Decidability of Order-Based Modal Logics

Inauguraldissertation
an der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Jonas Rogger

von Nebikon

Leiter der Arbeit
Prof. Dr. George Metcalfe
Mathematisches Institute, Universität Bern

Bern, 26. Juli 2016

Der Dekan:
Prof. Dr. Gilberto Colangelo

Acknowledgements

The journey towards this dissertation has been a fascinating and illuminating adventure. I have enjoyed so many incredible experiences, encountered almost too many demanding challenges, and met a great number of wonderful people. Many of these challenges I would not have been able to overcome without the generous help of others. I am thereby deeply indebted to everyone who accompanied me on this journey, but unfortunately I can only mention a few.

Let me first and foremost express my overwhelming gratitude towards George Metcalfe, who invited me on this adventure. It is no secret that without his wealth of knowledge and critical insights as well as his constant encouragement and patience, this thesis would never have been possible. Thank you for giving me the confidence and freedom needed not only to start, but also to survive and complete this adventurous journey. I cannot imagine a better supervisor.

Furthermore, I would like to thank my co-authors Xavier Caicedo and Ricardo Rodríguez for stimulating discussions and a very fruitful collaboration. Moreover, let me express my deep gratitude to Petr Cintula, Carles Noguera, and other members of the Czech Academy of Sciences, for giving me the opportunity to visit them in Prague for five months. The result was not only a fantastic collaboration, but also the extension of my knowledge on mathematical logic as well as Czech food, beer, and other culture. I would also like to thank my external referee Lluís Godo for his helpful comments and suggestions, and Gerhard Jäger for agreeing to be on my Phd-committee. Finally, let me acknowledge support by the Swiss National Science Foundation (SNF) grant 200021_146748.

My appreciation also goes to all the members of the Mathematical Institute as well as the Logic and Theory Group of the Institute of Computer Science at the University of Bern for providing me with the ideal environment for my scientific adventures, including interesting discussions, inspiring seminars, relaxing drinks after work, and the occasional barbecue by the river.

It goes without saying that this journey would not only have been dull and unfulfilling but blatantly impossible without the love, unconditional support, and continuous encouragement of my family and friends. Let me thank especially my parents for always being there and providing the necessary confidence as well as a proper meal when needed, a bed, or the appropriate shoes for a presentation. Moreover, my overwhelming gratitude belongs to Olivier, Helene, Ramon, Jody, Annina, Lory, and many other great friends for being there for me in the darker parts of this journey, lending an understanding ear or an encouraging embrace, as well as the bright parts, joining me for lunch, beer, coffee, dancing, dinner, barbecue, swimming, beach volleyball, sauna, football, hiking, cycling, surfing, camping, relaxing, and so many other activities that provided the necessary balance to complete this journey without losing sanity. Without all of you giving me the confidence that I sometimes lacked, I would not be where I am today.

Thank you!

Contents

Chapter 1

Introduction

Logical reasoning is something that humans and other animals often do intuitively. Given the information that Vladimir is taller than Nicolas and Nicolas is taller than Silvio, we naturally infer, using our knowledge that the relation "... is taller than ..." is transitive, that Vladimir is taller than Silvio.

Logic, as a field of study, concerns itself with inferences of this kind, but where the content is abstracted away and only the structure of the inference is considered. The abstract structure of the inference above is then as follows: if R is a transitive relation, a stands in relation R to b, and b stands in relation R to c, then a stands in relation R to c. We might even say that it has an even more abstract structure: if A, B, and from A and B follows C, then C. In this sense, logic can be seen as the study of the structure of valid inferences.

By studying logic, we aim to gain new insights into our way of reasoning, which in turn provides us with tools to advance our knowledge in many fields. In particular, logic has proved to be an especially powerful tool in sciences where there is no or very limited empirical input, such as philosophy and mathematics. Furthermore, by analysing our way of reasoning, capturing it formally, and automating it, logic has become a vital tool in the quest to advance technology further and push back the boundaries of artificial intelligence.

Unless stated otherwise, we will talk about *propositional logics* here, that is, logics based on a language where formulas are built inductively over a set of *propositional variables* p, q, r, ... by *propositional connectives*, usually including the binary connectives $\vee$ (weak disjunction), $\wedge$ (weak conjunction), & (strong conjunction; coincides with weak conjunction in classical logic), and $\to$ (implication) and the unary $\neg$ (negation). Furthermore, in this dissertation, we will use the terms "inference" and "logic" in a rather specific sense. An *inference* is a pair $\langle \Gamma, \varphi \rangle$, where Γ is a set of formulas, called the set of *premises*, and φ is a single formula, called the *conclusion*. A *logic* L is a consequence relation $\Vdash_{\mathsf{L}}$ relating sets of formulas and single formulas, that is, it is a set of inferences satisfying certain conditions. In this sense, an inference $\langle \Gamma, \varphi \rangle$ is called *valid in* L if

$\langle \Gamma, \varphi \rangle \in \mathsf{L}$, often written in infix notation: $\Gamma \Vdash_{\mathsf{L}} \varphi$. A formula φ is called a *theorem of* L if it is the conclusion of a valid inference without premises, i.e. if $\emptyset \Vdash_{\mathsf{L}} \varphi$.

The two most common approaches to specify a logic in this sense are by defining a semantics or an axiom system. The semantic approach usually specifies a class of models, such that an inference from Γ to φ is valid if φ is "true" in every model where all members of Γ are "true". The axiomatic approach specifies a set of axioms and deduction rules such that an inference from Γ to φ is valid if φ is "provable" using only those axioms, deduction rules, and members of Γ. These approaches to defining consequence relations reflect truth-oriented and deduction-oriented perspectives on logic.

Classical logic is the name we use nowadays for the logic developed around the turn of the twentieth century by Gottlob Frege, Betrand Russell, and others. One reason to call this logic classical is that it has roots reaching all the way back to ancient Greece and Aristotle, but perhaps a more important reason is that classical logic is very much the standard logic. One reason for this is that classical logic may be viewed as the strongest non-trivial propositional logic. *Non-classical logics*, are usually either weaker than classical logic (e.g. many-valued logics) or they make use of an expanded language (e.g. modal logics)

Let us to point out two aspects of classical logic from which the non-classical logics studied in this dissertation depart. Firstly, classical logic obeys the principle of *bivalence*, that is, a formula is either true or it is false, there is no middle ground. This is why classical logic is seen as a two-valued logic. Secondly, all propositional connectives of classical logic, such as negation $\neg$, conjunction $\wedge$, disjunction $\vee$, and implication $\to$, are *truth-functional*, i.e. the truth value of a compound formula is determined entirely by the truth values of its subformulas.

1.1 Many-Valued Logics

Many-valued logics depart from classical logic in the sense that they abandon the principle of bivalence. Characteristically, in many-valued logics, a formula is not always true or false, but might have one of at least three different truth values. The first to propose a many-valued logic was Jan Łukasiewicz in [89, 90], namely the three-valued logic $\mathsf{Ł}_3$ where the third truth value is called "possible". The name of the third value was inspired by Łukasiewicz's goal to model future contingents with this logic. Today, most logicians agree that this goal was not achieved, as future contingents and other modalities are widely held not to obey truth-functionality. Nevertheless, Łukasiewicz initiated in this way the still very much active field of many-valued logics.

In fact, Łukasiewicz considered many-valued logics with arbitrary numbers of truth-values, including the n-valued Łukasiewicz logics $\mathsf{Ł}_n$ and the infinite-valued $\mathsf{Ł}$, which he developed in [91]. Many other many-valued logics followed, famously including the three-valued *Kleene logic* K_3, where the third truth value is usually dubbed “unknown” or “undefined”. This logic was originally designed by Stephen Kleene in [82] to model partial functions and relations in mathematics, but later became popular in philosophy as a tool to determine fixed points in the revision theory of truth, a theory of truth introduced by Saul Kripke in [83]. A popular four-valued logic is *first-degree entailment logic* FDE, where there are two additional truth values: “neither true nor false” and “both true and false”. FDE is also known as *Dunn-Belnap logic*, as its usefulness was emphasised by Nuel Belnap in the context of information bases stored on a computer in [12] and by J. Michael Dunn in the context of relevant entailment in [51]. FDE is philosophically very interesting as it is both a *paraconsistent logic*, that is, a logic where contradictory formulas do not necessarily imply every other formula (cf. e.g. [106]), and a *relevance logic*, i.e. if an implication-formula is a theorem of FDE, its antecedent is “relevant” in a specific sense to the consequence (see e.g. [52]). During his investigation of *intuitionistic logic* (see e.g. [45]), Kurt Gödel came up in [62] with a whole family of many-valued logics, one for each natural number $n \geq 2$ of truth values. These logics are now called *n-valued Gödel logics*. Gödel used this logics to simultaneously prove that intuitionistic logic is not a finite-valued logic and that there are infinitely many so-called *intermediate logics*, i.e. logics intermediate in strength between intuitionistic logic and classical logic. Gödel’s approach was later generalized by Michael Dummett in [50], where he introduced the infinite-valued Gödel logic G (therefore also frequently called *Gödel-Dummett logic*) as the intersection of all finite-valued Gödel logics (seen as sets of valid formulas) and provided an axiomatization.

Logics are often designed with a certain *intended semantics* in mind, which is often an algebra or a class of algebras. For example, classical logic has as intended semantics the two-element Boolean algebra. In this sense, many-valued logics have as intended semantics an algebra (or a class of algebras) with three or more elements. E.g. $\mathsf{Ł}_3$ has as intended semantics the three-element chain and FDE the four-element diamond lattice. In particular, specific (classes of) residuated lattices (i.e. lattices with an additional monoidal operation that has unique left and right adjoints) are often chosen as intended semantics, as their monoidal operations provide suitable interpretations of strong conjunction and their residua (adjoints) are adequate interpretations of implication.

The term *fuzzy logics* is typically used to denote many-valued logics where truth values are to be understood as degrees of truth, e.g. when the intended semantics is based on the real unit interval $[0, 1]$. Unfortunately, as logicians do not always agree on what is the characteristic property of fuzzy logics, it is difficult to be more precise in their delimitation. Nevertheless, it is at least widely accepted that if a logic has as intended semantics a (class of) residuated lattice(s) with the real unit interval $[0, 1]$ as universe (often called *standard semantics*), then this is a fuzzy logic.

The development of fuzzy logics is motivated from many different perspectives, perhaps the most prominent originating from the context of *vagueness*. Natural languages contain vague concepts, i.e. concepts that lack clear boundaries in their denotation, such as “young”, “bald”, “green”, or “drunk”. It seems obvious that there are clear cases of people being either drunk or not drunk, but we might also have experienced cases where it was not clear if someone was drunk or not, we might in this case have referred to the person as slightly drunk or quite drunk. Such a classification becomes especially important when it comes to traffic laws, for example. One might argue that such imprecision is a necessary aspect of languages used in communication about reality, made unavoidable by the nature of reality itself and our sensory and cognitive abilities. Be this as it may, the undeniable existence of vague concepts poses challenges to the logical analysis of natural languages, for example when *Sorites paradoxes* are considered.

While classical logic seems unsuitable for dealing with vagueness, it has been proposed that assigning degrees of truth to propositions containing vague concepts, as we would do to formulas in fuzzy logics, can lead to a better understanding of the nature of vagueness in natural languages and help us overcome the challenges posed by it (see e.g. [117]). However, it is more widely held in the philosophical community today that fuzzy logics, for a variety of reasons, are inadequate tools to reach such an understanding of the nature of vagueness or to deal with Sorites paradoxes (see e.g. [122], [81], or [116]). This is not to say, however, that fuzzy and other many-valued logics are not very suitable tools for dealing with vagueness and imprecision in applications in fields other than philosophy.

With the development of *fuzzy set theory*, proposed by Lofti Zadeh in [123], a whole new research area evolved for dealing with imprecision, uncertainty, and gradual change in engineering and computer science. Many technological applications and tools have emerged from this line of research, including fuzzy controls (which e.g. fully automatically regulate the speed of the subway trains in Sendai, Japan), fuzzy image processing, methods in soft-computing, and applications in artificial intelligence. This application driven line of research is often denoted as *fuzzy logics in the broad sense* in order to distinguish it from the connected but more mathematically oriented field we will call *fuzzy logics in the narrow sense*, i.e. the mathematical investigation of logics based on residuated lattices with universe $[0,1]$.

Fuzzy logics in the narrow sense, or *mathematical fuzzy logics* (in short: MFL), emerged from the effort to provide solid mathematical foundations for the study of fuzzy logics in the broad sense. However, the study of MFL quickly evolved into a field of great interest in its own right with a very active research community, making it an important and well-studied subfield of mathematical logic. The state of the art of MFL can be found in the recent handbook series consisting of [39], [40], and [38].

Perhaps the most important monograph in the field of MFL was Petr Hájek's [67], where he laid the foundation for a systematic mathematical investigation of these logic. In this work, Hájek defines semantics for many different fuzzy logics by considering residuated lattices on the real unit interval $[0,1]$ where the monoidal operation (interpreting strong conjunction &) is a *continuous t-norm*, i.e. a continuous binary operation on $[0,1]$ that is commutative, associative, monotone, and has unit element 1. Fuzzy logics based on continuous t-norms include the infinite-valued *Łukasiewicz logic* Ł [91], *Gödel logic* G [50], and *product logic* P [72], as well as Hájek's *basic logic* BL [67], of which the three former logics are axiomatic extensions. While the intended semantics of BL is the class of all residuated lattices where the monoidal operation is a continuous t-norm, Ł, P, and G each has exactly one of these algebras as intended semantics.

Francesc Esteva and Lluís Godo generalized Hájek's approach by introducing *monoidal t-norm logic* MTL in [54], which has as intended semantics the class of all residuated lattices where the monoidal operation is a *left-continuous* t-norms, i.e. a t-norm that is only continuous with respect to suprema but not necessarily infima. A further generalization was presented by George Metcalfe and Franco Montagna in [94], where *uninorm logic* UL was introduced and proved to be sound and complete with respect to the class of all residuated lattices where the monoidal operation is a *residuated uninorm*, i.e. a binary operation on $[0,1]$ that is commutative, associative, monotone, has a unit element $\mathrm{e} \in [0,1]$ and a right adjoint. Note that a residuated uninorm with $e = 1$ is in fact a left-continuous t-norm.

While Ł and P are most regularly used for applications where the notion of *magnitude* is important, as their t-norms depend on how far the two arguments are apart, G is usually employed if the important notion is the *order* of values. This is because the Gödel t-norm depends only on the relative order of its arguments. With this in mind, we introduce the more general class of order-based logics in Chapter 3, of which G is a member.

For BL, Ł, P, and G, the validity problem is known to be coNP-complete (see e.g.[74] for an overview). While coNP-hardness is immediate, as classical propositional logic (for which the validity problem is also coNP-complete) can be interpreted in those logics, inclusion in coNP can be quite complicated to show. For MTL, upper complexity bounds are still unknown for the validity problem, while it is known that it is decidable (see e.g. [74]). Furthermore, decidability of UL is still open.

1.2 Modal Logics

Modal logics expand classical logic by adding non-truth-functional connectives, thus departing from the truth-functionality property of classical logic. Typically, two unary non-truth-functional connectives (often called *modal connectives* or *modal operators*) $\Box$

and $\Diamond$ are added to the language of propositional classical logic. Historically, in philosophy, the aims of expanding classical logic in this way was to logically model the notions of necessity ($\Box\varphi$ is interpreted as "φ is necessary") and possibility ($\Diamond\varphi$ is interpreted as "φ is possible"; see e.g. [60]). It seems clear that these notions cannot be appropriately modelled truth-functionally: while the sentences "the number 27 is smaller than the number 141'107" and "Bern had 141'107 citizen in April 2016" are both true, the sentence "it is necessary that the number 27 is smaller than the number 141'107" is true and "it is necessary that Bern had 141'107 citizen in April 2016 " is false.

Even though modal logics have a long tradition in philosophy, dating back far into past centuries, it was only with the development of *relational semantics* in the 1950s and 1960s by Saul Kripke and others (see [80, 84–87]), that the study of formal modal logics really took off. What is now called a *Kripke frame* is a relational structure comprised of a set of so-called *worlds* and a binary *accessibility relation* on this set of worlds. A *Kripke model* is a Kripke frame together with a *valuation function* assigning to each propositional variable a (classical) truth value at each world. On the one hand, formulas that do not contain modal connectives are then, locally at each world, assigned truth values truth-functionally as in classical logic. On the other hand, a formula $\Box\varphi$ is true at a world x if and only if φ is true at all worlds that are connected to x by the accessibility relation and a formula $\Diamond\varphi$ is true at a world x if and only if φ is true at some world that is connected to x. At this point, it becomes clear that $\Diamond$ can be defined by $\neg\Box\neg$. A formula is called *valid* in a Kripke model if it is true at all worlds of the model, and it is called *valid* in a class of Kripke models, if it is valid in all Kripke models in the class. An inference is called *(globally) valid* in a class of Kripke models if the conclusion is valid in all Kripke models of that class where all the premises are valid.

In fact, this only describes a specific kind of modal language and Kripke model, where only two unary modal connectives are added to the language and where both of them are interpreted by the same accessibility relation. It is straightforward to generalize to any number of modal connectives of any finite arity: we have an $(n+1)$-ary relation in the Kripke frame for each n-ary modal connective in the language. Modal logics including more modal connectives than just $\Box$ and $\Diamond$ are often called *multi-modal logics*.

Besides necessity and possibility, many other interesting notions have been modelled by modal logics. Among the possible readings of the $\Box$- and $\Diamond$-connectives are the following:

- Alethic Reading: $\Box\varphi$ stands for "φ is necessary" and $\Diamond\varphi$ stands for "φ is possible"
- Epistemic Reading: $\Box\varphi$ stands for "φ is known" and $\Diamond\varphi$ stands for "φ is consistent with the available information" (see e.g. [49])
- Temporal Reading: $\Box\varphi$ stands for "φ will always be true" and $\Diamond\varphi$ stands for "φ

will be true at some point in the future" (see e.g. [120])

- Deontic Reading: $\Box\varphi$ stands for "φ is obligatory" and $\Diamond\varphi$ stands for "φ is permitted" (see e.g. [93])
- Provability Reading: $\Box\varphi$ stands for "φ is provable" and $\Diamond\varphi$ stands for "φ is consistent" (see e.g. [20])

Understanding $\Box$ and $\Diamond$ in different ways might have implications for which formulas and inferences we would like to be valid in a modal logic. For example, if we adopt an epistemic reading, we will probably want that the formula $\Box\varphi \to \varphi$ is valid, as something that is known should also be true. On the other hand, the same formula should probably not be valid if we adopt a deontic reading. It would seem strange that everything that is obligatory is automatically also the case.

While the formula $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and the inference from φ to $\Box\varphi$ are valid in all Kripke models, the formula $\Box\varphi \to \varphi$ is only valid in Kripke models where the accessibility relation is reflexive. Furthermore, $\Box\varphi \to \Box\Box\varphi$ or $\varphi \to \Box\Diamond\varphi$ is valid in all Kripke models where the accessibility relation is transitive or symmetric, respectively. In fact, this natural correspondence between formulas and properties of Kripke models is one of the most important features of these relational semantics. The modal logic determined by the class of all Kripke models is called K, while T denotes the logic determined by the class of all Kripke models with a reflexive accessibility relation and $\mathsf{S4}$ or $\mathsf{S5}$ denotes the logic determined by the class of all Kripke models where the accessibility relation is either reflexive and transitive or an equivalence relation, respectively.

Even though Kripke models are very nice in many respects, they are not flexible enough to provide semantics for logics weaker than K. When we adopt an epistemic reading for example, we might not want the inference from φ to $\Box\varphi$ (which is valid in K) to be valid, as its validity would imply that all propositional tautologies are known, presupposing a logically omniscient knower. For this reason, other semantics have been developed for modal logics. Among the most popular of these are the so-called *neighbourhood semantics*, introduced independently by Richard Montague in [97] and Dana Scott in [115]. In this semantics, as in Kripke models, formulas not containing modal connectives are interpreted locally at worlds (the set of worlds denoted by W) as in classical logic, but the modal connectives are not interpreted via an accessibility relation on W, but rather via a function assigning to each world x a set of subsets of worlds $N(x) \subseteq \mathscr{P}(W)$, called the *neighbourhood* of x. A formula $\Box\varphi$ is true at a world x if and only if the set of worlds where φ is true is a member of $N(x)$. The modal logic determined by the class of all neighbourhood models is often called E and is considered to be the weakest sensible modal logic that expands classical logic, as the only thing we really know about $\Box$ in this setting is that the inference from $\varphi \leftrightarrow \psi$ to $\Box\varphi \leftrightarrow \Box\psi$ is valid in E.

Even though modal logics based on Kripke-style semantics can also be captured by the neighbourhood approach, it is still the former approach that is most popular. This is probably because of the natural correspondence between properties of Kripke frames and certain formulas and because most modal logics based on Kripke models enjoy the so-called *tree-model property*, i.e. a formula is valid in a certain class of Kripke models if and only if it is valid in all Kripke models of that class that have a tree structure.

With this in mind, it is understandable that even though relational semantics were developed with the aim of obtaining a better understanding of modal logics, today the view is often held in the mathematical logics community that conversely modal logics provide a better understanding of relational structures such as partially ordered sets, directed and undirected graphs, and tree structures (see e.g. [15]).

In computer science, modal logics have grown increasingly popular since the 1970s. They are applied for many purposes, including the verification of programs [44], databases [30], artificial intelligence [23], and knowledge representation, using so-called description logics (see e.g. [4]). *Description logics*, understood as multi-modal logics, take the set of worlds as the domain of discourse (worlds might then be objects, or people, or individual hairs on Mary's head) and the accessibility relations are taken to be binary relations on the domain (in this setting usually called *roles*) such as "... is the mother of ..." or "... is longer than ...". Formulas are taken to represent properties (in this setting usually called *concepts*) such as "... is a table", "... smells nice", or "if ... is a table then it smells nice". If a formula is true at a particular world x, this is taken to mean that the object x in the domain satisfies the property represented by the formula.

A large part of the popularity of modal logics in computer science is based on the fact that modal logics present an ideal compromise between expressive power and computational complexity. As classical propositional logic is a fragment of any propositional modal logic and any propositional modal logic can in turn be seen as a fragment of first-order logic (see e.g. [15]), propositional modal logics lie in between these two logics concerning expressibility and complexity. While first-order logic is undecidable, very many propositional modal logics are decidable, e.g. the validity problems for K, T, and $\mathsf{S4}$ are PSPACE-complete, and for $\mathsf{S5}$, it is coNP-complete (see [88]). This robustness in decidability is mostly due to the tree-model property (cf. [119]).

1.3 Many-Valued Modal Logics

Given the popularity of the two approaches towards non-classical logics presented above, it makes sense to combine them in order to obtain logics that enjoy the positive aspects of both approaches. In this spirit, *many-valued modal logics* abandon the principle of bivalence and at the same time add non-truth-functional connectives such as $\Box$ and $\Diamond$ to the language. The most favoured way in the literature to define such logics, stemming from the popularity of Kripke-style relational semantics, is by considering a many-valued logic based on a residuated lattice $\mathbf{A}$ and generalizing *Kripke frames* to consist of a set W of worlds and a *many-valued accessibility relation* R on W, that is, a function $R\colon W \times W \to A$ (A denoting the universe of $\mathbf{A}$), sometimes just called an accessibility relation. We also generalize *Kripke models* to consist of a Kripke frame together with a *many-valued valuation function* that assigns an element of A to each propositional variable at each world. Propositional connectives are then interpreted, locally at each world, truth-functionally by the operations of $\mathbf{A}$ and $\Box\varphi$ and $\Diamond\varphi$ are interpreted at a world x as infimum and supremum, respectively, of values of φ at worlds that are accessible from x to some degree. When a many-valued accessibility relation only takes values in $\{\bot^{\mathbf{A}}, \top^{\mathbf{A}}\} \subseteq A$, we call it *crisp*. We sometimes also call a Kripke frame or model many-valued or crisp if its accessibility relation is many-valued or crisp, respectively.

Among the first to consider many-valued modal logics in this way was Melvin Fitting in [57, 58], studying many-valued modal logics based on finite Heyting algebras. Numerous many-valued modal logics were subsequently developed to cater to the needs of applications where non-truth-functional notions such as knowledge, belief, tense, spatio-temporal relations, and program termination are to be modelled in the presence of vagueness, imprecision, or uncertainty. Such applications include modelling fuzzy belief [63, 66], spatial reasoning with with vague predicates [114], many-valued tense logic [47], fuzzy similarity measures [64], and substantial work on *many-valued description logics*, which are, understood as many-valued multi-modal logics, based on Kripke frames with arbitrarily many binary many-valued accessibility relations (see e.g. [18] for an overview).

With this growing landscape of logics designed for various applications, a more systematic investigation of many-valued modal logics was called for. At the heart of this investigation lies the article [21] by Felix Bou, Francesc Esteva, Lluís Godo, and Ricardo Rodríguez, where the box-fragments of many-valued modal logics are studied based on Kripke models over finite residuated lattices. While most earlier approaches considered crisp $\mathsf{S5}$-like modalities, that is, based on the class of crisp Kripke frames where the accessibility relation is an equivalence relation, [21] was one of the first to consider (non-crisp) K-like modalities, i.e. where the class of all Kripke frames is considered. Furthermore, after Hájek had considered many-valued crisp $\mathsf{S5}$ logics based on continuous t-norms in [67], many authors did the same for many-valued modal logics with K-like modalities. Xavier Caicedo and Ricardo Rodríguez axiomatize Gödel logic G expanded with K-,T-, $\mathsf{S4}$-, and $\mathsf{S5}$-like modalities (and others) in [28, 29] based on Kripke frames with a many-valued accessibility relation. In [95], George Metcalfe and Nicola Olivetti use analytic

Gentzen-style calculi to establish the decidability, indeed PSPACE-completeness, of the validity problem for the box- and diamond-fragments of Gödel modal logics based on Kripke frames with a crisp or a many-valued accessibility relation. Moreover, using tools from abstract algebraic logic, Georges Hansoul and Bruno Teheux consider finite- and infinite-valued Łukasiewicz modal logics based on the class of all crisp Kripke models in [75] and provide axiomatizations in the finite-valued cases and an infinitary axiomatization (including an infinitary deduction rule) for the infinite-valued case. Furthermore, in [121], Amanda Vidal, Francesc Esteva, and Lluís Godo consider infinite-valued product modal logics (expanded with rational constants and the Delta operator) based on different classes of crisp Kripke frames, providing infinitary axiomatizations and algebraizability results.

In [68], Hájek considers fuzzy description logics based on continuous t-norms and establishes decidability of the validity and satisfiability problem for these logics with respect to all witnessed interpretations, i.e. interpretations where each supremum or infimum is actually a maximum or minimum, respectively. As fuzzy modal logics can be understood as fragments of fuzzy description logics, these results imply the decidability of the validity and satisfiability problems for the fuzzy modal logics based on continuous t-norms determined by the class of all witnessed Kripke models, immediately implying the same for these logics determined by finite Kripke models (where the set of worlds is finite) and for many-valued modal logics based on a finite subalgebra of the standard semantics.

Furthermore, as Łukasiewicz modal logic (based on the class of all Kripke models) is complete with respect to the class of all witnessed Kripke models (see [68]), the decidability of the validity and satisfiability problem follows. Unfortunately, the same is not true for product modal logics and Gödel modal logics. However, for product modal logic based on all Kripke models with a many-valued accessibility relation, decidability of the validity and positive satisfiability problem was established in [31]. This was achieved by studying product description logic and reducing it to propositional product logic in exponential time. Moreover, Hájek was able to prove in [67] that the validity problem for the crisp $\mathsf{S5}$ version of Łukasiewicz modal logic is decidable using its correspondence to the one-variable fragment of first-order Łukasiewicz logic.

Interestingly, many-valued modal logics based on neighbourhood semantics, rather than Kripke's relational semantics, have received only very limited attention. After all, when considering non-truth-functional notions such as knowledge, belief, or high probability (see e.g. [103]) in the presence of imprecision or vagueness, one might not want to accept formulas and inferences valid in all many-valued Kripke models, such as e.g. the formula $(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$ or the inference from φ to $\Box\varphi$.

Given a residuated lattice $\mathbf{A}$, a *many-valued neighbourhood frame* consists of a set of worlds W and a *many-valued neighbourhood function* N, assigning to each world x a function $N(x)\colon A^W \to A$. A *many-valued neighbourhood model* consists of a many-valued neighbourhood frame and a valuation function V assigning values $V(p,x) \in A$ to each propositional variable p at each world x. This valuation function is extended to formulas inductively by interpreting the propositional connectives by the operations of $\mathbf{A}$ and by assigning to $\Box\varphi$, at a world x, the truth value $N(x)(V(\psi,\ldots)) \in A$, where $V(\psi,\ldots) \in A^W$ is understood as the function that maps each world y to $V(\varphi,y)$. Put in terms of fuzzy sets, the truth value of $\Box\varphi$ at a world x is the value of how much the *fuzzy truth-set of* φ (i.e. the fuzzy set to which a world y belongs to the degree $V(\varphi,y)$) belongs to the fuzzy set of fuzzy sets $N(x)$. In this sense, the definition of a many-valued neighbourhood model corresponds closely to its definition in the classical setting, except that the valuation function is now many-valued and sets are now fuzzy sets.

Rodríguez and Godo studied many-valued modal logics under neighbourhood semantics in [110, 111] with the intention of modelling non-truth-functional notions such as uncertainty and belief in a many-valued setting. Among many other results, they proved a correspondence between many-valued Kripke models and certain kinds of many-valued neighbourhood models and provided axiomatic systems for which they proved weak completeness with respect to certain classes of neighbourhood models.

Despite all of these results, there still remain many open question in the field of many-valued modal logics based on relational semantics or neighbourhood semantics. The following is an incomplete list of some of these questions that is inspired by the exposition so far:

(1) Is there an elegant finitary Hilbert-style axiomatization of the crisp counterparts of the Gödel modal logics studied in [29]?

(2) Are the Gödel modal logics with both $\Box$ and $\Diamond$ decidable? If yes, which complexity bounds can be obtained? (cf. [28, 29, 95])

(3) Are there suitable analytic proof systems for Gödel modal logics with both $\Box$ and $\Diamond$? (cf. [95])

(4) Can we find a broader framework to cover logics based on order and in this way generalize results obtained for Gödel (modal) logics?

(5) Are there elegant finitary axiomatizations of infinite-valued Łukasiewicz modal logics? (cf. [48, 75])

(6) Are there (finitary) axiomatizations of the product modal logics studied in [121] based on Kripke frames with a many-valued accessibility relation? Can these results also be obtained without adding rational constants and $\triangle$?

(7) Is the product or Gödel modal logic based on crisp Kripke frames where the accessibility relation is an equivalence relation decidable? Equivalently, are the one-variable fragments of their first-order counterparts decidable? If yes, which complexity bounds can be obtained? (cf. [67, Problem (13)])

(8) Are there other fragments of the first-order Łukasiewicz, product, or Gödel logic that are decidable?

(9) Is there a suitable correspondence between many-valued Kripke frames and certain kinds of many-valued neighbourhood frames (as opposed to the correspondence on the level of models)? (cf. [110])

(10) Are the axiomatizations of the many-valued modal logics based on many-valued neighbourhood models presented in [110] also strongly complete in certain cases? Can this be proved by algebraic methods?

(11) Are the many-valued modal logics based on many-valued neighbourhood models decidable?

1.4 Aims and Outline of the Dissertation

This dissertation mainly aims to extend the existing knowledge on decidability issues of many-valued modal logics based on Kripke models. In particular, it answers positively the open questions (2) and (7) for the case of Gödel logic. In fact, the results obtained are not restricted to Gödel modal logics, but they cover the larger class of order-based modal logics, of which Gödel modal logics are examples. In this way, we also answer question (4) positively. Moreover, question (3) is positively answered by providing tableau calculi for the Gödel modal logics GK, GK^{c}, and $\mathsf{GS5}^{\mathsf{c}}$. Furthermore, considering many-valued modal logics over MTL-chains based on many-valued neighbourhood models, we answer positively the questions (9) and (10) above for the box-fragments of these logics. Let us give a few more details below.

Decidability and complexity results are proved for modal expansions with $\Box$ and $\Diamond$ of so-called *order-based logics*, i.e. many-valued logics based on a sublattice of $\langle [0,1], \min, \max, 0, 1\rangle$ with additional operations defined based only on the order. More precisely, PSPACE-completeness is shown for the validity problem for order-based modal logics based on the classes of all Kripke frames with a crisp or many-valued accessibility relation, in cases where the underlying sublattice satisfies certain homogeneity properties. This broad class of logics includes the Gödel modal logics GK and GK^{c} (where for the latter only crisp accessibility relations are considered) and many-valued modal logics based on certain subalgebras of the standard Gödel algebra, e.g. subalgebras with the universe

$$G_{\downarrow} \;=\; \{0\} \cup \{\tfrac{1}{n+1} \mid n \in \mathbb{N}\} \qquad \text{or} \qquad G_{\uparrow} \;=\; \{1 - \tfrac{1}{n+1} \mid n \in \mathbb{N}\} \cup \{1\}\,.$$

Furthermore, if only many-valued Kripke frames are considered where the accessibility relations is a crisp equivalence relation, the complexity is reduced to coNP-completeness. This means e.g. that the validity problem for the Gödel modal logic $\mathsf{GS5^c}$ is coNP-complete, which in turn entails coNP-completeness for the validity problem for the one-variable fragment of first-order Gödel logic, solving a long standing open problem first explicitly formulated by Hájek in [67, Problem 13]. These results are based on joint work of the author of this dissertation with Xavier Caicedo, George Metcalfe, and Ricardo Rodríguez in [26, 27].

Based on independent work by the author of this thesis, we also provide tableau calculi for the sample cases GK, $\mathsf{GK^c}$, and $\mathsf{GS5^c}$. They could be extended to cover other order-based modal logics without much difficulty. We note that these tableau calculi do not provide algorithms of optimal complexity, but they do provide decision procedures that are easy to handle and might be implemented.

Moreover, we extend Rodríguez and Godo's results from [110] on many-valued neighbourhood semantics by providing a correspondence between many-valued Kripke frames and certain kinds of many-valued neighbourhood frames (while in [110], this correspondence was on the level of models) and providing axiomatizations that are proved by algebraic methods to be (finitely) strongly complete. Thus, we can answer three of the open problems posed in [110]. This is joint work of the author of this dissertation with Petr Cintula and Carles Noguera in [43].

Chapter 2 will set the stage by introducing the many-valued logics considered in this work as well as some closely related logics, giving a general framework of substructural logics in which they embedded. Moreover, many-valued modal logics are defined and a short survey on what is known about these logics concerning axiomatization, decidability, and complexity is provided.

In Chapter 3, order-based modal logics are defined and some crucial properties of these logics are established. This sets up Chapter 4, where an alternative semantics is defined for these order-based modal logics, based on "Kripke-like" models where the values of box- and diamond-formulas are restricted to certain subsets of the algebra. As one of the main contributions of this dissertation, it is shown that the new semantics determines the same valid formulas as the usual Kripke models, and by relying on the (bounded) finite model property with respect to this new semantics, decidability, indeed PSPACE-completeness, is proved for the validity problem for order-based modal logics based on all Kripke frames with a crisp and many-valued accessibility relation, in cases where the underlying sublattice satisfies some local homogeneity properties. This implies PSPACE-completeness of the Gödel modal logics GK and $\mathsf{GK^c}$.

A similar procedure to that followed in Chapter 4 is used in Chapter 5 to show coNP-completeness of the validity problem for order-based modal logics based on Kripke frames where the accessibility relation is a crisp equivalence relation. The coNP-completeness of the validity problem for the Gödel modal logic $\mathsf{GS5^c}$ follows and thus, by a standard translation, also the validity problem of the one-variable fragment of first-order Gödel logic is coNP-complete.

In Chapter 6, we present tableau calculi for the logics GK and $\mathsf{GS5^c}$. These calculi do not provide algorithms of optimal complexity (as opposed to the algorithms in the Chapters 4 and 5), but they provide decision procedures that are easy to handle and might be readily implemented.

In Chapter 7, neighbourhood semantics are studied for the box-fragments of many-valued modal logics over MTL-chains. A correspondence between many-valued Kripke frames and particular kinds of many-valued neighbourhood frames is presented and (finite) strong completeness is proved for certain axiomatizations using algebraic methods.

Concluding remarks can be found in Chapter 8, where we will mention ongoing work that was not ready to be included in this thesis, as well as suggestions for further work in the fascinating field of many-valued modal logics.

Chapter 2

Many-Valued Modal Logics

In this chapter, we set the stage for the new results presented in this dissertation. This includes embedding the relevant many-valued logics, which will later be studied under the expansion by modal operators, in a broader landscape of substructural logics.

In Section 2.1, we present a general framework of substructural logics, based on the one hand on axiomatic extensions of a basic Hilbert-style calculus, and on the other hand on varieties of certain algebras. This section also provides the opportunity to fix certain notations and conventions needed later in the thesis. In Section 2.2, we briefly discuss the most relevant many-valued logics for this dissertation, namely *monoidal t-norm logic* MTL and some of its most prominent axiomatic extensions, specifically, Petr Hájek's *basic logic* BL as well as *Łukasiewicz logic* $\mathsf{Ł}$, *product logic* P, and *Gödel logic* G. Their expansions with modal operators are defined in Section 2.3, along with a short survey of previously known results about these many-valued modal logics.

2.1 Substructural Logics

We denote by $\mathfrak{L}$ a finite algebraic language, containing a finite number of operation symbols of finite arity. $\mathrm{Fm}_{\mathfrak{L}}$ is the set of formulas for $\mathfrak{L}$, denoted by $\varphi, \psi, \chi \ldots$, defined inductively over a countably infinite set Var of propositional variables, denoted by $p, q, \ldots$. If the language is clear from the context, we drop the subscript $\mathfrak{L}$ from $\mathrm{Fm}_{\mathfrak{L}}$.

For a language $\mathfrak{L}$, we define an $\mathfrak{L}$-*substitution* to be a mapping $\sigma\colon \mathrm{Fm} \to \mathrm{Fm}$ such that for all n-ary $\star \in \mathfrak{L}$ and all formulas $\varphi_1, \ldots, \varphi_n \in \mathrm{Fm}$:

$$\sigma(\star(\varphi_1, \ldots, \varphi_n)) \;=\; \star(\sigma(\varphi_1), \ldots \sigma(\varphi_n))\,.$$

A *(single-conclusion) consequence relation* for $\mathfrak{L}$ is a binary relation $\mathsf{R} \subseteq \mathscr{P}(\mathrm{Fm}) \times \mathrm{Fm}$ ($\mathscr{P}$ denoting the power-set operator), also written $\Gamma \Vdash_{\mathsf{R}} \varphi$, that satisfies the following properties for all $\Gamma \cup \Sigma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$:

(i) reflexivity: $\{\varphi\} \Vdash_{\mathsf{R}} \varphi$, (*ii*) weakening: $\Gamma \Vdash_{\mathsf{R}} \varphi$ implies $\Gamma \cup \Sigma \Vdash_{\mathsf{R}} \varphi$,

(*iii*) transitivity: $\Gamma \Vdash_{\mathsf{R}} \varphi$ and $\Sigma \cup \{\varphi\} \Vdash_{\mathsf{R}} \psi$ together imply $\Gamma \cup \Sigma \Vdash_{\mathsf{R}} \psi$, and

(*iv*) structurality: $\Gamma \Vdash_{\mathsf{R}} \varphi$ implies $\sigma[\Gamma] \Vdash_{\mathsf{R}} \sigma(\varphi)$ for all $\mathfrak{L}$-substitutions σ.

Such a relation is called *finitary*, if for any (infinite) set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \Vdash_{\mathsf{R}} \varphi$ implies that $\Gamma' \Vdash_{\mathsf{R}} \varphi$ for some finite subset $\Gamma' \subseteq \Gamma$. For a general discussion about the notion of a consequence relation, see [78] by Rosalie Iemhoff.

We define a *logic* L for a language $\mathfrak{L}$ to be a consequence relation $\Vdash_{\mathsf{L}}$ for $\mathfrak{L}$, which can be specified in many ways, for example syntactically by a Hilbert-style calculus or semantically through a class of models, which in our case will either be a class algebras or a class of many-valued Kripke models.

While we postpone the discussion of many-valued modal logics to the following section, we now present a general framework which will cover many substructural logics and most many-valued logics relevant for this thesis. The framework is based on axiomatic extensions of a basic Hilbert-style calculus ($\mathcal{MAILL}$) on the syntactic side and on varieties of bounded pointed commutative residuated lattices on the semantic side. There is a natural interplay between these two sides and some very general completeness results can be obtained.

For this, let us fix an algebraic language $\mathfrak{L}$ containing the four binary propositional connectives, $\wedge$, $\vee$, $\&$, and $\to$, and the four constants e, f, $\bot$, and $\top$. Denote by Fm the set of formulas for $\mathfrak{L}$ defined inductively over Var. We frequently make use of two definable symbols: $\neg\varphi = \varphi \to \mathrm{f}$ and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$. Furthermore, we inductively define the following useful notation: let $\varphi^0 = \mathrm{e}$ and for every natural number $n \geq 1$, $\varphi^n = \varphi^{n-1} \,\&\, \varphi$.

The Syntactic Side

A *Hilbert-style calculus* $\mathcal{C}$ for $\mathfrak{L}$ is a set of axioms and derivation rules. *Axioms* are selected formula schemas. *Rules* are pairs $\langle \Gamma, \varphi \rangle$ consisting of a finite set of formula schemas Γ (called the premises of the rule) and a single formula schema φ (called the conclusion of the rule), often written as follows:

$$\frac{\varphi_1 \quad \cdots \quad \varphi_n}{\varphi},$$

if $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$. By *formula schema*, we understand formulas in Fm containing placeholders (e.g. φ or ψ) to be uniformly replaced by arbitrary formulas in Fm. When it is clear from the context, we will blur the distinction between formulas and formula schemas.

A *proof* of a formula φ from a set of formulas Γ in a given Hilbert-style calculus $\mathcal{C}$ is a finite sequence of formulas whose last member is φ and whose every member is either

(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(I)	$\varphi \to \varphi$		
(&1)	$\varphi \to (\psi \to (\varphi \,\&\, \psi))$	(e1)	e
(&2)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$	(e2)	$\varphi \to (\mathrm{e} \to \varphi)$
($\wedge$1)	$(\varphi \wedge \psi) \to \varphi$	($\vee$1)	$\varphi \to (\varphi \vee \psi)$
($\wedge$2)	$(\varphi \wedge \psi) \to \psi$	($\vee$2)	$\psi \to (\varphi \vee \psi)$
($\wedge$3)	$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$	($\vee$3)	$((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$
($\bot$)	$\bot \to \varphi$	($\top$)	$\varphi \to \top$
(MP)	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$	(ADJ)	$\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}$

Table 2.1: Axioms and Rules of $\mathcal{MAILL}$

(i) an axiom of $\mathcal{C}$, (ii) an element of Γ, or (iii) is derived from previous members of the sequence by a rule (R) of $\mathcal{C}$ (i.e. it is the conclusion of (R) and all premises of (R) are members of the initial sequence before the conclusion). If there is such a proof, we will say that φ is *provable* in $\mathcal{C}$ from Γ and write $\Gamma \vdash_{\mathcal{C}} \varphi$. If for some formula $\varphi \in \mathrm{Fm}$, $\emptyset \vdash_{\mathcal{C}} \varphi$, we will say that φ is a *theorem* of $\mathcal{C}$, written $\vdash_{\mathcal{C}} \varphi$, and denote the set of all theorems of $\mathcal{C}$ by $\mathsf{Thm}(\mathcal{C}) = \{\varphi \in \mathrm{Fm} \mid \vdash_{\mathcal{C}} \varphi\}$.

Note that any Hilbert-style calculus $\mathcal{C}$ for $\mathfrak{L}$ defines a finitary consequence relation $\vdash_{\mathcal{C}}$ for $\mathfrak{L}$, and thus a logic L for $\mathfrak{L}$ is defined by setting $\Vdash_{\mathsf{L}} = \vdash_{\mathcal{C}}$. In this case, we say that $\mathcal{C}$ *axiomatizes* L, and if $\varphi \in \mathsf{Thm}(\mathcal{C})$, we also say that φ is a *theorem* of L.

Given a Hilbert-style calculus $\mathcal{C}$ for $\mathfrak{L}$, we say that a Hilbert-style calculus $\mathcal{C}'$ for $\mathfrak{L}$ is an *axiomatic extension* of $\mathcal{C}$, if $\mathcal{C}'$ results from adding a (possibly empty) set of axioms $\mathcal{A}$ (no rules) to $\mathcal{C}$, that is $\mathcal{C}' = \mathcal{C} \cup \mathcal{A}$. Moreover, given a logic L for $\mathfrak{L}$ such that $\Vdash_{\mathsf{L}} = \vdash_{\mathcal{C}}$ for some Hilbert-style calculus $\mathcal{C}$, we also say that a logic L' is an *axiomatic extension* of L, if $\Vdash_{\mathsf{L}'} = \vdash_{\mathcal{C}'}$, for some axiomatic extension $\mathcal{C}'$ of $\mathcal{C}$.

Let us now define the Hilbert-style calculus $\mathcal{MAILL}$ (axiomatizing Multiplicative Additive Intuitionistic Linear Logic MAILL) as the set of axioms and rules in Table 2.1. Furthermore, in Table 2.2 we list some common axioms that are frequently added to $\mathcal{MAILL}$ in order to define axiomatic extensions of $\mathcal{MAILL}$.[1] Many prominent logics can be defined as axiomatic extensions of $\mathcal{MAILL}$, including the examples in Table 2.3. In this table, we list the Hilbert-style calculi with the additional axioms added to $\mathcal{MAILL}$, name the axiomatized logics and give some references. Note that many of the logics listed were originally formulated in different languages. Noting that e.g. in the presence of (W), the connectives e and $\top$, as well as f and $\bot$, collapse, we will ignore these differences here for the sake of a uniform presentation.

[1] The axioms and most of the labels are taken from [96].

Label	Axiom
(W)	$(\varphi \to \mathrm{e}) \wedge (\mathrm{f} \to \varphi)$
(DIS)	$(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
(INV)	$\neg\neg\varphi \to \varphi$
(PRL)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(DIV)	$(\varphi \wedge \psi) \to (\varphi \,\&\, (\varphi \to \psi))$
(C_n)	$\varphi^{n-1} \to \varphi^n$
(CAN)	$\neg\varphi \vee ((\varphi \to (\varphi \,\&\, \psi)) \to \psi)$
(NC)	$\neg(\varphi \wedge \neg\varphi)$

Table 2.2: Further Axioms

In Figure 2.1, we picture the logics of Table 2.3 in a diagram (except the logics MTL_n) where the lines represent a strict containment relation, such that the lower of two connected logics is contained in the one above.

The Semantic Side

After looking at the syntactic side, where we talked about provability and theoremhood, we now turn to the semantic side, where we talk about consequence and validity. For this, let us denote by $\mathbb{BPCRL}$ the class of bounded pointed commutative residuated lattices. That is, an algebra

$$\mathbf{A} \;=\; \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \&^{\mathbf{A}}, \to^{\mathbf{A}}, \mathrm{e}^{\mathbf{A}}, \mathrm{f}^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$$

belongs to $\mathbb{BPCRL}$ if

- $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ is a bounded lattice, defining a lattice order by setting for all $a, b \in A$: $a \leq^{\mathbf{A}} b$ iff $a \wedge^{\mathbf{A}} b = a$,
- $\langle A, \&^{\mathbf{A}}, \mathrm{e}^{\mathbf{A}} \rangle$ is a commutative monoid, and
- $\&^{\mathbf{A}}$ and $\to^{\mathbf{A}}$ form a residuated pair, i.e. $a \,\&^{\mathbf{A}}\, b \leq^{\mathbf{A}} c$ iff $a \leq^{\mathbf{A}} b \to^{\mathbf{A}} c$, for all $a, b, c \in A$.[2]

In fact, $\mathbb{BPCRL}$ is a variety, as these three conditions can be expressed by equations, thus defining an equational class (see e.g. [96]).

For an $\mathbf{A} \in \mathbb{BPCRL}$, we define an $\mathbf{A}$-*evaluation* to be a mapping $v \colon \mathrm{Var} \to A$ that extends to $v \colon \mathrm{Fm} \to A$ by interpreting the connectives in $\mathfrak{L}$ by the corresponding

[2] An excellent reference book for notions of universal algebra is [24].

Calculus	Additional Axioms	Name	Logic	References
$\mathcal{ML}$	(W)	Monoidal Logic	ML	[77, 101]
$\mathcal{UL}$	(DIS), (PRL)	Uninorm Logic	UL	[94]
$\mathcal{IUL}$	(DIS), (PRL), (INV)	Involutive Uninorm Logic	IUL	[94]
$\mathcal{IL}$	(W), (C_2)	Intuitionistic Logic	IL	
$\mathcal{MTL}$	(W), (PRL)	Monoidal T-Norm Logic	MTL	[54]
$\mathcal{SMTL}$	(W), (PRL), (NC)	Strict MTL	SMTL	[54]
$\mathcal{IMTL}$	(W), (PRL), (INV)	Involutive MTL	IMTL	[54]
$\mathcal{MTL}_n$	(W), (PRL), (C_n)	N-Contractive MTL	MTL_n	[33]
$\mathcal{BL}$	(W), (PRL), (DIV)	Hájek's Basic Logic	BL	[67]
$\mathcal{SBL}$	(W), (PRL), (DIV), (NC)	Strict BL	SBL	[55]
$\mathcal{Ł}$	(W), (PRL), (DIV), (INV)	Łukasiewicz Logic	$\mathsf{Ł}$	[25, 91]
$\mathcal{P}$	(W), (PRL), (DIV), (CAN)	Product Logic	P	[72]
$\mathcal{G}$	(W), (PRL), (CON)	Gödel Logic	G	[50]
$\mathcal{CL}$	(W), (INV), (C_2)	Classical Logic	CL	

Table 2.3: Prominent Axiomatic Extensions of $\mathcal{MAILL}$

operations of $\mathbf{A}$, that is,

$$\begin{array}{rclcrcl} v(\mathrm{e}) & = & \mathrm{e}^{\mathbf{A}} & \quad & v(\mathrm{f}) & = & \mathrm{f}^{\mathbf{A}} \\ v(\bot) & = & \bot^{\mathbf{A}} & & v(\top) & = & \top^{\mathbf{A}} \\ v(\varphi \wedge \psi) & = & v(\varphi) \wedge^{\mathbf{A}} v(\psi) & & v(\varphi \vee \psi) & = & v(\varphi) \vee^{\mathbf{A}} v(\psi) \\ v(\varphi \,\&\, \psi) & = & v(\varphi) \,\&^{\mathbf{A}}\, v(\psi) & & v(\varphi \to \psi) & = & v(\varphi) \to^{\mathbf{A}} v(\psi) \end{array}$$

In other words, an $\mathbf{A}$-evaluation is a homomorphism from the algebra of formulas to $\mathbf{A}$. If the algebra is clear from the context, we will omit superscript $\mathbf{A}$'s, but note that usually when we denote elements of the algebra, we will use lower-case letters a, b, c, $\ldots$, which make them easily distinguishable from formulas $\varphi, \psi, \chi, \ldots \in \mathrm{Fm}$.

Given a subset $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ and an $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \mathrm{e}, \mathrm{f}, \bot, \top \rangle \in \mathbb{BPCRL}$, we say that φ is an $\mathbf{A}$-*consequence* of Γ, written $\Gamma \models_{\mathbf{A}} \varphi$, if $\mathrm{e} \leq v(\varphi)$ for all $\mathbf{A}$-evaluations v such that $v[\Gamma] = \{v(\psi) \in A \mid \psi \in \Gamma\} \subseteq \{a \in A \mid \mathrm{e} \leq a\}$. Given a subclass of algebras $\mathbb{U} \subseteq \mathbb{BPCRL}$, we say that φ is a $\mathbb{U}$-*consequence* of Γ, written $\Gamma \models_{\mathbb{U}} \varphi$, if $\Gamma \models_{\mathbf{A}} \varphi$, for all algebras $\mathbf{A} \in \mathbb{U}$. If $\emptyset \models_{\mathbb{U}} \varphi$, we will say that φ is $\mathbb{U}$-*valid*, also denoted by $\models_{\mathbb{U}} \varphi$, and the set of $\mathbb{U}$-valid formulas will be denoted by $\mathsf{Val}(\mathbb{U})$, i.e. $\mathsf{Val}(\mathbb{U}) = \{\varphi \in \mathrm{Fm} \mid \models_{\mathbb{U}} \varphi\}$.

Noting that for any subclass $\mathbb{U} \subseteq \mathbb{BPCRL}$, $\models_{\mathbb{U}}$ is a consequence relation, we can define a logic L by setting $\Vdash_{\mathsf{L}} = \models_{\mathbb{U}}$. In this case, if a formula φ is in $\mathsf{Val}(\mathbb{U})$, we will also

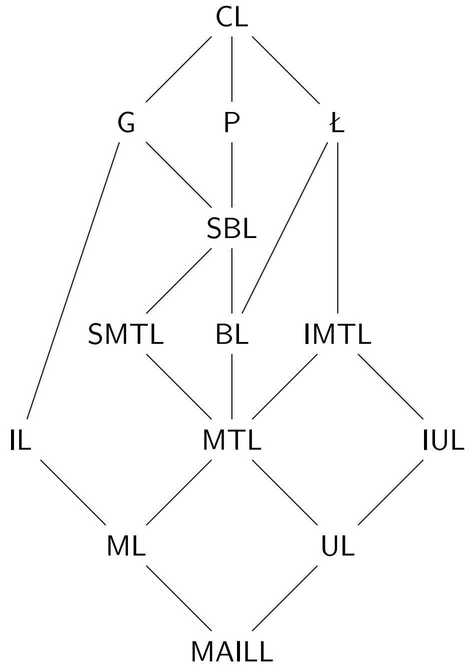

Figure 2.1: Some Axiomatic Extensions of MAILL

say that φ is *valid* in L or L-*valid*, setting $\mathsf{Val}(\mathsf{L}) = \mathsf{Val}(\mathbb{U})$.

Given a logic L for $\mathfrak{L}$, we say that L is *(finitely) strongly sound* with respect to a class $\mathbb{U} \subseteq \mathbb{BPCRL}$, if for any (finite) set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \Vdash_{\mathsf{L}} \varphi$ implies $\Gamma \models_{\mathbb{U}} \varphi$. L is called *(finitely) strongly complete* with respect to $\mathbb{U}$, if L is (finitely) strongly sound with respect to $\mathbb{U}$ and for any (finite) set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \models_{\mathbb{U}} \varphi$ implies $\Gamma \Vdash_{\mathsf{L}} \varphi$. If for some axiomatic extension $\mathcal{C}$ of $\mathcal{MAILL}$, $\Vdash_{\mathsf{L}} = \models_{\mathbb{U}} = \vdash_{\mathcal{C}}$, we also say that $\mathcal{C}$ *axiomatizes* $\mathbb{U}$, and if for all (finite) sets $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathcal{C}} \varphi$ iff $\Gamma \models_{\mathbb{U}} \varphi$, we call $\mathcal{C}$ *(finitely) strongly complete* with respect to $\mathbb{U}$. In both cases, it is obviously true that $\mathsf{Val}(\mathsf{L}) = \mathsf{Val}(\mathbb{U}) = \mathsf{Thm}(\mathcal{C})$. If the class $\mathbb{U}$ is a singleton class $\{\mathbf{A}\}$, we just replace $\{\mathbf{A}\}$ by $\mathbf{A}$ in the above definitions.

In this way, many substructural logics can be defined by subvarieties of $\mathbb{BPCRL}$, which in turn may be defined by restricting the variety $\mathbb{BPCRL}$ by adding further conditions (expressible as equations). A list of commonly added conditions is given in Table 2.4, where all conditions are to be understood as quantified over all $a, b, c \in A$, and $a^0 = \bot$ and for any natural number $n \geq 1$, $a^n = a \,\&\, \ldots \,\&\, a$ (n times). A list of the subvarieties defining the logics in Table 2.3 is given in Table 2.5, along with the required additional conditions. Note that each condition listed in Table 2.4, corresponds to an axiom in Table 2.2, as indicated in Table 2.4.

Remark 2.1. Let us note that many of the varieties listed in Table 2.5 were originally

Name	Label	Condition	Axiom
integrality	(int)	$\mathrm{e} \;\leq\; (a \to \mathrm{e}) \wedge (\mathrm{f} \to a)$	(W)
distributivity	(dis)	$a \wedge (b \vee c) \;=\; (a \wedge b) \vee (a \wedge c)$	(DIS)
involution	(inv)	$a \;=\; \neg\neg a$	(INV)
prelinearity	(prl)	$\mathrm{e} \;\leq\; (a \to b) \vee (b \to a)$	(PRL)
divisibility	(div)	$a \wedge b \;=\; a \,\&\, (a \to b)$	(DIV)
n-contraction	(c_n)	$a^n \;\leq\; a^{n+1}$	(C_n)
cancellation	(can)	$\mathrm{e} \;\leq\; \neg a \vee ((a \to (a \,\&\, b)) \to b)$	(CAN)
strictness	(str)	$\mathrm{e} \;\leq\; \neg(a \wedge \neg a)$	(NC)

Table 2.4: Further Conditinos

introduced over different algebraic languages. Moreover, we will discuss some of them later over a restricted language. This problem can be circumvented by the notion of term equivalence. Two algebras are *term equivalent* if the operations of one are definable by the operations of the other (e.g. if (int) is satisfied, e and $\top$ collapse, as well as f and $\bot$, and vice versa). In this sense, the varieties in Table 2.5 should be understood as being term-equivalent to the varieties usually denoted by these names in the literature.

Remark 2.2. Note that by considering subvarieties of $\mathbb{BPCRL}$, we cover almost all many-valued logics considered in this work[3] and many other interesting substructural logics. However, there are also many interesting substructural logics left out. While originally motivated by removing structural rules from Gentzen systems for intuitionistic logic or classical logic, it was suggested by Hiroakira Ono in [102] to delimit substructural logics as the logics of varieties of residuated lattices.

For some more examples, taking the present setting as the starting point and ignoring complications of different languages, we could remove conditions like boundedness of the lattice, commutativity of the monoidal operation, or the presence of the constant f in order to obtain substructural logics weaker than MAILL.

The choice to limit our framework to bounded pointed commutative residuated lattices was a compromise between ease of presentation and the range of logics we wanted to cover. For much broader overviews of substructural logics, see e.g. [102], [109], or [96].

[3]Note that the order-based logics defined in Chapter 3 might have an extended language. However, if the language is $\mathfrak{L}$, order-based algebras are members of $\mathbb{BPCRL}$.

Variety	Conditions	Logic
$\mathbb{ML}$	(int)	ML
$\mathbb{UL}$	(dis), (prl)	UL
$\mathbb{IUL}$	(dis), (prl), (inv)	IUL
$\mathbb{H}$	(int), (c_2)	IL
$\mathbb{MTL}$	(int), (prl)	MTL
$\mathbb{SMTL}$	(int), (prl), (str)	SMTL
$\mathbb{IMTL}$	(int), (prl), (inv)	IMTL
$\mathbb{MTL}_n$	(int), (prl), (c_n)	MTL_n
$\mathbb{BL}$	(int), (prl), (div)	BL
$\mathbb{SBL}$	(int), (prl), (div), (str)	SBL
$\mathbb{MV}$	(int), (prl), (div), (inv)	Ł
$\mathbb{P}$	(int), (prl), (div), (can)	P
$\mathbb{G}$	(int), (prl), (c_2)	G
$\mathbb{BOOL}$	(int), (inv), (c_2)	CL

Table 2.5: Prominent Subvarieties of $\mathbb{BPCRL}$

Completeness

We will now present a series of completeness results, properly relating the semantic and syntactic sides above, which will become increasingly stronger.

It is easy to check that the rules and axioms of $\mathcal{MAILL}$ are valid in any $\mathbf{A} \in \mathbb{BPCRL}$. For an axiomatic extension $\mathcal{C}$ of $\mathcal{MAILL}$, we will therefore call an $\mathbf{A} \in \mathbb{BPCRL}$ a *$\mathcal{C}$-algebra* if all axioms of $\mathcal{C}$ are valid in $\mathbf{A}$. We will then denote the class of all $\mathcal{C}$-algebras by $\mathsf{Gen}(\mathcal{C})$. The following completeness theorem with respect to general algebras can be proved via Lindenbaum constructions (see e.g. [96]):

Theorem 2.3 (Strong General Completeness)**.** *For any axiomatic extension $\mathcal{C}$ of $\mathcal{MAILL}$ and set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$:*

$$\Gamma \vdash_{\mathcal{C}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{Gen}(\mathcal{C})} \varphi\,.$$

Note also that $\mathsf{Gen}(\mathcal{MAILL}) = \mathbb{BPCRL}$ and, by the close correspondence between axioms in Table 2.2 and conditions in Table 2.4, we have that any variety $\mathbb{L}$ in Table 2.5 is the subvariety $\mathsf{Gen}(\mathcal{C}) \subseteq \mathbb{BPCRL}$ for the corresponding axiomatic extension $\mathcal{C}$ of $\mathcal{MAILL}$ in Table 2.3.

Remark 2.4. In general, any axiom φ added to $\mathcal{MAILL}$ can be translated into an equation $e \wedge \varphi = e$, and each equation $\varphi = \psi$ defining a subvariety of $\mathbb{BPCRL}$ can be translated into an axiom $\varphi \leftrightarrow \psi$, with the effect that for the resulting Hilbert-style calculus $\mathcal{C} \supseteq \mathcal{MAILL}$ and the resulting subvariety $\mathbb{U} \subseteq \mathbb{BPCRL}$, $\mathbb{U} = \mathsf{Gen}(\mathcal{C})$. In our case, however, we chose to present the axioms and conditions in formats more commonly used in the literature.

In fact, Theorem 2.3 can be generalized to much broader classes of Hilbert-style calculi (not just axiomatic extensions of $\mathcal{MAILL}$). This has been shown by Willem J. Blok and Don Pigozzi in [17], in which they develop the framework of *algebraizable logics*, where the presence of certain axioms on the syntactic side and the validity of certain equations on the semantic side guarantee the existence of a translation from one side to the other. A similar framework of *algebraic implicative logics* was developed by Petr Cintula and Carles Noguera in [41].

For axiomatic extensions of $\mathcal{MAILL}$ including (PRL) and (DIS), which incidentally are axiomatic extensions of $\mathcal{UL}$, a stronger result is folklore, namely completeness with respect to all *linearly ordered* $\mathcal{C}$-algebras, i.e. $\mathcal{C}$-algebras where the lattice order is total. Let $\mathsf{Lin}(\mathcal{C})$ denote the class of all linearly ordered $\mathcal{C}$-algebras, also called *$\mathcal{C}$-chains*. To prove linear completeness, we construct a linearly ordered Lindenbaum algebra, the existence of which is ensured by the presences of (PRL) in $\mathcal{C}$ and the proof-by-cases property enforced by (DIS) (see e.g. [96]).

