-1}(\{0\}) \to \mathscr{Y}$ is σ-proper, that is, there is a family $(V_n)_n$ with $\Phi^{-1}(\{0\}) = \bigcup_{n\in\mathbb{N}} V_n$ such that for each $n \in \mathbb{N}$ the map

$$(x,y) \mapsto y : V_n \to \mathscr{Y} \tag{5.10}$$

is proper.

Let $(x,y) \in \Phi^{-1}(\{0\})$ with $x \in \mathscr{X}_n$. Since $y(t) \to 0$ as $t \to \pm\infty$, x converges to an equilibrium as $t \to \pm\infty$ (Lemma 5.7). Hence,

$$\Phi^{-1}(\{0\}) = \bigcup_{(n,m)\in\mathbb{N}\times\mathbb{N}} \Big(\underbrace{\Phi^{-1}(\{0\}) \cap (\mathscr{X}_{n,m} \times \mathscr{Y})}_{=:V_{n,m}}\Big).$$

Let (x_n, y_n) be a sequence in $V_{n,m}$ with $y_n \to y_0$ in $\mathscr{Y}$. Using the compactness of the evolution operators on X^α defined by

$$u_t + Au = f(t,u) + y \qquad y \in \mathscr{Y},$$

it follows that there is a solution $x_0 : \mathbb{R} \to X^\alpha$ and a subsequence $(x_n')_n$ such that $x_n' \to x_0$ uniformly on bounded sets. Suppose that the convergence is not uniform with respect to $t \in \mathbb{R}$. In this case, there are a subsequence $(x_n'')_n$, a sequence $(t_n)_n$ and an $\eta > 0$ such that $\left\|x_n''(t_n) - x_0(t_n)\right\| \geq \eta$ for all $n \in \mathbb{N}$. Moreover, we can assume without loss of generality that $t_n \to \infty$ or $t_n \to -\infty$.

By the choice of $V_{n,m}$, there are equilibria $e^\pm$ with $x_n''(t) \in B_\varepsilon(e^\pm)$ for all t with $|t| \geq m$. Hence, one has $x_0(t) \in B_\varepsilon[e^\pm]$ for $|t| \geq m$. Using assumption (c) of Theorem 5.4 and [**26**, Theorem 47.5], it follows that there is a solution $u : \mathbb{R} \to B_\varepsilon[e]$ (either $e = e^+$ or $e = e^-$) of one of the limit equations such that $\|u(0) - e\|_\alpha \geq \eta > 0$. We can assume without loss of generality that $B_\varepsilon[e]$ is an isolating neighborhood for e, which means that $u \equiv e$. This is an obvious contradiction, so

$$\sup_{t\in\mathbb{R}} \|x_n(t) - x_0(t)\|_\alpha \to 0 \text{ as } n \to \infty.$$

By [**4**, Lemma 4.a.6], one has $x_n \to x_0 \in \mathscr{X}$, which proves that the map defined by (5.10) is proper.

Now, it follows from Theorem 5.11 that there is a residual subset $\mathscr{Y}_n \subset \mathscr{Y}$ such that for every $y \in \mathscr{Y}_n$, 0 is a regular value of $\Phi(.,y) : \mathscr{X} \to \mathscr{Z}$.

This completes the proof since a countable intersection of residual sets is residual. □

LEMMA 5.12. *For every $F \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ with $\det F > 0$, there is an $\hat{F} \in C^\infty([0,1], \mathscr{L}(\mathbb{R}^n,\mathbb{R}^n))$ such that $\hat{F}(0) = \text{id}$, $\hat{F}(1) = F$, and $\det F(t) > 0$ for all $t \in [0,1]$.*

The proof is omitted.

LEMMA 5.13. *Suppose that:*

(CH) *$B \in C^{0,\delta}(\mathbb{R}, \mathscr{L}(X^\alpha, X))$ with $B(t) \to B^+$ as $t \to \infty$ and $B(t) \to B^-$ as $t \to -\infty$. There further exists an $m^+ \in \mathbb{N}$ (resp. m^-) such that the evolution operator defined by solutions of*

$$u_t + Au = B^+ u \text{ (resp. } B^- u) \tag{5.11}$$

admits an exponential dichotomy P defined for $t \in \mathbb{R}^+$ (resp. $t \in \mathbb{R}^-$) with $|t|$ large and $\dim \mathscr{R}(P) = m^+$ (resp. m^-).

If $m^- = m^+ =: m$, then there exist $t_1 \leq t_2$ and an $R \in C^\infty([t_1, t_2], \mathscr{L}(X^\alpha, X))$ such that there does not exist a bounded non-trivial (mild) solution of

$$u_t + Au = B(t)u + \begin{cases} R(t)u & t \in [t_1, t_2] \\ 0 & \textit{otherwise.} \end{cases} \tag{5.12}$$

LEMMA 5.14. *Suppose that A is a positive sectorial operator having compact resolvent. Let $X_1 \subset X^1 = \mathscr{D}(A)$ be an arbitrary finite-dimensional subspace.*
Then, there are a closed subspace $X_2 \subset X$ and $B' \in \mathscr{L}(X, X)$ such that $X = X_1 \oplus X_2$, $(A - B')x = 0$ for all $x \in X_1$, and $(A - B')x \in X_2$ for all $x \in X_2 \cap \mathscr{D}(A)$.

PROOF. The claim is trivial for $X_1 = \{0\}$, so we will assume that $X_1 \neq \{0\}$.
Let $P \in \mathscr{L}(X, X_1)$ denote an otherwise arbitrary projection, and let $R(\mu, A) \in \mathscr{L}(X, X)$ denote the resolvent of $A + \mu I$. We have [**20**, Theorem 5.2 in Chapter 2]

$$\left\| R(\mu, A) \right\| \leq \frac{M}{|\mu|},$$

so every real $\mu > 0$ sufficiently large is in the resolvent set of

$$A + \mu I - AP = (A + \mu I)(I - R(\mu, A)AP). \tag{5.13}$$

Moreover, the resolvent $R'(\mu)$ of (5.13) is compact, and $\frac{1}{\mu}$ is an eigenvalue of $R'(\mu)$. Let $X = X_1' \oplus X_2'$ be the associated decomposition of X, where $X_1' \supset X_1$ is the generalized eigenspace associated with $\frac{1}{\mu}$ and X_2' is $R'(\mu)$ invariant.
Finally, let $Q \in \mathscr{L}(X, X_1')$ denote the projection with kernel X_2'. The operator

$$A - \underbrace{(AP + A(I - P)Q)}_{=: B'}$$

vanishes on X_1'. Let C satisfy the relation $X_1' = X_1 \oplus C$, and set $X_2 := C \oplus X_2'$. □

PROOF OF LEMMA 5.13. Let the evolution operator $T(t, s)$ be defined by

$$u_t + Au = B(t)u,$$

and consider the bundles

$$\begin{aligned} \mathscr{U} &:= \{(s, x) \in \mathbb{R} \times X^\alpha : \text{ there exists a solution } u : \mathbb{R}^- \to X \text{ with } u(s) = x \\ &\qquad \text{and } \sup_{t \in \mathbb{R}^-} \|u(t)\|_\alpha < \infty\} \\ \mathscr{S} &:= \{(s, x) \in \mathbb{R} \times X^\alpha : \sup_{t \in \mathbb{R}^+} \|T(t, s)x\|_\alpha < \infty\}. \end{aligned}$$

