Markowitz Theory–Based Asset Allocation Strategies with Special Regard to Private Wealth Management

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I would like to dedicate this thesis to my beloved parents and my sister...

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Abstract

This doctoral thesis concentrates on portfolio optimization problems with a special focus on asset allocation strategies for different types of idealized investors. While investors are classified according to their initial wealth, their attitude toward risk and their investment horizon, several asset allocation strategies are consequently constructed for them. Special attention is hereby dedicated to applications in Private Wealth Management, or, in other words, on portfolio management of High Net Worth Individuals (HNWIs). These investors are characterized by a rather high degree of risk aversion, a long investment horizon and a high initial wealth. Moreover, they are more flexible on adjusting the proportions of risky assets and riskless assets over time. It is shown that these wealthy investors can put a relatively high proportion of their initial wealth in risky assets, the latter nevertheless characterized by low volatilities. In order to control portfolio risk, Value at Risk (VaR) will be shown a useful instrument to monitor the loss probability. This refers especially to the “status problem” of wealthy investors. In the last main chapter, the impact of the uncertainty of the model’s estimates on the solutions of portfolio allocation problems is investigated.

The research is incorporated in the framework of both utility theory and the Markowitz model. Using monthly returns of ten different indices from seven asset classes recorded from 1996 to 2007, this dissertation shows that utility maximization for portfolio optimization problems based on quadratic utility and other popular but more difficult utility functions leads to similar results. Efficient portfolios

derived from the numerical solution of the classical Markowitz optimization problem are most often good approximations of maximizing expected utility. Furthermore, under some constraints, the concept of a naive diversification is shown to be a good strategy for direct utility maximizers.

Contents

List of Figures

List of Tables

List of Abbreviations

AAS:	Asset Allocation Strategy
APT:	Arbitrage Pricing Theory
CAPM:	Capital Asset Pricing Model
CDs:	Certificates of Deposit
CLA:	Critical Line Algorithm
CPI:	Consumer Price Index
CRRA:	Constant Relative Risk Aversion
CAGR:	Compound Annual Growth Rate
DAX:	Deutscher Aktienindex
EMH:	Efficient Market Hypothesis
GIC:	Guaranteed Investment Contracts
GMVP:	Global Minimum Variance Portfolio
GREI:	Germany Real Estate Index
HARA:	Hyperbolic Absolute Risk Aversion
HNWIs:	High Net Worth Individuals
IDC:	Indifference Curve
IFSL:	International Financial Services London
IMF:	International Monetary Fund
JB–Test:	Jarque–Bera Test

JPEC1ML:	JP Morgan Cash Index Euro Currency 1 Month
Lcd:	Lowest Common Divisor
MiFID:	Markets in Financial Instruments Directive
MSCI:	Morgan Stanley Capital International
MPT:	Modern Portfolio Theory
NYSE:	New York Stock Exchange
Pr:	Probability
REIT:	Real Estate Investment Trust
REX:	Deutscher Rentenmarktindex
TIAA/CREF:	Teachers Insurance and Annuity Association, College Retirement Equities Fund
VaR:	Value at Risk
WFE:	World Federation of Exchanges

Chapter 1

Introduction

1.1 Background of the Research

Individuals as well as institutional investors are confronted with basically the same set of problems when allocating their own financial funds or those of third parties. The asset allocation puzzle is – at least theoretically – actually of a huge dimension: There exist tens of thousands of listed companies (just to mention, e.g., 2,249 at the New York Stock Exchange (NYSE) and 1,088 at the Shanghai Stock Exchange in 2007), masses of government and commercial bonds with different risks and maturities, treasuries, currencies, commodities, arts and real estate. Moreover, there is an even much bigger number of financial derivatives on the mentioned asset classes and their representatives[1]such as different kinds of options, swaps, forwards and structured products. Practically every investor must thus undergo a *preselection* process. This starts usually with the selection of the "right" asset classes[2] and is followed by the identification of the "appropriate" elements of the chosen asset class.

The preselection process provides a reduced universe of assets where an actual investment is considered to be possible. The final decision mainly refers to how

[1]The technical terms *securities* and *assets* are interchangeably used in this work.

[2]This is often referred to by terms like saving– and dissaving periods in a human being's life; this is followed most often by rather qualitative recommendations such as the substitution during a person's life of risky portfolio components by others characterized by lower risk.

much to invest in the single selected assets, respectively. All these considerations are so far independent of the three input factors which play an important role in any structured investment process: the initial wealth, the attitude toward risk and the investment horizon. In this thesis, idealized investors will be classified and examined according to their expected return relationship, to their initial wealth and to their investment horizon (Behavioral or psychological aspects will not be considered.). From this starting point, reasonable asset allocation strategies will be derived.[1]

In this research is a specific focus on *High Net Worth Individuals*[2]. Although there are certainly individual exceptions, these investors are usually characterized by a high initial wealth, a comparably long investment horizon and a rather low risk tolerance.[3] While many rich families with industrial shares lost – at least virtually – a lot of money from the outbreak of the current financial crisis mid 2007, there were quite a few complaints about the administration of wealth by the private banks in Europe. That might, on the one hand, have to do with the high level of discretion of both customers and banks; but it is more likely, although it cannot here be proven, that it is because of the actions taken by these banks on their already conservative customers' portfolios at a time when the rise of the financial crisis was not yet recognized by the public.

Markowitz' theory provides a quite broad framework for optimal asset allocation. It is firstly – as are most mathematically–based capital market models – independent of whether they are normative or positive – a "one–period–model". This means that the model can be used for theoretically every investment period or

[1] There is, e.g., strong evidence that the asset allocation of individual as well as institutional investors explains a large part (up to 90 %) of the actual portfolio performance. See, e.g., Brinson *et al.* (1986), Brinson *et al.* (1991).

[2] A HNWI is usually understood to be an individual who holds at least US$ 1 million in financial assets, excluding collectibles, consumables, consumer durables and primary residences.

[3] The reader of this text will certainly realize that there are some assertions on "typical" HNWIs in this thesis which are not supported by appropriate references or academic research. These statments in questions have been developed during an internship of the author in a private bank. Despite serious efforts could in some cases no references been found.

horizon. Secondly, the computation of the investment weights (for more details see the following chapter) requires at least that the initially–invested budget is much larger than the price of a single asset of any element in the set of possible investment alternatives. This means that the initial investment sum cannot be “too small”, and is thus appropriate for dealing with “large” fortunes to be invested in risky securities. Thirdly, the Markowitz model provides a set of efficient portfolios, i.e., the (theoretically infinite) number of all portfolios with the highest achievable expected return for all (predefined) levels of (existing) risk. The better the preselection process is performed,[1] the better in statistical terms will be the set of efficient portfolios.[2]

The “neighborhood” of the *Global Minimum Variance Portfolio* toward the North–East is of a very special interest for asset allocation recommendations for HNWIs. They are motivated from the actual investment approach of a “typical” HNWI. This is: firstly, because the GMVP is the only portfolio which does not depend on the vector of expected returns; and secondly, because it is usually a very well–diversified portfolio. According to the fundamental insight into all risky investments, there is a positive trade–off between expected return and risk. The bad side of this insight is that, the more expected return an investor demands, the more he or she depends on “good” estimates or inputs in the models. The second, even worse side of this insight is that, if an investor, reinvesting the final amount at each period, repeats the investment game often enough, the expected “Future Value” of his portfolio is zero! Thirdly, the long investment horizon is directly linked to the fact that the portfolio should not be changed too much in time.[3] Very generally, an intertemporal Markowitz model in the long run works

[1] I.e., that it identifies low correlations between assets with “good” expected returns.

[2] The impact of a meltdown of correlations toward $+1$ as a characteristic of a financial crash will not be discussed in this thesis.

[3] We refer here to the problem of errors in the input estimates. The higher the expected return of the portfolio, the fewer are the assets which are usually dominating the portfolio, the more important are thus getting errors in the return distribution of these remaining assets. In the perfect case, the expected return of the portfolio, which has been constructed at $t = 0$, equals to the realized return at $t = 1$. The actual reliability of a portfolio can simply be derived as the difference between these two numbers.

very well for investors who ask for modestly increasing values of their investments, but not for investors with a high risk tolerance. It is thus very appropriate for HNWIs, who usually have most of their focus on simply preserving their fortunes.

The solution of the optimization problem according to Markowitz nevertheless depends on questionable assumptions. Either investors have quadratic utility, or portfolio returns are normally distributed; however, neither of these assumptions actually works fully in practice. The consequent question is: to what extent does this matter? H. Levy & H. M. Markowitz (1979) demonstrate that mean–variance approximations of utility correlate strongly with true utility. They show, for various utility functions and empirical return distributions, that the expected utility maximizer could typically do very well, if he or she acted knowing only the mean and the variance of the distribution under consideration. Kroll *et al.* (1984) examine the same question but for an infinite number of alternate distributions, and Samuelson (2003) finds that today investors have sufficient computational power to maximize expected utility based on a plausible utility function and the entire distribution of returns from empirical samples.[1]

From here on, we continue by comparing the results of mean–variance optimizations and direct expected utility optimizations of different utility functions, and intend to check which framework is more appropriate for different types of investors. Moreover, we aim to derive corresponding asset allocation strategies from different sets of available asset classes for these different types of investors, especially for the HNWIs. Representatives of the asset classes *arts* and *real estate* are, e.g., considered to be possible investment choices for the HNWIs, but not for "small" investors. Technically we must be more careful with some assumptions of the classical Markowitz model, such as normally distributed returns of the assets, and the infinite divisibility of the assets. We will deal with the impact of the uncertainty of the input parameters by treating it with sensitivity analyses. This shall provide us with information about the reliability and stability of the

[1] In this thesis only a part of the foundations of utility theory will be discussed, namely the one which is linked to portfolio choice theory.

portfolios which are derived from the asset allocation strategies mentioned before.

The structure of this thesis is as follows: After a concise overview of the important academic work related to the set of research questions discussed in this thesis, chapter 2 reviews the classical Markowitz model, fundamentals of utility theory, furthermore several extensions and some specific numerical techniques to solve portfolio optimization problems under different types of constraints. Chapter 3 provides a classification for some "typical" individual and institutional investors. It furthermore derives corresponding motivations for "reasonable" strategies of these investors according to their initial wealth, risk–return preferences and investment horizon, respectively, and then presents different asset allocation strategies for some types of investors. Chapter 4 shows the results of the empirical research. The influence of the initial wealth, the investment time horizon, and the attitude toward risk is there examined for some chosen strategies. Correspondingly, we will compare the results derived from Markowitz' mean–variance model and from Expected Utility Maximization by means of estimates derived from historical returns of different asset class indices. Furthermore, it investigates the impact of errors in the estimates, and provides two proposals for reducing the role of estimation errors on the asset allocation. Chapter 5 summarizes the main results of this research. The constrained optimization problems were treated numerically with the help of MATLAB–software. The used data and programme codes are attached to this thesis via a CD.

1.2 Literature Review

Asset allocation can be regarded as one of the most important applications of modern portfolio theory. Contrariwise and closer to investors' needs and intuition, portfolio theory is a set of tools and methods to identify "good" asset allocations. Tactical asset allocation assumes that the decision maker has a criterion defined over the "short" one–period rates of return on the portfolio. Correspondingly, strategic asset allocation has a long–term horizon. Two approaches are often used in the framework of strategic asset allocation. One is the stochastic optimal control problem which captures uncertainty by allowing for a continuum of states, which can be described at a given point in time by a small number of state variables following a joint Markov process[1]; and the other one is the stochastic programming approach which captures uncertainty by a branching event tree.[2]

Modern portfolio theory is a well–developed paradigm which is able to propose solutions to practically every asset allocation problem. Important milestones are Markowitz (1952a), Markowitz (1959), the Capital Asset Pricing Model (CAPM) derived by Sharpe (1964), Mossin (1966), Lintner (1965), the intertemporal CAPM by Merton (1973) and the Arbitrage Pricing Theory of Ross (1976).[3].

This section will provide a selection of important previous research results related to the topics discussed in this thesis. These are the Markowitz theory and its extensions, utility theory, and the results on theoretical and empirical aspects of asset allocation strategies. Markowitz (1952a, 1959) formulated the portfolio optimization problem in a $\mu - \sigma$–framework and proved the fundamental theorem of mean–variance–efficient portfolios. Depending on his or her risk–return preferences, an investor can then select an optimal portfolio.

There are obviously some important advantages compared to the competing, most often more technically sophisticated, models: Firstly, there is no evidence that adding additional statistical moments improves the properties of the portfolio selected. Secondly, and even more importantly from most investors' perspective, the implications of the mean–variance–portfolio theory are well developed, widely known, and have great intuitive appeal.[4]

Among the major theoretical problems – and practically corresponding to the

[1]See Merton (1971).

[2]More information on tactical asset allocation can be found in Brennan *et al.* (1997).

[3]There are also a number of excellent textbooks on this subject (See, e.g., Elton & M. J. Gruber (1997), Sharpe *et al.* (2008), Bodie & Crane, D. B. (1997)) A good review about the historical and further research for modern portfolio theory can be found in Elton & M. J. Gruber (1997).

[4]See Elton & M. J. Gruber (1997).

key question of how to rebalance a portfolio at not necessarily equidistant different discrete time points – that have been investigated by many researchers is how the classical one–period problem should be modified if the investor's true problem is a multi–period problem. Research by Fama (1970), Hakansson (1970), Hakansson (1974), Merton (1990) and Mossin (1969) has been executed by analyzing this problem under various assumptions. It was, e.g., found that under several sets of reasonable assumptions, the multi–period problem can be solved as a sequence of single–period problems. However, the optimum portfolio would be different from that selected if only one period was examined! This difference arises because the appropriate utility function in the multi–period case is a derived utility function that takes into account multiple periods and differs from the utility function that is appropriate for one single period.

Referring to the utility function, it is furthermore frequently asserted that mean–variance analysis applies exactly only when distributions are normal or utility functions are quadratic, suggesting that it gives nearly optimal results only when distributions are approximately normal or utility functions look almost like a parabola. Levy and Markowitz[1] showed empirically that the ordering of portfolios by the mean–variance rule was almost identical to the order obtained by using expected utility for various utility function and historical distributions of returns. Tobin[2] showed that the mean–variance model is consistent with the von Neumann–Morgenstern postulates of rational behavior if the utility of wealth is quadratic. Samuelson (1969)[3] and Merton (1971)[4] showed that a constant weight strategy is optimal if the investor's utility function displays constant relative risk aversion (CRRA) and asset prices follow geometric Brownian motion. Cox and Huang (1992)[5] demonstrated that for a broad class of utility functions (including the linear risk tolerance, or hyperbolic absolute risk aversion (HARA) functions), the optimal strategy converges to constant weights as the horizon increases.

[1] See Levy & H. M. Markowitz (1979).
[2] See Tobin (1958).
[3] See Samuelson (1969).
[4] See Merton (1971).
[5] See Cox & Huang, C. (1992).

As for uncertainty in expected returns and its implications for portfolio selection, there are extensive research contributions: the works of Barry (1974) and Bawa *et al.* (1979), for example. Many authors have addressed this problem, often resorting to a Bayesian framework F.(Black & R. Littermann (1992); J. D. Jobson & R. Korkie (1980); F. Jorion (1985); R. O. Michaud (1989); V. Chopra & W. T. Ziemba (1993)). Another popular approach is the one proposed by Michaud[6], which provides an attempt to maintain the advantages of the efficient frontier optimization framework, but accommodates parameter uncertainty by creating a *resampled frontier*. This approach, which regards an efficient frontier as one random realization, offers many more interesting insights, and has also been strongly supported by empirical research.[1]

[6]See Richard O. Michaud & Robert O. Michaud (1998).
[1]See Markowitz & N. Usmen (2003).

Chapter 2

The Markowitz Theory of Portfolio Optimization

2.1 Preliminaries

For *any* practical investment decision, there are three main questions which have to be answered by *every* investor:

1. In which asset classes should I invest?
2. Which representatives of these asset classes should I choose?
3. How much should I invest in these selected assets, respectively?

While the first two questions are associated with the aforementioned *preselection* process, the third question is directly linked to mathematical and statistical modeling. The question of *how much* to invest in individual assets is actually the central question of the Modern Portfolio Theory (MPT), which has been introduced into financial research by Harry Markowitz (1952a, 1952).

In order to build a reasonable model, we will abstract from i) different asset classes and different individual assets, ii) the investment horizon, and iii) the initial invested amount. Let $N \in \mathbb{N}$ denote the number of investment alternatives, or, in other words, different assets, which are taken into consideration for an investment after the preselection process. The initial amount to be invested is denoted by B. The budget B is assumed to be any large but fixed amount in, e.g., Euros. For all N assets, there exist the exogenous known market prices per unit, $P_1, P_2, \ldots, P_N$, and the decision variables for the investment are consequently the quantities of the assets to be purchased, $Q_1, Q_2, \ldots, Q_N$. Because the investor cannot spend more than B (and there could be some small cash remain), it holds the following inequality:

$$P_1Q_1 + P_2Q_2 + \ldots + P_{N-1}Q_{N-1} + P_NQ_N \leq B. \tag{2.1}$$

The prices $P_1, \ldots, P_N$ are supposed to be constant for acquiring all units Q_i, where $i = 1, \ldots, N$, of each asset, respectively.[1] Dividing (2.1) by B gives

$$\frac{P_1Q_1}{B} + \frac{P_2Q_2}{B} + \ldots + \frac{P_{N-1}Q_{N-1}}{B} + \frac{P_NQ_N}{B} \leq 1. \tag{2.2}$$

Defining $x_i := P_iQ_i/B$, $x_i, i = 1, \ldots, N$, results in $x_1 + \ldots x_N \leq 1$. The $x_i, i = 1, \ldots, N$, are the relative rates of the N investments, summing up to (almost) 1. Note that the inequality turns toward an equality if $B >> P_i \quad \forall i \in \{1, \ldots, N\}$. Under these assumptions, the equation

$$x_1 + \ldots x_N = 1 \tag{2.3}$$

expresses the intention of the investor to invest his or her initial budget or fortune completely.[2]

2.2 Efficient Portfolios

Defining a **portfolio** as a set of relative investment weights $(x_1, \ldots, x_N)^T$, $\sum_{i=1}^{N} x_i \leq 1$, an investment decision problem can basically be described by two *optimization* approaches:

[1]This is approximately about correct for very liquid assets, but does not hold for assets with a low number of traded units. For a given supply, the price would otherwise be bided up by an increasing demand.

[2]This assumption is often referred to as the infinite divisibility of the assets.

1. For a portfolio with a given initial wealth level and a predefined highest tolerable risk, maximize the expected return.

2. For a portfolio with a given initial wealth level and a predefined lowest level of expected return, minimize the risk.

A portfolio is called **efficient**, if, for a given level of risk, there is no portfolio with a higher expected return, or, for a given expected return equal or above the *Global Minimum Variance Portfolio*, there exists no portfolio with a lower risk.[1]

2.3 Inputs of the Model and Statistical Measures

Investors obviously prefer c.p. more expected return to less expected return and less risk to more risk.[2] Firstly, all (preselected) assets have now to be characterized by individual measures of potential return and risk, and secondly, these measures have to be derived for portfolios.

In a complete (i.e., the probabilities sum up to one) discrete *probability scenario*[3],[4] M denotes the number of states with associated probabilities $p_1, p_2, \ldots, p_M$, $\sum_{j=1}^{M} p_j = 1$. The investment alternatives are denoted by $a_1, a_2, \ldots, a_N$. Let $\mathbf{R} = (R_1, ..., R_N)^T$ denote a $N-$dimensional vector of random returns.

The **expected return** of an asset i, $i = 1, \ldots, N$ is defined by

$$\mathbb{E}[R_i] = \mu_i = p_1 R_{i1} + p_2 R_{i2} + \ldots + p_{M-1} R_{iM-1} + p_M R_{iM} = \sum_{j=1}^{M} p_j R_{ij}. \qquad (2.4)$$

[1]Graphically, the set of all efficient portfolios is represented as the upper branch of the efficiency curve or **efficient** frontier.

[2]Investors' preferences and utility functions will be discussed in section 2.5.

[3]It is supposed that all possible market expectations are in the complete scenario with the associated probabilities.

[4]For research on continuous cases, see Szegö (1980).

Table 2.1: Discrete Probability Scenario

	s_1	s_2	$\cdots$	s_{M-1}	s_M
a_1	R_{11}	R_{12}	$\cdots$	R_{1M-1}	R_{1M}
a_2	R_{21}	$\cdots$	$\cdots$		R_{2M}
$\vdots$					
a_{N-1}					
a_N	R_{N1}	R_{N2}	$\cdots$	$\cdots$	R_{NM}

R_{ij} is the realization of the random return of asset i in state j, $i = 1, ..., N; j = 1, ..., M$.

In practice, statistical parameters are often estimated from time series (such as historical data from financial markets), and after that, they are *somehow* adjusted.[1]

Whereas there is much consensus about measuring/describing the attractiveness of risky assets by their expected returns, the understanding of risk differs much more among investors and academics. The following risk measures are especially in practical use:

1. Variance or standard deviation,
2. Value at Risk,
3. Semi–Variance as a special case of Lower Partial Moments,
4. Skewness and Kurtosis.

[1]Note that any realization of a time series represents a special case of the general scenario shown here. While most of the statistical techniques dealing with real data are very precise, the overall dominating problem in putting a Capital Market Model into practice is the uncertainty of the parameters due either to errors in the estimates or to wrong or incomplete probability scenarios, both of which are equally serious.

As long as assets are supposed to be characterized by symmetric return distributions, variance or standard deviations are good measures. If there were normal distributions, however, variance or standard deviation would be perfect, because any normal distribution is completely described by its expected value and its variance.[1] The **variance** of the random return of asset i, $i = 1, \ldots, N$, in a complete discrete probability scenario is defined by

$$\mathbb{V}[R_i] = \sigma_i^2 = \sum_{j=1}^{M} p_j (R_{ij} - \mu_i)^2. \tag{2.5}$$

Although a return distribution cannot be fully normal[2], the variance is usually a good measure of risk as long as the underlying assets are bonds or stocks and the historical returns come from short equidistant time intervals (see, e.g., Artzner *et al.* (1999)). The variance is usually *not* a good risk measure if we deal with financial derivatives which are characterized by non–symmetric payoffs.

Mathematically, $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space and $\mathbf{R} : \Omega \to \mathbb{R}$ is a random vector. Let $\mathbf{R} = (R_1, R_2, \ldots, R_N)^T \in L^2(\Omega, \mathbf{F}, \mathbb{P})$. We denote

1. $\mathbb{E}[R - \mathbb{E}(R)]^k$ is the central moment of order k, $k \in \mathbb{N}$.
2. $\mathbb{V}(R) = \sigma_R^2 = \mathbb{E}[R - \mathbb{E}(R)]^2$ the variance of R.
3. $\mathbb{COV}(R_i, R_k) = \sigma_{ik} = \mathbb{E}[R_i - \mathbb{E}(R_i)][R_k - \mathbb{E}(R_k)]$ is the covariance between R_i and $R_k, i \neq k, i, k \in \{1, \ldots, N\}$.

The expected return μ_p of a portfolio is a linear combination of the portfolio $\mathbf{x} = (x_1, \ldots, x_N)^T$ and the vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)^T$.[3]

$$\mu_p = \boldsymbol{\mu}^T \mathbf{x} = \sum_{i=1}^{N} x_i \mu_i. \tag{2.6}$$

[1] Note that the use of the variance is equivalent to its positive square root, the standard deviation.

[2] The maximum loss can, e.g., not exceed 100 %.

[3] Notation: From now on, vectors and matrices are written in bold characters throughout this thesis.

The variance of a portfolio $\mathbf{x_p}$ is thus

$$\sigma^2(R(\mathbf{x_p})) = \sum_{i=1}^{N}\sum_{k=1}^{N} x_i x_k \sigma_{ik}$$

with $\sigma_{ik} = \mathbb{E}[(R_i - \mu_i)(R_k - \mu_k)]$. Using matrices we have $\sigma^2(R(\mathbf{x_p})) = \mathbf{x_p}^T \mathbf{\Sigma} \mathbf{x_p}$ with $\mathbf{\Sigma} = (\sigma_{ij})_{i,j=1}^N$ the variance–covariance matrix. From now on, the subscript P is omitted for the portfolio vectors and σ_p^2 denotes the variance of a portfolio P.

Theorem 2.1 summarizes the most important properties of the variance–covariance matrix $\mathbf{\Sigma} = \mathbb{E}(\mathbf{R} - \boldsymbol{\mu})(\mathbf{R} - \boldsymbol{\mu})^T$.

Theorem 2.1 *When $\mathbf{\Sigma}$ is the covariance matrix, then:*

1. *$\mathbf{\Sigma}$ is symmetric.*
2. *$\mathbf{\Sigma}$ is positive semi–definite.*
3. *From linearly independent expected returns it follows that $\mathbf{\Sigma}$ is regular.*

The proof of properties 1. to 3. can be found in Huang & R. H. Litzenberger (1988).[1]

We now make the assumption that all assets are risky, i.e., that each one has a positive variance. From a given $N-$dimensional vector of random returns

$$\mathbf{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix} : \Omega \to \mathbb{R}^N.$$

and a vector of expected returns

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} = \begin{pmatrix} \mathbb{E}(R_1) \\ \vdots \\ \mathbb{E}(R_N) \end{pmatrix}$$

[1] When applying time series models in practice, note that an estimate of Σ cannot be not regular, if the number of observed time periods is smaller than the number of the assets N (See, e.g., Wolfe (1959)).

and the variance–covariance matrix

$$\mathbb{COV}(\mathbf{R}) = \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \sigma_{N1} & \cdots & & \sigma_N^2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & & \vdots \\ \vdots & & \ddots & \\ \sigma_{N1} & \cdots & & \sigma_{NN} \end{pmatrix}$$

are derived. The return of a portfolio will be formulated as follows: $R_P = \mathbf{x}^T\mathbf{R}$. The optimization problem is now either

$$\min_{\mathbf{x}\in\mathbb{R}^N} \mathbb{V}(R) = \mathbf{x}^T\mathbf{\Sigma}\mathbf{x}$$

or

$$\max_{\mathbf{x}\in\mathbb{R}^N} \mathbb{E}(R) = \mathbf{x}^T\boldsymbol{\mu}$$

subject to

$$\mathbf{1}^T\mathbf{x} = \sum_{i=1}^{N} x_i = 1$$

and the additional constraint with respect to expected return or variance (cf.2.2.) In order to resolve the optimization problem, one has to choose the "right" objective function.[1] The expected return must to be fixed and the portfolio variance minimized, and the problem can thus be formulated as follows:

Unrestricted Standard Optimization Problem:

$$\min_{\mathbf{x}\in\mathbb{R}^N} \mathbb{V}(R) = \mathbf{x}^T\mathbf{\Sigma}\mathbf{x}$$

subject to

$$\mathbf{1}^T\mathbf{x} = \sum_{i=1}^{N} x_i = 1,$$

$$\boldsymbol{\mu}^T\mathbf{x} = \sum_{i=1}^{N} x_i\mu_i = \mu_p.$$

[1]For mathematical reasons it is easier to have a non–linear objective function with linear constraints instead of a linear objective function with non–linear constraints; see, e.g., D. G. Luenberger & Y. Ye (2008).

This is the standard form of the basic unrestricted Markowitz optimization model for *some* reasonable expected returns being at least as large as the expected return of the Global Minimum Variance Portfolio. The constraint

$$\mathbf{1}^T\mathbf{x} = \sum_{i=1}^{N} x_i = 1 \tag{2.7}$$

refers to a full investment and

$$\boldsymbol{\mu}^T\mathbf{x} = \sum_{i=1}^{N} x_i\mu_i = \mu_p$$

predefines the investor's lower bound with respect to his or her expected return. More generally we write the optimization problem as

$$\Phi(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\Sigma\mathbf{x} \to \min \quad \text{subject to} \quad B\mathbf{x} = \mathbf{c} \tag{2.8}$$

with a symmetric positively definite matrix $\Sigma \in \mathbb{R}^{N\times N}$, the matrix $B = (\mathbf{1}\ \boldsymbol{\mu})^{\mathbf{T}} \in \mathbb{R}^{2\times N}$ and the vector $\mathbf{c} = (1\ \ \mu_{\mathrm{p}})^{\mathbf{T}}$.
This problem has two linear constraints, i.e., the full investment condition and a given expected return. (Real investors will usually face more constraints, see section 2.6.)

2.3.1 Additional Constraints and some Practical Problems

As can easily seen, the above model is far from most practical applications, and the most important technical restrictions to the majority of individual and institutional investors are the following:

1. Short sales are often not possible, not desired or are not allowed; i.e., the $x_i \geq 0, i \in \{1, \ldots, N\}$.

2. There are often upper and lower limits for investment percentages (especially important are the legal restrictions for institutional investors):

$$x^i_{\text{lower}} \leq x_i \leq x^i_{\text{upper}}, i \in \{1, \ldots, N\}.$$

3. Integer constraints often apply, especially with regard to investments in real estate or pieces of art: $x_i \geq 0, i \in \{1, \ldots, N\}$: $x_i = c_i * z_i$, with integer z_i.

A directly associated problem of high importance for any actual asset allocation has already been referred to: The estimated probability distribution of the asset returns changes with new incoming information over time. There are thus necessarily errors in the model input estimates, and directly related consequences on the asset allocation and the deviation between expected and real performance of every portfolio.[1] In the following we will deal with quadratic optimization problems with different constraints.

2.3.2 Other Basic Assumptions of the Model

Applications of the Markowitz theory are widely discussed in academia, as well as among practitioners (see, e.g., Farrell (1997).). The Markowitz theory is among the core chapters of all important textbooks on Finance, Corporate Finance and Investments (see, e.g., Sharpe *et al.* (2008), Elton *et al.* (2007), Brealey *et al.* (2005)). There are usually several (not always the same) assumptions mentioned, the most important of which are summarized here:

- Only two parameters of asset return distributions are considered, namely the expected return and the variance or standard deviation. This actually implies a normal distribution, and empirical research shows rather small and acceptable deviations from this assumption when the assets are bonds or stock (see, e.g., Andersen & Ebens (2001)).