Theorem 2.5 (Strong Linear Completeness). *For any axiomatic extension $\mathcal{C}$ of $\mathcal{UL}$ and any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$:*

$$\Gamma \vdash_{\mathcal{C}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{Lin}(\mathcal{C})} \varphi\,.$$

Note that because (DIS) is provable in any extension of $\mathcal{MAILL}$ that includes (W) and (PRL), this theorem covers, among many others, all the logics in Table 2.3 except ML and IL. In particular, it covers all the logics in Table 2.3 which are considered to be fuzzy logics. In fact, enjoying linear completeness is often seen as the characteristic property of fuzzy logics (see e.g. [35]). However, fuzzy logics were often designed with a certain intended semantics in mind, which is usually a (class of) algebra(s) with the real unit interval $[0,1]$ as universe. This is why it is an interesting and important question, whether the mentioned logics enjoy completeness with respect to the so-called standard algebras, a question that is generally much harder to answer (compared to the question of linear completeness).

In order to define the standard algebras in $\mathbb{BPCRL}$, we make use of the notion of uninorms and t-norms. A *uninorm* is a binary operation $*\colon [0,1]^2 \to [0,1]$ on the closed real unit interval that is commutative, associative, monotone, and has a unit element

$e_* \in [0,1]$. A uninorm $*$ is called *left-continuous* or *right-continuous*, if for all $a, b \in [0,1]$,

$$\bigvee_{c<a}(c * b) \;=\; a * b \qquad \text{or} \qquad \bigwedge_{a<c}(c * b) \;=\; a * b\,,$$

respectively. It is called *continuous*, if it is left- and right-continuous. Furthermore, a uninorm $*$ is called *conjunctive* if for all $a \in [0,1]$, $a * 0 = 0$. Most notably, a uninorm $*$ is called a *t-norm* (short for *triangular norm*) if the unit element for $*$ is 1, i.e. $e_* = 1$.

Example 2.6. Let us list the three main examples of continuous t-norms:

- *Łukasiewicz t-norm*: $a *_{\mathsf{Ł}} b \;=\; \max(0, a + b - 1)\,,$
- *product t-norm*: $a *_{\mathsf{P}} b \;=\; a \cdot b$ (the usual product of reals),
- *Gödel t-norm*: $a *_{\mathsf{G}} b \;=\; \min(a, b)$ (also called the *mininum t-norm* or just min).

An example of a left-continuous t-norm that is not continuous is the *nilpotent minimum t-norm*:

$$a *_{\mathsf{NM}} b \;=\; \begin{cases} \min(a,b) & \text{if } a + b > 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

If a uninorm $*$ is left-continuous and conjunctive (or a left-continuous t-norm), the following condition determines a unique binary operation $\to_*\colon [0,1]^2 \to [0,1]$, called the *residuum* of $*$: for all $a, b, c \in [0,1]$,

$$a * b \le c \qquad \text{iff} \qquad a \le b \to_* c\,,$$

or, equivalently: for all $a, b \in [0,1]$,

$$a \to_* b \;=\; \max\{c \in [0,1] \mid a * c \le b\}\,.$$

In this case, the uninorm (or the t-norm) is called *residuated* (cf. [96]).

Example 2.7. The residual operations of the four t-norms in Example 2.6 are as follows:

$$a \to_* b = 1, \quad \text{if } a \le b,$$

and if $a > b$, then we have for the

- *Łukasiewicz implication*: $a \to_{\mathsf{Ł}} b \;=\; 1 - (a - b)\,,$
- *product implication*: $a \to_{\Pi} b \;=\; \frac{b}{a}\,,$
- *Gödel implication*: $a \to_{\mathsf{G}} b \;=\; b\,,$
- *nilpotent minimum implication*: $a \to_{\mathsf{NM}} b \;=\max(1 - a, b)\,.$

Remark 2.8. Let us mention that the continuous t-norms are completely classified by the Mostert-Shields Theorem [98]. It says that a t-norm is continuous if and only if it is isomorphic to an ordinal sum of the Gödel t-norm, the Łukasiewicz t-norm, and the product t-norm. In Section 2.2, we will present the logics determined by these three fundamental continuous t-norms.

An $\mathbf{A} \in \mathbb{BPCRL}$ is called *standard*, if its bounded lattice reduct is $\langle [0,1], \min, \max, 0, 1 \rangle$, & is a residuated uninorm, $\rightarrow$ is its residuum, and e its unit (f can be any value in $[0,1]$). For an axiomatic extension $\mathcal{C}$ of $\mathcal{MAILL}$, the class of all standard $\mathcal{C}$-algebras will be denoted by $\mathsf{Std}(\mathcal{C})$. In this case, we will say that a logic L, axiomatized by an axiomatic extension $\mathcal{C}$ of $\mathcal{MAILL}$, enjoys *(finite) strong standard completeness* if for any (finite) set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$:

$$\Gamma \vdash_{\mathcal{C}} \varphi \qquad \text{iff} \qquad \Gamma \models_{\mathsf{Std}(\mathcal{C})} \varphi\,.$$

Example 2.9. In the next section, we will see that MTL and G enjoy strong standard completeness, while BL, SBL (see [34]), $\mathsf{Ł}$, and P are finitely strongly complete with respect to their standard algebras. In fact, $\mathsf{Ł}$, P, and G, are finitely strongly complete with respect to just one standard algebra, respectively, namely the standard algebras determined by the Łukasiewicz t-norm, the product t-norm, and the Gödel t-norm. In the case of G, this is also true for infinite sets of premises.

Furthermore, the logics MTL_n, SMTL, and IMTL enjoy strong standard completeness. This was proved in [33] and [53] by extending the methods used in [79], where strong standard completeness of MTL was shown.

In [94], Metcalfe and Montagna prove strong standard completeness of UL, where $\mathsf{Std}(\mathcal{UL})$ is the class of all standard algebras in $\mathbb{BPCRL}$, thus justifying the choice of the name "uninorm logic". For their proof, the authors show in a first step the strong standard completeness of $\mathcal{UL}$ extended with a density rule (introduced in [118]) and in a second step, using hypersequent calculi, they prove that this density rule is admissible in UL. Whether also IUL enjoys (finite) strong standard completeness, however, is still an open question.

Another Syntactic Side

The approach taken by Metcalfe and Montagna in [94] represents a further popular way to introduce and study logics (next to specifying classes of algebras or Hilbert-style calculi), namely by specifying a set of axioms and rules in a Gentzen-style system (which includes sequent and hypersequent calculi). Sequent calculi were first introduced by Gerhard Gentzen, who presented the sequent calculi $\mathcal{LK}$ and $\mathcal{LJ}$ for first-order classical logic and intuitionistic logic in [61]. In fact, it was these calculi that gave the defining motivation for the introduction of substructural logics, namely by the removal of structural rules such as exchange, weakening, and contraction from either $\mathcal{LK}$ or $\mathcal{LJ}$ (see e.g. [109]). The first hypersequent calculus was then introduced by Arnon Avron in [3] as a generalization of sequent calculi in order to provide a Gentzen-style system for the relevance logic RM.

While Hilbert-style calculi provide very useful frameworks for classifying logics, establishing certain properties, and providing tight connections to algebraic semantics, they are less suitable as frameworks for theorem-proving and establishing algorithmic properties. This is where e.g. Gentzen-style systems come in handy, which in many cases can be shown to be *analytic*, that is, proofs in these systems can be built completely from subformulas of the formula to be proved, in which case proof-search usually presents an algorithmic decision procedure. The analyticity of a Gentzen-style system is usually proved by cut-elimination, i.e. a (preferably syntactic) proof that the only "non-analytic" rule, (CUT), is admissible in the system.

The monograph [96], by Metcalfe, Olivetti, and Gabbay, gives an overview of hypersequent calculi used for fuzzy logics, as in these cases, regular sequent calculi are less suitable. They provide proofs of the admissibility of (CUT) and certain density rules for many cases, thus being able to establish (not necessarily unknown) complexity bounds and strong standard completeness results for many of the fuzzy logics discussed in this chapter.

Note that Genzten-style systems are not the only kinds of calculi that can be analytic, also resolution, display logic, and tableau systems frequently have this property. In Chapter 6, we will present analytic tableau calculi for certain Gödel modal logics.

2.2 Monoidal T-Norm Logic and Axiomatic Extensions

After giving a very general view on some substructural logics in the last section, we will now take closer look at (some of) the logics we will be concerned with in the rest of this thesis, namely the fuzzy logic MTL and its axiomatic extensions BL, $\mathsf{Ł}$, P, and G.

Recall that in the presence of the axiom (W) $(\varphi \to \mathrm{e}) \wedge (\mathrm{f} \to \varphi)$, the connectives e and $\top$, as well as f and $\bot$, collapse. This is why, unless stated otherwise, we will fix the algebraic language $\mathfrak{L}$ to consist of the four binary connectives, $\wedge$ ("weak conjunction"), $\vee$ ("weak disjunction"), $\&$ ("strong conjunction"), and $\to$ ("implication"), and two constants, $\bot$ ("falsum") and $\top$ ("verum"), for the rest of this chapter. We will also use the defined symbols $\neg$ ("negation") and $\leftrightarrow$ ("equivalence"), defined by $\neg\varphi = \varphi \to \bot$ and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Remark 2.10. In fact, in the setting of MTL, also $\vee$ and $\top$ can be defined using the remaining connectives in $\mathfrak{L}$: $\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ and $\top = \bot \to \bot$. So, for our purposes, $\wedge$, $\&$, $\to$, and $\bot$ would be the only necessary symbols, but for the sake of a familiar presentation, we will treat all the connectives in $\mathfrak{L}$ as primitive.

Monoidal T-Norm Logic

Monoidal t-norm logic MTL was introduced in [54] by Francesc Esteva and Lluís Godo as the logic axiomatized by some Hilbert-style calculus (slightly different from $\mathcal{MTL}$ defined above), but the intention was that MTL should be the logic of left-continuous t-norms, which was later confirmed by Sándor Jenei and Franco Montagna in [79]. In fact, left-continuity is the weakest property that ensures that a t-norm has a residuum.

Let us recall that $\mathbb{MTL}$ is the subvariety of $\mathbb{BPCRL}$ where the conditions (int) and (prl) are satisfied, that is, it is the variety of prelinear commutative bounded integral residuated lattices. Put in terms of the restricted language $\mathfrak{L}$,

$$\mathbf{A} \;=\; \langle A, \wedge, \vee, \&, \to, \bot, \top\rangle$$

belongs to $\mathbb{MTL}$ if the following conditions are satisfied:

- $\langle A, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice,
- $\langle A, \&, \top\rangle$ is a commutative monoid,
- $\&$ and $\to$ form a residuated pair, i.e. $a \,\&\, b \leq c$ iff $a \leq b \to c$, for all $a, b, c \in A$,
- $\mathbf{A}$ is prelinear, i.e. $(a \to b) \vee (b \to a) = \top$ is satisfied for all $a, b \in A$.

As now the top element of the lattice corresponds with the unit of the monoid, the definition of consequence in an $\mathbf{A} \in \mathbb{MTL}$ simplifies as follows: given a subset $\Gamma \cup \{\varphi\} \subseteq$ Fm and an $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top\rangle \in \mathbb{MTL}$, $\Gamma \models_{\mathbf{A}} \varphi$ if and only if $v(\varphi) = \top$ for all $\mathbf{A}$-evaluations v such that $v[\Gamma] \subseteq \{\top\}$.

By general completeness, it makes sense to call algebras in $\mathbb{MTL}$ also MTL-*algebras*. Furthermore, recall that an $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top\rangle$ is a standard MTL-algebra, if $A = [0, 1]$, the closed real unit interval, $\wedge$ and $\vee$ are minimum and maximum, respectively, $\&$ is a left-continuous t-norm and $\to$ its residuum, and $\bot = 0$ and $\top = 1$. The following standard completeness result was proved in [79], making it clear that Esteva and Godo were justified in calling MTL the logic of left-continuous t-norms.

Theorem 2.11 (Strong Standard Completeness of MTL [79])**.** *For any* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,

$$\Gamma \vdash_{\mathcal{MTL}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{Std}(\mathcal{MTL})} \varphi\,.$$

Even though decidability of both the validity problem as well as the finitary consequence problem for MTL were established by Ono in [100], heavily relying on results in [16], determining upper complexity bounds for these decision problems remains open (cf. [74]).

Theorem 2.12 (Decidability of MTL [16, 100])**.** *The validity problem and the finitary consequence problem for* MTL *are decidable.*

An important property of MTL and its axiomatic extensions is that they validate the *local deduction theorem* (see e.g. [109]; recall that for any formula $\varphi \in \mathrm{Fm}$, $\varphi^0 = \top$, and for any natural number $n \geq 1$, $\varphi^n = \varphi^{n-1} \mathbin{\&} \varphi$).

Theorem 2.13 (Local Deduction Theorem)**.** *Let $\mathcal{C}$ be an axiomatic extension of $\mathcal{MTL}$, then for any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,*

$$\Gamma \cup \{\varphi\} \vdash_{\mathcal{C}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathcal{C}} \varphi^n \to \psi\,, \ \textit{for some } n \in \mathbb{N}\,.$$

Hajék's Basic Logic

Petr Hájek introduced the so called basic logic BL via a Hilbert-style axiomatization in his seminal work on mathematical fuzzy logics [67]. His intention, that BL is the logic of continuous t-norms, was later confirmed by Roberto Cignoli, Francesc Esteva, Lluís Godo, and Antoni Torrens in [34]. Continuous t-norms play a crucial role for the fuzzy logics studied in this thesis, enjoy nice properties, and were classified completely (see Remark 2.8).

Remark 2.14. Note that $\mathcal{BL}$ proves the formulas $(\varphi \wedge \psi) \leftrightarrow (\varphi \mathbin{\&} (\varphi \to \psi))$ and thus the connective $\wedge$ becomes definable. So, for BL and all its axiomatic extensions, the only necessary connectives would be &, $\to$, and $\bot$, but we will continue to treat all the connectives of $\mathfrak{L}$ as primitive.

Let us recall that an algebra $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top \rangle \in \mathbb{MTL}$ belongs to $\mathbb{BL}$ if it satisfies the condition (div), i.e. that for all $a, b \in A$:

$$a \wedge b \ = \ a \mathbin{\&} (a \to b)\,.$$

Thus, the class $\mathbb{BL}$ for $\mathfrak{L}$ can be defined as the variety of all *divisible* MTL-algebras, also called BL-*algebras*. Furthermore, recall that the standard BL-algebras $\mathbf{A} \in \mathsf{Std}(\mathcal{BL})$ for $\mathfrak{L}$ are of the form

$$\mathbf{A} \ = \ \langle [0,1], \min, \max, \&, \to, 0, 1 \rangle\,,$$

where & is a continuous t-norm and $\to$ its residuum. Note that while the consequence relations $\models_{\mathbb{BL}}$ and $\models_{\mathsf{Std}(\mathbb{MTL})}$ are both finitary, the consequence relation $\models_{\mathsf{Std}(\mathbb{BL})}$ is not. This is the reason that the following theorem, proved in [34], is only formulated for finite sets of formulas.

Theorem 2.15 (Finite Strong Standard Completeness for BL [34])**.** *For any finite set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,*

$$\Gamma \vdash_{\mathcal{BL}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{Std}(\mathcal{BL})} \varphi\,.$$

For BL, in contrast to MTL, not only the decidability of the validity and the finitary consequence problem are known, but also the complexity bounds for these decision problems. The complexity of the validity problem was obtained in [6] and the complexity of the problem of finitary consequence can be inferred from these results (see e.g. [74]).

Theorem 2.16 (Decidability and Complexity of BL [6]). *The validity problem and the finitary consequence problem for* BL *are* coNP*-complete.*

Furthermore, the fact that BL is an axiomatic extension of MTL ensures that BL satisfies the local deduction theorem (see Theorem 2.13 above).

Łukasiewicz Logic

Jan Łukasiewicz was the first person to mention publicly a three-valued logic in a speech in 1918 and thus marked the beginning of the study of many-valued logics. He later elaborated the topic in the published speech [89] and went on to define an infinite-valued version of his three-valued logic in [91]. In this work, he introduced a Hilbert-style calculus (quite different from $\mathcal{L}$ given in Table 2.3) that he conjectured to axiomatize the set of **Ł**-valid formulas, **Ł** being the algebra we now call the standard Łukasiewicz algebra (see below). The first published proof of this conjecture was given by Alan Rose and John Barkley Rosser in [112].

For historical reasons, we denote the class of algebras with respect to which Łukasiewicz logic is strongly complete by $\mathbb{MV}$, for "many-valued" algebras. This term was coined by Chen Chung Chang, who first introduced this class of algebras in [25] and proved that it determines the same logic as Łukasiewicz's Hilbert-style calculus.[4] This important class of algebras has received a great deal of attention in its own right. For a recent exposition on the subject, see e.g. [46].

Recall that we can define $\mathbb{MV}$ (for language $\mathfrak{L}$) as the variety of BL-algebras $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top \rangle$ such that for all $a \in A$, the following equation is satisfied:

$$\neg\neg a \;=\; a\,,$$

recalling that $\neg a = a \to \bot$. This condition is often called *double-negation elimination* and in this case, the negation is called *involutive*.

There is one algebra $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top \rangle \in \mathbb{MV} \subseteq \mathbb{BL} \subseteq \mathbb{MTL}$, which we will call the *standard* Ł*-algebra* (or *standard Łukasiewicz algebra*). It will be denoted by **Ł** and is defined as follows: $A = [0, 1]$, $\wedge$ and $\vee$ denote minimum and maximum, respectively, $\&$ is the Łukasiewicz t-norm $*_{\text{Ł}}$, $\to$ is the Łukasiewicz implication $\to_{\text{Ł}}$, $\bot = 0$, and $\top = 1$.

[4]Let us note that the language originally used to define $\mathbb{MV}$ in [25] was quite different than the languages we use in Sections 2.1 and 2.2

That is, the standard Ł-algebra is

$$\mathbf{Ł} \;=\; \langle [0,1], \min, \max, *_{Ł}, \to_{Ł}, 0, 1\rangle \;,$$

recalling that for all $a, b \in [0,1]$, $a *_{Ł} b = \max(0, a+b-1)$ and

$$a \to_{Ł} b \;=\; \begin{cases} 1 & \text{if } a \leq b\,, \\ 1-(a-b) & \text{otherwise}\,. \end{cases}$$

This algebra provides the intended semantics for Łukasiewicz logic and thus the finite strong completeness of $\mathcal{Ł}$ with respect to the standard Łukasiewicz algebra $\mathbf{Ł}$ is a very important result proved by Louise Hay in [76], noting that the consequence relation $\models_{\mathbf{Ł}}$ is not finitary.[5]

Theorem 2.17 (Finite Strong Standard Completeness of Ł [76]). *For any finite set* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*,*

$$\Gamma \vdash_{\mathcal{Ł}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbf{Ł}} \varphi\,.$$

While decidability was well-known, coNP-completeness of the validity problem for Ł was proved by Daniele Mundici in [99], essentially by reducing the problem to the validity problem for finite Ł-algebras. The complexity of the problem of finitary consequence for Ł was later obtained as a consequence of Mundici's work (cf. [74]).

Theorem 2.18 (Decidability and Complexity of Ł [99]). *The validity problem and the finitary consequence problem for* Ł *are* coNP*-complete.*

Product Logic

Product logic was introduce by Hájek, Godo, and Esteva in [72] as the logic of the product t-norm, which is just the operation of multiplication on the reals.[6] Product logic has since been studied intensively, as it is based on one of the three fundamental continuous t-norms (see Remark 2.8).

Recall that the variety $\mathbb{P}$ (in language $\mathfrak{L}$) can be defined as a subvariety of $\mathbb{BL}$ of all $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top\rangle \in \mathbb{BL}$ such that for all $a, b \in A$:

$$\neg a \vee ((a \to (a \,\&\, b)) \to b) \;=\; \top\,,$$

[5] Let us remark that it is an open discussion in the mathematical logic community, whether Łukasiewicz logic should be defined as the logic determined by the axiomatization given in [91] or by the standard Łukasiewicz algebra, which is widely held to be the intended semantics. As these consequence relations diverge when it comes to infinite sets of premises, this discussion is not idle. We chose the former approach in this thesis not so much as to take a stance in the discussion, but for the sake of uniform presentation in this chapter. Obviously, these discussions arise also for Hájek's basic logic, product logic, and many other logics.

[6] Similar to the case of Ł, we define product logic P as given by the finitary consequence relation $\vdash_{\mathcal{P}}$ (c.f. Table 2.3), which is a choice motivated by the benefits of a uniform presentation.

which is often called the *cancellation* condition. It is shown in [72] that P is finitely strongly complete with respect to the *standard* P*-algebra*

$$\mathbf{P} \;=\; \langle [0,1], \min, \max, *_{\mathsf{P}}, \to_{\mathsf{P}}, 0, 1 \rangle \,,$$

recalling that for all $a, b \in [0,1]$, $a *_{\mathsf{P}} b = a \cdot b$ and

$$a \to_{\mathsf{P}} b \;=\; \begin{cases} 1 & \text{if } a \leq b\,, \\ \frac{b}{a} & \text{otherwise}\,. \end{cases}$$

Theorem 2.19 (Finite Strong Standard Completeness of P [72]). *For a finite set of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*,*

$$\Gamma \vdash_{\mathcal{P}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbf{P}} \varphi\,.$$

Decidability and complexity results for the validity problem for P were established in [5], using a reduction to validity in Łukasiewicz logic $\mathsf{Ł}$. The complexity of the finitary consequence problem can be inferred from the complexity of the validity problem (see [74]).

Theorem 2.20 (Decidability and Complexity of P [5]). *The validity problem and the finitary consequence problem for* P *are* coNP*-complete.*

Gödel Logic

What we call propositional Gödel logic G in this work was not technically introduced by Kurt Gödel, contrary to what the name might suggest. Following his proof that the semantics of intuitionistic logic could not be finite-valued, Gödel noted that there are at least countably infinitely many logics intermediate in strength between intuitionistic propositional logic IL and classical propositional logic CL (see [62]). The logics he mentioned are what we now would call finite-valued Gödel logics. There is exactly one such logic G_n for each natural number $n \geq 2$, where n is the number of truth-values.

Michael Dummett developed Gödel's ideas further, introducing infinite-valued Gödel logic in [50]. This is why this infinite-valued variant is often called Gödel-Dummett logic or Dummett's logic LC.

To be precise, Gödel and Dummett both considered the logics they were talking about as sets of valid formulas. As in this work we understand a logic as a consequence relation, Gödel and Dummett were in fact talking about what we will denote here by $\mathsf{Val}(\mathsf{G})$, the set of G-valid formulas, also called tautologies or 1-tautologies of G. Making this distinction here is important, as Dummett, in some sense, defined $\mathsf{Val}(\mathsf{G})$ by taking the intersection of all finite-valued Gödel logics, that is

$$\mathsf{Val}(\mathsf{G}) \;=\; \bigcap_{n \geq 2} \mathsf{Val}(\mathsf{G}_n)\,.$$

This equality is not true when we consider Gödel logic as a consequence relation, i.e. $\mathsf{G} \neq \bigcap_{n\geq 2} \mathsf{G}_n$. However, in [9], Matthias Baaz and Richard Zach prove that at least

$$\mathsf{G} \subsetneq \bigcap_{n\geq 2} \mathsf{G}_n\,,$$

their proof being a consequence of a study of Gödel logic over different infinite sets of truth-values, something we will also look at later in this work.

We note that the variety of *Gödel algebras* $\mathbb{G}$ (for language $\mathfrak{L}$) can be defined as the subvariety of $\mathbb{MTL}$ of all algebras $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top\rangle \in \mathbb{MTL}$ with an *idempotent* monoidal operation, that is, the following identity is satisfied for all $a \in A$:

$$a \;=\; a \,\&\, a\,.$$

Remark 2.21. In fact, an MTL-algebra with an idempotent monoidal operation is divisible. For this reason, $\mathbb{G}$ is not only a subvariety of $\mathbb{MTL}$, but also of $\mathbb{BL}$.

The *standard* G*-algebra* (or *standard Gödel algebra*) $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top\rangle \in \mathbb{G}$ is defined as follows: $A = [0,1]$, $\wedge$ and $\vee$ denote minimum and maximum, respectively, $\&$ is the Gödel t-norm $*_{\mathsf{G}}$ and $\to$ the Gödel implication $\to_{\mathsf{G}}$, and $\bot = 0$ and $\top = 1$, recalling that for all $a, b \in [0,1]$, $a *_{\mathsf{G}} b = \min(a,b)$ and

$$a \to_{\mathsf{G}} b \;=\; \begin{cases} 1 & \text{if } a \leq b\,,\\ b & \text{otherwise}\,. \end{cases}$$

The standard G-algebra $\mathbf{A}$ will be denoted by $\mathbf{G}$, and because $\wedge$ and $\&$ both denote the minimum operation on the real unit interval $[0,1]$, we will omit writing $\&$. In other words, the standard G-algebra will be denoted by

$$\mathbf{G} \;=\; \langle [0,1], \min, \max, \to_{\mathsf{G}}, 0, 1\rangle\,.$$

Apart from the collapse of $\&$ and $\wedge$, the idempotency of the t-norm also has the effect that φ^n is equivalent to φ, for all $n \in \mathbb{N}^+$ and all $\varphi \in \mathrm{Fm}$ (recall that $\varphi^n = \varphi \,\&\, \ldots \,\&\, \varphi$, for n copies of φ). Therefore, the local deduction theorem for G can be strengthened to the classical form.

Theorem 2.22 (Deduction Theorem for G)**.** *For any subset* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$*,*

$$\Gamma \cup \{\varphi\} \vdash_{\mathcal{G}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathcal{G}} \varphi \to \psi\,.$$

The fact that $\mathsf{Thm}(\mathcal{G}) = \mathsf{Val}(\mathbf{G})$ was proved by Dummett in [50], which implies finite strong completeness of $\mathcal{G}$ with respect to $\mathbf{G}$, using the fact that G validates the (classical) deduction theorem. It is worth noting that in contrast to $\models_{\mathsf{Std}(\mathbb{BL})}$, $\models_{\mathbf{Ł}}$, and $\models_{\mathbf{P}}$, the consequence relation $\models_{\mathbf{G}}$ is finitary (see e.g. [67]), which allows for strong standard completeness even for infinite sets of premises.

Theorem 2.23 (Strong Standard Completeness of G [50]). *For any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,*

$$\Gamma \vdash_{\mathcal{G}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbf{G}} \varphi\,.$$

A proof of coNP-completeness of the validity problem for G can be found in [5]. The authors of that proof, however, claim that the result was already folklore. The fact that also the finitary consequence problem is coNP-complete then follows directly by the deduction theorem.

Theorem 2.24 (Decidability and Complexity of G [5]). *The validity problem and the finitary consequence problem for G are* coNP*-complete.*

An important feature of the standard Gödel algebra $\mathbf{G}$ is that all connectives in $\mathfrak{L}$, because the additive connective & coincides with the lattice connective $\wedge$, are interpreted by algebraic operations that depend solely on the lattice order of $\mathbf{G}$ or the constants, that is, these operations can be defined by quantifier-free first-order formulas using $\wedge$, $\vee$, and constants in $\mathfrak{L}$ as the only function symbols and identity as the only relation symbol. Logics where all connectives are interpreted by operations that depend only on the order will be called *order-based*. We will say more about them in Chapter 3. While G is an example of an order-based logic, $\mathsf{Ł}$ and P are not, as strong conjunction and implication in these logics depend also on addition and multiplication, respectively.

Another important feature of Gödel logic is the fact that it is a so-called *intermediate* or *superintuitionistic* logic, that is, G is intermediate in strength between intuitionistic logic IL and classical logic CL, i.e.

$$\mathsf{IL} \subsetneq \mathsf{G} \subsetneq \mathsf{CL}\,,$$

as we have already seen in Figure 2.1. Furthermore, it is well known that adding the *law of the excluded middle* (LEM) $\varphi \vee \neg\varphi$ as an axiom either to $\mathcal{IL}$ or to $\mathcal{G}$ would yield classical propositional logic CL, i.e. $\Vdash_{\mathsf{CL}} = \vdash_{\mathcal{IL}\cup\{(\mathrm{LEM})\}} = \vdash_{\mathcal{G}\cup\{(\mathrm{LEM})\}}$.

Furthermore, for each natural number $n \geq 2$, let us define the algebra

$$\mathbf{G}_n \;=\; \langle \{0, \tfrac{1}{n-1}, \ldots, \tfrac{n-2}{n-1}, 1\}, \min, \max, \to_{\mathsf{G}}, 0, 1\rangle\,,$$

which clearly is a subalgebra of $\mathbf{G}$. We will then denote by G_n the logic given by the finitary consequence relation $\models_{\mathbf{G}_n}$, which is the finite-valued Gödel logic with n different truth-values. For each $n \geq 2$, G_n can be axiomatized by adding the following axiom to $\mathcal{G}$:

$$(\mathrm{FIN}_n) \quad (\varphi_0 \to \varphi_1) \vee (\varphi_1 \to \varphi_2) \vee \ldots \vee (\varphi_{n-2} \to \varphi_{n-1}) \vee (\varphi_{n-1} \to \varphi_n)\,.$$

This shows that for any natural number $n \geq 2$, G_n is an axiomatic extension of IL and thus an intermediate logic between IL and CL. In fact it is even true that $\mathsf{G}_2 = \mathsf{CL}$, and we get the following inclusions:

$$\mathsf{IL} \subsetneq \mathsf{G} \subsetneq \ldots \subsetneq \mathsf{G}_n \subsetneq \ldots \subsetneq \mathsf{G}_4 \subsetneq \mathsf{G}_3 \subsetneq \mathsf{G}_2 = \mathsf{CL}\,.$$

Summing up, we can see that Gödel logic G has many nice properties which make it a logic well worth studying in detail:

- G is strongly complete with respect to the standard Gödel algebra **G**, even when infinite sets of premises are considered (in contrast to BL, Ł, and P)
- G is the logic of the minimum t-norm, one of the three fundamental continuous t-norms, which together, as an ordinal sum, compose every continuous t-norm,
- G validates the (classical) deduction theorem (in contrast to MTL, BL, Ł, and P),
- G is an order-based logic (in contrast to MTL, BL, Ł, and P),
- G is an intermediate logic, i.e. IL $\subsetneq$ G $\subsetneq$ CL (in contrast to MTL, BL, Ł, and P).

2.3 Adding Modal Operators

While in the last sections we looked at axiomatic extensions of MTL in the algebraic language $\mathfrak{L} = \{\wedge, \vee, \&, \to, \top, \bot\}$, the symbols of which we call *propositional connectives*, we will in this section study these logics under the expansion by two further unary connectives, the well-known modal operators box $\Box$ and diamond $\Diamond$, also called *modal connectives*. The box-connective is often understood as expressing the *necessity* of the formula it precedes, or that the succeeding formula is *known* or *provable*, while the diamond-connective is often taken to express *possibility* or that the formula succeeding it is *consistent with one's knowledge*.

There are different ways to specify many-valued modal logics. In this work, we choose a semantic approach, which in the present case is either through Kripke-style semantics, as studied by Saul Kripke and others for the classical case (see [80, 84–87]), or through neighbourhood semantics, introduced in the classical setting independently by Dana Scott in [115] and Richard Montague in [97].[7] In this section, we will only introduce the Kripke-style semantics for the many-valued setting, reserving the introduction of many-valued neighbourhood semantics for Chapter 7, where we will present some new results in that area.

Given an MTL-algebra **A**, for the Kripke-style semantics over **A**, we have a set of "(possible) worlds", an "accessibility relation" on this set, and at each "world", propositional connectives are interpreted by operations in **A**. That is, for each "world" x, there is a different **A**-evaluation $v_x\colon \mathrm{Var} \to A$, which is extended for the propositional connectives by operations in **A**, while the modal connectives $\Box$ and $\Diamond$ are understood as infimum and supremum in **A**, respectively, over all "worlds" which are "accessible" to some degree.

[7]Two excellent reference books for classical modal logics are [15] and [32]

One of the first to take this approach of defining many-valued modal logics over Kripke frames where locally at each world, formulas are interpreted in some algebra was Melvin Fitting in [57] and [58], where he considered Kripke models over finite Heyting algebras. The approach was further developed systematically by Felix Bou, Francesc Esteva, Lluís Godo, and Ricardo Rodríguez in [21], where classes of Kripke models over (finite) residuated lattices are studied. Kripke models over more specific algebras were subsequently studied by many authors, including most notably the standard Ł-, P-, and G-algebra. This line of research on many-valued modal logics will be roughly outlined in the present section.

Frames and Models

Fixing $\mathfrak{L} = \{\wedge, \vee, \&, \to, \bot, \top\}$, we consider the language $\mathfrak{L}_\mathsf{m} = \mathfrak{L} \cup \{\Box, \Diamond\}$ with the two additional unary *modal connectives* $\Box$ and $\Diamond$. The set of formulas $\mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}$ is then defined inductively for the language $\mathfrak{L}_\mathsf{m}$ over the countably infinite set Var, its members denoted by $\varphi, \psi, \chi \dots$. We will call formulas of the form $\Box\varphi$ and $\Diamond\varphi$ *box-formulas* and *diamond-formulas*, respectively. Furthermore, we let $\mathrm{Fm}_\Box$ denote the set of formulas for the language $\mathfrak{L} \cup \{\Box\}$ and similarly $\mathrm{Fm}_\Diamond$ denotes the set of formulas for $\mathfrak{L} \cup \{\Diamond\}$. The *box-fragment* or *diamond-fragment* of a many-valued modal logic L, denoted by $\mathsf{L}_\Box$ or $\mathsf{L}_\Diamond$, is L restricted to $\mathrm{Fm}_\Box$ or $\mathrm{Fm}_\Diamond$, respectively. When the language is clear from the context, we will write Fm instead of $\mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}$.

An algebra $\mathbf{A} \in \mathbb{MTL}$ is called *complete* if $\bigvee B$ and $\bigwedge B$ exist in A, for any (infinite) subset $B \subseteq A$. For the remainder of this chapter, we will restrict our discussion to complete MTL-algebras, and thus, unless stated otherwise, $\mathbf{A}$ will denote a complete MTL-algebra.

Remark 2.25. Note that the restriction to complete algebras is not so much a choice as a necessity, as our semantic definitions below would not make much sense without it (we would have formulas with undefined truth-values). This restriction follows some of the literature, while other authors restrict to so-called safe models, which might yield different logics in certain cases.

We define an $\mathbf{A}$-*frame* as a pair $\langle W, R\rangle$ such that W is a non-empty set of *worlds* and $R\colon W \times W \to A$ is an $\mathbf{A}$-*accessibility relation* on W. An $\mathbf{A}$-frame $\langle W, R\rangle$ is called

- *crisp*, if $Rxy \in \{\bot, \top\}$, for all $x, y \in W$,
- *reflexive*, if $Rxx = \top$, for all $x \in W$,
- *transitive*, if $Rxy \,\&\, Ryz \leq Rxz$, for all $x, y, z \in W$, and
- *symmetric*, if $Rxy = Ryx$, for all $x, y \in W$.

If $\langle W, R\rangle$ is a crisp $\mathbf{A}$-frame, we will also say that R is *crisp* and often write $R \subseteq W \times W$ and Rxy to mean $Rxy = \top$. Note that what we call a crisp $\mathbf{A}$-frame is exactly what is usually called a Kripke frame in the literature on classical modal logics (see e.g. [15]). We will denote by

- $\mathbb{K}$, the class of all $\mathbf{A}$-frames,
- $\mathbb{T}$, the class of all reflexive $\mathbf{A}$-frames,
- $\mathbb{S}\mathbf{4}$, the class of all reflexive and transitive $\mathbf{A}$-frames,
- $\mathbb{S}\mathbf{5}$, the class of all reflexive, transitive, and symmetric $\mathbf{A}$-frames, and
- given a class of $\mathbf{A}$-frames $\mathbb{F}$, $\mathbb{F}^{\mathsf{c}}$ will denote the subclass of its crisp members.

An $\mathbf{A}$-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$ such that $\langle W, R\rangle$ is an $\mathbf{A}$-frame and $V\colon \mathrm{Var} \times W \to A$ is a mapping, called an $\mathbf{A}$-*valuation*. V will be extended to $V\colon \mathrm{Fm} \times W \to A$ such that locally at each world $x \in W$, V acts as an $\mathbf{A}$-evaluation v_x for propositional formulas and interprets $\Box$ and $\Diamond$ as infimum and supremum over all worlds which are accessible from x to some degree, that is, for all $x \in W$:

$$
\begin{aligned}
V(\bot, x) &= \bot\,, \\
V(\top, x) &= \top\,, \\
V(\varphi \wedge \psi, x) &= V(\varphi, x) \wedge V(\psi, x)\,, \\
V(\varphi \vee \psi, x) &= V(\varphi, x) \vee V(\psi, x)\,, \\
V(\varphi \mathbin{\&} \psi, x) &= V(\varphi, x) \mathbin{\&} V(\psi, x)\,, \\
V(\varphi \to \psi, x) &= V(\varphi, x) \to V(\psi, x)\,, \\
V(\Box\varphi, x) &= \bigwedge\{Rxy \to V(\varphi, y) \mid y \in W\}\,, \\
V(\Diamond\varphi, x) &= \bigvee\{Rxy \mathbin{\&} V(\varphi, y)) \mid y \in W\}\,.
\end{aligned}
$$

For a class of $\mathbf{A}$-frames $\mathbb{F}$, we will say that an $\mathbf{A}$-model $\langle W, R, V\rangle$ is an $\mathbb{F}(\mathbf{A})$-*model*, if it is based on a $\mathbf{A}$-frame $\langle W, R\rangle \in \mathbb{F}$ and we will denote the class of all $\mathbb{F}(\mathbf{A})$-models by $\mathbb{F}(\mathbf{A})$.

Recall that a $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-*model* $\langle W, R, V\rangle$ satisfies the extra condition that $\langle W, R\rangle$ is a crisp $\mathbf{A}$-frame. In this case, the conditions for $\Box$ and $\Diamond$ simplify to

$$
\begin{aligned}
V(\Box\varphi, x) &= \bigwedge\{V(\varphi, y) \mid Rxy\}\,, \\
V(\Diamond\varphi, x) &= \bigvee\{V(\varphi, y) \mid Rxy\}\,.
\end{aligned}
$$

Let $\mathbb{F}$ be a class of $\mathbf{A}$-frames, $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ a set of formulas, and $\mathfrak{M} = \langle W, R, V\rangle$ an $\mathbb{F}(\mathbf{A})$-model. The formula φ is called an $\mathfrak{M}$-*consequence* of Γ, if $V(\varphi, x) = \top$ for all

$x \in W$ such that $V[\Gamma, x] = \{V(\psi, x) \mid \psi \in \Gamma\} \subseteq \{\top\}$. Moreover, φ will be called *valid in* $\mathfrak{M}$, if $V(\varphi, x) = \top$ for all $x \in W$, written $\mathfrak{M} \models_{\mathbb{F}(\mathbf{A})} \varphi$. We will also write $\mathfrak{M} \models_{\mathbb{F}(\mathbf{A})} \Gamma$, if $\mathfrak{M} \models_{\mathbb{F}(\mathbf{A})} \psi$, for all $\psi \in \Gamma$. We will say that φ is a *local* $\mathbb{F}(\mathbf{A})$*-consequence* of Γ, written $\Gamma \models^{l}_{\mathbb{F}(\mathbf{A})} \varphi$, if φ is an $\mathfrak{M}$-consequence of Γ, for all $\mathbb{F}(\mathbf{A})$-models $\mathfrak{M}$, and φ will be called a *global* $\mathbb{F}(\mathbf{A})$*-consequence* of Γ, written $\Gamma \models^{g}_{\mathbb{F}(\mathbf{A})} \varphi$, if $\mathfrak{M} \models_{\mathbb{F}(\mathbf{A})} \varphi$, for all $\mathbb{F}(\mathbf{A})$-models such that $\mathfrak{M} \models_{\mathbb{F}(\mathbf{A})} \Gamma$.

It is quite obvious that for any $\varphi \in \mathrm{Fm}$ and any class of $\mathbf{A}$-frames $\mathbb{F}$, $\emptyset \models^{l}_{\mathbb{F}(\mathbf{A})} \varphi$ iff $\emptyset \models^{g}_{\mathbb{F}(\mathbf{A})} \varphi$, in which case we will say that φ is $\mathbb{F}(\mathbf{A})$*-valid*, written $\models_{\mathbb{F}(\mathbf{A})} \varphi$. As we will almost exclusively deal with global consequence in the present work, we will often write $\Gamma \models_{\mathbb{F}(\mathbf{A})} \varphi$ to mean $\Gamma \models^{g}_{\mathbb{F}(\mathbf{A})} \varphi$.

We will denote by $\mathsf{K}(\mathbf{A})$ the logic defined by setting $\Vdash_{\mathsf{K}(\mathbf{A})} = \models_{\mathbb{K}(\mathbf{A})}$ and the set of all $\mathbb{K}(\mathbf{A})$-valid formulas by $\mathsf{Val}(\mathsf{K}(\mathbf{A})) = \{\varphi \in \mathrm{Fm} \mid \models_{\mathbb{K}(\mathbf{A})} \varphi\}$. As the logic $\mathsf{K}(\mathbf{A})$ is obtained by considering all $\mathbf{A}$-frames, it is the weakest modal logic based on $\mathbf{A}$-frames, also called in [21] the *minimum many-valued modal logic* over $\mathbf{A}$. Furthermore, let $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$, $\mathsf{T}(\mathbf{A})$, $\mathsf{T}^{\mathsf{c}}(\mathbf{A})$, $\mathsf{S4}(\mathbf{A})$, $\mathsf{S4}^{\mathsf{c}}(\mathbf{A})$, $\mathsf{S5}(\mathbf{A})$, and $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ denote the stronger modal logics over $\mathbf{A}$ defined by the consequence relations $\models_{\mathbb{K}^{\mathsf{c}}(\mathbf{A})}$, $\models_{\mathbb{T}(\mathbf{A})}$, $\models_{\mathbb{T}^{\mathsf{c}}(\mathbf{A})}$, $\models_{\mathbb{S}\mathbf{4}(\mathbf{A})}$, $\models_{\mathbb{S}\mathbf{4}^{\mathsf{c}}(\mathbf{A})}$, $\models_{\mathbb{S}\mathbf{5}(\mathbf{A})}$, and $\models_{\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})}$, respectively.

Given a class of $\mathbf{A}$-frames $\mathbb{F}$, an $\mathbb{F}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$ is called *finite* or *countable*, if its set of worlds W is finite or countable, respectively. Furthermore, for a logic L for $\mathfrak{L}_{\mathsf{m}}$, we will say that L has the *finite model property with respect to* $\mathbb{F}(\mathbf{A})$, if for all $\varphi \in \mathrm{Fm}$, φ is valid in L if and only if φ is valid in all finite $\mathbb{F}(\mathbf{A})$-models.

Remark 2.26. Let us point out a difficulty that arises with the increase of expressive power by expanding with modal connectives. For two algebras $\mathbf{A}, \mathbf{B} \in \mathbb{MTL}$ determining the same valid formulas on the propositional level, i.e. for any non-modal formula $\varphi \in \mathrm{Fm}_{\mathfrak{L}}$, $\models_{\mathbf{A}} \varphi$ iff $\models_{\mathbf{B}} \varphi$, we might obtain two different modal logics, i.e. for two modal formulas $\varphi, \psi \in \mathrm{Fm}_{\mathfrak{L}_{\mathsf{m}}}$: $\models_{\mathbb{K}(\mathbf{A})} \psi$ but $\not\models_{\mathbb{K}(\mathbf{B})} \psi$. This difficulty was pointed out in [22] and will become clear when we study different modal Gödel logics over different subalgebras of the standard Gödel algebra $\mathbf{G}$ in Section 3.2.

Remark 2.27. Another difficulty is that in the absence of involutive negation, the box and diamond connectives are not interdefinable (in contrast to the classical case). This is why both connectives are added as primitive in the present setting. Because of this difficulty, some of the discussions of many-valued modal logics in the literature consider the box- and diamond-fragments separately.

Modal Logics over MTL-Algebras

Let us present some known results about many-valued modal logics based on $\mathbb{K}(\mathbf{A})$-models over complete MTL-algebras $\mathbf{A}$. One of the most relevant studies of these logics is [21], where the minimum modal logics over complete bounded pointed commutative residuated lattices are considered (possibly with additional constants). For example, axiomatizations are presented for the box-fragments of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ for finite MTL-algebras $\mathbf{A}$ (and other algebras) with additional canonical constants, for the crisp logic, only in the case where $\mathbf{A}$ has a unique coatom. Even though these results are very interesting, they cannot easily be generalized to infinite MTL-algebras or MTL-algebras without canonical constants or a unique coatom. As the present work focuses mainly on infinite MTL-algebras, we will not present these results in detail here.

Nevertheless, let us highlight the following interesting $\mathsf{K}(\mathbf{A})$- and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$-validities, for any complete MTL-algebra $\mathbf{A}$, presented in [21]. Obviously, they are then also valid in every (crisp) many-valued modal logic over a complete BL-, $\mathsf{Ł}$-, P-, or G-algebra.

Proposition 2.28 ([21])**.** *Let $\mathbf{A}$ be a complete MTL-algebra and $\varphi, \psi \in \mathrm{Fm}$, then the following formulas are valid in $\mathsf{K}(\mathbf{A})$:*

- $(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$
- $\neg\neg\Box\varphi \to \Box\neg\neg\varphi$

Additionally, the following formulas are valid in $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$:

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $(\Box\varphi \mathbin{\&} \Box\psi) \to \Box(\varphi \mathbin{\&} \psi)$

This shows that the normality axiom (K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is valid in all crisp many-valued modal logics considered in this work. However, this is not true for some of their non-crisp versions.

Furthermore, in [68], Hájek established decidability of the validity and satisfiability problem for fuzzy description logics based on continuous t-norms with respect to *witnessed* interpretations, i.e. where each supremum or infimum is actually a maximum or minimum, respectively. As we can view many-valued modal logics as fragments of their description logic counterparts, these results imply the decidability of the validity and satisfiability problem for the logics determined by all witnessed $\mathbb{K}(\mathbf{A})$-models over standard BL-algebras $\mathbf{A}$. Note, however, that in general, $\mathsf{K}(\mathbf{A})$ is not complete with respect to witnessed $\mathbb{K}(\mathbf{A})$-models (except, for example, when $\mathbf{A}$ is $\mathbf{Ł}$ (see below)).

We now turn our attention to many-valued modal logics based on $\mathbb{K}(\mathbf{A})$-models where $\mathbf{A}$ is the standard Łukasiewicz, product, or Gödel algebra, respectively (or finite subalgebras thereof).

Łukasiewicz Modal Logics

The most extensive study of Łukasiewicz modal logics is [75] by Georges Hansoul and Bruno Teheux. They use methods from abstract algebraic logic to study Łukasiewicz modal logics based on crisp $\mathbb{K}(\mathbf{A})$-models, where $\mathbf{A}$ is $\mathbf{Ł}$, the standard Łukasiewicz algebra, or a finite subalgebra $\mathbf{Ł}_n$ of $\mathbf{Ł}$ with n evenly spaced elements. We will not go into detail on their results here, as even just presenting them would require an extensive elaboration of their algebraic framework. We will mention, however, that they are able to axiomatize the logic $\mathsf{K}^{\mathsf{c}}(\mathbf{Ł})$ in some sense, namely by using an infinitary rule, which will render some of the proofs in the calculus infinitely long. Furthermore, for any natural number $n \geq 2$, they axiomatize the logic $\mathsf{K}^{\mathsf{c}}(\mathbf{Ł}_n)$ by a Hilbert-style calculus without any infinitary rules.

Furthermore, we note that while the Gödel modal logic $\mathsf{K}(\mathbf{G})$ and the product modal logic $\mathsf{K}(\mathbf{P})$ are not complete with respect to witnessed Kripke models, the Łukasiewicz modal logic $\mathsf{K}(\mathbf{Ł})$ is. This fact can easily be inferred from Hájek's result that first-order Łukasiewicz logic is complete with respect to witnessed structures [69, 70]. Recalling that Hájek showed the decidability of the validity and satisfiability problem for Łukasiewicz modal logic (and others) based on all witnessed $\mathbb{K}(\mathbf{Ł})$-models, this implies the finite model property and the decidability of the satisfiability and validity problem for $\mathsf{K}(\mathbf{Ł})$.

Moreover, using results about the one-variable fragment of the first-order Łukasiewicz logic, Hájek was able to show in [67] that the validities in the logic $\mathsf{S5}^{\mathsf{c}}(\mathbf{Ł})$ are recursively enumerable, and thus he was able to obtain the decidability of the validity problem for this logic by proving that it has the finite model property. Let us sum up with the following two theorems:

Theorem 2.29 ([67, 68])**.** *$\mathsf{K}(\mathbf{Ł})$ and $\mathsf{S5}^{\mathsf{c}}(\mathbf{Ł})$ have the finite model property with respect to $\mathbb{K}(\mathbf{Ł})$ and $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{Ł})$, respectively.*

Theorem 2.30 ([67, 68])**.** *The validity problems for $\mathsf{K}(\mathbf{Ł})$ and $\mathsf{S5}^{\mathsf{c}}(\mathbf{Ł})$ are decidable.*

Finding finitary axiomatizations of infinite-valued Łukasiewicz modal logics or answering the questions of decidability for Łukasiewicz modal logics based on other classes of $\mathbb{K}(\mathbf{Ł})$-models remain open, however.

Product Modal Logics

The most extensive treatment of product modal logics is by Amanda Vidal, Francesc Esteva, and Lluís Godo in [121], where they provide axiomatization results and study the relationship between the Kripke-style semantics and the algebraic semantics of crisp product modal logics. For their results, however, these authors depend on strong standard completeness for infinite sets of premises, while for P, this is only given for fi-

(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	$(\mathrm{A}_\Box 1)$	$\Box(\overline{c} \to \varphi) \leftrightarrow (\overline{c} \to \Box\varphi)$
$(\mathrm{A}_\Diamond 1)$	$\Box(\varphi \to \overline{c}) \leftrightarrow (\Diamond\varphi \to \overline{c})$	$(\mathrm{A}_\Box 2)$	$\triangle\Box\varphi \leftrightarrow \Box\triangle\varphi$

$$(\mathrm{N}_\Box) \quad \frac{\varphi}{\Box\varphi}$$

Table 2.6: Modal Axioms and Rule for $\mathcal{PK}_l^{\mathsf{c}}$ and $\mathcal{PK}_g^{\mathsf{c}}$

nite sets. For this reason, they add the Delta operator $\triangle$ (also known as the Baaz-Monteiro operator, discussed in more detail in Chapter 3) to the algebraic language, along with a constant symbol for each rational number in $[0,1]$. In this way, for each $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top\rangle \in \mathbb{P}$, they obtain the expansion $\mathbf{A}^\infty = \langle A, \wedge, \vee, \&, \to, \triangle, \bot, \top, \{c \mid c \in [0,1] \cap \mathbb{Q}\}\rangle$ and set $\mathbb{P}^\infty = \{\mathbf{A}^\infty \mid \mathbf{A} \in \mathbb{P}\}$. Furthermore, they extend the calculus $\mathcal{P}$ with suitable axioms to deal with $\triangle$ and the extra constants as well as two infinitary rules (i.e. rules with infinite sets of premises) and denote this calculus by $\mathcal{P}^\infty$. Denoting by $\mathbf{P}^\infty$ the standard P-algebra $\mathbf{P} = \langle [0,1], \min, \max, *_{\mathsf{P}}, \to_{\mathsf{P}}, 0, 1\rangle$ expanded with $\triangle$ and a constant for any element of $[0,1] \cap \mathbb{Q}$, (infinite) strong standard completeness is achieved (a result proved by Petr Cintula in his PhD thesis [36]), that is, for any set of non-modal formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,

$$\Gamma \vdash_{\mathcal{P}^\infty} \varphi \qquad \text{iff} \qquad \Gamma \models_{\mathbb{P}^\infty} \varphi \qquad \text{iff} \qquad \Gamma \models_{\mathbf{P}^\infty} \varphi\,.$$

Slightly abusing notation, for an algebra $\mathbf{A} \in \mathbb{P}^\infty$, certain classes of $\mathbf{A}$-frames are denoted by $\mathbb{F}$ and the class of corresponding $\mathbf{A}$-models by $\mathbb{F}(\mathbf{A})$.

Subsequently, strongly complete infinitary axiomatizations (relying on infinite proofs) are provided in [121] for the local and global consequence relations $\models^l_{\mathbb{K}(\mathbf{P}^\infty)}$ and $\models^g_{\mathbb{K}(\mathbf{P}^\infty)}$. These axiomatizations are denoted by $\mathcal{PK}_l^{\mathsf{c}}$ and $\mathcal{PK}_g^{\mathsf{c}}$, respectively, and consist of adding to $\mathcal{P}^\infty$ the axioms and rule in Table 2.6, while for $\mathcal{PK}_l^{\mathsf{c}}$, the rule $(\mathrm{N}_\Box)$ can only be applied to theorems.

Theorem 2.31 (Strong Standard Completeness for $\mathcal{PK}_l^{\mathsf{c}}$ and $\mathcal{PK}_g^{\mathsf{c}}$ [121])**.** *For any set of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*,*

$$\Gamma \vdash_{\mathcal{PK}_l^{\mathsf{c}}} \varphi \quad \textit{iff} \quad \Gamma \models^l_{\mathbb{K}^{\mathsf{c}}(\mathbf{P}^\infty)} \varphi \qquad \textit{and} \qquad \Gamma \vdash_{\mathcal{PK}_g^{\mathsf{c}}} \varphi \quad \textit{iff} \quad \Gamma \models^g_{\mathbb{K}^{\mathsf{c}}(\mathbf{P}^\infty)} \varphi\,.$$

In fact, Theorem 2.31 is extended in [121] such that the left equivalence is also true for the subclasses of $\mathbb{T}^{\mathsf{c}}(\mathbf{P}^\infty)$-, $\mathbb{S}\mathbf{4}^{\mathsf{c}}(\mathbf{P}^\infty)$-, or $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{P}^\infty)$-models, if the axioms $(\mathrm{T}_\Box)$ $\Box\varphi \to \varphi$, or $(4_\Box)$ $\Box\varphi \to \Box\Box\varphi$ and $(\mathrm{T}_\Box)$, or (5) $\Diamond\Box\varphi \to \Box\varphi$ and $(4_\Box)$ and $(\mathrm{T}_\Box)$ are added to $\mathcal{PK}_l^{\mathsf{c}}$, respectively.

Furthermore, in [31], description logics based on the standard product algebra $\mathbf{P}$ are studied and decidability of the validity problem as well as the problem of positive

$(K_\Box)$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	$(K_\Diamond)$	$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$
$(Z_\Box)$	$\neg\neg\Box\varphi \to \Box\neg\neg\varphi$	$(Z_\Diamond)$	$\Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$
		$(F_\Diamond)$	$\neg\Diamond\bot$
(FS1)	$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	(FS2)	$(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$
$(N_\Box)$	$\dfrac{\varphi}{\Box\varphi}$	$(N_\Diamond)$	$\dfrac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$

Table 2.7: Modal Axioms and Rules of $\mathcal{GK}_\Box$, $\mathcal{GK}_\Diamond$, and $\mathcal{GK}$

satisfiability is established. The proof is based on an EXPTIME reduction to the propositional product logic. As in the case for Łukasiewicz modal logic, this implies the following theorem:

Theorem 2.32 ([31]). *The validity problem and the problem of positive satisfiability for* $\mathsf{K}(\mathbf{P})$ *are decidable.*

About product modal logics based on other classes of $\mathbb{K}(\mathbf{P})$-models (where $\mathbf{P}$ is the standard P-algebra without $\triangle$ or further constants), however, very little is known. Specifically, issues of axiomatization and decidability present many open questions.

Gödel Modal Logics

Because of the nice properties of propositional Gödel logic listed in Section 2.2, the study of Gödel modal logics is more advanced than for the other many-valued modal logics mentioned above. Previous to the results presented in the subsequent chapters of this work, many axiomatizability results, and even some decidability and complexity results were already known for Gödel modal logics.

In [28], Xavier Caicedo and Ricardo Rodríguez axiomatized separately the box- and the diamond-fragments of $\mathsf{X}(\mathbf{G})$, for $\mathsf{X} \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$. Consider the axioms and rules in Table 2.7. When the axioms $(K_\Box)$ and $(Z_\Box)$ and the rule $(N_\Box)$ are added to an axiomatization of G, e.g. $\mathcal{GK}_\Box = \mathcal{G} \cup \{(K_\Box), (Z_\Box), (N_\Box)\}$, a Hilbert-style axiomatization of the box-fragment of $\mathsf{K}(\mathbf{G})$ is obtained. In fact, $\mathcal{GK}_\Box$ also axiomatizes the box-fragment of $\mathsf{K}^c(\mathbf{G})$, as surprisingly, $\mathsf{K}(\mathbf{G})_\Box = \mathsf{K}^c(\mathbf{G})_\Box$.

On the other hand, the Hilbert-style calculus $\mathcal{GK}_\Diamond = \mathcal{G} \cup \{(K_\Diamond), (Z_\Diamond), (F_\Diamond), (N_\Diamond)\}$ axiomatizes the diamond-fragment of $\mathsf{K}(\mathbf{G})$. Moreover, the crisp diamond-fragment $\mathsf{K}^c(\mathbf{G})_\Diamond$ was axiomatized by George Metcalfe and Nicola Olivetti in [95] by adding to the calculus $\mathcal{GK}_\Diamond$ the rule

$$(N^*_\Diamond) \quad \frac{\varphi \vee (\psi \to \chi)}{\Diamond\varphi \vee (\Diamond\psi \to \Diamond\chi)}.$$

In [29], Caicedo and Rodríguez proved that the Hilbert-style calculus $\mathcal{GK}$, built from $\mathcal{G}$ by adding all the axioms and rules in Table 2.7, i.e.

$$\mathcal{GK} \;=\; \mathcal{G} \cup \{(\mathrm{K}_\Box), (\mathrm{Z}_\Box), (\mathrm{K}_\Diamond), (\mathrm{Z}_\Diamond), (\mathrm{F}_\Diamond), (\mathrm{FS1}), (\mathrm{FS2}), (\mathrm{N}_\Box), (\mathrm{N}_\Diamond)\}\,,$$

axiomatizes the full logic $\mathsf{K}(\mathbf{G})$. In fact, $\mathsf{K}(\mathbf{G})$ can also be axiomatized by adding the prelinearity axiom (PRL) $(\varphi \to \psi) \vee (\psi \to \varphi)$ to a Hilbert-style axiomatization of the intuitionistic modal logic IK (see [56]).

Furthermore, Gödel modal logics based on other classes of $\mathbf{G}$-frames are axiomatized by adding combinations of axioms in Table 2.8. Let $\mathcal{GT} = \mathcal{GK} \cup \{(\mathrm{T}_\Box), (\mathrm{T}_\Box)\}$, $\mathcal{GS4} = \mathcal{GT} \cup \{(4_\Box), (4_\Box)\}$, and $\mathcal{GS5} = \mathcal{GS4} \cup \{(\mathrm{B1}), (\mathrm{B2})\}$. We can summarize the above in the following theorem.

$(\mathrm{T}_\Box)$	$\Box\varphi \to \varphi$	$(\mathrm{T}_\Diamond)$	$\varphi \to \Diamond\varphi$
$(4_\Box)$	$\Box\varphi \to \Box\Box\varphi$	$(4_\Diamond)$	$\Diamond\Diamond\varphi \to \Diamond\varphi$
(B1)	$\varphi \to \Box\Diamond\varphi$	(B2)	$\Diamond\Box\varphi \to \varphi\,.$

Table 2.8: Further Modal Axioms

Theorem 2.33 (Strong Standard Completeness of GK and axiomatic extensions [29]). *For any set of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*,*

(a) $\Gamma \vdash_{\mathcal{GK}} \varphi$ *iff* $\Gamma \models_{\mathbb{K}(\mathbf{G})} \varphi$,

(b) $\Gamma \vdash_{\mathcal{GT}} \varphi$ *iff* $\Gamma \models_{\mathbb{T}(\mathbf{G})} \varphi$,

(c) $\Gamma \vdash_{\mathcal{GS4}} \varphi$ *iff* $\Gamma \models_{\mathbb{S}\mathbf{4}(\mathbf{G})} \varphi$,

(d) $\Gamma \vdash_{\mathcal{GS5}} \varphi$ *iff* $\Gamma \models_{\mathbb{S}\mathbf{5}(\mathbf{G})} \varphi$.

We will denote these logics, as is usually done in the literature, by GK, GT, $\mathsf{GS4}$, and $\mathsf{GS5}$, respectively, and their crisp counterparts by GK^c, GT^c, $\mathsf{GS4}^\mathsf{c}$, and $\mathsf{GS5}^\mathsf{c}$.

Decidability of the validity problem for the diamond-fragment $\mathsf{GX}_\Diamond$, for $\mathsf{X} \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$, was established in [28] by proving that it has the finite model property with respect to the appropriate subclass of $\mathbb{K}(\mathbf{G})$. However, neither $\mathsf{GK}^\mathsf{c}_\Diamond$ nor $\mathsf{GK}_\Box = \mathsf{GK}^\mathsf{c}_\Box$ has the finite model property with respect to $\mathbb{K}^\mathsf{c}(\mathbf{G})$. This failure of the finite model property is established by showing that the formulas $(\Diamond p \to \Diamond q) \to ((\Diamond p \to \bot) \vee \Diamond(p \to q)) \in \mathrm{Fm}_\Diamond$ (see [95]) and $\Box\neg\neg p \to \neg\neg\Box p \in \mathrm{Fm}_\Box$ (see [28]) are both valid in all finite $\mathbb{K}^\mathsf{c}(\mathbf{G})$-models, but not in some infinite $\mathbb{K}^\mathsf{c}(\mathbf{G})$-model (each formula in a different one). As the infinite models where these formulas fail are both in $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{G})$, the failure of the finite model property with respect to appropriate subclasses of $\mathbb{K}(\mathbf{G})$ extends to the Gödel modal logics $\mathsf{GX}^\mathsf{c}_\Diamond$, $\mathsf{GX}_\Box$, and $\mathsf{GX}^\mathsf{c}_\Box$, for $\mathsf{X} \in \{\mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$. Nevertheless, despite this failure of the finite model property, Metcalfe and Olivetti were able to establish decidability, indeed PSPACE-completeness, of the validity problem for $\mathsf{GK}^{\mathsf{c}}_{\Diamond}$ and $\mathsf{GK}_{\Box}$ in [95], using analytic Gentzen-style proof systems.

Obviously, as both $(\Diamond p \to \Diamond q) \to ((\Diamond p \to \bot) \vee \Diamond(p \to q))$ and $\Box\neg\neg p \to \neg\neg\Box p$ are formulas in Fm, also the full logics GK and GK^{c} do not have the finite model property with respect to $\mathbb{K}(\mathbf{G})$ and $\mathbb{K}^{\mathsf{c}}(\mathbf{G})$, respectively, and neither does GX, for $\mathsf{X} \in \{\mathsf{T}, \mathsf{T}^{\mathsf{c}}, \mathsf{S4}, \mathsf{S4}^{\mathsf{c}}, \mathsf{S5}, \mathsf{S5}^{\mathsf{c}}\}$, with respect to the appropriate subclasses of $\mathbb{K}(\mathbf{G})$. Furthermore, devising analytic proof calculi for Gödel modal logics with both modal connectives $\Box$ and $\Diamond$ seems to be very challenging (cf. [95]). For these reasons, decidability results for these Gödel modal logics have remained open.