$\mathscr{U}$ and $\mathscr{S}$ are positively invariant, that is, $(s,x) \in \mathscr{U}$ (resp. $\mathscr{S}$) implies $(t, T(t,s)x) \in \mathscr{U}$ (resp. $\mathscr{S}$) for all $t \geq s$.
It is well-known that, for small $t \in \mathbb{R}$ (resp. large $t \in \mathbb{R}$), $\dim \mathscr{U}(t) = m$ and $\operatorname{codim} \mathscr{S}(t) = m$ (see for instance [**4**, Lemma 4.a.11]). Choose $t_1 < t_2$ such that $\dim \mathscr{U}(t) = m$ for all $t \leq t_1$ and $\operatorname{codim} \mathscr{S}(t) = m$ for all $t \geq t_2$.
Let $X = \mathscr{S}(t_2) \oplus C_{\mathscr{S}}$, $X_1 := \mathscr{U}(t_1) + C_{\mathscr{S}}$, and $X = X_1 \oplus X_2$ with $X_2 \subset \mathscr{S}(t_2)$. For $t \geq s \geq t_2$, the evolution operator $T(t,s)$ induces an isomorphism $X/\mathscr{S}(s) \to X/\mathscr{S}(t)$, so $X = T(t,t_2)C_{\mathscr{S}} \oplus \mathscr{S}(t+t_2)$ for every $t \in \mathbb{R}^+$. By standard regularity results and choosing t_2 larger if necessary, we can thus assume without loss of generality that $C_{\mathscr{S}} \subset X^1$ so that $X_1 = \mathscr{U}(t_1) + C_{\mathscr{S}} \subset X^1$.
Let $F: X_1 \to X_1$ be a linear endomorphism with $\det F > 0$ which takes $\mathscr{U}(t_1)$ to $C_{\mathscr{S}}$, let $\hat{F}$ be given by Lemma 5.12, and set $G(t_1 + \xi(t_2 - t_1)) := \hat{F}(\xi)$ for $\xi \in [0,1]$. Let B' be defined by Lemma 5.14, and let $X = X_1 \oplus \tilde{X}_2$ with an $(A - B')$-invariant complement $\tilde{X}_2$. $\tilde{P} \in \mathscr{L}(X, X_1)$ denotes the projection along $\tilde{X}_2$. Consider the semigroup $S(t)$ defined by

$$\dot{u} + Au = B'u.$$

We can now define the modified evolution operator $\hat{T}(t,s)$ by

$$\hat{T}(t,s)(x) := \begin{cases} G(t)G(s)^{-1}x & x \in X_1 \text{ and } [s,t] \subset [t_1,t_2] \\ S(t-s)x & x \in \tilde{X}_2 \text{ and } [s,t] \subset [t_1,t_2] \\ T(t,s)x & [s,t] \cap [t_1,t_2] = \emptyset. \end{cases}$$

One has $\hat{T}(t_2,t_1)x = F(x)$ for all $x \in \mathscr{U}(t_1)$, so $\hat{T}(t_2,t_1)\mathscr{U}(t_1) \subset C_{\mathscr{S}}$, which proves that there does not exist a full bounded solution of $\hat{T}$.
Assume that u is a solution of $\hat{T}$ defined for $t \in]a,b[\subset [t_1,t_2]$. We have

$$\begin{aligned} \tilde{P}u_t &= \underbrace{(-A + B')\tilde{P}u}_{=0} + G_t(t)G(t)^{-1}\tilde{P}u \qquad (5.14) \\ (1-\tilde{P})u_t &= (-A + B')(1-\tilde{P})u, \end{aligned}$$

so every solution of $\hat{T}(t,s)$ is also a solution of 5.12, where

$$R(t) := B' + G_t(t)G(t)^{-1}\tilde{P} - B(t)$$

is obtained by comparison with (5.14). □

LEMMA 5.15. *Let $B \in L^\infty(\mathbb{R}, \mathscr{L}(X^\alpha, X))$ with $B(t) \to B^+$ as $t \to \infty$ and $B(t) \to B^-$ as $t \to -\infty$. Assume there exists an $m \in \mathbb{N}$ such that each of the evolution operators defined by solutions of*

$$u_t + Au = B^+u \text{ (resp. } B^-u)$$

admits an exponential dichotomy P defined for $t \in \mathbb{R}^+$ (resp. $t \in \mathbb{R}^-$) with $|t|$ large and $\dim \mathscr{R}(P) = m$.
Moreover, suppose that the only bounded mild solution $u: \mathbb{R} \to X^\alpha$ of

$$u_t + Au = B(t)u \qquad (5.15)$$

is $u \equiv 0$.

Then, for every $h \in L^\infty(\mathbb{R}, X)$, there is a unique mild solution $u_0 \in C_B(\mathbb{R}, X^\alpha)$ of

$$u_t + Au = B(t)u + h. \tag{5.16}$$

PROOF. It follows from [**26**, Theorem 44.3] that (5.16) generates a skew-product semiflow on a suitable phase space $W \times X^\alpha$, where $W := \mathrm{cl}\{B(t) : t \in \mathbb{R}\}$, p is a sufficiently large integer, and the closure is taken in $L^p_{\mathrm{loc}}(\mathbb{R}, \mathscr{L}(X^\alpha, X))$. Note that $W = \{\hat{B}^-, \hat{B}^+\} \cup \{B(t) : t \in \mathbb{R}\}$, where $\hat{B}^\pm(t) \equiv B^\pm$.
Now [**23**, Theorem C] implies that the evolution operator $T(t,s)$ defined by mild solutions of (5.15) admits an exponential dichotomy. Our claim follows using the same formula as [**12**, Theorem 7.6.3] (see also [**4**, Lemma 4.a.7] and [**4**, Lemma 4.a.8]). □

THEOREM 5.16. *Suppose that (CH) holds, and let $m := \max\{m^-, m^+\}$. Then there are $t_1 \le t_2$ and $w_1, \dots, w_m \in \mathscr{Y}$ with compact support such that $\tilde{L} : \mathscr{X} + \mathrm{span}\{w_1, \dots, w_m\} \to \mathscr{Z}$*

$$\tilde{L}(u, w) := L(u, w) = u_t + Au - B(t)u - w$$

is surjective.

PROOF. Consider the spaces

$$X' := \mathbb{R}^{|m^- - m^+|} \times X$$
$$(X')^\alpha := \mathbb{R}^{|m^- - m^+|} \times X^\alpha$$

and

$$\mathscr{X}' := C^{1,\delta}_B(\mathbb{R}, X') \cap C^{0,\delta}_B(\mathbb{R}, (X')^1)$$
$$\mathscr{Z}' := C^{0,\delta}_B(\mathbb{R}, X').$$

We define an operator $L' : \mathscr{X}' \to \mathscr{Z}'$, where

$$L'(x, u) := (x_t \pm \arctan(t)x, u_t + Au - B(t)u).$$

Choosing the sign in front of arctan appropriately, reduces the problem to the case where $m^- = m^+$. For the sake of simplicity, we will henceforth assume that $m^- = m^+$.
By Lemma 5.13, there are $t_1 \le t_2$ and $R \in C^\infty([t_1, t_2], \mathscr{L}(X^\alpha, X))$ such that

$$u_t + Au - B(t)u = \begin{cases} R(t)u & t \in [t_1, t_2] \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

does not have a non-trivial bounded solution. The evolution operator $T(t,s)$ defined by (5.17) has an exponentially stable subspace of finite codimension for $t \ge s \ge t_2$, that is, $X = X_1 \oplus X_2$ with $\mathrm{codim}\, X_2 = m^+$ and for some $M, \delta > 0$

$$\|T(t, t_2)x\|_\alpha \le M e^{-\delta(t-s)} \|x\|_\alpha \quad t \ge t_2. \tag{5.18}$$

Suppose that $\tilde{X} := T(t_2, t_1)X^1 \subset X^1$, $\tilde{X}_2 := \tilde{X} \cap X_2$ and $\tilde{X} = \tilde{X}_1 \oplus \tilde{X}_2$.

SUBLEMMA 5.17. *For every $\eta \in \tilde{X}_1$, there is a $w \in \mathscr{Y}$ and a solution $v : [t_1, t_2] \to X^\alpha$ of*

$$u_t + Au = B(t)u + w$$

with $v(t_1)=0$ and $v(t_2)=\eta$.

PROOF. Let $u:\ [t_1,t_2]$ be a solution of $T(t,s)$ with $u(t_2)=\eta\neq 0$. Note that, by standard regularity results, e.g. [**4**, Lemma 4.a.6], one has $u\in C^{0,\delta}([t_1,t_2],X^1)$. We have $u(t)\neq 0$ for all $t\in[t_1,t_2]$, so there is a one-dimensional $T(t,s)$-invariant subbundle spanned by $u(t)$. It can be described by

$$\varphi(t,x):=u(t)\cdot x \qquad t\in[t_1,t_2]\ x\in\mathbb{R}.$$

Let $x:\ \mathbb{R}\to\mathbb{R}$ be C^∞ with $x(t)=0$ for $t\le t_1$ and $x(t)=1$ for $t\ge t_2$.
Setting $v(t):=\varphi(t,x(t))=u(t)\cdot x(t)$, one has

$$\begin{aligned} v_t(s) &= \varphi_t(s,x(s))+\varphi_x(s,x(s))x_t(s)\\ &= (-A+B(s))\underbrace{u(s)x(s)}_{=v(s)}+\underbrace{u(s)x_t(s)}_{=:w(s)} \end{aligned}$$

and

$$\begin{aligned} v(t_1) &= x(t_1)\cdot\eta=0\\ v(t_2) &= x(t_2)\cdot\eta=\eta \end{aligned}$$