- The Markowitz model is a single–period model: This time period can be one day, one month, one year, three years, etc. We note that, the longer the period is, is the higher the uncertainty with respect to future outcomes. The variance is a linear function of the time (see, e.g., Harville (1977)).

- Investors are risk averse, objecting to maximizing expected utility (see, e.g., Farrell (1997)).

[1] This problem will be discussed in more detail in the empirical section of this thesis.

- It holds the Efficient Market Hypothesis in one of its three forms: The core argument is that prices of assets correspond to their economic value, because every investor can have sufficient information on the market and understand the possible probability distribution of the asset returns.[1]

- In the investment process, an individual or an organization can be the optimizing agent price taker. This means that their investment decisions are separate from the different asset classes.

- Inflation will not be considered, as the Markowitz model actually deals with nominal returns. Inflation is assumed to be one part of this overall nominal return.

- There are transaction costs and taxes. Trading on the market occurs without friction.[2]

2.4 Solution and Features of the Unrestricted Problem

There are different techniques to solve the quadratic optimization problem, depending on the set of constraints. While the classical, unrestricted problem can be solved analytically using Lagrange multipliers, efficient portfolios in the unrestricted class are characterized by a quadratic relation between variance and expected return. Solutions of problems with non–negativity constraints can be solved by using the Critical Line Algorithm proposed by Harry Markowitz[3] and Quadratic Programming used, e.g., by Philip Wolfe.[4] Special focus will be on two outstandingly important portfolios; namely, the Global Minimum Variance Portfolio and the Tangential Portfolio. In Section 2.6 we will investigate several

[1] See Fama (1970).

[2] Transaction costs are nowadays not very critical anymore, neither for individuals or for institutional investors, see, e.g., Hong (2004). Taxes actually have very different important possible impacts, e.g., with respect to the taxable income, legal status, tax authorities and citizenships, etc.

[3] See Markowitz (1956).

[4] See Wolfe (1959).

situations with non–negativity constraints. The development of the theory and the notation in this chapter follow, with some modifications, C. Huang & R. H. Litzenberger (1988).

2.4.1 Solution of the Unrestricted Problem

Theorem 2.2 *Suppose $\boldsymbol{\Sigma}$ is regular. The optimization problem*

$$\min_{\mathbf{x}\in\mathbb{R}^N} \mathbb{V}(R) = \mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}$$

subject to

$$\mathbf{1}^T\mathbf{x} = \sum_{i=1}^{N} x_i = 1$$

and

$$\boldsymbol{\mu}^T\mathbf{x} = \sum_{i=1}^{N} x_i\mu_i = \mu_p$$

has then the unique solution:

$$x^*(\mu_P) = \frac{c\mu_P - b}{ac-b^2}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right) + \frac{a - b\mu_P}{ac-b^2}\left(\boldsymbol{\Sigma}^{-1}\mathbf{1}\right), \tag{2.9}$$

where

$$\begin{aligned} a &:= \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \\ b &:= \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{1} = \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \\ c &:= \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}. \end{aligned}$$

Proof: Given the regular matrix $\boldsymbol{\Sigma}$, i.e., the inverse of this matrix also exists. With two linear constraints, the Lagrange function is now

$$\mathcal{L}(\mathbf{x},\lambda_1,\lambda_2) = \mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x} - \lambda_1(\mathbf{1}^T\mathbf{x}-1) - \lambda_2(\boldsymbol{\mu}^T\mathbf{x}-\mu_p). \tag{2.10}$$

A necessary condition for a solution to this problem is the existence of the first derivatives of $\mathcal{L}(\mathbf{x},\lambda_1,\lambda_2)$ with respect to $\mathbf{x}$,λ_1 and λ_2. It follows that

$$\frac{\partial\mathcal{L}(\mathbf{x},\lambda_1,\lambda_2)}{\partial\mathbf{x}} = \mathcal{L}_x(\mathbf{x},\lambda_1,\lambda_2) = 2\boldsymbol{\Sigma}\mathbf{x} - \lambda_1\mathbf{1} - \lambda_2\boldsymbol{\mu} \stackrel{!}{=} 0. \tag{2.11}$$

The derivatives of $\mathcal{L}(\mathbf{x}, \lambda_1, \lambda_2)$ with respect to λ_1 and λ_2 set equal to zero, giving us again, of course, the two initial constraints.

$$\begin{aligned}\frac{\partial \mathcal{L}(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_1} &= \mathbf{1}^T \mathbf{x} - 1 \overset{!}{=} 0.\\ \frac{\partial \mathcal{L}(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_2} &= \boldsymbol{\mu}^T \mathbf{x} - \mu_p \overset{!}{=} 0.\end{aligned}$$

There are now three linear equations with the three unknown variables $\mathbf{x}$, λ_1 and λ_2.

Using a matrix notation, (2.11) will be written with $\boldsymbol{\lambda} = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$ as

$$\begin{pmatrix}1 & \mu_1\\ \vdots & \vdots\\ 1 & \mu_N\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = [\mathbf{1}\,\boldsymbol{\mu}]\boldsymbol{\lambda} = 2\boldsymbol{\Sigma}\mathbf{x}.$$

Solving this with respect to $\mathbf{x}$ gives us

$$\mathbf{x} = \frac{1}{2}\boldsymbol{\Sigma}^{-1}[\mathbf{1}\,\boldsymbol{\mu}]\boldsymbol{\lambda}. \tag{2.12}$$

To substitute the Lagrange multipliers, (2.12) is now multiplied by $[\mathbf{1}\,\boldsymbol{\mu}]^T$ from the left side. It follows that

$$[\mathbf{1}\,\boldsymbol{\mu}]^T\mathbf{x} = \frac{1}{2}[\mathbf{1}\,\boldsymbol{\mu}]^T\boldsymbol{\Sigma}^{-1}[\mathbf{1}\,\boldsymbol{\mu}]\boldsymbol{\lambda}.$$

Defining now

$$\mathbf{A} := [\mathbf{1}\,\boldsymbol{\mu}]^T\boldsymbol{\Sigma}^{-1}[\mathbf{1}\,\boldsymbol{\mu}] = \begin{pmatrix}\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1} & \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\\ \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{1} & \boldsymbol{\mu}^T\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\end{pmatrix} =: \begin{pmatrix}c & b\\ b & a\end{pmatrix}.$$

We note that, as long as $\boldsymbol{\Sigma}$ is a positive definite or regular and not all expected returns μ_i, $i = 1, \ldots, N$, are equal, the unique inverse of $\mathbf{A}$ exists, namely

$$\mathbf{A}^{-1} = \frac{1}{ac - b^2}\begin{pmatrix}a & -b\\ -b & c\end{pmatrix}.$$

It follows that

$$[\mathbf{1}\,\boldsymbol{\mu}]^T\mathbf{x} = \frac{1}{2}\mathbf{A}\boldsymbol{\lambda}.$$

Using the two constraints we have

$$[\mathbf{1}\ \boldsymbol{\mu}]^T\mathbf{x} = \begin{pmatrix} \mathbf{1}^T\mathbf{x} \\ \boldsymbol{\mu}^T\mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 \\ \mu_P \end{pmatrix} = \frac{1}{2}\mathbf{A}\boldsymbol{\lambda}$$

and it consequently follows that

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \boldsymbol{\lambda} = 2\mathbf{A}^{-1}\begin{pmatrix} \mathbf{1}^T\mathbf{x} \\ \boldsymbol{\mu}^T\mathbf{x} \end{pmatrix}.$$

We get thus a closed solution for the variance minimizing portfolio under the constraints that we invest all, and that we predefine a given expected return, i.e., that 2.9 is equivalent to

$$\mathbf{x}^*(\mu_{\mathbf{p}}) = \boldsymbol{\Sigma}^{-1}[\mathbf{1}\ \boldsymbol{\mu}]\mathbf{A}^{-1}\begin{pmatrix} 1 \\ \mu_p \end{pmatrix}. \tag{2.13}$$

Remark: A very important relation between the expected return and the variance of all efficient portfolios follows from this result. By definition, the variance of the optimal portfolio equals $\sigma_p^2 = \mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}$. Taking the solution obtained in (2.13) and using it to replace the vector $\mathbf{x}$, it holds that

$$\begin{aligned}
\sigma_p^2(\mathbf{x}^*) = \mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x} &= (1\ \mu_p)\mathbf{A}^{-1}[\mathbf{1}\ \boldsymbol{\mu}]^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}[\mathbf{1}\ \boldsymbol{\mu}]\mathbf{A}^{-1}\begin{pmatrix} 1 \\ \mu_p \end{pmatrix} \\
&= (1\ \mu_p)\mathbf{A}^{-1}[\mathbf{1}\ \boldsymbol{\mu}]^T\boldsymbol{\Sigma}^{-1}[\mathbf{1}\ \boldsymbol{\mu}]\mathbf{A}^{-1}\begin{pmatrix} 1 \\ \mu_p \end{pmatrix} \\
&= (1\ \mu_p)\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1}\begin{pmatrix} 1 \\ \mu_p \end{pmatrix} \\
&= (1\ \mu_p)\mathbf{A}^{-1}\begin{pmatrix} 1 \\ \mu_p \end{pmatrix} \\
&= \frac{1}{ac-b^2}(1\ \mu_p)\begin{pmatrix} a & -b \\ -b & c \end{pmatrix}\begin{pmatrix} 1 \\ \mu_p \end{pmatrix}.
\end{aligned}$$

The last representation is equivalent to

$$\sigma_p^2(\mathbf{x}^*) = \frac{1}{ac-b^2}(c\mu_p^2 - 2b\mu_p + a). \tag{2.14}$$

2.4.2 The Global Minimum Variance Portfolio

Equation 2.14 leads to another important result concerning the **Global Minimum Variance Portfolio**. The comparison of this special portfolio with other efficient portfolios for different investors will be discussed in more detail in the following chapters.

Theorem 2.3 *The portfolio*

$$\mathbf{x_{GMVP}} = \frac{1}{c}\mathbf{\Sigma}^{-1}\mathbf{1}$$

is the overall risk minimum portfolio. The point $\left(\sqrt{\frac{1}{c}}, \frac{b}{c}\right)$ *denotes the coordinates of the GMVP in terms of* σ *and* μ*.*

Proof: The optimal solution must satisfy the following condition:

$$\frac{d\sigma_p^2}{d\mu_p} = \frac{1}{ac-b^2}(2c\mu_p - 2b) \stackrel{!}{=} 0.$$

The expected return of the variance minimizing portfolio follows directly from this equation. It is

$$\mu_{GMVP} = b/c. \tag{2.15}$$

From substituting μ_{GMVP} in (2.14), it follows that

$$\sigma_p^2(\mathbf{x}^*) = \frac{1}{ac-b^2}\left(c\left(\frac{b}{c}\right)^2 - 2b\left(\frac{b}{c}\right) + a\right) = \frac{ac-b^2}{ac-b^2}\frac{1}{c} = \frac{1}{c}. \tag{2.16}$$

Using 2.13, the vector of the overall risk minimum portfolio is now easily computed:

$$\mathbf{x}^* = \mathbf{x_{GMVP}} = \mathbf{\Sigma}^{-1}[\mathbf{1}\,\boldsymbol{\mu}]\mathbf{A}^{-1}\begin{pmatrix}1\\ b/c\end{pmatrix} = \frac{1}{c}\mathbf{\Sigma}^{-1}\mathbf{1}. \tag{2.17}$$

These results are actually of enormous importance regarding the GMVP: They have the following implications:

- If someone is, e.g., asking for the *best* portfolio with an expected return of say 5 %, one has to check first whether this target return is at least as large as the expected return of the global minimum variance portfolio!
- From the investors' perspectives, the GMVP should be taken as a lower benchmark, meaning that the investments under consideration should not underperform the GMVP.

- In the neighborhood toward North–East, the efficiency line has a very high positive slope. The investor thus gets a "good" exchange rate of return for taking more risk.

- The Global Minimum Variance Portfolio is the only portfolio which is independent of the vector $\boldsymbol{\mu}$, and it is thus less influenced by errors in the estimates.

The GMVP thus plays an outstanding role as a reference portfolio for the investments of especially risk averse investors.

2.4.3 The Risk–Free Asset and the Tangential Portfolio

So far we have taken only risky assets into consideration, i.e., $\sigma_i^2 > 0 \quad \forall i$. Now we will add a risk–free asset to the set of N risky investment alternatives, and it is further assumed that investors can borrow or lend as much as desired at the risk-free rate R_f.

Suppose x is the rate to be invested in the risky asset. Consequently, $1 - x$ is invested in the risk–free asset. Denoting the expected return of any risky asset by R and R_f as the guaranteed return of the risk–free asset, the expected return of the portfolio is now

$$\begin{aligned} \mu_p &= xR + (1-x)R_f \\ &= R_f + x(R - R_f) \end{aligned} \tag{2.18}$$

and for the variance holds

$$\begin{aligned} \sigma_p^2 &= x^2\sigma_R^2 + (1-x)^2\sigma_{R_f}^2 + 2x(1-x)\sigma_{(R,R_f)} \\ &= x^2\sigma_R^2. \end{aligned} \tag{2.19}$$

The standard deviation is thus a linear function of the rate invested in the risky asset: $\sigma_p = x\sigma_R$. In a σ–μ–framework, it follows consequently that

$$\mu_p = R_f + \frac{\sigma_p}{\sigma_R}(\mu_R - R_f). \tag{2.20}$$

It will now be shown that combinations of the risk–free asset and the **Tangential Portfolio** dominate all risky portfolios on the upper branch of the efficiency line. The Tangential Portfolio is the one with the highest achievable slope of a tangent connecting $(R_f, 0)$ and the upper branch of the efficiency line.

Theorem 2.4 *The portfolio weights of the Tangential Portfolio with* μ_{TP} *and* σ_{TP} *are:*

$$
\begin{aligned}
x_{TP} &= \frac{1}{ac-b^2}\boldsymbol{\Sigma}^{-1}[\boldsymbol{\mu}\ \mathbf{1}]\begin{pmatrix} c\mu_{TP}-b \\ -b\mu_{TP}+a \end{pmatrix} && (2.21)\\
&= \frac{1}{b-cR_f}\boldsymbol{\Sigma}^{-1}[\boldsymbol{\mu}-\mathbf{1}R_f]. && (2.22)
\end{aligned}
$$

Proof: From equation (2.14) it follows that

$$
\sigma_P(\mathbf{x}^*) = \left(\frac{1}{ac-b^2}(c\mu_p^2 - 2b\mu_p + a)\right)^{1/2}.
$$

Omitting the subscript p and taking the first derivative, it follows that

$$
\begin{aligned}
\frac{d\sigma}{d\mu} &= \frac{1}{2}\frac{1}{(ac-b^2)^{1/2}}(c\mu^2 - 2b\mu_p + a)^{-1/2}(2c\mu - 2b)\\
&= \frac{c\mu - b}{\sqrt{ac-b^2}\sqrt{c\mu^2 - 2b\mu + a}}.
\end{aligned}
$$

Inverting this expression leads to

$$
\frac{d\mu}{d\sigma} = \frac{\sqrt{ac-b^2}\sqrt{c\mu^2 - 2b\mu + a}}{c\mu - b}.
$$

This, however, is the exact slope $\frac{\mu_{TP}-R_f}{\sigma_{TP}}$ of the straight line connecting $(R_f, 0)$ and (μ_{TP}, σ_{TP}). We now obtain

$$
\begin{aligned}
\mu_{TP} &= R_f + \frac{\sqrt{ac-b^2}\sqrt{c\mu_{TP}^2 - 2b\mu_{TP} + a}}{c\mu - b}\sqrt{\frac{1}{ac-b^2}(c\mu_p^2 - 2b\mu_{TP} + a)}\\
&= R_f + \frac{c\mu_{TP}^2 - 2b\mu_{TP} + a}{c\mu_{TP} - b}.
\end{aligned}
$$

After some algebra it follows that

$$
\mu_{TP} = \frac{a - bR_f}{b - cR_f}. \qquad (2.23)
$$

Substituting with μ_{TP} we get the variance

$$\begin{aligned}\sigma_{TP}^2 &= \frac{1}{ac-b^2}\left(c\mu_{TP}^2 - 2b\mu_{TP} + a\right)\\ &= \frac{1}{ac-b^2}\left(c\left(\frac{a-bR_f}{b-cR_f}\right)^2 - 2b\left(\frac{a-bR_f}{b-cR_f}\right) + a\right).\end{aligned}$$

Substituting with μ_{TP} in (2.13) we obtain the following:

$$x_{TP} = \frac{1}{ac-b^2}\mathbf{\Sigma}^{-1}[\boldsymbol{\mu}\ \mathbf{1}]\begin{pmatrix} c\mu_{TP} - b \\ -b\mu_{TP} + a\end{pmatrix} \tag{2.24}$$

$$= \frac{1}{b-cR_f}\mathbf{\Sigma}^{-1}[\boldsymbol{\mu} - \mathbf{1}R_f]. \tag{2.25}$$

Letting x represent the proportion invested in the risky assets, and $1-x$ represent the proportion invested in the risk–free asset, we arrive at the following:

$$\mu_p = xR + (1-x)R_f.$$

$$\begin{aligned}\sigma_p^2 &= x^2\sigma_R^2 + (1-x)^2\sigma_{R_f}^2 + 2x(1-x)\sigma_{(R,R_f)}\\ &= x^2\sigma_R^2.\end{aligned}$$

The following list summarizes the most important features linked to the Tangential Portfolio:

- As for the efficiency line, the risk–free return is not stable over time. It is usually referred to as rates of central banks.
- All efficient portfolios are now mixes of the risk–free asset and the Tangential Portfolio; i.e., they are graphically on the straight line connecting the risk–free asset and the Tangential Portfolio. This is also referred to as the Two–Fund separation.[1]
- By mixing risky and riskless assets, an investor can (at least theoretically) select his or her expected return between the risk–free return and infinity. Depending on their attitude toward risk, investors can invest in only the risk–free asset, in mixes of non–negative proportions of the risk–free asset and the risky Tangential Portfolio, or in a levered Tangential Portfolio. The higher the expected return, the lower the rate invested in the risk–free asset.

[1] See C. Huang & R. H. Litzenberger (1988).

- The higher the expected return of the selected portfolio, the more does the actual performance depend on errors in the estimates.
- In practice, there are two riskless rates for borrowing and lending: R_b and R_l, $R_l < R_b$. These rates are not homogeneous and thus differ among investors according to, e.g., their initial wealth. The resulting graph therefore consists of two straight parts and a small arc of the efficiency line. The lower the difference is between the borrowing and the lending rate, the smaller is the part on the efficiency line, and the smaller the difference is between the slopes of both straight parts.
- The risk–free (lending) rate must be lower than $\mu_{GMVP} = b/c$ to guarantee an economically meaningful solution.

For more details on investors perspectives, see, e.g., Kleeberg (1995).

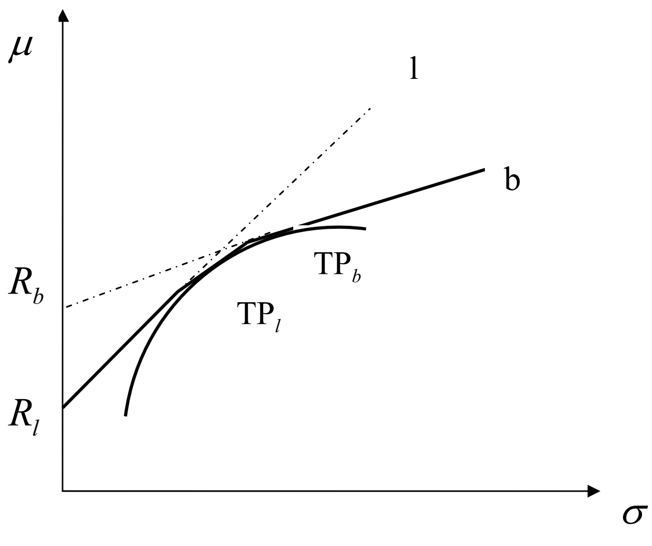

Figure 2.1: Borrowing and Lending

Figure 2.1 presents a pictorial representation of this much more practically relevant situation. Figure 2.2 shows the Minimum Variance Portfolio obtained from

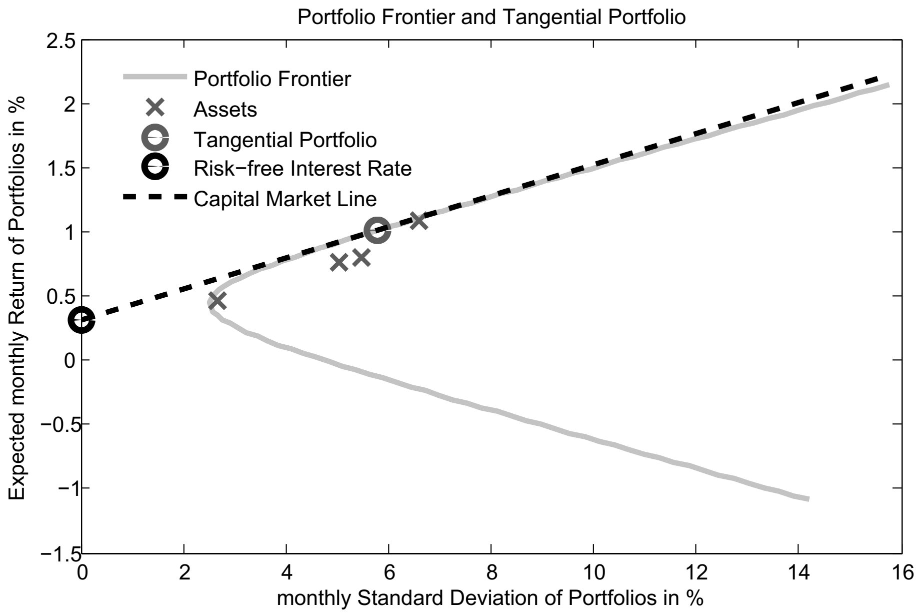

Figure 2.2: Risk–Free Asset and Tangential Portfolio

estimates based on time series data from December 1995 until December 2007, using the four indices DAX 30, GREI, S&P 500 and MSCIWRLD. The Global Minimum Variance Portfolio in this example is characterized by a monthly mean of 0.51% and a standard deviation of 2.38%. The Tangential Portfolio in this example has a monthly average return of 0.79% and a standard deviation of 3.6%. More details on these computations will be discussed in chapter 4.

2.5 The Utility Theory and Optimal Portfolios

Very generally, economics tries to help us understand *how* rational individiuals and the society as a whole allocate scarce resources and distribute wealth over time, whereas the utility theory aims at developing a better understanding of *how* people make choices when they face uncertainty. The utility theory models why an investor chooses some quantitities of different goods within a certain budget.[1]

[1]Cf., e.g., Varian (1992).

In their famous textbook *Theory of Games and Economic Behavior*, published in 1947, John von Neumann and Oskar Morgenstern stated that "We wish to find the mathematically complete principles which define 'rational behavior' for the participants in a social economy, and derive from them the general characteristics of that behavior."[1]

In this section, we will compose the general principles of the utility theory.[2] We will start by reviewing some issues of preference relationships, and then proceed to establish a link between some typical investors and utility functions. Already note that it is very difficult to choose a proper utility function for individual investors in practical applications (cf., e.g., Bell (1995)).

Suppose an individual or investor is faced with the outcomes of an entire set, S, of uncertain alternatives[3]:

$$S = \{x_1, x_2, \ldots, x_N\}.$$

A preference relationship is denoted by $\succeq$. For the entire set S of uncertain alternatives, an individual may regard outcome x_i as preferable to outcome x_j, x_j as preferable to x_i, or the individual may be indifferent regarding $x_j \sim x_i$. **Transitivity** is a feature of a set S if $\forall i, j, k \in \{1, ..., N\}$ holds that

$$(x_i \succ x_j \quad and \quad x_j \succ x_k) \Longrightarrow x_i \succ x_k, \forall \quad i, j, k.$$

Analogously, if an individual is indifferent as to x_i and x_j and to x_j and x_k, then he or she is indifferent as to x_i and x_k:

$$(x_i \sim x_j \quad and \quad x_j \sim x_k) \Longrightarrow x_i \sim x_k, \forall \quad i, j, k.$$

A utility function $U(\cdot) : R \rightarrow \mathbb{R}$ is a twice-differentiable function of a rate of return on an investment. The expected utility theory states that decision–makers choose between risky or uncertain prospects by comparing their expected utility

[1] See J. von Neumann & O. Morgenstern (1944).

[2] For a concise and readable overview of the concepts of utility function and expected utility, see, e.g., Varian (1992).

[3] See Copeland *et al.* (2005). Note that the notation x is not linked to optimal portfolios.

values. The utility of the expected return R via a Taylor series development of the function U at $\mathbb{E}[R]$ is

$$\begin{aligned} U(R) &= U(\mathbb{E}[R]) + U'(\mathbb{E}[R])(R - \mathbb{E}[R]) + \frac{1}{2}U''(\mathbb{E}[R])(R - \mathbb{E}[R])^2 \\ &+ \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)}(\mathbb{E}[R])(R - \mathbb{E}[R])^n. \end{aligned}$$

The expectation of the utility function is a linear combination of the central moments of the random variable R.

$$\begin{aligned} \mathbb{E}[U(R)] &= U(\mathbb{E}[R]) + \frac{1}{2}U''(\mathbb{E}[R])\mathbb{E}[(R - \mathbb{E}[R])^2] + \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)}(\mathbb{E}[R])\mathbb{E}[(R - \mathbb{E}[R])^n] \\ &= U(\mathbb{E}(R)) + \frac{U''(\mathbb{V}(R))}{2} + \frac{U'''(\mathbb{E}(R - \mathbb{E}R)^3)}{6} + \mathcal{O}((R - \mathbb{E}R)^4). \end{aligned} \tag{2.26}$$

For normal ditributions, $\mathbb{E}[U(R)]$ depends only on the first two moments, i.e., it is fully described by expected return and variance. The remaining moments are completely described by

$$\mathbb{E}[R - \mathbb{E}[R]]^n = \begin{cases} 0 & \text{, if } n = 2k - 1, \\ (\sigma^2)^k \prod_{l=1}^{k} (2l - 1) & \text{, if } n = 2k. \end{cases}$$

It follows that

$$\begin{aligned} \mathbb{E}[U(R)] &= U(\mathbb{E}[R]) + \sum_{n=1}^{\infty} \left[\frac{1}{2n!} U^{(2n)}(\mathbb{E}[R]) \sigma^{2n} \prod_{l=1}^{n} (2l - 1) \right] \\ &= U(\mathbb{E}[R]) + \sum_{n=1}^{\infty} \left[U^{(2n)}(\mathbb{E}[R]) \sigma^{2n} \prod_{l=1}^{n} (2l)^{-1} \right]. \end{aligned}$$

Quadratic utility is thus sufficient if both the expected return and the variance are finite. The mean–variance framework implicitly assumes that the investor's utility, defined by the portfolio return, is a function which depends only on the mean and variance. Therefore the legitimacy of any purely mean–variance optimization depends on the assumption that either investors have quadratic utility, or that portfolio returns are normally distributed.

Remark: W can also correspond to $1 + R$, with R being the rate of return on the investment. For normal distributions, the mean–variance model developed in chapter 1 can be motivated by assuming quadratic utility. An individual's expected utility is now characterized by the first two central moments of his or her end–of–period wealth W:

$$\begin{aligned} \mathbb{E}[U(W)] &= \mathbb{E}[W] - \frac{b}{2}\mathbb{E}[W^2] \\ &= \mathbb{E}[W] - \frac{b}{2}[(\mathbb{E}[W])^2 + \sigma^2(W)]. \end{aligned} \tag{2.27}$$

In the following we will discuss investors' attitude toward risk, and the link between risk aversion and utility functions.

2.5.1 Risk Aversion and Optimal Portfolios

Individuals differ with respect to their appreciation of expected return, the interrelation of exchange ratio between risk and expected return and their attitude toward risk. Following are the very general assumptions about the investor's beliefs:

1. Investors prefer more return to less return.
2. Investors know that the marginal utility of the return decreases.
3. Investors prefer less risk to more risk.

Let $U : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly monotonously increasing utility function. The first assumption is equivalent to $\frac{\partial U}{\partial R} > 0$, assumption two corresponds to the concavity of the utility function $\frac{\partial^2 U}{\partial R^2} < 0$, and $\frac{\partial U}{\partial \sigma^2} < 0$ corresponds to the third assumption.

The expected utility $\mathbb{E}[U(R)]$ can thus be maximized either by minimizing the variance or risk, or by maximizing the expected return.
The degree of *risk aversion* is usually measured as the marginal expected return

per additional unit of risk. An important measure of risk aversion is the **absolute risk aversion**, defined by Arrow (1970) and Pratt (1964):[1][2]

$$R_A(\cdot) = -\frac{U''(\cdot)}{U'(\cdot)}. \tag{2.28}$$

An individual's absolute risk aversion reveals whether he or or she treats a risky asset as a *normal* good when choosing between a single risky asset and a risk–less asset.[3] When $R_A(\cdot)$ is a strictly decreasing function, an individual's utility function displays decreasing absolute risk aversion. Table 2.2 gives an overview of several classical utility functions and their absolute and relative risk aversion measurements. As for the risk attitude, there are three increasing absolute risk aversion, constant absolute risk aversion, and decreasing absolute risk aversion. Note that all these types of functions are characterized by $U' > 0$, $U'' < 0$, and, except for quadratic utility and other utility functions, by $U''' > 0$[4].