As one of our main results, we establish decidability and PSPACE-completeness of the validity problems for GK and GK^{c} in Chapter 4. Furthermore, co-NP-completeness is proved for the validity problem for the Gödel modal logic $\mathsf{GS5}^{\mathsf{c}}$ in Chapter 5, which coincides with the one-variable fragment of first-order Gödel logic (see [67]). In fact, these results are generalized to the logics $\mathsf{K}(\mathbf{A})$, $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$, and $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ based on other complete G-chains $\mathbf{A}$. We note that these results are based on joint work of the author of this dissertation with Xavier Caicedo, George Metcalfe, and Ricardo Rodríguez in [26, 27].

Chapter 3

Order-Based Modal Logics

In this chapter, we introduce order-based logics and their modal counterparts. An order-based logic is defined as the consequence relation determined by an "order-based" algebras, which in turn is an algebra consisting of a complete sublattice of $\langle[0,1], \min, \max, 0, 1\rangle$ with additional operations defined based only on the order. Order-based modal logics will be defined in the same way as in Section 2.3 for the many-valued logics based on MTL-algebras, interpreting the box $\Box$ and the diamond $\Diamond$ as infimum and supremum over (to some degree) accessible worlds, respectively.

First we will define order-based logics and order-based modal logics in detail, presenting a Kripke-style semantics as in Section 2.3. We then recall that Gödel logic G is a significant example of an order-based logic and thus the logics GK and its extensions are order-based modal logics. We will then observe that, on the one hand, the defined order-based modal logics, like many classical modal logics, enjoy the bounded tree-model property, that is, a formula is valid in the logic if and only if it is valid in all tree-models of height bounded by a function of the length of the formula. On the other hand, order-based modal logics do not in many cases enjoy the finite model property with respect to the Kripke models introduced in the present chapter. Moreover, as perhaps the most characteristic property of order-based modal logics, we will show that validity in a Kripke model is preserved even when values of propositional variables and the accessibility relation are "moved about" in the algebra, as long as the order is preserved. This last fact will be used heavily in Chapter 4.

Unless stated otherwise, all of the results in this chapter originate from joint work of the author of this thesis with Xavier Caicedo, George Metcalfe, and Ricardo Rodríguez [26, 27].

3.1 Order-Based Propositional Logics

Before we can define order-based logics and their modal counterparts, we will need to say what we mean by operations being "defined based only on the order".

We reserve the symbols $\Rightarrow$, $\curlywedge$, $\sim$, and $\approx$ to denote implication, conjunction, negation, and equality, respectively, in classical first-order logic. We also recall an appropriate notion of first-order definability of operations for algebraic structures. Let $\mathfrak{L}$ be an algebraic language, $\mathbf{A}$ an algebra for $\mathfrak{L}$, and $\mathfrak{L}'$ a sublanguage of $\mathfrak{L}$. An operation $f\colon A^n \to A$ is *defined* in $\mathbf{A}$ by a first-order $\mathfrak{L}'$-formula $F(x_1, \ldots, x_n, y)$ with free variables $x_1, \ldots, x_n, y$ if for all $a_1, \ldots, a_n, b \in A$,

$$\mathbf{A} \models F(a_1, \ldots, a_n, b) \qquad \text{iff} \qquad f(a_1, \ldots, a_n) = b\,.$$

From now on, let $\mathfrak{L}$ be any finite algebraic language that includes the binary operation symbols $\wedge$ and $\vee$ and constant symbols $\bot$ and $\top$ (to be interpreted by the usual lattice operations), and denote the finite set of constants (nullary operation symbols) of this language by $\mathrm{C}_{\mathfrak{L}}$.

Remark 3.1. For convenience, we consider only finite algebraic languages, noting that to decide the validity of a formula we may in any case restrict to the language containing only operation symbols occurring in that formula.

An algebra $\mathbf{A}$ in language $\mathfrak{L}$ will be called *order-based* if it satisfies the following conditions:

- $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ is a complete sublattice of $\langle [0,1], \min, \max, 0, 1\rangle$; that is, $\{0,1\} \subseteq A \subseteq [0,1]$ and for all $B \subseteq A$, $\bigwedge^{[0,1]} B$ and $\bigvee^{[0,1]} B$ belong to A.
- For each operation symbol $\star$ of $\mathfrak{L}$, the operation $\star^{\mathbf{A}}$ is definable in $\mathbf{A}$ by a quantifier-free first-order formula in the algebraic language consisting of $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathfrak{L}}$.

We also let $\mathrm{C}_{\mathfrak{L}}^{\mathbf{A}}$ denote the finite set of constant operations $\{c^{\mathbf{A}} \mid c \in \mathrm{C}_{\mathfrak{L}}\}$ and define $R(\mathbf{A})$ and $L(\mathbf{A})$ to be the sets of *right* and *left accumulation points*, respectively, of $\mathbf{A}$ in the usual topology inherited from $[0,1]$; that is,

$a \in R(\mathbf{A})$	iff	there is a $c \in A$ such that $a <^{\mathbf{A}} c$ and for all such c, there is an $e \in A$ such that $a <^{\mathbf{A}} e <^{\mathbf{A}} c$.
$b \in L(\mathbf{A})$	iff	there is a $d \in A$ such that $d <^{\mathbf{A}} b$, and for all such d, there is an $f \in A$ such that $d <^{\mathbf{A}} f <^{\mathbf{A}} b$.

Note that, because $\mathbf{A}$ is a complete chain, an implication operation $\to^{\mathbf{A}}$ may always be introduced as the residual of $\wedge^{\mathbf{A}}$:

$$a \to^{\mathbf{A}} b \;=\; \bigvee\nolimits^{\mathbf{A}} \{c \in A \mid c \wedge^{\mathbf{A}} a \leq^{\mathbf{A}} b\} \;=\; \begin{cases} 1 & \text{if } a \leq^{\mathbf{A}} b\,, \\ b & \text{otherwise.} \end{cases}$$

Let $s \leq t$ stand for $s \wedge t \approx s$ and let $s < t$ stand for $(s \leq t) \curlywedge {\sim}(s \approx t)$. Then the implication operation $\rightarrow^{\mathbf{A}}$ is definable in $\mathbf{A}$ by the quantifier-free first-order formula

$$F^{\rightarrow}(x, y, z) \;=\; ((x \leq y) \Rightarrow (z \approx \top)) \curlywedge ((y < x) \Rightarrow (z \approx y))\,.$$

That is, for all $a, b, c \in A$,

$$\mathbf{A} \models F^{\rightarrow}(a, b, c) \qquad \text{iff} \qquad a \rightarrow^{\mathbf{A}} b = c\,.$$

Notice that the operation $\rightarrow^{\mathbf{A}}$ defined here is exactly the Gödel implication $\rightarrow_{\mathsf{G}}$ presented in Chapter 2. In this chapter, the symbol $\rightarrow$ will always be interpreted by $\rightarrow^{\mathbf{A}} = \rightarrow_{\mathsf{G}}$ in $\mathbf{A}$. As in the last chapter, will also make use of the negation connective $\neg$, defined by $\neg\varphi = \varphi \rightarrow \bot$, which is interpreted by the unary operation

$$\neg^{\mathbf{A}} a \;=\; \begin{cases} 1 & \text{if } a = 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

Examples of other useful operations (see e.g. [8]) covered by the order-based approach are the delta and nabla operators

$$\triangle^{\mathbf{A}} a \;=\; \begin{cases} 1 & \text{if } a = 1\,, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \triangledown^{\mathbf{A}} a \;=\; \begin{cases} 0 & \text{if } a = 0\,, \\ 1 & \text{otherwise,} \end{cases}$$

definable in $\mathbf{A}$ (noting also that $\triangledown^{\mathbf{A}} a = \neg^{\mathbf{A}}\neg^{\mathbf{A}} a$), by

$$\begin{aligned} F^{\triangle}(x, y) &= ((x \approx \top) \Rightarrow (y \approx \top)) \curlywedge ((x < \top) \Rightarrow (y \approx \bot))\,, \\ F^{\triangledown}(x, y) &= ((x \approx \bot) \Rightarrow (y \approx \bot)) \curlywedge ((\bot < x) \Rightarrow (y \approx \top))\,, \end{aligned}$$

and the dual-implication connective (the residual of $\vee^{\mathbf{A}}$)

$$a \leftarrow^{\mathbf{A}} b \;=\; \bigwedge^{\mathbf{A}} \{c \in A \mid b \leq^{\mathbf{A}} a \vee^{\mathbf{A}} c\} \;=\; \begin{cases} 0 & \text{if } b \leq^{\mathbf{A}} a\,, \\ b & \text{otherwise,} \end{cases}$$

definable in $\mathbf{A}$ by

$$F^{\leftarrow}(x, y, z) \;=\; ((y \leq x) \Rightarrow (z \approx \bot)) \curlywedge ((x < y) \Rightarrow (z \approx y))\,.$$

Remark 3.2. Note that for any n-ary operation $\star^{\mathbf{A}}$ defined in this way, it is the case that for all $a_1, \ldots, a_n \in A$, $\star^{\mathbf{A}}(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\} \cup \mathrm{C}^{\mathbf{A}}_{\mathfrak{L}}$.

For the remainder of this chapter, let us fix a finite language $\mathfrak{L}$ including the operational symbols $\top$, $\bot$, $\wedge$, $\vee$, and $\rightarrow$, and an order-based algebra $\mathbf{A}$ for $\mathfrak{L}$. The symbols $\wedge$, $\vee$, $\bot$, and $\top$ in $\mathfrak{L}$ will always be interpreted by the usual lattice operations on $[0, 1]$ (i.e. $\wedge^{\mathbf{A}}$

and $\vee^{\mathbf{A}}$ are minimum and maximum, respectively, and $\bot^{\mathbf{A}} = 0$ and $\top^{\mathbf{A}} = 1$) and $\to$ is interpreted by the Gödel implication (i.e. $\to^{\mathbf{A}} = \to_{\mathsf{G}}$). Furthermore, we will omit the superscript $\mathbf{A}$'s when the algebra or order is clear from the context.

Let $\mathrm{Fm}_{\mathfrak{L}}$ be the set of (propositional) formulas for the algebraic language $\mathfrak{L}$ defined inductively over a countably infinite set of propositional variables Var. Recall that an $\mathbf{A}$-evaluation e is a mapping $e\colon \mathrm{Var} \to A$ that is extended to $e\colon \mathrm{Fm}_{\mathfrak{L}} \to A$, as follows:

$$e(\star(\varphi_1,\ldots,\varphi_n)) \;=\; \star(e(\varphi_1),\ldots,e(\varphi_n))\,,$$

for each n-ary operation symbol $\star$ of $\mathfrak{L}$. Recall moreover that for a set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathfrak{L}}$, $\Gamma \models_{\mathbf{A}} \varphi$, if $e(\varphi) = 1$ for all $\mathbf{A}$-evaluations e such that $e[\Gamma] = \{e(\psi) \mid \psi \in \Gamma\} \subseteq \{1\}$, and φ is $\mathbf{A}$-valid, if $\models_{\mathbf{A}} \varphi$. A propositional logic L for the language $\mathfrak{L}$ will be called *order-based*, if $\Vdash_{\mathsf{L}} = \models_{\mathbf{A}}$ for some order-based algebra $\mathbf{A}$. In this case, we will also write $\mathsf{Val}(\mathsf{L})$ for the set of $\mathbf{A}$-validities $\{\varphi \in \mathrm{Fm}_{\mathfrak{L}} \mid \models_{\mathbf{A}} \varphi\}$.

Example 3.3. As mentioned in Section 2.2, it is clear that the standard Gödel algebra $\mathbf{G} = \langle [0,1], \min, \max, \to_{\mathsf{G}}, 0, 1\rangle$ is an order-based algebra and thus Gödel logic G is an order-based logic. Furthermore, any complete subalgebra of $\mathbf{G}$ is an order-based algebra. Specifically, the subalgebras $\mathbf{G}_{\downarrow} = \langle G_{\downarrow}, \wedge, \vee, \to, 0, 1\rangle$ and $\mathbf{G}_{\uparrow} = \langle G_{\uparrow}, \wedge, \vee, \to, 0, 1\rangle$ of $\mathbf{G}$, based on the universes

$$G_{\downarrow} \;=\; \{0\} \cup \{\tfrac{1}{n+1} \mid n \in \mathbb{N}\} \qquad \text{and} \qquad G_{\uparrow} \;=\; \{1 - \tfrac{1}{n+1} \mid n \in \mathbb{N}\} \cup \{1\}\,,$$

are order-based algebras and thus the consequence relations $\models_{\mathbf{G}_{\downarrow}}$ and $\models_{\mathbf{G}_{\uparrow}}$ define order-based logics. Clearly, order-based algebras with universes $G_{\downarrow}$ and $G_{\uparrow}$ are isomorphic to algebras with universes $\{-n \mid n \in \mathbb{N}\} \cup \{-\infty\}$ and $\mathbb{N} \cup \{\infty\}$, respectively.

3.2 Adding Modal Operators

We define order-based modal logics $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})$ based on crisp $\mathbf{A}$-frames and $\mathbf{A}$-frames with an accessibility relation taking values in A, respectively, where $\mathbf{A}$-frames for an order-based algebra $\mathbf{A}$ are defined similarly as for MTL-algebras.

Let us again denote by $\mathrm{Fm}_{\mathfrak{L}_{\mathsf{m}}}$ the set of formulas for the language $\mathfrak{L}_{\mathsf{m}}$, which is $\mathfrak{L}$ with additional unary operation symbols (modal connectives) $\Box$ and $\Diamond$, defined inductively over a countably infinite set Var of propositional variables. *Subformulas* are defined as usual, and the *length* of a formula φ, denoted by $\ell(\varphi)$, is the total number of occurrences of subformulas in φ. We also let $\mathrm{Var}(\varphi)$ denote the set of variables occurring in the formula φ. We will drop the subscript $\mathfrak{L}_{\mathsf{m}}$ if the language is clear from the context.

We define an $\mathbf{A}$-*frame* to be a pair $\langle W, R\rangle$ such that W is a non-empty set of *worlds* and $R\colon W \times W \to A$ is an $\mathbf{A}$-*accessibility relation* on W. If $Rxy \in \{0,1\}$ for all

$x, y \in W$, then R is *crisp* and $\langle W, R\rangle$ is called a *crisp* **A**-*frame*. In this case, we often write $R \subseteq W \times W$ and Rxy to mean $Rxy = 1$.

A $\mathbb{K}(\mathbf{A})$-*model* is defined as a triple $\mathfrak{M} = \langle W, R, V\rangle$ such that $\langle W, R\rangle$ is an **A**-frame and $V\colon \mathrm{Var} \times W \to A$ is a mapping, called an **A**-*valuation*, that is extended to $V\colon \mathrm{Fm} \times W \to A$ by

$$V(\star(\varphi_1, \ldots, \varphi_n), x) \;=\; \star(V(\varphi_1, x), \ldots, V(\varphi_n, x))\,,$$

for each n-ary operation symbol $\star$ of $\mathfrak{L}$, and

$$\begin{aligned} V(\Box\varphi, x) &= \bigwedge\{Rxy \to V(\varphi, y) \mid y \in W\} \\ V(\Diamond\varphi, x) &= \bigvee\{Rxy \wedge V(\varphi, y) \mid y \in W\}. \end{aligned}$$

A $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-*model* satisfies the extra condition that $\langle W, R\rangle$ is a crisp **A**-frame. In this case, the conditions for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigwedge\{V(\varphi, y) \mid Rxy\} \\ V(\Diamond\varphi, x) &= \bigvee\{V(\varphi, y) \mid Rxy\}. \end{aligned}$$

For a set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ and a $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$, φ is called an $\mathfrak{M}$-*consequence* of Γ, if $V(\varphi, x) = 1$ for all $x \in W$ such that $V[\Gamma, x] = \{V(\psi, x) \mid \psi \in \Gamma\} \subseteq \{1\}$. Moreover, φ will be called *valid in* $\mathfrak{M}$, if $V(\varphi, x) = 1$ for all $x \in W$, written $\mathfrak{M} \models_{\mathbb{F}(\mathbf{A})} \varphi$. We will say that φ is a $\mathbb{K}(\mathbf{A})$-*consequence* (a $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-*consequence*) of Γ, written $\Gamma \models_{\mathbb{K}(\mathbf{A})} \varphi$ ($\Gamma \models_{\mathbb{K}^{\mathsf{c}}(\mathbf{A})} \varphi$), if $\mathfrak{M} \models_{\mathbb{K}(\mathbf{A})} \varphi$, for all $\mathbb{K}(\mathbf{A})$-models $\mathfrak{M}$ (for all $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-models $\mathfrak{M}$) such that $\mathfrak{M} \models_{\mathbb{K}(\mathbf{A})} \Gamma$.[1] And if φ is valid in all $\mathbb{K}(\mathbf{A})$- or $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-models, then φ is said to be $\mathbb{K}(\mathbf{A})$-*valid* or $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-*valid*, respectively, written $\models_{\mathbb{K}(\mathbf{A})} \varphi$ or $\models_{\mathbb{K}^{\mathsf{c}}(\mathbf{A})} \varphi$. We will denote the logics defined by $\models_{\mathbb{K}(\mathbf{A})}$ and $\models_{\mathbb{K}^{\mathsf{c}}(\mathbf{A})}$ by $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$, respectively, define $\mathsf{Val}(\mathsf{K}(\mathbf{A})) = \{\varphi \mid\, \models_{\mathbb{K}(\mathbf{A})} \varphi\}$ and $\mathsf{Val}(\mathsf{K}^{\mathsf{c}}(\mathbf{A})) = \{\varphi \mid\, \models_{\mathbb{K}^{\mathsf{c}}(\mathbf{A})} \varphi\}$, and will call their members $\mathsf{K}(\mathbf{A})$- and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$-*valid*, respectively.

Recalling that the standard Gödel algebra $\mathbf{G} = \langle [0,1], \min, \max, \to_{\mathsf{G}}, 0, 1\rangle$ is an order-based algebra, we notice that the logics $\mathsf{K}(\mathbf{G})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{G})$ are just the Gödel modal logics GK and GK^{c} introduced in Section 2.3. More generally, we may consider the family of *Gödel modal logics* $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ where $\mathbf{A}$ is a complete subalgebra of $\mathbf{G}$, in particular, when $\mathbf{A}$ is $\mathbf{G}_{\downarrow}$ or $\mathbf{G}_{\uparrow}$, as presented in Example 3.3.

It is not hard to show that for a finite order-based algebra $\mathbf{A}$, the sets of valid formulas of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ depend only on the cardinality of A and are decidable (see below). Recall, moreover, that although all infinite subalgebras of $\mathbf{G}$ produce the same set of

[1] We only define global consequence here, as this is the only consequence relation we treat in this work. Local consequence for order-based modal logics could be defined similarly as for many-valued modal logics over MTL-algebras.

valid propositional formulas (see Chapter 2 or [50]), there are countably infinitely many different infinite-valued first-order Gödel logics (considered as sets of valid formulas; cf. [10]). Below, we show that this result holds also for Gödel modal logics.

Theorem 3.4. *There are countably infinitely many different logics* $\mathsf{K}(\mathbf{A})$ *(considered as sets of valid formulas), for different infinite subalgebras* $\mathbf{A}$ *of* $\mathbf{G}$*. Moreover, the same is true for* $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$*.*

Proof. It follows from the result in [10] that there can be at most countably many such logics, because for each infinite subalgebra $\mathbf{A}$ of $\mathbf{G}$, $\mathsf{K}(\mathbf{A})$ corresponds to a specific fragment of the first-order logic over $\mathbf{A}$, determined by the same standard translation π as in the classical setting, where box- and diamond-formulas are translated as follows:

$$\pi(\Box\varphi) \;=\; (\forall y)(Rxy \to \pi(\varphi)(y)) \qquad \text{and} \qquad \pi(\Diamond\varphi) \;=\; (\exists y)(Rxy \wedge \pi(\varphi)(y))\,.$$

To obtain the fragment in the crisp case, we may use the usual "crispification" of the relation symbol R by prefixing it with $\neg\neg$.

To show that there are infinitely such logics, let us fix, for each $n \in \mathbb{Z}^+$, a complete subalgebra $\mathbf{A}_n$ of $\mathbf{G}$ with exactly n right accumulation points. We then prove that for all $n, m \in \mathbb{Z}^+$ such that $n \neq m$, $\mathsf{K}(\mathbf{A}_n)$ and $\mathsf{K}(\mathbf{A}_m)$ are mutually distinct, even when only valid formulas are considered, and so are $\mathsf{K}^{\mathsf{c}}(\mathbf{A}_n)$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A}_m)$. For this, we define the formula

$$\varphi(p,q) \;=\; (\Box(q \to p) \wedge (q \to \Box q) \wedge \Box((p \to q) \to q)) \to ((\Box p \to q) \to q),$$

which detects right accumulation points, as is stated in the following claim.

Claim 1: For any $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$, $x \in W$, and $p, q \in \mathrm{Var}$: if $V(\varphi(p,q),x) < 1$ then $V(\varphi(p,q),x) = V(q,x) \in R(\mathbf{A})$.

Proof of Claim 1: Suppose that $V(\varphi(p,q),x) < 1$. Then $V(\varphi(p,q),x) = V((\Box p \to q) \to q, x) = (V(\Box p, x) \to V(q,x)) \to V(q,x) < 1$ and thus

$$V(\Box p, x) \;\leq\; V(q,x) \;=\; V(\varphi(p,q),x) \;<\; 1\,.$$

For a contradiction, we assume that $V(q,x) \notin R(\mathbf{A})$. In this case, there is a world $y \in W$ such that $Rxy \to V(p,y) \leq V(q,x) < 1$, and thus

$$Rxy \;>\; V(p,y) \;\leq\; V(q,x)\,. \tag{1}$$

Moreover, by the assumption that $V(\varphi(p,q),x) < 1$, we have that $V(\Box(q \to p) \wedge (q \to \Box q) \wedge \Box((p \to q) \to q), x) > V((\Box p \to q) \to q, x) = V(q,x)$, and thus

$$V(q,x) \;<\; V(\Box(q \to p), x) \tag{2}$$

$$V(q,x) \;<\; V(q \to \Box q, x) \tag{3}$$

$$V(q,x) \;<\; V(\Box((p \to q) \to q), x) \tag{4}$$

From inequality (2), we can infer that $V(q,x) < Rxy \to (V(q,y) \to V(p,y)) = (Rxy \wedge V(q,y)) \to V(p,y)$, the later equality following from a well-known property of the Gödel t-norm and its residuum. We thus have either $Rxy \wedge V(q,y) \leq V(p,y)$ or $Rxy \wedge V(q,y) > V(p,y) > V(q,x)$. As the latter case contradicts (1), we have $Rxy \wedge V(q,y) \leq V(p,y)$ and thus, also using (1), we obtain

$$V(q,y) \ \leq \ V(p,y)\,. \tag{5}$$

Furthermore, from inequality (3), we obtain $V(q,x) < V(q,x) \to V(\Box q,x)$ which implies that $V(q,x) \leq V(\Box q,x) \leq Rxy \to V(q,y)$. By (1) and (5), we then get $V(q,x) \leq Rxy \to V(q,y) = V(q,y) \leq V(p,y) \leq V(q,x)$ and thus

$$V(q,x) \ = \ V(q,y) \ = \ V(p,y)\,. \tag{6}$$

Finally, from inequality (4), we infer that $V(q,x) < Rxy \to ((V(p,y) \to V(q,y)) \to V(q,y))$ and by (6), $V(q,x) < Rxy \to ((V(q,x) \to V(q,x)) \to V(q,x)) = Rxy \to (1 \to V(q,x)) = Rxy \to V(q,x)$, from which we can infer, by (1) and (6),

$$V(q,x) \ < \ Rxy \to V(q,x) \ = \ V(q,x)\,,$$

which is a contradiction. It follows that $V(\varphi(p,q),x) = V(q,x) \in R(\mathbf{A})$ and Claim 1 is established.

Claim 2: Let $a \in R(\mathbf{A})$, $p,q \in \mathrm{Var}$, and define a $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-model $\mathfrak{M}_a = \langle W_a, R_a, V_a \rangle$ by $W_a = \mathbb{N}$, $R_a = \{0\} \times \mathbb{Z}^+$, and for all $k \in \mathbb{N}$, $V_a(q,k) = a$ and $V_a(p,k) = b_k$, for some strictly descending sequence $\{b_k\}_{k \in \mathbb{N}} \subseteq A$ such that $a = \bigwedge_{k \in \mathbb{N}} b_k$. We then have that

$$V(\varphi(p,q),0) \ = \ V(q,0) \ = \ a \ < \ 1\,.$$

Proof of Claim 2: This claim is easily verified by observing the following equalities:

$$\begin{aligned}
V_a(\Box(q \to p),0) &= \bigwedge\{V_a(q,k) \to V_a(p,k) \mid k \in \mathbb{Z}^+\} \\
&= \bigwedge\{a \to b_k \mid k \in \mathbb{Z}^+\} \\
&= 1 \\
V_a(q \to \Box q,0) &= V_a(q,0) \to \bigwedge\{V_a(q,k) \mid k \in \mathbb{Z}^+\} \\
&= a \to a \\
&= 1 \\
V_a(\Box((p \to q) \to q),0) &= \bigwedge\{(V_a(p,k) \to V_a(q,k)) \to V_a(q,k) \mid k \in \mathbb{Z}^+\} \\
&= \bigwedge\{(b_k \to a) \to a \mid k \in \mathbb{Z}^+\} \\
&= \bigwedge\{a \to a \mid k \in \mathbb{Z}^+\} \\
&= 1
\end{aligned}$$

$$\begin{aligned} V_a((\Box p \to q) \to q, 0) &= (\bigwedge\{V_a(p,k) \mid k \in \mathbb{Z}^+\} \to V_a(q,0)) \to V_a(q,0) \\ &= (a \to a) \to a \\ &= 1 \to a \\ &= a \end{aligned}$$

Having established Claim 1 and 2, we define for any $n \in \mathbb{Z}^+$, the formula

$$\varphi_n(p_1, q_1, \ldots, p_n, q_n) = \bigwedge_{i=1}^{n-1}((q_{i+1} \to q_i) \to q_i) \to \bigvee_{i=1}^{n} \varphi(p_i, q_i).$$

Claim 3: For each $n \in \mathbb{Z}^+$, the formula $\varphi_n(p_1, q_1, ..., p_n, q_n)$ is $\mathsf{K}(\mathbf{A})$-valid if and only if $|R(\mathbf{A})| < n$.

Proof of Claim 3: First, let us fix an $n \in \mathbb{Z}^+$. We then prove the right-to-left direction by contraposition. Let $V(\varphi_n(p_1, q_1, ..., p_n, q_n), x) < 1$ for some $\mathbb{K}(\mathbf{A})$-model $\langle W, R, V\rangle$ and $x \in W$. In this case, for all $i \in \{1, \ldots, n\}$, $V(\varphi(p_i, q_i), x) < 1$ and thus $V(\varphi(p_i, q_i), x) = V(q_i, x) \in R(\mathbf{A})$ by Claim 1. Moreover, for each $i \in \{1, \ldots, n\}$, $V((q_{i+1} \to q_i) \to q_i, x) > \bigvee_{i=1}^{n} V(q_i, x) \geq V(q_i, x)$ which implies that $V(q_i, x) < V(q_{i+1}, x)$. Hence we obtain a strictly increasing sequence of n right accumulation points and thus $|R(\mathbf{A})| \geq n$.

For the left-to-right direction, let $\{a_i \in R(\mathbf{A}) \mid 1 \leq i \leq n\}$ be a sequence in $R(\mathbf{A})$ such that $a_1 < \ldots < a_n$ and define a $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-model $\mathfrak{M} = \langle \mathbb{N}, \{0\} \times \mathbb{Z}^+, V\rangle$ such that $V(p_i, k) = V_{a_i}(p_i, k)$ and $V(q_i, k) = V_{a_i}(q_i, k)$, for each $i \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$, where V_{a_i} is defined as in Claim 2. We then have that

$$\bigwedge_{i=1}^{n-1} V((q_{i+1} \to q_i) \to q_i, 0) = \bigwedge_{i=1}^{n-1}((a_{i+1} \to a_i) \to a_i) = \bigwedge_{i=1}^{n-1}(a_i \to a_i) = 1.$$

Moreover, by Claim 2,

$$\bigvee_{i=1}^{n} V(\varphi(p_i, q_i), 0) = \bigvee_{i=1}^{n} V(q_i, 0) = \bigvee_{i=1}^{n} a_i = a_n < 1,$$

and thus $V(\varphi_n(p_1, q_1, ..., p_n, q_n), 0) = a_n < 1$ and Claim 3 is established.

Noting that the model $\mathfrak{M}$ in the proof of Claim 3 is crisp, we have shown that for each $n \in \mathbb{Z}^+$, φ_{n+1} is $\mathsf{K}(\mathbf{A}_n)$-valid and $\mathsf{K}^{\mathsf{c}}(\mathbf{A}_n)$-valid, but neither $\mathsf{K}(\mathbf{A}_m)$-valid nor $\mathsf{K}^{\mathsf{c}}(\mathbf{A}_m)$-valid for any $m \geq n+1$. $\Box$

The logics $\mathsf{K}(\mathbf{G})$, $\mathsf{K}(\mathbf{G}_\uparrow)$, and $\mathsf{K}(\mathbf{G}_\downarrow)$ and their crisp counterparts are all distinct. The formula $\Box\neg\neg p \to \neg\neg\Box p$ is valid in the logics based on $\mathbf{G}_\uparrow$, but not in those based on $\mathbf{G}$ or $\mathbf{G}_\downarrow$. To see this, note that 0 is an accumulation point in $[0,1]$ and $G_\downarrow$ (but not in $G_\uparrow$); hence for these sets there is an infinite strictly descending sequence of values $(a_i)_{i \in I}$ with limit 0, giving $\neg\neg a_i = 1$ for each $i \in I$ and $\inf_{i \in I} \neg\neg a_i = 1$, while $\neg\neg \inf_{i \in I} a_i = \neg\neg 0 = 0$

(see the proof of Theorem 3.9). Similarly, $(\Diamond p \to \Diamond q) \to (\neg\Diamond q \vee \Diamond(p \to q))$ is valid in the logics based on $\mathbf{G}_{\downarrow}$ but not those based on $\mathbf{G}$. Moreover, the formula $\neg\neg\Diamond p \to \Diamond\neg\neg p$ is valid in any of the crisp logics, but not in their non-crisp versions.

3.3 The Finite Model Property

Let us introduce some more useful notation and terminology. A subset $\Sigma \subseteq \mathrm{Fm}$ will be called a *fragment* if it contains all constants in $\mathrm{C}_{\mathfrak{L}}$ and is closed with respect to taking subformulas. For a formula $\varphi \in \mathrm{Fm}$, we let $\Sigma(\varphi)$ be the smallest (always finite) fragment containing φ. Also, for any $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$, subset $X \subseteq W$, and fragment $\Sigma \subseteq \mathrm{Fm}$, we let

$$V[\Sigma, X] \;=\; \{V(\varphi, x) \mid \varphi \in \Sigma,\; x \in X\}\,.$$

We shorten $V[\Sigma, \{x\}]$ to $V[\Sigma, x]$. For $\Sigma \subseteq \mathrm{Fm}$, we let $\Sigma_{\Box}$ and $\Sigma_{\Diamond}$ be the sets of all box-formulas in Σ and diamond-formulas in Σ, respectively.

We also consider many-valued analogues of some notions and results from classical modal logic (see e.g. [15]). For an $\mathbf{A}$-frame $\langle W, R\rangle$, we define the crisp relation R^+ and, for each $x \in W$, the set of worlds $R^+[x]$ as follows:

$$R^+ \;=\; \{(x, y) \in W^2 \mid Rxy > 0\} \qquad \text{and} \qquad R^+[x] \;=\; \{y \in W \mid R^+xy\}\,.$$

Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathbb{K}(\mathbf{A})$-model. We call $\mathfrak{M}' = \langle W', R', V'\rangle$ a $\mathbb{K}(\mathbf{A})$-*submodel* of $\mathfrak{M}$, written $\mathfrak{M}' \subseteq \mathfrak{M}$, if $W' \subseteq W$ and R' and V' are the restrictions to W' of R and V, respectively. In particular, given $x \in W$, the $\mathbb{K}(\mathbf{A})$-*submodel of* $\mathfrak{M}$ *generated by* x is the smallest $\mathbb{K}(\mathbf{A})$-submodel $\mathfrak{M}' = \langle W', R', V'\rangle$ of $\mathfrak{M}$ such that $x \in W'$ and for all $y \in W'$, whenever $z \in R^+[y]$, also $z \in W'$.

Lemma 3.5. *Let* $\mathfrak{M} = \langle W, R, V\rangle$ *be a* $\mathbb{K}(\mathbf{A})$*-model and* $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ *a generated* $\mathbb{K}(\mathbf{A})$*-submodel of* $\mathfrak{M}$*. Then* $\widehat{V}(\varphi, x) = V(\varphi, x)$ *for all* $x \in \widehat{W}$ *and* $\varphi \in \mathrm{Fm}$.

Proof. We proceed by induction on $\ell(\varphi)$. The base case is trivial for any submodel of $\mathfrak{M}$, so also for $\widehat{\mathfrak{M}}$. For the induction step, the case where $\varphi = \star(\varphi_1, \ldots, \varphi_n)$ for some operation symbol $\star$ follows immediately using the induction hypothesis.

Suppose now that $\varphi = \Box\psi$. Fix $x \in \widehat{W}$ and note that for any $y \in W \setminus \widehat{W}$, we have $Rxy = 0$. Observe also that $0 \to a = 1$ for all $a \in A$. Hence, excluding all worlds $y \in W$ such that $Rxy = 0$ does not change the value of $\bigwedge\{Rxy \to V(\psi, y) \mid y \in W\}$. So, using the induction hypothesis,

$$\begin{aligned} V(\Box\psi, x) \;&=\; \bigwedge\{Rxy \to V(\psi, y) \mid y \in \widehat{W}\} \\ &=\; \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y)\colon\; y \in \widehat{W}\} \\ &=\; \widehat{V}(\Box\psi, x)\,. \end{aligned}$$

The case where $\varphi = \Diamond\psi$ is very similar. $\square$

Following the usual terminology of modal logic, a *tree* is defined as a relational structure $\langle N, E\rangle$ such that (i) $E \subseteq N^2$ is irreflexive, (ii) there exists a unique *root* $x_0 \in N$ satisfying $E^* x_0 x$ for all $x \in N$ where E^* is the reflexive transitive closure of E, (iii) for each $x \in N \setminus \{x_0\}$, there is a unique $x' \in N$ such that $Ex'x$. A tree $\langle N, E\rangle$ has *height* $m \in \mathbb{N}$ if $m = \max\{|\{y \in N \mid E^* yx\}| \in \mathbb{N} \mid x \in N\}$. A $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$ is called a $\mathbb{K}(\mathbf{A})$-*tree-model* if $\langle W, R^+\rangle$ is a tree, and has finite *height* $\mathrm{hg}(\mathfrak{M}) = m$ if $\langle W, R^+\rangle$ has height m.

Lemma 3.6. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathbb{K}(\mathbf{A})$-model, $x_0 \in W$, and $k \in \mathbb{N}$. Then there exists a $\mathbb{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ with root $\widehat{x}_0$ and $\mathrm{hg}(\widehat{\mathfrak{M}}) \leq k$ such that $\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0)$ for all $\varphi \in \mathrm{Fm}$ with $\ell(\varphi) \leq k$. Moreover, if $\mathfrak{M}$ is a $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-model, then so is $\widehat{\mathfrak{M}}$.*

Proof. Consider the $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M}' = \langle W', R', V'\rangle$ obtained by "unravelling" $\mathfrak{M}$ at the world x_0; i.e. for all $n \in \mathbb{N}$ (noting that $0 \in \mathbb{N}$),

$$\begin{aligned} W' &= \bigcup_{n\in\mathbb{N}} \{(x_0, \ldots, x_n) \in W^{n+1} \mid R^+ x_i x_{i+1},\ i < n\}\,, \\ R'yz &= \begin{cases} Rx_n x_{n+1} & \text{if } y = (x_0, \ldots, x_n),\ z = (x_0, \ldots, x_{n+1})\,, \\ 0 & \text{otherwise}\,, \end{cases} \\ V'(p, (x_0, \ldots, x_n)) &= V(p, x_n)\,. \end{aligned}$$

Clearly, $\mathfrak{M}'$ is a $\mathbb{K}(\mathbf{A})$-tree-model with root $\widehat{x}_0 = (x_0)$ and R' is crisp if R is. Now let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ be the $\mathbb{K}(\mathbf{A})$-tree-submodel of $\mathfrak{M}'$ defined by cutting $\mathfrak{M}'$ at depth k; i.e. let $\widehat{W} = \{(x_0, \ldots, x_n) \in W' \mid n \leq k\}$ and let $\widehat{R}$ and $\widehat{V}$ be the restrictions of R' and V' to $\widehat{W} \times \widehat{W}$ and $\mathrm{Var} \times \widehat{W}$, respectively. A straightforward induction on $\ell(\varphi)$ shows that for all $\varphi \in \mathrm{Fm}$ and $n \in \mathbb{N}$ such that $\ell(\varphi) \leq k - n$, $\widehat{V}(\varphi, (x_0, \ldots, x_n)) = V(\varphi, x_n)$. In particular, $\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0)$ for all $\varphi \in \mathrm{Fm}$ with $\ell(\varphi) \leq k$. $\square$

A $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$ is called *finite* or *countable*, if its set of worlds W is finite or countable, respectively. A logic L for $\mathfrak{L}_{\mathsf{m}}$ has the *finite model property with respect to* $\mathbb{K}(\mathbf{A})$ ($\mathbb{K}^{\mathsf{c}}(\mathbf{A})$), if for all $\varphi \in \mathrm{Fm}$, φ is valid in L if and only if φ is valid in all finite $\mathbb{K}(\mathbf{A})$-models (all finite $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-models).

Lemma 3.7. *If $\mathbf{A}$ is a finite order-based algebra, then $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ have the finite model property with respect to $\mathbb{K}(\mathbf{A})$ and $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$, respectively.*

Proof. By Lemma 3.6, it suffices to show that for any finite fragment $\Sigma \subseteq \mathrm{Fm}$ and $\mathbb{K}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V\rangle$ of finite height with root x, there is a finite $\mathbb{K}(\mathbf{A})$-tree-model

$\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle \subseteq \mathfrak{M}$ with root x such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$. We prove this claim by induction on $\mathrm{hg}(\mathfrak{M})$. For the base case, $W = \{x\}$ and we let $\widehat{\mathfrak{M}} = \mathfrak{M}$.

For the induction step, consider for each $y \in R^{+}[x]$, the submodel $\mathfrak{M}_y = \langle W_y, R_y, V_y \rangle$ of $\mathfrak{M}$ generated by y. Each $\mathfrak{M}_y$ is a $\mathbb{K}(\mathbf{A})$-tree-model of finite height with root y and $\mathrm{hg}(\mathfrak{M}_y) < \mathrm{hg}(\mathfrak{M})$. Hence, by the induction hypothesis, for each $y \in R^{+}[x]$, there is a finite $\mathbb{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y \rangle \subseteq \mathfrak{M}_y \subseteq \mathfrak{M}$ with root $y \in \widehat{W}_y$ such that for all $\varphi \in \Sigma$, by Lemma 3.5, $\widehat{V}_y(\varphi, y) = V_y(\varphi, y) = V(\varphi, y)$.

Because $\mathbf{A}$ is finite, we can now choose for each $\varphi \in \Sigma_\Box \cup \Sigma_\Diamond$, a world y_φ such that $V(\varphi, x) = Rxy_\varphi \to \widehat{V}_y(\psi, y_\varphi)$ when $\varphi = \Box\psi$, and $V(\varphi, x) = Rxy_\varphi \wedge \widehat{V}_y(\psi, y_\varphi)$ when $\varphi = \Diamond\psi$. Define the finite set $Y = \{y_\varphi \in R^{+}[x] \mid \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\}$. We let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ where

$$\widehat{W} \;=\; \{x\} \cup \bigcup_{y \in Y} \widehat{W}_y,$$

and $\widehat{R}$ and $\widehat{V}$ are R and V, respectively, restricted to $\widehat{W}$. An easy induction on $\ell(\varphi)$ establishes that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$. □

Furthermore, we are able to establish the finite model property when the underlying (infinite) algebra is $\mathbf{G}_\uparrow$.

Theorem 3.8. *$\mathsf{K}(\mathbf{G}_\uparrow)$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{G}_\uparrow)$ have the finite model property with respect to $\mathbb{K}(\mathbf{G}_\uparrow)$ and $\mathbb{K}^{\mathsf{c}}(\mathbf{G}_\uparrow)$, respectively.*

Proof. By Lemmas 3.6 and 3.7, it suffices to show that if $\varphi \in \mathrm{Fm}$ is not valid in some $\mathbb{K}(\mathbf{G}_\uparrow)$-tree-model $\mathfrak{M}$ of finite height, then there is a finite subalgebra $\mathbf{B}$ of $\mathbf{G}_\uparrow$ and a $\mathbb{K}(\mathbf{B})$-model $\widehat{\mathfrak{M}}$ (that is crisp if $\mathfrak{M}$ is crisp) such that φ is not valid in $\widehat{\mathfrak{M}}$.

Suppose that $\beta = V(\varphi, x) < 1$ for some $\mathbb{K}(\mathbf{G}_\uparrow)$-tree-model $\mathfrak{M} = \langle W, R, V \rangle$ of finite height with root x. Let $\mathbf{B}$ be the finite subalgebra of $\mathbf{G}_\uparrow$ with universe $(G_\uparrow \cap [0, \beta]) \cup \{1\}$ and consider $h \colon \mathbf{G}_\uparrow \to \mathbf{B}$ defined by

$$h(a) = \begin{cases} a & \text{if } a \le \beta \\ 1 & \text{otherwise.} \end{cases}$$

We define a $\mathbb{K}(\mathbf{B})$-model $\widehat{\mathfrak{M}} = \langle W, \widehat{R}, \widehat{V} \rangle$ (that is crisp if $\mathfrak{M}$ is crisp) as follows. Let $\widehat{R}yz = h(Ryz)$ for all $y, z \in W$ and $\widehat{V}(p, y) = h(V(p, y))$ for all $y \in W$ and $p \in \mathrm{Var}$. We prove that $\widehat{V}(\psi, y) = h(V(\psi, y))$ for all $y \in W$ and $\psi \in \mathrm{Fm}$ by induction on $\ell(\psi)$.

The base case follows by definition (recalling that the only constants are $\bot$ and $\top$). For the induction step, the propositional cases follow by observing that h is a Heyting algebra homomorphism (i.e. preserves the operations $\wedge$, $\vee$, $\to$, $\bot$, and $\top$). The case of

$\psi = \Box\chi$ is also straightforward. If $\psi = \Diamond\chi$, then

$$\begin{aligned}
\widehat{V}(\Diamond\chi, y) &= \bigvee\{\widehat{R}yz \wedge \widehat{V}(\chi, z) \mid z \in W\} && (7)\\
&= \bigvee\{h(Ryz) \wedge h(V(\chi, z)) \mid z \in W\} && (8)\\
&= \bigvee\{h(Ryz \wedge V(\chi, z)) \mid z \in W\} && (9)\\
&= h(\bigvee\{Ryz \wedge V(\chi, z) \mid z \in W\}) && (10)\\
&= h(V(\Diamond\chi, y)). && (11)
\end{aligned}$$

The step from (7) to (8) follows using the induction hypothesis and the step from (8) to (9) follows because h is a Heyting algebra homomorphism. For the step from (9) to (10), note that for $\bigvee\{Ryz \wedge V(\chi, z) \mid z \in W\} \leq \beta$, the equality is immediate. Otherwise, $Ryz \wedge V(\chi, z) > \beta$ for some $z \in W$ and $h(Ryz \wedge V(\chi, z)) = 1$, so $h(\bigvee\{Ryz \wedge V(\chi, z) \mid z \in W\}) = 1 = \bigvee\{h(Ryz \wedge V(\chi, z)) \mid z \in W\}$. Hence $\widehat{V}(\varphi, x) = h(V(\varphi, x)) = h(\beta) = \beta < 1$ as required. $\Box$

The finite model property does not hold, however, for Gödel modal logics with universe $[0, 1]$ or $G_\downarrow$, or even $G_\uparrow$ if we add also the connective $\triangle$ to the language. The problem in these cases stems from the existence of accumulation points in the universe of truth values considered together with the non-continuous operation $\neg$ or $\triangle$. If infinitely many worlds are accessible from a world x, then the value taken by a formula $\Box\varphi$ (or $\Diamond\varphi$) at x will be the infimum (supremum) of values calculated from values of φ at these worlds, but may not be the minimum (maximum). A formula may therefore not be valid in such a model, but valid in all finite models where infima (suprema) and minima (maxima) coincide.

Theorem 3.9. *Suppose that either (i) the universe of* $\mathbf{A}$ *is* $[0, 1]$ *or* $G_\downarrow$*, or (ii) the universe of* $\mathbf{A}$ *is* $G_\uparrow$ *and the language contains* $\triangle$*. In either cases, neither* $\mathsf{K}(\mathbf{A})$ *nor* $\mathsf{K^c}(\mathbf{A})$ *has the finite model property with respect to* $\mathbb{K}(\mathbf{A})$ *or* $\mathbb{K}^\mathsf{c}(\mathbf{A})$*, respectively.*

Proof. For (i), we follow [28] where it is shown that the following formula provides a counterexample to the finite model property of GK and $\mathsf{GK^c}$:

$$\Box\neg\neg p \to \neg\neg\Box p\,.$$

Just observe that the formula is valid in all finite $\mathbb{K}(\mathbf{A})$-models, but not in the infinite $\mathbb{K}^\mathsf{c}(\mathbf{A})$-model $\langle \mathbb{N}, R, V\rangle$ where $Rmn = 1$ for all $m, n \in \mathbb{N}$ and $V(p, n) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Hence neither $\mathsf{K}(\mathbf{A})$ nor $\mathsf{K^c}(\mathbf{A})$ has the finite model property with respect to $\mathbb{K}(\mathbf{A})$ or $\mathbb{K}^\mathsf{c}(\mathbf{A})$, respectively.

Similarly, for (ii), the formula

$$\triangle\Diamond p \to \Diamond\triangle p$$

is valid in all finite $\mathbb{K}(\mathbf{A})$-models, but not in the infinite $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-model $\langle \mathbb{N}, R, V\rangle$ where $Rmn = 1$ for all $m, n \in \mathbb{N}$ and $V(p,n) = \frac{n}{n+1}$ for all $n \in \mathbb{N}$. $\square$

3.4 Order-Embeddings

Given a linearly ordered set $\langle P, \leq\rangle$ and $C \subseteq P$, a map $h\colon P \to P$ will be called a *C-order embedding* if it is an order-preserving embedding (i.e. $a \leq b$ if and only if $h(a) \leq h(b)$ for all $a, b \in P$) satisfying $h(c) = c$ for all $c \in C$. We will call an order embedding $h\colon P \to P$ *inflationary* or *deflationary* if for all $a \in P$, $a \leq h(a)$, or for all $a \in P$, $a \geq h(a)$, respectively. h will be called *B-complete* for $B \subseteq P$ if whenever $\bigvee D \in B$ or $\bigwedge D \in B$ for some $D \subseteq P$, respectively,

$$h(\bigvee D) \;=\; \bigvee h[D] \qquad \text{or} \qquad h(\bigwedge D) \;=\; \bigwedge h[D]\,.$$

The following lemma establishes the critical property of order-based modal logics for our purposes. Namely, it is only the relative order of the values taken by variables and the accessibility relation between worlds that plays a role in determining the values of formulas and checking validity.

Lemma 3.10. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathbb{K}(\mathbf{A})$-model and $\Sigma \subseteq \mathrm{Fm}$ a fragment, and let $h\colon A \to A$ be a $V[\Sigma_\Box \cup \Sigma_\Diamond, W]$-complete $\mathrm{C}_{\mathfrak{L}}$-order embedding. Consider the $\mathbb{K}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle W, \widehat{R}, \widehat{V}\rangle$ with $\widehat{R}xy = h(Rxy)$ and $\widehat{V}(p,x) = h(V(p,x))$ for all $p \in \mathrm{Var}$ and $x, y \in W$. Then for all $\varphi \in \Sigma$ and $x \in W$:*

$$\widehat{V}(\varphi, x) \;=\; h(V(\varphi, x))\,.$$

Proof. We proceed by induction on $\ell(\varphi)$. The case $\varphi \in \mathrm{Var} \cup \mathrm{C}_{\mathfrak{L}}$ follows from the definition of $\widehat{V}$. For the induction step, suppose that $\varphi = \star(\varphi_1, \ldots, \varphi_n)$ for some operation symbol $\star$ of $\mathfrak{L}$ and $\varphi_1, \ldots, \varphi_n \in \Sigma$. Recall that $\star$ is definable in $\mathbf{A}$ by some quantifier-free first-order formula $F^\star(x_1, \ldots, x_n, y)$ in the first-order language with $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathfrak{L}}$, i.e.

$$\star(a_1, \ldots, a_n) \;=\; b \qquad \text{iff} \qquad \mathbf{A} \models F^\star(a_1, \ldots, a_n, b)\,.$$

Because $F^\star(x_1, \ldots, x_n, y)$ is quantifier-free and h preserves $\wedge$, $\vee$, and $\mathrm{C}_{\mathfrak{L}}$,

$$\mathbf{A} \models F^\star(a_1, \ldots, a_n, b) \qquad \text{iff} \qquad \mathbf{A} \models F^\star(h(a_1), \ldots, h(a_n), h(b))\,.$$

So we may also conclude

$$\star(h(a_1), \ldots, h(a_n)) \;=\; h(\star(a_1, \ldots, a_n))\,.$$

Hence for all $x \in W$, using the induction hypothesis for the step from (1) to (2):

$$\begin{aligned}
\widehat{V}(\star(\varphi_1,\ldots,\varphi_n),x) &= \star(\widehat{V}(\varphi_1,x),\ldots,\widehat{V}(\varphi_n,x)) && (12)\\
&= \star(h(V(\varphi_1,x)),\ldots,h(V(\varphi_n,x))) && (13)\\
&= h(\star(V(\varphi_1,x),\ldots,V(\varphi_n,x))) && (14)\\
&= h(V(\star(\varphi_1,\ldots,\varphi_n),x))\,. && (15)
\end{aligned}$$

If $\varphi = \Diamond\psi$ for some $\psi \in \Sigma$, then we obtain for all $x \in W$:

$$\begin{aligned}
\widehat{V}(\Diamond\psi,x) &= \bigvee\{\widehat{R}xy \wedge \widehat{V}(\psi,y) \mid y \in W\} && (16)\\
&= \bigvee\{h(Rxy) \wedge h(V(\psi,y)) \mid y \in W\} && (17)\\
&= \bigvee\{h(Rxy \wedge V(\psi,y)) \mid y \in W\} && (18)\\
&= h(\bigvee\{Rxy \wedge V(\psi,y) \mid y \in W\}) && (19)\\
&= h(V(\Diamond\psi,x))\,. && (20)
\end{aligned}$$

(16) to (17) follows from the definition of $\widehat{R}$ and the induction hypothesis, (17) to (18) follows because h is an order embedding, and (18) to (19) follows because h is $V[\Sigma_\Box \cup \Sigma_\Diamond, W]$-complete and $\bigvee\{Rxy \wedge V(\psi,y) \mid y \in W\} = V(\Diamond\psi,x) \in V[\Sigma_\Diamond, W]$. The case $\varphi = \Box\psi$ is very similar. $\Box$

There is one more notion we will introduce before moving on to decidability and complexity issues for order-based modal logics. To ensure that the alternative semantics we will define in the next chapter accepts the same valid formulas as the original semantics, we restrict our attention to order-based algebras where the order satisfies a certain homogeneity property.

Recall that $R(\mathbf{A})$ and $L(\mathbf{A})$ are the sets of right and left accumulation points, respectively, of an order-based algebra $\mathbf{A}$ in the usual topology inherited from $[0,1]$. Note also that by (a,b), $[a,b)$, etc. we denote here the intervals $(a,b) \cap A$, $[a,b) \cap A$, etc. in $\mathbf{A}$. We say that $\mathbf{A}$ is *locally right homogeneous* if for any $a \in R(\mathbf{A})$, there is a $c \in A$ such that $a < c$ and for any $e \in (a,c)$, there is a complete deflationary order embedding $h\colon [a,c) \to [a,e)$ such that $h(a) = a$. In this case, c is called a *witness* of right homogeneity at a. Similarly, $\mathbf{A}$ is said to be *locally left homogeneous* if for any $b \in L(\mathbf{A})$, there is a $d \in A$ such that $d < b$ and for any $f \in (d,b)$, there is a complete inflationary order embedding $h\colon (d,b] \to (f,b]$ such that $h(b) = b$. In this case, d is called a *witness* of left homogeneity at b. We will call $\mathbf{A}$ *locally homogeneous* if it is both locally right homogeneous and locally left homogeneous.

Observe that if $c \in A$ is a witness of right homogeneity at a, then any $e \in (a,c)$ will also be a witness of right homogeneity at a. Hence c can be chosen sufficiently close to a so that (a,c) is disjoint to any given finite subset of A. A similar observation holds for witnesses of left homogeneity.

Example 3.11. Any finite $\mathbf{A}$ is trivially locally homogeneous. Also any $\mathbf{A}$ with $A = [0, 1]$ is locally homogeneous: for $a \in R(\mathbf{A}) = [0, 1)$, choose any $c > a$ to witness right homogeneity at a, and similarly for $b \in L(\mathbf{A}) = (0, 1]$, choose any $d < b$ to witness left homogeneity at b. In the case of $A = G_{\downarrow}$, $L(\mathbf{A}) = \emptyset$, $R(\mathbf{A}) = \{0\}$, and any $c > 0$ witnesses right homogeneity at 0. Similarly, for $A = G_{\uparrow}$, $R(\mathbf{A}) = \emptyset$, $L(\mathbf{A}) = \{1\}$, and any $d < 1$ witnesses left homogeneity at 1. Moreover, infinitely many more non-isomorphic examples can be constructed using the fact that any ordered sum or lexicographical product of two locally homogeneous ordered sets is locally homogeneous.

In the next chapter, we introduce a new kind of Kripke-style semantics for which we prove, if the underlying order-based algebra is locally homogeneous, that it renders valid the same formulas as the semantics introduced in the present chapter. We show that the modal logics based on such algebras do enjoy the finite model property with respect to this new semantics and obtain decidability and complexity results.

Chapter 4

Decidability and Complexity of Order-Based Modal Logics

Most order-based modal logics do not enjoy the finite model property with respect to the Kripke-style semantics introduced in Chapter 3. In the present chapter, we will introduce a new alternative semantics based on modified Kripke-models and show that an order-based modal logic based on a locally homogeneous order-based algebra does indeed enjoy the finite model property with respect to this alternative semantics. In fact, we will get a strong kind of finite model property where we can put an upper bound on the "size" of the models, which will give us decidability of the validity problem for these order-based modal logics in various cases. Furthermore, using the fact that we can restrict to finite tree-models, we obtain complexity bounds for these decision problems, namely PSPACE-completeness.

All of the ideas and results in this chapter were obtained as the result of joint work by the author of this thesis with Xavier Caicedo, George Metcalfe, and Ricardo Rodríguez [26, 27].

Before we begin, we fix a finite algebraic language $\mathfrak{L}$ including the operational symbols $\top$, $\bot$, $\wedge$, $\vee$, and $\to$, and recall that $\mathrm{C}_{\mathfrak{L}}$ denotes the finite set of constants, $\mathfrak{L}_{\mathfrak{m}} = \mathfrak{L} \cup \{\Box, \Diamond\}$, and the set of formulas $\mathrm{Fm}_{\mathfrak{L}_{\mathfrak{m}}}$ for $\mathfrak{L}_{\mathfrak{m}}$ will often be denoted by Fm. Furthermore, we fix a locally homogeneous order-based algebra $\mathbf{A}$ for $\mathfrak{L}$.

4.1 Alternative Kripke-Style Semantics

Let us consider again the failure of the finite model property of GK^{c} with respect to $\mathbb{K}^{\mathsf{c}}(\mathbf{G})$ (see Theorem 3.9). For a $\mathbb{K}^{\mathsf{c}}(\mathbf{G})$-model to render the formula $\Box\neg\neg p \to \neg\neg\Box p$ invalid at a world x, there must be values of p at worlds accessible to x that form an infinite descending sequence tending to but never reaching 0. This ensures that the infinite model falsifies the formula, but also that no particular world acts as a "witness" to the value of $\Box p$. In this section, we redefine models to restrict the values at each world that can be taken by box-formulas and diamond-formulas. A formula $\Box p$ can then be "witnessed" at a world where the value of p is merely "sufficiently close" to the value of $\Box p$.

Let us fix a finite algebraic language $\mathfrak{L}$ including the operational symbols $\top$, $\bot$, $\wedge$, $\vee$, and $\to$, and recall that $\mathrm{C}_{\mathfrak{L}}$ denotes the finite set of constants, $\mathfrak{L}_{\mathfrak{m}} = \mathfrak{L} \cup \{\Box, \Diamond\}$, and the set of formulas $\mathrm{Fm}_{\mathfrak{L}_{\mathfrak{m}}}$ for $\mathfrak{L}_{\mathfrak{m}}$ will often be denote by Fm. Furthermore, we fix a locally homogeneous order-based algebra $\mathbf{A}$ for $\mathfrak{L}$.

An $\mathbb{FK}(\mathbf{A})$-*model* is a five-tuple $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ such that $\langle W, R, V \rangle$ is a $\mathbb{K}(\mathbf{A})$-model and $T_\Box \colon W \to \mathscr{P}(A)$ and $T_\Diamond \colon W \to \mathscr{P}(A)$ are functions satisfying for each $x \in W$:

(i) $\mathrm{C}_{\mathfrak{L}}^{\mathbf{A}} \subseteq T_\Box(x) \cap T_\Diamond(x)$,

(ii) $T_\Box(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)$, for some finite $I \subseteq \mathbb{N}$ (possibly empty), where $a_i \in R(\mathbf{A})$, c_i witnesses right homogeneity at a_i, and the intervals (a_i, c_i) are pairwise disjoint,

(iii) $T_\Diamond(x) = A \setminus \bigcup_{j \in J}(d_j, b_j)$, for some finite $J \subseteq \mathbb{N}$ (possibly empty), where $b_j \in L(\mathbf{A})$, d_j witnesses left homogeneity at b_j, and the intervals (d_j, b_j) are pairwise disjoint.

The valuation V is extended to the mapping $V \colon \mathrm{Fm} \times W$ inductively as follows:

$$V(\star(\varphi_1, \dots, \varphi_n), x) \;=\; \star(V(\varphi_1, x), \dots, V(\varphi_n, x))$$

for each n-ary operational symbol $\star$ of $\mathfrak{L}$, and

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee \{ r \in T_\Box(x) \mid r \leq \bigwedge \{ Rxy \to V(\varphi, y) \mid y \in W \} \} \\ V(\Diamond\varphi, x) &= \bigwedge \{ r \in T_\Diamond(x) \mid r \geq \bigvee \{ Rxy \wedge V(\varphi, y) \mid y \in W \} \}. \end{aligned}$$

As before, an $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-*model* satisfies the extra condition that $\langle W, R \rangle$ is a crisp $\mathbf{A}$-frame, and the conditions for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee \{ r \in T_\Box(x) \mid r \leq \bigwedge \{ V(\varphi, y) \mid Rxy \} \} \\ V(\Diamond\varphi, x) &= \bigwedge \{ r \in T_\Diamond(x) \mid r \geq \bigvee \{ V(\varphi, y) \mid Rxy \} \}. \end{aligned}$$

A formula $\varphi \in \mathrm{Fm}$ is called *valid in an* $\mathbb{FK}(\mathbf{A})$-*model* $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$, if $V(\varphi, x) = 1$ for all $x \in W$, written $\mathfrak{M} \models_{\mathbb{FK}(\mathbf{A})} \varphi$. Furthermore, φ is called *valid* in $\mathbb{FK}(\mathbf{A})$ or $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$ written $\models_{\mathbb{FK}(\mathbf{A})} \varphi$ or $\models_{\mathbb{FK}^{\mathsf{c}}(\mathbf{A})} \varphi$, if φ is valid in all $\mathbb{FK}(\mathbf{A})$-models or φ is valid in all $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-models, respectively.

Example 4.1. Note that when A is finite, $T_\Box(x) = T_\Diamond(x) = A$. For $A = [0, 1]$, both $T_\Box(x)$ and $T_\Diamond(x)$ are obtained by removing finitely many arbitrary disjoint intervals (a, b) not containing constants. For $A = G_\downarrow$, the only possibilities are $T_\Diamond(x) = A$ and $T_\Box(x) = A$ or $T_\Box(x) = \{0, \frac{1}{n}, \frac{1}{n-1}, \dots, 1\}$ for some $n \in \mathbb{Z}^+$ respecting $\mathrm{C}_{\mathfrak{L}} \subseteq T_\Box(x)$. The case of $A = G_\uparrow$ is very similar.

Remark 4.2. We introduce this alternative semantics mainly as a tool to obtain decidability and complexity results for the logics $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$. Nevertheless, restricting the values assigned to box- and diamond-formulas might have other benefits, e.g. when we want to model certain notions as modal connectives which are crisp or have a different behaviour than the other connectives.

For example, in a crisp $\mathbb{FK}(\mathbf{A})$-model $\langle W, R, V, T_\Box, T_\Diamond\rangle$ where $T_\Box(x) = T_\Diamond(x) = \{0, 1\}$ for all $x \in W$, the formula $\Box\varphi$ represents the crisp property of the existence of an accessible world y such that $V(\varphi, y) < 1$, and the formula $\Diamond\psi$ represents the positive satisfiability of ψ at some accessible world y.

Furthermore, by defining $T_\Box(x) = T_\Diamond(x) = B \subseteq A$ for all worlds x, we would obtain semantics for a two-layered logic, where non-modal formulas are interpreted as in the order-based logic over $\mathbf{A}$ and purely modal formulas, where all propositional variables are preceded by a box or a diamond, are interpreted as in the order-based logic over the subalgebra of $\mathbf{A}$ generated by B. E.g. in the case where $T_\Box(x) = T_\Diamond(x) = \{0, 1\}$ for all worlds x, purely modal formulas would behave classically.

Remark 4.3. It is worth pointing out that in every $\mathbb{FK}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ and for any $x \in W$, $T_\Box(x)$ and $T_\Diamond(x)$ will be complete subsets of A. Hence, the supremum defining $V(\Box\varphi, x)$ and the infimum defining $V(\Diamond\varphi, x)$ will actually be a maximum and a minimum, respectively. Furthermore, we always have that $V(\Box\varphi, x) \in T_\Box(x)$ and $V(\Diamond\varphi, x) \in T_\Diamond(x)$.

The remainder of this section is devoted to proving that a formula $\varphi \in \mathrm{Fm}$ is valid in $\mathbb{FK}(\mathbf{A})$ or $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$ if and only if it is valid in all finite $\mathbb{FK}(\mathbf{A})$-(tree)-models or all finite $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-(tree)-models, respectively. In order to reach this goal, we first need to extend some previously introduced notions to $\mathbb{FK}(\mathbf{A})$-models and establish some crucial properties of these models.