as claimed. □

Let $\eta_1,\dots,\eta_n$ be a basis for $\tilde X_1$, and choose $w_1,\dots,w_n$ and $v_1,\dots,v_n$ according to Sublemma 5.17. It follows from Lemma 5.15 that for every $h\in\mathscr{X}$, there exists a unique mild solution $u_0\in C_B(\mathbb{R},X^\alpha)$ of

$$u_t+Au-B(t)u=R(t)u+h. \tag{5.19}$$

Let $v_1:\ [t_1,\infty[\ \to X^\alpha$ denote the solution of

$$v_t+Av-B(t)v=R(t)u_0 \quad v(t_1)=0,$$

and let $v_1(t_2)=\eta\oplus\eta'\in\tilde X_1\oplus\tilde X_2$.
There is a $w_0\in\operatorname{span}\{w_1,\dots,w_n\}$ such that the solution $v_2:\ [t_1,\infty[\ \to X^\alpha$ of

$$v_t+Av-B(t)v=-w_0 \quad v(t_1)=0$$

satisfies $v_2(t_2)=\eta$.
It follows that $v_0:=v_1-v_2$ is a solution of

$$v_t+Av-B(t)v=R(t)u_0+w_0 \quad v(t_1)=0$$

with $v_0(t_2)\in\tilde X_2\subset X_2$.
Using (5.18), one concluces that $\sup_{t\in\mathbb{R}}\|v_0(t)\|_\alpha<\infty$. Furthermore, u_0-v_0 is a bounded mild solution of

$$u_t+Au-B(t)u-w_0=h,$$

so by [**4**, Lemma 4.a.6], one has $u_0-v_0\in\mathscr{X}$ and thus $L(u_0-v_0,w_0)=h$, which completes the proof of Theorem 5.16. □

LEMMA 5.18. *Suppose that A is a sectorial operator having compact resolvent and B satisfies (CH). Let the operator $L:=L_B$ be defined by*

$$L_Bu:=u_t+Au-B(t)u$$

Then $\dim \mathcal{N}(L_B) \le m^-$.

PROOF. This is an immediate consequence of the existence of an exponentional dichotomy on an interval $]-\infty, t_0]$ for small t_0, which follows from [**4**, Lemma 4.a.11]. □

5.4. Adjoint Equations

Throughout this section, suppose that X is a reflexive Banach space, A is a positive sectorial operator defined on $X^1 \subset X$. As usual, we write $\langle x, x^* \rangle := x^*(x)$. The adjoint operator A^* with respect to this pairing is a positive sectorial operator on the dual space X^* [**20**, Theorem 1.10.6]. Let $A^{*,\alpha}$ denote the α-th fractional power of the operator A^* and $X^{*,\alpha}$ the α-th fractional power space defined by $A^{*,\alpha}$.
For the rest of this section, fix some $\alpha \in [0,1[$, and suppose that (CH) holds. Recall that (CH) means in particular that $B(t) \to B^\pm$ as $t \to \pm\infty$. We also write $B(\pm\infty)$ to denote $B^\pm$.
We will exploit the relationship between

$$u_t + Au = B(t)u \tag{5.20}$$

and its adjoint equation, where the adjoint is taken formally with respect to the pairing $(x, y) := \langle x, A^{*,\alpha} y \rangle$ between X and $X' := X^{*,\alpha}$. The adjoint equation for (5.20) reads as follows.

$$v_t + A^* v = (B(-t)A^{-\alpha})^* A^{*,\alpha} =: B'(t)v \tag{5.21}$$

LEMMA 5.19.

$$\langle x, A^{*,\alpha} y \rangle = \langle x, (A^\alpha)^* y \rangle = \langle x, (A^*)^\alpha y \rangle \quad \forall (x, y) \in X^\alpha \times X^{*,\alpha}$$

PROOF. We have [**20**, p. 70]

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} \, \mathrm{d}t,$$

where the integral is taken in $\mathcal{L}(X, X)$.
Hence, for $x \in X$ and $y \in X^*$, one has

$$\begin{aligned}
\langle A^{-\alpha} x, y \rangle &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \langle e^{-At} x, y \rangle \, \mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \langle x, e^{-A^* t} y \rangle \, \mathrm{d}t \\
&= \langle x, (A^*)^{-\alpha} y \rangle.
\end{aligned}$$

□

LEMMA 5.20. *Let* $B \in \mathcal{L}(X^\alpha, X)$. *Then* $B' := (BA^{-\alpha})^* A^{*,\alpha} \in \mathcal{L}(X^{*,\alpha}, X^*)$ *with* $\|B'\| \le \|B\|$.

PROOF. Let $(x, y) \in X \times X^{*,\alpha}$. We have

$$\begin{aligned}
\left| \langle x, B' y \rangle \right| &= \left| \langle BA^{-\alpha} x, A^{*,\alpha} y \rangle \right| \\
&\le \|B\|_{\mathcal{L}(X^\alpha, X)} \|x\|_X \left\| A^{*,\alpha} y \right\|_{X^*},
\end{aligned}$$

which shows that $\left\|B' y\right\|_{X^*} \le \|B\|_{\mathscr{L}(X^\alpha,X)}\left\|y\right\|_{X^{*,\alpha}}$. □

LEMMA 5.21. *Let $J \subset \mathbb{R}$ be an open interval, let $u: J \to X^\alpha$ be a solution of* (5.20) *and $v: -J \to X^{*,\alpha}$ be a solution of* (5.21). *Then*

$$(u(t), v(-t)) \equiv C \quad t \in J.$$

PROOF. We consider the function $h(t) := (u(t), v(-t))$, which is defined for all $t \in J$. Note that B is Hölder-continuous by (CH). Lemma 5.20 implies that B' is also Hölder-continuous. Therefore, u and v are continuously differentiable in X respectively X^*. One has

$$\begin{aligned} h_t(s) &= \lim_{h\to 0} \frac{1}{h}\left(\langle u(s+h)-u(s), A^{*,\alpha} v(-s-h)\rangle + \langle A^\alpha u(s), v(-s-h)-v(-s)\rangle\right) \\ &= \langle u_t(s), A^{*,\alpha} v(-s)\rangle + \langle A^\alpha u(s), -v_t(-s)\rangle \\ &= (Au(s)-B(t)u(s), v(-s)) - \langle A^\alpha u(s), A^* v(-s) - B'(-t)v(-s)\rangle = 0 \end{aligned}$$

□

LEMMA 5.22. *Let $J \subset \mathbb{R}$ be an interval and $P: J \to \mathscr{L}(X^\alpha, X^\alpha)$ an exponential dichotomy for the evolution operator $T(t,s)$ on X^α defined by* (5.20).
Then $P': -J \to \mathscr{L}(X^{*,\alpha}, X^{*,\alpha})$, $P'(t) := A^{*,-\alpha}P(-t)^* A^{*,\alpha}$, is an exponential dichotomy for the evolution operator $T'(t,s)$ defined by (5.21).

PROOF. It is easy to see that P' is well-defined and continuous (Lemma 5.20). We need to check the assumptions of an exponential dichotomy (Definition 5.2).
Suppose that $(x, y) \in X^\alpha \times X^{*,\alpha}$ and $[s,t] \subset J$.

(1) From Lemma 5.21, we obtain
$$\begin{aligned} (x, P'(-s)T'(-s,-t)y) &= (T(t,s)P(s)x, y) \\ &= (P(t)T(t,s)x, y) \\ &= (x, T'(-s,-t)P'(-t)y). \end{aligned}$$
Since $A^\alpha: X^\alpha \to X$ is an isomorphism, it follows that $P'(-s)T'(-s,-t) = T'(-s,-t)P'(-t)$.

(2) In order to show that $T'(-s,-t): \mathscr{R}(P'(-t)) \to \mathscr{R}(P'(-s))$ is an isomorphism, it is sufficient to show that it is injective. Suppose that $T'(-s,-t)y = 0$ for some $y \in \mathscr{R}(P'(-t))$. For $x \in X^\alpha$, we have
$$0 = (x, T'(-s,-t)y) = (T(t,s)x, P'(-t)y) = (T(t,s)P(s)x, y),$$
so $(x,y) = 0$ for all $x \in \mathscr{R}(P(t))$. This in turn implies $(x,y) = (x, P'(-t)y) = (P(t)x, y) = 0$ for all $x \in X^\alpha$, that is, $y = 0$.