An investor with a high degree of risk aversion will select investments with low variances and consequently rather low expected returns, whereas individuals with a higher risk tolerance will select investments with higher expected returns, taking into account the associated higher variances. Risk–averse investors will thus choose portfolios on the upper branch of the efficiency curve rather close to the GMVP, while the portfolios of the less risk–averse investors will tend more toward the North–East of the upper branch of the efficiency curve.[5] This entails deducing *what* tradeoff of return and risk constitutes utility for the individual under consideration. Following classical economic analysis, indifference curves are developed showing the magnitude and form of the risk–return trade-off in a

[1] Note that the Arrow–Pratt measurement requires a well–specified utility function, cf. Arrow 1970.

[2] The Arrow–Pratt measurement of relative risk aversion is $R_R(x) \equiv R_A(x)x$.

[3] Arrow showed that decreasing absolute risk aversion implies that the risky asset is a normal good; increasing absolute risk aversion implies that the risky asset is an inferior good; and constant absolute risk aversion implies that the individual's demand for the risky asset is invariant with respect to his initial wealth, cf., e.g., C. Huang & R. H. Litzenberger (1988), pp. 21 – 23.

[4] A positive U''' denotes decreasing risk aversion.

[5] On a more conceptual basis, the decision for an investment would be a matter of maximizing the individual's expected utility.

Table 2.2: Utility Functions and Measurements of Risk Aversion

Utility function	*Absolute Risk Aversion Measure*	*Proportional Risk Aversion Measure*
$W - \frac{\beta}{2}(W - W_0)^2$	Increasing	Increasing
$-l^{-\alpha W}, (\alpha > 0)$	Constant	Increasing
$(W + B)^\alpha, (0 < \alpha < 1)$	Decreasing	Increasing for $B > 0$ Constant for $B = 0$ Decreasing for $B < 0$
$ln(W + B), (B > 0)$	Decreasing	Increasing for $B > 0$ Constant for $B = 0$ Decreasing for $B < 0$

mean–variance framework (see figure 2.3). All combinations of expected return and variance along one indifference curve have the same utility for an investor; i.e., he or she is indifferent with repect to these $\mu - \sigma$ pairs. The optimum portfolio is now the one in the $\mu - \sigma$ space on the upper branch of the efficient frontier that is tangent to the highest achievable indifference curve (see 2.4).
In the upcoming subsection, we will discuss several utility functions which are often used in financial research.

2.5.2 Utility Function and Expected Utility Maximization

The most popular utility function depending on expected return μ and variance σ^2 is obviously[1]

$$U(\mu, \sigma^2) = \alpha\mu - \frac{1}{2}\beta\sigma^2, \tag{2.29}$$

[1] For a detailed discussion about approximation of expected utility under different functions, see Cremers *et al.* (2004) and Kroll *et al.* (1984).

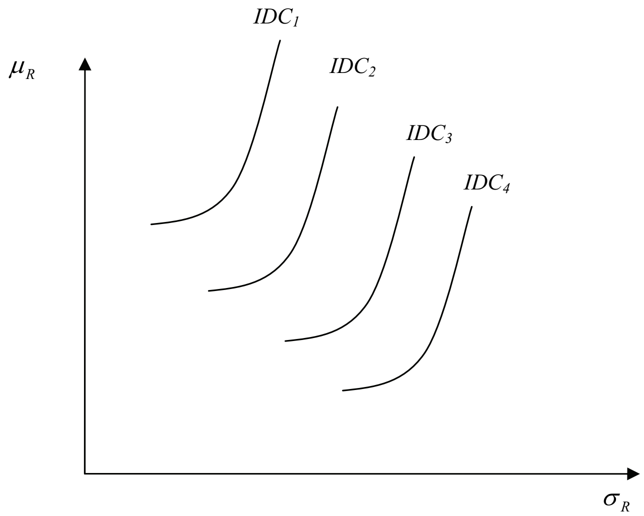

Figure 2.3: Indifference Curves

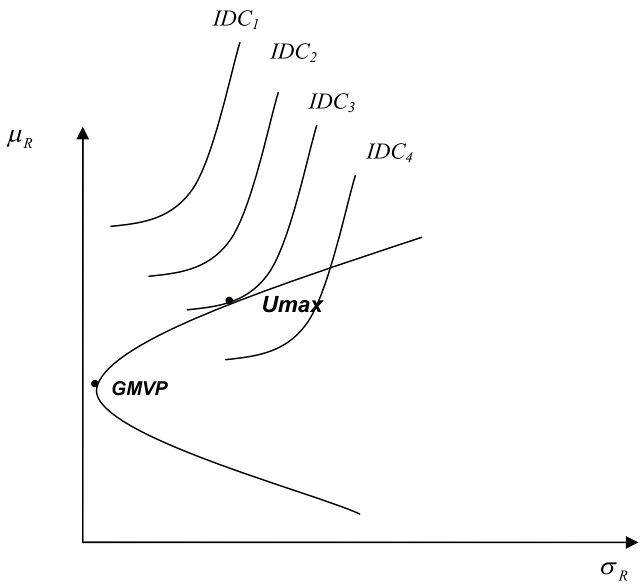

Figure 2.4: Indifference Curves and the Portfolio Frontier

with α and β being parameters which describe the investor's attitude toward risk (α is often assumed to be 1 and β is the only varying parameter.) Note that quadratic utility cannot be a fully realistic model of a typical investor's attitude toward risk as it assumes that investors are equally averse to deviations above the mean and to deviations below the mean, and that they sometimes prefer less wealth to more wealth. Quadratic utility displays the undesirable properties of satiation and increasing absolute risk aversion. The satiation property implies that an increase in wealth beyond the satiation point decreases utility, and the property of increasing absolute risk aversion implies that risky assets are inferior goods. The concavity indicates that investors derive less and less satisfaction with each subsequent unit of incremental wealth. Thus, economic conclusions based on the assumption of quadratic utility function are often counter–intuitive, and are not applicable to individuals who always prefer more wealth to less and who treat risky investments as normal goods (For more details on the utility theory and risk preference cf., e.g., C. Huang & R. H. Litzenberger (1988)).

Another utility function often applied by economists is the power utility function (A log wealth utility function is a special form of power utility function):[1]

$$U(W) = \frac{1}{\gamma} W^{\gamma}. \tag{2.30}$$

As γ converges to 0, the utility tends toward the natural logarithm of wealth. As γ equals 1/2, the utility function implies less risk aversion than log wealth, while a γ equal to -1 implies greater risk aversion.[2]
Using the Taylor series developments, we can approximate expected utility to be around μ.[3]
Table 2.3 summarizes these examples of utility functions and their corresponding approximated expected utilities.[4]

[1] See Cremers *et al.* (2004).
[2] Under this situation the utility is expressed as $1 - W^{-1}$.
[3] Levy & H. M. Markowitz (1979) show that the approximation based on a Taylor series around μ performed markedly better than the approximation around zero.
[4] See Cremers *et al.* (2004).

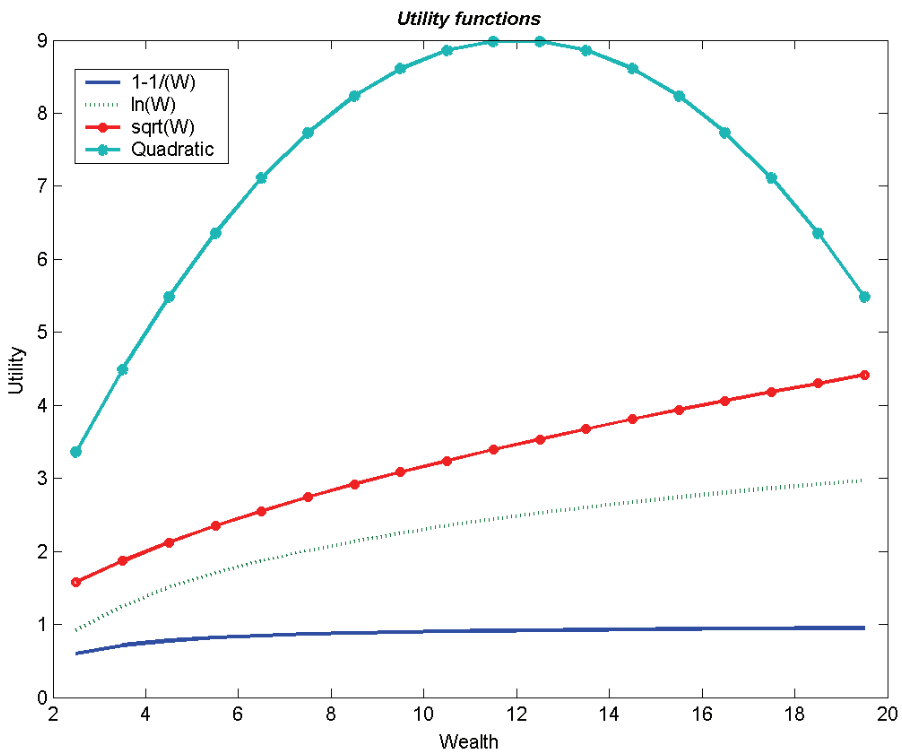

Figure 2.5: Utility Functions

Table 2.3: Utility Functions and Expected Utilities

Utility Function	*Expected Utility*	*Approximate Utility*
Quadratic	$W - \frac{\beta}{2}(W - W_0)^2$	$U_1(\mu, \sigma^2) = \mu - \frac{\beta}{2}\sigma^2$
Log wealth	$log(W)$	$U_2(\mu, \sigma^2) = log(1 + \mu) - \frac{\sigma^2}{2(1+\mu)^2}$
Power	$1 - \frac{1}{W}$	$U_3(\mu, \sigma^2) = 1 - \frac{1}{1+\mu} - \frac{\sigma^2}{(1+\mu)^3}$

Hence, under this definition of expected utility, the optimization problem 2.8 can be changed into the following form:

$$\Phi(\mathbf{x}) = \mathbf{E}[U(\mathbf{x})] \to \max \quad \text{subject to} \quad B\mathbf{x} = \mathbf{c}. \tag{2.31}$$

$U_1(\mu, \sigma^2)$ is the quadratic utility, and U_2 and U_3 are derived from power utility, also using only a portfolio's mean and variance. H. Levy & H. M. Markowitz (1979) show how to approximate these utility functions, and demonstrate that mean–variance approximations to utility based on plausible power utility functions performed exceptionally well for returns ranging from –30% to +60%. This is actually the reason that these different plausible utility functions were chosen for application in the further research of this thesis.

The three utility functions $U_i(\mu, \sigma^2), i \in (1, 2, 3)$ and their expected utilities from table 2.3 will be used in the following empirical research, and will furthermore be modified with respect to the initial wealth. We expect the following:

- There are differences between the expected utility of the optimum portfolios for the given utility functions and the expected utility of well–selected portfolios from the mean–variance efficient frontier. We will apply the expected utility maximization to asset allocation strategies of different types of investors.

- The three approximate utilities describe different types of investors with different risk aversion levels. Practically, this leads us to a classification of utility functions and certain types of investors.

In chapter 3, some typical types of investors with different initial wealth levels, degree of risk aversion and time horizons will be classified. Then we will construct optimal asset allocation strategies within the mean–variance framework. With respect to the different degrees of risk aversion of these typical investors, we will deal with the concept of expected utility maximization. We assume that a risk–averse investor will penalize the expected rate of return of a risky portfolio by a certain percentage, in order to account for the risk involved. The greater the risk the investor perceives, the larger the penalty. Next will follow a classification of different types of investors.

Before moving toward these goals, we will continue by discussing the impacts of more practically relevant constraints on the set of feasible portfolios, which constraints therefore also impact the set of optimal portfolios.

2.6 The Markowitz Model with Unilateral Restrictions

In this section we will discuss the influence of additional restrictions on the set of possible solutions of optimal portfolios x_i, $i = 1, \dots, N$. Such restrictions, which are highly relevant for practical investment decisions can, e.g., result from the law–based policy of mutual fund management, or from arbitrary investors' individual preferences.

We will firstly discuss the portfolio optimization problem under the restriction that short sales are prohibited; this means, that all components of the vector $\mathbf{x}$ are non–negative. Obviously, such a portfolio has an expected return in the range between the minimum and the maximum expected return of all N assets. Because short sales–driven leverage no longer exists, the feasible set of all possible solutions is now necessarily smaller than the feasible set of the unrestricted problem. Furthermore, in practical applications, the additional restrictions lead to a solution which is, for any feasible given expected return (as seen from the sole point of the total variance), usually worse (It cannot be better!) than the solution of the unrestricted optimization problem.

In subsection 2.6.3, even stronger restrictions will be introduced, namely that the shares of the assets are confined by a lower and by an upper limit. Realistically, we will not require that every asset participates in the portfolio with a certain minimal share, but will allow that the shares are in the range between a minimum and a maximum number, or that they are zero. In other words: the investor may buy at least some shares of an asset, or nothing. In section 2.7, we will show that the nature of such optimization problems differs substantially from the problems seen before.

2.6.1 Non–Negativity Restrictions

Consider the constrained optimization problem (2.8) together with the additional restrictions $x_i \geq 0$ for $i = 1, \ldots, N$. It is concisely described by

$$\Phi(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\Sigma\mathbf{x} \rightarrow \min \quad \text{subject to} \quad B\mathbf{x} = \mathbf{c} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}, \tag{2.32}$$

where $\mathbf{x} \geq \mathbf{0}$ means that every component $x_i, i = 1, \ldots, N$, of the vector $\mathbf{x}$ is non–negative.
This optimization problem with equality and inequality conditions requires computationally powerful quadratic optimization algorithms, such as the critical line algorithm of Markowitz (1959), and the extended simplex algorithm of Wolfe (1959). Newer research by Hirschberger *et al.* (2004) indicates that there still remains the need to improve the performance of the quadratic optimization algorithms.[1] A. Niedermayer & D. Niedermayer (2006) provide a recently developed operations research algorithm.[2]

Example: The following example, which will be continued in the upcoming section, is characterized by input that is rather close to practical applications. It is constructed in order to illustrate the differences with respect to the feasible set of unrestricted and restricted optimization problems. Assuming that there are 5 assets with the following characteristics:

$$\boldsymbol{\mu} = \begin{pmatrix} 0.06 \\ 0.04 \\ 0.09 \\ 0.01 \\ 0.07 \end{pmatrix}$$

and the variance–covariance matrix

$$\mathbb{COV}(\mathbf{R}) = \mathbf{\Sigma} = \begin{pmatrix} 0.070 & 0.025 & 0.014 & 0.017 & 0.017 \\ 0.025 & 0.050 & 0.020 & 0.040 & 0.020 \\ 0.014 & 0.020 & 0.080 & 0.016 & 0 \\ 0.017 & 0.040 & 0.016 & 0.050 & 0.020 \\ 0.017 & 0.020 & 0 & 0.020 & 0.050 \end{pmatrix}.$$

[1] For an application, see Wolf (2006).
[2] In this work we will use a Matlab quadratic optimization tool based on the critical line algorithm.

The figures 2.6 show the portfolio frontiers of the unrestricted problem (i.e., with short sales), and also the problem with non–negativity constraints (without short sales).

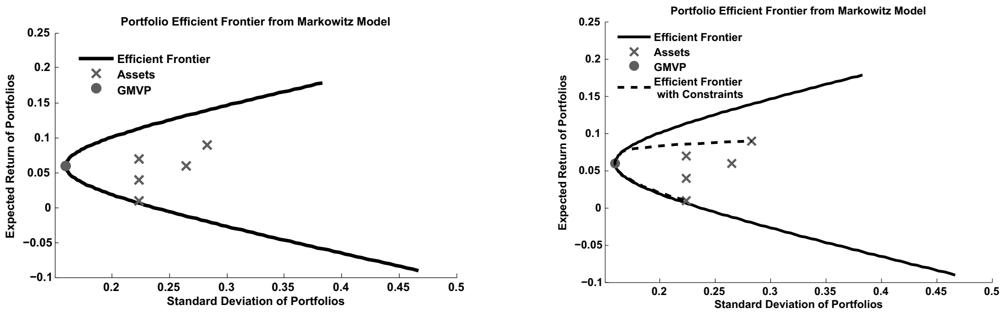

Figure 2.6: Portfolio Frontier with 5 Assets

2.6.2 The Kuhn–Tucker Conditions

Many quadratic programming algorithms are based on a technique from advanced calculus called **Kuhn–Tucker–conditions**.[1] For small–scale problems, these conditions can be used directly.[2]

The Lagrangian function of the constrained optimization problem (2.32)[3] is

$$L(\mathbf{x}, \boldsymbol{\lambda}) = -\mathbf{x}^{\mathbf{T}}\Sigma\mathbf{x} + \boldsymbol{\lambda}^{\mathbf{T}}(B\mathbf{x} - \mathbf{c})$$

with the Lagrangian multiplier being $\boldsymbol{\lambda} \in \mathbb{R}^2$. The Kuhn-Tucker conditions are a complete taxonomy of the first order necessary conditions of obtaining a saddle

[1]Although these conditions are popularly known as the Kuhn–Tucker conditions, it turns out that a master's student at the University of Chicago named Karush derived the same conditions as part of his M.S. thesis, several years prior to the 1952 work by Kuhn and Tucker at U.C. Berkeley.

[2]For more details see Karush (1939), H. W.Kuhn & A. W. Tucker (1951), and Dybvig (1984).

[3]A minimization problem can usually be transformed into a maximization problem by multiplying the objective function by − 1.

point for the Lagrangian function. These conditions are given by:

$$\frac{\partial}{\partial \mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) \;\leq\; 0\,, \mathbf{x} \geq 0\,, \quad \mathbf{x}\frac{\partial}{\partial \mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) = 0, \tag{2.33}$$

$$\frac{\partial}{\partial \boldsymbol{\lambda}} L(\mathbf{x}, \boldsymbol{\lambda}) \;\geq\; 0\,, \boldsymbol{\lambda} \geq 0\,, \text{and} \quad \boldsymbol{\lambda}\frac{\partial}{\partial \boldsymbol{\lambda}} L(\mathbf{x}, \boldsymbol{\lambda}) = 0. \tag{2.34}$$

The Kuhn–Tucker conditions state that $\mathbf{x}$ and $\boldsymbol{\lambda}$ must both be non–negative, and either the partial derivative of the Lagrangian function with respect to each of these variables must be zero, or the variable must be zero at the optimal solution. With these conditions can the quadratic optimization problem (2.32) be solved.

2.6.3 Upper and Lower Bounds

Furthermore, with respect to the decision making of most investors, upper and lower bounds apply. It is thus often not only reasonable to forbid short sales, but especially to restrict the maximum share of one asset in the portfolio. For individual investors, these barriers are most often linked to some minimum number of assets in their portfolio, or, to borrow the colloquialism, to the minimum number of "eggs in their basket".[1] Institutional investors are usually even more limited with respect to investment shares in individual assets and/or asset classes, and to avoid having too many small investments, it is desirable for these investors to have no investments in a portfolio which are below a certain percentage of the overall investment sum. This furthermore corresponds to the fact that the opportunity and transaction costs may be considerably high in comparison to a very small amount of one asset. In this subsection, the nature of these optimization problems (2.8) will be discussed, i.e., when we ask for a solution which is confined by an upper bound x_{upper} and a lower bound x_{lower} in each component.[2] That is

$$x^i_{\text{lower}} \leq x_i \leq x^i_{\text{upper}} \quad \text{for} \quad i = 1, \ldots, N\,.$$

The optimization problem can now be written as

$$\Phi(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\Sigma\mathbf{x} \to \min \quad \text{subject to} \quad B\mathbf{x} = \mathbf{c} \quad \text{and} \quad \mathbf{x} \in [x^i_{\text{lower}}, x^i_{\text{upper}}]^N \quad \forall i, \tag{2.35}$$

[1] For naive diversification strategies, cf., e.g., Malkiel (2008)'s *A Random Walk Down Wall Street*.

[2] Note that the upper and lower bounds are not necessarily equal for all potential investments.

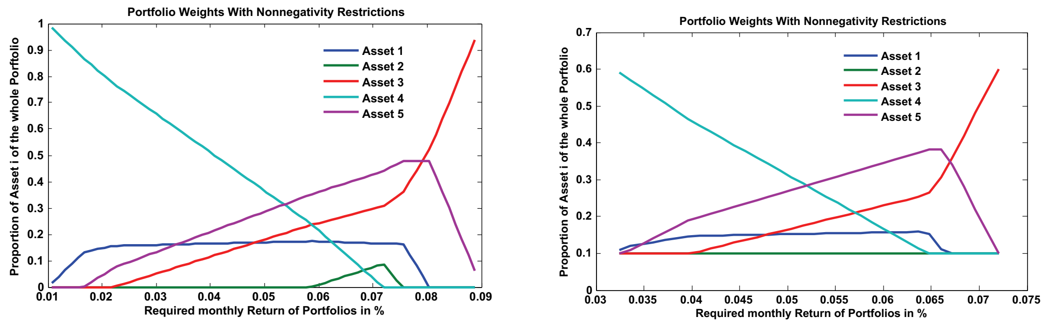

Figure 2.7: Portfolio Weights with Non–Negativity Constraints.

where $\tilde{\Omega} = [x^i_{\text{lower}}, x^i_{\text{upper}}]^N \subset \mathbb{R}^N$ is a N–dimensional cube. This cube is still a convex set. This is the mathematical reason for applying most of the methods from subsection 2.6.1 analogously. Note that a lower bound does not need to be zero.

Example continued: Sequentially we use the previous example to determine the portfolio weights with equal upper and lower bounds. The two figures in 2.7 illustrate the impact of upper and lower bounds, in the case of only non–negativity constraints. The left figure indicates the non–negativity constraint, and the right one the solutions, with upper and lower bounds of 0.1 and 0.6 $\forall i, i = 1, .., 5$. As can easily be imagined, the set of all feasible portfolios is further reduced.

2.7 Portfolio Optimization Problems with Integer Constraints

In the above section, we have discussed Markowitz optimization problems with unilateral restrictions, especially those with non–negativity restrictions. A further new restriction will be that of integer constraints. Note that integer constraints almost always apply for investment decisions (i.e., an investor can usually buy a flat, but not a share of the flat). The optimal quantities in 2.1 are rounded off and taken as correct because of the assumption $B >> P_i, i = 1, ..., N$. This is obviously violated in many investment cases, as, e.g., in real estate and art investments. This section is dedicated to such discrete optimization problems. In this section will we always refer to real estate as investments with integer constraints.

Real estate is, e.g., one of the most important asset classes, and is referred to often by integer restriction in portfolio optimization. Professionally–managed funds of real estate have attracted increasing interest during the last decade from individual investors, as well as from institutional investors. For example, some investors take equity of Real Estate Investment Trusts (REITs) as partial substitutes for conventional real estate investments. Despite the huge financial volumes of the funds[1], the use of quantitatively–based models for a “smart” asset allocation is far behind stock or bond portfolios. On the one hand, real estate fund managers are more conservative than their counterparts in stock or bond portfolios. The importance of intuition, expressed by the major quality criteria *location*, prevent many managers from looking at supporting quantitative models. This is supported by a usually long–lasting decision making process, a comparing of alternatives, etc. On the other hand, modeling is much more difficult because the budget comes back into play. Besides the estimation of expected returns and risk, attempts at measurements are hampered by the long–lasting investment horizons, and by economic cycles, integer constraints, lack of price information, and skewed and mixed return distributions. There is especially a subjective component of price determination, which is linked to the fact that real estate is often unique. Furthermore, big proportions of real estate portfolios are not liquid; that is, any optimization should be restricted on the liquid part. The decision whether or not to buy is essentially a 0–1 decision. This situation applies analogously to investments in pieces of art. In this section we will discuss integer optimization problems.

We refer again to an adjusted version of equation (2.1), where P_i, $i = 1, \ldots, N$, are the prices per unit of an asset, and $X_i \in \mathbb{N}$, $i = 1, \ldots, N$ (formerly denoted by Q_i) are the numbers to be purchased for the portfolio. If a single unit of, e.g.,

[1] According to Ernst & Young 2008, the total market capitalization of publicly–listed REITs around the world reached \$764 billion.

some real estate is considered, then $X_i \in \{0,1\}$, i. e. , the respective real estate is bought ($X_i = 1$) or not ($X_i = 0$). If the real estates are combined into classes according to how comparable they are, it follows that more real estates out of a certain class can be bought, and $X_i \in \mathbb{N}$ holds. The normalized variables

$$x_i = \frac{P_i}{B} X_i = c_i X_i\,,\ i = 1, \ldots, N \ \text{ with } \ X_i \in \mathbb{N}$$

are integer multiples of relative prices $c_i = P_i/B$. The condition that x_i is an integer multiple of the relative price is referred to as the "integer constraint". Using matrix notation, it follows that

$$\mathbf{x} = \mathbf{D}\mathbf{z} \ \text{ with } \ \mathbf{z} \in \mathbb{N}^N \ \text{ respectively } \ \mathbf{z} \in \{0,1\}^N,$$

with the diagonal matrix $\mathbf{D} = \operatorname{diag}(c_1, \ldots, c_N) \in \mathbb{R}^{N \times N}$ and the vector $\mathbf{z} = (X_1, \ldots, X_N)^{\mathsf{T}}$. This notation leads to an optimization problem, which is analogous to problem (2.8):

$$\mathbf{z}^{\mathsf{T}}(\mathbf{D}\boldsymbol{\Sigma}\mathbf{D})\mathbf{z} \to \min \ \text{ subject to } \ (\mathbf{D}\boldsymbol{\mu})^{\mathsf{T}}\mathbf{z} \geq \mu\,,\ (\mathbf{D}\mathbf{1})^{\mathsf{T}}\mathbf{z} \leq 1.$$

This implies, for instance, that modeling the optimization of a pure real estate portfolio differs from the original problem (2.8) of the classical Markowitz theory, just by the integer constraint $\mathbf{z} \in \mathbb{N}^N$ respectively $\mathbf{z} \in \{0,1\}^N$.

The integer constraint causes that the constraints, with respect to full investment and minimum target return in equalities (2.3, 2.7), cannot be satisfied in general. Furthermore, the solution is not necessarily a reasonable one if these constraints are fulfilled. Almost any attempt to invest a large amount in real estate or pieces of art will leave a cash remain which cannot be further invested in real estates, since this cash remain may be smaller than the lowest price of a possible real estate unit or a piece of art, respectively.

Assume now that this cash remain $x_{\mathrm{r}} = 1 - \mathbf{1}^{\mathsf{T}}\mathbf{x}$ is invested in other market instruments, e. g., in some stock, or in the risk–free asset. The investment of the remain x_{r} has the expected return μ_{r} and the variance σ_{r}^2. The expectations of the N real estate investments are now in the vector $\boldsymbol{\mu}$, and the variances and covariances are written into the matrix Σ. The covariance of the remaining investment and of the real estates are now in the vector $\mathbf{s} \in \mathbb{R}^N$.

Following is the optimization problem with integer constraints for the portfolio $\mathbf{x} \in \mathbb{R}^N$:

The Portfolio optimization problem with integer constraints:

$$(\mathbf{x}^{\mathrm{T}} x_{\mathrm{f}}) \begin{pmatrix} \Sigma & \mathbf{s} \\ \mathbf{s}^{\mathrm{T}} & \sigma_{\mathrm{r}}^2 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ x_{\mathrm{r}} \end{pmatrix} \rightarrow \min \tag{2.36}$$

subject to

$$\mathbf{1}^{\mathrm{T}}\mathbf{x} + x_{\mathrm{r}} = 1 \quad \text{and} \quad \boldsymbol{\mu}^{\mathrm{T}}\mathbf{x} + \mu_{\mathrm{r}} x_{\mathrm{r}} \geq \mu_P$$

and the additional integer constraint

$$x_i = c_i X_i \quad \text{with} \quad X_i \in \mathbb{N}.$$

For the reasonable case of an investment in the risk–free asset, the covariance to the real estate vector is obviously zero. The equations (2.7) translate the investor's desire to invest the whole fortune in real estate (e.g., in the risk–free investment), and show that the overall expected return is at least as large as the μ_{f}.

2.7.1 The Knapsack Problem

As seen, the requirement $\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x} + \mu_{\mathrm{f}} x_{\mathrm{f}} = \mu_P$ cannot be satisfied in general. Optimization problems with integer constraints can be solved using techniques such as brute force search and dynamical programming. The knapsack problem[1] is a famous problem in combinatorial optimization which often arises in resource allocation with financial constraints (see, e.g., Kellerer *et al.* (2005)). A set of items is given, each having a weight and a value. The problem is determining how many of each item to include in a collection so that the total weight is bounded by a given limit and the total value is as large as possible.

[1]This name is derived from the problem faced by a traveler who is constrained by a fixed–size knapsack and must fill it with the most useful items.

Mathematically, there are N possible objects to pack in. Each object has a weight $c_i > 0$, which is referred to as the cost, and an importance $v_i > 0$, which can be regarded as the value. The cost vector and the value vector are denoted by $\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}} \in \mathbb{R}^N$ and $\mathbf{v} = (v_1, \ldots, v_N)^{\mathrm{T}} \in \mathbb{R}^N$, respectively. The decision variables are in the vector $\mathbf{z} \in \{0,1\}^N$, which consists of N coefficients 0 or 1, where 1 at position k means to pack in the kth object, and 0 means to leave it out. Formally:

Knapsack Problem:

$$\mathbf{v}^{\mathrm{T}}\mathbf{z} \to \max$$

under all $\mathbf{z} \in \{0,1\}^N$ subject to the constraint

$$\mathbf{c}^{\mathrm{T}}\mathbf{z} \leq C,$$

where C is the maximal weight of the knapsack.

Remark: The analogy to portfolio optimization consists in the maximization of an objective function which is subject to a cost constraint and an integer constraint.

An often reasonable approach to solving the knapsack problem for small N is the **brute force search**. The main idea is to identify every possible combination of objects to be packed in by starting with the heaviest objects, and then looking to see what else can be packed in. Following this, the feasible combinations are compared. The amount of possible portfolios is reduced by sorting feasible portfolios in the sense mentioned above, which means starting with the most expensive real estate, and so on. The portfolios not fulfilling the required expected returns are then eliminated from the equation, allowing us to find the optimal portfolio, simply by comparing all feasible portfolios.