Given an $\mathbb{FK}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$, we call $\mathfrak{M}' = \langle W', R', V', T'_\Box, T'_\Diamond\rangle$ an $\mathbb{FK}(\mathbf{A})$-*submodel* of $\mathfrak{M}$, written $\mathfrak{M}' \subseteq \mathfrak{M}$, if $W' \subseteq W$ and R', V', $T'_\Box$, and $T'_\Diamond$ are the restrictions to W' of R, V, $T_\Box$, and $T_\Diamond$, respectively. As before, given $x \in W$, the $\mathbb{FK}(\mathbf{A})$-*submodel of* $\mathfrak{M}$ *generated by* x is the smallest $\mathbb{FK}(\mathbf{A})$-submodel $\mathfrak{M}' = \langle W', R', V', T'_\Box, T'_\Diamond\rangle$ of $\mathfrak{M}$ satisfying $x \in W'$ and for all $y \in W'$, $z \in R^+[y]$ implies $z \in W'$. Lemmas 3.5 and 3.6 then extend to $\mathbb{FK}(\mathbf{A})$-models as follows with minimal changes in the proofs.

Lemma 4.4. *Let $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ be an $\mathbb{FK}(\mathbf{A})$-model.*

(a) *Let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ be a generated $\mathbb{FK}(\mathbf{A})$-submodel of $\mathfrak{M}$. Then $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $x \in \widehat{W}$, and $\varphi \in \mathrm{Fm}$.*

(b) *Given any $x \in W$ and $k \in \mathbb{N}$, there exists an $\mathbb{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ with root $\widehat{x}$ and $\mathrm{hg}(\widehat{\mathfrak{M}}) \leq k$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \mathrm{Fm}$ with $\ell(\varphi) \leq k$, and if $\mathfrak{M}$ is an $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-model, then so is $\widehat{\mathfrak{M}}$.*

Example 4.5. There are very simple finite $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-counter-models for the formula $\Box\neg\neg p \to \neg\neg\Box p$ when $A = [0,1]$. For example for $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ defined by $W = \{a\}$, $Raa = 1$, $T_\Box(a) = T_\Diamond(a) = \mathrm{C}_{\mathfrak{L}}$, and $0 < V(p,a) < \min(\mathrm{C}_{\mathfrak{L}} \setminus \{0\})$:

$$\begin{aligned}
V(\Box\neg\neg p, a) &= \bigvee\{r \in \mathrm{C}_{\mathfrak{L}} \mid r \leq \bigwedge\{V(\neg\neg p, y) \mid Ray\}\} \\
&= \bigvee\{r \in \mathrm{C}_{\mathfrak{L}} \mid r \leq V(\neg\neg p, a)\} \\
&= \bigvee\{r \in \mathrm{C}_{\mathfrak{L}} \mid r \leq 1\} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
V(\neg\neg\Box p, a) &= \neg\neg\bigvee\{r \in \mathrm{C}_{\mathfrak{L}} \mid r \leq \bigwedge\{V(p, y) \mid Ray\}\} \\
&= \neg\neg\bigvee\{r \in \mathrm{C}_{\mathfrak{L}} \mid r \leq V(p, a)\} \\
&= \neg\neg 0 \\
&= 0
\end{aligned}$$

and thus

$$\begin{aligned}
V(\Box\neg\neg p \to \neg\neg\Box p, a) &= V(\Box\neg\neg p, a) \to V(\neg\neg\Box p, a) \\
&= 1 \to 0 \\
&= 0\,.
\end{aligned}$$

The same formula fails in a similar finite $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-model when $A = G_\downarrow$, and $\triangle\Diamond p \to \Diamond\triangle p$ fails in a similar $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-model when $A = G_\uparrow$.

Indeed, as shown below, given an $\mathbb{FK}(\mathbf{A})$-tree-model of finite height where $\varphi \in \mathrm{Fm}$ is not valid, we can always "prune" (i.e. remove branches from) the model in such a way that φ is still not valid in the resulting finite $\mathbb{FK}(\mathbf{A})$-tree-model. It then follows from part (b) of Lemma 4.4 that $\mathsf{FK}(\mathbf{A})$ and $\mathsf{FK}^{\mathsf{c}}(\mathbf{A})$ have the finite model property with respect to $\mathbb{FK}(\mathbf{A})$ and $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$, respectively.

Lemma 4.6. *Let $\Sigma \subseteq \mathrm{Fm}$ be a finite fragment. Then for any $\mathbb{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ of finite height with root x, there is a finite $\mathbb{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T_\Box}, \widehat{T_\Diamond} \rangle$ with $\langle \widehat{W}, \widehat{R}, \widehat{V} \rangle \subseteq \langle W, R, V \rangle$, root $x \in \widehat{W}$, and $|\widehat{W}| \leq |\Sigma|^{\mathrm{hg}(\mathfrak{M})}$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$.*

Proof. We prove the lemma by induction on $\mathrm{hg}(\mathfrak{M})$. For the base case, $W = \{x\}$ and it suffices to define $\widehat{\mathfrak{M}} = \mathfrak{M}$.

For the induction step $\mathrm{hg}(\mathfrak{M}) = n + 1$, consider for each $y \in R^+[x]$, the submodel $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y} \rangle$ of $\mathfrak{M}$ generated by y. Each $\mathfrak{M}_y$ is an $\mathbb{FK}(\mathbf{A})$-tree-model of finite height with root y and $\mathrm{hg}(\mathfrak{M}_y) \leq n$. Hence, by the induction hypothesis, for each $y \in R^+[x]$, there is a finite $\mathbb{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y, \widehat{T}_{\Box y}, \widehat{T}_{\Diamond y}\rangle$ with $\langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y, \rangle \subseteq \langle W_y, R_y, V_y\rangle$ and root $y \in \widehat{W}_y$, such that $|\widehat{W}_y| \leq |\Sigma|^n$ and for all $\varphi \in \Sigma$, using Lemma 4.4(a), $\widehat{V}_y(\varphi, y) = V_y(\varphi, y) = V(\varphi, y)$.

We choose a finite number of appropriate $y \in R^+[x]$ in order to build our finite $\mathbb{FK}(\mathbf{A})$-submodel $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ of $\mathfrak{M}$ as the "union" of these $\widehat{\mathfrak{M}}_y$ connected by the root world $x \in \widehat{W}$. First we define $\widehat{T}_\Box(x)$ and $\widehat{T}_\Diamond(x)$.

Consider $T_\Box(x) = A \setminus \bigcup_{i\in I}(a_i, c_i)$ for some finite $I \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I$, $a_i \in R(\mathbf{A})$, c_i witnesses right homogeneity at a_i, and the intervals (a_i, c_i) are pairwise disjoint. Consider also the finite (possibly empty) set $(V[\Sigma_\Box, x] \cap R(\mathbf{A})) \setminus \{a_i \mid i \in I\} = \{a_j \mid j \in J\}$ where $I \cap J = \emptyset$. For $j \in J$, choose a witness of right homogeneity c_j at a_j such that the intervals (a_i, c_i) are pairwise disjoint, for all $i \in I \cup J$, and $(V[\Sigma_\Box, x] \cup \mathrm{C}_\mathfrak{L}) \cap (\bigcup_{i\in I\cup J}(a_i, c_i)) = \emptyset$. We define $\widehat{T}_\Box(x) = A \setminus \bigcup_{i\in I\cup J}(a_i, c_i)$, satisfying conditions (i) and (ii) of the definition of an $\mathbb{FK}(\mathbf{A})$-model by construction. Note also that $V[\Sigma_\Box, x] \cup \mathrm{C}_\mathfrak{L} \subseteq \widehat{T}_\Box(x) \subseteq T_\Box(x)$.

Similarly, consider $T_\Diamond(x) = A \setminus \bigcup_{i\in I'}(d_i, b_i)$ for some finite $I' \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I'$, $b_i \in L(\mathbf{A})$, d_i witnesses left homogeneity at b_i, and the intervals (d_i, b_i) are pairwise disjoint. Consider also the finite (possibly empty) set $(V[\Sigma_\Diamond, x] \cap L(\mathbf{A})) \setminus \{b_i \mid i \in I'\} = \{b_j \mid j \in J'\}$. For $j \in J'$, choose a witness of left homogeneity d_j at b_j such that the intervals (d_i, b_i) are pairwise disjoint for all $i \in I' \cup J'$, and $(V[\Sigma_\Diamond, x] \cup \mathrm{C}_\mathfrak{L}) \cap (\bigcup_{i\in I'\cup J'}(d_i, b_i)) = \emptyset$. We define $\widehat{T}_\Diamond(x) = A \setminus \bigcup_{i\in I'\cup J'}(d_i, b_i)$, satisfying conditions (i) and (iii) of the definition of an $\mathbb{FK}(\mathbf{A})$-mode by construction. Note also that $V[\Sigma_\Diamond, x] \cup \mathrm{C}_\mathfrak{L} \subseteq \widehat{T}_\Diamond(x) \subseteq T_\Diamond(x)$.

Consider now $\varphi = \Box\psi \in \Sigma_\Box$ and let $a = V(\Box\psi, x) \in \widehat{T}_\Box(x)$. If $a \notin R(\mathbf{A})$, choose $y_\varphi \in R^+[x]$ such that $a = Rxy_\varphi \to V(\psi, y_\varphi)$. If $a \in R(\mathbf{A})$, there is an $i \in I \cup J$, such that $a = a_i$, and we choose $y_\varphi \in R^+[x]$ such that $Rxy_\varphi \to V(\psi, y_\varphi) \in [a_i, c_i)$. Similarly, for each $\varphi = \Diamond\psi \in \Sigma_\Diamond$, let $b = V(\Diamond\psi, x) \in T_\Diamond(x)$. If $b \notin L(\mathbf{A})$, choose $y_\varphi \in R^+[x]$ such that $b = Rxy_\varphi \wedge V(\psi, y_\varphi)$. If $b \in L(\mathbf{A})$, there is an $i \in I' \cup J'$, such that $b = b_i$ and we choose $y_\varphi \in R^+[x]$ such that $Rxy_\varphi \wedge V(\psi, y_\varphi) \in (d_i, b_i]$.

Now let $Y = \{y_\varphi \in R^+[x] \mid \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\}$, noting that $|Y| \leq |\Sigma_\Box \cup \Sigma_\Diamond| < |\Sigma|$. We define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ where

$$\widehat{W} \;=\; \{x\} \cup \bigcup_{y\in Y} \widehat{W}_y\,,$$

and $\widehat{R}$ and $\widehat{V}$ are R and V, respectively, restricted to $\widehat{W}$. $\widehat{T}_\Box(z)$ and $\widehat{T}_\Diamond(z)$ are defined as $\widehat{T}_{\Box y}(z)$ and $\widehat{T}_{\Diamond y}(z)$, respectively, if $z \in \widehat{W}_y$, for some $y \in Y$. $\widehat{T}_\Box(x)$ and $\widehat{T}_\Diamond(x)$ are defined as above.

Observe that $\langle \widehat{W}, \widehat{R}, \widehat{V}\rangle \subseteq \langle W, R, V\rangle$, $x \in \widehat{W}$ is the root of $\widehat{\mathfrak{M}}$, and $|\widehat{W}| \leq |Y||\Sigma|^n+1 < |\Sigma||\Sigma|^n = |\Sigma|^{\mathrm{hg}(\mathfrak{M})}$. Moreover, for each $y \in Y$, $\widehat{\mathfrak{M}}_y$ is an $\mathbb{FK}(\mathbf{A})$-submodel of $\widehat{\mathfrak{M}}$ generated by y. Hence, by Lemma 4.4(a) and the induction hypothesis, for all $\varphi \in \Sigma$,

$$\widehat{V}(\varphi, y) \;=\; \widehat{V}_y(\varphi, y) \;=\; V_y(\varphi, y) \;=\; V(\varphi, y)\,. \tag{21}$$

We show now that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$, proceeding by induction on $\ell(\varphi)$. The base case follows directly from the definition of $\widehat{V}$. For the inductive step, the non-modal cases follow directly using the induction hypothesis. For $\varphi = \Box\psi$, there are two cases. Suppose first that $V(\Box\psi, x) = a \notin R(\mathbf{A})$ and recall that

$$V(\Box\psi, x) \;=\; \bigvee\{r \in T_\Box(x) \mid r \leq \bigwedge\{Rxy \to V(\psi, y) \mid y \in W\}\} \;=\; a\,.$$

This implies that $Rxy \to V(\psi, y) \geq a$ for all $y \in Y \subseteq R^+[x]$. Hence, by (21), $\widehat{R}xy \to \widehat{V}(\psi, y) \geq a$ for all $y \in Y = \widehat{R}^+[x]$. Moreover, $\widehat{R}xy_\varphi \to \widehat{V}(\psi, y_\varphi) = a$ and hence, because $a \in V[\Sigma_\Box, x] \subseteq \widehat{T}_\Box(x)$,

$$\widehat{V}(\Box\psi, x) \;=\; \bigvee\{r \in \widehat{T}_\Box(x) \mid r \leq \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) \mid y \in \widehat{W}\}\} \;=\; a\,.$$

For the second case, suppose that $V(\Box\psi, x) = a \in R(\mathbf{A})$. Then $a = a_i$, for some $i \in I \cup J$, and we observe that

$$a_i \;=\; a \;\leq\; \bigwedge\{Rxy \to V(\psi, y) \mid y \in W\}\,.$$

By (21), we know that $\widehat{R}xy \to \widehat{V}(\psi, y) = Rxy \to V(\psi, y)$ for each $y \in \widehat{W}$, and because $\widehat{W} \subseteq W$, it follows that

$$a_i \;\leq\; \bigwedge\{Rxy \to V(\psi, y) \mid y \in W\} \;\leq\; \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) \mid y \in \widehat{W}\}\,.$$

By the choice of $y_\varphi \in \widehat{W}$,

$$\widehat{R}xy_\varphi \to \widehat{V}(\psi, y_\varphi) \;=\; Rxy_\varphi \to V(\psi, y_\varphi) \;<\; c_i\,.$$

Hence $a_i \leq \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) \mid y \in \widehat{W}\} < c_i$ and

$$\widehat{V}(\Box\psi, x) \;=\; \bigvee\{r \in \widehat{T}_\Box(x) \mid r \leq \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) \mid y \in \widehat{W}\}\} \;=\; a_i \;=\; a\,.$$

The case where $\varphi = \Diamond\psi$ is very similar. □

Remark 4.7. Let us suppose that in Lemma 4.6, for the $\mathbb{FK}(\mathbf{A})$-tree-model $\mathfrak{M}$ with root x, $T_\Box(x) = T_\Diamond(x) = A$. In this case, the number of intervals omitted from $\widehat{T}_\Box(x)$ and $\widehat{T}_\Diamond(x)$, defined in the proof, is smaller than or equal to the cardinality of $\Sigma_\Box$ and $\Sigma_\Diamond$, respectively, for the given finite fragment Σ. This is because the left endpoints of the intervals utilized in the proof to define $\widehat{T}_\Box(x)$ in the finite $\mathbb{FK}(\mathbf{A})$-tree-model belong to $V[\Sigma(\varphi)_\Box, x]$, and similarly for $\widehat{T}_\Diamond(x)$.

4.2 Relating the Two Semantics

Let us assume again that $\mathbf{A}$ is a locally homogeneous order-based algebra. We devote this section to establishing that a formula is valid in $\mathsf{K}(\mathbf{A})$ or $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ if and only if it is valid in $\mathsf{FK}(\mathbf{A})$ or $\mathsf{FK}^{\mathsf{c}}(\mathbf{A})$, respectively. Observe first that any $\mathbb{K}(\mathbf{A})$-model can be extended to an $\mathbb{FK}(\mathbf{A})$-model with the same valid formulas simply by defining $T_\Box$ and $T_\Diamond$ to be constantly A. Hence any $\mathsf{FK}(\mathbf{A})$-valid formula is also $\mathsf{K}(\mathbf{A})$-valid. We therefore turn our attention to the other (much harder) direction: proving that any $\mathsf{K}(\mathbf{A})$-valid formula is also $\mathsf{FK}(\mathbf{A})$-valid.

The main ingredient of the proof (see Lemma 4.10) is the construction of a $\mathbb{K}(\mathbf{A})$-tree-model taking the same values for formulas at its root as a given $\mathbb{FK}(\mathbf{A})$-tree-model. Note that the original $\mathbb{FK}(\mathbf{A})$-tree-model without the functions $T_\Box$ and $T_\Diamond$ cannot play this role in general; in $[0,1]$, for example, the infimum or supremum required for calculating the value of a box-formula or diamond-formula at the root x might not be in the set $T_\Box(x)$ or $T_\Diamond(x)$. This problem is resolved by taking infinitely many copies of an inductively defined $\mathbb{K}(\mathbf{A})$-model in such a way that certain parts of the intervals in A missing in $T_\Box(x)$ or $T_\Diamond(x)$ are "squeezed" closer to either their lower or upper bounds. The obtained infima and suprema will then coincide with the next smaller or larger member of $T_\Box(x)$ and $T_\Diamond(x)$: that is, the required values of the formulas at x in the original $\mathbb{FK}(\mathbf{A})$-tree-model. The following example illustrates this idea for the relatively simple case where $\mathbf{A} = \mathbf{G}$.

Example 4.8. Consider the $\mathbb{FK}^{\mathsf{c}}(\mathbf{G})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ with $W = \{x, y\}$, $R = \{(x, y)\}$, and $T_\Box(x) = [0, 1] \setminus (0.2, 0.8)$. Note that $0.2 \in R(\mathbf{G})$ and that 0.8 witnesses right homogeneity at 0.2. Suppose that $V(p, y) = 0.6$, so that

$$\begin{aligned} V(\Box p, x) &= \bigvee\{r \in T_\Box(x) \mid r \leq \bigwedge\{V(p, y) \mid Rxy\}\} \\ &= \bigvee\{r \in [0,1] \setminus (0.2, 0.8) \mid r \leq 0.6\} \\ &= 0.2\,. \end{aligned}$$

For each $k \geq 2$, we then consider the $\mathbb{K}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M}_k = \langle W_k, R_k, V_k\rangle$ with $W_k = \{y_k\}$, $R_k = \emptyset$, and $V_k(p, y_k) = h_k(V(p, y))$, for some deflationary $\{0,1\}$-order embedding $h_k\colon [0,1] \to [0,1]$, satisfying

$$h_k[[0.2, 0.8)] = [0.2, 0.2 + \tfrac{1}{k})\,.$$

Defining the $\mathbb{K}^{\mathsf{c}}(\mathbf{G})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$, with $\widehat{W} = \{x\} \cup \{y_k \mid k \geq 2\}$, $\widehat{R} = \{(x, y_k) \mid k \geq 2\}$, and $\widehat{V}(p, y_k) = V_k(p, y_k)$, we obtain (see Figure 4.1):

$$\begin{aligned} \widehat{V}(\Box p, x) &= \bigwedge\{\widehat{V}(p, y_k) \mid \widehat{R}xy_k\} \\ &= 0.2 \\ &= V(\Box p, x). \end{aligned}$$

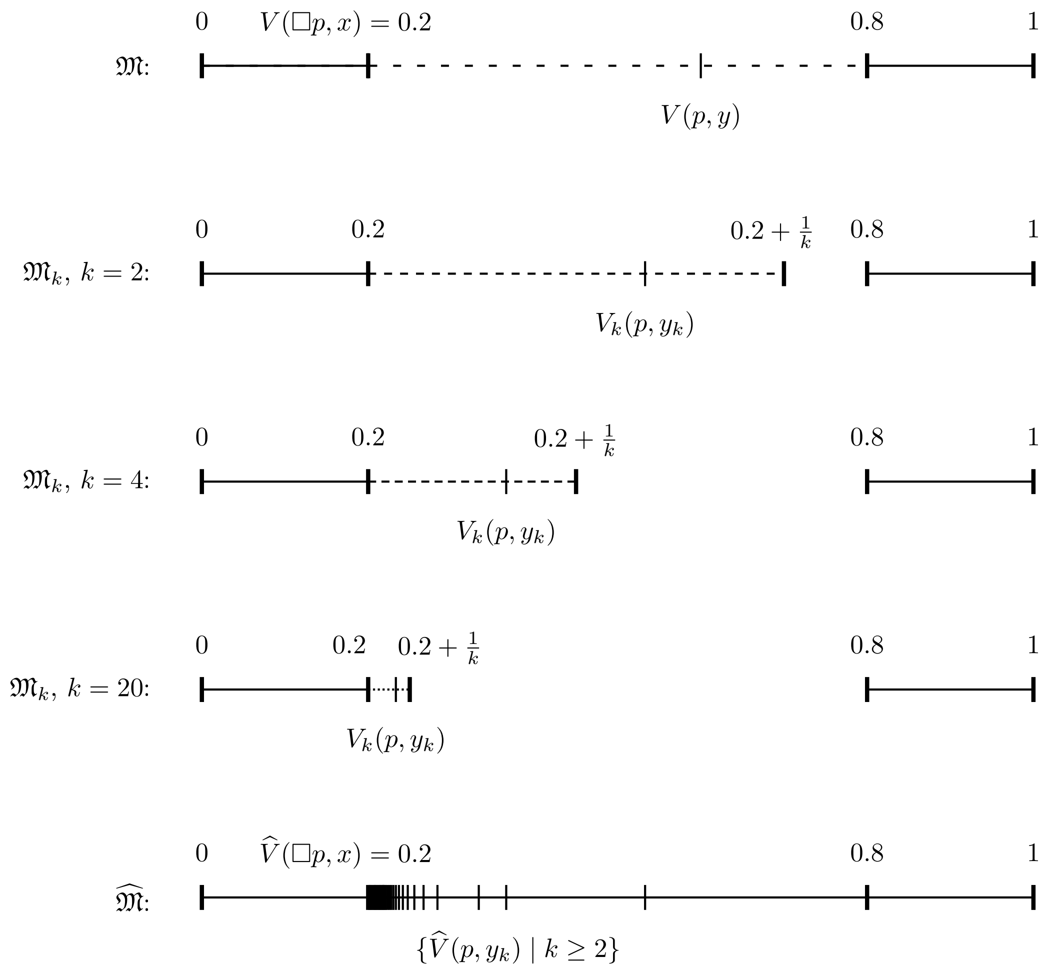

Figure 4.1: Squeezing models

A central tool in the proof of Lemma 4.10 below is the following technical result, which allows the "squeezing" of $\mathbb{K}(\mathbf{A})$-models so that the values of formulas are arbitrarily close to certain points (as in Example 4.8). Intuitively, in the proof of Lemma 4.10, the set B in Lemma 4.9 will be the set of values at the root world x of all box-formulas and diamond-formulas in some fragment Σ. Furthermore, the values a and c, in (a) of Lemma 4.9, will denote the endpoints of the removed interval and s will be the relevant value that we want to squeeze closer and closer towards a. The value t, the upper endpoint of the squeezed interval, will then be chosen in $A \setminus (B \cap L(\mathbf{A}))$ in order to ensure that all the suprema in B (relevant for determining the values of diamond-formulas in Σ) are preserved by the squeezing. Note that $u \in [0,1]$ can be any value as close to a as needed (e.g. $u = a + \frac{1}{k}$ for any $k \in \mathbb{Z}^+$) so as to squeeze the interval $[a,t)$ into $[a,u)$ by the B-complete deflationary order embedding h, with the intention that $s \in [a,t)$ and $h(s) \in [a,u)$. For (b) of Lemma 4.9, the ideas are very similar.

Lemma 4.9. *Let $B \subseteq A$ be countable.*

(a) *Given $a \in R(\mathbf{A})$, some witness $c > a$ of right homogeneity at a, and an $s \in [a,c)$, there is a $t \in (s,c]$ such that $t \notin B \cap L(\mathbf{A})$. Moreover, for all $u \in (a,t]$, there is a B-complete deflationary order embedding $h\colon A \to A$ such that*

$$h[[a,t)] \subseteq [a,u) \qquad \text{and} \qquad h|_{A\setminus(a,t)} = \mathrm{id}_A .$$

(b) *Given $b \in L(\mathbf{A})$, some witness $d < b$ of left homogeneity at b, and an $s \in (d,b]$, there is a $t \in [d,s)$ such that $t \notin B \cap R(\mathbf{A})$. Moreover, for all $u \in [t,b)$, there is a B-complete inflationary order embedding $h\colon A \to A$ such that*

$$h[(t,b]] \subseteq (u,b] \qquad \text{and} \qquad h|_{A\setminus(t,b)} = \mathrm{id}_A .$$

Proof. For (a), let $B \subseteq A$ be countable and consider $a \in R(\mathbf{A})$, a witness c of right homogeneity at a, and $s \in [a,c)$. We first prove that there is a $t \in (s,c]$ which is either in $A \setminus L(\mathbf{A})$ or in $A \setminus B$. If $c \notin L(\mathbf{A})$, choose $t = c$. If $c \in L(\mathbf{A})$, then $[s,c]$ is infinite. Recall that A is a complete sublattice of $[0,1]$ and that every non-empty perfect set of real numbers (closed and containing no isolated points) is uncountable. Hence if $[s,c]$ is countable, there must be an isolated point $t \in (s,c]$ such that $t \notin L(\mathbf{A})$. If $[s,c]$ is uncountable, then there is a $t \in (s,c] \setminus B$, as B is countable. Either way, there is a $t \in (s,c]$ such that $t \notin B \cap L(\mathbf{A})$.

Now we define the embedding. Because $t \leq c$ also witnesses right homogeneity at a, for each $u \in (a,t]$, there is a complete deflationary order embedding $g\colon [a,t) \to [a,u)$ with $g(a) = a$. Define h as g on $[a,t)$ and as the identity on $A \setminus [a,t)$. Then all arbitrary meets and joins in A are preserved except in the case where t is a join of elements in $[a,t)$ and so $t \in L(\mathbf{A})$. But in this case $t \notin B$. Hence (a) holds. For (b), we use a very similar argument. $\square$

Lemma 4.10. *Let Σ be a finite fragment and let $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ be a finite $\mathbb{FK}(\mathbf{A})$-tree-model with root x. Then there is a countable $\mathbb{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with root $\widehat{x}$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \Sigma$. Moreover, if $\mathfrak{M}$ is crisp, then so is $\widehat{\mathfrak{M}}$.*

Proof. The lemma is proved by induction on $\mathrm{hg}(\mathfrak{M})$. The base case is immediate, fixing $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with $\widehat{W} = W = \{x\}$, $\widehat{R} = R$, and $\widehat{V} = V$. For the induction step, given $y \in R^+[x]$, let $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y} \rangle$ be the submodel of $\mathfrak{M}$ generated by y. Then $\mathfrak{M}_y$ is a finite $\mathbb{FK}(\mathbf{A})$-tree-model with root y, $\mathrm{hg}(\mathfrak{M}_y) < \mathrm{hg}(\mathfrak{M})$, and, by Lemma 4.4(a), $V_y(\varphi, z) = V(\varphi, z)$ for all $z \in W_y$ and $\varphi \in \mathrm{Fm}$. So, by the induction hypothesis, there is a countable $\mathbb{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y \rangle$ (crisp if $\mathfrak{M}$ is crisp) with root $\widehat{y}$ such that $\widehat{V}_y(\varphi, \widehat{y}) = V_y(\varphi, y) = V(\varphi, y)$ for all $\varphi \in \Sigma$.

For each $\varphi \in \Sigma_\Box \cup \Sigma_\Diamond$, we will choose a world $y_\varphi \in R^+[x]$ as described below and then, using Lemma 4.9, define for each $k \in \mathbb{Z}^+$ a copy of the $\mathbb{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_{y_\varphi}$, denoted $\widehat{\mathfrak{M}}^k_\varphi$. Suppose that $\varphi = \Box\psi \in \Sigma_\Box$. Consider $T_\Box(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)$ for some finite $I \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I$, $a_i \in R(\mathbf{A})$, c_i witnesses right homogeneity at a_i, and the intervals (a_i, c_i) are pairwise disjoint. There are two cases.

(i) Suppose that $V(\Box\psi, x) = a_i$ for some $i \in I$. Recalling that

$$a_i \;=\; V(\Box\psi, x) \;=\; \bigvee\{r \in T_\Box(x) \mid r \leq \bigwedge\{Rxy \to V(\psi, y) \mid y \in W\}\},$$

there must be a world $y_\varphi \in R^+[x]$ such that

$$Rxy_\varphi \to V(\psi, y_\varphi) \;\in\; [a_i, c_i)\,.$$

We fix $B = \widehat{V}_{y_\varphi}[\Sigma_\Box \cup \Sigma_\Diamond, \widehat{W}_{y_\varphi}]$, which is countable because $\widehat{W}_{y_\varphi}$ is countable and $\Sigma_\Box \cup \Sigma_\Diamond$ is finite. Using Lemma 4.9, for some t satisfying

$$a_i \;\leq\; s \;=\; Rxy_\varphi \to V(\psi, y_\varphi) \;<\; t \;\leq\; c_i\,,$$

there exists for each $k \in \mathbb{Z}^+$, a B-complete deflationary order embedding $h_k \colon A \to A$ mapping $[a_i, t)$ into $[a_i, a_i + \frac{1}{k})$, and $h_k|_{A \setminus (a_i, t)} = \mathrm{id}_A$. Clearly, this implies that for all $k \in \mathbb{Z}^+$, h_k is a $\widehat{V}_{y_\varphi}[\Sigma_\Box \cup \Sigma_\Diamond, \widehat{W}_{y_\varphi}]$-complete deflationary $\mathrm{C}_\mathfrak{L}$-order embedding. We then define the copy $\widehat{\mathfrak{M}}^k_\varphi = \langle \widehat{W}^k_\varphi, \widehat{R}^k_\varphi, \widehat{V}^k_\varphi \rangle$ of $\widehat{\mathfrak{M}}_{y_\varphi}$ as follows:

- $\widehat{W}^k_\varphi$ is a copy of $\widehat{W}_{y_\varphi}$, denoting the copy of $\widehat{x}_{y_\varphi} \in \widehat{W}_{y_\varphi}$ by $\widehat{x}^k_\varphi$
- $\widehat{R}^k_\varphi \widehat{x}^k_\varphi \widehat{z}^k_\varphi = h_k(\widehat{R}_{y_\varphi} \widehat{x}_{y_\varphi} \widehat{z}_{y_\varphi})$ for $\widehat{x}_{y_\varphi}, \widehat{z}_{y_\varphi} \in \widehat{W}_{y_\varphi}$
- $\widehat{V}^k_\varphi(p, \widehat{x}^k_\varphi) = h_k(\widehat{V}_{y_\varphi}(p, \widehat{x}_{y_\varphi}))$ for $\widehat{x}_{y_\varphi} \in \widehat{W}_{y_\varphi}$.

Because h_k is a $\widehat{V}_{y_\varphi}[\Sigma_\Box \cup \Sigma_\Diamond, \widehat{W}_{y_\varphi}]$-complete deflationary $\mathrm{C}_\mathfrak{L}$-order embedding, it follows by Lemma 3.10 that $\widehat{V}^k_\varphi(\chi, \widehat{y}^k_\varphi) = h_k(\widehat{V}_{y_\varphi}(\chi, \widehat{y}_\varphi))$ for all $\chi \in \Sigma$. By the induction hypothesis,

$$\begin{aligned}(\dagger)\quad h_k(Rxy_\varphi) \to \widehat{V}^k_\varphi(\psi, \widehat{y}^k_\varphi) &= h_k(Rxy_\varphi) \to h_k(\widehat{V}_{y_\varphi}(\psi, \widehat{y}_\varphi)) \\ &= h_k(Rxy_\varphi \to \widehat{V}_{y_\varphi}(\psi, \widehat{y}_\varphi)) \\ &= h_k(Rxy_\varphi \to V(\psi, y_\varphi)) \\ &= h_k(s) \\ &\in [a_i, a_i + \tfrac{1}{k}).\end{aligned}$$

(ii) Suppose that $V(\Box\psi, x) \neq a_i$ for all $i \in I$. In this case, $V(\Box\psi, x) = \bigwedge\{Rxy \to V(\psi, y) \mid y \in W\}$ and, because W is finite, there is a $y_\varphi \in W$, such that, by the induction hypothesis,

$$V(\Box\psi, x) = Rxy_\varphi \to V(\psi, y_\varphi) = Rxy_\varphi \to \widehat{V}_{y_\varphi}(\psi, y_\varphi).$$

In this case, let h_k be the identity function on A and $\widehat{\mathfrak{M}}^k_\varphi = \langle \widehat{W}^k_\varphi, \widehat{R}^k_\varphi, \widehat{V}^k_\varphi \rangle = \widehat{\mathfrak{M}}_{y_\varphi}$.

In a similar fashion, when $\varphi = \Diamond\psi \in \Sigma_\Diamond$, we obtain for each $k \in \mathbb{Z}^+$, a $\mathbb{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}^k_\varphi$ as a copy of $\widehat{\mathfrak{M}}_{y_\varphi}$.

We now define the $\mathbb{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ by

$$\widehat{W} = \{\widehat{x}\} \cup \bigcup_{\varphi \in \Sigma_\Box \cup \Sigma_\Diamond} \bigcup_{k \in \mathbb{Z}^+} \widehat{W}^k_\varphi$$

$$\widehat{R}wz = \begin{cases} \widehat{R}^k_\varphi wz & \text{if } w, z \in \widehat{W}^k_\varphi \text{ for some } \varphi \in \Sigma_\Box \cup \Sigma_\Diamond,\ k \in \mathbb{Z}^+ \\ h_k(Rxy_\varphi) & \text{if } w = \widehat{x},\ z = \widehat{y}^k_\varphi \in \widehat{W}^k_\varphi \text{ for } \varphi \in \Sigma_\Box \cup \Sigma_\Diamond,\ k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$$

$$\widehat{V}(p, z) = \begin{cases} \widehat{V}^k_\varphi(p, z) & \text{if } z \in \widehat{W}^k_\varphi \text{ for some } \varphi \in \Sigma_\Box \cup \Sigma_\Diamond,\ k \in \mathbb{Z}^+ \\ V(p, x) & \text{if } z = \widehat{x}. \end{cases}$$

If $\mathfrak{M}$ is crisp, then for all $\varphi \in \Sigma_\Box \cup \Sigma_\Diamond$, $\widehat{\mathfrak{M}}_{y_\varphi}$ is crisp and so also are $\widehat{\mathfrak{M}}^k_\varphi$ for all $k \in \mathbb{Z}^+$. Hence, by construction, $\widehat{\mathfrak{M}}$ is crisp. Moreover, as there are only finitely many different countable $\widehat{\mathfrak{M}}_{y_\varphi}$, and we only take countably many copies of each one, $\widehat{\mathfrak{M}}$ is also countable.

Observe now that for each $\widehat{y}^k_\varphi \in \widehat{R}^+[\widehat{x}]$, we have that $\widehat{\mathfrak{M}}^k_\varphi$ is the submodel of $\widehat{\mathfrak{M}}$ generated by $\widehat{y}^k_\varphi$. Hence, by Lemma 3.5, for all $\chi \in \Sigma$ and $\widehat{y}^k_\varphi \in \widehat{R}^+[\widehat{x}]$,

$$(\ddagger)\quad \widehat{V}(\chi, \widehat{y}^k_\varphi) = \widehat{V}^k_\varphi(\chi, \widehat{y}^k_\varphi) = h_k(\widehat{V}_{y_\varphi}(\chi, \widehat{y}_\varphi)) = h_k(V_{y_\varphi}(\chi, y_\varphi)) = h_k(V(\chi, y_\varphi)).$$

Finally, we prove that $\widehat{V}(\chi, \widehat{x}) = V(\chi, x)$ for all $\chi \in \Sigma$, proceeding by induction on $\ell(\chi)$. The base case follows directly from the definition of $\widehat{V}$. For the induction step, the cases for the non-modal connectives follow easily using the induction hypothesis. Let us just consider the case $\chi = \varphi = \Box\psi$ (a formula in $\Sigma_\Box$), the case $\chi = \Diamond\psi$ being very similar. There are two possibilities.

(i) Suppose that $V(\Box\psi, x) = a_i$ for some $i \in I$. Then for all $z \in W$, we have $Rxz \to V(\psi, z) \geq a_i$. Note that it is not possible for any $a \in A$ and h_k defined above that $h_k(a) < a_i \leq a$, as h_k is either the identity on $T_\Box(x)$ or is inflationary on A. So by construction, for all $\widehat{z} \in \widehat{W}$,

$$\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) \ \geq \ a_i\,.$$

Moreover, for $y_\varphi \in W$,

$$Rxy_\varphi \to V(\psi, y_\varphi) \ \in \ [a_i, c_i)\,,$$

and by (†) and (‡),

$$\begin{aligned}
a_i \ &\leq \ \bigwedge\{\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) \mid \widehat{z} \in \widehat{W}\} \\
&\leq \ \bigwedge\{\widehat{R}\widehat{x}\widehat{y}^k_\varphi \to \widehat{V}(\psi, \widehat{y}^k_\varphi) \mid k \in \mathbb{Z}^+\} \\
&= \ \bigwedge\{h_k(Rxy_\varphi) \to \widehat{V}^k_\varphi(\psi, \widehat{y}^k_\varphi) \mid k \in \mathbb{Z}^+\} \\
&= \ \bigwedge\{h_k(Rxy_\varphi \to V(\psi, y_\varphi)) \mid k \in \mathbb{Z}^+\} \\
&\leq \ \bigwedge\{a_i + \tfrac{1}{k} \mid k \in \mathbb{Z}^+\} \\
&= \ a_i\,.
\end{aligned}$$

So $\widehat{V}(\Box\psi, \widehat{x}) = \bigwedge\{\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) : \widehat{z} \in \widehat{W}\} = a_i = V(\Box\psi, x)$ as required.

(ii) Suppose that $V(\Box\psi, x) \neq a_i$ for all $i \in I$. Again, for all $z \in W$, we have that $Rxz \to V(\psi, z) \geq V(\Box\psi, x) \in T_\Box(x)$. As h_k is either the identity on $T_\Box(x)$ or is inflationary on A, by construction, for all $\widehat{z} \in \widehat{W}$,

$$\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) \ \geq \ V(\Box\psi, x)\,.$$

Moreover, as in (ii) above, because W is finite, there is a $y_\varphi \in W$ such that

$$Rxy_\varphi \to V(\psi, y_\varphi) \ = \ V(\Box\psi, x)\,.$$

Using (‡) and the fact that h_k is either the identity on $T_\Box(x)$ or inflationary on A,

$$\begin{aligned}
\widehat{V}(\Box\psi, \widehat{x}) \ &= \ \bigwedge\{\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) \mid \widehat{z} \in \widehat{W}\} \\
&= \ \bigwedge\{\widehat{R}\widehat{x}\widehat{y}^k_\varphi \to \widehat{V}(\psi, \widehat{y}^k_\varphi) \mid k \in \mathbb{Z}^+\} \\
&= \ \bigwedge\{h_k(Rxy_\varphi) \to \widehat{V}^k_\varphi(\psi, \widehat{y}^k_\varphi) \mid k \in \mathbb{Z}^+\} \\
&= \ \bigwedge\{h_k(Rxy_\varphi \to V(\psi, y_\varphi)) \mid k \in \mathbb{Z}^+\} \\
&= \ Rxy_\varphi \to V(\psi, y_\varphi) \\
&= \ V(\Box\psi, x)\,.
\end{aligned}$$

So $\widehat{V}(\Box\psi, \widehat{x}) = V(\Box\psi, x)$ as required. $\Box$

We then obtain the following equivalences.

Theorem 4.11.

(a) *For any formula* $\varphi \in \mathrm{Fm}$*, the following are equivalent:*

(*i*) $\models_{\mathbb{K}(\mathbf{A})} \varphi$

(*ii*) $\models_{\mathbb{FK}(\mathbf{A})} \varphi$

(*iii*) φ *is valid in all finite* $\mathbb{FK}(\mathbf{A})$*-tree-models* $\langle W, R, V, T_\Box, T_\Diamond \rangle$ *such that* $|W| \leq (\ell(\varphi) + |\mathrm{C}_{\mathfrak{L}}|)^{\ell(\varphi)}$

(b) *For any formula* $\varphi \in \mathrm{Fm}$*, the following are equivalent:*

(*i*) $\models_{\mathbb{K}^{\mathsf{c}}(\mathbf{A})} \varphi$

(*ii*) $\models_{\mathbb{FK}^{\mathsf{c}}(\mathbf{A})} \varphi$

(*iii*) φ *is valid in all finite* $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$*-tree-models* $\langle W, R, V, T_\Box, T_\Diamond \rangle$ *such that* $|W| \leq (\ell(\varphi) + |\mathrm{C}_{\mathfrak{L}}|)^{\ell(\varphi)}$

Proof. For (a), the step from (*ii*) to (*i*) is immediate using the fact that every $\mathbb{K}(\mathbf{A})$-tree-model can be extended to an $\mathbb{FK}(\mathbf{A})$-tree-model with the same valid formulas by setting $T_\Box$ and $T_\Diamond$ to be constantly A. For the steps from (*i*) to (*iii*) and from (*iii*) to (*ii*), suppose that $\not\models_{\mathbb{FK}(\mathbf{A})} \varphi$. By Lemmas 4.4 and 4.6, there is a finite $\mathbb{FK}(\mathbf{A})$-tree-model $\langle W, R, V, T_\Box, T_\Diamond \rangle$ with root x such that $V(\varphi, x) < 1$ and $|W| \leq (\ell(\varphi) + |\mathrm{C}_{\mathfrak{L}}|)^{\ell(\varphi)}$. By Lemma 4.10, we obtain a $\mathbb{K}(\mathbf{A})$-tree-model $\langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with root $\widehat{x}$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x) < 1$. So $\not\models_{\mathbb{K}(\mathbf{A})} \varphi$.

The proof of (b) is very similar, using the fact that Lemmas 4.4, 4.6, and 4.10 preserve crisp models. $\Box$

Let us extend the notion of finite model property as follows: a logic L for $\mathfrak{L}_{\mathfrak{m}}$ has the *finite model property with respect to* $\mathbb{FK}(\mathbf{A})$ ($\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$), if for all $\varphi \in \mathrm{Fm}$, φ is valid in L if and only if φ is valid in all finite $\mathbb{FK}(\mathbf{A})$-models (all finite $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-models). The following corollary is then an immediate consequence of Theorem 4.11.

Corollary 4.12. *The logics* $\mathsf{K}(\mathbf{A})$ *and* $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ *have the finite model property with respect to* $\mathbb{FK}(\mathbf{A})$ *and* $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$*, respectively.*

4.3 Decidability and Complexity

Let us assume again that $\mathbf{A}$ is a locally homogeneous order-based algebra. In this section, we will use the finite model property of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ with respect to $\mathbb{FK}(\mathbf{A})$ and $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$, respectively, to obtain decidability and complexity results for the validity problem of these logics in various cases. We prove, in particular, that the validity problem of the Gödel modal logics GK and GK^{c} (i.e. where $\mathbf{A}$ is $\mathbf{G}$) are both PSPACE-complete and that the same is true for $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ in case $\mathbf{A}$ is $\mathbf{G}_{\downarrow}$ or $\mathbf{G}_{\uparrow}$. These and other results in this section contrast with the fact that no first-order expansion of a Gödel logic based on a countably infinite set of truth values is recursively axiomatizable (see [8]).

A standard reference book for notions in complexity theory is [104]. Moreover, the more recent text book [2] covers the state of art of complexity theory.

For simplicity of exposition, we will assume that the only constants are $\top$ and $\bot$. To explain the ideas involved in the proofs, consider $\varphi \in \mathrm{Fm}$ and $n = |\Sigma(\varphi)| = \ell(\varphi) + |\mathrm{C}_{\mathfrak{L}}| = \ell(\varphi) + 2$. To check that φ is not $\mathsf{K}(\mathbf{A})$-valid, it suffices, by Lemmas 4.4, 4.6, and 4.10, to find a finite $\mathbb{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond} \rangle$ of height $\leq \ell(\varphi)$ with root x and $|W| \leq |\Sigma(\varphi)|^{\ell(\varphi)} \leq n^n$ such that $V(\varphi, x) < 1$.

If $\mathbf{A}$ is infinite, then $T_{\Box}(x)$ and $T_{\Diamond}(x)$ may also be infinite, and hence $\mathfrak{M}$ may not be a computational object. We therefore introduce a modified version of $\mathfrak{M}$:

$$\mathfrak{M}^* \;=\; \langle W, R, V, \{\Phi(x)\}_{x \in W}, \{\Psi(x)\}_{x \in W} \rangle,$$

where for each $x \in W$, $\Phi(x) \subseteq A^2$ is the set of ordered pairs for which $T_{\Box}(x) = A \setminus \bigcup_{\langle r,s \rangle \in \Phi(x)} (r, s)$, and $\Psi(x) \subseteq A^2$ is the set of ordered pairs defining $T_{\Diamond}(x)$. Using the proof of Lemma 4.6 applied to a $\mathbb{K}(\mathbf{A})$-model, we may assume that $|\Phi(x)|, |\Psi(x)| \leq |\Sigma(\varphi)| = n$ for all $x \in W$ (see Remark 4.7). Let us define inductively in $\mathfrak{M}^*$, for all $x \in W$ and $\psi \in \mathrm{Fm}$,

$$V(\Box\psi, x) \;=\; \begin{cases} r & \text{if } \bigwedge\{Rxy \to V(\psi, y) \mid y \in W\} \in (r, s) \\ & \text{for some } \langle r, s \rangle \in \Phi(x) \\ \bigwedge\{Rxy \to V(\psi, y) \mid y \in W\} & \text{otherwise,} \end{cases}$$

$$V(\Diamond\psi, x) \;=\; \begin{cases} s & \text{if } \bigvee\{Rxy \wedge V(\psi, y) \mid y \in W\} \in (r, s) \\ & \text{for some } \langle r, s \rangle \in \Psi(x) \\ \bigvee\{Rxy \wedge V(\psi, y) \mid y \in W\} & \text{otherwise.} \end{cases}$$

Then $\mathfrak{M}^*$ and $\mathfrak{M}$ assign the same values to a formula at any world. Moreover, for $\chi \in \Sigma(\varphi)$, the computation of $V(\chi, x)$ in $\mathfrak{M}^*$ involves only the set of values

$$N \;=\; V[\Sigma(\varphi), W] \cup \{Rxy \mid x, y \in W\} \cup \{r, s \mid \langle r, s \rangle \in \Phi(x) \cup \Psi(x),\ x \in W\}.$$

Note that $|N| \leq 4n^{2n} = e_n$. Hence, we may assume that R and V take values in the fixed set $A(e_n)$, where for $m \in \mathbb{Z}^+$,

$$A(m) \;=\; \{0, \tfrac{1}{m}, \ldots, \tfrac{m-1}{m}, 1\}.$$

We can also assume that W is $W_n \subseteq \{0, 1, \ldots, n^n\}$, yielding a finite structure

$$\mathfrak{M}^* \;=\; \langle W_n, R, V, \{\Phi(i)\}_{i \in W_n}, \{\Psi(i)\}_{i \in W_n} \rangle,$$

where $\langle W_n, R^+\rangle$ is a tree with root 0 of height $\leq n$ and branching $\leq n$, and the sets $\Phi(i), \Psi(i)$, for $i \in W_n$, determine the endpoints of a family of disjoint open intervals in $A(e_n)$. We will call this kind of structure a (*crisp* if R is crisp) $\mathbb{FK}(e_n)$-*tree-model.* In order to recover the connection with the original $\mathbb{FK}(\mathbf{A})$-model, we introduce the following convenient notion.

A *finite system* is a triple $\mathbf{A}(m) = \langle A(m), \Phi, \Psi\rangle$ where $\Phi, \Psi \subseteq A(m)^2$. We call $\mathbf{A}(m)$ *consistent with* $\mathbf{A}$ if for some order-preserving embedding $h\colon A(m) \to A$, satisfying $h(0) = 0$ and $h(1) = 1$,

- $h(c)$ witnesses right homogeneity at $h(a) \in R(\mathbf{A})$ for all $\langle a, c\rangle \in \Phi$,
- $h(d)$ witnesses left homogeneity at $h(b) \in L(\mathbf{A})$ for all $\langle d, b\rangle \in \Psi$.

Then we obtain from the previous discussion:

Theorem 4.13. *The validity problems of* $\mathsf{K}(\mathbf{A})$ *and* $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ *are decidable if the problem of consistency of finite systems* $\mathbf{A}(m)$ *with* $\mathbf{A}$ *is decidable. Moreover, the validity problems of* $\mathsf{K}(\mathbf{A})$ *and* $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ *are* coNEXPTIME-*reducible (in the length of the formula) to the problem of consistency of finite systems* $\mathbf{A}(m)$ *with* $\mathbf{A}$.

Proof. As observed above, $\varphi \in \mathrm{Fm}$ with $n = \ell(\varphi) + 2$ is not $\mathsf{K}(\mathbf{A})$-valid ($\mathsf{K}^{\mathsf{c}}(\mathbf{A})$-valid) if and only if there is a (crisp) $\mathbb{FK}(e_n)$-tree-model of the form $\mathfrak{M}^* = \langle W_n, R, V, \{\Phi(i)\}_{i\in W_n}, \{\Psi(i)\}_{i\in W_n}\rangle$ for which $V(\varphi, 0) < 1$ and the finite system $\mathbf{A}(e_n) = \langle A(e_n), \bigcup_{i\in W_n} \Phi(i), \bigcup_{i\in W_n} \Psi(i)\rangle$ is consistent with $\mathbf{A}$.

Choose non-deterministically $V\colon \mathrm{Var}(\varphi) \to A(e_n)$, $R\colon W_n^2 \to A(e_n)$, and $\Phi(i), \Psi(i) \subseteq A(e_n)^2$ for all $i \in W_n$ to obtain the (crisp) $\mathbb{FK}(e_n)$-tree-model $\mathfrak{M}^*$, and compute $V(\varphi, 0)$ to verify $V(\varphi, 0) < 1$. This takes a number of steps bounded by a constant multiple of e_n. Then utilize an oracle to verify the consistency of $\mathbf{A}(e_n)$ with $\mathbf{A}$. $\square$

Example 4.14. Any finite system $\mathbf{A}(m) = \langle A(m), \Phi, \Psi\rangle$ is consistent with $\mathbf{G}$. Also $\mathbf{A}(m)$ is consistent with $\mathbf{G}_{\downarrow}$ if and only if $\Psi = \emptyset$ and $\Phi = \{(0, \frac{k_1}{m}), \ldots, (0, \frac{k_l}{m})\}$ for some $l \in \mathbb{Z}^+$ and $k_1, \ldots, k_l \in \mathbb{N}$, or is $\emptyset$, and $\mathbf{A}(m)$ is consistent with $\mathbf{G}_{\uparrow}$ if and only if $\Phi = \emptyset$ and $\Psi = \{(\frac{k_1}{m}, 1), \ldots, (\frac{k_l}{m}, 1)\}$ for some $l \in \mathbb{Z}^+$ and $k_1, \ldots, k_l \in \mathbb{N}$, or is $\emptyset$. Hence in these cases the consistency problem is obviously decidable in linear time and space (null-space if the size of the input tape is not considered).

Moreover, it is easy to verify inductively that any algebra $\mathbf{A}$ obtained from $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, and finite order-based algebras as a finite combination of ordered sums, lexicographical products, and fusion of consecutive points has a (PTIME) decidable consistency problem. In all of these cases, validity in $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ is (coNEXPTIME) decidable. This includes the case when A, as an ordered set, is isomorphic to an ordinal $\alpha + 1 < \omega^{\omega}$ or its reverse.

The algebras $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, and finite order-based algebras have the additional property that if the finite systems $\langle A(m), \Phi_i, \Psi_i \rangle$, for $i = 0, \ldots, k$, are consistent with $\mathbf{A}$, then the same holds for $\langle A(m), \bigcup_{i \leq k} \Phi_i, \bigcup_{i \leq k} \Psi_i \rangle$. This will allow us to improve the decidability result in these cases to PSPACE-completeness. First, however, we need a result about $\mathbb{FK}(e_n)$-tree-models.

Lemma 4.15. *The following problem is* PSPACE*-reducible (in n) to the consistency of finite systems with $\mathbf{A}$:*

Given $\Sigma = \{\varphi_1, \ldots, \varphi_k\} \subseteq$ Fm *(not necessarily distinct formulas) such that $k \leq n$ and $\ell(\varphi_j) \leq n$ for $j = 1, \ldots, k$, and given intervals $I_1, \ldots, I_k \subseteq A(e_n)$ (closed or open at their endpoints), determine if there exists a (crisp) $\mathbb{FK}(e_n)$-tree-model $\mathfrak{M}^* = \langle W_n, R, V, \{\Phi(i)\}_{i \in W_n}, \{\Psi(i)\}_{i \in W_n} \rangle$ with root* 0 *and height $\leq n$ such that $V(\varphi_j, 0) \in I_j$, for $j = 1, \ldots, k$, and for $i \in W_n$, the system $\langle A(e_n), \Phi(i), \Psi(i) \rangle$ is consistent with $\mathbf{A}$.*

Proof. As PSPACE = NPSPACE (see [113]), it suffices to give a non-deterministic polynomial space algorithm to produce the $\mathbb{FK}(e_n)$-tree-mode $\mathfrak{M}^*$. Because the full model may need exponential space to be displayed, our strategy is to search sequentially the branches of $\mathfrak{M}^*$, from the root down, so that all branches are built in the same polynomial space. This is the basic idea of Richard Ladner's proof in [88] of the PSPACE complexity of the classical modal logic K. We do not try to optimize the space bound but show that $22n^5$ does the job.

Input. Each value in $A(e_n)$ may be represented by a binary word of length at most $\log e_n \leq 2n^2$, and the only information we need from the input, besides Σ, is the maximum (strictly smaller than 1) of $A(e_n)$ and the endpoints of the intervals I_j, indicating if they are included or not in the intervals. We consider also as part of the input a particular world $x \in W_n$, written in binary notation (length $\leq \log n^n \leq n^2$). At the initial stage, $x = 0$. With appropriate markings in the formulas, we may also assume that each φ_j appears decomposed in the form:

$$\varphi_j \;=\; \chi_j(p_1, \ldots, p_l, \Box\psi_1^j, \ldots, \Box\psi_{n_j}^j, \Diamond\theta_1^j, \ldots, \Diamond\theta_{m_j}^j)\,,$$

where $P = \{p_1, \ldots, p_l\} \subseteq$ Var and $\chi_j(p_1, \ldots, p_l, q_1, \ldots, q_{n_j}, s_1, \ldots, s_{m_j})$ is a non-modal formula. Set:

$$S_{\Box} \;=\; \{\Box\psi_1^j, \ldots, \Box\psi_{n_j}^j : j = 1, \ldots, k\}\,, \qquad S_{\Diamond} \;=\; \{\Diamond\theta_1^j, \ldots, \Diamond\theta_{m_j}^j : j = 1, \ldots, k\}\,,$$

$$F_{\Box} \;=\; \{\psi_1^j, \ldots, \psi_{n_j}^j : j = 1, \ldots, k\}\,, \qquad F_{\Diamond} \;=\; \{\theta_1^j, \ldots, \theta_{m_j}^j : j = 1, \ldots, k\}\,.$$

Note that the input may be displayed in space at most $3n^2 + (1 + 2n)2n^2 \leq 9n^3$.

Step 1. Choose values $V(\rho, x) \in A(e_n)$, for all $\rho \in P \cup S_{\Box} \cup S_{\Diamond}$, and verify that $V(\varphi_j, x) \in I_j$ for each $j \leq k$.

Choose partial functions $\Phi(x) = \{\langle a, c_a\rangle : a \in G\} \subseteq V[S_\Box, x] \times A(e_n)$ and $\Psi(x) = \{\langle d_b, b\rangle : b \in H\} \subseteq A(e_n) \times V[S_\Diamond, x]$ and verify that the finite system $\langle A(e_n), \Phi(x), \Psi(x)\rangle$ is consistent with $\mathbf{A}$. Each $a \in G$ plays the role of a "right accumulation point" and c_a plays the role of a "witness of right homogeneity" at a; similarly, each $b \in H$ plays the role of a "left accumulation point" and d_b plays the role of a "witness of left homogeneity" at b. An oracle for the consistency problem must certify that this distribution can be realized in $\mathbf{A}$.

Choose also worlds $y_1, \ldots, y_m \in W_n$ for $m \leq n$ in the next level of the tree and values $Rxy_t \in A(e_n)$ for $t = 1, \ldots, m$.

Note that the space required to perform this step and store the data produced is at most $3n \cdot 2n^2 + n \cdot n^2 = 7n^3$. The values of the desired tree-model $\mathfrak{M}^*$ are guessed at the root. Hence, this model exists if and only if it is possible to find further (crisp, if necessary) $\mathbb{FK}(e_n)$-tree-models $\mathfrak{M}_t^*$ of height $\leq n-1$ with respective roots y_t, for $t = 1, \ldots, m$, such that for any $\rho \in F_\Box \cup F_\Diamond$,

$$
\begin{array}{llcll}
1. & \bigwedge_{t=1}^{m} (Rxy_t \to V(\rho, y_t)) & \in & [V(\Box\rho, x), c_a), & \text{if } \rho \in F_\Box \text{ and } V(\Box\rho, x) = a \in G, \\
2. & \bigwedge_{t=1}^{m} (Rxy_t \to V(\rho, y_t)) & = & V(\Box\rho, x), & \text{if } \rho \in F_\Box \text{ and } V(\Box\rho, x) \notin G, \\
3. & \bigvee_{t=1}^{m} (Rxy_t \wedge V(\rho, y_t)) & \in & (d_b, V(\Diamond\rho, x)], & \text{if } \rho \in F_\Diamond \text{ and } V(\Diamond\rho, x) = b \in H, \\
4. & \bigvee_{t=1}^{m} (Rxy_t \wedge V(\rho, y_t)) & = & V(\Diamond\rho, x), & \text{if } \rho \in F_\Diamond \text{ and } V(\Diamond\rho, x) \notin H.
\end{array}
$$

If $F_\Box^t$ ($F_\Diamond^t$) denotes the set of $\rho \in F_\Box$ ($\rho \in F_\Diamond$) for which the minimum (maximum) associated to ρ above is realized at y_t, then the situation $\rho \in F_\Diamond^t$, $V(\Diamond\rho, x) = b \in H$ and $Rxy_t \leq d_b$ does not arise and, similarly, the situation $\rho \in F_\Diamond^t$, and $Rxy_t < V(\Diamond\rho, x) \notin H$ is impossible. Moreover, the above conditions are equivalent to asking for all t and ρ:

$$
\begin{array}{llcll}
1. & Rxy_t \to V(\rho, y_t) & \geq & V(\Box\rho, x) & \text{if } \rho \in F_\Box \\
2. & Rxy_t \to V(\rho, y_t) & \in & [V(\Box\rho, x), c_a) & \text{if } \rho \in F_\Box^t \text{ and } V(\Box\rho, x) \in G, \\
3. & Rxy_t \to V(\rho, y_t) & = & V(\Box\rho, x) & \text{if } \rho \in F_\Box^t \text{ and } V(\Box\rho, x) \notin G, \\
4. & Rxy_t \wedge V(\rho, y_t) & \leq & V(\Diamond\rho, x) & \text{if } \rho \in F_\Diamond \\
5. & Rxy_t \wedge V(\rho, y_t) & \in & (d_b, V(\Diamond\rho, x)] & \text{if } \rho \in F_\Diamond^t \text{ and } V(\Diamond\rho, x) \in H, \\
6. & Rxy_t \wedge V(\rho, y_t) & = & V(\Diamond\rho, x) & \text{if } \rho \in F_\Diamond^t \text{ and } V(\Diamond\rho, x) \notin H.
\end{array}
$$

These conditions are equivalent, in turn, to asking that for each model $\mathfrak{M}_t^*$ and $\rho \in F_\Box \cup F_\Diamond$, the value $V(\rho, y_t)$ belongs to the interval $I_{\rho,t}$, fixed to be

$$
\begin{array}{lll}
1. & [V(\Box\rho, x), Rxy_t) & \text{if } \rho \in F_\Box \text{ and } V(\Box\rho, x) < 1, \\
 & [Rxy_t, 1] & \text{if } \rho \in F_\Box \text{ and } V(\Box\rho, x) = 1,
\end{array}
$$

2.	$[V(\Box\rho,x), c_a \wedge Rxy_t)$	if $\rho \in F^t_\Box$ and $V(\Box\rho,x) = a \in G$,
3.	$[V(\Box\rho,x), V(\Box\rho,x)]$	if $\rho \in F^t_\Box$, $V(\Box\rho,x) \notin G$, and $V(\Box\rho,x) < 1$,
	$[Rxy_t, 1]$	if $\rho \in F^t_\Box$, $V(\Box\rho,x) \notin G$, and $V(\Box\rho,x) = 1$,
4.	$[0, Rxy_t \to V(\Diamond\rho,x)]$	if $\rho \in F_\Diamond$,
5.	$(d_b, Rxy_t \to V(\Diamond\rho,x)]$	if $\rho \in F^t_\Diamond$, $V(\Diamond\rho,x) = b \in H$,
6.	$[V(\Diamond\rho,x), Rxy_t \to V(\Diamond\rho,x)]$	if $\rho \in F^t_\Diamond$, $V(\Diamond\rho,x) \notin H$.

But this amounts to the original problem: the existence of $\mathfrak{M}^*_t$ with root y_t satisfying the conditions of the lemma for the input $\Sigma' = F_\Box \cup F_\Diamond$ and intervals $I_{\rho,t}$, $\rho \in \Sigma'$. This justifies the next steps of the algorithm.

Step 2. Find coverings $F_\Box = \bigcup_{t\in(1,m]} F^t_\Box$ and $F_\Diamond = \bigcup_{t\in(1,m]} F^t_\Diamond$, verify that the situations $\rho \in F^t_\Diamond$, $V(\Diamond\rho,x) = b \in H$, and $Rxy_t \leq d_b$, or $\rho \in F^t_\Diamond$ and $Rxy_t < V(\Diamond\rho,x) \notin H$ do not arise, and compute for each t and $\rho \in F_\Box \cup F_\Diamond$ the interval $I_{\rho,t}$.

Note that computing and storing the data produced in this step requires space at most $2n \cdot n^2 + 2n^2 \cdot 2n^2 \leq 6n^4$.

Step 3. For $t = 1, \ldots, m$, return consecutively to Step 1 with input: $\Sigma' = F_\Box \cup F_\Diamond$, $\{I_{\rho,t} : \rho \in \Sigma'\}$, and $x = y_t$, traversing the resulting tree of worlds in pre-order; that is, the leftmost branch is exhausted before passing to the next unexplored sub-branch at the right.