(3) The estimates for $y \in \mathscr{R}(P'(-t))$ and $y \in \mathscr{R}(I - P'(-t))$ can be deduced using roughly the same arguments. Hence, we will treat only the case $y \in \mathscr{R}(P'(-t))$.
Suppose that
$$\|T(t,s)x\|_\alpha \le M e^{-\gamma(s-t)}\|x\|_\alpha \quad s > t \quad x \in \mathscr{R}(P(s)).$$

We have

$$\begin{aligned}\langle x, A^{*,\alpha} T'(-s,-t)y\rangle &= (x, T'(-s,-t)P'(-t)y)\\ &= (P(s)x, T'(-s,-t)P'(-t)y)\\ &= (T(t,s)P(s)x, y)\\ &\le CMe^{-\gamma(s-t)}\|x\|_X \left\|A^{*,\alpha} y\right\|_{X^*}.\end{aligned}$$

Thus, $\left\|A^{*,\alpha}T'(-s,-t)y\right\|_{X^*} \le CMe^{-\gamma(s-t)}\left\|A^{*,\alpha}y\right\|_{X^*}$, where the constant C is determined by the projections $P(.)$.

□

To sum it up, we have proved that (5.21) satisfies (CH). Compared to (5.20), the Morse indices m^- and m^+ are obviously swapped. This is caused by the reversal of the time variable.
Let the spaces $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be defined as in the previous section, and let $\mathscr{X}'$, $\mathscr{Y}'$, $\mathscr{Z}'$ denote their dual counterparts, that is,

$$\begin{aligned}\mathscr{X}' &:= C_B^{1,\delta}(\mathbb{R}, X^*) \cap C_B^{0,\delta}(\mathbb{R}, X^{*,1})\\ \mathscr{Z}' &:= C_B^{0,\delta}(\mathbb{R}, X^*).\end{aligned}$$

We consider the operators $L \in \mathscr{L}(\mathscr{X}, \mathscr{Z})$ (resp. $L' \in \mathscr{L}(\mathscr{X}', \mathscr{Z}')$) defined by

$$Lu := u_t + Au - B(t)u$$

and

$$L'v := v_t + A^*v - B'(t)v.$$

LEMMA 5.23. *If $\mathscr{R}(L) \supset \mathscr{X}$, then $\mathscr{N}(L') = \{0\}$. Analogously, if $\mathscr{R}(L') \supset \mathscr{X}'$, then $\mathscr{N}(L) = \{0\}$.*

PROOF. Assume that $L'v = 0$ for some $v \in \mathscr{X}'$ and let $u \in \mathscr{X}$ satisfy $u(t) \to 0$ as $|t| \to \infty$. Integration by parts shows that

$$\begin{aligned}&\int_{-a}^{a} \langle Lu(s), A^{*,\alpha} v(-s)\rangle \,\mathrm{d}s\\ &= \int_{-a}^{a} \langle u_t(s), A^{*,\alpha} v(-s)\rangle + \langle A^\alpha u(s), A^* v(-s) - B'(-s)v(-s)\rangle \,\mathrm{d}s\\ &= (u(a), v(-a)) - (u(-a), v(-a)) + \int_{-a}^{a} \langle A^\alpha u(s), \underbrace{v_t(-s) + (A^* - B'(-s))v(-s)}_{=(L'v)(-s)\equiv 0}\rangle \,\mathrm{d}s.\end{aligned}$$

Consequently for all $u \in \mathscr{X}$ with $u(t) \to 0$ as $|t| \to \infty$, one has

$$\int_{-a}^{a} \langle (Lu)(s), A^{*,\alpha} v(-s)\rangle \,\mathrm{d}s \to 0 \text{ as } a \to \infty. \tag{5.22}$$

Arguing by contradiction, suppose that $v(t_0) \neq 0$ for some $t_0 \in \mathbb{R}$. Since [**20**, Theorem 2.6.8] X^1 is dense in X, there is an $x_0 \in X^1$ such that $\langle x_0, A^{*,\alpha} v(t_0)\rangle \neq 0$. Choose $w \in C_B^{1,\delta}(\mathbb{R}, X^1)$ such that $w(t_0) = x_0$ and $w(t) = 0$ for all $t \in \mathbb{R}$ with $|t - t_0| \geq \varepsilon$. For small $\varepsilon > 0$, we have

$$C := \int_{-\infty}^{\infty} \langle \underbrace{w(s)}_{\in \mathscr{R}(L)}, A^{*,\alpha} v(-s) \rangle \, \mathrm{d}s \neq 0.$$

We further have $w \in \mathscr{X} \subset \mathscr{R}(L)$, that is, $w = Lu$ for some $u \in \mathscr{X}$. Since $w(t) = 0$ for $|t|$ sufficiently large, it follows from (CH) respectively from the existence of exponential dichotomies at ∞ and $-\infty$ that $u(t) \to 0$ as $|t| \to 0$. Hence, one has $C = 0$ by (5.22), which is a contradiction.

Using the Hahn-Banach theorem, the second claim can be treated similarly. □

LEMMA 5.24. *Suppose that L is surjective. Then:*

(1) *$m^- \geq m^+$;*
(2) *if $m^- = m^+$, then L is also injective.*

PROOF. (1) Assume to the contrary that $m^- < m^+$. Let P^- (resp. P^+) denote the projections associated with the exponential dichotomy at $-\infty$ (resp. $+\infty$), which are given by (CH). Let $(P^-)'$ and $(P^+)'$ defined by Lemma 5.22. Note that $\dim \mathscr{R}(P^-)' = m^-$ and $\dim \mathscr{R}(P^+)' = m^+$.

By $T'(t,s)$, we mean the evolution operator on $X^{\alpha,*}$ defined by (5.21). Let $t_1 < 0 < t_2$ such that $(P^+)'(t_1)$ and $(P^-)'(t_2)$ are defined. Since $m^- < m^+$, the operator $(P^-(t_2))' T'(t_2, t_1) : \mathscr{R}(P^+(t_1))' \to \mathscr{R}(P^-(t_2))'$ is not injective. Therefore, there exists a non-trivial bounded solution of (5.21), in contradiction to Lemma 5.23.

(2) L' is injective by Lemma 5.23. We can now apply Lemma 5.15 to L', showing that L' is also surjective. Finally, Lemma 5.23 implies that L is injective as claimed.

□

CHAPTER 6

Applications to Asymptotically Autonomous Equations

Isolated invariant sets of generic gradient flows or reaction diffusion equations exhibit a remarkably simple structure. As a consequence, the homology of certain invariant manifolds or the homology Conley index of certain invariant sets can be described by means of an algebraic construction known as Morse complex [**2**, **15**, **24**]. Roughly speaking, the homology respectively the homology index – both notions coincide in this context – is determined by some kind of hyperbolic zeros forming a Morse decomposition as well as connections between them.

Passing on to asymptotically autonomous[1] equations of the same type, additional difficulties arise. While invariant sets are generically still composed of hyperbolic zeros and connections (see Chapter 5) forming a Morse-decomposition, a meaningful index is not available.

An index can be obtained by embedding an asymptotically autonomous equation into a cycle[2] of asymptotically autonomous equations since these cycles can be represented as ω-limes sets of appropriate initial elements (see Chapter 4). In other words, we construct nonautonomous evolution equations which are an approximation of a cycle of asymptotically autonomous equations.

In this setting, the role of the zeros is played by cycles of connections between zeros of the limiting (autonomous) equations. Imagine the fixed points of each limiting equation as nodes and the solutions of the nonautonomous equations as edges connecting them. A circle in this graph corresponds to an isolated invariant subset, and its index is the same as that of its autonomous counterpart – a fixed point.

In this chapter, we consider semilinear parabolic equations, the nonlinearity being asymptotically autonomous. Applying the results of the previous chapters, we will obtain several theorems concerning the existence of solutions.

6.1. One Solution

We are interested in bounded solutions of

$$\begin{array}{lcll} u_t + \Delta u & = & f(t,x,u,\nabla u) & (t,x)\in\mathbb{R}\times\Omega \\ u(t,x) & = & 0 & x\in\partial\Omega \end{array} \tag{6.1}$$

where $\Omega\subset\mathbb{R}^N$ is a (sufficiently) smooth domain.

For the general abstract setting, the reader is referred to Section 2.5.3. In addition to (2.24), suppose that $f(t,x,u,v)$ is C^∞ in u and v all $(t,x)\in\mathbb{R}\times\Omega$. As usual, let $A:\ X^1\to X^0$ be

[1] in time

[2] or circle

defined by $-\Delta$. Recall that $X := X^\alpha$ instead of $X = X^0$, which might be the more common choice.