If the computational time is not a restrictive factor in general, the brute force search technique works well, up to the range of approximately $N = 20$. Since the numerical effort grows exponentially, it is barely applicable for a slightly larger N. There are some better implementations, cf., e.g., S. Martello & P. Toth (1990), which organize the search more efficiently. However, the knapsack problem is NP–hard (see, e.g., M. R. Garey & D. S. Johnson (1979)), which means that there is no essentially better algorithm for an exact solution. These difficulties can thus be avoided only by dealing with approximate solution strategies.

In summarization, the brute force search technique can be usefully applied to the optimization of smaller real estate portfolios, especially during the inauguration of a fund. Furthermore, this technique is applicable for portfolio adjustments or best next trade problems, where only a part of the portfolio has to be replaced, and the number of feasible combinations is then substantially reduced.

2.7.2 Dynamic Programming

An example of a more efficient search strategy is the so–called **dynamic programming**, which works by an iterative augmentation of the cost. The augmentation steps are multiples of a common base d for the costs of all objects. If these costs are integers, then the base is the largest common divisor $\mathrm{lcd}\{c_1, \ldots, c_N\}$. Dynamic programming can be applied in a given case if $c_i = dz_i$ with some $z_i \in \mathbb{N}$ for $i = 1, \ldots, N$. Since the last case can be reduced to the integer case by dividing all c_i and the maximal cost by C, we restrict ourselves here to the **algorithm for integer costs c_i.**

Let $A(i)$ be the maximum value which can be attained for the cost i. Obviously, it holds that $A(0) = 0$. We set

$$A(i) = \max_{j, j \text{ not used in } A(i-c_j)} \{v_j + A(i - c_j) \,:\, c_j \leq i\}. \tag{2.37}$$

The numerical effort depends on the necessary number of steps C/d. If this quotient is “small”, then dynamic programming is powerful. With respect to applications of the optimization of, e.g., real estate funds, it holds that dynamic programming is a good technique, if the prices of the objects under consideration are small multiples of a common base. Since the prices are inaccurate prior to the acquisition (i. e. , in the moment that the calculation has to be done), dynamic programming can here be applied with rounded price estimates only, cf., e.g., Adda & R. Cooper (2003); Bertsekas (2000); Stokey *et al.* (1989).

2.7.3 The Greedy Algorithm

The next presented algorithm can be done quickly but is only approximate, and is therefore called a *greedy* algorithm, cf., e.g., S. Martello & P. Toth (1990).[1] The standard greedy algorithm for the knapsack problem is based on a certain order of the objects. Firstly, they are ordered according to their individual values, in decreasing order, and are then inserted into the knapsack, until there no longer remains any space for more. It can be shown that if k is the number of objects in the optimal knapsack solution, then the greedy algorithm inserts at least $k/2$ of them. Secondly and alternatively, the objects can be ordered according to their effectivity v_i/c_i, i. e. , value per cost. Note that intuitively–constructed greedy algorithms do not need to produce good solutions in every setting. Bang-Jensen *et al.* (2004) shows, e.g., that even greedy algorithms, which work well for a large number of realistic applications, may generate the worst possible solutions for particularly chosen input values.

Greedy algorithms are reasonably applied to portfolios with a large number of small assets that have integer constraints. The essential precondition for the use of the presented greedy algorithms, and for dynamic programming, is the additivity of the selected sets, i. e. , the union of two best disjoint sets for certain total costs is the best (or at least a good) set of the summed cost, possibly with respect to a restriction of the single costs. This is not the case, however, in portfolio optimization with correlated assets; in some sense, the best two real estates do not need to form a better portfolio than two other real estates which are, e.g., uncorrelated.[2]

[1] Similar greedy algorithms are used in a wide range of optimization problems (Cormen *et al.* (2001)).

[2] For geographically well–diversified real estate portfolios, the assumption of pair wise low

2.7.4 Modifications to the Knapsack Problem

Another practically relevant case is when there are several preselected and equal units of one assets under integer constraint. With respect to real estate funds, we would look, e.g., at the fact that there are rather homogeneous flats in large buildings. If putting an object more than once into the knapsack is allowed, then the solution is generally to use only the most effective object.
The discrete optimization problem with $\mathbf{z} \in \mathbb{N}^N$ (2.36) is intuitively easier than the binary variant $\mathbf{z} \in \{0,1\}^N$. Indeed, it is easier to find an approximate "guess", but the refinement of that guess leads to difficulties which are identical to those in the binary variant.

Consider the following modified knapsack problem: Fill a knapsack with objects of value v_i, $i = 1, \ldots, N$ and costs c_i, $i = 1, \ldots, N$ of multiplicity $z_i \in \mathbb{N}$ under the condition that the total costs are restricted by C, and maximize an objective function $f(z_1, \ldots, z_n)$.

Typical examples of objective functions are linear functions

$$f(z_1, \ldots, z_N) = z_1 v_1 + \ldots + z_N v_N,$$

which are dealt with in the theory of linear programming or linear optimization (see, e.g., B. Gärtner & J. Matousek (2006)), or non–linear functions, cf., e. g. ,

$$f(z_1, \ldots, z_N) = z_1 z_2 \cdot \ldots \cdot z_N,$$

which are handled by non–linear programming or optimization in J. Nocedal & S. J. Wright (2006) and Schrijver (1998).[1]

The deficiency of a brute force search in the neighborhood of the optimal solution in the $\mathbb{R}^N$ can be overcome by a spectral transformation of the symmetric matrix

or zero correlation is often reasonable (see, e.g., D. Linowski & S. Hartmann (2007).)

[1]As already seen, quadratic programming, where the objective function is a multivariate parabola, is a particular case of non–linear programming. The discrete portfolio optimization problem is thus a quadratic optimization problem with binary integer constraints.

$\mathbf{\Sigma} = \mathbf{Q}^{\mathrm{T}}\mathbf{D}\mathbf{Q}$, with the diagonal matrix $\mathbf{D}$ containing the positive eigenvalues of the positively definite covariance matrix $\mathbf{\Sigma}$. The optimization problem

$$\mathbf{x}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{x} \to \min \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \le \mathbf{b}$$

now reads

$$\mathbf{y}^{\mathrm{T}}\mathbf{y} \to \min \quad \text{subject to} \quad \mathbf{A}\mathbf{Q}^{\mathrm{T}}\sqrt{\mathbf{D}^{-1}}\mathbf{y} \le \mathbf{b},$$

with the transformation $\mathbf{y} = \sqrt{\mathbf{D}}\mathbf{Q}\mathbf{x}$. This optimization problem consists in optimizing a parabola over a multifocal domain, and the solution consists in finding the nearest admissible corner of a parallelogram in $\mathbf{y}$–coordinates, the corners of which are integers in the $\mathbf{x}$–coordinates. Some geometrical considerations restrict the checking of the set of neighbored parallelogram corners. Heuristic search methods and heuristic optimization techniques, which usually incorporate stochastic elements, have been discussed more and more in recent times. For more information about heuristic optimization, see Maringer (2005).

Chapter 3

Investor Classification and Asset Allocation Strategies

Making decisions under uncertainty is a central challenge for investors. A major area of financial research is the *positive* question of *how* people actually make decisions when faced with risk; for example, some people like buying lottery tickets, although the expected value of such an "investment" is less than the cost of the ticket. Harry Markowitz (1952)[1] proposed one of the earliest solutions to this problem by suggesting that investor attitudes toward gambles of different amounts were implicitly relevant to the "customary wealth" of each investor, and gambles for large amounts related to customary wealth are treated more conservatively. In other words, a willingness to *gamble* depends very much on the status quo. The initial wealth level here draws upon our interest in developing a deeper understanding of what it is that motivates investors.

In this chapter, investors will be classified normatively; however, before doing so, we will provide an overview of different regulated capital markets. While, e.g., stocks, bonds, commodities and most financial derivatives are traded at regulated exchanges, note here that these markets cover only a part of all investments (we mention here two very important asset classes: traded real estate, a very small proportion of which is, e.g., linked to some products traded at regu-

[1] See Markowitz (1952b).

lated exchanges; and pieces of art, which are, e.g., either traded bilaterally or via auctions). Figure 3.1 provides some relevant data: The global stock market capitalization at the end of the year 2007 accounted, e.g., for approximately 60.1 trillion USD.[1] Compared to bond and stock markets, alternative asset markets (such as private equity and hedge funds) are still comparably small, but seem – despite the financial crisis – to be becoming more and more important. High Net Worth Individuals are estimated to possess 40.7 trillion USD, with around one third of this sum incorporated into conventional investment management such as pension funds, mutual funds and insurance assets[2]. Investors in or the clients of financial

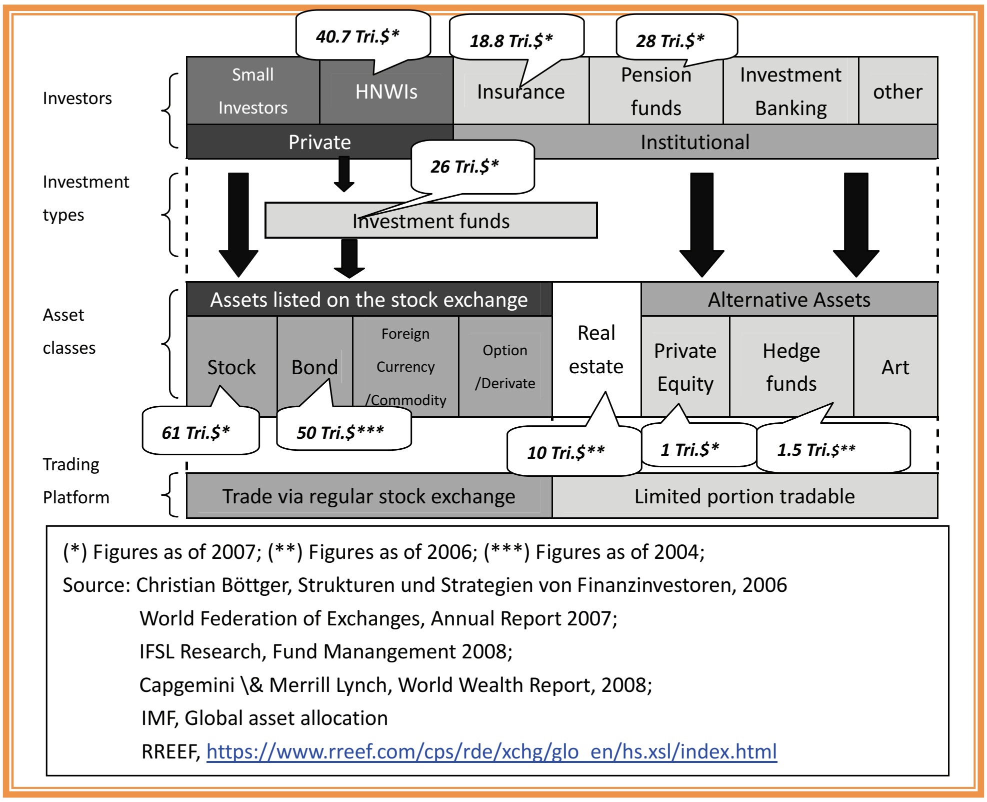

Figure 3.1: An Overview on the Capital Market

institutions are characterized by different levels of wealth, knowledge, skills and expertise. According to the European Union's Markets in Financial Instruments

[1]See the World Federation of Exchanges Annual Report (2007).
[2]See Capgemini & Merrill Lynch, World Wealth Report 2008.

Directive (MiFID)[1], investors are classified into three types, namely into retail clients, professional clients and eligible counterparties. This thesis concentrates on reasonable asset allocation strategies of the first two groups of investors, which are usually referred to as **individual** and **institutional investors**.[2].

3.1 Individual and Institutional Investors

3.1.1 Individual Investors

We now consider investors who purchase securities for themselves, as opposed to institutional investors. We will furthermore distinguish the investors according to the amount of available wealth that each one has which can be invested in the financial markets. Individual investors are here divided in two groups: *small* investors and *wealthy* investors.[3]

Several investment guidelines[4] suggest that equity holdings increase with wealth. Wealthier individuals should invest more in risky securities because they can bear more risk. This, however, does not often hold in practice. Firstly, wealthy individuals invest in representatives of more asset classes than just the trades, and thus benefit from the effect of diversification. Secondly, the risk appetite (or, a willingness to take risks) is mainly restricted to their ability to generate wealth through business endeavors, while wealthy people are usually rather conservative with respect to their investments. According to the *World Wealth Report* of Capgemini and Merril Lynch from 2008, about 60 percent of the High Net Worth Individuals agree with the statement “high risk in business, low risk in investments”, compared to 36 per cent among investors who are below the barrier of 1 million USD for investments. In another bank survey[5], many wealthy individuals stated that they became more risk averse after having realized their wealth.

[1]See http://eur-lex.europa.eu/.

[2]The third group will not be considered in this research.

[3]Note that the arbitrary boundary of 1 million USD liquidity between both groups is rather fuzzy.

[4]See, e.g., Bodie & Crane, D. B. (1997).

[5]See Barclay's Wealth Insights (2007).

With respect to financial decisions, Tilmes (2006) shows that the main goals of wealthy people are wealth preservation, wealth accumulation and wealth transfer.

The last decade was characterized by a significant development of equity markets around the globe (fueled to a large extent by the European and the US markets), and the size of the wealth of the High Net Worth Individuals has dramatically increased from about US$ 16.6 trillion in 1996 to more than US$ 40.7 trillion in 2007. The average annual return (see figure 3.2) was thus 7.76 %. The rise in the size of wealth can partly be explained by the significant development of the number of HNWIs, which estimates expanded according to the World Wealth Report from 4.5 million in 1996 to 10.1 million in 2007.

Table 3.1: Development of Global HNWIs

HNWIs[1] Global	96	97	98	99	00	01	02	03	04	05	06	07	CAGR[2] (96-07)
Financial Wealth (USD trillion)	16.6	19.1	21.6	25.5	27.0	26.2	26.7	28.5	30.7	33.3	37.2	40.7	7.76%
Number of HNWIs (million)	4.5	5.2	5.9	7.0	7.2	7.1	7.3	7.7	8.2	8.7	9.5	10.1	6.97%

Small investors are here all investors who are below the barrier of 1 million USD liquidity for investments. Although this is clearly a strong generalization, small investors are assumed to be characterized by a higher risk tolerance than the HNWIs.

[1] See Capgemini and Merrill Lynch (2006, 2007, 2008).

[2] The compound annual growth rate is calculated by taking the n–th root of the total percentage growth rate, where n is the number of years in the period being considered: CAGR=$(\frac{V_{end}}{V_{beg}} - 1)^{(\frac{1}{N})}$.

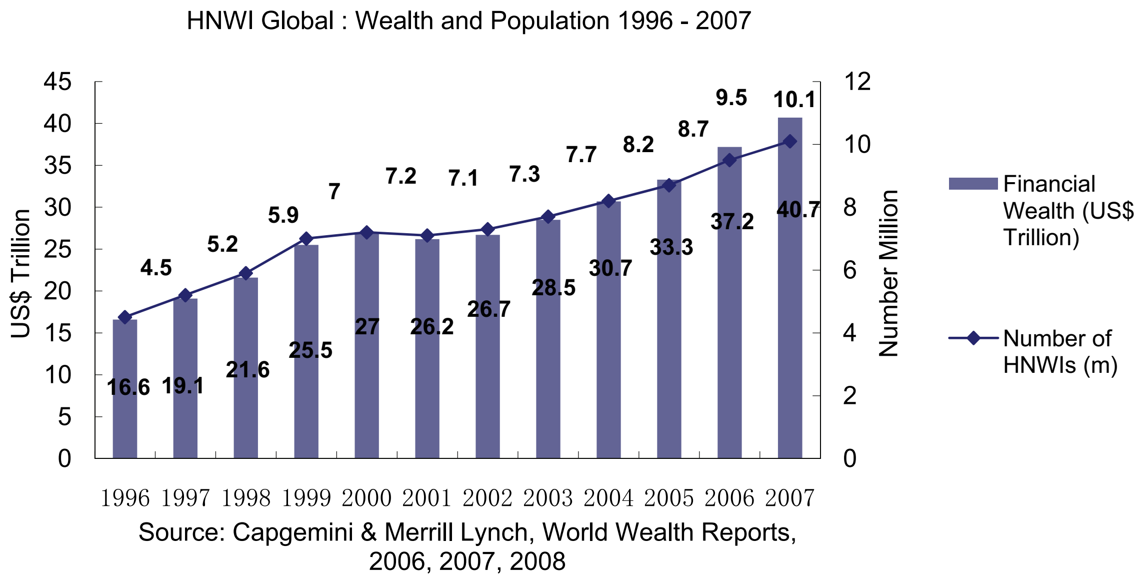

Figure 3.2: Development of Global HNWIs

3.1.2 Institutional Investors

Institutional investors can be defined as specialized financial institutions that manage savings collectively on behalf of small investors toward a specific objective in terms of acceptable risk, return maximization, and maturity of claims (cf., e.g., E. P. Davis & B. Steil (2004).)[1] A general feature common to all institutional investors is that they provide a form of risk pooling for small investors, i.e., that they provide a good tradeoff between risk and return via diversification.

Institutional investors are furthermore characterized by a liquidity preference. They hence use large and liquid capital markets, in order to be able to adjust holdings in pursuit of objectives, in response to new information. Holdings of illiquid assets such as property typically accounts only for a relatively small share of their portfolios. Important for them is the ability to absorb and process information, which is much more relevant, in comparison to individual investors.

[1] This definition does not cover financial institutions which service other financial institutions, such as pension funds or banks, with respect to their investment process.

Moreover, institutional investors rely stronger on public information rather than private, which links strongly to their desire for liquidity. The size of the institutions has important implications. There may be economies of scale, which result in lower average costs for investors. These arise, e.g., from the ability to transact in large volumes, lowering, e.g., commission fees for the customers. The asset management process is furthermore undertaken by the institutional investor itself, or by a separate institution such as a specialist fund manager, a life insurer, etc. In this thesis we will not distinguish between different types of institutional investors, but regard them only as dependent investors with large assets under their management.

3.2 Classification of Investors According to their Attitude toward Risk, Initial Wealth and Time Horizon

According to their attitude toward risk, their initial wealth and their investment horizon, individual and institutional investors are now classified into "typical" groups. In the next section this will be followed by the providing of corresponding "reasonable" asset allocation strategies for these "typical" investors. Investors are characterized by different

- Risk attitudes (μ, σ^2).
- Initial wealth levels (W_0).
- Time horizons (T).

As derived in part 2.5.1, it is assumed that all investors prefer more to less wealth, and that they are risk averse (i.e., they satisfy the conditions $\frac{\partial U}{\partial R} > 0$ and $\frac{\partial^2 U}{\partial R^2} < 0$). The attitude toward risk can be described by the trade–off between expected return and variance. The variance of the return distribution is furthermore accepted as an appropriate measure of risk[1]. Table 3.2 categorizes investors

[1]As already pointed out, this is reasonable as long as the portfolio is characterized by a symmetric return distribution.

according to their expected return and their risk attitude. Next, we classify

Table 3.2: Classification of Investors According to their Attitude Toward Risk

μ_R&σ_R	lower σ_R	higher σ_R
lower μ_R	HNWIs Institutional Investors	Nobody
higher μ_R	Do Not Exist	Small Investors, Investment Banking

investors according to their initial wealth level.

Table 3.3: Classification of Investors According to Risk Tolerance and Initial Wealth

μ_R/σ_R&W_0	Small W_0	High W_0
Low μ_R/σ_R	Some Small Investors	HNWIs, Institutional Investors
High μ_R/σ_R	Small Investors	Investment Banking

Investors are furthermore characterized by different time horizons for their investments. Investment professionals usually link investments with an expected holding period of less than one year to a short time horizon. Here, we find that especially risk–seeking investors come into play. The strategic asset allocation of HNWIs is usually characterized by a long time horizon, as in table 3.4 shown.

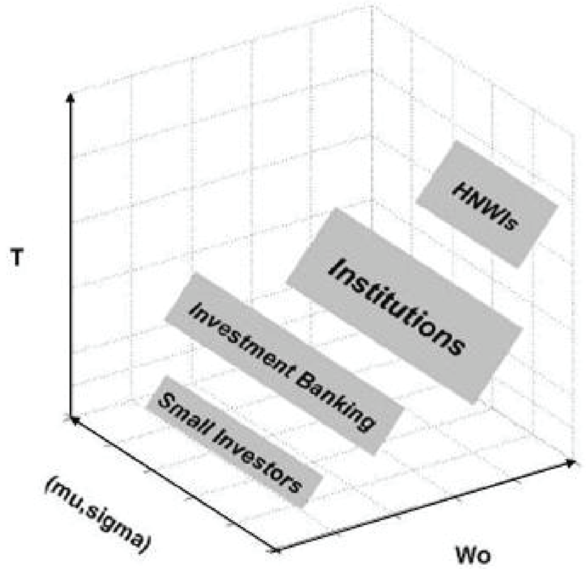

Figure 3.3: "Typical" Investors

Table 3.4: Classification of Investors According to their Investment Horizon

Short–term (day, month, up to 1 year)	Middle–term (1 to 3 years)	Long–term (more than 3 years)
Small Investors	HNWIs	HNWIs
Investment Banking	Institutional Investors	

Figure 3.3 shows some of the derived "typical" investors according to the three dimensions of risk tolerance, initial wealth and investment time horizon. Although some tough generalizations with respect to the special groups of investors have certainly been made, the chosen characteristics can imply further useful applications for different types of investors.

3.3 Motivations to Asset Allocation Strategies for Different Types of Investors

3.3.1 Risk Aversion and Expected Utility

In the following, we will look again at utility functions and expected utilities, as shown in table 2.3. Consider now the different parameters in

$$U(\mu, \sigma^2) = \alpha\mu - \beta\sigma^2. \tag{3.1}$$

The parameter β will be taken as the level of risk aversion, and the higher β is, the higher is the level of risk aversion. Referring again to investors with different *Wealth Levels* and different *Risk Attitudes*, 3.6 can be rewritten as follows:

$$U_1(\mu, \sigma^2) = \alpha(W_0 + A\mu) - \beta(A\sigma)^2. \tag{3.2}$$

For $R_f \neq 0$, we have:

$$U_1(\mu, \sigma^2) = \alpha(W_0(1 + R_f) + A(\mu - R_f)) - \beta(A\sigma)^2.$$

Denoting $t = A/W_0$, it holds for $R_f = 0$ that

$$U_1(\mu, \sigma^2) = \alpha W_0(1 + t\mu) - \beta W_0^2 t^2 \sigma^2, \tag{3.3}$$

and for $R_f \neq 0$, we have

$$U_{1f}(\mu, \sigma^2) = \alpha W_0(1 + t\mu + R_f - tR_f) - \beta W_0^2 t^2 \sigma^2. \tag{3.4}$$

Similarly, we get the expected utility under the forms of $U_2(\mu, \sigma^2)$ and $U_3(\mu, \sigma^2)$ as follows: We substitute the mean and variance of terminal wealth in the approximate utility U_2 and U_3 in Table 2.3, then we get:

$$U_2(\mu, \sigma^2) = \log(1 + W_0(1 + t\mu + R_f - tR_f)) - \frac{(tW_0\sigma)^2}{2(1 + W_0(1 + t\mu + R_f - tR_f))^2},$$

Analogously U_3 can be transformed into:

$$U_3(\mu, \sigma^2) = 1 - \frac{1}{1 + W_0(1 + t\mu + R_f - tR_f)} - \frac{(tW_0\sigma)^2}{(1 + W_0(1 + t\mu + R_f - tR_f))^3}.$$

All these three types of utility functions can simulate the behavior of investors with different attitudes towards risk. On a more conceptual basis, the decision for an investment is a matter of maximizing the individual's expected utility. This, in turn, entails deducing what trade–off of return and risk constitutes utility for the individual. In the empirical research, the optimization problem under expected utility will be based on the utility functions derived in this part.

3.3.2 Wealth Level and Probability of Loss

As discussed in the previous sections, the initial wealth level may play an important role in determining an investor's decision with respect to an appropriate asset allocation. Next, we will analyze how the initial wealth level can affect investors' decisions.

Consider a portfolio choice problem of a risk averse investor who strictly prefers more to less (i.e., he or she has a strictly increasing utility function). Suppose now that one part of the initial wealth, denoted by W_0, is invested in risky securities, while the remaining part is put into a risk–less asset. We denote the contribution of risky assets by $A = \sum_j A_j$. Now we use the following notations:

W_0	: *Initial Wealth.*
$A = \sum_j A_j$	: *Part of Wealth Invested in Risky Assets.*
$W_0 - A = W_0 - \sum_j A_j$	: *Part of Wealth Invested in a Risk–Free Asset.*

The uncertain, or stochastic, wealth at the end of the investment period is thus:

$$W = (W_0 - A)(1 + R_f) + A(1 + R). \tag{3.5}$$

Here R denotes the stochastic return of the weighted risky assets, and R_f is the return of the risk–free asset. For simplification, we will in the following assume that $R_f = 0$, and furthermore, that R is supposed to be normally distributed: $R \sim N(\mu_R, \sigma_R^2)$. At the end–of–period holds consequently

$$W \sim N(W_0 + A\mu_R, (A\sigma_R)^2). \tag{3.6}$$

The investor's wealth at the end of the investment period thus follows a normal distribution with expected return $(W_0 + A\mu_R)$ and variance $(A\sigma_R)^2$.

An important question, which is directly linked to the *Status Problem* discussed in the following subsection, is now the following: How is the certainty to maintain the wealth on a certain level taking the initial wealth as a reference level? This question of course is not new, and reflection on it can, e.g., be traced back to Roy's[1] *Safety–First* reasoning. Consider now the probability that the wealth falls below a certain predefined level (to be fixed by a *degree of wealth maintenance* $b \in (0, 1)$[2]) over the period of the investment:

$$Pr(W \leq bW_0).$$

This idea can also be linked to the conception of the **Value at Risk** (VaR). For a given portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability that the mark–to–market loss on the portfolio over the given time horizon exceeding this value (assuming normal markets and no trading) is the given probability level, that is $Pr(r_p < -VaR) \leq 1 - c$, where c is the confidence level (see Jorion (1992)).[3]

We extend this concept here by taking into account the initial wealth level, and the percentage of this wealth that is invested in risky assets, and hence derive the following relationship between the probability of a predefined level of wealth maintenance and the proportion of risky assets:

[1] See Roy (1952).

[2] This probability can be regarded as a probability of loss.

[3] Although VaR gained high popularity among practitioners and some academics during the last decade, it has also always been heavily criticized: VaR cannot make a statement on the loss distribution. Moreover, VaR is not coherent, that is, it is not sub–additive. See, e.g., A. Guthoff & F. Rüter (1999) and J. Kremer (2008). In this research VaR is taken as a constraint which focuses on controlling the probability of loss.

$$\begin{aligned} Pr(W \leq bW_0) &= Pr(W_0 - A + A(1+R) \leq bW_0) \\ &= Pr(R \leq (b-1)\tfrac{W_0}{A}) \\ &= 1 - \Phi\left(\tfrac{\mu_R + (1-b)\frac{W_0}{A}}{\sigma}\right). \end{aligned}$$

The "adjusted" VaR is now:

$$VaR = (1-b)\tfrac{W_0}{A} = -(\mu_R - \Phi^{-1}(c)\sigma_R),$$

with $\Phi(\cdot)$ being the cumulative distribution function of the standard normal distribution.

In order to get a visual impression of this (see below), we assume now that $W_0 = 1, \quad \mu = 1\%, \quad \sigma = 10\% \quad and \quad b = 0.95$. The loss probability for $A = 0.3$ equals now 3.86% (cf. figure 3.4), while it grows to a remarkable 20.77% for $A = 0.7$ (see 3.5). For the given input, the loss probability as a function of the percentage invested in the risky assets is shown in figure 3.6. The graph is not linear; the slope of the tangential lines becomes smaller with increasing percentage of the risky assets.