Note that the cyclic repetition of Steps 1 and 2 (an exponential number of times), if successful at each stage, runs through a tree of height less than n, so the space needed to guess a branch of the tree is at most $22n^5$. The key point is that having verified successfully the existence of a branch we may utilize the same space for the next one, and thus the total space required is bounded by $22n^5$. Informally, returning to Step 1 with $t = 1$ starts a search for $\mathfrak{M}^*_1$, after finishing it successfully, we return to Step 1 with $t = 2$ and utilize the same space, bounded by $22n^4(n-1)$, to search for $\mathfrak{M}^*_2$, etc. Adding to this common space the space of the first cycle, we obtain $22n^5$. □

Theorem 4.16. *The validity problems for* $\mathsf{K}(\mathbf{A})$ *and* $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ *are* PSPACE*-complete for the algebras* $\mathbf{G}$*,* $\mathbf{G}_\downarrow$*, and* $\mathbf{G}_\uparrow$*.*

Proof. Lemma 4.15 applied to a formula φ and the interval $I = [0,1)$ yields a PSPACE algorithm in the length of φ to determine for these algebras, whether there is an $\mathbb{FK}(e_n)$-tree-model for which $V(\varphi,0) < 1$ and $\langle A(e_n), \Phi(i), \Psi(i)\rangle$ is consistent with $\mathbf{A}$, for each $i \in W_n \subseteq \{0,1,\ldots,n^n\}$. The latter condition is equivalent to consistency with $\mathbf{A}$ of $\langle A(e_n), \bigcup_{i\in W_n} \Phi(i), \bigcup_{i\in W_n} \Psi(i)\rangle$. The existence of this model is equivalent, recalling the earlier discussion in this section, to the existence of a $\mathbb{K}(\mathbf{A})$-counter-model for φ. The lower bound follows from the fact that classical modal logic K is PSPACE-hard (cf. [88])

and can be interpreted faithfully in $\mathsf{K}(\mathbf{A})$ or $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$ by the double negation interpretation which adds $\neg\neg$ in front of any subformula of a formula. $\square$

Remark 4.17. Note that the last theorem applies to any algebra for which the consistency problem is (PSPACE) decidable and the union of consistent finite systems is consistent. Examples of these algebras are finite algebras (trivially), the ordinals $\omega^n + 1$, $n \in \mathbb{N}^+$, and their reverse orders. We also expect that PSPACE-completeness holds for all finite combinations of $\mathbf{G}$, $\mathbf{G}_\downarrow$, $\mathbf{G}_\uparrow$, and finite algebras built via ordered sums, lexicographical products, and fusion of consecutive points, but will not prove this here.

To generalize the results in this section to languages with a finite set of constants $\mathrm{C}_{\mathfrak{L}} = \{c_1 < \ldots < c_l\}$, utilize a set of values $A'(e_n)$ containing an isomorphic copy $\mathrm{C}'_{\mathfrak{L}} = \{c'_1 < \ldots < c'_l\}$ of $\mathrm{C}_{\mathfrak{L}}$ such that $|[c'_i, c'_{i+1}]_{A'(e_n)}| = |[c_i, c_{i+1}]_{\mathbf{A}}|$, if $|[c_i, c_{i+1}]_{\mathbf{A}}| < e_n$, and $|[c'_i, c'_{i+1}]_{A'(e_n)}| = e_n$, otherwise. This allows V and R to take values in any possible interval of consecutive constants. Moreover, $|A'(e_n)| \leq |\mathrm{C}_{\mathfrak{L}}| e_n$ and all bounds are multiplied by a constant. Finite systems must have now the form $\langle A(m), \Phi, \Psi, \{c'\}_{c \in \mathrm{C}_{\mathfrak{L}}} \rangle$ and the embeddings granting consistency must send c' to c.

Chapter 5

One-Variable Fragments of Order-Based First-Order Logics

Similarly to the many-valued modal logics over complete MTL-algebras, further order-based modal logics may be defined as logics of particular classes of $\mathbb{K}(\mathbf{A})$-models, for a given order-based algebra $\mathbf{A}$. In particular, in this chapter, we will study the class of $\mathbb{K}(\mathbf{A})$-models where the accessibility relation is a crisp equivalence relation, which defines crisp order-based $\mathsf{S5}$ logics. We first turn our attention to proving decidability and coNP-completeness of the validity problem for these logics in various cases and show later that these logics may be understood also as one-variable fragments of order-based first-order logics. In particular, we give a positive answer to the open decidability problem (and establish coNP-completeness) for validity in the one-variable fragment of first-order Gödel logic (see e.g. [67, Chapter 9, Problem 13]).

Unless stated otherwise, the results in the present chapter are based on joint work of the author of this dissertation with Xavier Caicedo, George Metcalfe, and Ricardo Rodríguez [26, 27].

Before we start, let us fix again a finite algebraic language $\mathfrak{L}$ including the operational symbols $\top$, $\bot$, $\wedge$, $\vee$, and $\to$, and an order-based algebra $\mathbf{A}$ for $\mathfrak{L}$. Fm will denote the set of formulas in $\mathrm{Fm}_{\mathfrak{L}_{\mathfrak{m}}}$, where $\mathfrak{L}_{\mathfrak{m}} = \mathfrak{L} \cup \{\Box, \Diamond\}$.

5.1 Alternative Semantics for Order-Based Crisp $\mathsf{S5}$ Logics

In this section, we will define crisp $\mathsf{S5}$ versions of order-based modal logics and relate them to a similar alternative semantics as in Chapter 4. The proofs are similar to those in the previous chapter, but considerably less complicated, as we can restrict to universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-models, where R is a universal crisp relation.

We define an $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-*model* to be a $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-model $\mathfrak{M} = \langle W, V, R\rangle$ (see Section 3.2) such that R is an equivalence relation. We call $\mathfrak{M}$ *universal* if $R = W \times W$ and in this case write $\mathfrak{M} = \langle W, V \rangle$, noting that the clauses for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigwedge\{V(\varphi, y) \mid y \in W\} \\ V(\Diamond\varphi, x) &= \bigvee\{V(\varphi, y) \mid y \in W\}. \end{aligned}$$

Let us denote by $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ the logic defined by $\models_{\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})}$. The following lemma is an immediate corollary of Lemma 3.5 and the fact that an $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-submodel generated by a world is universal.

Lemma 5.1. *A formula $\varphi \in \mathrm{Fm}$ is valid in $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ if and only if φ is valid in all universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-models.*

The infinite $\mathbb{K}(\mathbf{A})$-model defined in the proof of Theorem 3.9 for the formula $\Box\neg\neg p \to \neg\neg\Box p$ is in fact a universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model. Hence, if the universe of $\mathbf{A}$ is $[0, 1]$ or $G_{\downarrow}$, then $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ does not have the finite model property with respect to $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$. Also, as in Theorem 3.8, the logic $\mathsf{S5}^{\mathsf{c}}(\mathbf{G}_{\uparrow})$ has the finite model property, but not if $\triangle$ is added to the language. We will prove decidability for these and other cases here using again a new equivalent semantics.

Let us assume from now on that $\mathbf{A}$ is a locally homogeneous order-based algebra for $\mathfrak{L}$. We define an $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$*-model* as an $\mathbb{FK}^{\mathsf{c}}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond} \rangle$ such that $\langle W, R, V \rangle$ is an $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model, and for all $x, y \in W$,

(i) $T_{\Box}(x) = T_{\Box}(y)$ and $T_{\Diamond}(x) = T_{\Diamond}(y)$ whenever Rxy,

(ii) $\{V(\Diamond p, x) \mid p \in \mathrm{Var}\} \subseteq T_{\Box}(x)$ and $\{V(\Box p, x) \mid p \in \mathrm{Var}\} \subseteq T_{\Diamond}(x)$.

We call $\mathfrak{M}$ *universal* if $R = W \times W$ and in this case write $\mathfrak{M} = \langle W, V, T_{\Box}, T_{\Diamond} \rangle$, where $T_{\Box}$ and $T_{\Diamond}$ may now be understood as fixed subsets of A, and the clauses for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\{r \in T_{\Box} \mid r \leq \bigwedge\{V(\varphi, y) \mid y \in W\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\{r \in T_{\Diamond} \mid r \geq \bigvee\{V(\varphi, y) \mid y \in W\}\}. \end{aligned}$$

Note in particular that, by condition (i), in universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$- and $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-models, the truth values of box-formulas and diamond-formulas are independent of the world.

The new condition (ii) for $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-models reflects the fact that we deal here with universal models not tree-models and must therefore take into account the values of diamond-formulas and box-formulas when fixing the values in $T_{\Box}$ and $T_{\Diamond}$, respectively. We will now show that (ii) extends to all diamond- and box-formulas.

Lemma 5.2. *For any universal $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model $\mathfrak{M} = \langle W, V, T_{\Box}, T_{\Diamond} \rangle$ and $x \in W$,*

$$\{V(\Diamond\varphi, x) \mid \varphi \in \mathrm{Fm}\} \subseteq T_{\Box} \qquad \textit{and} \qquad \{V(\Box\varphi, x) \mid \varphi \in \mathrm{Fm}\} \subseteq T_{\Diamond}.$$

Proof. Let $\mathfrak{M} = \langle W, V, T_\Box, T_\Diamond \rangle$ be a universal $\mathbb{FS}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model, $x \in W$, and $\varphi \in \mathrm{Fm}$. We prove that $V(\Diamond\varphi, x) \in T_\Box$ and $V(\Box\varphi, x) \in T_\Diamond$ by an induction on the length of φ.

For the base case, let φ be a propositional variable or a constant in $\mathrm{C}_{\mathfrak{L}}$. Then the statements follow, for the first case, from condition (ii) in the definition of an $\mathbb{FS}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model, and for the second case, from condition (i) in the definition of an $\mathbb{FK}(\mathbf{A})$-model in Chapter 4 (i.e. in the present case, $\mathrm{C}_{\mathfrak{L}} \subseteq T_\Box \cap T_\Diamond$).

The induction step for the non-modal connectives follows easily from the induction hypothesis and the fact that the operations of an order-based algebra either map to a constant or to one of its arguments (see Remark 3.2).

Let us therefore consider the case where $\varphi = \Box\psi$. For the first part, because the values of box-formulas are independent of the world and because clearly $V(\Box\psi, x) \in T_\Box$, we obtain that

$$\begin{aligned} V(\Box\Box\psi, x) &= \bigvee\{r \in T_\Box \mid r \leq \bigwedge\{V(\Box\psi, y) \mid y \in W\}\} \\ &= \bigvee\{r \in T_\Diamond \mid r \leq V(\Box\psi, x)\} \\ &= V(\Box\psi, x)\,, \end{aligned}$$

and thus, by the induction hypothesis, $V(\Box\varphi, x) = V(\Box\psi, x) \in T_\Diamond$. For the second part, we note that by the induction hypothesis ($V(\Box\psi, x) \in T_\Diamond$) we have that

$$\begin{aligned} V(\Diamond\Box\psi, x) &= \bigwedge\{r \in T_\Diamond \mid r \geq \bigvee\{V(\Box\psi, y) \mid y \in W\}\} \\ &= \bigwedge\{r \in T_\Diamond \mid r \geq V(\Box\psi, x)\} \\ &= V(\Box\psi, x)\,, \end{aligned}$$

and thus, $V(\Diamond\varphi, x) = V(\Box\psi, x) \in T_\Box$. The case for $\varphi = \Diamond\psi$ is very similar. $\Box$

We now show that $\mathsf{S5}^\mathsf{c}(\mathbf{A})$-validity is equivalent to validity in finite universal $\mathbb{FS}\mathbf{5}^\mathsf{c}(\mathbf{A})$-models, following fairly closely the corresponding proofs in Chapter 4.

Lemma 5.3. *Let $\Sigma \subseteq \mathrm{Fm}$ be a finite fragment, $\mathfrak{M} = \langle W, V \rangle$ a universal $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model, and $x \in W$. Then there is a finite universal $\mathbb{FS}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V}, \widehat{T_\Box}, \widehat{T_\Diamond} \rangle$ with $x \in \widehat{W} \subseteq W$ and $|\widehat{W}| \leq |\Sigma|$ such that $\widehat{V}(\varphi, y) = V(\varphi, y)$ for all $\varphi \in \Sigma$ and $y \in \widehat{W}$.*

Proof. The proof is similar to the proof of Lemma 4.6. Let us fix a finite fragment $\Sigma \subseteq \mathrm{Fm}$, a universal $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model $\mathfrak{M} = \langle W, V \rangle$, and $x \in W$. Consider the finite (possibly empty) sets

$$V[\Sigma_\Box, x] \cap R(\mathbf{A}) \;=\; \{a_i \mid i \in I\} \qquad \text{and} \qquad V[\Sigma_\Diamond, x] \cap L(\mathbf{A}) \;=\; \{b_j \mid j \in J\}\,,$$

noting that these sets are independent of the choice of the world $x \in W$. For each $i \in I$, choose a witness of right homogeneity c_i at a_i such that the intervals (a_i, c_i) are pairwise disjoint for all $i \in I$, and

$$(V[\Sigma_\Box, x] \cup \{V(\Diamond p, x) \mid p \in \mathrm{Var} \cap \Sigma\} \cup \mathrm{C}_{\mathfrak{L}}) \cap (\bigcup_{i \in I}(a_i, c_i)) \ = \ \emptyset\,.$$

Similarly, for each $j \in J$, choose a witness of left homogeneity d_j at b_j such that the intervals (d_j, b_j) are pairwise disjoint for all $j \in J$, and

$$(V[\Sigma_\Diamond, x] \cup \{V(\Box p, x) \mid p \in \mathrm{Var} \cap \Sigma\} \cup \mathrm{C}_{\mathfrak{L}}) \cap (\bigcup_{j \in J}(d_j, b_j)) \ = \ \emptyset\,.$$

We define

$$\widehat{T}_\Box \ = \ A \setminus \bigcup_{i \in I}(a_i, c_i) \qquad \text{and} \qquad \widehat{T}_\Diamond \ = \ A \setminus \bigcup_{j \in J}(d_j, b_j)\,.$$

Now consider $\varphi = \Box\psi \in \Sigma_\Box$ and $a = V(\Box\psi, x) \in \widehat{T}_\Box$. If $a \notin R(\mathbf{A})$, then we choose $y_\varphi \in W$ such that $a = V(\psi, y_\varphi)$. If $a \in R(\mathbf{A})$, then there is an $i \in I$ such that $a = a_i$, and we choose $y_\varphi \in W$ such that $V(\psi, y_\varphi) \in [a_i, c_i)$. Suppose now that $\varphi = \Diamond\psi \in \Sigma_\Diamond$ and $b = V(\Diamond\psi, x) \in \widehat{T}_\Diamond$. If $b \notin L(\mathbf{A})$, then we choose $y_\varphi \in W$ such that $b = V(\psi, y_\varphi)$. If $b \in L(\mathbf{A})$, then there is a $j \in J$ such that $b = b_j$, and we choose $y_\varphi \in W$ such that $V(\psi, y_\varphi) \in (d_j, b_j]$.

Now let $\widehat{W} = \{x\} \cup \{y_\varphi \in W : \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\}$, noting that $|\widehat{W}| \leq 1 + |\Sigma_\Box \cup \Sigma_\Diamond| \leq |\Sigma|$. Define for each $y \in \widehat{W}$ and $p \in \mathrm{Var}$:

$$\widehat{V}(p, y) \ = \ \begin{cases} V(p, y) & \text{if } p \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ is a finite $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model satisfying $x \in \widehat{W} \subseteq W$ and $|\widehat{W}| \leq |\Sigma|$. It then follows by an easy induction on $\ell(\varphi)$ that $\widehat{V}(\varphi, y) = V(\varphi, y)$ for all $y \in \widehat{W}$ and $\varphi \in \Sigma$. $\Box$

Remark 5.4. Note that the number of intervals omitted from $\widehat{T}_\Box$ and $\widehat{T}_\Diamond$, defined in the proof of Lemma 5.3, is smaller than or equal to the cardinality of $\Sigma_\Box$ and $\Sigma_\Diamond$, respectively, for the given fragment Σ.

Lemma 5.5. *Let $\mathfrak{M} = \langle W, V, T_\Box, T_\Diamond \rangle$ be a finite universal $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model. Then there is a universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V} \rangle$ with $W \subseteq \widehat{W}$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \mathrm{Fm}$ and $x \in W$.*

Proof. Given a finite universal $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model $\mathfrak{M}$, we construct our universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model $\widehat{\mathfrak{M}}$ directly by taking infinitely many copies of $\mathfrak{M}$.

Consider $T_\Box = A \setminus \bigcup_{i \in I}(a_i, c_i)$ and $T_\Diamond = A \setminus \bigcup_{j \in J}(d_j, b_j)$ for finite (possibly empty) sets I, J, where for each $i \in I$, right homogeneity at $a_i \in R(\mathbf{A})$ is witnessed by c_i such that the intervals (a_i, c_i) are pairwise disjoint, and, similarly, for each $j \in J$, left homogeneity at $b_j \in L(\mathbf{A})$ is witnessed by d_j such that the intervals (d_j, b_j) are pairwise disjoint. We define a family of $\mathrm{C}_{\mathfrak{L}}$-order embeddings $\{h_k \colon A \to A\}_{k \in \mathbb{Z}^+}$ such that

- for each even $k \in \mathbb{Z}^+$, h_k is the identity function on $T_\Box$ and for each $i \in I$,

$$h_k[[a_i, c_i)] \subseteq [a_i, a_i + \tfrac{1}{k}),$$

- for each odd $k \in \mathbb{Z}^+$, h_k is the identity function on $T_\Diamond$ and for each $j \in J$,

$$h_k[(d_j, b_j]] \subseteq (b_j - \tfrac{1}{k}, b_j].$$

Note that Lemma 5.2 ensures for all $x \in W$ that $\{V(\Box\varphi, x), V(\Diamond\varphi, x) \colon \varphi \in \mathrm{Fm}\} \subseteq T_\Box \cap T_\Diamond$ and hence that for all $k \in \mathbb{Z}^+$ (even and odd), h_k is the identity function on $\{V(\Box\varphi, x), V(\Diamond\varphi, x) \colon \varphi \in \mathrm{Fm}\}$. Let h_0 be the identity on A, let $\widehat{W_0} = W$, and for each $k \in \mathbb{Z}^+$, let $\widehat{W_k}$ be a copy of W with a distinct copy $\widehat{x}_k$ of each $x \in W$; also let $\widehat{x}_0 = x$ for each $x \in W$. We define the universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V} \rangle$ where

$$\widehat{W} = \bigcup_{k \in \mathbb{N}} \widehat{W_k} \qquad \text{and} \qquad \widehat{V}(p, \widehat{x}_k) = h_k(V(p, x)), \text{ for } p \in \mathrm{Var},\ x \in W, \text{ and } k \in \mathbb{N}.$$

It suffices now to prove that for all $\varphi \in \mathrm{Fm}$, $x \in W$, and $k \in \mathbb{N}$,

$$\widehat{V}(\varphi, \widehat{x}_k) = h_k(V(\varphi, x)),$$

proceeding by induction on $\ell(\varphi)$. The base case follows by definition, while for the non-modal connectives, the argument is the same as in the proof of Lemma 3.10. Consider $\varphi = \Diamond\psi$. Fix $x \in W$ and $k \in \mathbb{N}$. There are two cases.

For the first case, suppose that $V(\Diamond\psi, x) = b_j$ for some $j \in J$. Note first that by Lemma 5.2, $V(\Diamond\psi, x) = b_j \in T_\Diamond \cup T_\Box$ and hence $h_k(b_j) = b_j$. Clearly $V(\psi, z) \leq b_j$ for all $z \in W$. Hence, by the induction hypothesis and the construction of $\{h_n \colon A \to A\}_{n \in \mathbb{N}}$, for all $n \in \mathbb{N}$ and $\widehat{z}_n \in \widehat{W}$,

$$\widehat{V}(\psi, \widehat{z}_n) = h_n(V(\psi, z)) \leq b_j.$$

Also, for some $y \in W$,

$$V(\psi, y) \in (d_j, b_j].$$

Hence for any odd $n \in \mathbb{N}$,

$$h_n(V(\psi, y)) \in (b_j - \tfrac{1}{n}, b_j].$$

Using the induction hypothesis,

$$\begin{aligned} \widehat{V}(\Diamond\psi, \widehat{x}_k) &= \bigvee \{\widehat{V}(\psi, \widehat{y}_n) \mid y \in W, n \in \mathbb{N}\} \\ &= \bigvee \{h_n(V(\psi, y)) \mid y \in W, n \in \mathbb{N}\} \\ &= \bigvee \{b_j - \tfrac{1}{n} \mid n \in \mathbb{Z}^+\} \\ &= b_j \\ &= h_k(V(\Diamond\psi, x)). \end{aligned}$$

For the second case, suppose that $V(\Diamond\psi, x) = b \neq b_j$ for all $j \in J$. Note again that by Lemma 5.2, $V(\Diamond\psi, x) = b \in T_\Diamond \cup T_\Box$ and hence $h_k(b) = b$. Clearly, $V(\psi, z) \leq b$ for all $z \in W$. It follows again by the induction hypothesis and the construction of $\{h_n \colon A \to A\}_{n\in\mathbb{N}}$ that for all $n \in \mathbb{N}$ and $\widehat{z}_n \in \widehat{W}$,

$$\widehat{V}(\psi, \widehat{z}_n) \;=\; h_n(V(\psi, z)) \leq b\,.$$

Moreover, because W is finite, there is a $y \in W$ such that $V(\psi, y) = b = V(\Diamond\psi, x)$. Using the induction hypothesis and the fact that h_n is the identity function on $\{V(\Box\varphi, z), V(\Diamond\varphi, z) \mid \varphi \in \mathrm{Fm}\}$ for all $n \in \mathbb{N}$ and $z \in W$, it follows that

$$\begin{aligned}
\widehat{V}(\Diamond\psi, \widehat{x}) \;&=\; \bigvee\{\widehat{V}(\psi, \widehat{z}_n) \mid z \in W, n \in \mathbb{N}\}\\
&=\; \bigvee\{h_n(V(\psi, z)) \mid z \in W, n \in \mathbb{N}\}\\
&=\; \bigvee\{h_n(b) \mid n \in \mathbb{N}\}\\
&=\; b\\
&=\; h_k(V(\Diamond\psi, x)).
\end{aligned}$$

The case $\varphi = \Box\psi$ is very similar. □

Combining Lemmas 5.1, 5.3, and 5.5, we obtain the following equivalence.

Theorem 5.6. *Let* $\varphi \in \mathrm{Fm}$*, then*

$$\models_{\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})} \varphi \qquad \textit{iff} \qquad \mathfrak{M} \models_{\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})} \varphi, \textit{ for all finite universal } \mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})\textit{-models } \mathfrak{M}.$$

Extending the notion of the finite model property with respect to the class $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$ of $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-models in the obvious way, we obtain the following corollary.

Corollary 5.7. *The logic* $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ *has the finite model property with respect to* $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$.

5.2 Decidability and Complexity

The desired decidability and complexity results are now obtained by considering the number of truth values needed to check validity of formulas in finite universal $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-models. Recall (see Section 4.3) that if $A(m) = \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\}$, then a finite system $\langle A(m), \Phi, \Psi, \{c'\}_{c\in \mathrm{C}_{\mathfrak{L}}}\rangle$, where $\Phi, \Psi \subseteq A(m)^2$, is consistent with $\mathbf{A}$ if there exists an order-preserving embedding $h \colon A(m) \to A$ such that $h(c') = c$ for all $c \in \mathrm{C}_{\mathfrak{L}}$, $h(c)$ witnesses right homogeneity at $h(a) \in R(\mathbf{A})$, for all $\langle a, c\rangle \in \Phi$, and $h(d)$ witnesses left homogeneity at $h(b) \in L(\mathbf{A})$, for all $\langle d, b\rangle \in \Phi$.

Theorem 5.8. *Let* $\mathbf{A}$ *be a locally homogeneous order-based algebra for* $\mathfrak{L}$*. Then the validity problem for* $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ *is* coNP*-reducible to the problem of consistency of finite systems with* $\mathbf{A}$.

Proof. Consider $\varphi \in \mathrm{Fm}$ and let $n = |\Sigma(\varphi)| = \ell(\varphi) + |\mathrm{C}_{\mathfrak{L}}|$. To check if φ is not $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$-valid, it suffices, by Lemmas 5.3 and 5.5, to check that φ is not valid in a finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, V, T_{\Box}, T_{\Diamond}\rangle$ with $|W| \leq |\Sigma(\varphi)| = n$. To compute $V(\varphi, x)$ in such a model, we need to know only the values $V[\Sigma(\varphi), W]$ (that is, fewer than n^2 values) and the endpoints of the intervals defining $T_{\Box}$ and $T_{\Diamond}$ (that is, considering Remark 5.4, fewer than $2n$ values). So, we need at most $3n^2$ distinct values. Therefore, we may assume that these values are in a fixed finite set $A_n = A(p(n)) = \{0, \frac{1}{p(n)}, \ldots, \frac{p(n)-1}{p(n)}, 1\}$, containing properly spaced copies of constants, where $p(n) = 3|\mathrm{C}_{\mathfrak{L}}|n^2$. We may assume also that $W = W_n \subseteq \{0, 1, \ldots, n-1\}$. Then checking non-deterministically that φ is not valid amounts to performing the following steps:

1. Guessing the values $V(p, i)$ in A_n for each $p \in \mathrm{Var}(\varphi)$ and $i \in W_n$ (no more than $np(n)$ steps).
2. Guessing the sets $\Phi, \Psi \subseteq A_n^2$ such that Φ and Ψ define families of disjoint open intervals and using them to define, respectively, the two subsets $T_{\Box}^*$, $T_{\Diamond}^* \subseteq A_n$ (at most $2p(n)^2$ steps).
3. Checking that the system $\langle A_n, \Phi, \Psi, \{c'\}_{c \in \mathrm{C}_{\mathfrak{L}}}\rangle$ is consistent with $\mathbf{A}$.
4. Computing $V(\varphi, 0)$ in the model $\langle W_n, V, T_{\Box}^*, T_{\Diamond}^*\rangle$ and checking $V(\varphi, 0) < 1$ (essentially n^3 steps).

Hence a counter-model for φ may be guessed in polynomial time if we have an oracle for the consistency problem. $\Box$

Corollary 5.9. *The validity problems for* $\mathsf{S5}^{\mathsf{c}}(\mathbf{G})$, $\mathsf{S5}^{\mathsf{c}}(\mathbf{G}_{\downarrow})$, *and* $\mathsf{S5}^{\mathsf{c}}(\mathbf{G}_{\uparrow})$ *are* coNP-*complete. The same is true for* $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ *if* $\mathbf{A}$ *is a finite combination of* $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, *and finite algebras via ordered sums, lexicographical products, and fusion of consecutive points.*

Proof. The validity problem is coNP-hard already for the pure propositional logic over any $\mathbf{A}$, because classical propositional logic is interpretable in these logics. Moreover, for $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, and $\mathbf{G}_{\uparrow}$, the consistency problem is checked in null or linear time. In the other cases, the consistency problem is solvable in polynomial time. $\Box$

5.3 Order-Based First-Order Logics

In this section, we introduce the syntax and semantics of order-based first-order logics. We then present a standard translation between modal formulas and formulas in the one-variable fragment of the first-order expansions and show that a modal formula is valid in $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ if and only if its translation – a first-order formula in the one-variable fragment – is valid in all first-order structures over $\mathbf{A}$. Considering this translation, we can infer coNP-completeness of the validity problem for the one-variable fragments of the first-order expansions of order-based logics over certain locally homogeneous order-based algebras $\mathbf{A}$ from the coNP-completeness of the validity problem for $\mathsf{S5^c}(\mathbf{A})$ over these algebras $\mathbf{A}$ (see Corollary 5.9). In particular, we can infer that the validity problem for the one-variable fragment of first-order Gödel logic is decidable and coNP-complete.

We define a *first-order language* $\mathfrak{F}$ to be a triple $\langle \mathrm{Pr}, \mathrm{Fu}, \mathrm{Ar}\rangle$ where Pr is a non-empty set of *predicate symbols*, Fu is a set (disjoint with Pr) of *function symbols*, and Ar is an *arity function*, assigning to each predicate or function symbol a natural number called the *arity* of the symbol.

We fix a countable set $\mathrm{FVar} = \{x_n \mid n \in \mathbb{N}\}$ of *first-order variables* and define for a first-order language $\mathfrak{F} = \langle \mathrm{Pr}, \mathrm{Fu}, \mathrm{Ar}\rangle$ the set $\mathrm{Tm}(\mathfrak{F})$ of $\mathfrak{F}$-*terms* inductively over FVar such that $\mathrm{FVar} \subseteq \mathrm{Tm}(\mathfrak{F})$ and for each n-ary $f \in \mathrm{Fu}$ and any $t_1, \ldots, t_n \in \mathrm{Tm}(\mathfrak{F})$, $f(t_1, \ldots, f_n) \in \mathrm{Tm}(\mathfrak{F})$. Moreover, the set $\mathrm{Fm}(\mathfrak{F})$ of $\mathfrak{F}$-*formulas* is defined inductively as follows:

- for any n-ary relation symbol $P \in \mathrm{Pr}$ and $t_1, \ldots, t_n \in \mathrm{Tm}(\mathfrak{F})$,

$$P(t_1, \ldots, t_n) \ \in \ \mathrm{Fm}(\mathfrak{F}),$$

- for any n-ary operation symbol $\star \in \mathfrak{L}$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{Fm}(\mathfrak{F})$,

$$\star(\varphi_1, \ldots, \varphi_n) \ \in \ \mathrm{Fm}(\mathfrak{F}),$$

- for any $x \in \mathrm{FVar}$ and any $\varphi \in \mathrm{Fm}(\mathfrak{F})$,

$$(\forall x)\varphi \ \in \ \mathrm{Fm}(\mathfrak{F}) \qquad \text{and} \qquad (\exists x)\varphi \ \in \ \mathrm{Fm}(\mathfrak{F}).$$

Given a first-order language $\mathfrak{F} = \langle \mathrm{Pr}, \mathrm{Fu}, \mathrm{Ar}\rangle$, let us define an $\mathbf{A}$-*structure* $\mathfrak{S}$ for $\mathfrak{F}$ to be a triple $\langle D, \{P^{\mathfrak{S}}\}_{P\in\mathrm{Pr}}, \{f^{\mathfrak{S}}\}_{f\in\mathrm{Fu}}\rangle$, where D is a non-empty set called the *domain*, for each n-ary predicate symbol $P \in \mathrm{Pr}$, $P^{\mathfrak{S}}$ is an n-ary mapping from D^n to the universe of the algebra $\mathbf{A}$, i.e. $P^{\mathfrak{S}} \colon D^n \to A$, and for each n-ary function symbol $f \in \mathrm{Fu}$, $f^{\mathfrak{S}}$ is a function $f^{\mathfrak{S}} \colon D^n \to D$.

Let $\mathfrak{F} = \langle \mathrm{Pr}, \mathrm{Fu}, \mathrm{Ar}\rangle$ be a first-order language and $\mathfrak{S} = \langle D, \{P^{\mathfrak{S}}\}_{P\in\mathrm{Pr}}, \{f^{\mathfrak{S}}\}_{f\in\mathrm{Fu}}\rangle$ an $\mathbf{A}$-structure for $\mathfrak{F}$, we define an $\mathfrak{S}$-*assignment* to be a mapping $h \colon \mathrm{FVar} \to D$ assigning an element of the domain D to each variable in FVar, extended to a mapping $h \colon \mathrm{Tm}(\mathfrak{F}) \to D$ inductively such that

$$h(f(t_1, \ldots, t_n)) \ = \ f^{\mathfrak{S}}(h(t_1), \ldots, h(t_n)),$$

for any n-ary function symbol $f \in \mathrm{Fu}$ and $t_1, \ldots, t_n \in \mathrm{Tm}(\mathfrak{F})$. Let h be an $\mathfrak{S}$-assignment, $x \in \mathrm{FVar}$, and $d \in D$, then we define the $\mathfrak{S}$-assignment $h(x \to d)$ as follows: for all

$y \in \mathrm{FVar}$,

$$h(x \to d)(y) \;=\; \begin{cases} d & \text{if } y = x\,, \\ h(y) & \text{otherwise.} \end{cases}$$

Given an $\mathfrak{S}$-assignment h, we define the map $V_h \colon \mathrm{Fm}(\mathfrak{F}) \to A$, called an $\mathfrak{S}$-*valuation* for h, inductively as follows:

- for any n-ary relation symbol $P \in \mathrm{Pr}$ and $t_1, \ldots, t_n \in \mathrm{Tm}(\mathfrak{F})$,

$$V_h(P(t_1, \ldots, t_n)) \;=\; P^{\mathfrak{S}}(h(t_1), \ldots, h(t_n))\,,$$

- for any n-ary operation symbol $\star \in \mathfrak{L}$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{Fm}(\mathfrak{F})$,

$$V_h(\star(\varphi_1, \ldots, \varphi_n)) \;=\; \star(V_h(\varphi_1), \ldots, V_h(\varphi_n))\,,$$

- for any $x \in \mathrm{FVar}$ and any $\varphi \in \mathrm{Fm}(\mathfrak{F})$,

$$\begin{aligned} V_h((\forall x)\varphi) &= \bigwedge\{V_{h(x\to d)}(\varphi) \mid d \in D\}\,, \\ V_h((\exists x)\varphi) &= \bigvee\{V_{h(x\to d)}(\varphi) \mid d \in D\}\,. \end{aligned}$$

Let $\mathfrak{S} = \langle D, \{P^{\mathfrak{S}}\}_{P\in\mathrm{Pr}}, \{f^{\mathfrak{S}}\}_{f\in\mathrm{Fu}}\rangle$ be an $\mathbf{A}$-structure for a first-order language $\mathfrak{F} = \langle \mathrm{Pr}, \mathrm{Fu}, \mathrm{Ar}\rangle$ and $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\mathfrak{F})$ a set of $\mathfrak{F}$-formulas, we will say that φ is an $\mathfrak{S}$-*consequence* of Γ, written $\Gamma \models_{\mathfrak{S}} \varphi$, if $V_h(\varphi) = 1$ for all $\mathfrak{S}$-assignments h such that $V_h[\Gamma] = \{V_h(\psi) \mid \psi \in \Gamma\} \subseteq \{1\}$. Let us denote by $\mathbb{V}(\mathbf{A})$ the class of all $\mathbf{A}$-structures for $\mathfrak{F}$, we will say that φ is a $\mathbb{V}(\mathbf{A})$-*consequence*, written $\Gamma \models_{\mathbb{V}(\mathbf{A})} \varphi$, if $\Gamma \models_{\mathfrak{S}} \varphi$ for all $\mathbf{A}$-structures $\mathfrak{S} \in \mathbb{V}(\mathbf{A})$ for $\mathfrak{F}$. A formula φ will be called *valid in* $\mathfrak{S}$ if $\emptyset \models_{\mathfrak{S}} \varphi$, and φ is *valid in* $\mathbb{V}(\mathbf{A})$ if $\emptyset \models_{\mathbb{V}(\mathbf{A})} \varphi$, also written $\models_{\mathbb{V}(\mathbf{A})} \varphi$. We will denote the logic defined by the consequence relation $\models_{\mathbb{V}(\mathbf{A})}$ on $\mathrm{Fm}(\mathfrak{F})$ by $\forall(\mathbf{A})$ and will say that an $\mathfrak{F}$-formula φ is *valid in* $\forall(\mathbf{A})$ if it is valid in $\mathbb{V}(\mathbf{A})$.

One-Variable Fragments and the Standard Translation

In order to be able to define a suitable translation between modal formulas and first-order formulas with only one-variable, we fix $\mathrm{FVar} = \{x\}$ and a first-order language $\mathfrak{F}_1 = \langle \mathrm{Pr}, \mathrm{Fu}, \mathrm{Ar}\rangle$, such that $\mathrm{Pr} = \{P_i \mid i \in \mathbb{N}\}$ is enumerable, $\mathrm{Fu} = \emptyset$, and for all $P \in \mathrm{Pr}$, $\mathrm{Ar}(P) = 1$. The set $\mathrm{Fm}(\mathfrak{F}_1)$ of $\mathfrak{F}_1$-formulas is then defined inductively over the singleton-set $\mathrm{FVar} = \{x\}$ of one first-order variable. Given an order-based algebra $\mathbf{A}$, we will call the logic $\forall_1(\mathbf{A})$, resulting from restricting the logic $\forall(\mathbf{A})$ to $\mathrm{Fm}(\mathfrak{F}_1)$, the *one-variable fragment* of $\forall(\mathbf{A})$.

Remark 5.10. Note that usually the one-variable fragment of a first-order logic L for any first-order language $\mathfrak{F}$ denotes the restriction of L to the set of formulas in $\mathrm{Fm}(\mathfrak{F})$ which contain no function symbols and at most one variable (but possibly several occurrences of it). However, restricting to a fixed first-order language with countably many unary predicate symbols, such as $\mathfrak{F}_1$, is not essential in this case. Firstly, as we are only interested in validity, we may in any case restrict to the finitely many predicate symbols occurring in the formula in question. Secondly, in the presence of only one variable, each n-ary predicate symbol P in any formula φ can be uniformly replaced by a unary predicate symbol $\widehat{P}$, resulting in the formula $\widehat{\varphi}$, such that for any $\mathbf{A}$-structure $\mathfrak{S}$: $\models_{\mathfrak{S}} \varphi$ if and only if $\models_{\mathfrak{S}} \widehat{\varphi}$.

Recall that $\mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}$ denotes the set of formulas for the language $\mathfrak{L}_\mathsf{m} = \mathfrak{L} \cup \{\Box, \Diamond\}$ defined inductively over the enumerable set Var of propositional variables. We then notice that because both $\mathrm{Pr} = \{P_i \mid i \in \mathbb{N}\}$ and $\mathrm{Var} = \{p_i \mid i \in \mathbb{N}\}$ are enumerable sets, we can assume that there is a bijection mapping p_i to P_i for all $i \in \mathbb{N}$.

The *translation* π between formulas in $\mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}$ and $\mathfrak{F}_1$-formulas in $\mathrm{Fm}(\mathfrak{F}_1)$ is a map $\pi\colon \mathrm{Fm}_{\mathfrak{L}_\mathsf{m}} \to \mathrm{Fm}(\mathfrak{F}_1)$ defined inductively as follows:

$$\begin{aligned}
\pi(p_i) &= P_i(x), \text{ for all } p_i \in \mathrm{Var}, \\
\pi(\star(\varphi_1, \ldots, \varphi_n)) &= \star(\pi(\varphi_1), \ldots, \pi(\varphi_n)), \text{ for all } n\text{-ary } \star \in \mathfrak{L} \text{ and } \varphi_1, \ldots, \varphi_n \in \mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}, \\
\pi(\Box\varphi) &= (\forall x)\pi(\varphi), \text{ for all } \varphi \in \mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}, \\
\pi(\Diamond\varphi) &= (\exists x)\pi(\varphi), \text{ for all } \varphi \in \mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}.
\end{aligned}$$

In fact, π is a bijection between $\mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}$ and $\mathrm{Fm}(\mathfrak{F}_1)$ and thus there is also an inverse translation $\pi^{-1}\colon \mathrm{Fm}(\mathfrak{F}_1) \to \mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}$.

We associate an $\mathbf{A}$-structure for $\mathfrak{F}_1$ to each universal $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model and vice versa. Given a universal $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model $\mathfrak{M} = \langle W, V\rangle$, define the $\mathbf{A}$-structure $\mathfrak{S}_\mathfrak{M} = \langle D, \{P_i^{\mathfrak{S}_\mathfrak{M}}\}_{P_i \in \mathrm{Pr}}, \emptyset\rangle$ for $\mathfrak{F}_1$ as follows:

$$D = W \qquad \text{and} \qquad P_i^{\mathfrak{S}_\mathfrak{M}}(d) = V(p_i, d), \text{ for all } i \in \mathbb{N} \text{ and all } d \in W.$$

Conversely, given an $\mathbf{A}$-structure $\mathfrak{S} = \langle D, \{P_i^{\mathfrak{S}}\}_{P_i \in \mathrm{Pr}}, \emptyset\rangle$ for $\mathfrak{F}_1$, we define the universal $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{A})$-model $\mathfrak{M}_\mathfrak{S} = \langle W, V\rangle$ as follows:

$$W = D \qquad \text{and} \qquad V(p_i, d) = P_i^{\mathfrak{S}}(d), \text{ for all } i \in \mathbb{N} \text{ and all } d \in D.$$

We will omit the proof of the following lemma, as it is just a routine induction on the length of the formula. It can be found for example in [67]. The following theorem is an obvious consequence.

Lemma 5.11 ([67]).

(a) *Let* $\mathfrak{M} = \langle W, V\rangle$ *be a universal* $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{A})$*-model and* $\mathfrak{S}_\mathfrak{M} = \langle D, \{P_i^{\mathfrak{S}_\mathfrak{M}}\}_{P_i \in \mathrm{Pr}}, \emptyset\rangle$ *the associated* $\mathbf{A}$*-structure for* $\mathfrak{F}_1$*. Then for all modal formulas* $\varphi \in \mathrm{Fm}_{\mathfrak{L}_\mathsf{m}}$*, all worlds* $d \in W$*, and all* $\mathfrak{S}_\mathfrak{M}$*-assignments* h*:*

$$V(\varphi, d) = V_{h(x \to d)}(\pi(\varphi))\,.$$

(b) *Let $\mathfrak{S} = \langle D, \{P_i^{\mathfrak{S}}\}_{P_i \in \mathrm{Pr}}, \emptyset\rangle$ be an $\mathbf{A}$-structure for $\mathfrak{F}_1$ and $\mathfrak{M}_{\mathfrak{S}} = \langle W, V\rangle$ the associated universal $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$-model. Then for all $\mathfrak{F}_1$-formulas $\varphi \in \mathrm{Fm}(\mathfrak{F}_1)$, all $d \in D$, and all $\mathfrak{S}$-assignments h:*

$$V_{h(x\to d)}(\varphi) \; = \; V(\pi^{-1}(\varphi), d)\,.$$

Theorem 5.12. *Let $\mathbf{A}$ be any order-based algebra. Then, a modal formula $\varphi \in \mathrm{Fm}_{\mathcal{L}_{\mathrm{m}}}$ is valid in $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ if and only if $\pi(\varphi)$ is valid in $\forall_1(\mathbf{A})$ (and $\forall(\mathbf{A})$), and an $\mathfrak{F}_1$-formula $\psi \in \mathrm{Fm}(\mathfrak{F}_1)$ is valid in $\forall_1(\mathbf{A})$ (and $\forall(\mathbf{A})$) if and only if $\pi^{-1}(\psi)$ is valid in $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$.*

Recalling Corollary 5.9, we then easily conclude the main result of this chapter, which answers a long-standing open problem posed by Hájek in [67].

Theorem 5.13. *The validity problems for the one-variable fragments of first-order Gödel logics based on $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, and $\mathbf{G}_{\uparrow}$ are* coNP*-complete. The same is true for the one-variable fragments of first-order Gödel logics based on a finite combination of $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, and $\mathbf{G}_{\uparrow}$, and finite algebras via ordered sums, lexicographical products, and fusion of consecutive points.*

Remark 5.14. It was pointed out to us by Lluís Godo that part of this result was already known. In [7], it is shown that the validity problem for the untangled monadic fragment of $\forall(\mathbf{G}_{\uparrow})$ (i.e. where only unary predicates are allowed and no subformula has more than one free variable) is decidable, indeed coNP-complete. As the one-variable fragment is strictly weaker than the untangled monadic fragment, the results in [7] imply the coNP-completeness of the validity problem for $\forall_1(\mathbf{G}_{\uparrow})$. This is made explicit in [71], where it is mentioned that the results in [7] imply the coNP-completeness of the validity problem for $\mathsf{S5}^{\mathsf{c}}(\mathbf{G}_{\uparrow})$.

Chapter 6

Tableaux Calculi for Gödel Modal Logics

In this chapter, we introduce tableau calculi for validity in the Gödel modal logics GK, $\mathsf{GK^c}$, and $\mathsf{GS5^c}$, which were developed independently by the author of this dissertation. In fact, these are just sample cases, it would be rather straightforward to adapt the calculi for $\mathsf{K}(\mathbf{A})$, $\mathsf{K^c}(\mathbf{A})$, and $\mathsf{S5^c}(\mathbf{A})$ for many other locally homogeneous order-based algebras $\mathbf{A}$.

The calculi presented in this chapter provide decision procedures that can be applied and implemented more easily than the decision procedures introduced in Sections 4.3 and 5.1. They do not, however, provide algorithms of optimal complexity. For the calculi for GK and $\mathsf{GK^c}$, this is not surprising, as the same is the case for the usual tableau calculi for the classical modal logic K (see e.g. [107]). On the other hand, the tableau calculus for $\mathsf{GS5^c}$ could be slightly modified - by removing repetitions of formulas - such that it yields a coNP-algorithm. This would make the exposition more cumbersome, however, and is therefore left out.

As is common for tableau calculi for modal logics, a tableau is designed to reflect the construction of a Kripke model for the given modal logic. In our case, the tableaux reflect the modified Kripke models from the alternative semantics (i.e. $\mathbb{FK}(\mathbf{G})$-, and $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-models), because GK, $\mathsf{GK^c}$, and $\mathsf{GS5^c}$ enjoy the finite model property with respect to these models.

For convenience, we simplify $\mathbb{FK}(\mathbf{G})$- and $\mathbb{FS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-models slightly while not changing the set of valid formulas they determine. This is possible because of the nice topological properties of the algebra $\mathbf{G}$, in particular, the fact that every point except 0 and 1 is a right and left accumulation point and any other point witnesses local homogeneity. These simplified models will be introduced in Section 6.1, before we go on to present a tableau calculus for $\mathsf{GS5^c}$ in Sections 6.2 and similar but more complicated calculi for GK and $\mathsf{GK^c}$ in Section 6.3.

Let us fix the algebraic language $\mathfrak{L}$ consisting of the binary operational symbols $\wedge$, $\vee$, and $\to$, and the nullary constants $\top$ and $\bot$, and let $\mathfrak{L}_{\mathfrak{m}} = \mathfrak{L} \cup \{\Box, \Diamond\}$. The set of formulas

$\mathrm{Fm}_{\mathfrak{L}_\mathfrak{m}}$ for $\mathfrak{L}_\mathfrak{m}$ will be denoted by Fm. Recall that $\mathbf{G}$ denotes the standard Gödel algebra $\langle [0,1], \min, \max, \to_\mathsf{G}, 0, 1\rangle$ for $\mathfrak{L}$.

6.1 Simplified Alternative Semantics

We define an $\mathbb{SK}(\mathbf{G})$-*model* as a quadruple $\mathfrak{M} = \langle W, R, V, T\rangle$, where $\langle W, R, V\rangle$ is a $\mathbb{K}(\mathbf{G})$-model (see Section 2.3) and $T\colon W \to \mathscr{P}_{<\omega}([0,1])$ is a function from worlds to finite sets of truth values satisfying $\{0,1\} \subseteq T(x) \subseteq [0,1]$ for all $x \in W$.

The valuation V is extended to the mapping $V\colon \mathrm{Fm} \times W$ inductively as follows:

$$V(\star(\varphi_1, \dots, \varphi_n), x) \;=\; \star(V(\varphi_1, x), \dots, V(\varphi_n, x))$$

for each n-ary operation symbol $\star$ of $\mathfrak{L}$, and

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\{r \in T(x) \mid r \leq \bigwedge\{Rxy \to V(\varphi, y) \mid y \in W\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\{r \in T(x) \mid r \geq \bigvee\{Rxy \wedge V(\varphi, y) \mid y \in W\}\}. \end{aligned}$$

We denote the class of all $\mathbb{SK}(\mathbf{G})$-models by $\mathbb{SK}(\mathbf{G})$ and the class of all $\mathbb{SK}(\mathbf{G})$-models with a crisp accessibility relation by $\mathbb{SK}^\mathsf{c}(\mathbf{G})$ (sometimes calling them crisp $\mathbb{SK}(\mathbf{G})$-models). As before, a formula $\varphi \in \mathrm{Fm}$ is *valid* in an $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$, written $\mathfrak{M} \models_{\mathbb{SK}(\mathbf{G})} \varphi$. And for a subclass $\mathbb{U} \subseteq \mathbb{SK}(\mathbf{G})$, we will say that φ is *valid in* $\mathbb{U}$, written $\models_\mathbb{U} \varphi$, if $\mathfrak{M} \models_{\mathbb{SK}(\mathbf{G})} \varphi$ for all $\mathfrak{M} \in \mathbb{U}$.

An $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$, where R is a crisp equivalence relation and $T(x) = T(y)$ for all $x, y \in W$ such that Rxy, is called an $\mathbb{SS}\mathbf{5}^\mathsf{c}(\mathbf{G})$-*model*. Let us denote the classes of all $\mathbb{SS}\mathbf{5}^\mathsf{c}(\mathbf{G})$-models by $\mathbb{SS}\mathbf{5}^\mathsf{c}(\mathbf{G})$.

We note that again validity in $\mathbb{SS}\mathbf{5}^\mathsf{c}(\mathbf{G})$ amounts to validity in all *universal* $\mathbb{SS}\mathbf{5}^\mathsf{c}(\mathbf{G})$-*models*, written $\mathfrak{M} = \langle W, V, T\rangle$, where $R = W \times W$ and thus T may be understood as a single fixed finite subset of $[0,1]$ (cf. Chapter 5). In this case, the conditions for box- and diamond-formulas simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\{r \in T \mid r \leq \bigwedge\{V(\varphi, y) \mid y \in W\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\{r \in T \mid r \geq \bigvee\{V(\varphi, y) \mid y \in W\}\}. \end{aligned}$$

Noting that $\mathbb{SK}(\mathbf{G})$-, $\mathbb{SK}^\mathsf{c}(\mathbf{G})$-, and $\mathbb{SS}\mathbf{5}^\mathsf{c}(\mathbf{G})$-models can be understood as special cases of $\mathbb{FK}(\mathbf{G})$-, $\mathbb{FK}^\mathsf{c}(\mathbf{G})$-, and $\mathbb{FS}\mathbf{5}^\mathsf{c}(\mathbf{G})$-models, respectively, we can prove the following theorem by using theorems from previous chapters and slightly adopting the proofs of some lemmas.

Theorem 6.1. *For any formula $\varphi \in \mathrm{Fm}$,*

(a) *φ is valid in GK if and only if it is valid in all finite $\mathbb{SK}(\mathbf{G})$-models,*

(b) *φ is valid in $\mathsf{GK^c}$ if and only if it is valid in all finite $\mathbb{SK}^c(\mathbf{G})$-models,*

(c) *φ is valid in $\mathsf{GS5^c}$ if and only if it is valid in all finite universal $\mathbb{SS}\mathbf{5}^c(\mathbf{G})$-models.*

Proof. Note that an $\mathbb{SK}(\mathbf{G})$-model $\langle W, R, V, T\rangle$ can be understood as a special case of an $\mathbb{FK}(\mathbf{G})$-model $\langle W, R, V, T_\Box, T_\Diamond\rangle$ where for all $x \in W$:

$$T(x) \;=\; T_\Box(x) \;=\; T_\Diamond(x) \;=\; [0,1] \setminus \bigcup_{i<n} (a_i, a_{i+1})$$

for some $n \in \mathbb{N}$ and $\{a_0, \dots, a_n\} \subseteq [0,1]$ satisfying $0 = a_0 < \dots < a_n = 1$. A similar inclusion holds for $\mathbb{SK}^c(\mathbf{G})$- and $\mathbb{FK}^c(\mathbf{G})$-models as well as for universal $\mathbb{SS}\mathbf{5}^c(\mathbf{G})$- and $\mathbb{FS}\mathbf{5}^c(\mathbf{G})$-models, as in the latter case also

$$\{V(\Diamond\varphi, x), V(\Box\varphi, x) \mid \varphi \in \mathrm{Fm}\} \;\subseteq\; T\,.$$

Using Theorems 4.11 and 5.6, this establishes the left-to-right directions of (a), (b), and (c), respectively.

For the right-to-left direction in (a) and (b), consider Lemma 4.6 but instead of a (crisp) $\mathbb{FK}(\mathbf{A})$-tree-model we are given a (crisp) $\mathbb{K}(\mathbf{G})$-tree-model and show that we can find a finite (crisp) $\mathbb{SK}(\mathbf{G})$-tree-model where φ fails at the root. This can be done by defining for each $x \in W$,

$$T(x) \;=\; V[\Sigma_\Box \cup \Sigma_\Diamond, x] \cup \{0,1\}\,,$$

and then adapting the proof of Lemma 4.6 in a straightforward manner.

In almost the same way, we can adapt the proof of Lemma 5.3 to prove the right-to-left direction in (c). Instead of laboriously defining $T_\Box$ and $T_\Diamond$, we just define T as follows:

$$T \;=\; V[\Sigma_\Box \cup \Sigma_\Diamond, x] \cup \{0,1\}\,. \qquad \Box$$

For a tableau calculus to decide validity in GK, $\mathsf{GK^c}$ or $\mathsf{GS5^c}$, it is therefore enough to decide validity in all finite $\mathbb{SK}(\mathbf{G})$-models, finite $\mathbb{SK}^c(\mathbf{G})$-models, or universal $\mathbb{SS}\mathbf{5}^c(\mathbf{G})$-models, respectively.

6.2 A Tableau Calculus for $\mathsf{GS5^c}$

We start by defining a tableau calculus for $\mathsf{GS5^c}$, as it is considerably simpler than the calculi for GK and $\mathsf{GK^c}$. A proof in the tableau calculus will be a tree, called a tableau, where intuitively, the nodes represent (in)equalities between values of formulas at the given world. These notions will be made clear below.

Tableaux

In order to define nodes, we not only make use of formulas in Fm as symbols, but also the symbols $\leq$, $<$, $=$, $:$, and *world-symbols* $w \in \mathrm{WS}$. Furthermore, we make use of a set TS of *T-symbols* γ, which is defined inductively over a set bTS of *basic* T-symbols t such that $\mathrm{bTS} \cup \{\overline{0}, \overline{1}\} \subseteq \mathrm{TS}$ and whenever $\gamma \in \mathrm{TS}$, then also $s(\gamma) \in \mathrm{TS}$. Note that the set TS of T-symbols is distinct from the set Var of propositional variables and that members of TS cannot occur in formulas. We then define a *node* to be a string of the form

$$\begin{aligned} &w\colon \varphi \lhd \psi\,, && \text{where } \varphi, \psi \in \mathrm{Fm} \cup \mathrm{TS},\ w \in \mathrm{WS}, \text{ and } \lhd \in \{\leq, <\}, \\ &\gamma = \Box\psi\,, && \text{where } \psi \in \mathrm{Fm} \text{ and } \gamma \in \mathrm{TS}, \text{ or} \\ &\Diamond\varphi = \gamma\,, && \text{where } \varphi \in \mathrm{Fm} \text{ and } \gamma \in \mathrm{TS}. \end{aligned}$$

A node will often be denoted by N and nodes of the form $w\colon \varphi \lhd \psi$, where $\varphi, \psi \in \{\bot, \top\} \cup \mathrm{Var} \cup \mathrm{TS}$, will be called *atomic*.

The idea is that for an $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\langle W, V, T\rangle$, each world-symbol w will be associated with a world in W and each T-symbol γ will be associated with a value a in T such that $s(\gamma)$ is associated with the next larger value b in T (except when $a = 1$, then also $b = 1$). The T-symbols $\overline{0}$ and $\overline{1}$ will always be associated with 0 and 1, respectively. A node $w\colon \varphi \lhd \psi$ then "states" that at the world $x \in W$ associated with w, $V(\varphi, x) \lhd V(\psi, x)$ (if $\varphi, \psi \in \mathrm{Fm}$). A node of the form $\Diamond\varphi = \gamma$ "states" for the value $a \in T$ associated with γ and any world $y \in W$, $V(\Diamond\psi, y) = a$. This makes sense because in $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-models, box- and diamond-formulas are assigned truth values independently of worlds.

A *tableau* is a pair $\mathfrak{T} = \langle D, E\rangle$ with D being a set of nodes and $E \subseteq D^2$ such that $\langle D, E\rangle$ is a tree (see Section 3.3). A *branch* of a tableau $\langle D, E\rangle$ is a sequence of nodes $\langle N_0, \ldots, N_k\rangle \in D^{k+1}$, for $k \in \mathbb{N}$, such that N_0 is the root of $\langle D, E\rangle$ and $\langle N_i, N_{i+1}\rangle \in E$ for all $i < k$.

The rules of $\mathcal{TGS}5^{\mathsf{c}}$ are displayed in Figure 6.1, where $\lhd \in \{<, \leq\}$, t is a new basic T-symbol (i.e. not occurring above on the branch), and v is a new world-symbol. Furthermore, "(u occurs on b)" means that the rule can be applied for any world-symbol u that occurs anywhere on the current branch b.

A *$\mathcal{TGS}5^{\mathsf{c}}$-tableau* is a tableau that is built top-down (starting with the root) according to the rules of $\mathcal{TGS}5^{\mathsf{c}}$ without repetition, that is, to each node of the form $w\colon \varphi \lhd \psi$, for $\varphi, \psi \in \mathrm{Fm} \cup \mathrm{TS}$, a rule is applied at most once and to nodes $\gamma = \Box\psi$ and $\Diamond\varphi = \gamma$ the rules $(=\Box)$ and $(\Diamond=)$ are applied at most once per world-symbol occurring on the current branch.

Remark 6.2. Every $\mathcal{TGS}5^{\mathsf{c}}$-tableau is finite. This is because the branching factor is at most 2 and, building the tableau top-down, every rule decomposes the formulas occurring at its root, resulting in branches of finite length. The only cases where a formula is not

$$(\wedge\lhd): \quad \begin{array}{c} w\colon \varphi\wedge\psi\lhd\chi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} w\colon \varphi\lhd\chi & w\colon \psi\lhd\chi \end{array} \end{array} \qquad (\lhd\wedge): \quad \begin{array}{c} w\colon \varphi\lhd\psi\wedge\chi \\ | \\ w\colon \varphi\lhd\psi \\ w\colon \varphi\lhd\chi \end{array}$$

$$(\vee\lhd): \quad \begin{array}{c} w\colon \varphi\vee\psi\lhd\chi \\ | \\ w\colon \varphi\lhd\chi \\ w\colon \psi\lhd\chi \end{array} \qquad (\lhd\vee): \quad \begin{array}{c} w\colon \varphi\lhd\psi\vee\chi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} w\colon \varphi\lhd\psi & w\colon \varphi\lhd\chi \end{array} \end{array}$$

$$(\to\lhd): \quad \begin{array}{c} w\colon \varphi\to\psi\lhd\chi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} w\colon \top\lhd\chi & w\colon \psi<\varphi \\ & w\colon \psi\lhd\chi \end{array} \end{array} \qquad (\lhd\to): \quad \begin{array}{c} w\colon \varphi\lhd\psi\to\chi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} w\colon \varphi\lhd\chi & w\colon \psi\leq\chi \\ & w\colon \varphi\lhd\top \end{array} \end{array}$$

$$(\Box\lhd): \quad \begin{array}{c} w\colon \Box\varphi\lhd\psi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} w\colon \top\lhd\psi & w\colon t\lhd\psi \quad (t \text{ new}) \\ & v\colon \varphi<s(t) \quad (v \text{ new}) \end{array} \end{array} \qquad (\lhd\Diamond): \quad \begin{array}{c} w\colon \varphi\lhd\Diamond\psi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} w\colon \varphi\lhd\bot & w\colon \varphi\lhd s(t) \quad (t \text{ new}) \\ & v\colon t<\psi \quad (v \text{ new}) \end{array} \end{array}$$

$$(\lhd\Box): \quad \begin{array}{c} w\colon \varphi\lhd\Box\psi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} \overline{1}=\Box\psi & t=\Box\psi \quad (t \text{ new}) \\ w\colon \varphi\lhd\overline{1} & w\colon \varphi\lhd t \\ & v\colon \psi<s(t) \quad (v \text{ new}) \end{array} \end{array} \qquad (\Diamond\lhd): \quad \begin{array}{c} w\colon \Diamond\varphi\lhd\psi \\ \swarrow \qquad \searrow \\ \begin{array}{cc} \Diamond\varphi=\overline{0} & \Diamond\varphi=s(t) \quad (t \text{ new}) \\ w\colon \overline{0}\lhd\psi & w\colon s(t)\lhd\psi \\ & v\colon t<\varphi \quad (v \text{ new}) \end{array} \end{array}$$

$$(=\Box): \quad \begin{array}{c} \gamma=\Box\psi \\ (u \text{ occurs on } b) \\ | \\ u\colon \gamma\leq\psi \end{array} \qquad (\Diamond=): \quad \begin{array}{c} \Diamond\varphi=\gamma \\ (u \text{ occurs on } b) \\ | \\ u\colon \varphi\leq\gamma \end{array}$$

Figure 6.1: $\mathcal{TGS}5^{\mathsf{c}}$-Rules

$$(\neg\neg\leq):\quad \begin{array}{c} w\colon \neg\neg\varphi \leq \psi \\ \diagup \qquad \diagdown \\ w\colon \top \leq \psi \quad w\colon \varphi \leq \bot \end{array} \qquad\qquad (\leq\neg\neg):\quad \begin{array}{c} w\colon \varphi \leq \neg\neg\psi \\ \diagup \qquad \diagdown \\ w\colon \bot < \psi \quad w\colon \varphi \leq \bot \end{array}$$

$$(\neg\neg<):\quad \begin{array}{c} w\colon \neg\neg\varphi < \psi \\ | \\ w\colon \bot < \psi \\ w\colon \varphi \leq \bot \end{array} \qquad\qquad (<\neg\neg):\quad \begin{array}{c} w\colon \varphi < \neg\neg\psi \\ | \\ w\colon \bot < \psi \\ w\colon \varphi < \top \end{array}$$

Figure 6.2: Derived $\mathcal{TGS5}^{\mathsf{c}}$-Rules for Double Negation

directly decomposed are the rules $(\lhd\Box)$ and $(\Diamond\lhd)$, but in those cases there are no other choices than to stop, to decompose the box- or diamond formula by using $(=\Box)$ or $(\Diamond=)$, respectively, or to decompose the other formula by another rule. At some point, there are no more formulas to decompose and the building of the tableau is stopped.

Example 6.3. The following is an example of a $\mathcal{TGS5}^{\mathsf{c}}$-tableau:

$$\begin{array}{c} w\colon (\varphi \to \bot) \to \bot \leq \psi \\ \diagup \qquad\qquad \diagdown \\ \begin{array}{cc} w\colon \top \leq \psi \quad & \begin{array}{c} w\colon \bot < \varphi \to \bot \\ w\colon \bot \leq \psi \\ \diagup \qquad \diagdown \\ w\colon \bot < \bot \quad \begin{array}{c} w\colon \varphi \leq \bot \\ w\colon \bot < \top \end{array} \end{array} \end{array} \end{array}$$

As negation is defined by $\neg\varphi = \varphi \to \bot$, double negation is $\neg\neg\varphi = (\varphi \to \bot) \to \bot$. Considering the fully decomposed nodes in Example 6.3 which are neither "trivial" nor "contradictory", namely $w\colon \top \leq \psi$ and $w\colon \varphi \leq \bot$, we can formulate the derived rule $(\neg\neg\leq)$ for double negation in Figure 6.2. The other derived $\mathcal{TGS5}^{\mathsf{c}}$-rules in Figure 6.2 can all be justified in a similar way.

Tableaux Satisfaction and Proofs

For a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M} = \langle W, V, T\rangle$, we define a mapping $f\colon \mathrm{WS} \cup \mathrm{TS} \to W \cup [0,1]$, called an $\mathfrak{M}$-*assignment*, that assigns to each world-symbol $w \in \mathrm{WS}$ a world $x \in W$ and to each T-symbol $\gamma \in \mathrm{TS}$ a value $a \in T = \{a_0, \ldots, a_n\}$, where $0 = a_0 < \ldots < a_n = 1$, such that if for some $i < n$, $f(\gamma) = a_i$, then $f(s(\gamma)) = a_{i+1}$ (otherwise $f(s(\gamma)) = 1$), and always $f(\overline{0}) = 0$ and $f(\overline{1}) = 1$.