We further require that f is asymptotically autonomous (2) in t and asymptotically linear (3) in u. More precisely, let F denote the set of all functions $f : \mathbb{R}\times\bar\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ which satisfy:

(1) For every $k \in \mathbb{N}\cup\{0\}$, there is a continuous derivative $D^k_{u,v} f$, and the following estimates hold: for every $K \in \mathbb{R}^+$, there are constants $C_k := C_k(K)$ such that that

$$\begin{array}{lcl} \left|D^k_{u,v} f(t,x,u,v)\right| & \leq & C_k \\ \left|D^k_{u,v} f(t,x,u,v) - D^k_{u,v} f(t',x,u',v')\right| & \leq & C_k\left(\left|t-t'\right|^\delta + \left|u-u'\right| + \left|v-v'\right|\right) \end{array}$$

whenever $|u|, \left|u'\right|, |v|, \left|v'\right| \leq K$.

(2) There are $f^\pm \in C^0(\bar\Omega\times\mathbb{R},\mathbb{R})$ such that $f(x,.) \in C^1(\mathbb{R},\mathbb{R})$ for all $x \in \Omega$ and

$$\begin{array}{lcl} \left|f(t,x,u,v) - f^\pm(x,u)\right| & \to & 0 \\ \left|D_{u,v} f(t,x,u,v) - D_{u,v} f^\pm(x,u)\right| & \to & 0 \end{array}$$

as $t \to \pm\infty$ uniformly on sets B_C of the form $B_C := \{(t,x,u,v) \in \mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^N : |u|,|v| \leq C\}$.

(3) There is a constant $a \in \mathbb{R}$ such that $\lambda^{-1} f(t,x,\lambda u,\lambda v) \to a\cdot u$ as $\lambda \to \infty$, uniformly on sets B'_C of the form $B'_C := \{(t,x,u,v) \in \mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^N : |u|,|v| \leq C\}$.

Let the parameter space $(\mathscr{Y}^1, d_1)$ be defined as in Section 4.1, and set $(Y,d) := (\mathscr{Y}^1, d_1)$. To each function f, occuring as right-hand side in (6.1), we associate its Nemitsky operator $y := y(f) : \mathbb{R}\times X^\alpha \to X^0$ by

$$[y(f)(t,x)](\omega) := f(t,\omega,x(\omega),\nabla x(\omega)) \quad \omega\in\Omega.$$

By (2.24) and hypotheses (1)–(2), one has $\{y(f) : f \in F\} \subset Y$ (cf. Lemma 2.51).

It follows from (3) that $y^t \to y^{\pm\infty}$ as $t \to \pm\infty$, where we set $y := y(f)$ and $y^{\pm\infty}(t,x)(\omega) := f^\infty(\omega, x(\omega))$. It is clear that $\Sigma(y) = \{y^t : t \in \mathbb{R}\} \cup \{y^{-\infty}, y^\infty\}$.

In order to apply the results developed previously, we need an initial element y_0 such that $\omega(y_0) = \Sigma(y)$. In general (if $y^- \neq y^+$), an initial element y_0 need not exist. We therefore define $\tilde y(t,x) := y(-t,x)$. It follows that $(y,\tilde y)$ is an asymptotically autonomous cycle (H1).

Let the initial element $y_0 \in Y_c$ be given by (4.4) respectively Lemma 4.11.

In addition to our previous assumptions, we need to avoid a thin[3] set of "bad" parameters:

(G1) Letting $L_\infty(t,x) := a\cdot x$, it is clear that $L_\infty \in Y_c$. Suppose that L_∞ is weakly hyperbolic.

(G2) Suppose that every solution[4] $(v,u) :\ \mathbb{R} \to \Sigma(y)\times X$ converges to equilibria $e^\pm$ as $t \to \pm\infty$ and $m(e^+) \leq m(e^-)$.

If $m(e^+) = m(e^-)$, assume additionally that (v,u) is weakly hyperbolic.

DEFINITION 6.1. If y, y' satisfy (G2), and if $K \subset (\Sigma(y)\cup\Sigma(y'))\times X$ is a compact invariant set, we are given a Morse-decomposition by setting $(y,x) \in M_k$ if and only if there is a solution $(v,u) :\ \mathbb{R} \to K$ such that u converges to equilibria $e^\pm$ of Morse-index $m(e^+) = m(e^-) = k$ as $t \to \pm\infty$.

[3] The reader is referred to [**4**] as well as Chapter 5.

[4] It is important to make this assumption not only for y but also for y^+ and y^-.

The next, immediate consequence of the above assumptions is that there exists a largest compact invariant set. Moreover, we are able to compute its index.

LEMMA 6.2. *Let $f \in F$, and set $y := y(f)$. Additionally, suppose that (G1) holds for y (as well as for $\tilde{y}(t,x) := y(-t,x)$), and let y_0 be defined by* (4.4) *with respect to the asymptotically autonomous cycle $(y, \tilde{y})$.*
Then there exists a $k_\infty \in \mathbb{N}$ such that the codimension of the stable subbundle (Section 2.4) given by

$$u_t + Au = a \cdot u$$

satisfies $\operatorname{codim} \mathscr{S} = k_\infty$.
Furthermore, there exists a largest compact invariant subset $K_{\max}$ of $\Sigma^+(y_0) \times X$, and one has $\mathrm{h}(y_0, K_{\max}) = \Sigma^{k_\infty}$.

PROOF. It is sufficient (see the remark following Theorem 2.36) to prove that $\operatorname{codim} \mathscr{S} < \infty$, which follows from A having compact resolvent.
Our second claim then follows from Lemma 2.57 and Theorem 2.58. □

We are now in a position to state the main theorem of the first section.

THEOREM 6.3. *Let $f \in F$, and set $y := y(f)$. Additionally, assume that (G1) and (G2) hold.*
Then:

(1) *There exists an equilibrium e^- (respl e^+) of $u_t + Au = y^{-\infty}(t,u)$ (resp. $u_t + Au = y^{\infty}(t,u)$) of Morse-index k_∞ (as defined in Lemma 6.2), and*
(2) *a hyperbolic solution $u_0 : \mathbb{R} \to X$ of $u_t + Au = y(t,u)$ with $u_0(t) \to e^{\pm\infty}$ as $t \to \pm\infty$.*

The claim of the above theorem is not the existence of two equilibria, which is well-known, but of a connection between two of these equilibria. It is helpful to prove an auxiliary lemma.

LEMMA 6.4. *Let $y_0 \in Y_c$, and let $K \subset \Sigma^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighborhood. Suppose that $(M_k)_{k \in \mathbb{Z}}$ is a Morse-decomposition[5] of K ordered by the natural order $\leq$ on $\mathbb{Z}$.*
Let $\partial_k = (\partial_{k,q})_{q \in \mathbb{Z}}$, $k \in \mathbb{Z}$ denote the connecting homomorphism of the following long exact attractor-repeller sequence[6].

$$\longrightarrow \mathrm{H}_q\,\mathscr{C}(y_0, M(\{k-1,k\})) \longrightarrow \mathrm{H}_q\,\mathscr{C}(y_0, M_k) \xrightarrow{\partial_{k,q}} \mathrm{H}_{q-1}\,\mathscr{C}(y_0, M_{k-1}) \longrightarrow$$

Finally, assume that[7]

$$\mathrm{H}_q\,\mathscr{C}(y_0, M_k) = 0 \textit{ for all } q \neq k$$

and define

$$\delta_q := \partial_{q,q} \quad q \in \{1, \dots, n\}.$$

Then for all $q \in \mathbb{Z}$:

$$\mathrm{H}_q\,\mathscr{C}(y_0, K) \simeq \ker \delta_q / \operatorname{im} \delta_{q+1}$$

---

[5] $M_k = \emptyset$ for almost all $k \in \mathbb{Z}$
[6] Concerning the notation, the reader is referred to Section 3.4.1
[7] This is the crucial assumption of the lemma.

The proof is omitted. The proof of [**15**, Proposition 5.9] can be used almost verbatimly. Alternatively, with the necessary modifications, a proof can also be found in Chapter V.1 of [**7**].