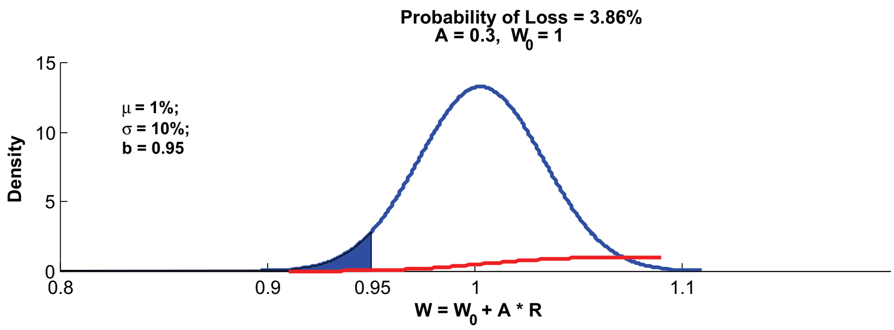

Figure 3.4: Low Risk Case: Small Proportion of A and Loss Probability

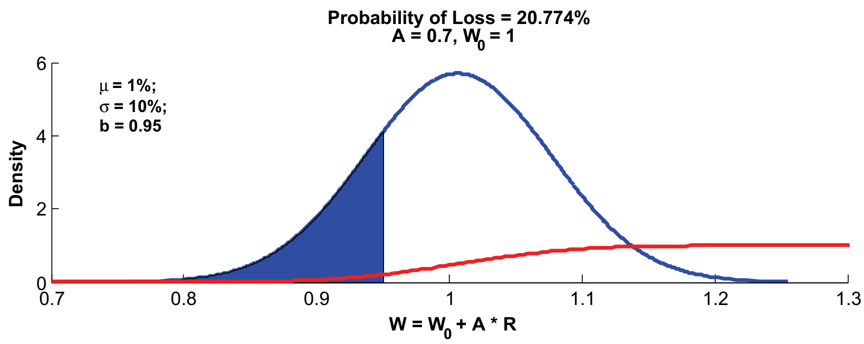

Figure 3.5: Low Risk Case: Large Proportion of A and Loss Probability

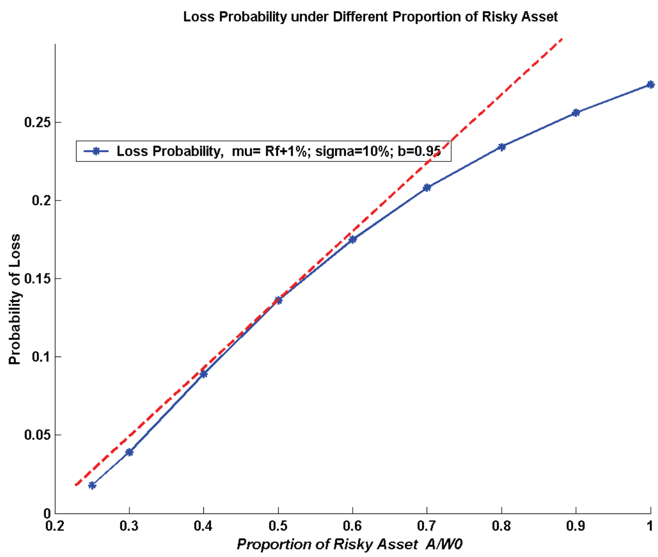

Figure 3.6: Low Risk Case: Loss Probability and Proportions of the Risky Asset

Next, we change the input so that it is more volatile, as in the previous example. Assume now that $\mu = 10\%$, $\sigma = 30\%$, and $b = 0.95$. Figures 3.7 and 3.8 show the loss probabilities for the more volatile assets. The loss probability for $A = 0.3$ now equals to 18.69%, while it grows to 28.37 % for A = 0.7. Figure 3.9 shows the dependency of the loss probability on A for the riskier example.

Table 3.5 explains how the probability of loss for both increasing initial wealth

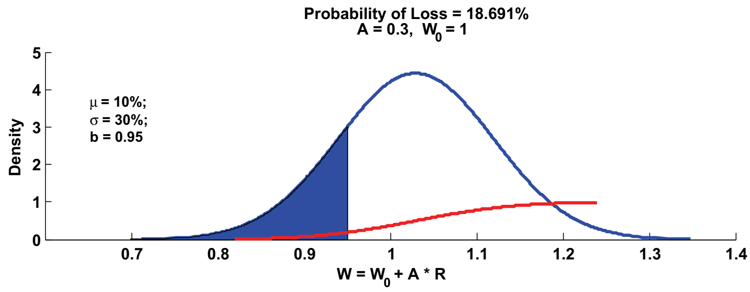

Figure 3.7: High Risk Case: Small Proportion of A and Loss Probability

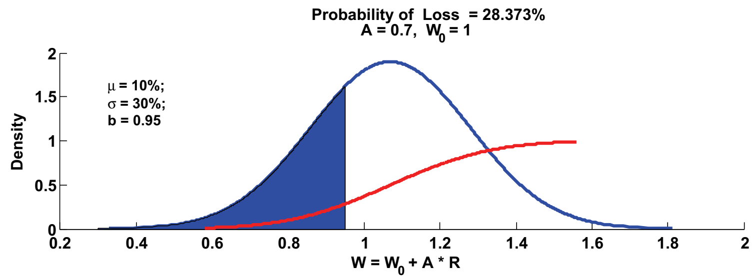

Figure 3.8: High Risk Case: Large Proportion of A and Loss Probability

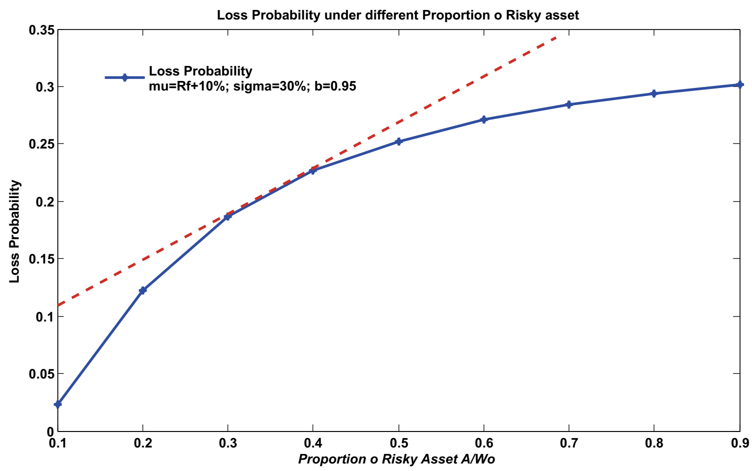

Figure 3.9: High Risk Case: Loss Probability and Proportions of the Risky Asset

and decreasing percentage invested in the risky assets eventually decreases.

Table 3.5: Loss Probability under Different Initial Wealth Levels

W_0	1	1.125	1.286	1.5	1.8	2.25	3	4.5	9
A/W_0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
$Pr(W \leq bW_0)$	0.256	0.234	0.208	0.175	0.136	0.089	0.039	0.005	0.000

The next question which arises is how should a portfolio which limits the short–fall probability to a certain "acceptable" region look? Optimization problem 2.8 will now be transformed into the following form:

$$\Phi(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\Sigma\mathbf{x} \rightarrow \min \quad \text{subject to} \quad B\mathbf{x} = \mathbf{c}, \quad Pr(W \leq bW_0) \leq 1 - c. \tag{3.7}$$

3.3.3 Time Horizon and Uncertainty in the Estimates

Time is one of the most important dimensions of any investment, and in particular of wealth management processes. While many small individual investors usually restrict their investment time horizons to be within their own lifetimes, many HNWIs are characterized by an investment approach which bridges generations. Questions of heritage and taxes play an outstandingly important role. If an asset allocation strategy is determined with respect to time, HNWIs do not tend to change their portfolios in a shorter investment horizon.

We begin by reviewing some theoretical and practical aspects with respect to time and investment decisions. There still seems to be some controversy surrounding this subject of investment time horizon and risk, and its effect on portfolio diversification. Samuelson (1969)[1] and Merton (1971)[2] both showed that investors

[1]See Samuelson (1969).
[2]See Merton (1971).

with a standard differentiable utility function that exhibits constant relative risk aversion will allocate a fixed proportion of their wealth to a risky asset, regardless of the investment horizon under consideration. On the other hand and consistent with intuition, there's a lot of research indicating that the incorporation of an investor's time horizon in an optimization model has a significant impact on the optimal proportions invested in the different asset classes.

Del Prete (1997) advises investors with long time horizons to allocate most of their liquid wealth to stocks, and then to decrease the stock allocation as time passes. Using data from TIAA/CREF, Bodie & Crane, D. B. (1997) show that individual asset allocations are consistent with the recommendations of expert practitioners. Barberis (2000) investigates three strategies of power utility investors: a buy–and–hold strategy, a deterministic strategy with constant weights, and an optimal rebalancing strategy for a rebalancing interval of one year. The comparison of the speculative and the hedging demand in stocks shows, for example, that their magnitudes become equivalent for investment time horizons greater than 2 years. Moreover, while the optimal allocation in stocks increases with the time horizon, it seems to stabilize for time horizons longer than 6 years. The effect of the time horizon on the optimal allocation in bonds is even subtler.[1]

The use of Monte–Carlo–Simulations will now help us to investigate the influence of arbitrary time horizons and assets' return distribution on the optimization results. Using empirical data of four asset classes recorded during 13 years[2], and computing expected returns, variances and covariances[3], simulations based on time series with different lengths were carried out. As can easily be seen, portfolios with longer simulated time horizons come closer to the portfolios on the efficiency line when estimated with all of the historical data. Long–term investments are expected to generate a higher expected terminal wealth[4]. For more details, cf. chapter 4. We summarize:

[1]See G. Lenoir & N. S. Tuchschmid (2001).
[2]See Chapter 4 for more details.
[3]The returns of the asset indices were supposed to follow a multivariate normal distribution.
[4]See Z. Li & J. Yao (2004)

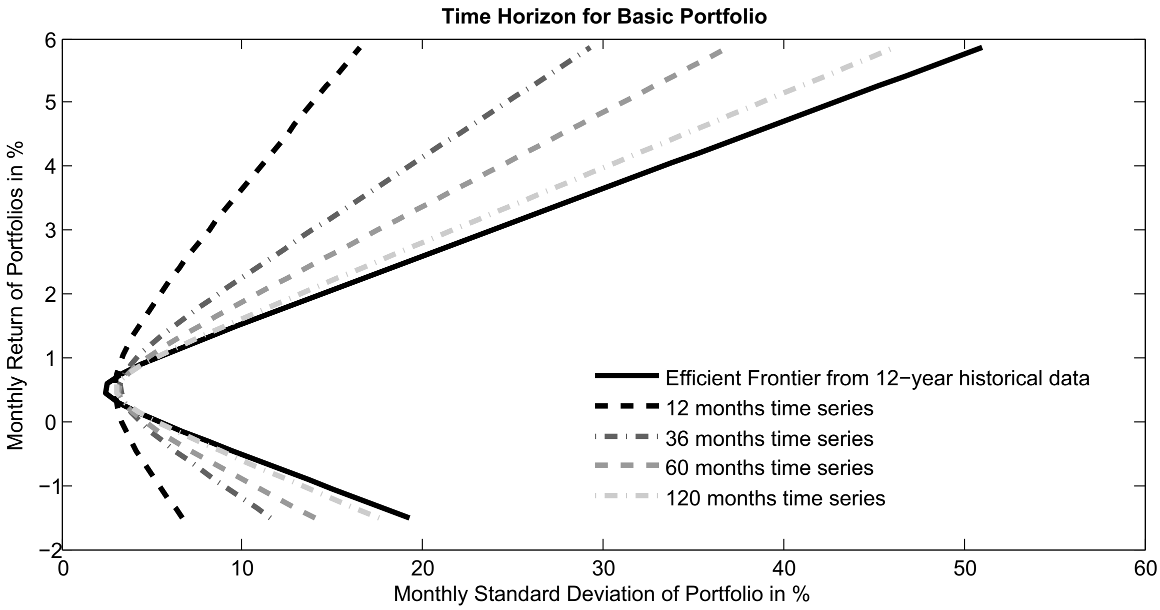

Figure 3.10: Portfolio Frontier with Different Time Horizon Simulation

- The time horizon is a key input variable in determining an appropriate balance of interest–generating (cash or bonds) versus equity (stocks and other alternative investments) investments in a portfolio.
- The longer the time horizon is, the smaller the deviation from the "true" efficient frontier becomes.
- The lower the investors' aversion toward risk is, the lower is the importance of the time horizon for actual investment decisions.

On the one hand, it has been demonstrated that an individual who is concerned with maximizing a Safety–First utility function will choose an asset allocation that is invariant to the investment's time horizon. Moreover, the risk of a portfolio, which is defined as the probability of earning a cumulative rate of return that is less than the risk–free asset, declines exponentially with the time horizon.[1] See McEnally (1985) for another form of argumentation against time diversification

[1]The underlying assumption here is that the risk–free rate remains constant, regardless of the time horizon in question.

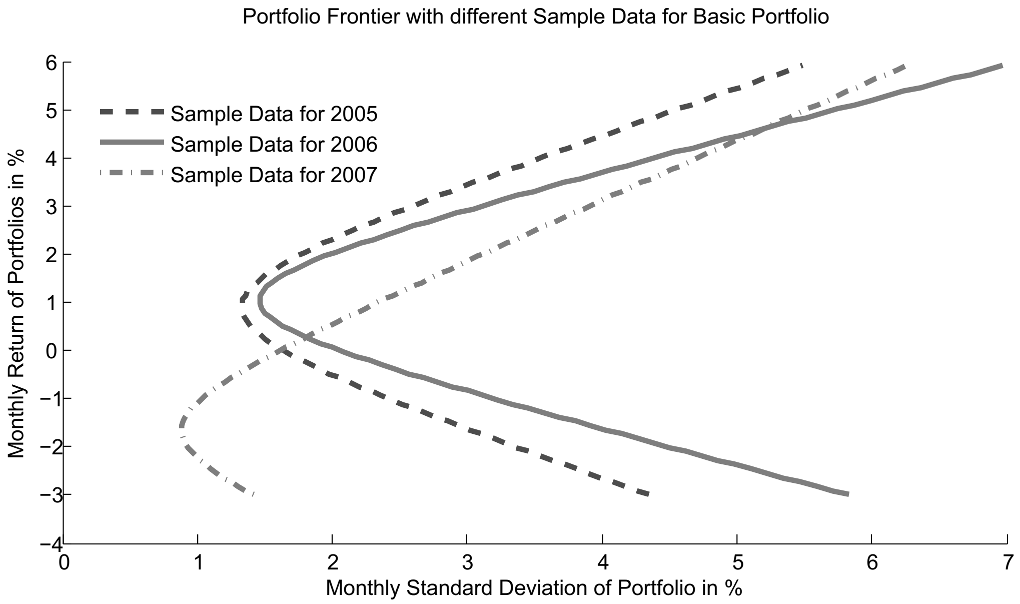

Figure 3.11: Portfolio Frontier with Different Sample Data

on the basis that total variance increases with the investment horizon. On the other hand, we have noticed the impact of the time interval of sample data.

In an economy characterized by an upward–sloping term structure, the Safety–First investor will, in the long run, have to take more risks in order to beat the return from the risk–free asset. This does not, however, dilute the actuality of the invariance of time horizon on the pure (constant return) Safety–First portfolio. *How* to choose the “proper” length of historical data for deriving practical investment recommendations thus remains a crucial and open problem for investors in the decision–making progress.

3.3.4 Status Problem of HNWIs and Implications of their Asset Allocation Strategies

As we have already pointed out, many individual investors are much concerned with their potential loss relative to their initial wealth level, and this holds especially for the HNWIs. As can easily be observed, many HNWIs consider their wealth to be a status symbol. This corresponds to their desire to maintain a (rather high) minimum level of wealth, since falling below a certain threshold would correspond to a loss of their social status level. These investors' decisions are often driven by – sometimes smart – intuition, which is based on the desire to have a guaranteed maximum loss, and this leads us to incorporate the VaR approach in reasonable asset allocation strategies for this type of investor. This leads to optimization problem 3.7 :

$$\Phi(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\mathbf{\Sigma}\mathbf{x} \rightarrow \min \quad \text{subject to} \;\; \mathbf{Bx} = \mathbf{c}, \quad Pr(W \leq bW_0) \leq 1 - c.$$

In the empirical part problem 2.8 will be solved without using the Value at Risk constraint in the computations, but, by checking afterwards whether a solution satisfies the Value at Risk constraint. If not is it excluded.

Considering the initial wealth and the proportion invested in the risky assets, it follows that

$$VaR = (1-b)\tfrac{W_0}{A} = -(\mu_R - \Phi^{-1}(c)\sigma_R).$$

This, however, is a linear constraint, under which the situation in a Markowitz–$\mu - \sigma$–framework looks like the following:
From this conceptual framework follow some important implications with respect to reasonable asset allocation strategies:

1. With an increasing A/W_0, the loss probability (i.e., that the wealth at the end of a period falls below a given level) will increase (cf. figures 3.6 and 3.9). The proportion which is invested in the risky assets as well as the initial wealth level, can be used as a lever to restrict the loss probability within a certain range.

2. For a fixed A will the loss probability decrease with an increasing initial wealth W_0. This sounds trivial, but it means practically for the HNWIs that they can limit potential losses by fixed amounts, instead of by percentages invested in risky securities.

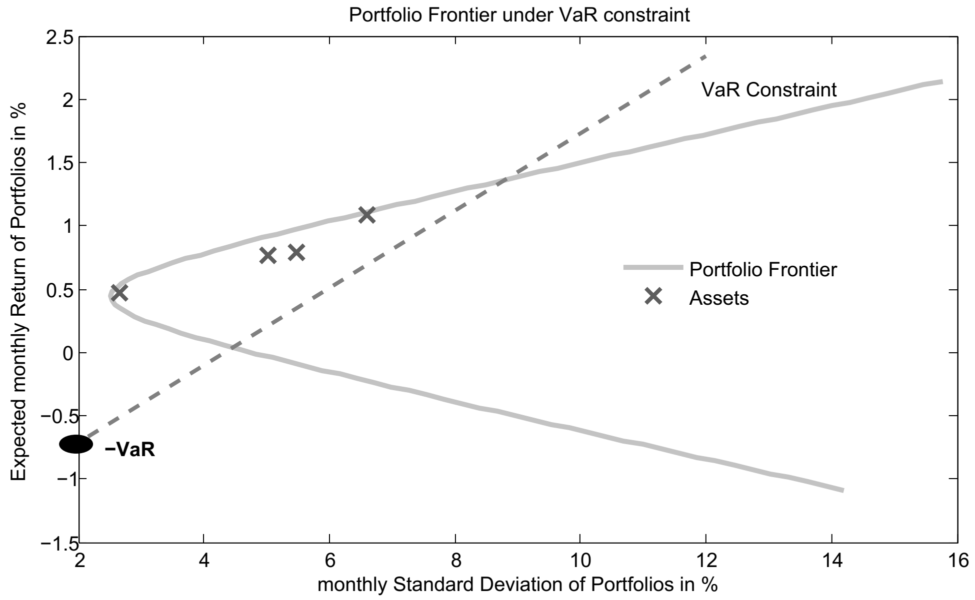

Figure 3.12: Portfolio Frontier under VaR constraint

3. For high A/W_0 (here in the range of 0.6 to 0.9), the loss probability will slowly increase, especially when investors choose less volatile assets (i.e., risk can substantially be reduced by investment in less volatile assets, even though the proportion of risky assets is large).

3.4 Types of Asset Classes

Asset allocation refers to how an investor distributes his or her investments among various classes of investment vehicles. Referring to the classification from section 3.2, we will now formulate *reasonable* asset allocation strategies for the last two of our "typical" investors based on the following considerations from the previous sections.

- The low level of initial wealth and the high degree of risk aversion of *Small Investors* investing in, e.g., Mutual Funds.
- The low level of initial wealth and the low degree of risk aversion of, e.g., *aggressive* Small Investors investing in risky assets via debt.

- The high level of initial wealth and the high degree of risk aversion of the *HNWIs* and Institutional Investors.
- The high level of initial wealth and the low degree of risk aversion of in some parts of *Investment Banking*.

We continue with a description of the characteristics of the most important asset classes, the representatives of which will be incorporated later in the empirical research. There we will address cash and bonds (German and US markets), stocks (German, US and global markets), real estate (German and US markets), hedge funds, gold and art markets.

There are many ways to categorize asset classes, and not only financial instruments. The following table (3.6) shows one possible categorization (it is rather freely oriented at Farrell (1997)) of asset classes for the portfolio construction available to the participants in the portfolio management process. Money market instruments, bonds, and equities are the major large asset categories that are generally highly marketable, and which traditionally have been used extensively by long–term portfolio investors. These three classical investment asset classes are extended to real estate and alternative investments[1], as shown in Table 3.6.

Note here that the distinction between "domestic" and "international" assets is characteristic of earlier American research and textbooks, but is not always an appropriate distinction today. However, most investors in all parts of the (Western) world distinguish between domestic assets (of which investors are intuitively under the impression are associated with a lower risk) and assets from abroad. In this thesis we are using an asset classification which distinguishes between domestic and foreign financial instruments, and which is consequently suitable for all investors worldwide. We distinguish between money market instruments (cash, deposits), bonds (domestic and international government bonds and corporate bonds), stocks (domestic equities and international equities), real estate (Real

[1]Note that there is no unambiguous definition of alternative investments. Some authors count, e.g., commodities as a subset of these investments, while others don't, and rather regard them as a separate asset class.

Estate Investment Trusts, which are usually invested in commercial real estate and residential property[1]), and alternative investments (such as structured products, hedge funds, financial derivatives, foreign currencies, commodities, private equity/venture capital, and investments of passion).

3.4.1 Basic Asset Allocation

Cash market securities are short–term debt instruments sold by governments, financial institutions and corporations. The important characteristic of these securities is that they have maturities when issued in one year or less.[2] For an example of the historical performance of cash market securities, see the JP Morgan 1 Month Cash Index (JPEC1ML). Figure 3.13 shows the performance development during the period from December 1995 to December 2007[3].

Most traditional **bonds** (also called fixed–income securities) promise to pay specific financial amounts at specific times.[4] In most countries, the government issues fixed income securities over a broad range of the maturity spectrum. They are generally considered to be safe from default, and thus differences in expected returns ought to be due to differences in maturity, differences in liquidity, or the presence or absence of a call provision.[5] Nevertheless, there are remarkable differences in these securities according to country risks.[6] Bonds issued by business entities are called corporate bonds, and compared to government and agency bonds, they are generally characterized by a risk of default. Corporate bonds are rated as to quality by several agencies, the best known of which are Standard and

[1]To avoid optimization problems under integer constraints, we are supposing here that the price of a REIT unit is much smaller than the budget to be invested (cf. equation 2.1.)

[2]More details can be found in, e.g., Elton *et al.* (2007).

[3]The data will be discussed in detail in the following chapter.

[4]For detailed information, cf. any good textbook on Fixed Income or Corporate Finance, as, e.g., Brealey *et al.* (2005).

[5]This problem implies tax effects on different coupon rates, and also on differences in yield. More details about tax implications can be found in Elton *et al.* (2007).

[6]The development of the Greek government bonds, although denominated in Euros, proved this again impressively in mid 2009.

Table 3.6: Asset Classes

Common Stocks	Domestic Equities
	Large–Capitalization
	Small–Capitalization
	International Equities
	Major–Country Markets
	Emerging Markets
Bonds	Governments and Agencies
	Corporate Bonds
	AAA–rated
	Baa–rated
	High–yield (junk) Bonds
	Mortgage–Backed Securities
	International Bonds
Money Market Instruments	Treasury Bills
	CDs and Commercial Papers
	Guaranteed Investment Contracts (GIC)
Real Estate	Real Estate Investment Trusts (REITs)
	Commercial Real Estate
	Other Property
Alternative Investments	Structured Products
	Hedge Funds, Derivatives, Foreign Currencies
	Commodities, Private Equity/Venture Capital
	Investments of Passion
	Art Collections
	Luxury Cars, Yachts, ...

Poor's, and Moody's and Fitch.

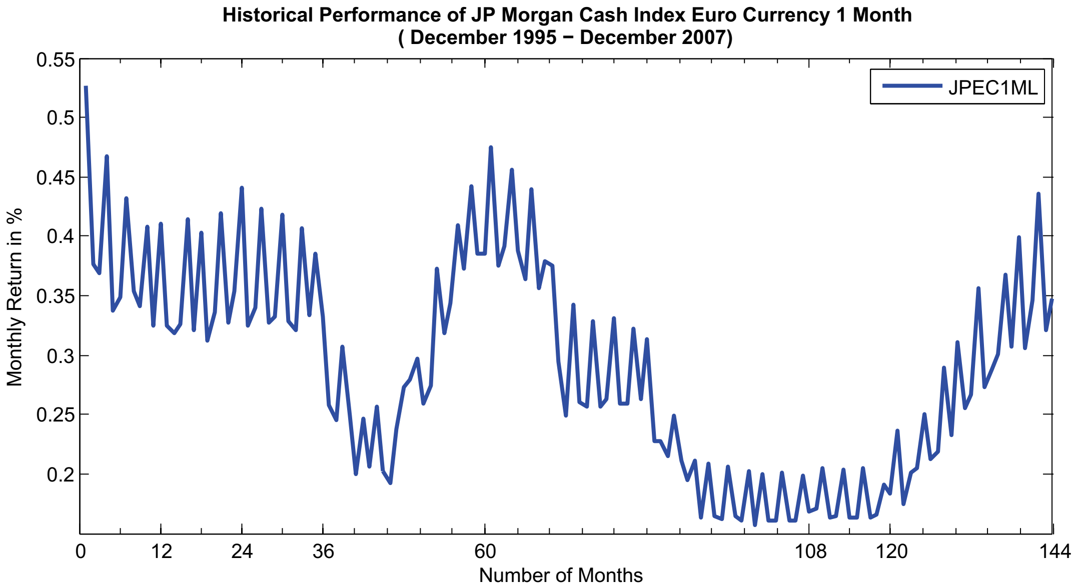

Figure 3.13: Historical Performance of JP Morgan Euro Currency 1 Month from December 1995 to December 2007

Figure 3.14 shows the performance development of the German bond market index REX during the period from December 1995 to December 2007.

Besides the bond market, the **stock** market is the other most important source for companies to raise capital. It allows businesses to go public, or to raise additional capital for expansion. The liquidity that a stock exchange provides affords investors the ability to quickly and easily sell shares. This is an attractive feature of investing in stocks, compared to other less liquid investments, such as real estate. Figure 3.15 shows the performance development of the German stock market index DAX 30 over the period from December 1995 to December 2007.

Figure 3.16 shows the performance of the three indices JPEC1ML, REX and DAX 30 from 1996 to 2007. The index JPEC1ML has the lowest standard deviation, the German bond index REX [1] is characterized by a moderate deviation, while

[1]As we will see in the next chapter, REX is a special index which has a slightly negative correlation with the other indices chosen for empirical research.

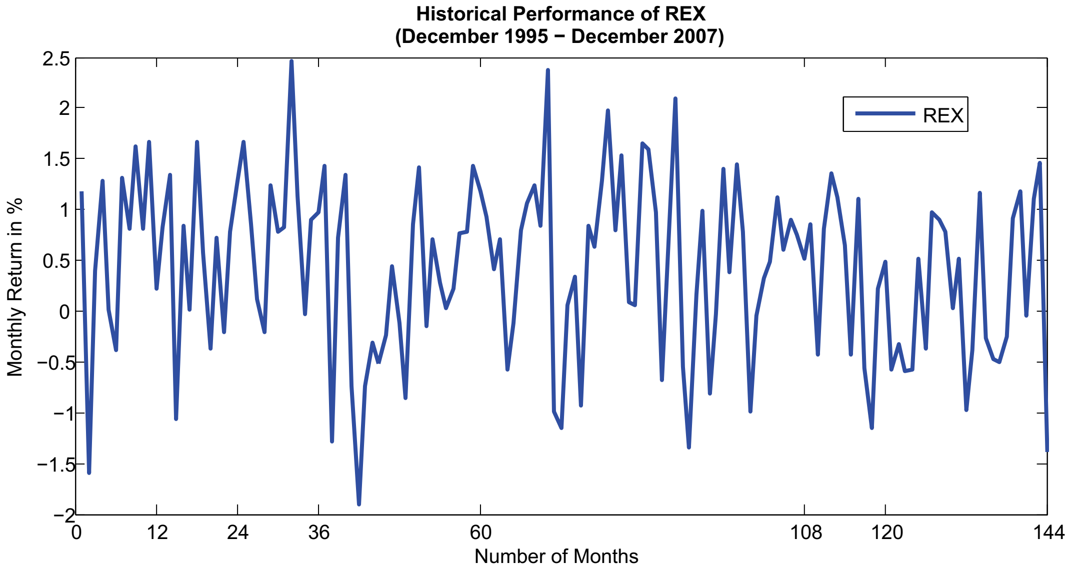

Figure 3.14: Historical Performance of REX from December 1995 to December 2007

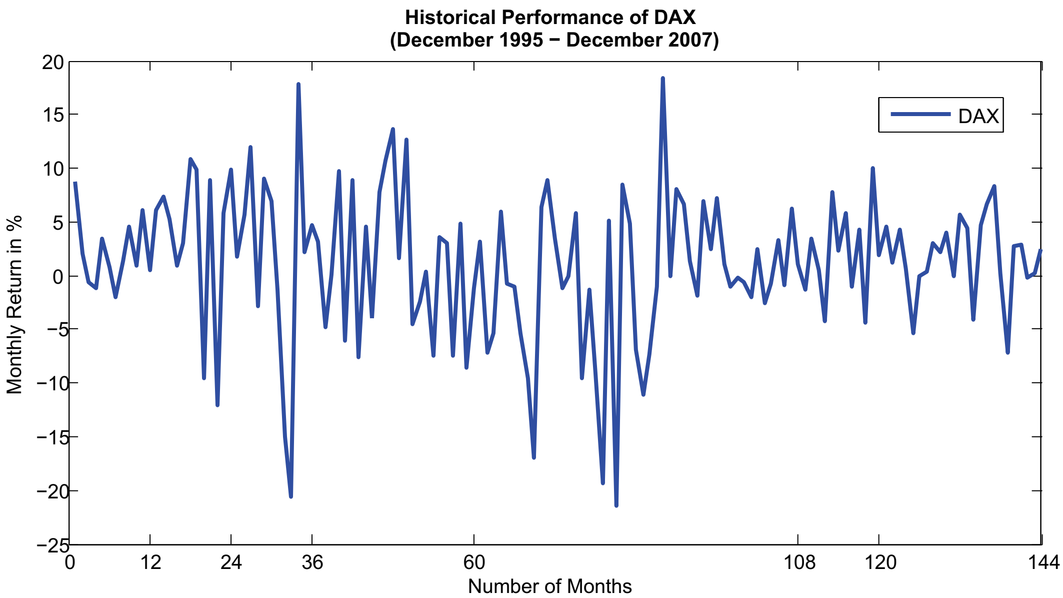

Figure 3.15: Historical Performance of DAX 30 from December 1995 to December 2007

the German stock market index DAX 30 is the most volatile.[1]

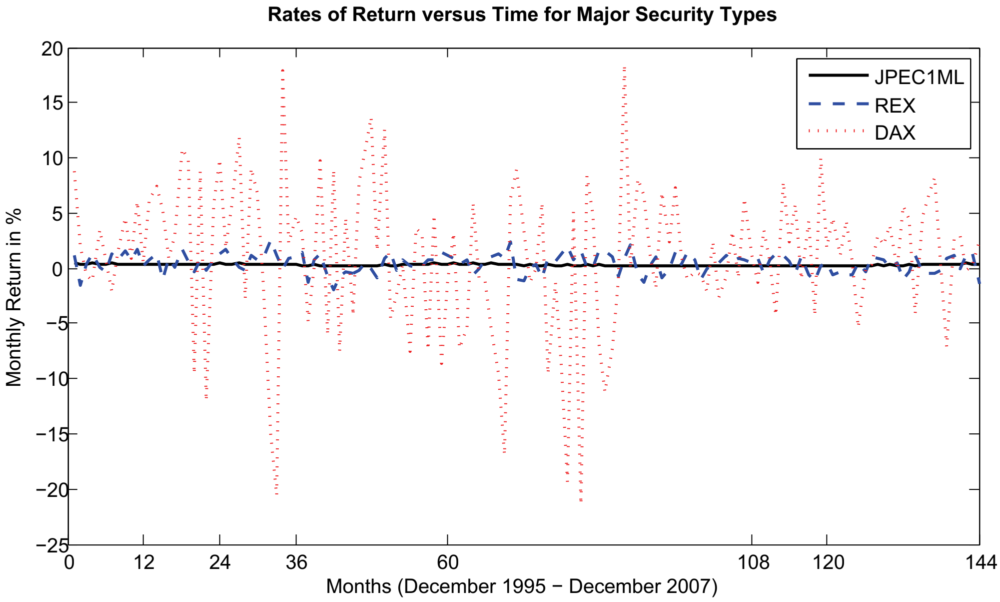

Figure 3.16: Rates of Return versus Time for the Three Classical Assets from December 1995 to December 2007

[1]This is the most important index for German stocks, while it includes 30 stocks of the largest listed German companies. For composition rules, cf., e.g., www.exchange.de. Note that DAX 30 has a dramatic fluctuation compared with other asset indices.