A node of the form $w\colon \varphi \lhd \psi$ will be called *satisfied* by a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M} = \langle W, V, T\rangle$ under an $\mathfrak{M}$-assignment f if

$$\begin{array}{ll} V(\varphi, f(w)) \lhd V(\psi, f(w)), & \text{if } \varphi, \psi \in \mathrm{Fm}, \\ V(\varphi, f(w)) \lhd f(\psi), & \text{if } \varphi \in \mathrm{Fm} \text{ and } \psi \in \mathrm{TS}, \\ f(\varphi) \lhd V(\psi, f(w)), & \text{if } \varphi \in \mathrm{TS} \text{ and } \psi \in \mathrm{Fm}, \text{ or} \\ f(\varphi) \lhd f(\psi), & \text{if } \varphi, \psi \in \mathrm{TS}. \end{array}$$

Nodes of the form $\gamma = \Box\psi$ or $\Diamond\varphi = \gamma$ will be called *satisfied* by a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M} = \langle W, V, T\rangle$ under an $\mathfrak{M}$-assignment f if, respectively, for all $x \in W$, $f(\gamma) = V(\Box\psi, x)$, or for all $x \in W$, $V(\Diamond\varphi, x) = f(\gamma)$.

A branch b of a tableau is called *closed* if the atomic nodes on b cannot all be jointly satisfied by any universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M}$ under any $\mathfrak{M}$-assignment f, i.e. the atomic nodes on b represent some inconsistent collection of inequalities. In a tableau, we will indicate that a branch is closed by $\bigotimes$ and usually write the inconsistent collection of inequalities just below (writing "t" or "$s(t)$" for "$f(t)$" or "$f(s(t))$", respectively, and "p_w" for "$V(p, f(w))$"). A branch that is not closed, we call *open*, and if no more rules can be applied to nodes of a branch, it is called *complete*.

If all its branches are closed, a tableau is called *closed*, and *open* otherwise. A tableau is called *complete*, if each of its branches is either a closed branch or a complete open branch. A formula $\varphi \in \mathrm{Fm}$ is called *provable in $\mathcal{TGS}5^{\mathsf{c}}$*, abbreviated by $\vdash_{\mathcal{TGS}5^{\mathsf{c}}} \varphi$, if there is a closed $\mathcal{TGS}5^{\mathsf{c}}$-tableau with root $w\colon \varphi < \top$, for some world-symbol w.

Remark 6.4. Note that a branch of an $\mathcal{TGS}5^{\mathsf{c}}$-tableau is complete if and only if it is closed or to each non-atomic node of the form $w\colon \varphi \lhd \psi$ a rule has been applied exactly once and to each node of the form $\gamma = \Box\psi$ or $\Diamond\varphi = \gamma$ the rule $(=\Box)$ or $(\Diamond=)$, respectively, has been applied exactly once for each world-symbol occurring on the branch.

It is also worth noticing that every $\mathcal{TGS}5^{\mathsf{c}}$-tableau can be extended to a complete $\mathcal{TGS}5^{\mathsf{c}}$-tableau by applying all applicable rules that have not been applied yet on every open branch.

Later in this section, we will prove soundness (Theorem 6.9) and completeness (Theorem 6.11) for the tableau calculus $\mathcal{TGS}5^{\mathsf{c}}$ with respect to validity in $\mathsf{GS5}^{\mathsf{c}}$ (using Theorem 6.1), but in fact, the completeness lemma (Lemma 6.10) lets us infer a slightly stronger statement. Lemma 6.10 implies that if there is a complete open $\mathcal{TGS}5^{\mathsf{c}}$-tableau with the root $w\colon \varphi < \top$, we can read off each complete open branch a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\langle W, V, T\rangle$ satisfying each node on the branch and thus $V(\varphi, x) < 1$ for some world $x \in W$. This implies that there is no closed $\mathcal{TGS}5^{\mathsf{c}}$-tableau with the same root. Recalling that each $\mathcal{TGS}5^{\mathsf{c}}$-tableau is finite (see Remark 6.2), we obtain the following decision procedure:

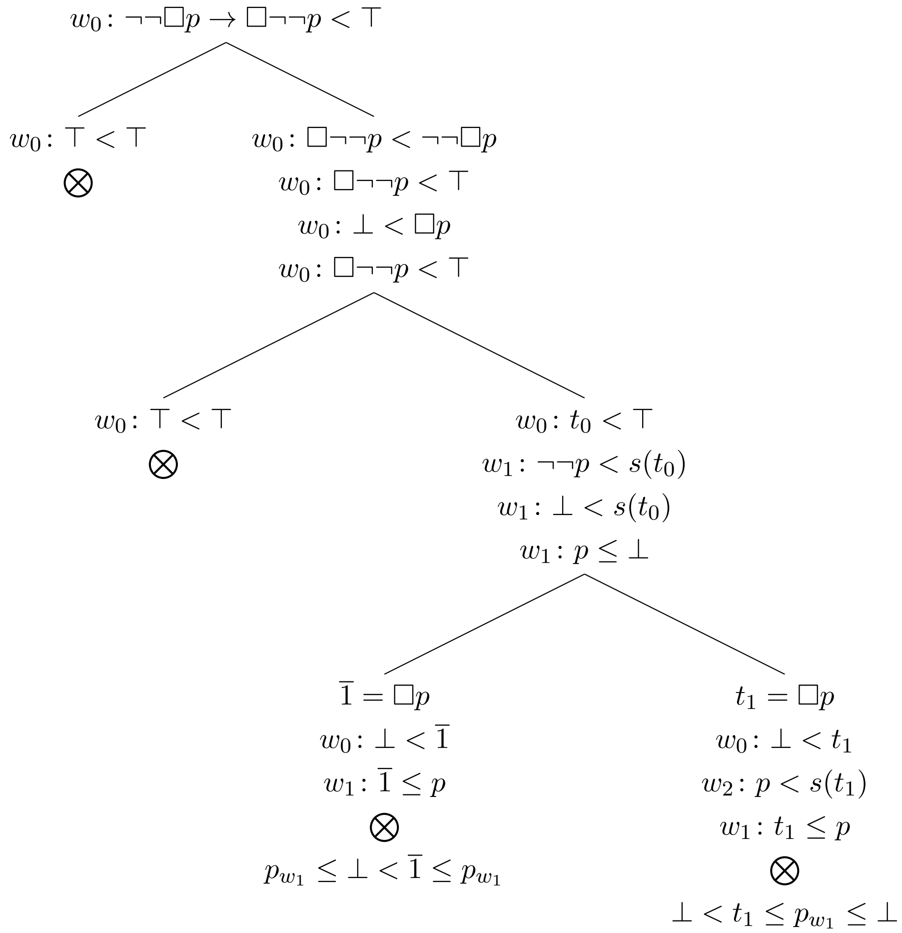

Figure 6.3: A Closed $\mathcal{TGS5}^{\mathsf{c}}$-Tableau

Theorem 6.5 (Decision Procedure)**.** *Let $\varphi \in \mathrm{Fm}$ and let $\mathfrak{T}$ be a complete $\mathcal{TGS5}^{\mathsf{c}}$-tableau with root $w\colon \varphi < \top$, then*

(a) *if $\mathfrak{T}$ is closed then φ is valid in* $\mathsf{GS5^c}$*,*

(b) *if $\mathfrak{T}$ is open then φ is not valid in* $\mathsf{GS5^c}$*.*

We will now present two examples where the validity in $\mathsf{GS5^c}$ of two interesting formulas are decided by using the tableau calculus $\mathcal{TGS5}^{\mathsf{c}}$.

Example 6.6. In Figure 6.3, an example of a closed $\mathcal{TGS5}^{\mathsf{c}}$-tableau is displayed, showing that $\vdash_{\mathcal{TGS5}^{\mathsf{c}}} \neg\neg\Box p \to \Box\neg\neg p$ and thus, by Theorem 6.9, the formula $\neg\neg\Box p \to \Box\neg\neg p$ is valid in $\mathsf{GS5^c}$.

We first use the rule $(\to<)$ on the root node, then $(<\neg\neg)$ on the node $w_0\colon \Box\neg\neg p < \neg\neg\Box p$, and $(\Box<)$ on $w_0\colon \Box\neg\neg p < \top$. Subsequently, we use $(\neg\neg<)$ on $w_1\colon \neg\neg p < s(t_0)$

and $(<\Box)$ on $w_0\colon \bot < \Box p$. On the left branch, in order to close the branch, we apply $(=\Box)$ on $\overline{1} = \Box p$ for the world-symbol w_1. The right branch is closed by applying $(=\Box)$ on $t_1 = \Box p$ for the world-symbol w_1.

Example 6.7. Figure 6.4 is an example of a complete open tableau, establishing that the formula used to disprove the finite model property of GK with respect to $\mathbb{K}(\mathbf{G})$ is not provable in $\mathcal{TGS5}^{\mathsf{c}}$, i.e. $\nvdash_{\mathcal{TGS5}^{\mathsf{c}}} \Box\neg\neg p \to \neg\neg\Box p$, and thus, by Theorem 6.11, $\Box\neg\neg p \to \neg\neg\Box p$ is not valid in $\mathsf{GS5}^{\mathsf{c}}$.

We first use $(\to<)$ on the root node, then $(\neg\neg<)$ on $w_0\colon \neg\neg\Box p < \Box\neg\neg p$ and $(\neg\neg<)$ on $w_0\colon \neg\neg\Box p < \top$, followed by $(\Box\leq)$ on the upper most occurrence of $w_0\colon \Box p \leq \bot$. Next, we apply again $(\Box\leq)$ on the next lower occurrence of $w_0\colon \Box p \leq \bot$. After using $(<\Box)$ on $w_0\colon \bot < \Box\neg\neg p$, we first apply on the right branch the rule $(\neg\neg<)$ to $w_3\colon \neg\neg p < s(t_2)$, then $(=\Box)$ to $t_2 = \Box\neg\neg p$ for the world-symbol w_3, and finally $(\leq\neg\neg)$ to $w_3\colon t_2 \leq \neg\neg p$, closing both resulting branches. On the left branch, we apply $(=\Box)$ to $\overline{1} = \Box\neg\neg p$ for w_0 and then $(\leq\neg\neg)$ to $w_0\colon \overline{1} \leq \neg\neg p$. These last two steps are then repeated for w_1 and w_2. Having applied all possible rules at the open branch b (marked with an $\Uparrow$), b is complete, and thus also the tableau is open and complete.

Because branch b (marked with an $\Uparrow$) is open, we know that there is a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M} = \langle W, V, T\rangle$ and an $\mathfrak{M}$-assignment f that satisfy all the atomic nodes on b. By Lemma 6.10, we can infer that $\mathfrak{M}$ also satisfies all the other nodes, including the root node $w_0\colon \Box\neg\neg p \to \neg\neg\Box p < \top$, which implies that $V(\Box\neg\neg p \to \neg\neg\Box p, f(w_0)) < 1$. Let us find such a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M} = \langle W, V, T\rangle$ and $\mathfrak{M}$-assignment f.

Assuming that $\mathfrak{M}$ satisfies all the atomic nodes on b under f, we read off of b the following constraints on $\mathfrak{M}$ and f:

(i)	$V(p, f(w_1)) < f(s(t_0))\,,$	from the node $w_1\colon p < s(t_0)$,
(ii)	$V(p, f(w_2)) < f(s(t_1))\,,$	from the node $w_2\colon p < s(t_0)$,
(iii)	$V(p, f(w_0)) > 0\,,$	from the node $w_0\colon \bot < p$,
(iv)	$V(p, f(w_1)) > 0\,,$	from the node $w_1\colon \bot < p$,
(v)	$V(p, f(w_2)) > 0\,,$	from the node $w_2\colon \bot < p$, and
(vi)	$f(t_0) = f(t_1) = 0\,,$	from the nodes $w_0\colon t_0 \leq \bot$ and $w_0\colon t_1 \leq \bot$.

Let us therefore define $\mathfrak{M} = \langle W, V, T\rangle$ and f as follows:[1]

- $W = \{x_0\}$ and $f(w_0) = f(w_1) = f(w_2) = x_0$,
- $V(p, x_0) = \frac{1}{2}$, and

[1] There are of course uncountably many different universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-models $\mathfrak{M}$ and $\mathfrak{M}$-assignments f satisfying these constraints. We choose one example.

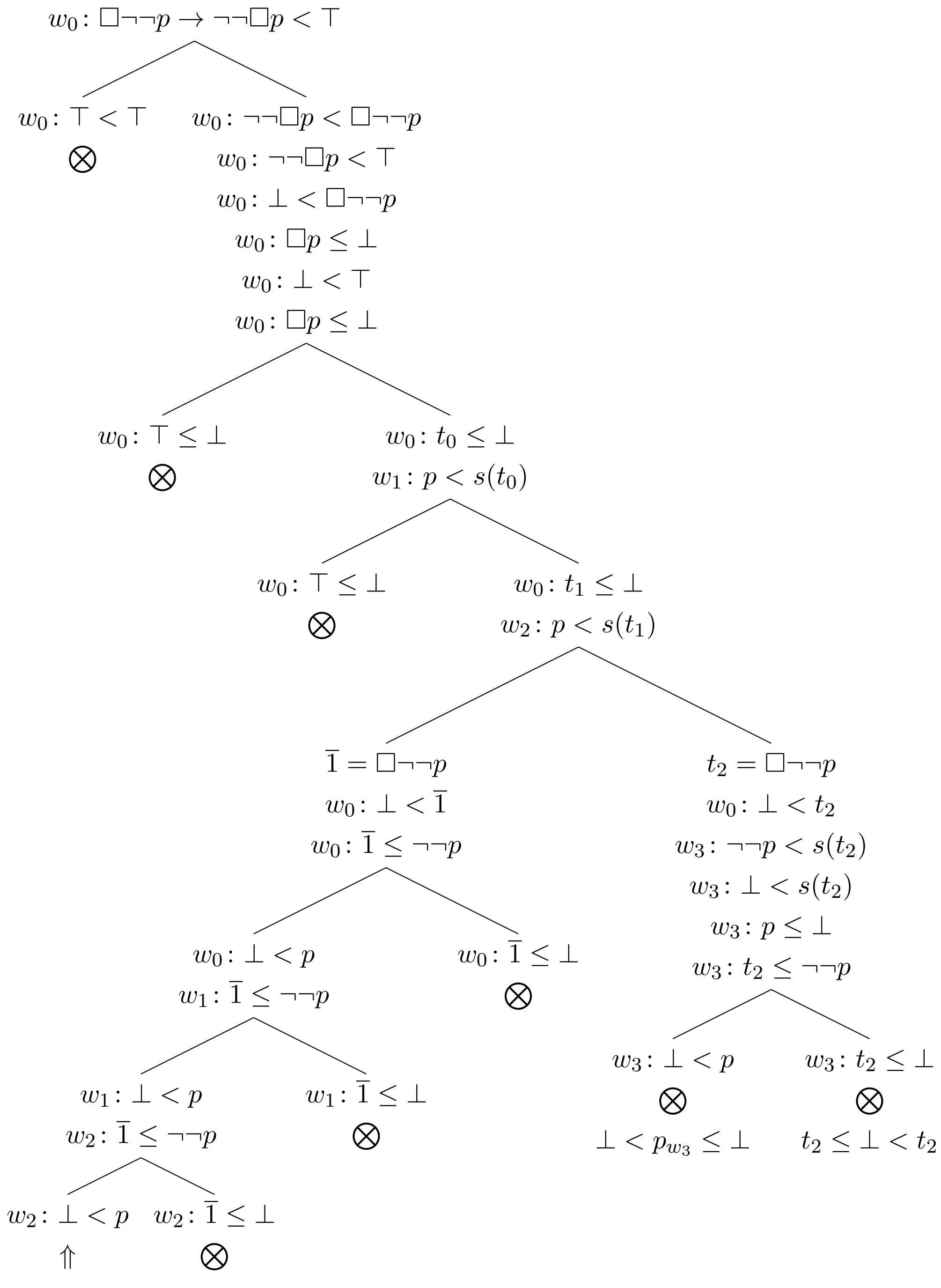

Figure 6.4: A Complete Open $\mathcal{TGS}5^{\mathsf{c}}$-Tableau

- $T = \{0, 1\}$ and $f(t_0) = f(t_1) = 0$ and $f(s(t_0)) = f(s(t_1)) = 1$.

We confirm that f is an $\mathfrak{M}$-assignment and that $\mathfrak{M}$ satisfies all the atomic nodes of b under f. Moreover, we note that

$$\begin{aligned}
V(\Box\neg\neg p, f(w_0)) &= \bigvee\{r \in T \mid r \leq \bigwedge\{V(\neg\neg p, y) \mid y \in W\}\} \\
&= \bigvee\{r \in \{0,1\} \mid r \leq \neg\neg\tfrac{1}{2}\} \\
&= \bigvee\{r \in \{0,1\} \mid r \leq 1\} \\
&= 1\,,
\end{aligned}$$

$$\begin{aligned}
V(\neg\neg\Box p, f(w_0)) &= \neg\neg\bigvee\{r \in T \mid r \leq \bigwedge\{V(p, y) \mid y \in W\}\} \\
&= \neg\neg\bigvee\{r \in \{0,1\} \mid r \leq \tfrac{1}{2}\} \\
&= \neg\neg 0 \\
&= 0\,,
\end{aligned}$$

and thus

$$\begin{aligned}
V(\Box\neg\neg p \to \neg\neg\Box p, f(w_0)) &= V(\Box\neg\neg p, f(w_0)) \to V(\neg\neg\Box p, f(w_0)) \\
&= 1 \to 0 \\
&= 0 \\
&< 1\,.
\end{aligned}$$

Soundness and Completeness

In order to prove soundness of our tableau calculus $\mathcal{TGS}5^{\mathsf{c}}$, we introduce the following notions. Let $\mathfrak{M}$ be a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model, f an $\mathfrak{M}$-assignment, and b a branch of a $\mathcal{TGS}5^{\mathsf{c}}$-tableau, then $\mathfrak{M}$ is called *faithful* to b under f if and only if all nodes on b are satisfied by $\mathfrak{M}$ under f. For a branch b of a $\mathcal{TGS}5^{\mathsf{c}}$-tableau $\mathfrak{T}$, we say that b' is an *extension* of b if b' is a branch of $\mathfrak{T}$ and b is an initial segment of b'.

Lemma 6.8 (Soundness Lemma)**.** *Let $\mathfrak{T}$ be a $\mathcal{TGS}5^{\mathsf{c}}$-tableau, b a branch of $\mathfrak{T}$, and let $\mathfrak{M}$ be a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model faithful to b under some $\mathfrak{M}$-assignment f. If a $\mathcal{TGS}5^{\mathsf{c}}$-rule is applied to a node on b, at least one extension of b, b', is produced such that $\mathfrak{M}$ is faithful to b' under some $\mathfrak{M}$-assignment f'.*

Proof. We will prove this lemma by considering all the different rules of $\mathcal{TGS}5^{\mathsf{c}}$ that could be applied to a node on b.

First, consider the case where $(\to\leq)$ is applied to a node of the form $w\colon \varphi \to \psi \leq \chi$ on b where $\varphi, \psi, \chi \in \mathrm{Fm}$. In this case, two extensions of b, b_1' and b_2', are produced:

$$
b_1' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ w\colon \top \le \chi \qquad \end{array}
\qquad\qquad
b_2' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ \qquad w\colon \psi < \varphi \\ \qquad w\colon \psi \le \chi \end{array}
$$

As $\mathfrak{M} = \langle W, V, T\rangle$ is faithful to b under f, we know $V(\varphi \to \psi, f(w)) \le V(\chi, f(w))$. Recall that

$$V(\varphi \to \psi, f(w)) \;=\; \begin{cases} 1 & \text{if } V(\varphi, f(w)) \le V(\psi, f(w)) \\ V(\psi, f(w)) & \text{if } V(\varphi, f(w)) > V(\psi, f(w)), \end{cases}$$

and thus either $1 = V(\varphi \to \psi, f(w)) \le V(\chi, f(w))$, in which case $\mathfrak{M}$ is faithful to the extension b_1' under $f' = f$, or $V(\psi, f(w)) < V(\varphi, f(w))$ and $V(\psi, f(w)) = V(\varphi \to \psi, f(w)) \le V(\chi, f(w))$, in which case $\mathfrak{M}$ is faithful to b_2' under $f' = f$.

When we consider the same case as above, except that $\chi \in \mathrm{TS}$, we replace all occurrences of "$V(\chi, f(w))$" in the above argument by "$f(\chi)$" and it works the same way. Furthermore, the other rules including propositional connectives are treated very similar.

Consider now the case where $(\le\Box)$ has been applied to a node on b of the form $w\colon \varphi \le \Box\psi$. We investigate first the case (i) where $\varphi, \psi \in \mathrm{Fm}$. In this case, two extensions of b, b_1' and b_2', are produced:

$$
b_1' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ \overline{1} = \Box\psi \qquad \\ w\colon \varphi \le \overline{1} \qquad \end{array}
\qquad\qquad
b_2' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ \qquad t = \Box\psi \\ \qquad w\colon \varphi \le t \\ \qquad v\colon \psi < s(t) \end{array}
$$

As $\mathfrak{M} = \langle W, V, T\rangle$ is faithful to b under f, we know that $V(\varphi, f(w)) \le V(\Box\psi, f(w))$. Recall that

$$V(\Box\psi, f(w)) \;=\; \bigvee\{r \in T \mid r \le \bigwedge\{V(\psi, z) \mid z \in W\}\}\,.$$

This means that either $V(\Box\psi, x) = 1$, for all $x \in W$, and thus also $V(\varphi, f(w)) \le V(\Box\psi, f(w)) = 1$, in which case $\mathfrak{M}$ is clearly faithful to b_1' under $f' = f$, or we have the following: there are $a_i, a_{i+1} \in T = \{a_0, \ldots, a_n\}$, with $0 = a_0 < \ldots < a_n = 1$, such that for all $x \in W$ (as box-formulas are invariant over worlds),

$$V(\varphi, f(w)) \;\le\; a_i \;=\; V(\Box\psi, f(w)) \;=\; V(\Box\psi, x) \qquad \text{and} \qquad V(\psi, y) \;<\; a_{i+1},$$

for some $y \in W$. We then define an $\mathfrak{M}$-assignment f' such that $f'(v) = y$, $f'(t) = a_i$, $f(s(t)) = a_{i+1}$, and otherwise f' agrees with f (implying that $f(w) = f'(w)$). It follows that $\mathfrak{M}$ is faithful to b'_2 under f', as also for all $x \in W$:

$$V(\varphi, f'(w)) \ \leq \ f'(t) \ = \ V(\Box\psi, x) \qquad \text{and} \qquad V(\psi, f'(v)) \ < \ f'(s(t))\,.$$

Secondly we consider the case (ii) where $\varphi \in \mathrm{TS}$. In this case, we replace all occurrences of "$V(\varphi, f(w))$" by "$f(\varphi)$" and all occurrences of "$V(\varphi, f'(w))$" by "$f'(\varphi)$" in the argument above. The argument then also goes through as now $f'(\varphi) = f(\varphi)$.

Let us next consider the case where $(=\Box)$ has been applied to a node on b of the form $\gamma = \Box\psi$ for some world-symbol u occurring on b, noting that the only possibility is that $\gamma \in \mathrm{TS}$ and $\psi \in \mathrm{Fm}$. Then the extensions b' of b is produced:

$$b' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ | \\ u\colon \gamma \leq \psi \end{array}$$

As $\mathfrak{M} = \langle W, V, T\rangle$ is faithful to b, we know that $f(u) \in W$ and $f(\gamma) = V(\Box\psi, x)$ for all $x \in W$. It then follows that $\mathfrak{M}$ is faithful to b' under $f' = f$ because

$$f(\gamma) \ = \ V(\Box\psi, x) \ \leq \ \bigwedge\{V(\psi, z) \mid z \in W\} \ \leq \ V(\psi, f(u))\,.$$

As the argument for the rule $(<\Box)$ is very similar to the argument for $(\leq\Box)$ and because the rules for diamond are exactly symmetrical to the box rules, the last rule we will consider here is $(\Box\leq)$. Consider the case where $(\Box\leq)$ is applied to the node $w\colon \Box\varphi \leq \psi$ on the branch b, where for case (i): $\varphi, \psi \in \mathrm{Fm}$. In this case, two extensions of b, b'_1 and b'_2, are produced:

$$b'_1 : \quad \begin{array}{ccc} & \vdots & \\ & b & \\ & \vdots & \\ / & & \backslash \\ w\colon \top \leq \psi & & \end{array} \qquad\qquad b'_2 : \quad \begin{array}{ccc} & \vdots & \\ & b & \\ & \vdots & \\ / & & \backslash \\ & & w\colon t \leq \psi \\ & & v\colon \varphi < s(t) \end{array}$$

As $\mathfrak{M} = \langle W, V, T\rangle$ is faithful to b under f, we know that $V(\Box\varphi, f(w)) \leq V(\psi, f(w))$. This means that either $V(\psi, f(w)) = 1$, in which case $\mathfrak{M}$ is faithful to b'_1 under $f' = f$, or we have that there are $a_i, a_{i+1} \in T = \{a_0, \ldots, a_n\}$, with $0 = a_0 < \ldots < a_n = 1$, such that

$$a_i \ = \ V(\Box\varphi, f(w)) \ \leq \ V(\psi, f(w)) \qquad \text{and} \qquad V(\varphi, y) \ < \ a_{i+1}, \text{ for some } y \in W.$$

In this case, we define an $\mathfrak{M}$-assignment f' such that f' agrees with f except that $f'(v) = y$, $f'(t) = a_i$, and $f'(s(t)) = a_{i+1}$. It then follows that $\mathfrak{M}$ is faithful to b'_2 under f', as also

$$f'(t) \ \le \ V(\psi, f'(w)) \qquad \text{and} \qquad V(\varphi, f'(v)) \ < \ f'(s(t))\,.$$

In case (ii) where $\psi \in \mathrm{TS}$, as above, we replace all occurrences of "$V(\psi, f(w))$" by "$f(\psi)$" and "$V(\psi, f'(w))$" by "$f'(\psi)$" in the argument. $\square$

Theorem 6.9 (Soundness of $\mathcal{TGS}5^{\mathsf{c}}$)**.** *For all* $\varphi \in \mathrm{Fm}$*, if*

$$\vdash_{\mathcal{TGS}5^{\mathsf{c}}} \varphi \qquad \textit{then} \qquad \models_{\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})} \varphi\,.$$

Proof. Let $\not\models_{\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})} \varphi$. Then there is a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M} = \langle W, V, T\rangle$ and a world $x \in W$ such that $V(\varphi, x) < 1$.

Now consider any complete $\mathcal{TGS}5^{\mathsf{c}}$-tableau $\mathfrak{T}$ with the root $w\colon \varphi < \top$. Clearly, $\mathfrak{M}$ is faithful to the branch b consisting only of the root node under any $\mathfrak{M}$-assignment f with $f(w) = x$. Applying the soundness lemma, we know that $\mathfrak{M}$ is faithful to at least one extension b' of b under some $\mathfrak{M}$-assignment f'. Applying the soundness lemma finitely many times, we find a complete branch $\widehat{b}$ such that $\mathfrak{M}$ is faithful to $\widehat{b}$ under $\mathfrak{M}$-assignment $\widehat{f}$. Because a closed branch is not satisfied by any universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M}$ under any $\mathfrak{M}$-assignment, it follows that $\widehat{b}$ must be open.

We have thus shown that any complete $\mathcal{TGS}5^{\mathsf{c}}$-tableau with the root $w\colon \varphi < \top$ must have an open branch. Therefore, there cannot exist a closed $\mathcal{TGS}5^{\mathsf{c}}$-tableau with the root $w\colon \varphi < \top$, which means that $\not\vdash_{\mathcal{TGS}5^{\mathsf{c}}} \varphi$. $\square$

For the proof of completeness of $\mathcal{TGS}5^{\mathsf{c}}$, we make use of one more notion. Given an open branch b on a $\mathcal{TGS}5^{\mathsf{c}}$-tableau, we call a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model $\mathfrak{M} = \langle W, V, T\rangle$ *induced by* b if and only if there is an $\mathfrak{M}$-assignment f such that each atomic node on b is satisfied by $\mathfrak{M}$ under f and for $\mathrm{WS}_b = \{w \in \mathrm{WS} \mid w \text{ occurs on } b\}$, $f[\mathrm{WS}_b] = W$.

Lemma 6.10 (Completeness Lemma)**.** *Let b be a complete open branch of a $\mathcal{TGS}5^{\mathsf{c}}$-tableau $\mathfrak{T}$ with root of the form $w\colon \phi < \top$, let $\mathfrak{M} = \langle W, V, T\rangle$ be a universal $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model induced by b, and let f be an $\mathfrak{M}$-assignment such that each atomic node on b is satisfied by $\mathfrak{M}$ under f. Then all nodes on b are satisfied by $\mathfrak{M}$ under f.*

Proof. The lemma is proved by an induction on the lexicographic ordering on $\{\langle \ell(N), e(N)\rangle \mid N \text{ is a node on } b\}$, where for any $\varphi, \psi \in \mathrm{Fm} \cup \mathrm{TS}$ and any node N of the form $w\colon \varphi \lhd \psi$ or $\varphi = \psi$, $\ell(N) = \ell(\varphi) + \ell(\psi)$ (where $\ell(t) = 1$ for all $t \in \mathrm{TS}$) and

$$e(N) \ = \ \begin{cases} 1 & \text{if } N \text{ is of the form } w\colon \varphi \lhd \psi\,, \\ 0 & \text{if } N \text{ is of the form } \varphi = \psi\,. \end{cases}$$

For the base case, consider a node N with $\langle \ell(N), e(N)\rangle = \langle 2, 1\rangle$. This is the smallest element of the ordering defined above because if it were the case that $e(N) = 0$, then N must be of the form $\varphi = \psi$ and thus either φ is a diamond-formula or ψ is a box-formula, implying that $\ell(N) \geq 3$. From $\langle \ell(N), e(N)\rangle = \langle 2, 1\rangle$ it follows that N is an atomic node of the form $w\colon \varphi \lhd \psi$ and therefore N is satisfied by $\mathfrak{M}$ under f by the definition of induced models.

For the inductive step, consider a node N on b with $\langle \ell(N), e(N)\rangle = \langle n, m\rangle$. There are many different cases to consider; let us start with the case where N is of the form $w\colon \varphi < \psi \to \chi$ with $\varphi, \psi, \chi \in \mathrm{Fm}$ (the case where $\varphi \in \mathrm{TS}$ can be treated by replacing "$V(\varphi, f(w))$" by "$f(\varphi)$" in the argument below). In this case, as b is complete and open, either $(<\!\to)$ has been applied to N or some other rule decomposing φ. For the case on hand, we assume that $(<\!\to)$ has been applied to N, thus b is either of the form b_1 or b_2.

$$
b_1 : \quad
\begin{array}{c}
\vdots \\
w\colon \varphi < \psi \to \chi \\
\vdots \\
\diagup \qquad \diagdown \\
\begin{array}{cc}
\begin{array}{c} w\colon \varphi < \chi \\ \vdots \end{array} & \qquad
\end{array}
\end{array}
\qquad\qquad
b_2 : \quad
\begin{array}{c}
\vdots \\
w\colon \varphi < \psi \to \chi \\
\vdots \\
\diagup \qquad \diagdown \\
\begin{array}{cc}
\qquad & \begin{array}{c} w\colon \psi \leq \chi \\ w\colon \varphi < \top \\ \vdots \end{array}
\end{array}
\end{array}
$$

In the case where $b = b_1$, as $\ell(\chi) < \ell(\psi \to \chi)$ and thus $\langle \ell(\varphi) + \ell(\chi), e(w\colon \varphi < \chi)\rangle < \langle n, m\rangle$, the induction hypothesis yields that $w\colon \varphi < \chi$ is satisfied by $\mathfrak{M}$ under f and thus $V(\varphi, f(w)) < V(\chi, f(w))$. Then $V(\varphi, f(w)) < V(\chi, f(w)) \leq V(\psi \to \chi, f(w))$ is established, as the latter inequality always holds in any $\mathbb{SS}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model. We conclude that N is satisfied by $\mathfrak{M}$ under f.

If $b = b_2$, it follows by the induction hypothesis that $V(\psi, f(w)) \leq V(\chi, f(w))$ and $V(\varphi, f(w)) < 1$. Therefore $V(\psi \to \chi, f(w)) = 1$ and thus $V(\varphi, f(w)) < V(\psi \to \chi, f(w))$, so N is satisfied by $\mathfrak{M}$ under f.

The other cases where rules for propositional connectives are applied are very similar. For this reason let us next consider the case where N is of the form $w\colon \varphi \leq \Diamond\psi$ with $\varphi, \psi \in \mathrm{Fm}$ (while the case where $\varphi \in \mathrm{TS}$ can be treated again by replacing "$V(\varphi, f(w))$" by "$f(\varphi)$" in the argument below). As b is complete, we assume that $(\leq \Diamond)$ has been applied to N to yield either b_1 or b_2:

$$
b_1: \quad \begin{array}{c} \vdots \\ w\colon \varphi \leq \Diamond\psi \\ \vdots \\ \begin{array}{cc} w\colon \varphi \leq \bot & \\ \vdots & \end{array} \end{array}
\qquad\qquad
b_2: \quad \begin{array}{c} \vdots \\ w\colon \varphi \leq \Diamond\psi \\ \vdots \\ \begin{array}{cc} & w\colon \varphi \leq s(t) \\ & v\colon t < \psi \\ & \vdots \end{array} \end{array}
$$

If $b = b_1$, it follows by the induction hypothesis that $V(\varphi, f(w)) = 0$. In this case, $V(\varphi, f(w)) \leq V(\Diamond\psi, f(w))$ is trivially true and thus N is satisfied by $\mathfrak{M}$ under f.

If $b = b_2$, consider $T = \{a_0, \ldots, a_k\}$ with $0 = a_0 < \ldots < a_k = 1$. Furthermore, note that $\ell(t) = \ell(s(t)) = 1 < \ell(\Diamond\psi)$ and thus $\langle \ell(\varphi) + \ell(s(t)), e(w\colon \varphi \leq s(t))\rangle < \langle n, m\rangle > \langle \ell(t) + \ell(\psi), e(v\colon t < \psi)\rangle$. It therefore follows by the induction hypothesis for some $i < k$, $V(\varphi, f(w)) \leq f(s(t)) = a_{i+1}$ and $a_i = f(t) < V(\psi, f(v))$. Therefore, $a_i < V(\psi, f(v)) \leq \bigvee\{V(\psi, z) \mid z \in W\}$, and thus

$$
V(\varphi, f(w)) \;\leq\; a_{i+1} \;\leq\; \bigwedge\{r \in T \mid r \geq \bigvee\{V(\psi, z) \mid z \in W\}\} \;=\; V(\Diamond\psi, f(w)),
$$

which yields that N is satisfied by $\mathfrak{M}$ under f.

Let us next consider the case where N is of the form $w\colon \Diamond\varphi \leq \psi$, with $\varphi, \psi \in \mathrm{Fm}$ (while the case where $\psi \in \mathrm{TS}$ can be treated by replacing "$V(\psi, f(w))$" by "$f(\psi)$" in the argument), and assume that $(\Diamond \leq)$ has been applied to N to yield either b_1 or b_2:

$$
b_1: \quad \begin{array}{c} \vdots \\ w\colon \Diamond\varphi \leq \psi \\ \vdots \\ \begin{array}{cc} \Diamond\varphi = \overline{0} & \\ w\colon \overline{0} \leq \psi & \\ \vdots & \end{array} \end{array}
\qquad\qquad
b_2: \quad \begin{array}{c} \vdots \\ w\colon \Diamond\varphi \leq \psi \\ \vdots \\ \begin{array}{cc} & \Diamond\varphi = s(t) \\ & w\colon s(t) \leq \psi \\ & v\colon t < \varphi \\ & \vdots \end{array} \end{array}
$$

In this case, $e(N) = 1$, $\ell(\Diamond\varphi = \overline{0}) = \ell(\Diamond\varphi) + 1 \leq \ell(N)$, and $e(\Diamond\varphi = \overline{0}) = 0 < 1 = e(N)$ and thus $\langle \ell(\Diamond\varphi = \overline{0}), e(\Diamond\varphi = \overline{0})\rangle \leq \langle n, 0\rangle < \langle n, 1\rangle = \langle n, m\rangle$. Therefore, if $b = b_1$, it follows by the induction hypothesis that $V(\Diamond\varphi, f(w)) = 0 \leq V(\psi, f(w))$ and thus N is satisfied by $\mathfrak{M}$ under f.

In the case where $b = b_2$, it follows by the induction hypothesis that $V(\Diamond\varphi, f(w)) = f(s(t)) \leq V(\psi, f(w))$ (and $f(t) < V(\varphi, f(v))$). Therefore N is satisfied by $\mathfrak{M}$ under f.

As the box-rules are just symmetrical to the diamond-rules and are thus treated similarly, we end our proof by considering the case where N is of the form $\Diamond\varphi = \gamma$ with

$\varphi \in \mathrm{Fm}$ and $\gamma \in \mathrm{TS}$. Note that the only possibilities in which N can occur in a $\mathcal{TGS}5^{\mathsf{c}}$-tableau $\mathfrak{T}$ are either when N is the root of $\mathfrak{T}$ or when N was produced by an application of $(\Diamond\lhd)$ to a node of the form $w\colon \Diamond\varphi \leq \psi$.

The first case does not need to be considered here, as we only consider $\mathcal{TGS}5^{\mathsf{c}}$-tableaux with a root of the form $w : \phi < \top$. For the second case, there are two possible forms of b: either γ is $\overline{0}$ and $b = b_1$, or γ is $s(t)$ and $b = b_2$, where b_1 and b_2 are as follows:

$$b_1 : \quad \begin{array}{c} \vdots \\ \diagup\quad\diagdown \\ \Diamond\varphi = \overline{0} \\ w\colon \overline{0} \leq \psi \\ \vdots \\ u\colon \varphi \leq \overline{0} \\ \vdots \end{array} \qquad\qquad b_2 : \quad \begin{array}{c} \vdots \\ \diagup\quad\diagdown \\ \Diamond\varphi = s(t) \\ w\colon s(t) \leq \psi \\ v\colon t < \varphi \\ \vdots \\ u\colon \varphi \leq s(t) \\ \vdots \end{array}$$

Note that in both cases, as b is complete, the node $u\colon \varphi \leq \gamma$ occurs on b for every world-symbol u occurring on b (including v in the case of $b = b_2$).

If $b = b_1$, then it follows by the induction hypothesis that $V(\varphi, f(u)) = 0$ for all $u \in \mathrm{WS}_b \in \{u \in \mathrm{WS} \mid u \text{ occurs on } b\}$. This implies, as $f[\mathrm{WS}_b] = W$, that $V(\varphi, z) = 0$ for all $z \in W$. Therefore, $V(\Diamond\varphi, x) = 0$ for all $x \in W$ and N is satisfied by $\mathfrak{M}$ under f.

If $b = b_2$, considering $T = \{a_0, \ldots, a_k\}$ with $0 = a_0 < \ldots < a_k = 1$. As $f[\mathrm{WS}_b] = W$, it follows by the induction hypothesis that for some $i < k$, $V(\varphi, z) \leq f(s(t)) = a_{i+1}$, for all $z \in W$, and $a_i = f(t) < V(\varphi, f(v))$. Therefore, we have that $a_i < \bigvee\{V(\varphi, z) \mid z \in W\} \leq a_{i+1}$ and thus for all $x \in W$,

$$V(\Diamond\varphi, x) \;=\; \bigwedge\{r \in T \mid r \geq \bigvee\{V(\varphi, z) \mid z \in W\}\} \;=\; a_{i+1} \;=\; f(s(t)),$$

so N is satisfied by $\mathfrak{M}$ under f. □

Theorem 6.11 (Completeness of $\mathcal{TGS}5^{\mathsf{c}}$)**.** *For all* $\varphi \in \mathrm{Fm}$*, if*

$$\models_{\mathbb{S}\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{G})} \varphi \qquad \textit{then} \qquad \vdash_{\mathcal{TGS}5^{\mathsf{c}}} \varphi\,.$$

Proof. Let $\nvdash_{\mathcal{TGS}5^{\mathsf{c}}} \varphi$. Then any complete $\mathcal{TGS}5^{\mathsf{c}}$-tableau with the root $w\colon \varphi < \top$ is open. Choose one and call it $\mathfrak{T}$, then choose a complete open branch b of $\mathfrak{T}$ and let $\mathfrak{M} = \langle W, V, T\rangle$ be a universal $\mathbb{S}\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model induced by b. Such a model exists, as the atomic nodes on b are jointly satisfiable by a universal $\mathbb{S}\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{G})$-model (otherwise b would be closed). By the completeness lemma it follows that the root node $w\colon \varphi < \top$ of $\mathfrak{T}$ is satisfied by $\mathfrak{M}$ under an $\mathfrak{M}$-assignment f and thus $V(\varphi, x) < 1$, for $x = f(w) \in W$. Therefore we have established that $\not\models_{\mathbb{S}\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{G})} \varphi$. □

6.3 Tableau Calculi for GK and GK^{c}

In this section, we define tableau calculi for the Gödel modal logics GK and GK^{c} which will rely on the simplified alternative semantics of (crisp) $\mathbb{SK}(\mathbf{G})$-models. The calculi are more complicated, as we have to take care of the accessibility relation and the fact that T is not a constant function, but they will be based on the same principles as $\mathcal{TGS}5^{\mathsf{c}}$.

Tableaux

In order to define the different forms of nodes, we make use of some more symbols. We still use formulas in Fm as symbols, world-symbols $w \in \mathrm{WS}$, and the symbols $\leq$, $<$, $=$, and $:$. Additionally, for each world-symbol $w \in \mathrm{WS}$, we have a set bTS_w of *basic T-symbols* for w, denoted by $t(w) \in \mathrm{bTS_w}$, and define inductively the set TS_w of *T-symbols* for w such that $\mathrm{bTS}_w \subseteq \mathrm{TS}_w$ and if $\gamma \in \mathrm{TS}_w$, then $s(\gamma) \in \mathrm{TS}_w$. We then define $\mathrm{TS} = \{\overline{0}, \overline{1}\} \cup \bigcup_{w \in \mathrm{WS}} \mathrm{TS}_w$. Moreover, for each pair of world-symbols $\langle w, v\rangle \in \mathrm{WS}^2$, we have a *relation-symbol* $rwv \in \mathrm{RS}$. Note that the sets TS and RS are distinct from each other and the set Var of propositional variables. *Nodes* are then defined to be a strings of the form

$$w\colon \varphi \lhd \psi\,, \qquad \text{where } \varphi, \psi \in \mathrm{Fm} \cup \mathrm{RS} \cup \mathrm{TS},\ w \in \mathrm{WS}, \text{ and } \lhd \in \{\leq, <\},$$

$$w\colon \gamma = \Box\psi\,, \quad \text{where } \psi \in \mathrm{Fm},\ \gamma \in \mathrm{TS}, \text{ and } w \in \mathrm{WS}, \text{ or}$$

$$w\colon \Diamond\varphi = \gamma\,, \quad \text{where } \varphi \in \mathrm{Fm},\ \gamma \in \mathrm{TS}, \text{ and } w \in \mathrm{WS}.$$

Nodes of the form $w\colon \varphi \lhd \psi$, where $\varphi, \psi \in \{\bot, \top\} \cup \mathrm{Var} \cup \mathrm{RS} \cup \mathrm{TS}$, will be called *atomic*.

A *tableau* is now defined to be a tableau $\langle D, E\rangle$ as defined in the last section, except that the D is a set of nodes as defined just above. The rules of $\mathcal{TGK}$ are displayed in Figure 6.5, where $\lhd \in \{<, \leq\}$, $t(w)$ is a new basic T-symbol, and v is a new world-symbol. The instruction "(u occurs on b)" means that the rule can be applied for any world-symbol u that occurs anywhere on the current branch b. Note that the rules for propositional connectives are exactly the same as in $\mathcal{TGS}5^{\mathsf{c}}$, except that the symbols φ, ψ, and χ can now also stand for relation-symbols.

A *$\mathcal{TGK}$-tableau* is a tableau that is built top-down (starting with the root) according to the rules in $\mathcal{TGK}$ without repetition, that is, to each node of the form $w\colon \varphi \lhd \psi$, for $\varphi, \psi \in \mathrm{Fm} \cup \mathrm{RS} \cup \mathrm{TS}$, a rule is applied at most once and to nodes $w\colon \gamma = \Box\psi$ and $w\colon \Diamond\varphi = \gamma$ the rules $(=\Box)$ and $(\Diamond =)$ are applied at most once per world-symbol occurring on the current branch.

Remark 6.12. $\mathcal{TGK}$-tableaux are finite for the same reasons as those ensuring that $\mathcal{TGS}5^{\mathsf{c}}$-tableaux are finite.

Obviously, we again have the derived rules in Figure 6.6.

$$(\wedge\lhd):\quad \begin{array}{c} w\colon \varphi \wedge \psi \lhd \chi \\ \hline \begin{array}{c|c} w\colon \varphi \lhd \chi & w\colon \psi \lhd \chi \end{array} \end{array} \qquad\qquad (\lhd\wedge):\quad \begin{array}{c} w\colon \varphi \lhd \psi \wedge \chi \\ | \\ w\colon \varphi \lhd \psi \\ w\colon \varphi \lhd \chi \end{array}$$

$$(\vee\lhd):\quad \begin{array}{c} w\colon \varphi \vee \psi \lhd \chi \\ | \\ w\colon \varphi \lhd \chi \\ w\colon \psi \lhd \chi \end{array} \qquad\qquad (\lhd\vee):\quad \begin{array}{c} w\colon \varphi \lhd \psi \vee \chi \\ \hline \begin{array}{c|c} w\colon \varphi \lhd \psi & w\colon \varphi \lhd \chi \end{array} \end{array}$$

$$(\rightarrow\lhd):\quad \begin{array}{c} w\colon \varphi \rightarrow \psi \lhd \chi \\ \hline \begin{array}{c|c} w\colon \top \lhd \chi & \begin{array}{c} w\colon \psi < \varphi \\ w\colon \psi \lhd \chi \end{array} \end{array} \end{array} \qquad\qquad (\lhd\rightarrow):\quad \begin{array}{c} w\colon \varphi \lhd \psi \rightarrow \chi \\ \hline \begin{array}{c|c} w\colon \varphi \lhd \chi & \begin{array}{c} w\colon \psi \leq \chi \\ w\colon \varphi \lhd \top \end{array} \end{array} \end{array}$$

$$(\Box\lhd):\quad \begin{array}{c} w\colon \Box\varphi \lhd \psi \\ \hline \begin{array}{c|c} w\colon \top \lhd \psi & \begin{array}{c} w\colon t(w) \lhd \psi \quad (t(w) \text{ new}) \\ v\colon \varphi < rwv \quad (v \text{ new}) \\ v\colon \varphi < s(t(w)) \end{array} \end{array} \end{array} \qquad\qquad (\lhd\Diamond):\quad \begin{array}{c} w\colon \varphi \lhd \Diamond\psi \\ \hline \begin{array}{c|c} w\colon \varphi \lhd \bot & \begin{array}{c} w\colon \varphi \lhd s(t(w)) \quad (t(w) \text{ new}) \\ v\colon t(w) < rwv \quad (v \text{ new}) \\ v\colon t(w) < \psi \end{array} \end{array} \end{array}$$

$$(\lhd\Box):\quad \begin{array}{c} w\colon \varphi \lhd \Box\psi \\ \hline \begin{array}{c|c} \begin{array}{c} w\colon \overline{1} = \Box\psi \\ w\colon \varphi \lhd \overline{1} \end{array} & \begin{array}{c} w\colon t(w) = \Box\psi \quad (t(w) \text{ new}) \\ w\colon \varphi \lhd t(w) \\ v\colon \psi < rwv \quad (v \text{ new}) \\ v\colon \psi < s(t(w)) \end{array} \end{array} \end{array} \qquad\qquad (\Diamond\lhd):\quad \begin{array}{c} w\colon \Diamond\varphi \lhd \psi \\ \hline \begin{array}{c|c} \begin{array}{c} w\colon \Diamond\varphi = \overline{0} \\ w\colon \overline{0} \lhd \psi \end{array} & \begin{array}{c} w\colon \Diamond\varphi = s(t(w)) \quad (t(w) \text{ new}) \\ w\colon s(t(w)) \lhd \psi \\ v\colon t(w) < rwv \quad (v \text{ new}) \\ v\colon t(w) < \varphi \end{array} \end{array} \end{array}$$

$$(=\Box):\quad \begin{array}{c} w\colon \gamma = \Box\psi \\ (u \text{ occurs on } b) \\ \hline \begin{array}{c|c} u\colon \gamma \leq \psi & u\colon rwu \leq \psi \end{array} \end{array} \qquad\qquad (\Diamond=):\quad \begin{array}{c} w\colon \Diamond\varphi = \gamma \\ (u \text{ occurs on } b) \\ \hline \begin{array}{c|c} u\colon rwu \leq \gamma & u\colon \varphi \leq \gamma \end{array} \end{array}$$

Figure 6.5: $\mathcal{TGK}$-Rules

$$(\neg\neg\leq): \quad \begin{array}{c} w\colon \neg\neg\varphi \leq \psi \\ \hline w\colon \top \leq \psi \quad \mid \quad w\colon \varphi \leq \bot \end{array} \qquad\qquad (\leq\neg\neg): \quad \begin{array}{c} w\colon \varphi \leq \neg\neg\psi \\ \hline w\colon \bot < \psi \quad \mid \quad w\colon \varphi \leq \bot \end{array}$$

$$(\neg\neg<): \quad \begin{array}{c} w\colon \neg\neg\varphi < \psi \\ \mid \\ w\colon \bot < \psi \\ w\colon \varphi \leq \bot \end{array} \qquad\qquad (<\neg\neg): \quad \begin{array}{c} w\colon \varphi < \neg\neg\psi \\ \mid \\ w\colon \bot < \psi \\ w\colon \varphi < \top \end{array}$$

Figure 6.6: Derived $\mathcal{TGK}$-Rules for Double Negation

Tableaux Satisfaction and Proofs

For an $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$, we define a mapping $f\colon \mathrm{WS} \cup \mathrm{RS} \cup \mathrm{TS} \to W \cup [0,1]$, called an $\mathfrak{M}$-*assignment*, that assigns to each world-symbol $w \in \mathrm{WS}$ a world $x \in W$, to each relation-symbol $rwv \in \mathrm{RS}$ the value $Rf(w)f(v) \in [0,1]$, and to each T-symbol $\gamma \in \mathrm{TS}_w$ a value $a_i \in T(f(w)) = \{a_0, \dots, a_n\}$ (where $0 = a_0 < \dots < a_n = 1$) for some $i \leq n$, such that if $i < n$, then $f(s(\gamma)) = a_{i+1}$, otherwise $f(s(\gamma)) = 1$, and always $f(\overline{0}) = 0$ and $f(\overline{1}) = 1$.

A node of the form $w\colon \varphi \lhd \psi$ is called *satisfied* by an $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$ under an $\mathfrak{M}$-assignment f if

$$\begin{array}{ll} V(\varphi, f(w)) \lhd V(\psi, f(w)), & \text{if } \varphi, \psi \in \mathrm{Fm}, \\ V(\varphi, f(w)) \lhd f(\psi), & \text{if } \varphi \in \mathrm{Fm} \text{ and } \psi \in \mathrm{RS} \cup \mathrm{TS}, \\ f(\varphi) \lhd V(\psi, f(w)), & \text{if } \varphi \in \mathrm{RS} \cup \mathrm{TS} \text{ and } \psi \in \mathrm{Fm}, \text{ or} \\ f(\varphi) \lhd f(\psi), & \text{if } \varphi, \psi \in \mathrm{RS} \cup \mathrm{TS}. \end{array}$$

Nodes of the form $w\colon \gamma = \Box\psi$ or $w\colon \Diamond\varphi = \gamma$ are called *satisfied* by an $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$ under $\mathfrak{M}$-assignment f if, respectively, $f(\gamma) = V(\Box\psi, f(w))$ or $V(\Diamond\varphi, f(w)) = f(\gamma)$.

A branch b of a tableau is called *(crisply) closed* if the atomic nodes on b cannot all be jointly satisfied by any (crisp) $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M}$ under any $\mathfrak{M}$-assignment f, i.e. the atomic nodes on b represent some contradicting series of inequalities. We will again indicate that a branch is (crisply) closed by a $\bigotimes$ and might write the contradicting series of inequalities just below. A branch that is not (crisply) closed we call *(crisply) open*, and if no more rules can be applied to nodes of a branch, the branch is called *complete*.

Remark 6.13. Note that if a branch is closed, it is also crisply closed, and if a branch is crisply open, it is also open. In both cases, the reverse direction does not hold.

As in Section 6.2, a tableau is called *closed* if all its branches are closed and it is called *open* otherwise. Furthermore, a tableau is called *crisply closed* if all its branches are crisply closed and it is called *crisply open* otherwise. A tableau is called *complete*, if each of its branches is either a closed branch, a crisply closed branch, or a complete crisply open branch. A formula $\varphi \in \mathrm{Fm}$ is called *provable in* $\mathcal{TGK}$, abbreviated by $\vdash_{\mathcal{TGK}} \varphi$, if there is a closed $\mathcal{TGK}$-tableau with the root $w\colon \varphi < \top$, for some world-symbol w and φ is called *provable in* $\mathcal{TGK}^{\mathsf{c}}$ if there is a crisply closed $\mathcal{TGK}$-tableau with the root $w\colon \varphi < \top$, for some world-symbol w, written $\vdash_{\mathcal{TGK}^{\mathsf{c}}} \varphi$,

Remark 6.14. Note, as in Remark 6.4, that a branch of a $\mathcal{TGK}$-tableau is complete if and only if to each non-atomic node of the form $w\colon \varphi \lhd \psi$ a rule has been applied exactly once and to each node of the form $w\colon \gamma = \Box\psi$ or $w\colon \Diamond\varphi = \gamma$ the rule $(= \Box)$ or $(\Diamond =)$, respectively, has been applied exactly once for each world-symbol occurring on the branch. We also note that each $\mathcal{TGK}$-tableau can be extended to a complete $\mathcal{TGK}$-tableau.

Again, we will prove soundness (Theorem 6.19) and completeness (Theorem 6.21) below for the tableau calculi $\mathcal{TGK}$ and $\mathcal{TGK}^{\mathsf{c}}$ with respect to validity in GK and GK^{c} respectively. In fact, the completeness lemma (Lemma 6.20) implies for each complete (crisply) open $\mathcal{TGK}$-tableau with the root $w\colon \varphi < \top$ and complete (crisply) open branch b, the existence of a (crisp) $\mathbb{SK}(\mathbf{G})$-model $\langle W, V, R, T\rangle$ satisfying each node on b and thus $V(\varphi, x) < 1$ for some world $x \in W$. Recalling that each $\mathcal{TGK}$-tableau is finite, we obtain the following decision procedure:

Theorem 6.15 (Decision Procedure)**.** *Let $\varphi \in \mathrm{Fm}$ and let $\mathfrak{T}$ be a complete $\mathcal{TGK}$-tableau with root $w\colon \varphi < \top$, then*

(a) *if $\mathfrak{T}$ is (crisply) closed then φ is valid in* GK $(\mathsf{GK}^{\mathsf{c}})$*,*

(b) *if $\mathfrak{T}$ is (crisply) open then φ is not valid in* GK $(\mathsf{GK}^{\mathsf{c}})$*.*

Example 6.16. Figure 6.7 is an example of a closed $\mathcal{TGK}$-tableau. It establishes that $\vdash_{\mathcal{TGK}} \neg\neg\Box p \to \Box\neg\neg p$ and thus $\neg\neg\Box p \to \Box\neg\neg p$ is valid in GK. Obviously, this also means that it is a crisply closed tableau and therefore $\neg\neg\Box p \to \Box\neg\neg p$ is also GK^{c}-valid.

We first use the rule $(\to<)$ on the root node and then $(<\neg\neg)$ on the node $w_0\colon \Box\neg\neg p < \neg\neg\Box p$. Subsequently, we use $(\Box<)$ on the lower occurrence of $w_0\colon \Box\neg\neg p < \top$, producing two branches of which the left closes immediately while on the right, we introduce the new world-symbol w_1. Furthermore, we apply $(\neg\neg<)$ to $w_1\colon \neg\neg p < rw_0w_1$ and subsequently split the branch in two by using $(<\Box)$ on the node $w_0\colon \bot < \Box p$ (on the fourth line), introducing the new world-symbol w_2 on the resulting right branch. On the resulting left branch, we apply $(=\Box)$ to $w_0\colon \bar{1} = \Box p$ for the world-symbol w_1 and obtain two branches

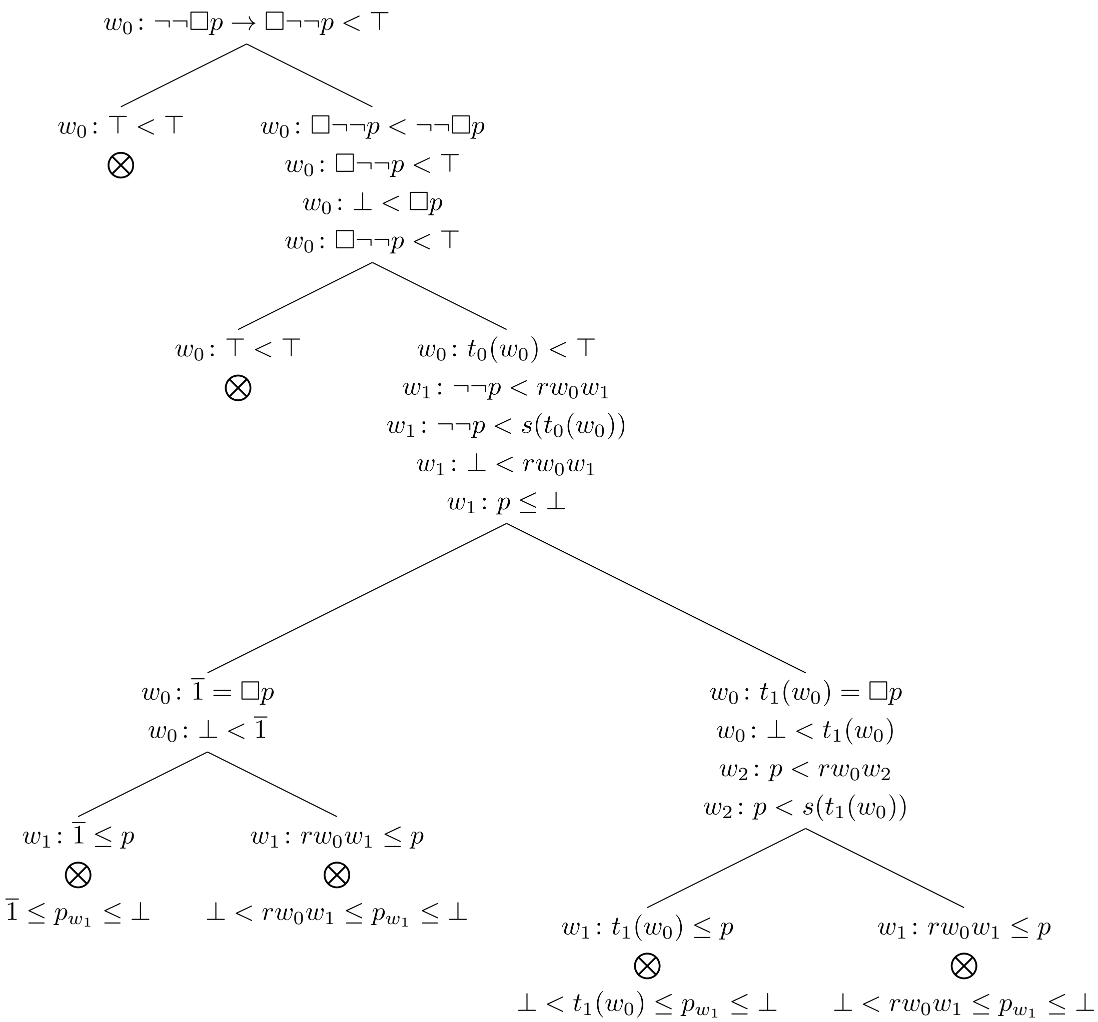

Figure 6.7: A Closed $\mathcal{TGK}$-Tableau

that close immediately. For the right branch, we use $(=\Box)$ on the node $w_0\colon t_1(w_0) = \Box p$ with the world-symbol w_1 to obtain two branches that both close immediately.

Example 6.17. In Figure 6.8, we present an example of a complete open $\mathcal{TGK}$-tableau for the formula $\neg\neg\Diamond p \to \Diamond\neg\neg p$, the distinguishing formula between GK and $\mathsf{GK^c}$, establishing that $\nvdash_{\mathcal{TGK}} \neg\neg\Diamond p \to \Diamond\neg\neg p$ and thus $\neg\neg\Diamond p \to \Diamond\neg\neg p$ is not valid in GK.

On the other hand, the tableau is in fact crisply closed, as all open branches (marked by $\Uparrow_1$ and $\Uparrow_2$) are crisply closed. Therefore, $\vdash_{\mathcal{TGK^c}} \neg\neg\Diamond p \to \Diamond\neg\neg p$ and thus $\neg\neg\Diamond p \to \Diamond\neg\neg p$ is valid in $\mathsf{GK^c}$.

In the tableau, we first use $(\to<)$ on the root node, followed by $(<\neg\neg)$ on $w_0\colon \Diamond\neg\neg p < \neg\neg\Diamond p$ and $(<\Diamond)$ on $w_0\colon \bot < \Diamond p$, resulting in a split into two branches of which the left branch closes immediately and on the right branch we introduce the new symbols $s(t_0(w_0))$, w_1, $t_0(w_0)$, and rw_0w_1. We subsequently use $(\Diamond<)$ on the lower occurrence of $w_0\colon \Diamond\neg\neg p < \bot$, resulting again in a split into two branches, b_1 on the left and b_2 on the right, where the new symbols $s(t_1(w_0))$, w_2, $t_1(w_0)$, and rw_0w_2 occur.

On b_1, we use $(\Diamond=)$ on the node $w_0\colon \Diamond\neg\neg p = \overline{0}$ for the world-symbol w_1, resulting in a split of which the left branch closes immediately while on the right branch we apply $(\neg\neg\leq)$. Both resulting branches close immediately as well.

On b_2, we first use $(<\neg\neg)$ on $w_2\colon t_1(w_0) < \neg\neg p$. We then use $(\Diamond=)$ on $w_0\colon \Diamond\neg\neg p = s(t_1(w_0))$ for the world-variable w_2, resulting in a split into two branches. On the right branch, we use $(\neg\neg\leq)$ and both resulting branches close. We then repeat applying $(\Diamond=)$ to $w_0\colon \Diamond\neg\neg p = s(t_1(w_0))$ for the world-variables w_1 and w_0, and also repeat using $(\neg\neg\leq)$ on the resulting right branch.

We now have two open branches, marked by $\Uparrow_1$ and $\Uparrow_2$, to which no more rules can be applied. As the rest of the branches are closed, we have obtained a complete open tableau, establishing that $\nvdash_{\mathcal{TGK}} \neg\neg\Diamond p \to \Diamond\neg\neg p$.