PROOF OF THEOREM 6.3. As in Lemma 6.2, we define $\tilde{y}(t,x) := y(-t,x)$. $(y,\tilde{y})$ is now obviously an asymptotically autonomous cycle, but it is unknown whether (G2) holds with respect to $\tilde{y}$.
As a consequence of assumption (3) on F, every parameter $y(f)$ with $f \in F$ particularly satisfies hypothesis (d) of Theorem 5.4. It is easy to see that $\tilde{y} = y(\tilde{f})$, where $\tilde{f}$ is defined by inverting the time variable. Hence, by Theorem 5.4, there exist a $g \in C^{0,\delta}_{B,0}(\mathbb{R}, X^\alpha)$ such that (G2) holds for $\tilde{\tilde{y}}(t,x) := \tilde{y}(t,x) + g(t)$.
Let y_0 be defined with respect to the asymptotically autonomous cycle $(y, \tilde{\tilde{y}})$. It follows from Lemma 6.2 that for every $n \in \mathbb{N}$, there exists a largest compact invariant subset $K_{\max}$ of $\omega(y_0) \times X$, and for some $k_\infty \in \mathbb{N}$ determined by the linearization at infinity

$$\mathrm{h}(y_0, K_{\max}) = \Sigma^{k_\infty}.$$

A Morse-decomposition $(M_0, \ldots, M_n)$ of $K_{\max}$ is given by Definition 6.1. By (G2), (H1), (H2) and (H3) hold. Hence, by Theorem 4.12, for all $k \in \{0, \ldots, n\}$ and all $q \in \mathbb{Z}$:

$$\mathrm{H}_q\, \mathscr{C}(y_0, M_k) \simeq 0 \quad q \neq k$$

Therefore, it follows from Lemma 6.4 that

$$\mathrm{H}_{k_\infty}\, \mathscr{C}(y_0, M_{k_\infty}) \neq 0.$$

Consequently, Corollary 2.22 implies that there exists a solution $(v_0, u_0) : \mathbb{R} \to M_{k_\infty}$ with $v_0(0) = y$. Due to the choice of the sets M_k, one must have $(v_0, u_0)(t) \to (y^{\pm\infty}, e^{\pm\infty})$ as $t \to \pm\infty$, where $e^{\pm\infty}$ are equilibria of Morse-index k_∞. □

6.2. Another Solution

Assume that the hypotheses and hence the conclusions of Theorem 6.3 hold. Suppose additionally that we are given a cycle formed by weakly hyperbolic solutions of a prescribed Morse-index k_0 distinct from k_∞. In case of a gradient-like dynamical system there would exist an equlibrium solution of Morse-index k_0+1 or k_0-1 as well as a heteroclinic connection. In the setting of the previous section, a third possibility emerges as formulated in the theorem below.

THEOREM 6.5. *Let $f, g \in F$ such that (G1) and (G2) hold. Moreover, suppose that (f,g) is an asymptotically autonomous cycle. Let k_∞ be given by Lemma 6.2, and assume that for every $y \in \Sigma(y(f)) \cup \Sigma(y(g))$, $u \equiv 0$ is a weakly hyperbolic solution*[8] *of Morse-index $k_0 \neq k_\infty$.*
Then, there is the following alternative:

(a) *There is a non-trivial solution $u_0 : \mathbb{R} \to X$ of $\Phi_{y(g)}$ with $u_0(t) \to 0$ as $|t| \to \infty$.*
(b) *There is a nontrivial, bounded*[9]*, weakly hyperbolic solution $u_0 : \mathbb{R} \to X$ of $\Phi_{y(f)}$ having Morse index k_0-1, k_0 or k_0+1.*

[8]Equivalently: $u \equiv 0$ is a solution, and both equilibria are hyperbolic having Morse-index k_0.
[9]Consequently, its α– and ω-limes set each contain a single equilibrium having the same Morse-index as u_0.

Furthermore, if there does not exist another weakly hyperbolic solution with Morse-index k_0 of $\Phi_{y(f)}$, then there are either equilibria e^-, e^+ of Morse-index k_0+1 and a solution of $\Phi_{y(f)}$ connecting e^- to e^+ as well as a solution of $\Phi_{y(f)}$ connecting e^- to 0, or there are equilibria e^-, e^+ of Morse-index k_0-1 and a solution of $\Phi_{y(f)}$ connecting e^- to e^+ as well as a solution of $\Phi_{y(f)}$ connecting 0 to e^+.

If neither of the evolution operators $\Phi_{y(f)}$ and $\Phi_{y(g)}$ referred to in the above theorem has a non-trivial connection from 0 to 0, then the remaining alternative applies to both equations respectively evolution operators.

PROOF OF THEOREM 6.5. Let $y_0 = y_0(f,g)$ denote the initial element, which is given by Lemma 4.11. Note that by Lemma 6.2, there is a largest compact invariant set $K_{\max}$ and:

$$\mathrm{H}_* \mathscr{C}(y_0, K_{\max}) \simeq \begin{cases} \mathbb{Z} & q = k_\infty \\ 0 & q \neq k_\infty \end{cases}$$

Suppose that (a) does not hold and that there also does not exist a non-trivial solution of $\Phi_{y(f)}$ with Morse-index k_0. By using Lemma 4.18, we conclude that $M'_{k_0} := \omega(y_0) \times \{0\}$ is an isolated invariant set and:

$$\mathrm{H}_q \mathscr{C}(y_0, M'_{k_0}) \simeq \begin{cases} \mathbb{Z} & q = k_0 \\ 0 & q \neq k_0 \end{cases}$$

Let $E^- \subset X$ (resp. $E^+ \subset X$) denote the set of all equilibria associatiated with $f^+ = g^-$ (resp. $f^- = g^+$) having Morse-index k_0. Define:

$$\begin{aligned} R_{k_0} &:= \{(y(g^-), e) : 0 \neq e \in E^-\} \\ A_{k_0} &:= \{(y(g^+), e) : 0 \neq e \in E^+\} \end{aligned}$$

It is easy to see that $(R_{k_0}, M'_{k_0}, A_{k_0})$ is a Morse-decomposition of M_{k_0}. Since

$$R_{k_0}(y(f)) = A_{k_0}(y(f)) = \emptyset$$

by assumption, it follows from Corollary 2.22 that $\mathrm{H}_q \mathscr{C}(y_0, R_{k_0}) \simeq \mathrm{H}_q \mathscr{C}(y_0, A_{k_0}) \simeq 0$ for all $q \in \mathbb{Z}$. Hence, by using Theorem 3.27, one calculates that $\mathrm{H}_* \mathscr{C}(y_0, M_{k_0}) \simeq \mathrm{H}_* \mathscr{C}(y_0, M'_{k_0})$, so in particular,

$$\mathrm{H}_{k_0} \mathscr{C}(y_0, M_{k_0}) \neq 0. \tag{6.2}$$

Let δ_k be defined as in Lemma 6.4, and suppose that $\delta_{k_0} = 0$ and $\delta_{k_0+1} = 0$. As $\mathrm{H}_{k_0} \mathscr{C}(y_0, M_{k_0}) \neq 0$, it follows from Lemma 6.4 that $\mathrm{H}_{k_0}(y_0, K) \simeq \mathrm{H}_{k_0}(y_0, M_{k_0})$, which is a contradiction.
Therefore, $\delta_{k_0} \neq 0$ or $\delta_{k_0+1} \neq 0$. This is only possible if $\mathrm{H}_* \mathscr{C}(y_0, M_{k_0+1}) \neq 0$ or $\mathrm{H}_* \mathscr{C}(y_0, M_{k_0-1}) \neq 0$. As in the proof of Theorem 6.3, it follows, using Corollary 2.22, that there exists a solution $u_0 : \mathbb{R} \to X$ of $\Phi_{y(f)}$ having Morse-index k_0+1 or Morse-index k_0-1.
Finally, suppose that $\delta_k \neq 0$ for $k \in \{k_0+1, k_0\}$ (and thus also $\partial_k \neq 0$) and consider the set $M(\{k, k-1\})$, which consists of the Morse sets M_k, M_{k-1} and all connecting orbits $C_k \subset \Sigma^+(y_0) \times X$ between them. The set $M(\{k, k-1\})$ is not uniformly connected if $C_k(y(f))$ is empty. Hence, it follows from Theorem 3.58 and the fact that $\delta_k \neq 0$ that $x \in C_k(y(f))$ for some $x \in X$.