3.4.2 Real Estate, Commodities and Alternative Investments

In a wider sense, investments in real estate may include the acquisition and development of commercial and residential real estate objects. As a tangible asset, **real estate**, especially land, has long been viewed as a conservative investment class. Mixes of cheap loans and a general lack of supply in the past actually changed to overheated markets, making investments in real estate considerably risky. Many developed countries, such as the USA, Japan and the Scandinavian countries, and notably most East European countries, have experienced the blow–up of the real estate bubble when the amount of mortgage loans grew dramatically over the last decades. This growth was due primarily to foreign capital inflows and cheap mortgage credits at low collateral. However, when interest rates increased, many borrowers could not afford monthly payments. As a result, a huge amount of pledged real estate was sold in the markets, which caused decreases of prices by 15 - 40% in the medium term of two to four years.

According to Capgemini/Merrill Lynch World Worth Report (April 2008), real estate experienced record returns in 2006 across various categories. Many HNWIs took profits from these increased values during 2007. But with the rise of the crisis, they pulled out of real estate investments[1] more significantly than anticipated, finishing 2007 with only 14% of their financial assets allocated to real estate, a 10–percentage–point drop from 2006 levels. However, globally–direct commercial real estate investments rose by 8.4% (US$ 59 billion) during 2007, amounting to a total of US$ 759 billion[2], and thus remained one of the important asset classes for HNWIs. Figure 3.22 shows the performance development of the German real estate conjuncture index during the period from December 1995 to December 2007.

[1] Includes commercial real estate, REITs, and other investment properties.
[2] See Jones Lang LaSalle (2008).

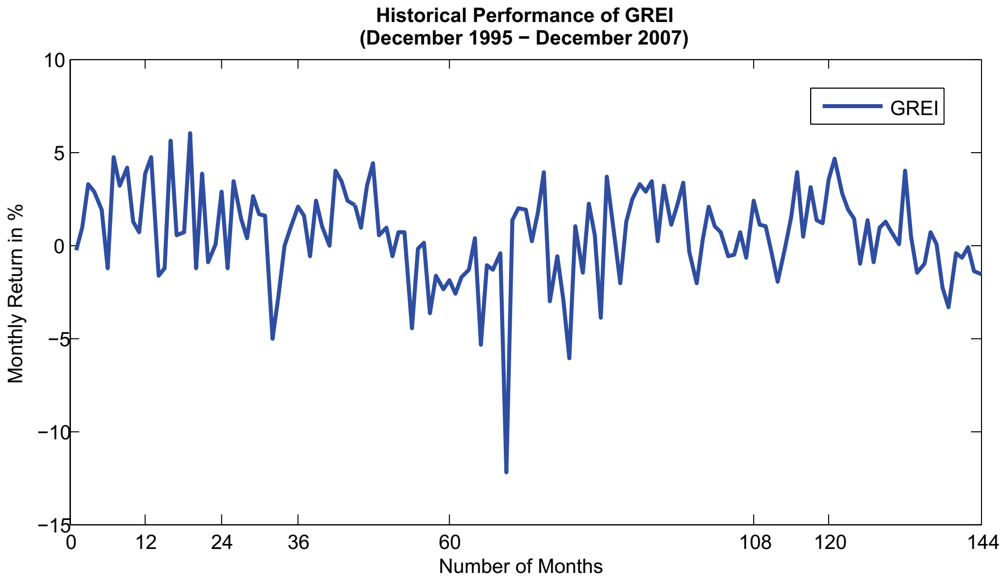

Figure 3.17: Historical Performance of the German Real Estate Index from December 1995 to December 2007

The term **alternative asset** refers to any nontraditional asset with potential economic value, that would not be found in a standard investment portfolio. Examples of alternative assets include hedge funds, private equity, pieces of art, antiques, stamps, luxury cars, and other commodities.

In order to be extra distinguished, **commodity markets** have to be where raw (or "primary") products are exchanged. These raw commodities are traded on regulated commodities exchanges such as the Chicago Mercantile Exchange, in which they are bought and sold in standardized contracts. Most commodity markets across the world trade in agricultural products and other raw materials (e.g., wheat, barley, sugar, cotton, coffee, milk products, pork bellies, oil, metals, etc.), and future contracts based on them. A special case is **gold**. Gold is often used as a hedge against inflation, political risk, and currency exchange risk. Investments in gold can be made by buying gold jewellery, gold coins, gold bullion, futures and options contracts on gold bullion, the stocks of gold mining companies, and gold mutual funds (which invest in gold stock and gold bullion). However, gold does not earn any interest, and furthermore the storage and transaction costs for investing in gold jewellery, gold coins and gold bullion can be substantial.[1]

Commodity prices have different features: figure 3.18 shows, e.g., that the price of gold has a smaller volatility, in comparison with the Brent Crude Oil price.

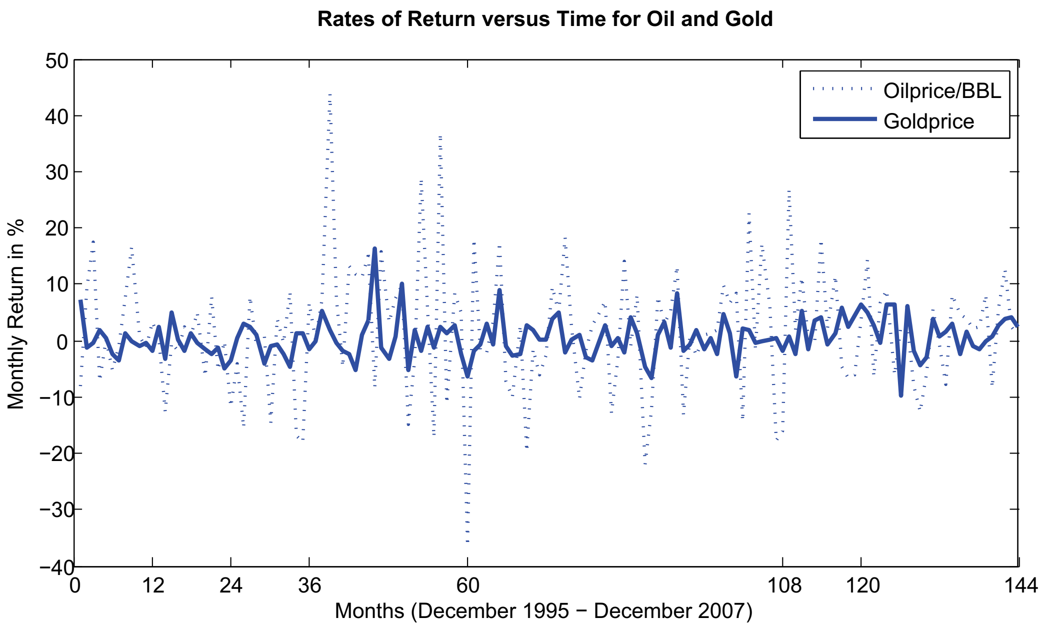

Figure 3.18: Rates of Return versus Time for Gold and Oil from December 1995 to December 2007

Another important subclass of alternative assets are investments of passion, such as pieces of **art**, which are, by the way, often linked to the status of HNWIs. For wealthy individuals, art is a very important alternative asset class. Many HNWIs believe that the art market yields huge long–term profits, in comparison to other investment markets. This is based mainly on information in mass media concerning record prices of paintings. In 2007, the seventh consecutive year of increasing prices, luxury collectibles accounted for 16.2% of passion investments, and fine art, representing 15.9%, continued to be the most popular choice for HNWIs worldwide.[2] The rising prices were accompanied by a higher total art

[1]See Blose (1996).
[2]Capgemini/Merrill Lynch World Worth Report, 2008.

market revenue at 9.2 billion dollars, up 43.8% in comparison with 2006, and driven by a substantially higher number of sales above the threshold of the one million dollar line, namely 1254 in 2007 compared with 810 in 2006. Although there do not exist standardized financial derivatives, the (primary) art market can be taken to be a market like any other, and one in which the attainable financial returns to other investments can be compared. There are many research findings suggesting that the correlations of art returns with other financial assets are low[1]. However, how diversification gains (in a portfolio with Contemporary Masters, 19th Century European, Old Masters and 20th Century English paintings) affect the efficient frontier during the investment period is still an open question.[2]

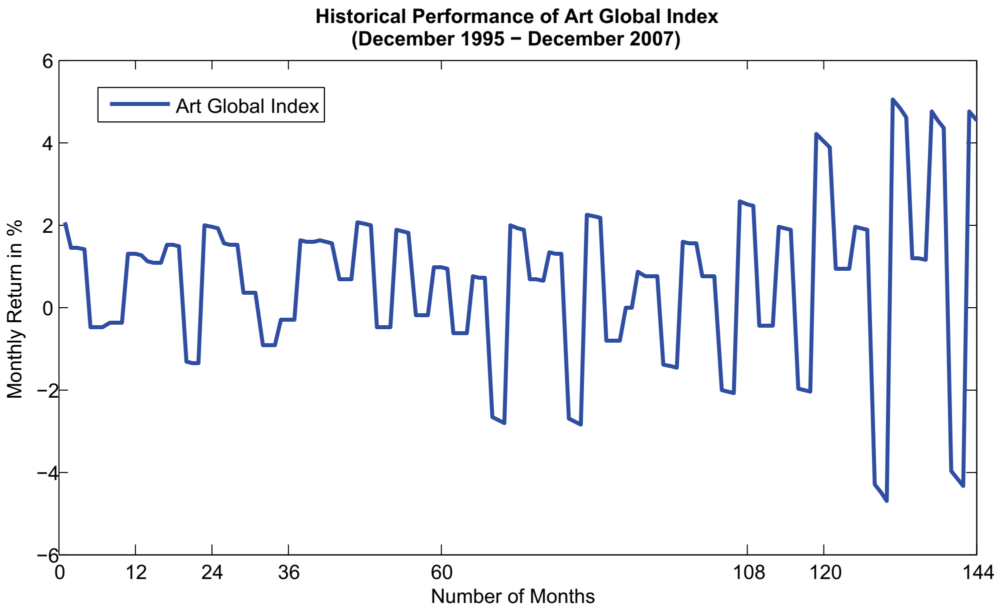

Figure 3.19: Historical Performance of Art Price Global Indices from December 1995 to December 2007

There remain nevertheless some major problems which hamper the research on art investment. As B. S. Frey & R. Eichenberger (1995) have pointed out, these problems concern the data, which is often biased by the fact that the main sources

[1]For the empirical results of this research, cf. chapter 4.

[2]See A. C. Worthington & H. Higgs (2004).

are auctions; and also by transaction costs, which significantly lower the rate of return, but are difficult to register. The same also holds for taxes, which differ among countries and art objects. Another problem is that the empirical outcomes are partly the result of the approach employed, and therefore less robust than desirable. In this research we will make some strong simplifications by not targeting individual art objects, and also by not considering the transaction costs and other fees. For the empirical research in art investment[1], the Art Global Index will be taken as the performance indicator for meaningful practical applications, this would require tradability of the index, as with REITs.

There is no unambiguous definition of hedge funds either. Hedge funds differ remarkably with respect to investment style and invested amounts. A **hedge fund** is basically an investment fund that is permitted by regulators to undertake a wider range of activities than other investment funds, and which also pays a performance fee to its investment manager. Hedge funds as an asset class invest in a broad range of investments which extend over equities, bonds commodities, and money market instruments[2]. The typical hedge fund investor is a HNWI or an institution, and these investors are presumed to be financially sophisticated. Hedge funds often seek to offset potential losses in the principal markets in which they invest in by hedging their investments using, e.g., short selling. For many years there has been an ongoing debate concerning in the financial sector benefits and dangers of hedge funds. Figure 3.1 shows that the global volume of hedge funds was US$ 1.5 trillion in 2006, when it was estimated that there were over 8,000 hedge funds worldwide. With a similar form of argumentation as we used for art, we will later choose an index for hedge funds, namely the Credit Suisse/Tremont hedge fund indices[3] for the period December 1995 to December 2007, as shown in Figure 3.20.[4]

[1]The index measures a weighted average of prices of paintings, sculptures, prints, drawings, and photographs, but excludes antiques and furniture (Base July 1990 = 100).

[2]There is a lot of research on hedge funds strategies in existence, see, e.g., T. Schneeweis & Spurgin, R. (2000), P. Boyle & S. S. Liew (2007), and D. Indjic & F. Partners (2002).

[3]See www.hedgeindex.com.

[4]The Credit Suisse/Tremont hedge fund database contains monthly data on 14 different hedge fund indices. These indices correspond to different styles of hedge fund investments, or

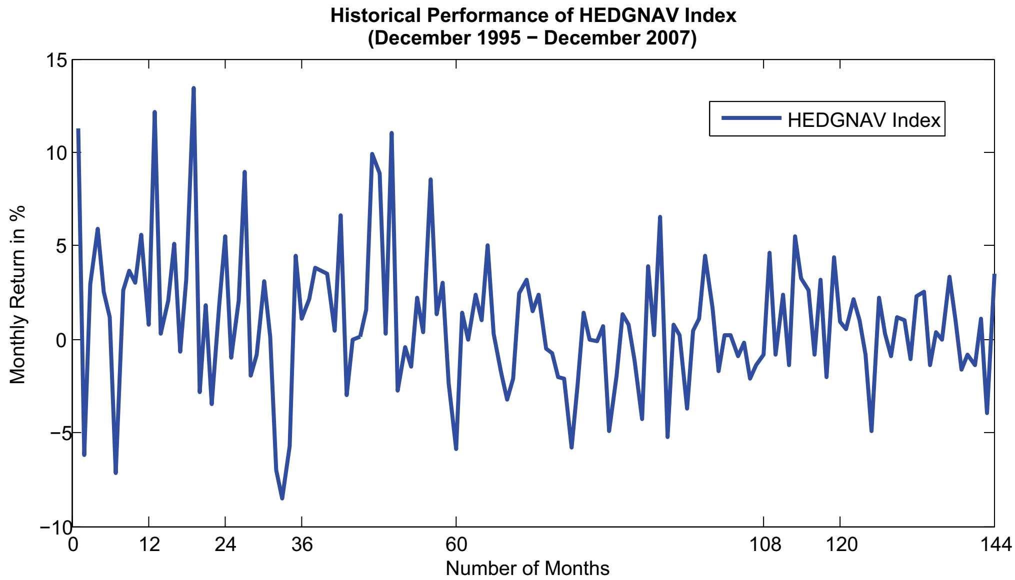

Figure 3.20: Historical Performance of the Credit Suisse/Tremont Hedge Fund Index from December 1995 to December 2007

3.5 Asset Allocation Strategies of Investors with High Initial Wealth and Different Risk Preferences

On behalf of mixing different asset classes, the goal of this section consists in describing the impact of different asset allocation strategies of wealthy investors on the set of efficient portfolios on the upper branch of the efficiency curve. In the next chapter, we will use market indices of asset classes to indicate the application of each asset allocation strategy. Referring to the classification shown in table 3.3, we distinguish, e.g., the following qualitative asset allocation strategies (AASs), which are based on the empirical statistical measures indicated in Table 4.2, Table 4.3 and Figure 4.1.

- AAS 1: For investors with low W_0 and low risk tolerance in terms of (μ, σ^2) using the basic asset allocation (i.e., mixing cash and, e.g., mutual funds).

combinations of these styles.

- AAS 2: For investors with low W_0 and high risk tolerance (μ, σ^2) who could, e.g., invest in levered mutual funds.
- AAS 3: For investors with high W_0 and low risk tolerance (μ, σ^2) using, e.g., the basic asset allocation of cash, bonds and stocks.
- AAS 4: For investors with high W_0 and high risk tolerance (μ, σ^2) who could, e.g., invest in domestic and international stocks, real estate, commodities and art.

In the following we will concentrate exclusively AAS 3 and AAS 4.

AAS 3: Strategy of Investors with High Initial Wealth Level and Low Risk Tolerance: The basic asset classes cash, bonds and stocks are easily available, liquid, and are associated with low transaction, or opportunity cost for everybody. We assume here that investors with high W_0 and low (μ, σ^2) will hold this basic portfolio.[1]

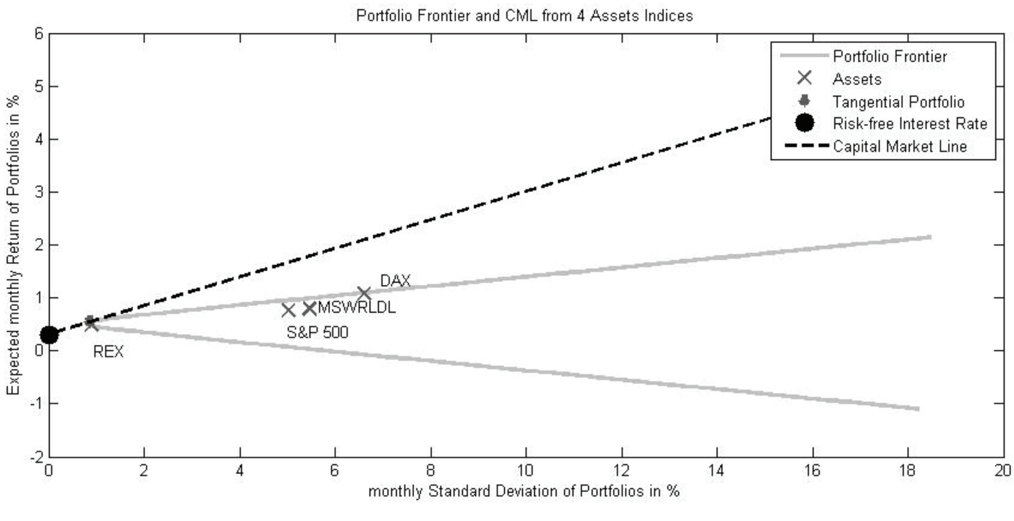

Figure 3.21: Portfolio Frontier with German Bond REX from December 1995 to December 2007

[1]Note that most wealthy people separate their wealth into two parts: one part for investments in residential real estate, and the other part for free use in other investments.

The efficient frontier is almost linear because of the negative correlations of the REX and the two other asset class indices. According to portfolio theory (cf. chapter 2), portfolios on the upper branch of the efficiency line are characterized by *favourable* combinations, in terms of expected return and standard deviation. Bonds serve here as a risk insuring asset class.

AAS 4: Example of a Strategy of Investors with High Wealth Level and High Risk Tolerance: Investors with a high level of initial wealth naturally have many choices of different assets from which to choose. Besides the basic portfolio, they can invest in real estate, art, private equity, hedge funds, and some other asset classes which have not been specified here. As an example of an aggressive asset allocation, we here derive the efficiency line of a mix of domestic and international stock and real estate. All index returns used in this research are characterized (cf. table 4.3) by positive long–term correlations. As a consequence, the efficiency line moves toward North–East. Especially note here the difference of the GMVP in figure 3.21 and 3.22.

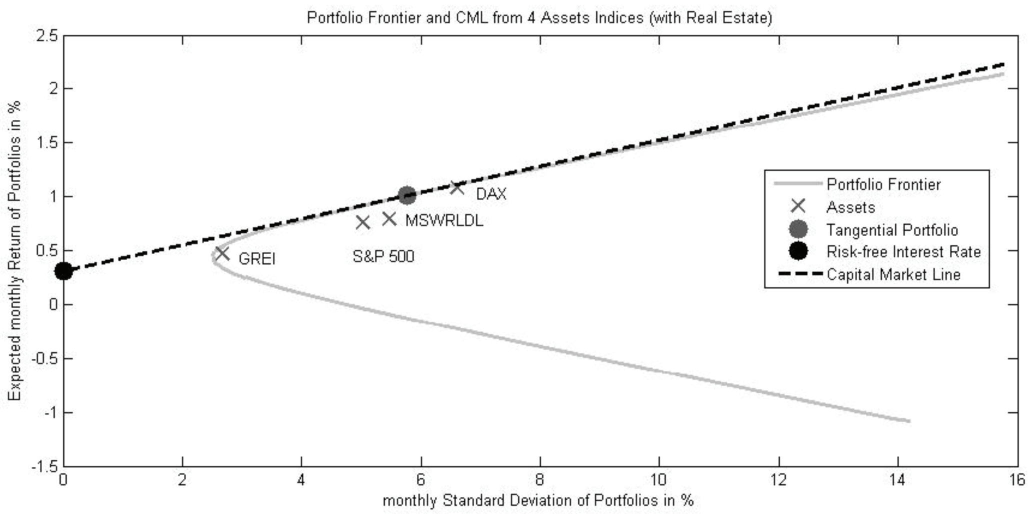

Figure 3.22: Portfolio Frontier with Real Estate from December 1995 to December 2007

Chapter 4

Portfolio Optimization for Special Asset Allocation Strategies

We will now start to construct, and in the following to execute, asset allocation strategies for different types of investors, based on the principles of portfolio selection. Investors' behavior with respect to risk attitude and initial wealth will be distinguished according to the asset classes in which they invest. We will investigate the impact that different asset allocation approaches have on risk–return–criteria of several specific portfolios. All portfolios are formed from the perspective of a German investor, and returns are consequently recorded in the currency of the Euro.

Two portfolio optimization approaches will be applied and analyzed in this chapter. The first one is a two–stage portfolio optimization starting with a mean–variance optimization and followed by a maximization of an expected utility; in this approach we must firstly identify the efficient frontiers for different asset allocation strategies, and then apply expected utility optimizations in order to locate optimal portfolios on the efficient frontiers. The second approach refers to the considerations in subsection 3.3.4, specifically to limiting the probability of loss. This approach is specifically designed for use with the asset allocation strategies of HNWIs, and thus deals simultaneously with the mean–variance and expected utility optimization, and with the Value at Risk approach. Note that both approaches implicitly take the initial wealth into account by the proportion of risky assets in the overall investment sum.

4.1 Data

The database consists of monthly returns from 10 different asset indices from Germany, the USA, and global financial markets for the period from December 1995 to December 2007. These indices cover cash markets, bond markets, stock markets, commodities, hedge funds, real estate and art[1].

Table 4.1: Indices for Asset Classes from December 1995 to December 2007

Cash Market	JPM EURO CASH 1M (JPEC1ML)
Bond Market	REX
Stock Market	DAX 30, S&P 500, MSWRLD
Commodities	OILBREN, GOLD
Hedge Funds	HEDGNAV CS Tremont[2]
Real Estate	Germany Real Estate Index (GREI)[3]
Art Market	Art Global Index[4]

The following tables and figure 4.1 show the statistical long–term measures return, standard deviation, skewness and kurtosis of all 10 indices, and the correlations among the 9 risky asset indices (excluding the JPEC1ML) on a monthly basis.

[1]Other than the data of the hedge funds, the arts, and the German real estate index, all data of the remaining indices was drawn from Datastream.

[2]The Credit Suisse/Tremont Hedge Fund Index is an unmanaged asset–weighted index of hedge funds which uses a rules–based construction methodology. Performance data used in the index is net of all fees. The index is calculated and rebalanced monthly.

[3]Deutscher Immobilienkonjunkturindex, BulwienGesa AG, Berlin, München & Hamburg.

[4]*Art price* analyzes the trends in 72 countries based on information from 2,900 auctioneers. Art price owns and exploits the world's largest data bank of fine art auction records (covering the categories of paintings, prints–posters, drawings, miniatures, sculptures–installations, photography, and tapestry), with about 4 million auction entries since 1700. Cf. http://web.artprice.com/start.aspx.

Table 4.2: Descriptive Statistics on a Monthly Base for Asset Class Indices from December 1995 to December 2007

Indices	Mean[1]	Std. Deviation	Skewness	Std. Kurtosis
JPEC1ML	0.29%	0.09%	0.204	-0.847
REX	0.42%	0.87%	-0.299	-0.398
DAX 30	1.12%	6.77%	-0.703	1.474
S&P500	0.69%	5.16%	-0.137	0.527
MSWRLD	0.80%	5.58%	-0.458	1.894
OILBREN	1.65%	11.11%	0.279	1.967
GOLD	0.51%	3.51%	0.725	2.660
HEDGNAV Index	0.92%	3.71%	0.580	1.509
GREI	0.57%	2.50%	-1.075	4.146
Art Global Index	0.58%	1.97%	-0.327	0.562

Table 4.3: Correlations on a Monthly Base for Asset Class Indices from December 1995 to December 2007

	REX	DAX30	S&P500	MS WRLD	OIL	GOLD	HEDG NAV	GREI	ArtG
REX	1.00								
DAX30	**(-0.22)**	1.00							
S&P 500	**(-0.20)**	0.66	1.00						
MSWRLD	**(-0.27)**	0.56	0.76	1.00					
OIL	**(-0.07)**	**(-0.04)**	**0.04**	**0.03**	1.00				
GOLD	**(-0.09)**	0.08	0.09	0.19	**0.18**	1.00			
HEDGNAV	**(-0.13)**	0.51	0.61	0.61	**0.21**	0.32	1.00		
GREI	**(-0.23)**	0.54	0.41	0.37	**0.16**	(-0.01)	0.36	1.00	
ArtG	**(-0.20)**	**0.13**	**0.09**	**0.12**	**0.12**	**0.11**	**0.18**	**0.09**	1.00

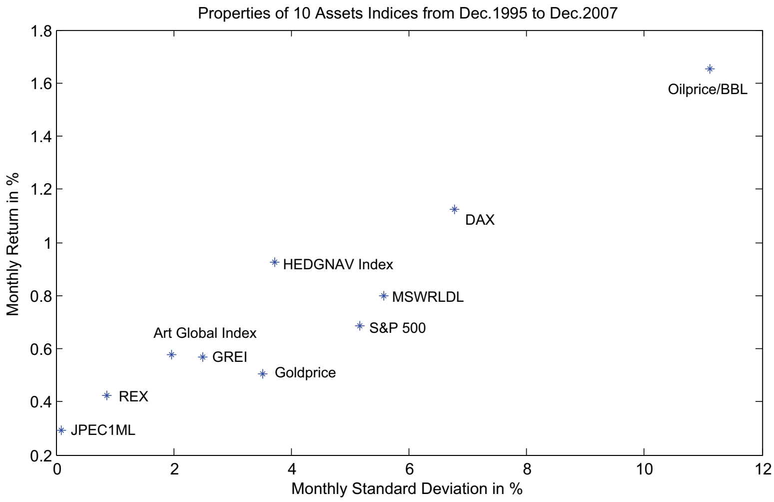

Figure 4.1: Empirical Long–Term Risk–Return Characteristics of the 10 Asset Indices (Data from December 1995 to December 2007)

Figure 4.1 indicates, e.g., that the Brent Crude Oil price was not only characterized by the highest long–term mean but also by the highest standard deviation while, e.g., the art global index had a relatively low mean, and also a low standard deviation.

Normality Test

Before we begin dealing with the optimization problems, it is important to remember that proper application of Markowitz theory requires (nearly) normal distribution of the assets' returns. As already pointed out, historical data is not and cannot be normally distributed[1]. Empirical research has shown that most empirical return distributions are right–skewed and leptokurtic (see A. Kraus & R. Litzenberger (1976), V. Chopra & W. T. Ziemba (1993).), so we will check here whether the returns are *approximately* normally distributed. The first reference is the third standard statistical moment, the *skewness*, which is, due to symmetry,

[1] The monthly returns – dividends that are taken into account – are computed using arithmetic. A detailed discussion about the differences of calculations of returns can be found in Benninga (2005).

[1]Remember that quadratic utility depends only on mean and variance.

zero for every normal distribution. A negative skewness corresponds graphically to a longer left tail and a bigger mass of the distribution, concentrated on the right of the figure. The distribution is then said to be left–skewed. Positive skewness is equated with right–skewed. The skewness is defined by $s = \frac{\mu_3}{\sigma^3}$, where μ_3 is the third moment about the mean, and σ is the standard deviation.[1] The kurtosis is the statistical measure of the "peakedness" of a distribution of a real–valued random variable, and a distribution with a positive kurtosis is called leptokurtic. In terms of shape, a leptokurtic distribution has a more acute "peak" around the mean (i.e., a higher probability than a normally distributed variable of values near the mean) and "fat tails" (i.e., a higher probability than a normally distributed variable of extreme values). The kurtosis is the fourth standard moment (i.e., $k = \frac{\mu_4}{\sigma^4}$), and is often also defined as $k = \frac{\mu_4}{\sigma^4} - 3$, which is also commonly known as excess kurtosis. The "minus 3" at the end of this formula is often explained to be a correction, which makes the kurtosis of the normal distribution equal to zero[2].