Note that strictly speaking, the tableau $\mathfrak{T}$ in Figure 6.8 is not complete. We have not applied any rule to the upper most occurrence of the node $w_0\colon \Diamond\neg\neg p < \bot$. However, for simplicity, we ignore this node and abuse notation in calling the branches marked by $\Uparrow_1$ and $\Uparrow_2$ complete. We do this because the node has an identical twin two lines below, to which a rule has been applied. For this reason, we do not lose any information by ignoring the node, as clearly any $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M}$ and $\mathfrak{M}$-assignment f satisfying all other nodes on a "complete" open branch of $\mathfrak{T}$ will satisfy also the upper most occurrence of the node $w_0\colon \Diamond\neg\neg p < \bot$, as it satisfies its twin.

As in Example 6.7, we know for example the branch b marked by $\Uparrow_2$ that there is an $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$ and an $\mathfrak{M}$-assignment f that satisfy all the atomic nodes on b, i.e. $\mathfrak{M}$is induced by b. By Lemma 6.20, we can infer that $\mathfrak{M}$ also satisfied all the other nodes, including the root node $w_0\colon \neg\neg\Diamond p \to \Diamond\neg\neg p < \top$, which implies that $V(\neg\neg\Diamond p \to \Diamond\neg\neg p, f(w_0)) < 1$. Let us give an example of such an $\mathbb{SK}(\mathbf{G})$-model

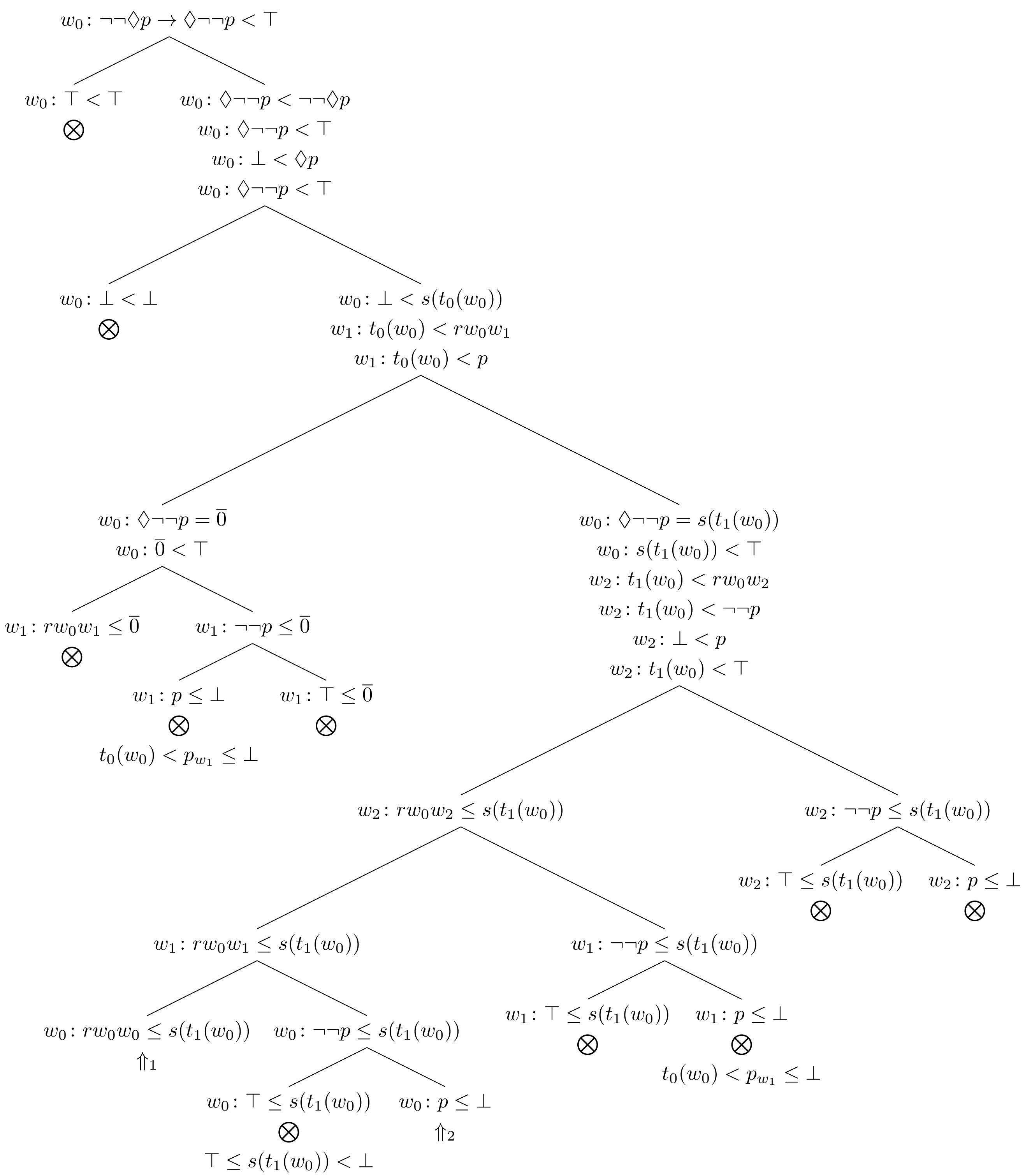

Figure 6.8: A Complete Open $\mathcal{TGK}$-Tableau

$\mathfrak{M} = \langle W, R, V, T\rangle$ and $\mathfrak{M}$-assignment f. Omitting some redundant requirements, we obtain from the atomic nodes on b the following constraints on $\mathfrak{M}$ and f:

(i)	$Rf(w_0)f(w_1) > f(t_0(w_0))$,	from the node $w_0 \colon t_0(w_0) < rw_0w_1$,
(ii)	$Rf(w_0)f(w_2) > f(t_1(w_0))$,	from the node $w_2 \colon t_1(w_0) < rw_0w_2$,
(iii)	$Rf(w_0)f(w_2) \leq f(s(t_1(w_0)))$,	from the node $w_2 \colon rw_0w_2 \leq s(t_1(w_0))$,
(iv)	$Rf(w_0)f(w_1) \leq f(s(t_1(w_0)))$,	from the node $w_1 \colon rw_0w_1 \leq s(t_1(w_0))$,
(v)	$V(p, f(w_2)) > 0$,	from the node $w_2 \colon \bot < p$,
(vi)	$V(p, f(w_0)) = 0$,	from the node $w_0 \colon p \leq \bot$,
(vii)	$V(p, f(w_1)) > f(t_0(w_0))$,	from the node $w_1 \colon t_0(w_0) < p$,
$(viii)$	$f(s(t_0(w_0))) > 0$,	from the node $w_0 \colon \bot < s(t_0(w_0))$,
(ix)	$f(s(t_1(w_0))) < 1$,	from the node $w_0 \colon s(t_1(w_0)) < \top$,
(x)	$f(t_1(w_0)) < 1$,	from the node $w_2 \colon t_1(w_0) < \top$.

There are uncountably many different $\mathbb{SK}(\mathbf{G})$-models $\mathfrak{M}$ and $\mathfrak{M}$-assignments f satisfying these constraints. Let e.g. $\mathfrak{M} = \langle W, R, V, T\rangle$ be defined by

- $W = \{x_0, x_1\}$,
- $Rx_0x_0 = Rx_0x_1 = \frac{1}{2}$ and $Rx_1x_0 = Rx_1x_0 = 0$,
- $V(p, x_0) = 0$ and $V(p, x_1) = \frac{1}{2}$, and
- $T(x_0) = \{0, \frac{1}{4}, \frac{3}{4}, 1\}$ and $T(x_1) = \{0, 1\}$.

Furthermore, let f be a mapping $f \colon \mathrm{WS} \cup \mathrm{RS} \cup \mathrm{TS} \to W \cup [0,1]$ that satisfies

- $f(w_0) = x_0$, $f(w_1) = f(w_2) = x_1$,
- $f(rx_0x_0) = f(rx_0x_1) = f(rx_0x_2) = \frac{1}{2}$, and
- $f(t_0(w_0)) = f(t_1(w_0)) = \frac{1}{4}$ and $f(s(t_0(w_0))) = f(s(t_1(w_0))) = \frac{3}{4}$.

We notice that f is an $\mathfrak{M}$-assignment and that $\mathfrak{M}$ satisfies all the atomic nodes of b under f, meaning that $\mathfrak{M}$ is induced by b. Moreover, we note that

$$\begin{aligned}
V(\neg\neg\Diamond p, f(w_0)) &= \neg\neg \bigwedge\{r \in T(f(w_0)) \mid r \geq \bigvee\{Rf(w_0)y \wedge V(p,y) \mid y \in W\}\} \\
&= \neg\neg \bigwedge\{r \in T(x_0) \mid r \geq \bigvee\{Rx_0y \wedge V(p,y) \mid y \in W\}\} \\
&= \neg\neg \bigwedge\{r \in \{0, \tfrac{1}{4}, \tfrac{3}{4}, 1\} \mid r \geq (\tfrac{1}{2} \wedge 0) \vee (\tfrac{1}{2} \wedge \tfrac{1}{2})\} \\
&= \neg\neg \bigwedge\{r \in \{0, \tfrac{1}{4}, \tfrac{3}{4}, 1\} \mid r \geq \tfrac{1}{2}\}
\end{aligned}$$

$$
\begin{aligned}
&= \neg\neg \tfrac{3}{4} \\
&= 1\,,
\end{aligned}
$$

$$
\begin{aligned}
V(\Diamond\neg\neg p, f(w_0)) &= \textstyle\bigwedge\{r \in T(f(w_0)) \mid r \geq \bigvee\{Rf(w_0)y \wedge V(\neg\neg p, y) \mid y \in W\}\} \\
&= \textstyle\bigwedge\{r \in T(x_0) \mid r \geq \bigvee\{Rx_0y \wedge V(\neg\neg p, y) \mid y \in W\}\} \\
&= \textstyle\bigwedge\{r \in \{0, \frac{1}{4}, \frac{3}{4}, 1\} \mid r \geq (\frac{1}{2} \wedge \neg\neg 0) \vee (\frac{1}{2} \wedge \neg\neg \frac{1}{2})\} \\
&= \textstyle\bigwedge\{r \in \{0, \frac{1}{4}, \frac{3}{4}, 1\} \mid r \geq \frac{1}{2}\} \\
&= \tfrac{3}{4}\,,
\end{aligned}
$$

and thus

$$
\begin{aligned}
V(\neg\neg\Diamond p \to \Diamond\neg\neg p, f(w_0)) &= V(\neg\neg\Diamond p, f(w_0)) \to V(\Diamond\neg\neg p, f(w_0)) \\
&= 1 \to \tfrac{3}{4} \\
&= \tfrac{3}{4} \\
&< 1\,.
\end{aligned}
$$

Finally, note that the following collection of inequalities cannot be satisfied by a crisp $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M}$ under any $\mathfrak{M}$-assignment f: $t_1(w_0) < rw_0w_2 \leq s(t_1(w_0)) < \top$. That is, the atomic nodes on the branches marked by $\Uparrow_1$ and $\Uparrow_2$ cannot be jointly satisfied by a crisp $\mathbb{SK}(\mathbf{G})$-model, and therefore these branches are crisply closed. This means that the tableau is crisply closed and thus $\vdash_{\mathcal{TGK}^{\mathsf{c}}} \neg\neg\Diamond p \to \Diamond\neg\neg p$.

Soundness and Completeness

In order to prove soundness and completeness of the tableau calculi $\mathcal{TGK}$ and $\mathcal{TGK}^{\mathsf{c}}$ with respect to validity in GK and GK^{c}, respectively, we need to adapt the notions of faithfulness and inducement, introduced in Section 6.2, to $\mathbb{SK}(\mathbf{G})$-models. Let $\mathfrak{M}$ be an $\mathbb{SK}(\mathbf{G})$-model, f an $\mathfrak{M}$-assignment, and b a branch of a $\mathcal{TGK}$-tableau, then $\mathfrak{M}$ is called *faithful* to b under f if and only if all nodes on b are satisfied by $\mathfrak{M}$ under f. For a branch b of a $\mathcal{TGK}$-tableau $\mathfrak{T}$, we again say that b' is an *extension* of b if b' is a branch of $\mathfrak{T}$ and b is an initial segment of b'.

Furthermore, let b be a open branch of a $\mathcal{TGK}$-tableau. We denote the finite set of all world-symbols occurring on b by WS_b and call an $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$ *induced by* b if and only if there is an $\mathfrak{M}$-assignment f such that each atomic node on b is satisfied by $\mathfrak{M}$ under f and $f[\mathrm{WS}_b] = W$.

Lemma 6.18 (Soundness Lemma). *Let b be a branch of a $\mathcal{TGK}$-tableau and let $\mathfrak{M}$ be an $\mathbb{SK}(\mathbf{G})$-model faithful to b under some $\mathfrak{M}$-assignment f. If a $\mathcal{TGK}$-rule is applied to a node on b, at least one extension of b, b', is produced such that $\mathfrak{M}$ is faithful to b' under some $\mathfrak{M}$-assignment f'.*

Proof. Similarly to Lemma 6.8, this lemma is proved by considering all the different rules of $\mathcal{TGK}$ that could be applied to a node on b. As the rules for the propositional connectives are the same as in $\mathcal{TGS}5^{\mathsf{c}}$, we just consider the rules for $\Box$, the rules for $\Diamond$ being treated similarly.

We first consider the case where $(\leq\Box)$ has been applied to a node on b of the form $w\colon \varphi \leq \Box\psi$ and investigate the subcase (i) where $\varphi, \psi \in \mathrm{Fm}$. In this case, two extensions of b, b'_1 and b'_2, are produced:

$$b'_1 : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ \begin{array}{c} w\colon \overline{1} = \Box\psi \\ w\colon \varphi \leq \overline{1} \end{array} \end{array} \qquad\qquad b'_2 : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ \begin{array}{c} w\colon t(w) = \Box\psi \\ w\colon \varphi \leq t(w) \\ v\colon \psi < rwv \\ v\colon \psi < s(t(w)) \end{array} \end{array}$$

As $\mathfrak{M} = \langle W, R, V, T\rangle$ is faithful to b under f, we know that $f(w) \in W$, let us call it x, and $V(\varphi, x) \leq V(\Box\psi, x)$. In the case where $V(\Box\psi, x) = 1$, we note that $V(\varphi, x) \leq V(\Box\psi, x) = 1$ and thus $\mathfrak{M}$ is clearly faithful to b'_1 under $f' = f$. Let us thus assume that $V(\Box\psi, x) < 1$. Recalling that

$$V(\Box\psi, x) \;=\; \bigvee\{r \in T(x) \mid r \leq \bigwedge\{Rxz \to V(\psi, z) \mid z \in W\}\},$$

we note that there must be a world $y \in W$ and values $a_i, a_{i+1} \in T(x) = \{a_0, \ldots, a_n\}$, with $0 = a_0 < \ldots < a_n = 1$, such that

$$a_i \;=\; V(\Box\psi, x) \;\leq\; Rxy \to V(\psi, y) \qquad \text{and} \qquad Rxy \to V(\psi, y) \;<\; a_{i+1}\,.$$

This implies also that $V(\psi, y) < Rxy$ and $V(\psi, y) < a_{i+1}$. We then define an $\mathfrak{M}$-assignment f' such that $f'(v) = y$, $f'(t(w)) = a_i$, $f(s(t(w))) = a_{i+1}$, and otherwise agrees with f (implying that $f(w) = f'(w) = x$ and $f'(rwv) = f(rwv) = Rxy$). It follows that $\mathfrak{M}$ is faithful to b'_2 under f', as

- $f'(t(w)) \;=\; a_i \;=\; V(\Box\psi, x) \;=\; V(\Box\psi, f'(w))\,,$
- $V(\varphi, f'(w)) \;=\; V(\varphi, x) \;\leq\; V(\Box\psi, x) \;=\; f'(t(w))\,,$
- $V(\psi, f'(v)) \;=\; V(\psi, y) \;<\; Rxy \;=\; f'(rwv)$, and
- $V(\psi, f'(v)) \;=\; V(\psi, y) \;<\; a_{i+1} \;=\; f'(s(t(w)))\,.$

In order to treat the case (ii) where $\varphi \in \mathrm{RS} \cup \mathrm{TS}$, we replace all occurrences of "$V(\varphi, f(w))$" and "$V(\varphi, x)$" in the argument above by "$f(\varphi)$" and also replace all occurrences of "$V(\varphi, f'(w))$" by "$f'(\varphi)$". The argument then also goes through as now $f'(\varphi) = f(\varphi)$.

Let us consider next the case where $(=\Box)$ has been applied to a node on b of the form $w\colon \gamma = \Box\psi$ for some world-symbol u occurring on b, noting that the only possibility is that $\gamma \in \mathrm{TS}$ and $\psi \in \mathrm{Fm}$. Then the extensions b_1' and b_2' of b are produced:

$$
b_1' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ u\colon \gamma \leq \psi \qquad \end{array}
\qquad\qquad
b_2' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ \qquad u\colon rwu \leq \psi \end{array}
$$

As $\mathfrak{M} = \langle W, R, V, T\rangle$ is faithful to b, we know that $f(w) = x$ and $f(u) = y$ for some $x, y \in W$, and that $f(\gamma) = V(\Box\psi, x)$. It follows that

$$f(\gamma) \;=\; V(\Box\psi, x) \;\leq\; \bigwedge\{Rxz \to V(\psi, z) \mid z \in W\} \;\leq\; Rxy \to V(\psi, y)\,,$$

and we either have $Rxy > V(\psi, y)$ or $Rxy \leq V(\psi, y)$. In the former case, $f(\gamma) \leq Rxy \to V(\psi, y) = V(\psi, y) = V(\psi, f(u))$, and thus $\mathfrak{M}$ is faithful to b_1' under $f' = f$. In the latter case, $\mathfrak{M}$ is faithful to b_2' under $f' = f$, as $f(rwu) = Rxy \leq V(\psi, y) = V(\psi, f(u))$.

As the argument for the rule $(<\Box)$ is very similar to the argument for $(\leq\Box)$, the last rule we will consider here is $(\Box\leq)$. Consider the case where $(\Box\leq)$ is applied to the node $w\colon \Box\varphi \leq \psi$ on the branch b, where for case (i): $\varphi, \psi \in \mathrm{Fm}$. In this case, two extensions of b, b_1' and b_2', are produced:

$$
b_1' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ w\colon \top \leq \psi \qquad \end{array}
\qquad\qquad
b_2' : \quad \begin{array}{c} \vdots \\ b \\ \vdots \\ \diagup \quad \diagdown \\ \qquad w\colon t(w) \leq \psi \\ \qquad v\colon \varphi < rwv \\ \qquad v\colon \varphi < s(t(w)) \end{array}
$$

As $\mathfrak{M} = \langle W, R, V, T\rangle$ is faithful to b under f, we let $f(w) = x \in W$ and infer that $V(\Box\varphi, x) \leq V(\psi, x)$. This means that either $V(\psi, x) = 1$, in which case $\mathfrak{M}$ is faithful to b_1' under $f' = f$, or we have that there are $a_i, a_{i+1} \in T(x) = \{a_0, \ldots, a_n\}$, with $0 = a_0 < \ldots < a_n = 1$, such that

$$a_i \;=\; V(\Box\varphi, x) \;\leq\; V(\psi, x) \qquad \text{and} \qquad Rxy \to V(\varphi, y) \;<\; a_{i+1}, \text{ for some } y \in W.$$

In this case, $Rxy > V(\varphi, y) < a_{i+1}$ and we define an $\mathfrak{M}$-assignment f' such that it agrees with f except that $f'(v) = y$, $f'(t(w)) = a_i$, and $f'(s(t(w))) = a_{i+1}$. It then follows that $\mathfrak{M}$ is faithful to b_2' under f', as also

- $f'(t(w)) \;=\; a_i \;=\; V(\Box\varphi, x) \;\leq\; V(\psi, x) \;=\; V(\psi, f'(w))\,,$
- $V(\varphi, f'(v)) \;=\; V(\varphi, y) \;<\; Rxy \;=\; f'(rwv)\,,$ and
- $V(\varphi, f'(v)) \;=\; V(\varphi, y) \;<\; a_{i+1} \;=\; f'(s(t(w)))\,.$

In case (ii) where $\psi \in \mathrm{TS}$, as above, we replace all occurrences of "$V(\psi, x)$" by "$f(\psi)$" and "$V(\psi, f'(w))$" by "$f'(\psi)$" in the argument. $\square$

Theorem 6.19 (Soundness of $\mathcal{TGK}$ and $\mathcal{TGK}^{\mathsf{c}}$)**.** *For all* $\varphi \in \mathrm{Fm}$*,*

$$\textit{if} \quad \vdash_{\mathcal{TGK}} \varphi \quad \textit{then} \quad \models_{\mathbb{SK}(\mathbf{G})} \varphi \qquad \textit{and} \qquad \textit{if} \quad \vdash_{\mathcal{TGK}^{\mathsf{c}}} \varphi \quad \textit{then} \quad \models_{\mathbb{SK}^{\mathsf{c}}(\mathbf{G})} \varphi\,.$$

Proof. For the first implication, let $\not\models_{\mathbb{SK}(\mathbf{G})} \varphi$. Then there is an $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V, T\rangle$ and a world $x \in W$ such that $V(\varphi, x) < 1$.

Now consider any complete $\mathcal{TGK}$-tableau $\mathfrak{T}$ with the root $w\colon \varphi < \top$. Clearly, $\mathfrak{M}$ is faithful to the branch b consisting only of the root node under any $\mathfrak{M}$-assignment f with $f(w) = x$. Applying the soundness lemma repeatedly, we find a complete branch b' such that $\mathfrak{M}$ is faithful to b' under $\mathfrak{M}$-assignment f'. Because a closed branch is not satisfied by any $\mathbb{SK}(\mathbf{G})$-model $\mathfrak{M}$ under any $\mathfrak{M}$-assignment, it follows that b' must be open.

This means that any complete $\mathcal{TGK}$-tableau with the root $w\colon \varphi < \top$ must have an open branch. Therefore, there is no closed $\mathcal{TGK}$-tableau with the root $w\colon \varphi < \top$, which means that $\not\vdash_{\mathcal{TGK}} \varphi$.

For the second implication, replace "$\mathbb{SK}(\mathbf{G})$" by "$\mathbb{SK}^{\mathsf{c}}(\mathbf{G})$", "closed" by "crisply closed", and "open" by "crisply open" in the argument above and it works in the same way. $\square$

Lemma 6.20 (Completeness Lemma)**.** *Let b be a complete (crisply) open branch of a $\mathcal{TGK}$-tableau with root of the form $w\colon \phi < \top$, let $\mathfrak{M} = \langle W, R, V, T\rangle$ be a (crisp) $\mathbb{SK}(\mathbf{G})$-model induced by b, and let f be an $\mathfrak{M}$-assignment such that each atomic node on b is satisfied by $\mathfrak{M}$ under f. Then all nodes on b are satisfied by $\mathfrak{M}$ under f.*

Proof. Similarly to the proof of the completeness lemma for $\mathcal{TGS5}^{\mathsf{c}}$, this lemma will be proved by an induction on the lexicographic ordering on $\{\langle \ell(N), e(N)\rangle \mid N \text{ is a node on } b\}$, where for any $\varphi, \psi \in \mathrm{Fm} \cup \mathrm{RS} \cup \mathrm{TS}$ and any N of the form $w\colon \varphi \lhd \psi$ or $w\colon \varphi = \psi$, $\ell(N) = \ell(\varphi) + \ell(\psi)$ (where $\ell(\chi) = 1$ for all $\chi \in \mathrm{RS} \cup \mathrm{TS}$) and

$$e(N) \;=\; \begin{cases} 1 & \text{if } N \text{ is of the form } w\colon \varphi \lhd \psi\,, \\ 0 & \text{if } N \text{ is of the form } w\colon \varphi = \psi\,. \end{cases}$$

For the base case, consider a node N with $\langle \ell(N), e(N)\rangle = \langle 2, 1\rangle$. This implies that N is an atomic node of the form $w\colon \varphi \lhd \psi$ and thus is satisfied by $\mathfrak{M}$ under f by the definition of induced models.

For the inductive step, consider a node N on b with $\langle \ell(N), e(N)\rangle = \langle n, m\rangle$. There are many different cases to consider, let us start with the case where N is of the form $w\colon \varphi \to \psi < \chi$ with $\varphi, \psi, \chi \in \mathrm{Fm}$ (the case where $\chi \in \mathrm{RS} \cup \mathrm{TS}$ can be treated by replacing "$V(\chi, f(w))$" by "$f(\chi)$" in the argument below). In this case, as b is complete, either $(\to\leq)$ has been applied to N or some other rule decomposing χ. For the case on hand, we assume that $(\to\leq)$ has been applied to N, thus b is either of the form b_1 or b_2.

$$b_1: \quad \begin{array}{c} \vdots \\ w\colon \varphi \to \psi \leq \chi \\ \vdots \\ \begin{array}{cc} w\colon \top \leq \chi & \\ \vdots & \end{array} \end{array} \qquad\qquad b_2: \quad \begin{array}{c} \vdots \\ w\colon \varphi \to \psi \leq \chi \\ \vdots \\ \begin{array}{cc} & w\colon \psi < \varphi \\ & w\colon \psi \leq \chi \\ & \vdots \end{array} \end{array}$$

In the case where $b = b_1$, as $\ell(\top) < \ell(\varphi \to \psi)$ and thus $\langle \ell(\top) + \ell(\chi), e(w\colon \top \leq \chi)\rangle < \langle n, m\rangle$, the induction hypothesis yields that $w\colon \top \leq \chi$ is satisfied by $\mathfrak{M}$ under f and thus $V(\chi, f(w)) = 1$. Then clearly $V(\varphi \to \psi, f(w)) \leq V(\chi, f(w))$ and so N is satisfied by $\mathfrak{M}$ under f.

If $b = b_2$, it follows by the induction hypothesis that $V(\psi, f(w)) < V(\varphi, f(w))$ and $V(\psi, f(w)) \leq V(\chi, f(w))$. Therefore we can conclude that $V(\varphi \to \psi, f(w)) = V(\psi, f(w)) \leq V(\chi, f(w))$ and N is satisfied by $\mathfrak{M}$ under f.

The other cases where rules for propositional connectives are applied are very similar. For this reason let us next consider the case where N is of the form $w\colon \varphi \leq \Diamond\psi$ with $\varphi, \psi \in \mathrm{Fm}$ (while the case where $\varphi \in \mathrm{RS} \cup \mathrm{TS}$ can be treated again by replacing "$V(\varphi, f(w))$" and "$V(\varphi, x)$" with "$f(\varphi)$" in the argument below). We assume that $(\leq \Diamond)$ has been applied to N to yield either b_1 or b_2:

$$b_1: \quad \begin{array}{c} \vdots \\ w\colon \varphi \leq \Diamond\psi \\ \vdots \\ \begin{array}{cc} w\colon \varphi \leq \bot & \\ \vdots & \end{array} \end{array} \qquad\qquad b_2: \quad \begin{array}{c} \vdots \\ w\colon \varphi \leq \Diamond\psi \\ \vdots \\ \begin{array}{cc} & w\colon \varphi \leq s(t(w)) \\ & v\colon t(w) < rwv \\ & v\colon t(w) < \psi \\ & \vdots \end{array} \end{array}$$

If $b = b_1$, it follows by the induction hypothesis that $V(\varphi, f(w)) = 0$. In this case, $V(\varphi, f(w)) \leq V(\Diamond\psi, f(w))$ is trivially the case and thus N is satisfied by $\mathfrak{M}$ under f.

For $b = b_2$, note that $\ell(\varphi) \geq \ell(t(w)) = \ell(s(t(w))) = \ell(rwv) = 1 < \ell(\Diamond\psi)$, and thus it follows by the induction hypothesis that the nodes $w\colon \varphi \leq s(t(w))$, $v\colon t(w) < rwv$, and $v\colon t(w) < \psi$ are satisfied by $\mathfrak{M}$ under f. Considering $f(w) = x$ and $f(v) = y$ for some $x, y \in W$ and $T(x) = \{a_0, \dots, a_k\}$ with $0 = a_0 < \dots < a_k = 1$, we can infer that for some $i < k$, $V(\varphi, x) \leq f(s(t(w))) = a_{i+1}$, $a_i = f(t(w)) < f(rwv) = Rxy$, and $a_i < V(\psi, y)$. It follows that $a_i < Rxy \wedge V(\psi, y) \leq \bigvee\{Rxz \wedge V(\psi, z) \mid z \in W\}$, and thus

$$V(\varphi, x) \ \leq\ a_{i+1} \ \leq\ \bigwedge\{r \in T \mid r \geq \bigvee\{Rxz \wedge V(\psi, z) \mid z \in W\}\} \ =\ V(\Diamond\psi, x)\,.$$

It follows that N is satisfied by $\mathfrak{M}$ under f.

Let us next consider the case where N is of the form $w\colon \Diamond\varphi \leq \psi$, with $\varphi, \psi \in \mathrm{Fm}$ (while the case where $\psi \in \mathrm{RS} \cup \mathrm{TS}$ can be treated by replacing "$V(\psi, f(w))$" by "$f(\psi)$" in the argument), and we assume that $(\Diamond\leq)$ has been applied to N to yield either b_1 or b_2:

$$b_1 : \quad \begin{array}{c} \vdots \\ w\colon \Diamond\varphi \leq \psi \\ \vdots \\ \begin{array}{cc} \begin{array}{c} w\colon \Diamond\varphi = \overline{0} \\ w\colon \overline{0} \leq \psi \\ \vdots \end{array} & \end{array} \end{array} \qquad\qquad b_2 : \quad \begin{array}{c} \vdots \\ w\colon \Diamond\varphi \leq \psi \\ \vdots \\ \begin{array}{cc} & \begin{array}{c} w\colon \Diamond\varphi = s(t(w)) \\ w\colon s(t(w)) \leq \psi \\ v\colon t(w) < rwv \\ v\colon t(w) < \varphi \\ \vdots \end{array} \end{array} \end{array}$$

In this case, $\ell(w\colon \Diamond\varphi = \overline{0}) = \ell(\Diamond\varphi) + 1 \leq \ell(N) = n$, and $e(w\colon \Diamond\varphi = \overline{0}) = 0 < 1 = e(N)$ and thus $\langle \ell(w\colon \Diamond\varphi = \overline{0}), e(w\colon \Diamond\varphi = \overline{0})\rangle \leq \langle n, 0\rangle < \langle n, 1\rangle = \langle n, m\rangle$. Therefore, if $b = b_1$, it follows by the induction hypothesis that $V(\Diamond\varphi, f(w)) = 0 \leq V(\psi, f(w))$, thus N is satisfied by $\mathfrak{M}$ under f.

In the case where $b = b_2$, it follows by the induction hypothesis, among other facts, that $V(\Diamond\varphi, f(w)) = f(s(t(w))) \leq V(\psi, f(w))$ and thus N is satisfied by $\mathfrak{M}$ under f. Notice that the satisfaction of the other nodes on b_2 is only needed when we consider the rule $(\Diamond{=})$.

As the box-rules are just symmetrical to the diamond-rules and are thus treated similarly, we end our proof by considering the case where N is of the form $w\colon \Diamond\varphi = \gamma$ with $\varphi \in \mathrm{Fm}$ and $\gamma \in \mathrm{TS}$. Note that the only possibilities in which N can occur in a $\mathcal{TGK}$-tableau are either when N is the root or when N was produced by an application of $(\Diamond\lhd)$ (where $\lhd \in \{<, \leq\}$).

The first case does not need to be considered here, as the $\mathcal{TGK}$-tableau considered has a root of the form $w : \varphi < \bot$. For the second case, there are two possible forms of b: either γ is $\overline{0}$ and $b = b_1$, or γ is $s(t(w))$ and $b = b_2$, where b_1 and b_2 are as follows:

$$
b_1 : \quad \begin{array}{c}
\vdots \\
w\colon \Diamond\varphi \lhd \psi \\
\vdots \\
\begin{array}{cc}
\begin{array}{c} w\colon \Diamond\varphi = \overline{0} \\ w\colon \overline{0} \lhd \psi \\ \vdots \end{array} &
\end{array}
\end{array}
\qquad\qquad
b_2 : \quad \begin{array}{c}
\vdots \\
w\colon \Diamond\varphi \lhd \psi \\
\vdots \\
\begin{array}{cc}
 & \begin{array}{c} w\colon \Diamond\varphi = s(t(w)) \\ w\colon s(t(w)) \lhd \psi \\ v\colon t(w) < rwv \\ v\colon t(w) < \varphi \\ \vdots \end{array}
\end{array}
\end{array}
$$

Furthermore, in case $b = b_1$, as b is complete, there is for each $u \in \mathrm{WS}_b = \{u \in \mathrm{WS} \mid u \text{ occurs on } b\}$ either the node $u\colon rwu \leq \overline{0}$ or the node $u\colon \varphi \leq \overline{0}$ on b. As for any $u \in \mathrm{WS}_b$, $\ell(rwu) + \ell(\overline{0}) = 2 \leq \ell(\varphi) + \ell(\overline{0}) < \ell(\Diamond\varphi) + \ell(\overline{0})$, it follows by the induction hypothesis that for each $u \in \mathrm{WS}_b$ either $Rf(w)f(u) = 0$ or $V(\varphi, f(u)) = 0$. As $f[\mathrm{WS}_b] = W$, this implies that $Rf(w)z \wedge V(\varphi, z) = 0$ for all $z \in W$. Therefore, $V(\Diamond\varphi, f(w)) = 0$ and N is satisfied by $\mathfrak{M}$ under f.

If $b = b_2$, we consider $f(w) = x$ and $f(v) = y$ for some $x, y \in W$ and $T(x) = \{a_0, \ldots, a_k\}$ with $0 = a_0 < \ldots < a_k = 1$. It follows by the induction hypothesis that for some $i < k$,

$$a_{i+1} \;=\; f(s(t(w))) \;\lhd\; V(\psi, x)\,, \qquad a_i \;=\; f(t(w)) \;<\; Rxy\,, \text{ and } \qquad a_i \;<\; V(\varphi, y)\,.$$

Moreover, similar to the case where $b = b_1$, there is for each $u \in \mathrm{WS}_b$ either the node $u\colon rwu \leq s(t(w))$ or the node $u\colon \varphi \leq s(t(w))$ on $b = b_2$. As $f[\mathrm{WS}_b] = W$, we therefore have by the induction hypothesis that $Rxz \wedge V(\varphi, z) \leq f(s(t(w))) = a_{i+1}$ for each $z \in W$. As from the above it also follows that $a_i < Rxy \wedge V(\varphi, y)$, we can therefore infer that $a_i < \bigvee\{Rxz \wedge V(\varphi, z) \mid z \in W\} \leq a_{i+1}$ and thus

$$V(\Diamond\varphi, x) \;=\; \bigwedge\{r \in T(x) \mid r \geq \bigvee\{Rxz \wedge V(\varphi, z) \mid z \in W\}\} \;=\; a_{i+1}\,.$$

It follows that N is satisfied by $\mathfrak{M}$ under f. $\square$

Theorem 6.21 (Completeness of $\mathcal{TGK}$ and $\mathcal{TGK}^{\mathrm{c}}$)**.** *For all* $\varphi \in \mathrm{Fm}$*,*

$$\text{if} \quad \models_{\mathbb{SK}(\mathbf{G})} \varphi \quad \text{then} \quad \vdash_{\mathcal{TGK}} \varphi \qquad\quad \text{and} \qquad\quad \text{if} \quad \models_{\mathbb{SK}^{\mathrm{c}}(\mathbf{G})} \varphi \quad \text{then} \quad \vdash_{\mathcal{TGK}^{\mathrm{c}}} \varphi\,.$$

Proof. For the first implication, we assume that $\nvdash_{\mathcal{TGK}} \varphi$. Then any complete $\mathcal{TGK}$-tableau with the root $w\colon \varphi < \top$ is open. Choose one, then choose a complete open branch b on it, and let $\mathfrak{M} = \langle W, R, V, T\rangle$ be an $\mathbb{SK}(\mathbf{G})$-model induced by b. By the completeness lemma it follows that the root node $w\colon \varphi < \top$ of $\mathfrak{T}$ is satisfied by $\mathfrak{M}$ under some $\mathfrak{M}$-assignment f and therefore $V(\varphi, x) < 1$, for $x = f(w) \in W$. Thus we have that $\not\models_{\mathbb{SK}(\mathbf{G})} \varphi$.

For the second implication, simply replace "$\mathbb{SK}(\mathbf{G})$" by "$\mathbb{SK}^{\mathsf{c}}(\mathbf{G})$" and "open" by "crisply open" in the argument above. $\square$

Chapter 7

Neighbourhood Semantics for Many-Valued Modal Logics

In classical modal logic, neighbourhood semantics, introduced independently by Richard Montague in [97] and Dana Scott in [115], is a more general and thus more flexible framework than Kripke-style relational semantics that provides semantics for a broader class of modal logics, including modal logics strictly weaker than K. It is a framework in which a plethora of different notions can be modelled as "modal" connectives, such as knowledge, obligation, belief, evidence, high probability, and even negation and generalized quantifiers (see e.g. [105]). For a recent introduction with motivating examples, see [103].

The goal in the present chapter is to propose a form of neighbourhood semantics (as already considered in [110, 111] by Rodríguez and Godo) that provides semantics for a broad class of many-valued modal logics, namely the class of axiomatic extensions of MTL. After recalling the main notions from the classical case and some notation inspired by fuzzy class theory in Section 7.1, we introduce neighbourhood frames for many-valued logics in Section 7.2. We then show how such frames relate to Kripke frames in Section 7.3 and we obtain an axiomatization of the logics given by all neighbourhood frames in Section 7.4.

The content of this chapter originates from joint work of the author of this dissertation with Petr Cintula and Carles Noguera [43].

7.1 Preliminaries

In this section, we briefly introduce classical neighbourhood semantics and some helpful notation inspired by the syntax of fuzzy class theory (see e.g. [11]). First, we fix the algebraic language $\mathfrak{L} = \{\wedge, \vee, \&, \to, \Box, \top\}$ and let $\mathfrak{L}_\Box = \mathfrak{L} \cup \{\Box\}$.[1] We will drop the subscript from the set of formulas $\mathrm{Fm}_{\mathfrak{L}}$ and $\mathrm{Fm}_{\mathfrak{L}_\Box}$ if the language is clear from the context.

[1] Recall that in the absence of double-negation elimination, as is the case for MTL, the diamond modality $\Diamond$ is not definable from the box $\Box$.

Classical Neighbourhood Semantics

In the classical setting, a *neighbourhood model*, or $\mathbb{SM}$-*model*, is a triple $\mathcal{M} = \langle W, N, V\rangle$, where W is a non-empty set of *worlds* and N is a function $N\colon W \to \mathscr{P}(\mathscr{P}(W))$ ($\mathscr{P}$ denoting the powerset operator) that assigns to each world $x \in W$ a set of subsets $N(x) \subseteq \mathscr{P}(W)$, called the *neighbourhood* of x. V is a *valuation* $V\colon \mathrm{Var} \times W \to \{0,1\}$ that is extended to all formulas inductively as in classical propositional logic (where & and $\wedge$ coincide and both denote classical conjunction), while for a box-formula:

$$V(\Box\varphi, x) \ = \ 1 \qquad \text{iff} \qquad [\![\varphi]\!]_{\mathcal{M}} \ \in \ N(x)\,,$$

where $[\![\varphi]\!]_{\mathcal{M}} = \{y \in W \mid V(\varphi, y) = 1\}$, the set of worlds where "φ is true".

We say that a formula $\varphi \in \mathrm{Fm}$ is *valid in an* $\mathbb{SM}$-*model* $\mathcal{M} = \langle W, N, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$ (which we can equivalently formulate as $[\![\varphi]\!]_{\mathcal{M}} = W$), written $\mathcal{M} \models_{\mathbb{SM}} \varphi$. For a set of formulas $\Gamma \subseteq \mathrm{Fm}$, we use the shorthand notation $\mathcal{M} \models_{\mathbb{SM}} \Gamma$, if for all $\psi \in \Gamma$, $\mathfrak{M} \models_{\mathbb{SM}} \psi$. Furthermore, a formula $\varphi \in \mathrm{Fm}$ is called an $\mathbb{SM}$-*consequence* of a set of formulas $\Gamma \subseteq \mathrm{Fm}$, if for all $\mathbb{SM}$-models $\mathcal{M}$, such that $\mathcal{M} \models_{\mathbb{SM}} \Gamma$, also $\mathcal{M} \models_{\mathbb{SM}} \varphi$, written $\Gamma \models_{\mathbb{SM}} \varphi$.[2] If $\emptyset \models_{\mathbb{SM}} \varphi$, we write $\models_{\mathbb{SM}} \varphi$ and say that φ is *valid* in $\mathbb{SM}$.

We note that a classical Kripke model, or shortly $\mathbb{K}$-*model*, is a $\mathbb{K}(\mathbf{A})$-model (as defined in Chapter 2) where $\mathbf{A}$ is the two-element Boolean algebra. Recall that $R[x] = \{y \in W \mid Rxy\}$ and V is extended box-formulas as follows:

$$V(\Box\varphi, x) \ = \ 1 \qquad \text{iff} \qquad V(\varphi, y) \ = \ 1, \ \text{for all } y \in R[x]\,.$$

Note that we can equivalently write this condition as $R[x] \subseteq [\![\varphi]\!]_{\mathfrak{M}}$, where $[\![\varphi]\!]_{\mathfrak{M}}$ is defined as in the case of the $\mathbb{SM}$-semantics.

It is not hard to see, that given any $\mathbb{K}$-model $\mathfrak{M} = \langle W, R, V\rangle$, we obtain an $\mathbb{SM}$-model $\mathcal{M}_{\mathfrak{M}} = \langle W, N_R, V\rangle$ by setting for all $x \in W$,

$$N_R(x) \ = \ \{X \in \mathscr{P}(W) \mid R[x] \subseteq X\}\,.$$

Conversely, given any $\mathbb{SM}$-model $\mathcal{M} = \langle W, N, V\rangle$, we can define a $\mathbb{K}$-model $\mathfrak{M}_{\mathcal{M}} = \langle W, R_N, V\rangle$ by setting for all $x, y \in W$,

$$R_N xy \qquad \text{iff} \qquad y \in X, \ \text{for each } X \in N(x)\,.$$

Note that this entails that $R_N[x] = \bigcap N(x) = \bigcap_{X \in N(x)} X$. However, in order to preserve valid formulas in the latter case (i.e. ensuring that for all $\varphi \in \mathrm{Fm}$, $\mathcal{M} \models_{\mathbb{SM}} \varphi$ iff $\mathfrak{M}_{\mathcal{M}} \models_{\mathbb{K}} \varphi$), we need the original $\mathbb{SM}$-model $\mathcal{M}$ to satisfy the following two additional conditions for each $x \in W$:

[2] Note that we consider the so-called *global* consequence relations. The reformulations of all our definitions to the *local* variant is straightforward.

- $N(x)$ contains its core, i.e. the set $(\bigcap_{X \in N(x)} X) \in N(x)$,
- $N(x)$ is closed under taking supersets, i.e. if $X \in N(x)$ and $X \subseteq Y$, then $Y \in N(x)$.

In this case, $\mathcal{M}$ is called *augmented.* The following results about these transitions and the axiomatization can be found for example in [32] or [103].

Theorem 7.1.

(a) *Let* $\mathfrak{M} = \langle W, R, V\rangle$ *be a* $\mathbb{K}$*-model. Then* $R_{N_R} = R$ *and* $\mathcal{M} = \langle \widehat{W}, N_R, \widehat{V}\rangle$*, where* $\widehat{W} = W$ *and* $\widehat{V} = V$*, is an augmented* $\mathbb{SM}$*-model, and for all* $\varphi \in \mathrm{Fm}$ *and* $x \in W$*,*

$$\widehat{V}(\varphi, x) \;=\; V(\varphi, x)\,.$$

(b) *Let* $\mathcal{M} = \langle W, N, V\rangle$ *be an augmented* $\mathbb{SM}$*-model. Then* $N_{R_N} = N$ *and* $\mathfrak{M} = \langle \widehat{W}, R_N, \widehat{V}\rangle$*, where* $\widehat{W} = W$ *and* $\widehat{V} = V$*, is a* $\mathbb{K}$*-model, and for all* $\varphi \in \mathrm{Fm}$ *and* $x \in W$*,*

$$\widehat{V}(\varphi, x) \;=\; V(\varphi, x)\,.$$

Corollary 7.2. *For any subset* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*,*

$$\Gamma \models_{\mathbb{K}} \varphi \quad \textit{iff} \quad \mathcal{M} \models_{\mathbb{SM}} \varphi \textit{ for all augmented } \mathbb{SM}\textit{-models } \mathcal{M} \textit{ such that } \mathcal{M} \models_{\mathbb{SM}} \Gamma\,.$$

Furthermore, let $\mathcal{CL}$ denote any Hilbert-style axiomatization of classical propositional logic CL, and define the following rule:

$$\text{(E)} \qquad \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}\,.$$

Letting $\mathcal{SM} = \mathcal{CL} \cup \{\text{(E)}\}$, we obtain the following completeness result:

Theorem 7.3. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*, then*

$$\Gamma \vdash_{\mathcal{SM}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbb{SM}} \varphi\,.$$

Fuzzy Sets and Notation

In order to formulate neighbourhood semantics over MTL-algebras, rather than the two-element Boolean algebra, we need to talk about fuzzy subsets of worlds and fuzzy sets of fuzzy subsets. To do this efficiently, we introduce a convenient notation, inspired by the syntax of fuzzy class theory (see e.g. [11]), where what we call “third-order formulas” will denote functions into a given MTL-chain. This notation considerably improves the readability of the proofs in the following sections and makes the connections to the classical setting more obvious.

Recall that an MTL-algebra $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \bot, \top \rangle$ is defined for the language $\mathfrak{L}$. Let $\mathfrak{L}_\triangle = \mathfrak{L} \cup \{\triangle\}$ be the algebraic language $\mathfrak{L}$ with an additional unary connective $\triangle$, which will be interpreted by the Delta operation, also called the Baaz-Monteiro operation. An algebra $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \triangle, \bot, \top \rangle$ for $\mathfrak{L}_\triangle$ will then be called an $\mathsf{MTL}_\triangle$*-chain* if its $\mathfrak{L}$-reduct $\langle A, \wedge, \vee, \&, \to, \bot, \top \rangle$ is an MTL-chain and for all $a \in A$:

$$\triangle a \;=\; \begin{cases} \top & \text{if } a = \top\,, \\ \bot & \text{otherwise,} \end{cases}$$

By $\mathbf{A}_\triangle$, we denote the algebra resulting from adding $\triangle$ to the MTL-chain $\mathbf{A}$ and say that $\mathbf{A}_\triangle$ is the $\triangle$*-expansion* of $\mathbf{A}$. Furthermore, we will call $\mathbf{A}_\triangle$ a *complete* $\mathsf{MTL}_\triangle$*-chain* if it is the $\triangle$-expansion of a complete MTL-chain $\mathbf{A}$ (i.e. $\bigvee B$ and $\bigwedge B$ are in $\mathbf{A}$ for all $B \subseteq A$).

Given a complete $\mathsf{MTL}_\triangle$-chain $\mathbf{A}_\triangle = \langle A, \wedge, \vee, \&, \to, \triangle, \bot, \top \rangle$ and a (classical) set of worlds W, a *fuzzy subset* X of W is defined as a function $X\colon W \to A$. Intuitively, a world $x \in W$ is a member of X to the degree $X(x) \in A$, for this reason, we also write $x \,\varepsilon\, X$ to denote the value $X(x)$ in A. A *fuzzy set* $\mathcal{X}$ *of fuzzy subsets* of W is a function $\mathcal{X}\colon A^W \to A$ and, for all $X \in A^W$, we also write $X \,\varepsilon\, \mathcal{X}$ for the value $\mathcal{X}(X)$ in A. We use lower case letters $x, y, z, \ldots$ to denote members of W, upper case letters $X, Y, Z, \ldots$ to denote members of A^W and upper case calligraphic letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ to denote members of A^{A^W}.

Note that we can view expressions like $x \,\varepsilon\, X$ and $X \,\varepsilon\, \mathcal{X}$, figuratively speaking, as atomic formulas of some "third-order language" (with just one binary predicate symbol ε) where x, X, and $\mathcal{X}$ act as "free variables", also called "first-order variables", "second-order variables", and "third-order variables", respectively. We then use the connectives in $\mathfrak{L}_\triangle$ to form more complicated "third-order formulas", e.g. $x \,\varepsilon\, X \to X \,\varepsilon\, \mathcal{X}$, and quantifier symbols $(\forall x)$, $(\forall X)$, $(\exists x)$ and $(\exists X)$ to bind "first-order variables" and "second-order variables".

For an $\mathsf{MTL}_\triangle$-chain $\mathbf{A}_\triangle$ and a (classical) set of worlds W, we will interpret the connectives in $\mathfrak{L}_\triangle$ by the corresponding operations of $\mathbf{A}_\triangle$, and the quantifiers $\forall$ and $\exists$ stand for infima and suprema, respectively. In this sense, "third-order formulas" are just short-hand notations for functions mapping into A where "free first-order variables" x, y, and z are argument-positions for worlds in W, "free second-order variables" X, Y, and Z are argument-positions for fuzzy subsets in A^W, and "free third-order variables" $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are argument-positions for fuzzy sets of fuzzy subsets in A^{A^W}. If all "variables" are "bound" by a quantifier symbol, the "third-order formulas" stands for a constant function into A, that is, an element of A. For example, the "third-order formula" $(\forall X)(x \,\varepsilon\, X)$ stands for the function

$$f\colon W \to A \qquad \text{defined by} \qquad f(x) \;=\; \bigwedge_{X \in A^W} X(x)\,,$$

and the "third-order formula" $(\forall X)(\exists Y)(\forall x)(x \mathrel{\varepsilon} X \to x \mathrel{\varepsilon} Y)$ stands for the following constant function into (or element of) A:

$$\bigwedge_{X \in A^W} \bigvee_{Y \in A^W} \bigwedge_{x \in W} (X(x) \to Y(x)) \,.$$

Furthermore, given a "third-order formula" φ where the only "free variable" (if there is one) is a "first-order variable" (recall that in this case, φ stands for a function $\varphi\colon W \to A$), we define a fuzzy set $X = \{x \mid \varphi(x)\} \in A^W$ to which each world $x \in W$ belongs exactly to the same degree as is the value of $\varphi(x)$ in A, i.e. for all $x \in W$:

$$\varphi(x) \;=\; x \mathrel{\varepsilon} \{x \mid \varphi(x)\} \;=\; X(x) \,.$$

Using the same idea, we introduce fuzzy sets of fuzzy subsets of W by "comprehension terms", i.e. for a "third-order formula" where the only "free variable" (if there is one) is a "second-order variable" (recall that then φ stands for a function $\varphi\colon A^W \to A$), we let $\mathcal{X} = \{X \mid \varphi(X)\} \in A^{A^W}$ denote the fuzzy set of fuzzy sets such that for all $X \in A^W$:

$$\varphi(X) \;=\; X \mathrel{\varepsilon} \{X \mid \varphi(X)\} \;=\; \mathcal{X}(X) \,.$$

Finally, for fixed fuzzy subsets $X, Y \in A^W$, we write $X \sqsubseteq Y$ to denote the "third-order formula" $(\forall x)(x \mathrel{\varepsilon} X \to x \mathrel{\varepsilon} Y)$, which stands for $\bigwedge_{x \in W}(X(x) \to Y(x))$, which in this case, as X and Y are fixed, is a constant function into (or element of) A.

7.2 Many-Valued Neighbourhood Semantics

Let us fix an MTL-chain $\mathbf{A}$ for $\mathfrak{L}$, recalling that $\triangle \notin \mathfrak{L}$. We define an $\mathbf{A}$-*neighbourhood frame* (for short: $\mathbb{SM}(\mathbf{A})$-*frame*) to be a pair $\langle W, N\rangle$ such that W is a non-empty (classical) set of worlds while N is a function $N\colon W \to A^{A^W}$ that assigns to each world $x \in W$ a fuzzy set of fuzzy subsets of W, called the $\mathbf{A}$-*neighbourhood* of $x \in W$.

We define an $\mathbf{A}$-*neighbourhood model* (short: $\mathbb{SM}(\mathbf{A})$-model) to be a triple $\mathcal{M} = \langle W, N, V\rangle$, where $\langle W, N\rangle$ is an $\mathbb{SM}(\mathbf{A})$-frame and V is an $\mathbf{A}$-valuation $V\colon \mathrm{Var} \times W \to A$ that is extended to formulas $\varphi \in \mathrm{Fm}$ inductively as follows: for all n-ary connectives $\star \in \mathfrak{L}$ and all $\varphi_1, \ldots, \varphi_n, \varphi \in \mathrm{Fm}$:

$$\begin{aligned} V(\star(\varphi_1, \ldots, \varphi_n), x) &= \star(V(\varphi_1, x), \ldots, V(\varphi_n, x)) \,, \\ V(\Box\varphi, x) &= [\![\varphi]\!]_{\mathcal{M}} \mathrel{\varepsilon} N(x) \,, \end{aligned}$$

where $[\![\varphi]\!]_{\mathcal{M}}$ denotes the fuzzy subset of W to which $y \in W$ belongs to the degree $V(\varphi, y)$, i.e. the fuzzy subset $\{y \mid V(\varphi, y)\} \in A^W$.

Furthermore, if $\mathbf{A}$ is a complete MTL-chain, recall that an $\mathbf{A}$-frame (we will also call it a $\mathbb{K}(\mathbf{A})$-frame) is a pair $\langle W, R\rangle$ such that W is a non-empty (classical) set of worlds while R is a function $R\colon W \times W \to A$ (see Chapter 2). For any $x \in W$ we define the fuzzy subset $R[x] = \{y \mid Rxy\} \in A^W$, i.e. the fuzzy subset of W to which y belongs to the degree Rxy. Also recall that a $\mathbb{K}(\mathbf{A})$-model is a triple $\mathfrak{M} = \langle W, R, V\rangle$, where $\langle W, R\rangle$ is a $\mathbb{K}(\mathbf{A})$-frame and V is an $\mathbf{A}$-valuation $V\colon \mathrm{Var} \times W \to A$ that extends to formulas $\varphi \in \mathrm{Fm}$ inductively as follows:

$$\begin{aligned} V(\star(\varphi_1, \dots, \varphi_n), x) &= \star(V(\varphi_1, x), \dots, V(\varphi_n, x)), \\ V(\Box\varphi, x) &= \bigwedge\{Rxy \to V(\varphi, y) \mid y \in W\}. \end{aligned}$$

Note that when we define the fuzzy subset $[\![\varphi]\!]_{\mathfrak{M}} = \{y \mid V(\varphi, y)\} \in A^W$, the value $\bigwedge\{Rxy \to V(\varphi, y) \mid y \in W\}$ in A can also be expressed by the constant function denoted by the "third-order formula" $R[x] \sqsubseteq [\![\varphi]\!]_{\mathfrak{M}}$. We can thus easily recognize the tight connection to the classical setting.

Given an $\mathbb{SM}(\mathbf{A})$-model $\mathcal{M} = \langle W, N, V\rangle$, a formula $\varphi \in \mathrm{Fm}$ is *valid in* $\mathcal{M}$, if $V(\varphi, x) = \top$ for all $x \in W$, written $\mathcal{M} \models_{\mathbb{SM}(\mathbf{A})} \varphi$. For a subset $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, we say that φ is an $\mathbb{SM}(\mathbf{A})$-*consequence* of Γ, written $\Gamma \models_{\mathbb{SM}(\mathbf{A})} \varphi$, if for all $\mathbb{SM}(\mathbf{A})$-models $\mathcal{M}$ such that $\mathcal{M} \models_{\mathbb{SM}(\mathbf{A})} \Gamma$, also $\mathcal{M} \models_{\mathbb{SM}(\mathbf{A})} \varphi$. The notions of validity in a $\mathbb{K}(\mathbf{A})$-model and $\mathbb{K}(\mathbf{A})$-consequence ($\Gamma \models_{\mathbb{K}(\mathbf{A})} \varphi$) are defined in Chapter 2. The logics defined by the consequence relations $\models_{\mathbb{SM}(\mathbf{A})}$ and $\models_{\mathbb{K}(\mathbf{A})}$ will be denoted by $\mathsf{SM}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})$, respectively.

Given a complete MTL-chain $\mathbf{A}$, an $\mathbb{SM}(\mathbf{A})$-frame $\langle W, N\rangle$ will be called $\mathbf{A}$-*augmented* if the function $f_{aug}\colon W \to A$, represented by the "third-order formula"

$$(\exists X)\triangle(\forall Y)(X \sqsubseteq Y \leftrightarrow Y \mathrel{\varepsilon} N(x)),$$

maps each $x \in W$ to $\top$ in the $\triangle$-expansion $\mathbf{A}_\triangle$ of $\mathbf{A}$.

Remark 7.4. Note that for a given world $x \in W$, $f_{aug}(x) = \top$ means that there is a fuzzy subset $C_x \in A^W$ such that $(C_x \sqsubseteq Y) = (Y \mathrel{\varepsilon} N(x))$ for every $Y \in A^W$. This implies that $(C_x \mathrel{\varepsilon} N(x)) = \top$, because for each fuzzy subset $X \in A^W$ we have $(X \sqsubseteq X) = \top$. Furthermore, if there would be two such fuzzy subsets $C_x, C'_x \in A^W$, we would have $C_x = C'_x$ for the following reason: consider $X = C_x$ and $Y = C'_x$, and vice-versa, implying that $(C_x \sqsubseteq C'_x) = \top = (C'_x \sqsubseteq C_x)$. This fuzzy subset C_x, for a given x, is called the *core of* $N(x)$. Clearly, an $\mathbb{SM}(\mathbf{A})$-frame $\langle W, N\rangle$ is $\mathbf{A}$-augmented if and only if for each world $x \in W$, $N(x)$ has a core.

Example 7.5. Let us consider a simple example. We define an $\mathbb{SM}(\mathbf{A})$-frame $\langle W, N\rangle$ for $\mathbf{A} = \mathbf{G}$, the standard Gödel algebra, by setting $W = \{x\}$ and for each fuzzy subset $X \in [0,1]^W$ of W:

$$(X \mathrel{\varepsilon} N(x)) = (x \mathrel{\varepsilon} X).$$

In fact, there is exactly one fuzzy subset $X \in [0,1]^W$ for every real number r in $[0,1]$, thus, let X_r denote the fuzzy subset such that $(x \mathrel{\varepsilon} X_r) = r$. In this case, $(X_r \mathrel{\varepsilon} N(x)) = r$ for each $r \in [0,1]$.

We note that $\langle W, N\rangle$ is $\mathbf{G}$-augmented with the core C_x of $N(x)$ being X_1. For this, note first that for each $r \in [0,1]$, $(X_1 \sqsubseteq X_r) = (x \,\varepsilon\, X_1 \to x \,\varepsilon\, X_r) = 1 \to r = r$, and thus

$$\begin{aligned}
\triangle(\forall Y)(X_1 \sqsubseteq Y \leftrightarrow Y \,\varepsilon\, N(x)) &= \triangle\,(\bigwedge_{r\in[0,1]} (X_1 \sqsubseteq X_r \leftrightarrow X_r \,\varepsilon\, N(x))) \\
&= \triangle\,(\bigwedge_{r\in[0,1]} (r \leftrightarrow r)) \\
&= \triangle\, 1 \\
&= 1\,.
\end{aligned}$$

In order to show that X_1 is the unique core of $N(x)$, let us suppose for a contradiction that there is an $s \in [0,1)$, such that $(\forall Y)(X_s \sqsubseteq Y \leftrightarrow Y \,\varepsilon\, N(x)) = 1$. In this case, there is a $t \in [0,1]$, such that $s < t < 1$, and

$$(X_s \sqsubseteq X_t) \;=\; (x \,\varepsilon\, X_s \to x \,\varepsilon\, X_t) \;=\; s \to t \;=\; 1 \;>\; t \;=\; (x \,\varepsilon\, X_t) \;=\; (X_t \,\varepsilon\, N(x))\,,$$

contradicting the assumption that $\bigwedge_{r\in[0,1]}(X_s \sqsubseteq X_r \leftrightarrow X_r \,\varepsilon\, N(x)) = 1$.

Finally, let us point out that in fact this examples works the same way for any standard MTL-chain $\mathbf{A}$, not just $\mathbf{G}$.

7.3 Relating Neighbourhood and Kripke Semantics

For this section, let $\mathbf{A}$ be a complete MTL-chain for $\mathfrak{L}$. We show that also in the many-valued setting, there is, analogously to the classical case, a close relationship between many-valued neighbourhood semantics and many-valued Kripke semantics. While the (many-valued) neighbourhood function N allows more flexibility, it becomes equivalent to the more restricted (many-valued) binary relation R when it is required to be ($\mathbf{A}$-)augmented.

Similarly to the classical case, given a $\mathbb{K}(\mathbf{A})$-frame $\langle W, R\rangle$, we define an $\mathbb{SM}(\mathbf{A})$-frame $\langle W, N_R\rangle$ as follows. For all $x \in W$ let

$$N_R(x) \;=\; \{X \mid (\forall y)(Rxy \to y \,\varepsilon\, X)\}\,.$$

and notice that for all $x \in W$ and $X \in A^W$, $(\forall y)(Rxy \to y \in X) = (R[x] \sqsubseteq X)$. On the other hand, given an $\mathbb{SM}(\mathbf{A})$-frame $\langle W, N\rangle$, we define a $\mathbb{K}(\mathbf{A})$-frame $\langle W, R_N\rangle$ as follows:

$$R_N[x] \;=\; \{y \mid (\forall X)(X \,\varepsilon\, N(x) \to y \,\varepsilon\, X)\}\,.$$

Example 7.6. Recall Example 7.5, where we considered the $\mathbb{SM}(\mathbf{G})$-frame $\langle W, N\rangle$ with

$W = \{x\}$ and $(X_r \,\varepsilon\, N(x)) = r$ for all $r \in [0,1]$. In this case,

$$\begin{aligned} R_N[x] &= \{y \mid (\forall X)(X \,\varepsilon\, N(x) \to y \,\varepsilon\, X)\} \\ &= \{y \mid \bigwedge_{r\in[0,1]} (X_r \,\varepsilon\, N(x) \to y \,\varepsilon\, X_r)\} \\ &= \{y \mid \bigwedge_{r\in[0,1]} (r \to r)\} \\ &= \{y \mid 1\} \\ &= X_1\,, \end{aligned}$$

which means that $R_N xx = (x \,\varepsilon\, R_N[x]) = (x \,\varepsilon\, X_1) = 1$.

As in the classical case, the goal is to prove that for any $\mathbb{K}(\mathbf{A})$-frame $\langle W, R\rangle$, we have $R_{N_R} = R$, and if an $\mathbb{SM}(\mathbf{A})$-frame $\langle W, N\rangle$ is $\mathbf{A}$-augmented, then $N_{R_N} = N$. These proofs follow the same ideas as in the classical case (see e.g. [103]), but obviously an adaptation to deal with fuzzy sets of fuzzy subsets of W is needed.