Each $x \in C_k(y(f))$ corresponds to a solution $u_x : \mathbb{R} \to X$ of $\Phi_{y(f)}$ with $d((y^t, u_x(t)), M_k) \to 0$ as $t \to -\infty$ and $d((y^t, u_x(t)), M_{k-1}) \to 0$ as $t \to \infty$, whence the last claim follows. $\square$

In view of Theorem 6.5, one is naturally interested in conditions which rule out alternative (a). Trivially, this can be achieved if the left-hand and the right-hand limit of $y(f)$ agree, that is, $f^- = f^+$. We are going to relax that assumption by proving that is sufficient to assume that

$$(\operatorname{sgn} u) f^+(x,u) \le (\operatorname{sgn} u) f^-(x,u) \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{6.3}$$

Let F_0 denote the set of all $f \in F$ which are independent of v. Given $f \in C(\mathbb{R} \times \bar{\Omega} \times \mathbb{R})$, we define $V_f : X \to \mathbb{R}$ by

$$V_f(t,u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x - \int_\Omega \int_0^{u(x)} f(t,x,s) \, \mathrm{d}s \, \mathrm{d}x. \tag{6.4}$$

REMARK 6.1. *One has $X \subset H_0^1(\Omega)$, so V_f is well-defined. Furthermore, it is easy to see that $f_n \to f$ in F_0 that is, uniformly on bounded sets, implies that $V_{f_n} \to V_f$ pointwise on $\mathbb{R} \times X^\alpha$.*

LEMMA 6.6. *Let $f^+, f^- \in C(\bar{\Omega} \times \mathbb{R})$ satisfy* (6.3)*, and define*

$$g(t,x,u) := \frac{1}{\pi}\Big(\frac{\pi}{2} - \arctan t\Big) f^+(x,u) + \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan t\Big) f^-(x,u). \tag{6.5}$$

Let V_g be defined by (6.4)*, and let $u : \mathbb{R} \to X$ be a non-constant solution of $\Phi_{y(g)}$.*
Then, $t \mapsto V_g(t, u(t))$ is strictly monotone decreasing.

PROOF. One has $u \in C^1(\mathbb{R}, X^0) \cap C^0(\mathbb{R}, X^1)$, where $X^1 \subset H^2(\Omega) \cap H_0^1(\Omega)$, by standard regularity results, and

$$u_t(t,x) = \Delta u(t,x) + g(t, u(t,x))$$

in $X^0 = L^p(\Omega)$.
Therefore, $V_g(t, u(t))$ is differentiable and

$$\begin{aligned}
\partial_t V_g(t,u(t)) &= -\int_\Omega \Delta u(t,x) u_t(x) \, \mathrm{d}x - \int_\Omega g(t,x,u(t,x)) u_t(t,x) + \int_0^{u(t,x)} g_t(t,x,s) \, \mathrm{d}s \, \mathrm{d}x \\
&= -\int_\Omega |u_t(t,x)|^2 \, \mathrm{d}x - \int_\Omega \int_0^{u(t,x)} g_t(t,x,s) \, \mathrm{d}s \, \mathrm{d}x.
\end{aligned}$$

We further have

$$g_t(t,x,s) = \frac{1}{\pi} \frac{1}{1+t^2} \big(f^-(x,s) - f^+(x,s)\big),$$

so $\operatorname{sgn}(s) g_t(t,x,s) \ge 0$ for all $s \in \mathbb{R}$ by (6.3). Thus,

$$\partial_t V_g(t,u(t)) \le -\int_\Omega |u_t(t,x)|^2 \, \mathrm{d}x,$$

whence the claim follows immediately. $\square$

LEMMA 6.7. *Let $f \in F$ such that (G1) and (G2) hold, and let f^+, f^- denote the respective limit parameter for $t \to \pm\infty$. Suppose that* (6.3) *holds, and let $g \in F$ be defined by* (6.5)*.*
Then, there exists a largest compact invariant subset K_g of $\Sigma(y(g)) \times X$.

Now that K_g is defined, let $\delta > 0$, and set

$$M_+ := \{(y(h), x) \in K_g : \ V_h(0, x) \geq \delta\}$$
$$M_0 := \{(y(h), x) \in K_g : \ V_h(0, x) = 0\}$$
$$M_- := \{(y(h), x) \in K_g : \ V_h(0, x) \leq -\delta\}$$

For sufficiently small δ, (M_+, M_0, M_-) is a Morse-decomposition of K_g. Moreover, M_0 can be written as a finite union

$$M_0 = \bigcup_x \big(\Sigma(y(g)) \times \{x\}\big) \cup \bigcup_x \big(\{y(f^+)\} \times \{x\}\big) \cup \bigcup_x \big(\{y(f^-)\} \times \{x\}\big).$$

In other words, if $(v, u) : \mathbb{R} \to M_0$ is a solution, then u is constant, and its value is called an equilibrium of $\Phi_{y(f)}$. As the union above is finite, there are only finitely many of these equilibria.

PROOF. First of all, the assignment $F \ni f \mapsto y(f) \in Y$ is one-one, so M_+, M_0 and M_- are well-defined. In view of (6.5), one can easily see that $g \in F$.
By Lemma 6.2, there is largest compact invariant subset K of $\big(\Sigma(y(f)) \cup \Sigma(y(g))\big) \times X$. Thus, $K_g := K \cap (\Sigma(y(g)) \times X)$ is the largest compact invariant subset of $\Sigma(y(g)) \times X$.
Let $u : \mathbb{R} \to X$ be a solution of Φ_y for some $y \in \Sigma(y(g))$. It is a consequence of (G2) that there are equilibria $e^\pm$ (of one of the limit equations given by $y(f^+)$ or $y(f^-)$) such that $u(t) \to e^\pm$ as $t \to \pm\infty$.
By the hyperbolicity assumption in (G2) and the compactness of K_g, we may choose $\delta > 0$ sufficiently small such that there does not exist an equilibrium e of $\Phi_{y(f^+)}$ (resp. $\Phi_{y(f^-)}$) with $V_{f^+}(0, e) \in]-\delta, 0[\cup]0, \delta[$ (resp. $V_{f^-}(0, e) \in]-\delta, 0[\cup]0, \delta[$).
Let $M \in \{M_+, M_0, M_-\}$. In order to prove that M is closed, let $(y(f_n), x_n) \to (y(f_0), x_0)$ be a sequence. It is clear[10] that $f_n \to f_0$, so $V_{f_n}(0, x_n) \to V_{f_0}(0, x_0)$ and $(y(f_0), x_0) \in M$.
Let $(v, u) : \mathbb{R} \to K$ be a solution with $v(0) = y(f_0)$. One has $V_{f^t}(0, u(t)) = V_f(t, u(t))$ for all $t \in \mathbb{R}$, so (M_+, M_0, M_-) is a Morse-decomposition by Lemma 6.6.
Finally, if $(v(t), u(t)) \in M_0$ for all $t \in \mathbb{R}$, then $u(t) \equiv x_0$ by Lemma 6.6, which proves our last claim. □

THEOREM 6.8. *Let $f \in F$ such that (G1), (G2) and* (6.3) *hold.*
Suppose that[11] *for all $t \in \mathbb{R}$, $f(t, x, 0, 0) = 0$, $f_u(t, x, 0, 0) = b(x)$, and $f_v(t, x, 0, 0) = 0$. Set $L_0(t, u)(x) := b(x)u(x)$, assume that L_0 is weakly hyperbolic, and let $k_0 := \operatorname{codim} \mathscr{S}_{L_0}$.*
Let k_∞ be given by Lemma 6.2, and assume that for every $y \in \Sigma(y(f)) \cup \Sigma(y(g))$, $u \equiv 0$ is a weakly hyperbolic solution of Morse-index $k_0 \neq k_\infty$.
Then, there is a nontrivial, bounded, weakly hyperbolic solution $u_0 : \mathbb{R} \to X$ of $\Phi_{y(f)}$ having Morse index $k_0 - 1$, k_0 or $k_0 + 1$.
Furthermore, if there does not exist another weakly hyperbolic solution with Morse-index k_0 of $\Phi_{y(f)}$, then there are either equilibria e^-, e^+ of Morse-index $k_0 + 1$ and a solution of $\Phi_{y(f)}$ connecting e^- to e^+ as well as a solution of $\Phi_{y(f)}$ connecting e^- to 0, or there are equilibria e^-, e^+ of Morse-index $k_0 - 1$ and a solution of $\Phi_{y(f)}$ connecting e^- to e^+ as well as a solution of $\Phi_{y(f)}$ connecting 0 to e^+.

[10] $\Sigma(g)$ is compact, and $g \mapsto y(g)$ is continuous and one-one, hence a homeomorphism $\Sigma(g) \to \Sigma(y(g))$.
[11] In other words, the linearization at 0 is that of a hyperbolic equilibrium of Morse-index k_0.