In this work we will use the **Jarque–Bera** test, a non–parametric test of normality that is based on skewness and kurtosis.[3] It is non–parametric in the sense that it tests normality without specifying a particular mean or variance. The test statistic is defined as

$$JB = \frac{n}{6}(s^2 + \frac{(k-3)^2}{4}),$$

where s is the skewness, k is the kurtosis, and n is the sample size. With an $\alpha = 0.05$, we obtain the following results of this two–sided test: the test does not reject the normal distribution hypothesis for the stock indices DAX 30, S&P 500 and MSWRLD and for the German Real Estate Index, but it refuses the normal distribution hypothesis for the Hedge Fund Index, the Art Global Index, Gold,

[1] See, e.g., Panik (2005).

[2] See, e.g., Panik (2005).

[3] Other tests which could be applied here are, e.g., the Kolmogoroff–Smirnoff test, or a Chi-Square goodness–of–fit test.

Brent Crude Oil, the REX and the JPEC1ML. This implies that one has at least to be very careful when applying the Markowitz mean–variance analysis directly!

4.2 Maximization of Expected Utility on Efficient Frontiers vs. Direct Utility Maximization

4.2.1 The Optimization Problems

The first optimization approach to be discussed in this section consists of two consecutive parts. First, the efficient frontiers will be computed, and secondly, an expected utility optimization will be executed in order to locate an optimal portfolio on the efficient frontiers. This procedure will be performed for different asset allocation strategies leading to distinguished efficient frontiers, and, in the second step, for different utility functions with varying parameters.

Step 1: Recall the problem formulation 2.8 without the permission of short sales from subsection 2.2.1:

$$\min_{\mathbf{x}\in\mathbb{R}^N} \mathbb{V}(R) = \mathbf{x}^T \mathbf{\Sigma} \mathbf{x}$$

subject to

$$\begin{aligned} \mathbf{1}^T\mathbf{x} &= \sum_{i=1}^{N} x_i = 1, \\ \boldsymbol{\mu}^T\mathbf{x} &= \sum_{i=1}^{N} x_i \mu_i = \mu_p, \\ \mathbf{x} &\geq 0. \end{aligned}$$

The non–negativity constraint will later be strengthened further by adding upper bounds for the investment percentages. In the following we derive numerically a fine mesh of points along the mean-variance efficient frontier.

Step 2: For each mean–variance efficient portfolio, we calculate its expected utility by using the chosen "plausible" utility function.

$$E[U(\boldsymbol{\mu}^T\mathbf{x}, \mathbf{x}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{x})]. \tag{4.1}$$

By calculating the expected utility for all efficient portfolios, we thus select the one efficient portfolio which maximizes the expected utility for a specific utility function $U(\mu_p, \sigma_p^2)$. The maximum expected utility obtained by the mean–variance efficient set is now denoted by $E*U(\cdot)$. Note that the approximation of the expected utility uses only mean and variance. We refer again to the three types of utility functions and their expected utilities from Table 2.3; the expected utilities of these different functions will finally be compared.

In the second optimization approach, we use the same set of data of individual indices for asset classes, and select the portfolio which maximizes expected utility of the given utility function, i.e., not just the maximum among mean–variance efficient portfolios, but among all feasible portfolios. The value obtained by this maximization, which we shall refer to as the direct maximization, will be denoted by $EU(\cdot)$, as distinguished from $E*U(\cdot)$. Referring again to the notation of section 2.3, we thus get the maximization problem

$$\Phi(\mathbf{x}) = \mathbf{E}[U(\mathbf{x})] \rightarrow \max \quad \text{subject to} \quad B\mathbf{x} = \mathbf{c}. \tag{4.2}$$

We deal here with an additional VaR constraint in order to limit the loss probability in a certain region. We recall from subsection 3.3.4 the phenomenon that especially HNWIs are much concerned with their potential losses while investing in risky securities. Incorporating the VaR approach in an appropriate optimization problem for this type of investor, we formulate 3.3.4 as follows[1]:

$$\Phi(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\Sigma\mathbf{x} \rightarrow \min \quad \text{subject to} \quad B\mathbf{x} = \mathbf{c}, \quad Pr(W \leq bW_0) \leq 1 - c \tag{4.3}$$

with

$$VaR = (1-b)\frac{W_0}{A} = -(\mu_R - \Phi^{-1}(c)\sigma_R).$$

[1]Because of computation limits, we cannot here directly resolve the problem with VaR constraints. This idea, however, can be taken as a useful instrument to limit the probability of loss.

As already mentioned, the VaR–constraint is not incorporated in the computations, but, by checking afterwards whether a solution of problem 2.8 satisfies the Value at Risk constraint.

4.2.2 The Selected Utility Functions

In this section we will try to construct different asset allocation strategies for HNWIs. After having computed an efficient frontier for an asset allocation strategy, we continue by calculating the expected utilities of a discrete number of elements on the upper branch of the efficiency line. Referring to Table 2.3, we use the approximations of the expected utilities of the utility functions $U_i(\mu, \sigma^2)$and$i \in \{1, 2, 3\}$ in the empirical tests.

Quadratic utility:

$$U_1(\mu, \sigma^2) = \mu - \frac{\beta}{2}\sigma^2.$$

Log wealth utility:

$$U_2(\mu, \sigma^2) = log(1 + \mu) - \frac{\sigma^2}{2(1 + \mu)^2}.$$

Power utility:

$$U_3(\mu, \sigma^2) = 1 - \frac{1}{1 + \mu} - \frac{\sigma^2}{(1 + \mu)^3}.$$

We expand this $\mu - \sigma$ framework now by incorporating the initial wealth W_0, and a parameter $t = A/W_0$ describing the overall proportion of risky assets in the portfolio. The coordinates of efficient pairs μ and σ can now be rewritten as

$$\mu_p = W_0(1 + R_f) + A(\mu - R_f),$$

$$\sigma_p^2 = A^2\sigma^2.$$

U_1 is now transformed into

$$U_1(\mu, \sigma) = W_0(1 + R_f) + tW_0(\mu - R_f) - \frac{\beta}{2}W_0^2 t^2\sigma^2.$$

The direct utility optimization problem is formulated as follows:

$$\begin{aligned} \max_j \quad & E[U_j(\textstyle\sum_{i=1}^N x_i \mu_i)] \\ \text{s.t.} \quad & \mathbf{1}^T \mathbf{x} = \sum_{i=1}^N x_i = 1 \\ & i = 1, \ldots, N; \quad j = 1, 2, 3. \end{aligned}$$

4.3 The Empirical Results

In this section we present the empirical results from the chosen database under the framework described in the previous section. In the following, the initial wealth W_0 is fixed, and we assume without loss of generalization that it equals 1. The following computations have been executed for portfolios consisting of 4 and 7 asset class indices[1] as examples, respectively:

1. Determination of optimal portfolios on the efficient frontier, followed by computation of the utility values for various representatives of the three different types of utility functions.

2. Direct maximization of utility functions.

Table 4.4 refers to the unbounded optimal strategy under quadratic utility for investors with different risk aversion levels. Figure 4.3 shows that the portfolios on the efficient frontier with maximal expected quadratic utility remain the same for those less risk–averse investors; and the optimal portfolio moves in the direction of global minimum variance portfolio (GMVP) for more risk–averse investors. In this 4–assets example, the portfolio with maximal utility is unequal to the global minimum variance portfolio, and to the tangential portfolio. The maximal utilities on the efficient frontier are clearly larger than those of GMVP and the tangential portfolio. The portfolio utilities differentiate with parameter β; there's higher utility with lower β, and vice versa. Practically, this means

[1] The 4 asset class indices include DAX 30, S&P 500, MSWRLD and GREI; besides these there are the following indices: the Gold price, the HEDGNAV Index and the Art Global Index.

Table 4.4: Optimal Strategy with Quadratic Utility on the Efficient Frontier of the Unrestricted Optimization Problem

U_1	$\beta = 0.6$			$\beta = 1$			$\beta = 2$		
	t=0.3	0.7	0.9	t=0.3	0.7	0.9	t=0.3	0.7	0.9
u_{GMVP}	1.0036	1.0044	1.0049	1.0035	1.0043	1.0046	1.0035	1.0039	1.0040
u_{TP}	1.0044	1.0064	1.0074	1.0043	1.0060	1.0068	1.0042	1.0052	1.0053
$\max(U_1)$	1.0087	1.0162	1.0200	1.0072	1.0089	1.0092	1.0048	1.0053	1.0053
μ	0.0087	0.0164	0.0202	0.0087	0.0135	0.0141	0.0063	0.0072	0.0078
σ^2	0.0024	0.0132	0.0219	0.0024	0.0078	0.0083	0.0008	0.0010	0.0012
x_{DAX}	2.8396	2.8396	2.8396	2.8396	2.1306	1.6579	1.4806	0.5352	0.4171
$x_{S\&P500}$	-2.1018	-2.1018	-2.1018	-2.1018	-1.5793	-1.2309	-1.1003	-0.4035	-0.3164
$x_{MSWRLDL}$	1.3645	1.3645	1.3645	1.3645	1.0519	0.8435	0.7654	0.3486	0.2965
x_{GREI}	-1.1023	-1.1023	-1.1023	-1.1023	-0.6032	-0.2705	-0.1457	0.5197	0.6029

that the more risk averse investors, especially the wealthy ones, should hold their portfolios near to the global minimum variance portfolio.

While an elegant analytical solution in the unrestricted case exists (cf. section 2.4), restricted problems have to be solved numerically (cf. section 2.6). The constrained optimization problems will here be treated with the help of MATLAB–software. In the following we compare results obtained for a quadratic utility function U_1 with $\alpha = 1$ and β taking values of 0.01, 0.6 and 2, respectively, and for the log–wealth utility function U_2 and the power utility function U_3. Note again that U_2 and U_3 do not depend on any other parameters, but only on μ and σ^2. Table 4.4 refers to an optimization using U_1, and shows the weights of the four assets DAX 30, S&P 500, MSWRLD, and GREI for different parameters of β and different proportions t invested in risky assets. From this table we derive that there are almost no differences between the varying t and $\beta = 0.01$ and $\beta = 0.6$ on the weights of optimal portfolios. For $\beta = 2$ and $t = 0.3$ we find that the optimal portfolio is the same as those portfolios with $\beta = 0.01$ and $\beta = 0.6$; however, with an increasing proportion of risky assets, the optimal portfolio changes. Investors must then put more wealth into the MSWRLD and the Germany Real Estate Index.

Figure 4.3 shows the portfolio weights of the four assets depending on the

Table 4.5: Optimal Asset Allocation on the Efficient Frontier with Quadratic Utility and Non–Negativity Constraints for varying β and t

U_1	$\beta = 0.01$			$\beta = 0.6$			$\beta = 2$		
	t=0.3	0.7	0.9	t=0.3	0.7	0.9	t=0.3	0.7	0.9
max(U_1)	1.0054	1.0086	1.0102	1.0051	1.0074	1.0082	1.0046	1.0051	1.0052
μ	0.0054	0.0086	0.0103	0.0054	0.0086	0.0103	0.0054	0.0068	0.0073
σ^2	0.0004	0.0021	0.0035	0.0004	0.0021	0.0035	0.0004	0.0009	0.0010
x_{DAX30}	0.9536	0.9536	0.9536	0.9536	0.9536	0.9536	0.9536	0.4609	0.3283
$x_{S\&P500}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
x_{MSWRLD}	0.0464	0.0464	0.0464	0.0464	0.0464	0.0464	0.0464	0.1275	0.1218
x_{GREI}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4116	0.5498

expected return under non–negativity constraints, but without additional upper and lower bounds.[1] Note that the weight of DAX 30 increases as the required monthly return of the portfolio increases. The weight of the MSWRLD remains rather stable for a medium range of expected returns.

Table 4.5 gives the results of the computations for U_2 and U_3 with varying t. An interesting result is that the optimal portfolio according to U_2 is actually identical to the one obtained using quadratic utility with $\beta = 0.01$ or $\beta = 0.6$ and the portfolio resulting from U_3 is the same as the portfolio from quadratic utility with $\beta = 2$. The following results refer to the maximal expected utility optimization on the efficient frontier.

Now we will concentrate our investigations on direct expected utility maximization. The following tables are the results from direct utility maximization of the three expected utility functions. Note again that this approach doesn't focus only on efficient portfolios, but on all available portfolios. Because the expected utility optimization problem is equivalent to the direct utility maximization problem for quadratic utility functions, the results are the same; there is thus no difference between these two optimization problems.

For U_2 and U_3, however, there are obvious differences between the solutions of the maximal expected utility optimization and the direct utility maximization

[1] This will be analyzed in the 7 assets case, see below.

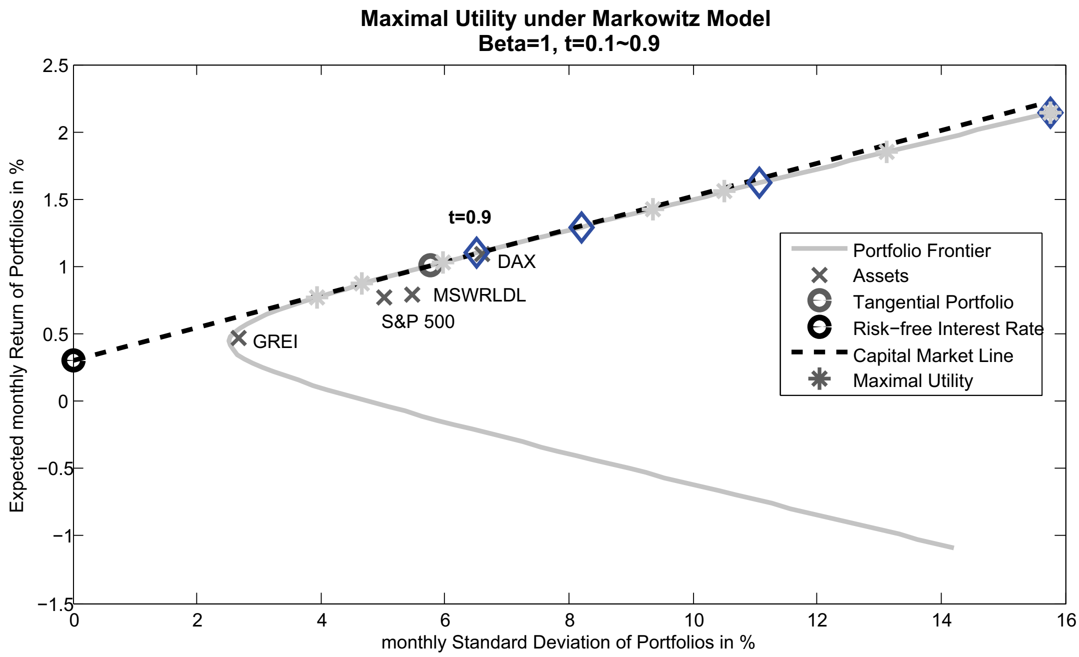

Figure 4.2: Maximal Quadratic Utility under the Markowitz Model, Beta = 1

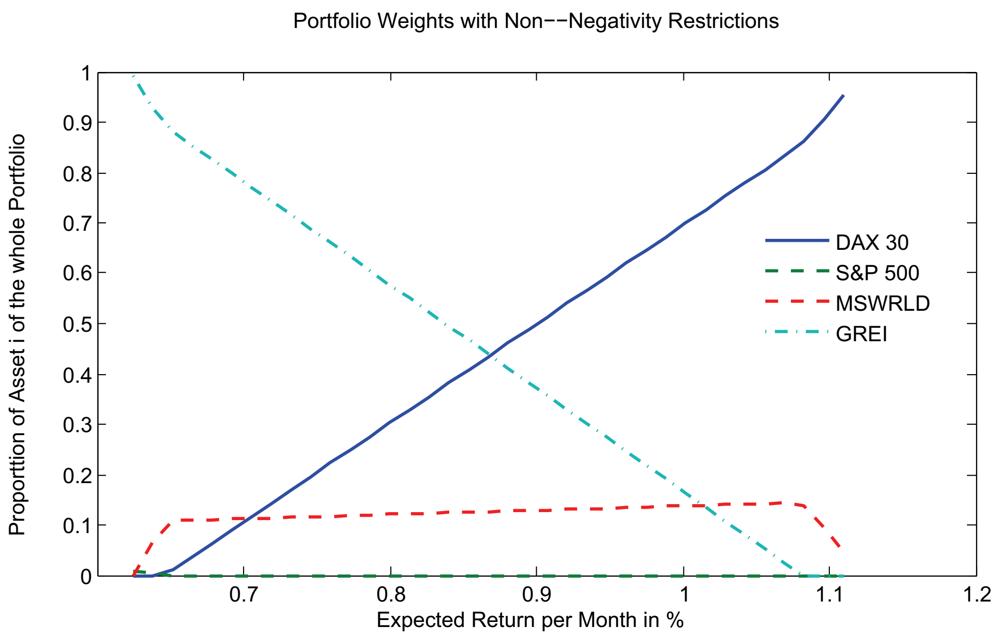

Figure 4.3: Portfolio Weights with Non–Negativity Constraints

Table 4.6: Optimal Asset Allocation on the Efficient Frontier with U_2 and U_3 and Non–Negativity Constraints and varying t

	U_2			U_3		
	t=0.3	0.7	0.9	t=0.3	0.7	0.9
$\max(U(\mu,\sigma^2))$	0.6958	0.6972	0.6972	0.5011	0.5013	0.5013
μ	0.0054	0.0086	0.0103	0.0054	0.0068	0.0073
σ^2	0.0004	0.0021	0.0035	0.0004	0.0009	0.0010
x_{DAX30}	0.9536	0.9536	0.9536	0.9536	0.4609	0.3283
$x_{S\&P500}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
x_{MSWRLD}	0.0464	0.0464	0.0464	0.0464	0.1275	0.1218
x_{GREI}	0.0000	0.0000	0.0000	0.0000	0.4116	0.5498

Table 4.7: Direct Utility Maximization of U_2 and U_3 with Non–Negativity Constraints

	U_2			U_3		
	t=0.3	0.7	0.9	t=0.3	0.7	0.9
$\max(U(\mu,\sigma^2))$	0.6953	0.6962	0.6976	0.5010	0.5015	0.5017
μ	0.0044	0.0064	0.0098	0.0044	0.0064	0.0074
σ^2	0.0005	0.0012	0.0033	0.0005	0.0012	0.0016
x_{DAX30}	0.2509	0.2522	0.7873	0.2504	0.2510	0.2513
$x_{S\&P500}$	0.2497	0.2492	0.0000	0.2498	0.2496	0.2495
$x_{MSWRLDL}$	0.2500	0.2500	0.2127	0.2500	0.2500	0.2500
x_{GREI}	0.2494	0.2486	0.0000	0.2497	0.2494	0.2492

problem. Table 4.7 shows the results from direct U_2 and U_3 utility maximization. Optimal portfolios tend to be rather *naive* portfolios with U_3 and U_2, and with lower proportions in risky assets. This result implies that, for more risk–averse investors with relatively complicated utility forms, naively diversified portfolios are a good choice.

Another interesting result is that the optimal portfolio under U_2 with a higher proportion in the risky part ($t = 0.9$) deviates strongly from other optimal portfolios. The expected return of this portfolio is much higher than that of the portfolio with $t = 0.7$, and it has a consequently a higher weight of the DAX 30. Here we thus derive the following question from Table 4.7: up to which proportion

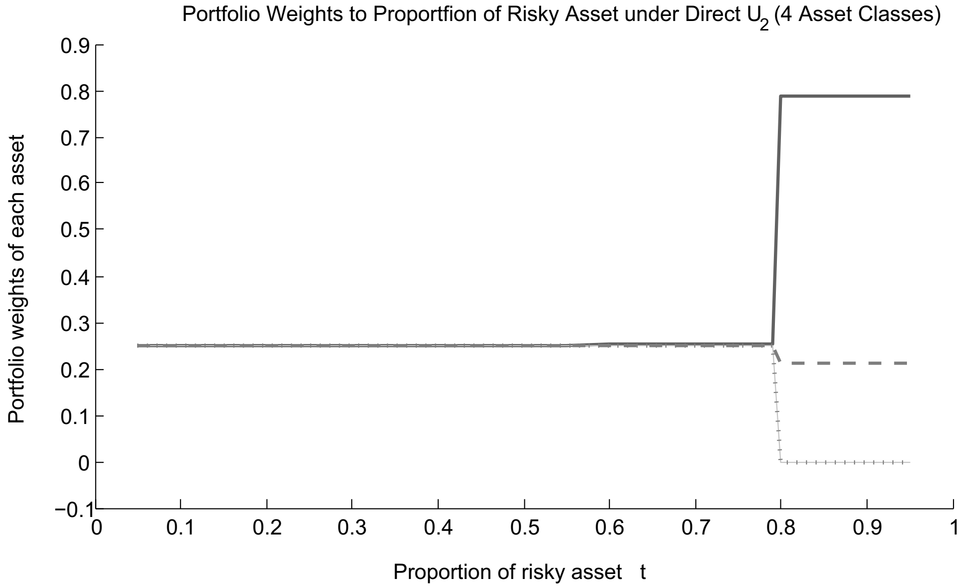

Figure 4.4: Portfolio Weights of the 4 Assets as a Function of t under Direct Utility Maximization of U_2

of the risky part should an investor hold a naively diversified portfolio? Computations visible in Figure 4.4 indicate that from $t = 0.790885$, the composition of the optimal portfolio weights will be changing dramatically. The bold black line represents here the weight of the DAX 30 while the dashed line represents the weight of the MSCWRLD.

From Table 4.5 and 4.6 we see that, for practical optimization cases as computed here, the use of quadratic utility functions is sufficient. Log wealth and power utility functions deliver results of the same quality. Moreover, risk–conscious investors such as HNWIs (modeled here by $\beta = 2$) can choose portfolios with a rather large t, if the risky part is dominated by contributions with individually low risks.

After having compared these two methods we conclude that, for quadratic utility functions, there is no difference between maximal expected utility and the direct utility maximization problem; and for more complicated forms of utility functions (such as U_2 and U_3), maximal expected utility leads toward a genuine diversification while direct utility maximization tends toward a naively diversified portfolio for a t that is not *too* big. This is actually in line with *good* intuition.

We will now expand the universe of assets by enlarging it from 4 asset indices to 7 asset indices. The three additional classes we will now add as options for investors are the Gold Price Index, the Hedge Fund Index and the Art Global Index. The optimization problems will again be treated under non–negativity constraints. Furthermore, we will investigate the practically interesting case of an upper bound, here identical for all assets to be set to 0.25.

Figure 4.5 shows the portfolio weights of the 7 asset indices as a function of the portfolio's expected return under maximization of quadratic utility on the efficient frontier with non–negativity constraints, where $\beta = 2$, and $t = 0.9$. With increasing required expected return, the weight of the DAX 30 rises again. The weight of gold changes dramatically when the required return varies from 4.5% to 5.5%. Figure 4.6 gives the portfolio weights for these 7 asset indices with upper bounds of 25%, and in this case, the portfolio weights of S&P 500 and MSWRLD should be reinvested with increasing required expected return.

Table 4.8: Optimal Strategy for 7 Assets with Quadratic Utility and Non–Negativity Constraints

U_1	$\beta = 0.01$			$\beta = 0.6$			$\beta = 2$		
	t=0.3	0.7	0.9	t=0.3	0.7	0.9	t=0.3	0.7	0.9
$\max(U_1)$	1.0054	1.0086	1.0102	1.0051	1.0075	1.0085	1.0047	1.0062	1.0067
μ	0.0054	0.0086	0.0103	0.0054	0.0083	0.0096	0.0051	0.0077	0.0084
σ^2	0.0004	0.0020	0.0033	0.0004	0.0014	0.0018	0.0002	0.0007	0.0009
x_{DAX30}	0.9248	0.9248	0.9248	0.9248	0.6992	0.5487	0.5487	0.2479	0.1824
$x_{S\&P500}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{MSWRLDL}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{Goldprice}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{HEDGNAVIndex}$	0.0752	0.0752	0.0752	0.0752	0.3008	0.4513	0.4513	0.7521	0.6394
x_{GREI}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{ArtGlobalIndex}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1781

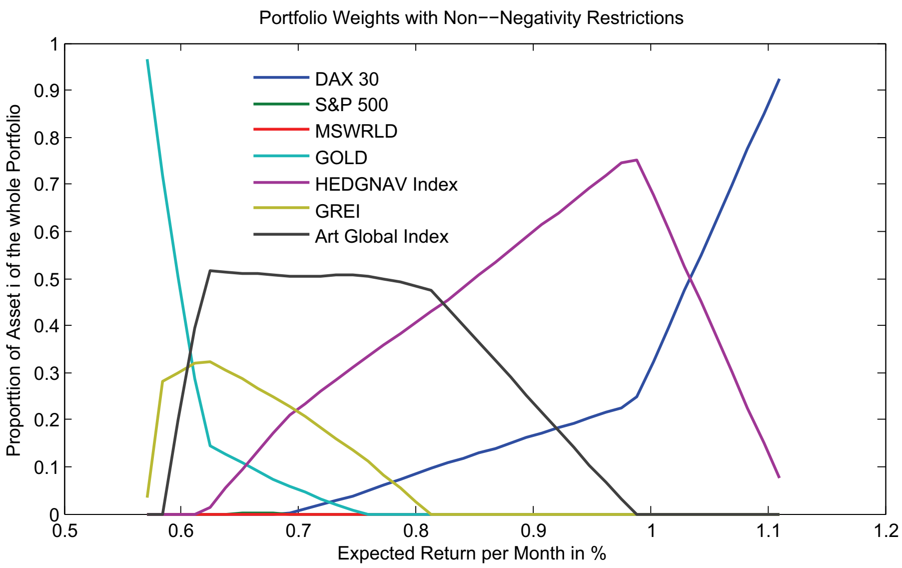

Figure 4.5: Portfolio Weights for 7 Asset Indices with Non–Negativity Constraints

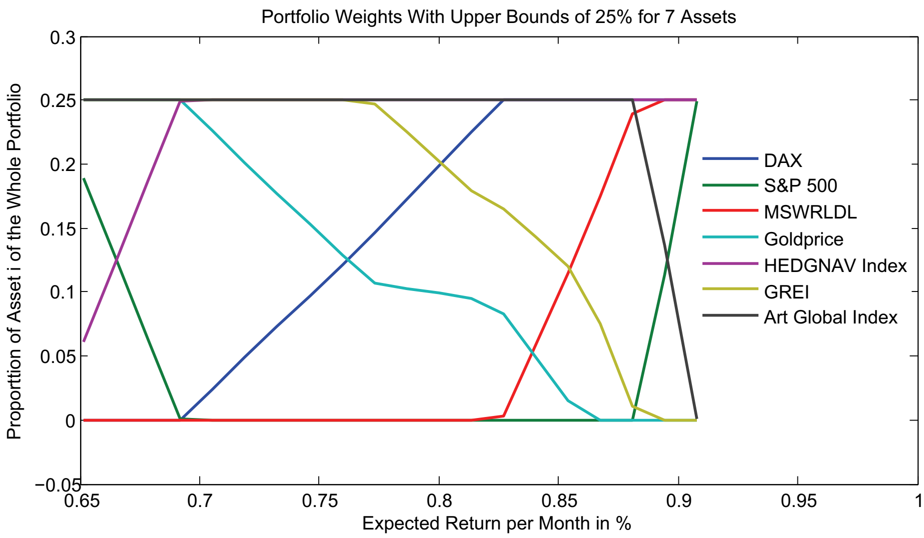

Figure 4.6: Portfolio Weights for 7 Asset Indices with Upper Bounds of 25%

Table 4.9: Optimal Strategy for 7 Assets with U_2 and U_3 with Non–Negativity Constraints

	U_2			U_3		
	t=0.3	0.7	0.9	t=0.3	0.7	0.9
$\max(U(\mu,\sigma^2))$	0.6958	0.6972	0.6979	0.5012	0.5015	0.5017
μ	0.0054	0.0086	0.0103	0.0051	0.0077	0.0085
σ^2	0.0004	0.0020	0.0033	0.0002	0.0007	0.0009
x_{DAX30}	0.9248	0.9248	0.9248	0.5487	0.2479	0.1931
$x_{S\&P500}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{MSWRLDL}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{Goldprice}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{HEDGNAVIndex}$	0.0752	0.0752	0.0752	0.4513	0.7521	0.6658
x_{GREI}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_{ArtGlobalIndex}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.1411

We remember the results for the situation with 4 assets; Table 4.8 now shows the results for 7 assets under U_1, and Table 4.9 the results under U_2 and U_3. As Figure 4.5 points out, for a higher required expected return there are only three asset classes, namely the DAX 30, The HEDGNAV and Art, in which investment is possible. Similar to the former results, the maximal expected utility U_2 has the same optimal portfolio as U_1, when $\beta = 0.01$; and the maximal expected utility U_3 leads to results similar to those of the optimal portfolio U_1, when $\beta = 2$. There is a small difference between U_1 and U_3 when a higher risky–asset proportion t is invested: Under U_1, 17.81% of the wealth in risky assets is invested in the Art Global Index, while it is 14.11% under U_3.