PROOF. Let g be defined by (6.5). The strategy of the proof is to apply Theorem 6.5. There is main open is that (G2) might not hold for g. However, by using Theorem 5.4, there is an $h \in C^{0,\delta}_{B,0}(\mathbb{R}, X)$ of arbitrarily small norm such that (G2) holds for $y(g)+h$.
By Lemma 6.7, we are given a Morse-decomposition (M_+, M_0, M_-) of K_g, which can be further refined by setting $M_0 = M_0' \dot{\cup} (\Sigma(y(g)) \times \{0\})$. The resulting Morse-decomposition and its associated partial order are depicted below.

$$\begin{tikzcd}
 & M_+ \arrow[dl, "\prec"'] \arrow[dr, "\succ"] & \\
M_0' \arrow[dr, "\succ"'] & & \Sigma(y(g)) \times \{0\} \arrow[dl, "\prec"] \\
 & M_- &
\end{tikzcd}$$

Let $(h_n)_n$ be a sequence in $C^{0,\delta}_{B,0}(\mathbb{R}, X)$ such that (G2) holds with respect to each h_n and $h_n \to 0$ (in $C^{0,\delta}_{B,0}$ i.e., $\sup_{t\in\mathbb{R}} \|h_n(t)\| \to 0$ as $n \to \infty$).
Letting $K_0 := \Sigma^+(y_0) \times \{0\}$, it follows from our assumptions and Theorem 2.56 that $\mathrm{h}(y_0, K_0) = \Sigma^{k_0} \neq \bar{0}$, so by Corollary 2.22 and for all $n \in \mathbb{N}$ sufficiently large, there is a solution $u_n : \mathbb{R} \to X$ of $\Phi_{y(g)+h_n}$ converging to 0 as $|t| \to \infty$.
For large indices $n \in \mathbb{N}$, we define:

$$g_n'(t, x, u, v) := g(t, x, u + u_n(t), v) - g(t, x, u_n(t), v)$$

For each asymptotically autonomous cycle $(y(f), y(g_n'))$ (resp. $(y(f), y(g))$), let y_n (resp. y_0) denote the initial element given by Lemma 4.11. It follows from the construction of y_n that

$$\sup_{t\in\mathbb{R}} d(y_n^t, y_0^t) \to 0$$

as $n \to \infty$.
u is a solution of $\Phi_{y(g_n')}$ if and only if $u + u_n$ is a solution of $\Phi_{y(g)}$. Because (LIN0) holds for $y(g)$, Lemma 2.54 implies that there exist a real constant $\eta_0 > 0$ and an $n_0 \in \mathbb{N}$ such that $Y \times B_{\eta_0}[0, X]$ is an isolating neighborhood for $(y_n, \Sigma^+(y_n) \times \{0\})$ for all $n \in \mathbb{N}$ sufficiently large.
Furthermore, by Theorem 3.39, there exist an $n_0 \in \mathbb{N}$, a family $K_n \subset \Sigma^+(y_n) \times X$, and a family of Morse-decompositions $(M_{n,-}, M_{n,0}, M_{n,+})$ of K_n such that $(y_n, K_n, (M_{n,-}, M_{n,0}, M_{n,+})) \to (y_0, K_g, (M_- \cup M_0', \Sigma(y(g)) \times \{0\}, M_+))$ as $n \to \infty$.
Therefore, given a sequence $u_n' : \mathbb{R} \to X$ such that each u_n' solves the respective evolution operator $\Phi_{y(g_n')}$ and $u_n'(t) \to 0$ as $|t| \to 0$, we must have $u_n' \to 0$ uniformly in t. However, for large $n \in \mathbb{N}$, $Y \times B_{\eta_0}[0, X]$ is an isolating neighborhood for $(y_n, \Sigma^+(y_n) \times \{0\})$, which is a contradiction.
We have proved that the assumptions of Theorem 6.5 are satisfied, while alternative (a) of its conclusions cannot hold. Thus, alternative (b) must hold, which is precisely the claim of this theorem. $\square$

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Notation Index

$\diamond$	value an evolution operator or semiflow, means "undefined"	
$\mathbb{R}^+$	$\{x \in \mathbb{R} : x \geq 0\}$	
$\mathbb{N}$	$\{x \in \mathbb{Z} : x \geq 1\}$	
$\mathbb{Z}^+$	$\mathbb{Z} \cap \mathbb{R}^+$	
$\mathbb{R}/N\mathbb{Z}$	$\{\{x + Nk : k \in \mathbb{Z}\} : x \in \mathbb{R}\}$	
$\mathrm{cl}_X M$	closure of a set M in a topological space X	
$\mathrm{int}_X M$	interior of a set M in a topological space X	
$\bar{\Omega}$	closure of Ω in $\mathbb{R}^N$	
$B_\varepsilon(x, X)$	open ε-ball in a metric space X centered at x	
$B_\varepsilon[x, X]$	closed ε-ball in a metric space X centered at x	
$B_\varepsilon(x)$	$B_\varepsilon(x, X)$ when the space X can be deduced from the context	
$B_\varepsilon[x]$	$B_\varepsilon[x, X]$ when the space X can be deduced from the context	
$U_\varepsilon(K)$	ε-neighborhood of a compact subset K	
$C(X, Y)$	continuous mappings from a space X to a space Y	
$C^k(X, Y)$	mappings $X \to Y$ which are k-times continuously differentiable	
$C^{k,\delta}(X, Y)$	mappings in $C^k(X, Y)$ with k-th derivative being Hölder-continuous of exponent δ	
$C_b(X, Y)$	continuous mappings $X \to Y$ which map bounded sets into bounded sets	p. 35
$C_b^k(X, Y)$	mappings $X \to Y$ with derivatives up to order k in C_b	
$C_b^{k,\delta}(X, Y)$	mappings in $C_b^k(X, Y)$, Hölder-continuous with exponent δ, uniformly on bounded sets	p. 103
$C^{k,\delta}(\Omega)$	$C^{k,\delta}(\Omega, \mathbb{R})$	
$C^{k,\delta}(\bar{\Omega})$	$C^{k,\delta}(\mathrm{cl}\,\Omega, \mathbb{R})$	
$C_B(X, Y)$	continuous mappings $X \to Y$, globally bounded	
$C_B^k(X, Y)$	mappings $X \to Y$ with derivatives up to order k in C_B	
$C_B^{k,\delta}(X, Y)$	mappings in $C_B^k(X, Y)$, uniformly Hölder-continuous with exponent δ	p. 103
$C_{B,0}^{k,\delta}(\mathbb{R}, X)$	mappings in $u \in C_B^{k,\delta}$ with $u(t) \to 0$ as $\lvert t \rvert \to \infty$	
$\mathrm{h}(A, x)$	homotopy type of the pointed space (A, x), $x \in A$	
$\mathrm{h}(A, B)$	homotopy type of the pair (A, B), $B \subset A$	
$\mathrm{H}_*(A, B)$	homology functor, associates a graded abelian group with a pair of topological spaces	p. 1
$\mathrm{H}_q(A, B)$	q-th homology of the pair (A, B) of topological spaces	p. 1
$\mathrm{H}_*[A, B]$	$\mathrm{H}_*(A/B, B/B)$	p. 2
$\mathrm{H}_q[A, B]$	$\mathrm{H}_q(A/B, B/B)$	p. 2

$\mathrm{Inv}_\pi N$	largest invariant subset of N with respect to π	p. 3
$\mathrm{Inv}_\pi^- N$	largest negatively invariant subset of N with respect to π	p. 3
$\mathrm{Inv}_\pi^+ N$	largest positively invariant subset of N with respect to π	p. 3
$\Sigma^+(y_0)$	positive hull of (a parameter) y_0 with respect to translation	p. 4
$\Sigma(y_0)$	(complete) hull of y_0 with respect to translation	p. 84
Y_c	set of all $y \in Y$ for which $\Sigma^+(y_0)$ is compact	p. 4
Y_{cl}	set of all $y \in Y_c$ which are linear and satisfy an additional admissibility assumption	p. 28
Y_{aa}	set of all $y \in Y$ which are asymptotically autonomous	p. 83
$V(y)$	$\{x : (y,x) \in V\}$ where $V \subset Y \times X$ or, in particular, $V \subset \mathbb{R}^+ \times X$	p. 74
$V(U)$	$V \cap (U \times X)$ where V as above and $U \subset Y$ or $U \subset \mathbb{R}^+$	p. 74
X^α	α-th fraction power space with respect to an operator A	p. 38
$[a,b]_{\mathbb{R}/N\mathbb{Z}}$	projection of the interval $[a,b] \subset \mathbb{R}$ to $\mathbb{R}/N\mathbb{Z}$	
$X \oplus Y$	direct sum of two vector spaces, i.e. $X \oplus Y = X + Y$, and $X \cap Y = \{0\}$	
$\mathscr{U}$	a subset of $\Sigma^+(y_0) \times X$, denoting the unstable subbundle given by a linear evolution operator	p. 28
$\mathscr{S}$	a subset of $\Sigma^+(y_0) \times X$, denoting the stable subbundle given by a linear evolution operator	p. 28

Subject Index