As mentioned before, the quadratic functions expected utility optimization problem is identical to the direct utility maximization problem. For U_2 and U_3 there are obvious differences between maximal expected utility optimization and direct utility maximization; Table 4.10 shows the results from direct U_2 and U_3 utility maximization. Here we find that optimal portfolios tend to be naively diversified portfolios under U_3 and U_2 with a low overall proportion of risky assets ($t = 0.3$). This result again implies that, for more risk averse investors with relatively complicated (or no) utility forms, naive portfolios can be regarded as a good choice. The optimal portfolio (under U_2 with $t = 0.7$ and $t = 0.9$) appears to be very different from other optimal portfolios; the expected return is much

Table 4.10: Optimal Strategy for 7 Assets with Direct U_2 and U_3 Maximization with Non–Negativity Constraints

	U_2			U_3		
	t=0.3	0.7	0.9	t=0.3	0.7	0.9
$\max(U(\mu,\sigma^2))$	0.6952	0.6969	0.6977	0.5010	0.5014	0.5016
μ	0.0043	0.0080	0.00983	0.0043	0.0061	0.0070
σ^2	0.0002	0.0016	0.0024	0.0002	0.0006	0.0008
x_{DAX30}	0.1439	0.5285	0.6564	0.1434	0.1441	0.1444
$x_{S\&P500}$	0.1427	0.0000	0.0000	0.1427	0.1426	0.1425
$x_{MSWRLDL}$	0.1430	0.1384	0.0000	0.1429	0.1430	0.1430
$x_{Goldprice}$	0.1422	0.0000	0.0000	0.1425	0.1421	0.1419
$x_{HEDGNAVIndex}$	0.1432	0.3331	0.3436	0.1431	0.1435	0.1437
x_{GREI}	0.1424	0.0000	0.0000	0.1426	0.1423	0.1422
$x_{ArtGlobalIndex}$	0.1424	0.0000	0.0000	0.1427	0.1424	0.1423

higher than for the portfolio with $t = 0.3$, and the differences between the $t = 0.7$ and $t = 0.9$ cases exist, but are comparably small. Again, a higher proportion of assets has been applied to DAX 30 and HEDGNAV Index. Similar to the case of four asset indices are these results which we have gained, a risky proportion to which the investor may hold a "naive" portfolio. In this seven asset indices case, $t = 0.577558$ is a corresponding threshold, from which the weights of the optimal start to change substantially. Other than these differences, all previously mentioned conclusions from the 4 assets case hold. The bold black line in Figure 4.7. represents the weight of the DAX 30 while the dashed line represents the weight of the HEDGNAV index.

Implicit Check of the VaR–Constraint

For HNWIs we proposed a check using a VaR constraint. We will now see whether the solutions of the problem 2.8. satisfy also 3.3.4 with the additional VaR–constraint. Given the portfolios from the above results, a probability level c of 95% and a time horizon of one month, we assume further that wealthy investors can endure a 10% wealth loss. We deal here with $b = 0.9$ and as before with $t = 0.3$ and $t = 0.9$. Obviously, the larger the t is, the higher the VaR

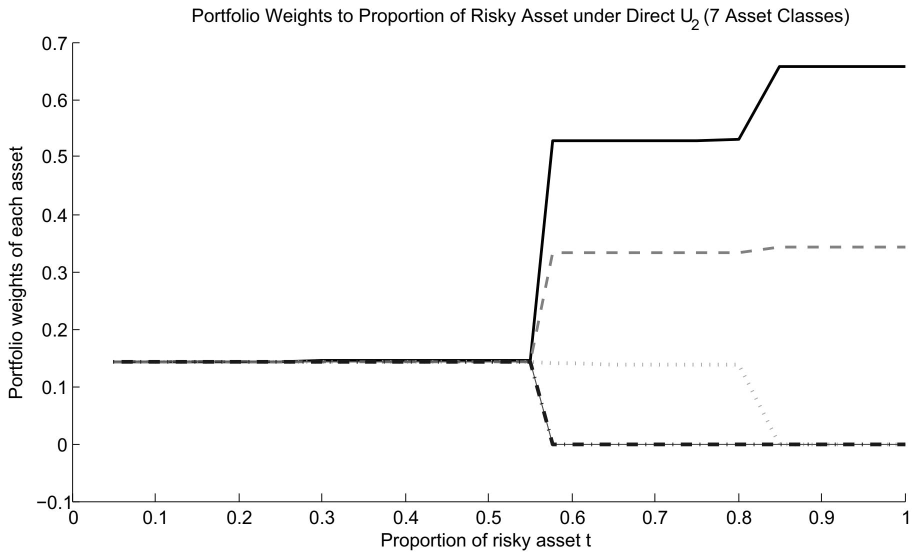

Figure 4.7: Portfolio Weights to Proportion of Risky Asset under Direct U_2 for 7 Asset Indices

is. We check whether the portfolios satisfy the VaR constraints or not;[1] those portfolios which do not should not be considered anymore. The following table shows that this constraint is not violated for all optimizations with $t = 0.3$ and $t = 0.9$, respectively. In Table 4.11 are all the VaRs smaller than 10%; this means that all these optimal portfolios are principle acceptable under the given set of assumptions.[2]

[1]As already mentioned, we don't solve the optimization problem directly with VaR constraint, but rather as a benchmark in our empirical research.

[2]The quality of the approximation can be examined using a criterion according to Kroll *et al.* (1984):

$$I = \frac{E * U(\cdot) - E_N U(\cdot)}{EU(\cdot) - E_N U(\cdot)},$$

where $E_N U(\cdot)$ is the expected utility of a naively diversified portfolio, in which $\frac{1}{n}$ is invested in each asset class; that is:

$$E_N U(\cdot) = EU\left(\sum_{i=1}^{N} \frac{1}{n}\mu_i\right).$$

Table 4.11: Results for Direct and Approximated Expected Utility with Non–Negativity Constraints and Value–at–Risk

	t=0.3				t=0.9			
U_i	$E*U(\cdot)$[1]	$EU(\cdot)$[2]	$EU_N(\cdot)$[3]	VaR	$E*U(\cdot)$	$EU(\cdot)$	$EU_N(\cdot)$	VaR
4 Asset Indices								
U_1								
$\beta = 0.01$	1.0054	1.0054	1.0044	2.75%	1.0102	1.0102	1.0074	8.7%
$\beta = 0.6$	1.0051	1.0051	1.0043	2.75%	1.0082	1.0082	1.0066	8.7%
$\beta = 2$	1.0046	1.0046	1.0041	2.75%	1.0052	1.0052	1.0046	4.5%
U_2	0.6958	0.6953	0.6953	2.75%	0.6972	0.6976	0.6965	8.7%
U_3	0.5011	0.5010	0.5011	2.75%	0.5013	0.5017	0.5017	4.5%
7 Asset Indices								
U_1								
$\beta = 0.01$	1.0054	1.0054	1.0043	2.75%	1.0102	1.0102	1.0070	8.4%
$\beta = 0.6$	1.0051	1.0051	1.0042	2.75%	1.0085	1.0085	1.0065	6.0%
$\beta = 2$	1.0047	1.0047	1.0041	1.8%	1.0067	1.0067	1.0056	4.1%
U_2	0.6958	0.6952	0.6953	2.75%	0.6979	0.6977	0.6964	8.4%
U_3	0.5012	0.5010	0.5011	1.8%	0.5017	0.5017	0.5016	4.1%

4.4 Uncertainty of the Estimates and Back testing

4.4.1 Uncertainty of the Model Input

The value of a portfolio optimization model for investors is linked directly to the relation of expected returns at the beginning to the realized return at the end of the investment period. The perfect model is model wherein all expected returns are equal to the realized returns one period later. In his work from 1952 (which eventually lead to his winning a Nobel prize), Modern Portfolio Theory founder Harry Markowitz pointed out that "The process of selecting a portfolio may be divided in two stages. The first stage starts with observations and experience and ends with beliefs about the future performance of available securities. The second stage starts with the relevant beliefs about future performance and ends with the choice of the portfolio. This paper is concerned with the second stage."

[1] The highest value of the expected utility from all efficient portfolios.
[2] The expected utility from direct maximization.
[3] Expected utility of the naive portfolio.

In other words, Markowitz theory provides the optimization techniques but not the input or parameters, respectively. It is thus clear that the estimation of the model input is of crucial importance.

The Markowitz model requires estimates (cf. chapter 2) for expected returns, variances and covariances. As known from theory and from empirical research, most important are the errors in the expected returns, followed by the covariances.[1] Estimates for returns and risks in a portfolio optimization are typically a combination of current and historical information. Statisticians today use a variety of techniques that are designed to improve from historical data[2] the forecast value of estimates of return and risk. Strategic or long–term asset allocators especially use historical return data, but adjust the derived statistical parameters using current information. From this point of view, asset managers are natural Bayesian; however, few use formal Bayesian procedures in optimization estimates. A number of methods have been suggested to handle with risk–estimation problems within a portfolio context (see Richard O. Michaud & Robert O. Michaud (2008) for a review). Estimates based on different historical data differ and will necessarily lead to different efficient frontiers.

On the basis of the research by M. J. Best & R. R. Grauer (1991) we will now illustrate the impact of a change in (only) one input. Figure 4.8 illustrates, e.g., the change of the efficient frontier in the case with our 4–asset–indices case if only the expected return of the DAX 30 decreases by 0.2 (corresponding to 20 %). We find that the efficient frontier moves remarkably toward South–East as a result of this single change of an asset's expected return as an input parameter. It is easy to imagine which impact changes of more expected asset returns in the same direction would have.

Figure 4.9 provides different efficiency curves for the 4–asset–indices case which are estimated on the base of consecutive yearly historical data. The graph illus-

[1] Cf. e.g., Chopra & W. T. Ziemba (1993).

[2] See J. D. Jobson *et al.* (1979), J. D. Jobson *et al.* (1980) and J. D. Jobson & R. Korkie (1980) which use, e.g., James Stein Estimators to improve the Markowitz optimization.

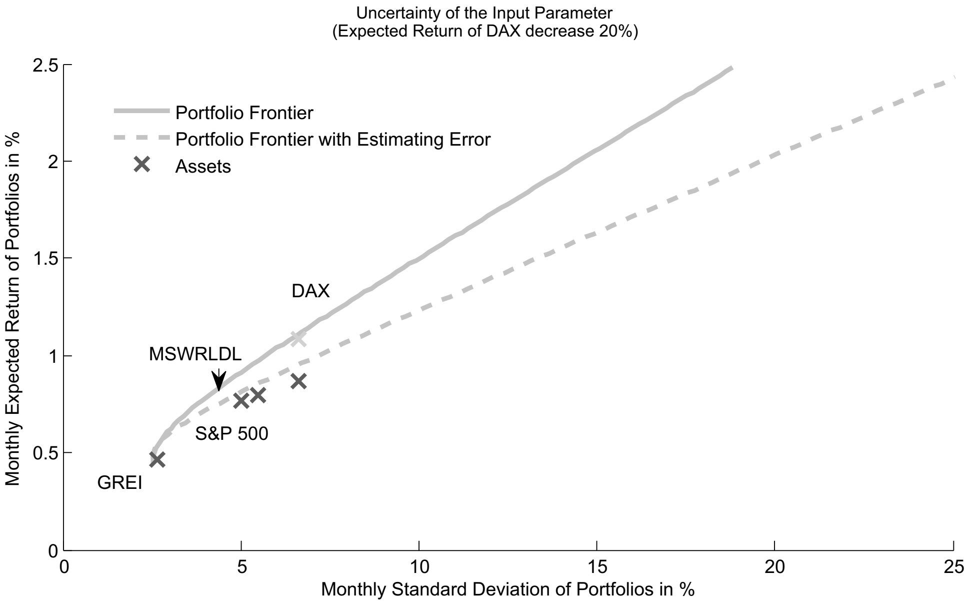

Figure 4.8: Uncertainty of the Input Parameters

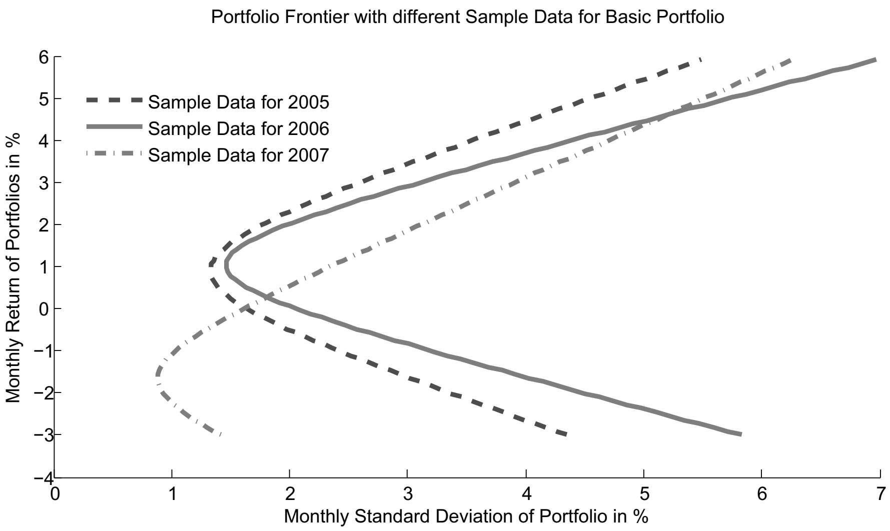

Figure 4.9: Impact of a Single Change of an Expected Return on an Efficiency Line

trates the dynamics of the efficient frontier, which were generated by the historical returns from the years 2005, 2006 and 2007. The efficiency line based on the data from 2005 would thus be the line for indicating risk–return characteristics for investments in 2006, the 2006 line for investments in 2007, etc. Note here that the efficiency lines move from 2006 to 2007 toward South–West; this corresponds with the generally improved economic perceptions and the recovery of the (four) markets from 2006 to 2007.

As Figures 4.8 and 4.9 point out, the efficient frontier can vary greatly under different input parameters and under different time horizons. One reason for this is that under the multivariate normality assumption we use maximum likelihood estimators for μ and Σ and therefore, R_{ij} is the realization of the random return of asset i at the point j. For the input parameter we take the *mean* as the estimator of the expected return, assuming that every p_j equals $\frac{1}{k}$; i.e., these estimators do not take into account the weights for some sub periods. Possible methods that can be used which improve the estimation of the input parameter refer, e.g., to the weights of sample data from different time points.

The first proposed approach now used is that the different recorded returns for different periods will be assigned by different weights, respectively. We let

$$\mu_{ik} \approx \sum_{j=0}^{k} r_{k-j} p_j,$$

where $\sum p_j = 1$, and take this μ_{ik} as a weighted return of asset i for the past return. Similarly we have the variance of the return:

$$\sigma_{ik}^2 \approx \sum_{j=0}^{k} p_j (r_{k-j} - \mu_{ik})^2.$$

Figure 4.10 is, e.g., derived from $q = 0.9$ with $p_j = q^j(1-q)$, $q = 0.9$ [1]. The "right" choice of q depends on the length of the time series; the nearer one comes

[1] In practice we can also assume that $q = 0.95$, or any or another value.

to the point k, the greater is the weight of an observation in the model.[2]

A second method to improve the estimation of the inputs here is named "time–window"; with this method we differentiate between the estimations with respect to the number of data points entering the estimation process. We now define a "time–window" of, in our case, two years, including 1 year of overlap length. Under this method we run a sensitivity analysis with historical data to find out how the efficient frontier moves. We now have a new portfolio efficient frontier every year. Except for the first estimated efficient frontier, every efficient frontier is computed taking into account a one–year overlap. In doing so, we can investigate the shifting of the portfolio frontier as it depends on the time–window.

4.4.2 Backtesting with In–Sample and Out–of–Sample Data

We will now perform backtesting procedures to investigate to what extend the two methods applied above are time–dependent. In the first one, we will consider the weights of all historical data; and in the second one we will focus on the length of the "time–window". For backtesting we need *in–sample* and *out–of–sample*[1] data and an investment period. We refer again to Table 4.2 and Table 4.3. which indicate the historical long–term average returns, and correlations.

Consider a sample of data $R_{i1}, R_{i2}, \ldots, R_{iT}$ drawn from N assets and T periods. The first k periods will be taken as in–sample data, and the remaining are consequently for the investment period. Here we take $T = 144$ months, and we assume (e.g.,) that $k = 72$. The *in–sample* data is thus from the first 6 years, and the investment period lasts 6 years, too.

In order to back test the second method we use the whole database. Each 2–year time window will be taken as in–sample data, and each next "time window" will be an investment period.

[2]Cf. the application in the next subsection.

[1]See, e.g., Inoue & L. Kilian (2002), Zumbach (2006) and Campbell (2006).

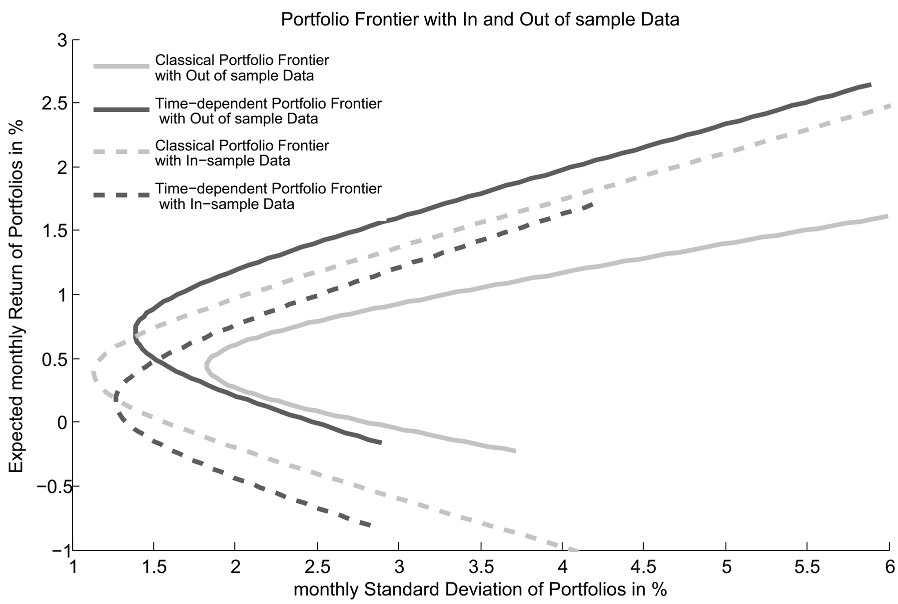

Figure 4.10: Efficient Frontier with In–Sample and Out–of–Sample Data

Figure 4.10 shows us the different efficient frontiers with in–sample and out–of–sample data. In this figure the dashed lines represent efficient frontiers from in–sample data, while the solid lines represent the efficient frontiers during the period of investment with out–of–sample data. The black lines are the time–dependent input estimations, and the grey ones are from classical Markowitz frontiers. The black solid line is the time–dependent frontier, and the grey solid line is the efficient frontier from classical Markowitz mean–variance. It is clear that the time–dependent efficient frontiers (black lines) have relatively similar variances (i.e., time–dependent frontiers can keep the risk of investment in a small range), although they have differences in expected return. The grey lines (i.e., those efficient frontiers deduced from equal–weighted mean) have a large fluctuation of risk. By this standard the time–dependent efficient frontiers are reasonable choices for practical use for most risk–averse investors.

Figures 4.11 and 4.12 provide nice smoothed pictures of the changed perceptions of the market from 2001 – 2007. A movement to the left and/or downwards corresponds to improved market perceptions and a movement of the efficiency

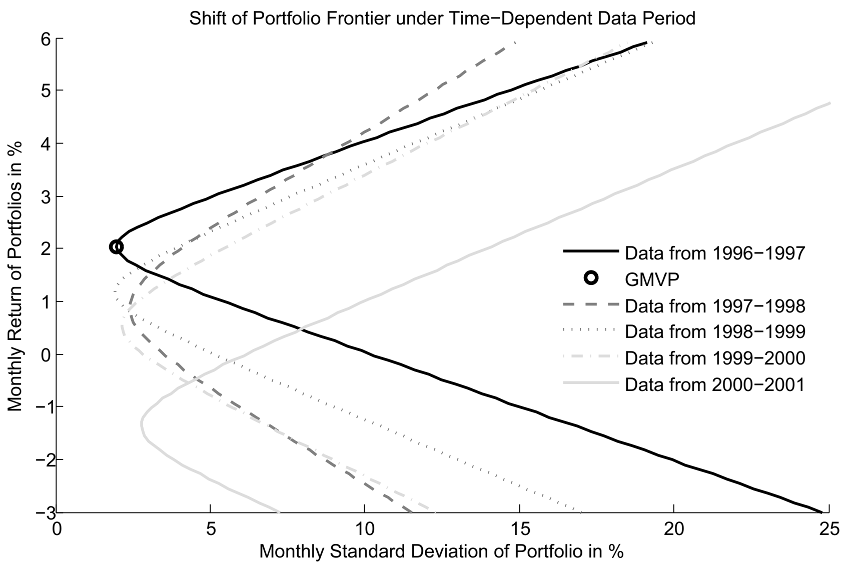

Figure 4.11: Shift of the Portfolio Frontier in the First Half Sample Period with Time Windows Equals 2 Years

curve to the right and/or upward indicates a worsening of the expected asset performances. From the figures we can see that the portfolio frontier of 2000 – 2001 is very far from the efficiency lines based on estimates from data of other sub periods. It clearly refers to the dotcom bubble with its climax on March 10, 2000 (when the NASDAQ index reached its peak of 5132.52 points), and the following burst. Figure 4.12 thus shows that the dotcom bubble popped, and that the market perceptions then started to improve beginning in 2003. From these two figures, then, we draw the conclusion that a time–window portfolio frontier can actually do best by reflecting changes of the investors perceptions on future performances.

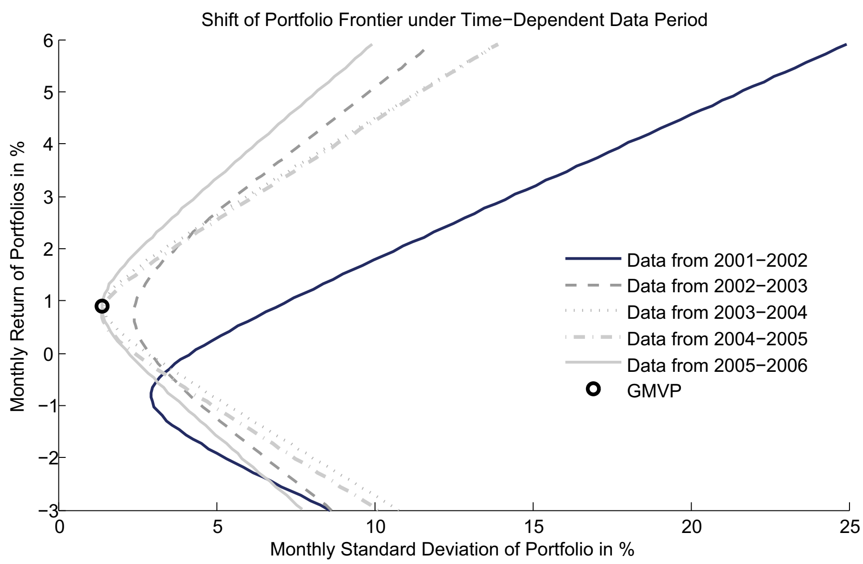

Figure 4.12: Shift of the Portfolio Frontier in the Last Half Sample Period with Time Windows Equals 2 Years

In order to apply this method pragmatically we run the sensitivity analysis of the 2–year time–window with non–negativity constraints. Figures 4.13 and 4.14 show the portfolio frontiers based on the same database, only without short sales.

Note here that the finding of *right* time windows or weights for the estimation procedures are not stable over time and that there can't be only true time–independent estimates; they have regularly to be checked and adjusted for practical applications.

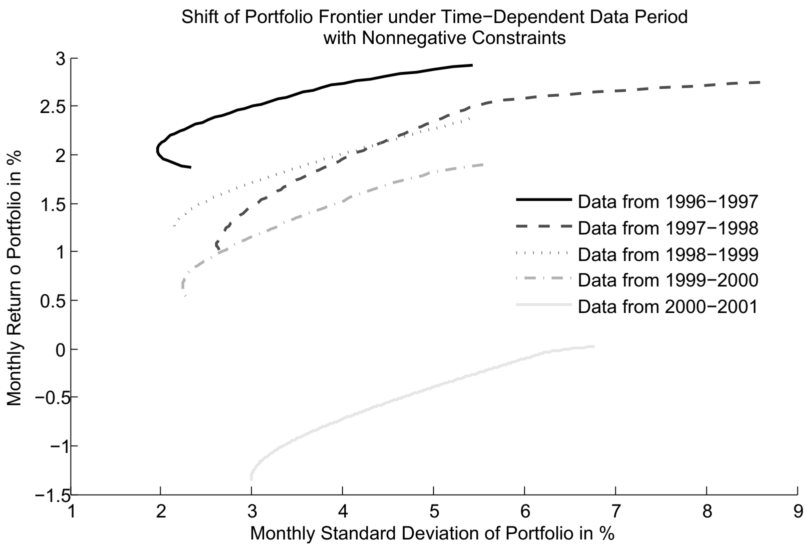

Figure 4.13: Shift of the Portfolio Frontier in the First Half Sample Period with Non–Negativity Constraints

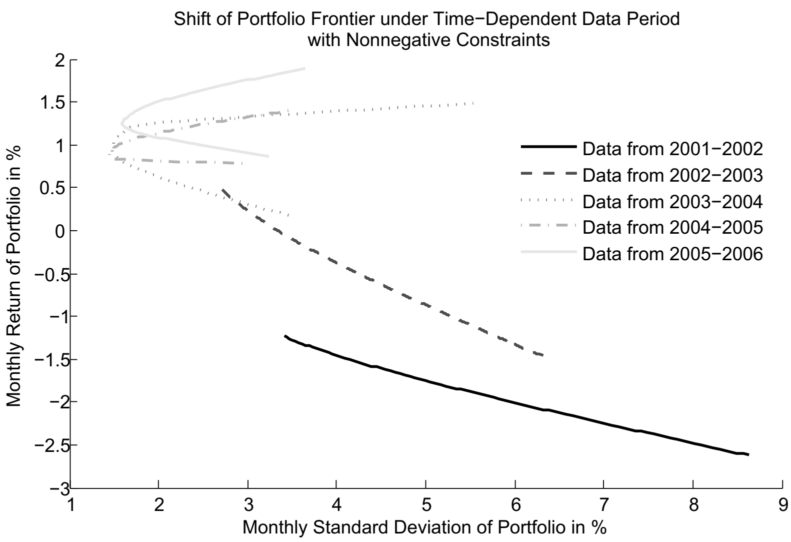

Figure 4.14: Shift of the Portfolio Frontier in the Last Half Sample Period with Non–Negativity Constraints

Chapter 5

Summary and Conclusions

The research executed in this thesis was motivated mainly by highly relevant determinants for any investment decision in the practice, namely an investor's attitude toward risk, the investor's initial wealth level and the investment's expected time horizon. While a lot of research on intertemporal capital models has been executed, neither these models nor the more commonly–used and simpler one–period capital market models or time series models deal simultaneously with these three determinants.

We have established here a classification leading to some "typical" investors with respect to their initial wealth, attitude toward risk and investment horizon, and linked some of them successfully to actual players in the markets. Special attention will be given in this work to High Net Worth Individuals (HNWIs), who are investors usually characterized by a high initial wealth level, a long investment horizon and a low aversion toward risk. From this classification we derived normative investment, or asset allocation strategies for investors with *low* and *high* scaled attitudes toward risk and levels of initial wealth, respectively. Investors were assumed to have the choice between investments in indices representing either four or seven asset classes as money market instruments: German bonds, German stock, American stock, German Real Estate, Hedge funds and Art. The third input, time, is investigated via "selecting" appropriate time windows for the estimation process producing the input for the portfolio optimization models. It is shown on behalf of historical data covering 10 indices over 12 years that time windows for estimating the models' inputs of 2 years are rather stable, and seem to thus be appropriate for HNWI portfolio optimization.

The theoretical frameworks with respect to portfolio optimization are the Markowitz theory and the Expected Utility theory. It is shown that Markowitz theory is appropriate for our purposes even though it is a one–period model that does not specify or restrict the length of the investment period. Markowitz theory deals only with the first two central statistical moments: expected return and variance. A normality test of the returns of the indices representing the asset classed mentioned earlier supported this assumption: or, in other words, this test supported the reasonable applicability of the Markowitz theory, with some restrictions. Thus, no higher moments are taken into account for the portfolio optimizations.

In the empirical part of this thesis we computed efficient frontiers for different sets of asset classes, and taking the initial wealth into account and accordingly using some utility functions, we then selected a "best one". Also we dealt alternatively with direct utility maximization. Here we have shown that quadratic utility seems to be powerful enough to model all possible real investors, and can thus appropriately substitute for log–wealth or power utility. If an investor is assumed to divide his or her fortune into a risk–free part and a risky part, direct utility maximization can be used to generate naively diversified portfolios up to a rather high threshold of the risky percentage of the overall investment sum. This threshold depends on the number of assets, their expected returns and their correlation structure. Further research on theses interrelations seems very promising.

In this empirical research we obtained the following main results:

1. The results of utility maximization for the optimization problem with quadratic utility with a risk parameter $beta$, and the log–wealth and power utility are very similar.

2. Direct maximization of the utility leads to practical and rational results; and for the log–wealth utility function we found that, to a certain proportion of

risky assets, the investor could simply hold a naively diversified portfolio.

Well–working models require good estimates for the input parameters, and we found that small input errors can lead to large differences in the optimized portfolio. From the empirical back–testing we conclude that:

1. Different sample–period data is very important to estimators. Time–dependent estimators work better than normal estimators, without consideration of the time period.

2. Time–dependent estimators can be obtained by working with "time–windows", an idea which basically corresponds to the concept of a sensitivity analysis, and also helps us to get a better understanding of the shifting of portfolio frontiers from different sample periods.

Long–term efficient portfolios can usefully be considered as a subset and special case of mean–variance efficiency. Although there exist multi–period models, a properly applied and extended classical Markowitz mean–variance model remains a very broadly applicable theoretical framework, even for investors, with a long–term investment horizons of about 5 or more years.

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Eidesstattliche Versicherung

Ich erkläre hiermit, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

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