Lemma 7.7. *Let $\langle W, N\rangle$ be an $\mathbf{A}$-augmented $\mathbb{SM}(\mathbf{A})$-frame, $x \in W$, and let C_x be the core of $N(x)$. Then $C_x = R_N[x]$.*

Proof. We prove that $C_x = R_N[x]$ by showing that for all $y \in W$, $(y \,\varepsilon\, R_N[x]) \le (y \,\varepsilon\, C_x)$ and $(y \,\varepsilon\, C_x) \le (y \,\varepsilon\, R_N[x])$. First note that because C_x is the core of $N(x)$, it is the case that $(C_x \,\varepsilon\, N(x)) = \top$ (see Remark 7.4). Fixing a world $y \in W$, it follows that

$$\begin{aligned} (y \,\varepsilon\, R_N[x]) &= (\forall Y)(Y \,\varepsilon\, N(x) \to y \,\varepsilon\, Y) \\ &\le (C_x \,\varepsilon\, N(x) \to y \,\varepsilon\, C_x) \\ &= (y \,\varepsilon\, C_x)\,, \end{aligned}$$

as $(\forall Y)$ stands for $\bigwedge_{Y\in A^W}$ and its instantiation by $C_x \in A^W$ is greater. The last equality is justified by the fact that the equation $\top \to a = a$ is satisfied in any MTL-algebra.

For the other inequality, note first that for all $y \in W$ and all $Y \in A^W$,

$$(C_x \sqsubseteq Y) \;=\; (\forall z)(z \,\varepsilon\, C_x \to z \,\varepsilon\, Y) \;\le\; (y \,\varepsilon\, C_x \to y \,\varepsilon\, Y)\,.$$

By residuation and commutativity of the & operation, it follows that

$$(y \,\varepsilon\, C_x \;\&\; C_x \sqsubseteq Y) \;=\; (C_x \sqsubseteq Y \;\&\; y \,\varepsilon\, C_x) \;\le\; (y \,\varepsilon\, Y)\,,$$

for all $y \in W$ and $Y \in A^W$, and thus, by residuation again,

$$(y \,\varepsilon\, C_x) \;\le\; (C_x \sqsubseteq Y \to y \,\varepsilon\, Y)\,.$$

From this, the fact that C_x is the core of $N(x)$ (and thus $(C_x \sqsubseteq Y) = (Y \,\varepsilon\, N(x))$ for all $Y \in A^W$), and by the definition of R_N, we can complete the proof using the following chain of (in)equalities

$$\begin{aligned}
(y \,\varepsilon\, C_x) &= (\forall Y)(y \,\varepsilon\, C_x) \\
&\leq (\forall Y)(C_x \sqsubseteq Y \to y \,\varepsilon\, Y) \\
&= (\forall Y)(Y \,\varepsilon\, N(x) \to y \,\varepsilon\, Y) \\
&= (y \,\varepsilon\, R_N[x])\,.
\end{aligned}$$
□

Lemma 7.8. *If $\langle W, R\rangle$ is a $\mathbb{K}(\mathbf{A})$-frame, then the $\mathbb{SM}(\mathbf{A})$-frame $\langle W, N_R\rangle$ is $\mathbf{A}$-augmented.*

Proof. We prove that for each world $x \in W$, $R[x]$ is the core of $N_R(x)$ and so $\langle W, N_R\rangle$ is $\mathbf{A}$-augmented. We fix a world $x \in W$ and, recalling that $(R[x] \sqsubseteq Y) = (\forall y)(Rxy \to y \,\varepsilon\, X)$, we note that

$$(Y \,\varepsilon\, N_R(x)) \;=\; (Y \,\varepsilon\, \{X \mid (\forall y)(Rxy \to y \,\varepsilon\, X)\}) \;=\; (R[x] \sqsubseteq Y)\,.$$

Thus, we obtain for $X = R[x] \in A^W$:

$$\triangle(\forall Y)(X \sqsubseteq Y \leftrightarrow Y \,\varepsilon\, N_R(x)) \;=\; \top\,.$$
□

Theorem 7.9. *Let $\langle W, N\rangle$ be an $\mathbb{SM}(\mathbf{A})$-frame. Then $\langle W, N\rangle$ is $\mathbf{A}$-augmented iff $N_{R_N} = N$.*

Proof. For the direction from left to right, let $\langle W, N\rangle$ be an $\mathbf{A}$-augmented $\mathbb{SM}(\mathbf{A})$-frame. Then notice for all $x \in W$:

$$\begin{aligned}
N_{R_N}(x) &= \{Y \mid (\forall y)(R_N xy \to y \,\varepsilon\, Y)\} && (22)\\
&= \{Y \mid R_N[x] \sqsubseteq Y\} && (23)\\
&= \{Y \mid C_x \sqsubseteq Y\} && (24)\\
&= \{Y \mid Y \,\varepsilon\, N(x)\} && (25)\\
&= N(x)\,. && (26)
\end{aligned}$$

While the first two and the last equalities are just notational facts, step (23) to (24) is justified by Lemma 7.7, and we get from (24) to (25) by Remark 7.4. The right to left direction is an easy consequence of Lemma 7.8. □

Theorem 7.10. *If $\langle W, R\rangle$ is a $\mathbb{K}(\mathbf{A})$-frame, then $R_{N_R} = R$.*

Proof. Let $\langle W, R\rangle$ be a $\mathbb{K}(\mathbf{A})$-frame and fix an $x \in W$, then

$$\begin{aligned}
R_{N_R}[x] &= \{y \mid (\forall Y)(Y \,\varepsilon\, N_R(x) \to y \,\varepsilon\, Y)\} \\
&= \{y \mid (\forall Y)(Y \,\varepsilon\, \{Z \mid (\forall y)(Rxy \to y \,\varepsilon\, Z)\} \to y \,\varepsilon\, Y)\} \\
&= \{y \mid (\forall Y)(Y \,\varepsilon\, \{Z \mid R[x] \sqsubseteq Z\} \to y \,\varepsilon\, Y)\} \\
&= \{y \mid (\forall Y)(R[x] \sqsubseteq Y \to y \,\varepsilon\, Y)\}\,.
\end{aligned}$$

It then remains to be shown that for all worlds $y \in W$, $(\forall Y)(R[x] \sqsubseteq Y \to y \,\varepsilon\, Y) = (y \,\varepsilon\, R[x])$. For this, note first that for all $y \in W$ and all $Y \in A^W$,

$$(R[x] \sqsubseteq Y) \;=\; (\forall z)(z \in R[x] \to z \,\varepsilon\, Y) \;\leq\; (y \,\varepsilon\, R[x] \to y \,\varepsilon\, Y)\,.$$

By residuation and commutativity of the & operation, we obtain $(y \,\varepsilon\, R[x]) \leq (R[x] \sqsubseteq Y \to y \,\varepsilon\, Y)$ for all $y \in W$ and $Y \in A^W$, and thus also for all $y \in W$,

$$(y \,\varepsilon\, R[x]) \;\leq\; (\forall Y)(R[x] \sqsubseteq Y \to y \,\varepsilon\, Y)\,.$$

On the other hand, by instantiation,

$$(\forall Y)(R[x] \sqsubseteq Y \to y \,\varepsilon\, Y) \;\leq\; (R[x] \sqsubseteq R[x] \to y \,\varepsilon\, R[x]) \;=\; (y \,\varepsilon\, R[x])\,,$$

and thus $R_{N_R}[x] = \{y \mid (\forall Y)(R[x] \sqsubseteq Y \to y \,\varepsilon\, Y)\} = \{y \mid y \,\varepsilon\, R[x]\} = R[x]$. ☐

Having established a tight connection between **A**-neighbourhood and **A**-Kripke frames, the extension of this connection to the level of models does not come as a surprise.

Theorem 7.11.

(a) *Given a* $\mathbb{K}(\mathbf{A})$*-model* $\mathfrak{M} = \langle W, R, V\rangle$*, define the* $\mathbb{SM}(\mathbf{A})$*-model* $\mathcal{M} = \langle \widehat{W}, \widehat{N}, \widehat{V}\rangle$ *with* $\widehat{W} = W$*,* $\widehat{N} = N_R$*, and* $\widehat{V} = V$*. Then for all* $\varphi \in \mathrm{Fm}$ *and* $x \in W$*:*

$$\widehat{V}(\varphi, x) \;=\; V(\varphi, x)\,.$$

(a) *Given an* **A***-augmented* $\mathbb{SM}(\mathbf{A})$*-model* $\mathcal{M} = \langle W, N, V\rangle$*, define the* $\mathbb{K}(\mathbf{A})$*-model* $\mathfrak{M} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ *with* $\widehat{W} = W$*,* $\widehat{R} = R_N$*, and* $\widehat{V} = V$*. Then for all* $\varphi \in \mathrm{Fm}$ *and* $x \in W$*:*

$$\widehat{V}(\varphi, x) \;=\; V(\varphi, x)\,.$$

Proof. We proceed by induction on the length of the formula $\varphi \in \mathrm{Fm}$. For (a) and (b), the case where $\varphi \in \mathrm{Var}$ or φ is a constant follows by the definition of $\widehat{V}$ while the case where φ is not a box-formula follows trivially from the induction hypothesis (as only box-formulas depend on R or N). Let $\varphi = \Box\psi$ for some $\psi \in \mathrm{Fm}$.

For (a), note that by the induction hypothesis, for any $x \in \widehat{W} = W$,

$$\begin{aligned}
\widehat{V}(\Box\psi, x) &= ([\![\psi]\!]_{\mathcal{M}} \,\varepsilon\, N_R(x)) \\
&= ([\![\psi]\!]_{\mathcal{M}} \,\varepsilon\, \{Y \mid (\forall y)(Rxy \to y \,\varepsilon\, Y)\}) \\
&= (\forall y)(Rxy \to y \,\varepsilon\, [\![\psi]\!]_{\mathcal{M}}) \\
&= \bigwedge_{y \in \widehat{W}} (Rxy \to \widehat{V}(\psi, y)) \\
&= \bigwedge_{y \in W} (Rxy \to V(\psi, y)) \\
&= V(\Box\psi, x)\,.
\end{aligned}$$

For (b), we first note that for any $x \in \widehat{W} = W$, $R_N[x]$ is the core of $N(x)$ by Lemma 7.7 (and thus $(R_N[x] \sqsubseteq Y) = (Y \mathrel{\varepsilon} N(x))$ for all $Y \in A^W$), as $\mathcal{M}$ is $\mathbf{A}$-augmented. We can then use the induction hypothesis to conclude the proof by the following equalities:

$$\begin{aligned} V(\Box\psi, x) &= (\llbracket\psi\rrbracket_{\mathcal{M}} \mathrel{\varepsilon} N(x)) \\ &= (R_N[x] \sqsubseteq \llbracket\psi\rrbracket_{\mathcal{M}}) \\ &= (\forall y)(y \mathrel{\varepsilon} R_N[x] \to y \mathrel{\varepsilon} \llbracket\psi\rrbracket_{\mathcal{M}}) \\ &= \bigwedge_{y \in W} (R_N xy \to V(\psi, y)) \\ &= \bigwedge_{y \in \widehat{W}} (R_N xy \to \widehat{V}(\psi, y)) \\ &= \widehat{V}(\Box\psi, x)\,. \end{aligned}$$

$\square$

Corollary 7.12. *For all sets of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*,*

$\Gamma \models_{\mathbb{K}(\mathbf{A})} \varphi$ *iff* $\mathcal{M} \models_{\mathbb{SM}(\mathbf{A})} \varphi$ *for all* $\mathbf{A}$*-augmented* $\mathbb{SM}(\mathbf{A})$*-models* $\mathcal{M}$ *such that* $\mathcal{M} \models_{\mathbb{SM}(\mathbf{A})} \Gamma$.

Proof. For the contraposition of the right-to-left direction, let us assume $\Gamma \not\models_{\mathbb{K}(\mathbf{A})} \varphi$, that is, there is a $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$ such that $V[\Gamma, W] = \{V(\psi, x) \mid \psi \in \Gamma, x \in W\} \subseteq \{\top\}$ and $V(\varphi, y) < \top$, for some world $y \in W$. Define an $\mathbb{SM}(\mathbf{A})$-model $\mathcal{M} = \langle \widehat{W}, \widehat{N}, \widehat{V}\rangle$ by $\widehat{W} = W$, $\widehat{N} = N_R$, and $\widehat{V} = V$ and notice that by Lemma 7.8, $\mathcal{M}$ is $\mathbf{A}$-augmented, and thus for all $x \in W$ and all $\psi \in \mathrm{Fm}$, $\widehat{V}(\psi, x) = V(\psi, x)$, by Theorem 7.11(a). It therefore follows that $\widehat{V}[\Gamma, \widehat{W}] = V[\Gamma, W] \subseteq \{\top\}$ and $\widehat{V}(\varphi, y) = V(\varphi, y) < \top$ and thus the right-hand side of the claim is false.

For the contraposition of the left-to-right direction, let us assume that there is an $\mathbf{A}$-augmented $\mathbb{SM}(\mathbf{A})$-model $\mathcal{M} = \langle W, N, V\rangle$, such that $V[\Gamma, W] \subseteq \{\top\}$ and $V(\varphi, y) < \top$, for some world $y \in W$. Define the $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ by $\widehat{W} = W, \widehat{R} = R_N$, and $\widehat{V} = V$ and notice that for all $x \in W$ and all $\psi \in \mathrm{Fm}$, $\widehat{V}(\psi, x) = V(\psi, x)$, by Theorem 7.11(b). It therefore follows that $\widehat{V}[\Gamma, \widehat{W}] = V[\Gamma, W] \subseteq \{\top\}$ and $\widehat{V}(\varphi, y) = V(\varphi, y) < \top$ and thus $\Gamma \not\models_{\mathbb{K}(\mathbf{A})} \varphi$. $\square$

7.4 An Axiomatization of $\mathsf{SM}(\mathbf{A})$

For the current section, let us denote by L an axiomatic extension of MTL, that is, L is either MTL, BL, $\mathsf{Ł}$, P, G, or some other logic axiomatized by adding axioms to the Hilbert-style calculus $\mathcal{MTL}$.

For a Hilbert-style calculus $\mathcal{L}$ for the language $\mathfrak{L}$ that axiomatizes L, we define the Hilbert-style calculus $\mathcal{LSM} = \mathcal{L} \cup \{(\mathrm{E})\}$ for the language $\mathfrak{L}_\Box$, recalling that (E) is the following rule:

$$\text{(E)} \qquad \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}.$$

Let us define the modal logic LSM in the language $\mathfrak{L}_\Box$ by setting $\Vdash_{\mathsf{LSM}} = \vdash_{\mathcal{LSM}}$.

We recall that we call an axiomatic extension L of MTL, axiomatized by $\mathcal{L}$, (finitely) strongly complete with respect to an L-chain $\mathbf{C}$ if for every (finite) set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathfrak{L}}$: $\Gamma \vdash_{\mathcal{L}} \varphi$ iff $\Gamma \models_{\mathbf{C}} \varphi$.

Our goal in this section is to prove the following statement: if L is (finitely) strongly complete with respect to an L-chain $\mathbf{C}$ then LSM is (finitely) strongly complete with respect to the class of all $\mathbb{SM}(\mathbf{C})$-models. In order to reach this goal, we need to recall some facts and definitions from (abstract) algebraic logic.

Firstly, we have seen in Chapter 2 that MTL (and hence any axiomatic extensions L of MTL) is *algebraizable* in the sense of Blok and Pigozzi's [17] (see also [42]), which means, in loose terms, that it has an algebraic semantics in (a subvariety) of $\mathbb{MTL}$. Furthermore, from the fact that L is algebraizable it follows that LSM is algebraizable, as (E) clearly ensures the preservation of the *congruence law* (see [42]), that is, the presence of (E) lets us infer from the fact that for all non-modal formulas $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathfrak{L}}$, $\{\varphi \leftrightarrow \psi\} \vdash_{\mathcal{L}} \chi(\varphi) \leftrightarrow \chi(\psi)$, we also have that for all modal formulas $\varphi', \psi', \chi' \in \mathrm{Fm}_{\mathfrak{L}_\Box}$, $\{\varphi' \leftrightarrow \psi'\} \vdash_{\mathcal{LSM}} \chi'(\varphi') \leftrightarrow \chi'(\psi')$, where $\chi(\varphi)$ stands for χ where a specific propositional variable has been uniformly substituted by φ. It therefore makes sense to speak of LSM-*algebras* in the language $\mathfrak{L}_\Box$, i.e. algebras $\mathbf{A}_\Box = \langle A, \wedge, \vee, \&, \to, \Box, \bot, \top\rangle$ such that for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathfrak{L}_\Box}$: $\Gamma \models_{\mathbf{A}_\Box} \varphi$ whenever $\Gamma \vdash_{\mathcal{LSM}} \varphi$. We then obviously have that LSM is strongly complete with respect to the class of all LSM-algebras and for each LSM-algebra $\langle A, \wedge, \vee, \&, \to, \Box, \bot, \top\rangle$, the $\mathfrak{L}$-reduct $\langle A, \wedge, \vee, \&, \to, \bot, \top\rangle$ is an L-alegbra.

Furthermore, as L is an axiomatic extension of MTL and thus satisfies prelinearity, we have that every L-algebra $\mathbf{A}$ is *representable as a subdirect product of* L*-chains* (see [54] and e.g. [37]), that is, there is a family of L-chains $\{\mathbf{A}_i\}_{i\in I}$ and an embedding $\alpha\colon \mathbf{A} \to \prod_{i\in I} \mathbf{A}_i$ such that $(\pi_i \circ \alpha)[A] = A_i$, where $\prod_{i\in I} \mathbf{A}_i$ denotes the direct product and π_i the i-th projection.

Moreover, for two MTL-algebras $\mathbf{A}$ and $\mathbf{B}$, we will say that $\mathbf{A}$ *partially embeds into* $\mathbf{B}$, if for each finite subset $F \subseteq A$, there is a one-to-one mapping $h_F\colon F \to B$, called a *partial* F*-embedding*, such that for all n-ary connectives $\star \in \mathfrak{L}$ and all $a_1, \ldots, a_n \in F$:

$$h_F(\star^{\mathbf{A}}(a_1, \ldots, a_n)) \;=\; \star^{\mathbf{B}}(h_F(a_1), \ldots, h_F(a_n))\,.$$

Finally, we recall two facts about L-chains that we will be crucial in the completeness prove below (see [37, Theorems 3.5 and 3.8]).

Theorem 7.13 ((Partial) Embeddability of Chains [37]). *For any axiomatic extension* L *of* MTL *and any* L*-chain* $\mathbf{C}$:

(a) L *is strongly complete with respect to* $\mathbf{C}$ *if and only if every countable* L*-chain embeds into* $\mathbf{C}$.

(b) L *is finitely strongly complete with respect to* $\mathbf{C}$ *if and only if every* L*-chain partially embeds into* $\mathbf{C}$.

We now have everything we need to prove the following completeness theorem.

Theorem 7.14. *Let* L *be an axiomatic extension of* MTL*, let* $\mathcal{L}$ *be a Hilbert-style calculus axiomatizing* L*, and let* $\mathbf{C}$ *be an* L*-chain. If* L *is (finitely) strongly complete with respect to* $\mathbf{C}$*, then for each (finite)* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathfrak{L}_\Box}$ *we have:*

$$\Gamma \vdash_{\mathcal{LSM}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbb{SM}(\mathbf{C})} \varphi\,.$$

Proof. For the left-to-right directions, we only need to check the soundness of the rule (E). Let us assume that for some $\mathbb{SM}(\mathbf{C})$-model $\mathcal{M} = \langle W, N, V\rangle$ and some formulas $\psi, \chi \in \mathrm{Fm}_{\mathfrak{L}_\Box}$, $\mathfrak{M} \models_{\mathsf{SM}(\mathbf{C})} \psi \leftrightarrow \chi$, then

$$\begin{aligned}
\mathcal{M} \models_{\mathbb{SM}(\mathbf{C})} \psi \leftrightarrow \chi \qquad &\Rightarrow \qquad V(\psi, x) = V(\chi, x), \text{ for all } x \in W,\\
&\Rightarrow \qquad [\![\psi]\!]_{\mathcal{M}} = [\![\chi]\!]_{\mathcal{M}}\\
&\Rightarrow \qquad ([\![\psi]\!]_{\mathcal{M}} \,\varepsilon\, N(x)) = ([\![\chi]\!]_{\mathcal{M}} \,\varepsilon\, N(x)), \text{ for all } x \in W,\\
&\Rightarrow \qquad V(\Box\psi, x) = V(\Box\chi, x), \text{ for all } x \in W,\\
&\Rightarrow \qquad \mathcal{M} \models_{\mathbb{SM}(\mathbf{C})} \Box\psi \leftrightarrow \Box\chi.
\end{aligned}$$

For the reverse implication in the finite strong completeness case, assume that $\Gamma \nvdash_{\mathcal{LSM}} \varphi$ for a finite set $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathfrak{L}_\Box}$. By the algebraizability of LSM, there is an LSM-algebra $\mathbf{A}_\Box = \langle A, \wedge, \vee, \&, \to, \Box, \bot, \top\rangle$ and an $\mathbf{A}_\Box$-evaluation $e\colon \mathrm{Fm}_{\mathfrak{L}_\Box} \to A$ such that $e[\Gamma] \subseteq \{\top^{\mathbf{A}_\Box}\}$ and $e(\varphi) \neq \top^{\mathbf{A}_\Box}$. In this case, the $\mathfrak{L}$-reduct of $\mathbf{A}_\Box$, denoted by $\mathbf{A}$, is an L-algebra. $\mathbf{A}$ is therefore representable as a subdirect product of L-chains $\{\mathbf{A}_i\}_{i\in I}$, denoting the embedding involved by α. Moreover, because L is finitely strongly complete with respect to $\mathbf{C}$, we have that each $\mathbf{A}_i$ partially embeds into $\mathbf{C}$ by Theorem 7.13(b).

Let Σ be the finite set of the subformulas of $\Gamma \cup \{\varphi\}$. In this case, also the subsets $e[\Sigma] \subseteq A$, $(\alpha \circ e)[\Sigma] \subseteq \prod_{i\in I} A_i$, and $(\pi_i \circ \alpha \circ e)[\Sigma] =: B_i \subseteq A_i$ are finite for all $i \in I$. By partial embeddability of $\mathbf{A}_i$ into $\mathbf{C}$, it follows that for each $i \in I$ there is a partial B_i-embedding $h_{B_i}\colon B_i \to C$. For notational convenience, let us finally define, for each $i \in I$, a map $f_i\colon \Sigma \to C$ such that $f_i(\psi) = (h_{B_i} \circ \pi_i \circ \alpha \circ e)(\psi)$ for all $\psi \in \Sigma$.

Now we have all the ingredients to build the $\mathbb{SM}(\mathbf{C})$-counter-model $\mathcal{M} = \langle W, N, V\rangle$. Let $W = I$ and for all $p \in \mathrm{Var}$ and $j \in W$:

$$\begin{aligned}
V(p, j) &= \begin{cases} f_j(p) & \text{if } p \in \Sigma, \\ \perp^{\mathbf{C}} & \text{otherwise,} \end{cases} \\
(\langle a_i\rangle_{i\in W} \,\varepsilon\, N(j)) &= \begin{cases} f_j(\Box\psi) & \text{if there is a } \Box\psi \in \Sigma \text{ s.t. } \langle a_i\rangle_{i\in W} = \langle f_i(\psi)\rangle_{i\in W}, \\ \perp^{\mathbf{C}} & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that if for two formulas $\psi, \psi' \in \Sigma$ it holds that $\Box\psi, \Box\psi' \in \Sigma$ and $\langle f_i(\psi)\rangle_{i\in W} = \langle f_i(\psi')\rangle_{i\in W}$, it follows by the definition of f_i that $e(\psi) = e(\psi')$. By the validity of the rule (E) in $\mathbf{A}_\Box$, this implies that $e(\Box\psi) = e(\Box\psi')$ and thus also $f_j(\Box\psi) = f_j(\Box\psi')$ for each $j \in I$. Therefore, the neighbourhood function N above is well-defined.

We prove that $V(\psi, j) = f_j(\psi)$ for all $\psi \in \Sigma$ and $j \in W$ by an induction on the length of ψ. The base case follows immediately by definition.

For the induction step, let for the first case $\psi = \psi_1 \star \psi_2$ for some binary $\star \in \mathfrak{L}$. In this case, we have the following series of equalities:

$$\begin{aligned}
f_j(\psi_1 \star \psi_2) &= h_{B_j}(\pi_j(\alpha(e(\psi_1 \star \psi_2)))) && (27)\\
&= h_{B_j}(\pi_j(\alpha(e(\psi_1) \star^{\mathbf{A}_\Box} e(\psi_2)))) && (28)\\
&= h_{B_j}(\pi_j(\alpha(e(\psi_1)) \star^{\prod_{i\in I}\mathbf{A}_i} \alpha(e(\psi_2)))) && (29)\\
&= h_{B_j}(\pi_j(\alpha(e(\psi_1))) \star^{\mathbf{A}_j} \pi_j(\alpha(e(\psi_2)))) && (30)\\
&= h_{B_j}(\pi_j(\alpha(e(\psi_1)))) \star^{\mathbf{C}} h_{B_j}(\pi_j(\alpha(e(\psi_2)))) && (31)\\
&= f_j(\psi_1) \star^{\mathbf{C}} f_j(\psi_2) && (32)\\
&= V(\psi_1, j) \star^{\mathbf{C}} V(\psi_2, j) && (33)\\
&= V(\psi_1 \star \psi_2, j)\,. && (34)
\end{aligned}$$

The steps from (27) to (31) are justified, respectively, by the facts that e is an $\mathbf{A}_\Box$-evaluation, α is an embedding, π_j is a projection and thus a homomorphism, and h_{B_j} is a partial B_j-embedding and clearly $\pi_j(\alpha(e(\psi_1)))$ and $\pi_j(\alpha(e(\psi_2)))$ are in $(\pi_i\circ\alpha\circ e)[\Sigma] = B_j$. Furthermore, the step from (32) to (33) follows by the induction hypothesis.

For the second case, let $\psi = \Box\chi$. By the induction hypothesis and the definition of $N(j)$, we justify the following chain of equalities and thus the induction is finished:

$$\begin{aligned}
V(\Box\chi, j) &= ([\![\chi]\!]_{\mathcal{M}} \,\varepsilon\, N(j)) \\
&= (\langle V(\chi, j)\rangle_{i\in W} \,\varepsilon\, N(j)) \\
&= (\langle f_i(\chi)\rangle_{i\in W} \,\varepsilon\, N(j)) \\
&= f_j(\Box\chi)\,.
\end{aligned}$$

Therefore, we have that $\mathcal{M} \models_{\mathbb{SM}(\mathbf{C})} \Gamma$ because for each $\psi \in \Gamma$, $e(\psi) = \top^{\mathbf{A}_\Box}$, and so for each $j \in W$:

$$V(\psi, j) \;=\; f_j(\psi) \;=\; (h_{B_j} \circ \pi_j \circ \alpha \circ e)(\psi) \;=\; \top^{\mathbf{C}}.$$

On the other hand, it is also true that $\mathcal{M} \not\models_{\mathbb{SM}(\mathbf{C})} \varphi$. This is because $e(\varphi) \neq \top^{\mathbf{A}_\Box}$ and therefore there has to be a world $j \in W$ such that $\pi_j(\alpha(e(\varphi))) \neq \top^{\mathbf{A}_j}$ and so

$$V(\varphi, j) \;=\; f_j(\varphi) \;=\; (h_{B_j} \circ \pi_j \circ \alpha \circ e)(\varphi) \;\neq\; \top^{\mathbf{C}}.$$

The proof of the right-to-left direction for the strong completeness case is very similar. Note that the set Σ of all subformulas in $\Gamma \cup \{\varphi\}$ is countable and thus, for each $i \in I$, by restricting the L-chain $\mathbf{A}_i$ to the universe $(\pi_i \circ \alpha \circ e)[\Sigma] =: B_i \subseteq A_i$, we obtain a countable subalgebra $\mathbf{B}_i$ of $\mathbf{A}_i$ which is itself an L-chain. It then follows by Theorem 7.13(a) that for each $i \in I$ the countable L-chain $\mathbf{B}_i$ embeds into the L-chain $\mathbf{C}$. If we denote the resulting embedding by h_{B_i}, then the argument above establishes also the right-to-left direction for the strong completeness case, only that we now justify the step from (30) to (31) by the fact that h_{B_j} is an embedding from $\mathbf{B}_j$ into $\mathbf{C}$. $\Box$

Chapter 8

Concluding Remarks

In this chapter, we present a short summary of the results achieved in this thesis and recall which questions listed in the introduction (Chapter 1) we were able to (partially) answer. We will also mention some other problems that are left open and give some suggestions on how they might be tackled. Furthermore, some connections to other logics and other fields are drawn and ideas are given on how our results might be applied and extended in those settings.

8.1 Summary of the Thesis

In this work, we have mainly considered many-valued logics based on order-based algebras $\mathbf{A}$, i.e. subalgebras of $\mathbf{G} = \langle [0,1], \min, \max, \to_{\mathsf{G}}, 0, 1\rangle$[1] with additional operations defined based only on the order, and their expansions with the modal connectives $\Box$ and $\Diamond$, interpreted over the class of all $\mathbb{K}(\mathbf{A})$-models (yielding the logics $\mathsf{K}(\mathbf{A})$), the class of all crisp $\mathbb{K}(\mathbf{A})$-models (yielding the logics $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$), and the class of all crisp $\mathbb{K}(\mathbf{A})$-models where the accessibility relation is an equivalence relation (yielding the logics $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$).

The investigation of these logics was motivated by the questions (2), (3), (4), and (7) formulated in Chapter 1. Moreover, the methodology we have used was inspired by the fact that decidability was already established for Gödel modal logics based on *witnessed* $\mathbb{K}(\mathbf{G})$-models, i.e. $\mathbb{K}(\mathbf{G})$-models $\langle W, R, V\rangle$ where for each world $x \in W$, box-formula $\Box\varphi$, and diamond-formula $\Diamond\psi$, there are (witnessing) worlds $y, z \in W$ such that $V(\Box\varphi, x) = Rxy \to V(\varphi, y)$ and $V(\Diamond\varphi, x) = Rxz \wedge V(\varphi, z)$ (see e.g. [18]). Despite the fact that GK is not complete with respect to witnessed $\mathbb{K}(\mathbf{G})$-models (e.g. the formula $\Box\neg\neg p \to \neg\neg\Box p$ is valid in all witnessed $\mathbb{K}(\mathbf{G})$-models, but not in GK (see Theorem 3.9)), there is something to learn from the witnessed case. This lead to the idea of restricting the

[1]Note that order-based algebras were defined in Chapter 3 as sublattices of $\langle [0,1], \min, \max, 0, 1\rangle$ with additional order-based operations. In fact, however, we have subsequently only considered order-based algebras containing the Gödel implication (as we have used it to define the interpretation of $\Box$ and $\Diamond$). This is why we just mention expanded subalgebras of $\mathbf{G}$ here.

values of box- and diamond-formulas such that a world y "witnesses" the value $V(\Box\varphi, x)$ if the value $Rxy \to V(\varphi, y)$ is merely sufficiently close to the value $V(\Box\varphi, x)$. This idea was realized by the introduction of $\mathbb{FK}(\mathbf{G})$-models and the proof that these models determine the same set of valid formulas as $\mathbb{K}(\mathbf{G})$-models. There are two main ingredients essential for this completeness that can be isolated (cf. the proof of Lemma 4.10):

(1) In a $\mathbb{K}(\mathbf{G})$-model $\mathfrak{M}$, we can "move" values of propositional variable around without changing the set of valid formulas in $\mathfrak{M}$, as long as the relative order is not changed and certain infinite meets and joins are preserved (see Lemma 3.10).

(2) In $\mathbf{G}$, each interval can be "squeezed" into a smaller interval by an order-embedding (see Lemma 4.9). This property is captured by the definition of local homogeneity.

The isolation of these two properties made it possible to generalize our results to the much larger class of order-based modal logics, as long as the underlying order-based algebra is locally homogeneous. These logics of order represent natural antagonists to logics of magnitude, of which Łukasiewicz logic and product logic are two main examples, and thus are of great interest when it comes to applications where the relative order of values is the essential factor.

In Chapters 4 and 5, we have shown that for validity in some order-based modal logic, we can restrict to $\mathbb{FK}(\mathbf{A})$-models, where the values assigned to box- and diamond-formulas are restricted to certain subsets of A and can therefore be "witnessed" more easily. More precisely, we have proved that if $\mathbf{A}$ is locally homogeneous, then the logics $\mathsf{K}(\mathbf{A})$, $\mathsf{K^c}(\mathbf{A})$, and $\mathsf{S5^c}(\mathbf{A})$ enjoy the finite model property with respect to $\mathbb{FK}(\mathbf{A})$-, $\mathbb{FK}^\mathsf{c}(\mathbf{A})$-, and $\mathbb{FS}\mathbf{5}^\mathsf{c}(\mathbf{A})$-models, respectively, using our result that the classes $\mathbb{FK}(\mathbf{A})$, $\mathbb{FK}^\mathsf{c}(\mathbf{A})$, and $\mathbb{FS}\mathbf{5}^\mathsf{c}(\mathbf{A})$ validate the same formulas as the classes $\mathbb{K}(\mathbf{A})$, $\mathbb{K}^\mathsf{c}(\mathbf{A})$, and $\mathbb{S}\mathbf{5}^\mathsf{c}(\mathbf{A})$, respectively (see Theorems 4.11 and 5.6). Taking advantage of the fact that the size of the finite models involved in testing a formula φ for validity can be bounded by a function on the length of φ, we have provided algorithms that decide validity in certain cases. In particular, for the case where the order-based locally homogeneous algebra $\mathbf{A}$ is either $\mathbf{G}$, $\mathbf{G}_\downarrow$, or $\mathbf{G}_\uparrow$ (or some other algebra for which a specific consistency problem is decidable (see Remark 4.17)), we have proved that the validity problem for $\mathsf{K}(\mathbf{A})$ and $\mathsf{K^c}(\mathbf{A})$ is PSPACE-complete (Theorem 4.16) and for $\mathsf{S5^c}(\mathbf{A})$ coNP-complete (Corollary 5.9). Using the fact that the Gödel modal logic $\mathsf{GS5^c}$ corresponds to the one-variable fragment of first-order Gödel logic (Theorem 5.12), we have been able to present the following major consequences of our results.

- The validity problems for the Gödel modal logics GK and $\mathsf{GK^c}$ are PSPACE-complete (cf. Theorem 4.16).
- The validity problems for the Gödel modal logic $\mathsf{GS5^c}$ and the one-variable fragment of first-order Gödel logic are coNP-complete (see Corollary 5.9 and Theorem 5.13).

With this, we have answered positively questions (2) and (4) posed in the introduction and have given positive answers to question (7) for the case of Gödel logic. In particular, we note that the second result above solves a long-standing open problem that was first explicitly formulated by Hájek in [67, Problem (13)].

Furthermore, using the finite model property with respect to the alternative semantics, we have presented tableau calculi in Chapter 6 for the cases GK, $\mathsf{GK^c}$, and $\mathsf{GS5^c}$, thus answering positively question (3) of the introduction. Even though these calculi do not deliver optimal complexity, they provide useful decision procedures that are easy to handle and are suitable for implementation.

In Chapter 7, we have considered the box-fragment of many-valued modal logics over MTL-chains based on neighbourhood semantics. Firstly, for any complete MTL-chain $\mathbf{A}$, we have presented a correspondence between $\mathbb{K}(\mathbf{A})$-frames (Kripke frames) and $\mathbf{A}$-augmented $\mathbb{SM}(\mathbf{A})$-frames (neighbourhood frames) (Theorems 7.9 and 7.10). Furthermore, given a Hilbert-style axiomatization $\mathcal{L}$ of an axiomatic extension L of MTL, we have proved by algebraic methods that for the rule

$$\text{(E)} \qquad \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi},$$

$\mathcal{LSM} = \mathcal{L} \cup \{(\mathrm{E})\}$ (finitely) strongly completely axiomatizes the box-fragment of the many-valued modal logic determined by the class of all $\mathbb{SM}(\mathbf{C})$-models, if $\mathbf{C}$ is an L-chain such that L is (finitely) strongly complete with respect to $\mathbf{C}$ (Theorem 7.14).

With our results and methods in Chapter 7, we have answered questions (9) and (10) posed in Chapter 1, which are based on three questions formulated by Rodríguez and Godo in [110], one of the first studies of neighbourhood semantics in the setting of MTL.

Neighbourhood semantics provide a more general and thus more flexible framework for modal logics, a framework in which a plethora of different notions can be modelled as modal connectives, including the notions of knowledge, obligation, belief, evidence, and high probability (see e.g. [103]). In fact, even negation and generalized quantifiers have been modelled as modal connectives in this setting (see e.g. [105]). With our study of these semantics in the setting of MTL and its axiomatic extensions, we have thus taken a further step towards a more general theory of many-valued modal logics, which is a very promising field, especially when we look at how far modal logics have been developed in the classical setting.

8.2 Open Problems and Further Work

To add to the unanswered questions listed in the introduction, that is, questions (1), (5), (6), (7) (partially answered), (8) and (11), we now sketch some of the problems that are left open by our work and speculate how they might be tackled, possibly using approaches developed in previous chapters.

Consequence Relations

In Chapters 4 and 5, we proved decidability and complexity results for the validity problems for $\mathsf{K}(\mathbf{A})$, $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$, and $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ for certain order-based algebras $\mathbf{A}$. We did not, however, consider the problems of consequence (or entailment). This is because our results do not always easily extend to these problems. Nevertheless, we will discuss the consequence relations of the considered logics roughly below.

There are two different natural consequence relations defined over $\mathbb{K}(\mathbf{A})$-models that are often considered in the literature. Given an order-based algebra $\mathbf{A}$, a $\mathbb{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V \rangle$, a set of formulas $\Gamma \subseteq \mathrm{Fm}$, and a world $x \in W$, let $\mathfrak{M}, x \models_{\mathbb{K}(\mathbf{A})} \Gamma$ denote the fact that $V[\Gamma, x] \subseteq \{1\}$, and let us write $\mathfrak{M} \models_{\mathbb{K}(\mathbf{A})} \Gamma$ to denote the fact that for all $x \in W$, $\mathfrak{M}, x \models_{\mathbb{K}(\mathbf{A})} \Gamma$. For a set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, we then define global and local consequence as follows:

- φ is a *global* $\mathbb{K}(\mathbf{A})$*-consequence* of Γ, written $\Gamma \models^{g}_{\mathbb{K}(\mathbf{A})} \varphi$, if $\mathfrak{M} \models_{\mathbb{K}(\mathbf{A})} \{\varphi\}$ for all $\mathbb{K}(\mathbf{A})$-models $\mathfrak{M}$ such that $\mathfrak{M} \models_{\mathbb{K}(\mathbf{A})} \Gamma$.
- φ is a *local* $\mathbb{K}(\mathbf{A})$*-consequence* of Γ, written $\Gamma \models^{l}_{\mathbb{K}(\mathbf{A})} \varphi$, if $\mathfrak{M}, x \models_{\mathbb{K}(\mathbf{A})} \{\varphi\}$ for all $\mathbb{K}(\mathbf{A})$-models $\mathfrak{M} = \langle W, R, V \rangle$ and all $x \in W$ such that $\mathfrak{M}, x \models_{\mathbb{K}(\mathbf{A})} \Gamma$.

These two consequence relation are identical when we restrict to empty sets of premises (i.e. if $\Gamma = \emptyset$), that is, they produce the same set of valid formulas. However, in general, the local consequence relation is strictly stronger than the global one, i.e. $\models^{l}_{\mathbb{K}(\mathbf{A})} \subsetneq \models^{g}_{\mathbb{K}(\mathbf{A})}$ (see e.g. [21]). Note also that the logics we have called $\mathsf{K}(\mathbf{A})$, $\mathsf{K}^{\mathsf{c}}(\mathbf{A})$, and $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ are all determined by a global consequence relation.

In the case where $\mathbf{A}$ is $\mathbf{G}$, we have the deduction theorem for $\models^{l}_{\mathbb{K}(\mathbf{A})}$ (see [29]), i.e. for any set of formulas $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$:

$$\Gamma \cup \{\varphi\} \models^{l}_{\mathbb{K}(\mathbf{G})} \psi \qquad \text{iff} \qquad \Gamma \models^{l}_{\mathbb{K}(\mathbf{G})} \varphi \to \psi\,.$$

From our results about validity in GK, we can thus directly infer decidability and PSPACE-completeness for the problem of finite local consequence for GK, i.e. where the set of premises is finite. In fact, this result can even be extended to countable local consequence. This is due to the strong completeness of the axiomatization presented in [29] with respect to countable sets of premises.

While these results certainly extend to axiomatic extensions of GK, such as, GT, $\mathsf{GS4}$, and $\mathsf{GS5}$ (see [29]), we also expect them to extend to the crisp counterparts of these logics and to the case where underlying order-based algebra is not $\mathbf{G}$ (at least with respect to finite sets of premises). For these cases, however, no axiomatizations have been found and thus it is not immediately clear how the deduction theorem in this strong form can be established. Nevertheless, semantic arguments might establish the deduction theorem in these cases (cf. [8]).

For the global consequence relation, the deduction theorem in the strong form above generally fails, even in the case of classical modal logics (i.e. when $\mathbf{A}$ is the two-element Boolean algebra; see e.g. [15]). Extending our results to finite global consequence might therefore be more challenging. To achieve it, we might consider again the alternative semantics defined in Chapters 4 and 5 and check whether for finite set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,

$$\Gamma \models^{g}_{\mathbb{FK}(\mathbf{A})} \varphi \qquad \text{iff} \qquad \Gamma \models^{g}_{\mathbb{K}(\mathbf{A})} \varphi\,.$$

While the left-to-right direction follows easily from the fact every $\mathbb{K}(\mathbf{A})$-model $\langle W, R, V\rangle$ can be understood as an $\mathbb{FK}(\mathbf{A})$-model $\langle W, R, V, T_{\Box}, T_{\Diamond}\rangle$ by setting $T_{\Box}(x) = T_{\Diamond}(x) = A$ for all $x \in W$, the right-to-left direction is slightly less obvious. We are confident, however, that by a careful analysis of the proof of Lemma 4.10 this direction could be obtained as well. If this is possible, we would then need to check whether our techniques for proving decidability and PSPACE-completeness extend to global consequence as well. We expect this to be straightforward for finite sets of premises. Nevertheless, working out the details for this and other cases (including GK^{c} and $\mathsf{GS5}^{\mathsf{c}}$) has to be left for future work.

Order-Based Modal Logics Over Other Classes of Frames

In this thesis, we focused on order-based modal logics over the class of all $\mathbb{K}(\mathbf{A})$-models, the class of all crisp $\mathbb{K}(\mathbf{A})$-models, and the class of all crisp $\mathbb{K}(\mathbf{A})$-models where accessibility is an equivalence relation. However, it would also be interesting to investigate other classes of $\mathbb{K}(\mathbf{A})$-models, as was done extensively for classical modal logics. Interesting classes of $\mathbb{K}(\mathbf{A})$-models include the following (cf. Section 2.3):

- $\mathbb{T}(\mathbf{A})$ ($\mathbb{T}^{\mathsf{c}}(\mathbf{A})$) is the class of all (crisp) $\mathbb{K}(\mathbf{A})$-models where the accessibility relation is reflexive (i.e. $Rxx = 1$),
- $\mathbb{S}\mathbf{4}(\mathbf{A})$ ($\mathbb{S}\mathbf{4}^{\mathsf{c}}(\mathbf{A})$) is the class of all (crisp) $\mathbb{K}(\mathbf{A})$-models where the accessibility relation is reflexive and transitive (i.e. $Rxy \wedge Ryz \leq Rxz$), and
- $\mathbb{S}\mathbf{5}(\mathbf{A})$ ($\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$) is the class of all (crisp) $\mathbb{K}(\mathbf{A})$-models where the accessibility relation is reflexive, transitive, and symmetric (i.e. $Rxy = Ryx$).

Let $\mathsf{T}(\mathbf{A}) = \models_{\mathbb{T}(\mathbf{A})}$, $\mathsf{T}^{\mathsf{c}}(\mathbf{A}) = \models_{\mathbb{T}^{\mathsf{c}}(\mathbf{A})}$, $\mathsf{S4}(\mathbf{A}) = \models_{\mathbb{S}\mathbf{4}(\mathbf{A})}$, $\mathsf{S4}^{\mathsf{c}}(\mathbf{A}) = \models_{\mathbb{S}\mathbf{4}^{\mathsf{c}}(\mathbf{A})}$, $\mathsf{S5}(\mathbf{A}) = \models_{\mathbb{S}\mathbf{5}(\mathbf{A})}$, and $\mathsf{S5}^{\mathsf{c}}(\mathbf{A}) = \models_{\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})}$ (the global consequence relations). It is straightforward to adopt our techniques from Chapter 4 to establish PSPACE-completeness of the validity problem for $\mathsf{T}(\mathbf{A})$ and $\mathsf{T}^{\mathsf{c}}(\mathbf{A})$ (if certain consistency problems for $\mathbf{A}$ are in PSPACE (cf.

Remark 4.17)), as slight changes to the proofs (and the definition of a $\mathbb{K}(\mathbf{A})$-tree-model) can assure the preservation of reflexivity in Lemmas 4.4, 4.6, and 4.10.

When it comes to transitivity or symmetry, an adaptation of our techniques seem to be less straightforward. To prove the correspondence between $\mathbb{K}(\mathbf{A})$- and $\mathbb{FK}(\mathbf{A})$-models as well as PSPACE-completeness of the validity problem for $\mathsf{K}(\mathbf{A})$, we relied heavily on the (bounded) tree-model property of $\mathsf{K}(\mathbf{A})$ (Lemma 3.6). While clearly no $\mathbb{K}(\mathbf{A})$-model in $\mathbb{T}(\mathbf{A})$, $\mathbb{S}\mathbf{4}(\mathbf{A})$, or $\mathbb{S}\mathbf{5}(\mathbf{A})$ is a $\mathbb{K}(\mathbf{A})$-tree-model, it is easy to suitably adjust the definition of a $\mathbb{K}(\mathbf{A})$-tree-model for $\mathsf{T}(\mathbf{A})$ and $\mathsf{T}^{\mathsf{c}}(\mathbf{A})$ (just allow nodes in a tree to be connected to themselves). Whether this is possible also for $\mathsf{S4}(\mathbf{A})$, $\mathsf{S4}^{\mathsf{c}}(\mathbf{A})$, $\mathsf{S5}(\mathbf{A})$, or $\mathsf{S5}^{\mathsf{c}}(\mathbf{A})$ are interesting questions that remain to be answered.

On a larger scale, it would be useful to develop a more general approach, perhaps using the alternative semantics developed in Chapter 4, to investigate questions of decidability for order-based modal logics over different subclasses of $\mathbb{K}(\mathbf{A})$.

Order-Based Multi-Modal Logics

We have only focussed on the expansions by the unary modal connectives $\Box$ and $\Diamond$ in this dissertation. It would also be interesting to investigate order-based modal logics (or others) expanded by an arbitrary number of modal connectives of any finite arity.

Let $\mathfrak{L}_{\mathfrak{m}}^{\star} = \mathfrak{L} \cup \boxtimes$, where $\boxtimes$ is a set of modal connectives of finite arity and let $\mathrm{Fm}^{\star}$ denote the set of formulas defined inductively in $\mathfrak{L}_{\mathfrak{m}}^{\star}$ over Var. For convenience, we will assume that the modal connectives in $\boxtimes$ come in pairs of the same arity, i.e. let $\boxtimes = \{[d], \langle d\rangle \mid d \in I\}$, for some non-empty index-set I, and let $d^{\star} \in \mathbb{N}$ denote the arity of $[d]$ and $\langle d\rangle$.

A way to generalize $\mathbb{K}(\mathbf{A})$-models to accommodate for such expansions is the following. A $\mathbb{K}^{\star}(\mathbf{A})$-*model* for $\mathfrak{L}_{\mathfrak{m}}^{\star}$ is a triple $\langle W, \{R_d\}_{d\in I}, V\rangle$ where W is a set of worlds, for each $d \in I$, R_d is a $(d^{\star}+1)$-*ary* $\mathbf{A}$-*accessibility relation*, i.e. it is a function $R_d\colon W^{(d^{\star}+1)} \to A$, and V is an $\mathbf{A}$-valuation $V\colon \mathrm{Var} \times W \to A$. V is then extended to $\mathrm{Fm}^{\star}$ as for $\mathbb{K}(\mathbf{A})$-models except that for every $d \in I$:

$$\begin{aligned} V([d]\varphi, x) &= \bigwedge\{R_d x y_1 \dots y_{d^{\star}} \to \bigwedge_{i\leq d^{\star}} V(\varphi, y_i) \mid y_1, \dots, y_{d^{\star}} \in W\}, \\ V(\langle d\rangle\varphi, x) &= \bigvee\{R_d x y_1 \dots y_{d^{\star}} \wedge \bigwedge_{i\leq d^{\star}} V(\varphi, y_i) \mid y_1, \dots, y_{d^{\star}} \in W\}. \end{aligned}$$

It is an open question whether order-based modal logics for $\mathfrak{L}_{\mathfrak{m}}^{\star}$ based on $\mathbb{K}^{\star}(\mathbf{A})$-models are decidable. Noting that for classical modal logic, many important notions easily generalize to the multi-modal case (cf. [15]), we expect that our approach to order-based modal logics, i.e. restricting the possible truth values of box- and diamond-formulas, generalizes without much difficulty to order-based multi-modal logics also. We thus expect that our approach might be useful for answering questions of decidability for many of these logics.

Order-Based Description Logics

The special cases of order-based multi-modal logics where all modal connectives are unary are of particular interest, as they are closely related to fuzzy description logics. Fuzzy description logics generalize (classical) description logics, a family of knowledge representation formalisms (for an overview, see e.g. [4] for classical description logics and [18] for fuzzy description logics). Languages for (fuzzy) description logics are based on individuals, concepts, and roles, which are interpreted semantically as elements of a domain, (fuzzy) subsets of the domain, and (fuzzy) binary relations on the domain, respectively. Complex concepts can be built from atomic concepts via concept constructors, often including intersection, union, complementation, implication, value restriction, and existential quantification (the last two making use of roles).

For a suitable $\mathfrak{L}^\star_{\mathfrak{m}} = \mathfrak{L} \cup \boxtimes$ with $\boxtimes = \{[d], \langle d \rangle \mid d \in I \text{ and } d^\star = 1\}$, a $\mathbb{K}^\star(\mathbf{A})$-model $\langle W, \{R_d\}_{d \in I}, V\rangle$ for $\mathfrak{L}^\star_{\mathfrak{m}}$ is a natural interpretation of a (fuzzy) description language, where W is the domain, individuals are interpreted as worlds, atomic concepts as propositional variables (such that an individual belongs to the atomic concept to the degree the propositional variable is true at the corresponding world), and roles as binary $\mathbf{A}$-accessibility relations. Concept constructors are then interpreted by the propositional and modal connectives in $\mathfrak{L}^\star_{\mathfrak{m}}$, e.g. intersection by $\wedge$, union by $\vee$, complementation by $\neg$, implication by $\to$, and for each role interpreted by an $\mathbf{A}$-accessibility relations R_d, the value restriction using this roles is interpreted by $[d]$ and the existential quantification using it by $\langle d \rangle$. In this sense, an individual belongs to a complex concept to the degree the corresponding formula is true at the corresponding world.

For order-based description logics where the implication constructor is removed from the language or if they are restricted to witnessed $\mathbb{K}^\star(\mathbf{A})$-models, many decision problems (including the problems of validity, satisfiability, and subsumption) are known to be decidable and in EXPTIME, as they can be reduced (in linear time) to the respective problem in the classical setting via crispification by a double-negation interpretation (see e.g. [18, 19]).[2] Order-based description logics based on all $\mathbb{K}^\star(\mathbf{A})$-models are generally not complete with respect to witnessed $\mathbb{K}^\star(\mathbf{A})$-models, however, and $\to$ usually cannot be defined by the other connectives. As mentioned above, it is exactly the former "deficiency" we work around with the approach in Chapter 4 of restricting the truth values a box- or diamond-formula can be assigned. In this approach, it is enough for a world y to "witness" the value of $V(\Box\varphi, x)$ if the value $Rxy \to V(\varphi, y)$ is just merely "close enough"

[2] In fact, these problems have only been considered for $\mathbf{A} = \mathbf{G}$, but we expect them to easily generalize to any order-based algebra $\mathbf{A}$ for the language $\mathfrak{L} = \{\wedge, \vee, \to, \bot, \top\}$.

to $V(\Box\varphi, x)$.

In order to adapt our approach to the case at hand, we define an $\mathbb{FK}^\star(\mathbf{A})$-*model*, for a locally homogeneous order-based algebra $\mathbf{A}$, to be a five-tuple $\langle W, \{R_d\}_{d\in I}, V, T_\Box, T_\Diamond\rangle$ such that for each $d \in I$, $\langle W, R_d, V, T_\Box, T_\Diamond\rangle$ is an $\mathbb{FK}(\mathbf{A})$-model and for each $x \in W$:

$$\begin{aligned} V([d]\varphi, x) &= \bigvee\{r \in T_\Box(x) \mid r \leq \bigwedge\{R_d xy \to V(\varphi, y) \mid y \in W\}\}\,, \\ V(\langle d\rangle\varphi, x) &= \bigwedge\{r \in T_\Box(x) \mid r \geq \bigvee\{R_d xy \wedge V(\varphi, y) \mid y \in W\}\}\,. \end{aligned}$$

It is then possible to minimally change the proofs in Chapter 4 to accommodate $\mathbb{K}^\star(\mathbf{A})$- and $\mathbb{FK}^\star(\mathbf{A})$-models instead of $\mathbb{K}(\mathbf{A})$- and $\mathbb{FK}(\mathbf{A})$-models. For example, the definition of a tree-model needs adjusting, that is, a $\mathbb{K}^\star(\mathbf{A})$-model $\langle W, \{R_d\}_{d\in I}, V\rangle$ or an $\mathbb{FK}^\star(\mathbf{A})$-model $\langle W, \{R_d\}_{d\in I}, V, T_\Box, T_\Diamond\rangle$ is called a *tree-model*, if $\langle W, \bigcup_{d\in I} R_d^+\rangle$ is a tree, recalling that for each $d \in I$, $R_d^+ = \{\langle x, y\rangle \in W^2 \mid R_d xy > 0\}$.

It is then straightforward to infer the finite model property with respect to $\mathbb{FK}^\star(\mathbf{A})$-models as well as PSPACE-completeness of the validity problems for order-based multi-modal logics with only unary modal connectives (in case certain consistency problems for $\mathbf{A}$ are in PSPACE (cf. Remark 4.17)).[3] In particular, this implies PSPACE-completeness of the validity problem of Gödel multi-modal logic expanded with arbitrarily many unary modal connectives and of fuzzy description logics based on Gödel semantics (see e.g. [18]).

As they are usually motivated by specific applications, researchers working in the field of fuzzy description logics are typically more interested in the problem of satisfiability of specific sets of formulas (called knowledge bases), the problem of consequence (or entailment), and other decision problems. In the case of order-based description logics based on all $\mathbb{K}^\star(\mathbf{A})$-models, decidability and complexity results for the validity problem cannot always be easily transferred to these problems and thus they are still open in most cases (depending on the expressivity of the description language).

Order-Based Epistemic Logics

The notion of knowledge is often modelled by multi-modal logics, so-called epistemic logics, where there is a set I of agents, and for each agent $d \in I$, there are two unary modal connectives $[d]$ ("d knows that …") and $\langle d\rangle$ ("d considers it possible that …"). To model the notion of knowledge appropriately, these multi-modal logics are often based on class of Kripke models where each accessibility relation is an equivalence relation (see e.g. [49]).

If we want to study many-valued epistemic logics based on an order-based algebra $\mathbf{A}$, we might consider $\mathbb{K}^\star(\mathbf{A})$-models $\langle W, \{R_d\}_{d\in I}, V\rangle$ where for each $d \in I$, R_d is a

[3] We note that if there are infinitely many modal connectives in $\mathfrak{L}_{\mathfrak{m}}^\star$, even a finite $\mathbb{K}^\star(\mathbf{A})$-model for $\mathfrak{L}_{\mathfrak{m}}^\star$ is an uncomputable object. However, when testing a formula for validity, we can restrict to the modal connectives occurring in that formula.

crisp equivalence relation. In this case, we expect that our approach from Chapter 5 would work to tackle open problems of decidability for such order-based epistemic logics. However, as we cannot restrict our attention to universal models, adapting the approach might not be entirely straightforward.

Optimizing and Extending the Tableau Calculi

We already mentioned that the algorithms provided by the tableau calculi for GK, $\mathsf{GK^c}$, and $\mathsf{GS5^c}$ in Chapter 6 are not of optimal complexity. As also the usual tableau calculi for the classical modal logic K do not provide algorithms of optimal complexity (see e.g. [107]), we do not expect there to be easy adaptations of our tableau calculi $\mathcal{TGK}$ and $\mathcal{TGK}^{\mathsf{c}}$ that provide PSPACE-algorithms. In the case of $\mathsf{GS5^c}$, however, it would be rather straightforward to remedy this deficiency. Note that the major problem in this case lies in rules like the following:

$$(\leq\wedge): \quad \begin{array}{c} w\colon \varphi \leq \psi \wedge \chi \\ | \\ w\colon \varphi \leq \psi \\ w\colon \varphi \leq \chi \end{array}$$

By applying the rule $(\leq\wedge)$, we decompose the formula $\psi \wedge \chi$ is decomposed, but we also repeat the formula φ twice. This has the effect that in a complete tableau, the formula φ would have to be decomposed twice, making the tableau larger without adding more information.

This could, for example, be remedied by introducing a new propositional variable p (not occurring anywhere above on the branch) which at w gets assigned the same value as φ and replaces φ in the two resulting nodes. That is, an alternative rule might look like this:

$$(\leq\wedge)': \quad \begin{array}{c} w\colon \varphi \leq \psi \wedge \chi \\ | \\ w\colon \varphi = p \quad (p \text{ new}) \\ w\colon p \leq \psi \\ w\colon p \leq \chi \end{array}$$

Because in this case φ only needs to be decomposed once and p does not need to be decomposed at all, the resulting tableau would be smaller. To work out the details of this approach is left for future work.

Furthermore, for stepping from the calculus $\mathcal{TGK}$ to the calculus $\mathcal{TGK}^{\mathsf{c}}$, we have adjusted the closure conditions for a branch of a $\mathcal{TGK}$-tableau to the class of all *crisp* $\mathbb{SK}(\mathbf{G})$-models. Note that we could have instead added the following rule to the set $\mathcal{TGK}$:

$$\text{(CR)}: \quad \begin{array}{c} (rwv \text{ occurs on } b) \\ \diagup \quad \diagdown \\ rwv \leq \bot \quad \top \leq rwv \end{array}$$

The set of rules in Figure 6.5 together with (CR) provides a calculus that is sound and complete with respect to validity in $\mathsf{GK^c}$, no matter whether we define the closure of a branch with respect to all $\mathbb{SK}(\mathbf{G})$-models or with respect to $\mathbb{SK}^{\mathsf{c}}(\mathbf{G})$-models.

Moreover, if we add the following rule to $\mathcal{TGK}$, the resulting tableau calculus – depending on the definition of the closure of a branch – is sound and complete with respect to GT and $\mathsf{GT^c}$:

$$\text{(RE)}: \quad \begin{array}{c} (u \text{ occurs on } b) \\ | \\ \top \leq ruu \end{array}$$

We might also want to consider tableau calculi for order-based modal logics based on algebras other than $\mathbf{G}$. For example if $\mathbf{A}$ is $\mathbf{G}_{\downarrow}$ or $\mathbf{G}_{\uparrow}$, we would have to work with $\mathbb{FK}(\mathbf{A})$-models (as opposed to $\mathbb{SK}(\mathbf{A})$-models) and thus would need to distinguish between T-symbols for $T_{\Box}$ and $T_{\Diamond}$. We do not expect major difficulties with such an approach, but the more complicated formalisms still need to be worked out.

Finally, it would be interesting to implement these calculi in order to obtain automated decision procedures for the order-based modal logics in question.

Many-Valued Neighbourhood Semantics

In Chapter 7, we study the box-fragments of many-valued modal logics over MTL-algebras based on $\mathbf{A}$-neighbourhood models.

We first note that we did not consider issues of decidability and complexity for these logics. However, given how weak the modal counterpart LSM of a logic L is, we expect that in many cases, decision problems for LSM are PTIME-reducible to corresponding problems in L, which would yield the same complexity bounds as in L. Working out the details of this is left for future work, however.

Furthermore, in contrast to the classical case, $\Diamond$ cannot suitably be defined by $\neg\Box\neg$. It is therefore an interesting question how to interpret a formula $\Diamond\varphi$ based on some sort of $\mathbb{SM}(\mathbf{A})$-like models such that the "meaning" of $\Diamond$ reflects that of $\Diamond$ in classical modal logics based on neighbourhood frames (where $\Diamond$ is defined by $\neg\Box\neg$).

One way to do this, as was done by Rodríguez and Godo in [111], is to extend $\mathbb{SM}(\mathbf{A})$-model with an extra neighbourhood function P. That is, an $\mathbb{SM}_{\Diamond}(\mathbf{A})$-*model* is a quadruple $\mathcal{M} = \langle W, N, P, V\rangle$ where $\langle W, N, V\rangle$ is an $\mathbb{SM}(\mathbf{A})$-model and P is an $\mathbf{A}$-neighbourhood function $P\colon W \to A^{A^W}$, which is then used to interpret $\Diamond$ in the same way as $\Box$, i.e.

$$V(\Diamond\varphi, x) \;=\; [\![\varphi]\!]_{\mathcal{M}}\, \varepsilon\, P(x)\,.$$

This approach has the drawback that if we consider all $\mathbb{SM}_\Diamond(\mathbf{A})$-models, $\Box$ and $\Diamond$ are completely independent, which does not reflect the classical case. In fact, $\Diamond$ can then be understood just as another box-connective. Moreover, in this case, it is unclear how a connection between classes of $\mathbb{SM}_\Diamond(\mathbf{A})$-models and $\mathbb{K}(\mathbf{A})$-models can be established.

In [111], this drawback is remedied by restricting to $\mathbb{SM}_\Diamond(\mathbf{A})$-models that satisfy certain conditions. Unfortunately, the conditions given are just trivial rewritings of the desired axioms and do not provide deeper information on the $\mathbb{SM}_\Diamond(\mathbf{A})$-models considered. It would therefore be interesting to find "more semantical" conditions on N and P that are needed to obtain a suitable interpretation of $\Diamond$. In fact, it would be even more desirable to find an interpretation of $\Diamond$ that relies on N rather than P, such that Theorem 7.11 can be extended to a modal language including $\Box$ and $\Diamond$.

It might also be fruitful to study the connections between $\mathbf{A}$-augmented $\mathbb{SM}(\mathbf{A})$-frames and $\mathbb{K}(\mathbf{A})$-frames more deeply. The goal would be to find conditions on $\mathbf{A}$-augmented $\mathbb{SM}(\mathbf{A})$-frames to obtain correspondences between certain classes of $\mathbf{A}$-augmented $\mathbb{SM}(\mathbf{A})$-frames with interesting classes of $\mathbb{K}(\mathbf{A})$-frames, such as $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$, $\mathbb{T}(\mathbf{A})$, $\mathbb{T}^{\mathsf{c}}(\mathbf{A})$, $\mathbb{S}\mathbf{4}(\mathbf{A})$, $\mathbb{S}\mathbf{4}^{\mathsf{c}}(\mathbf{A})$, $\mathbb{S}\mathbf{5}(\mathbf{A})$, and $\mathbb{S}\mathbf{5}^{\mathsf{c}}(\mathbf{A})$. More generally, we might be interested in classes of $\mathbb{SM}(\mathbf{A})$-frames (not necessarily $\mathbf{A}$-augmented) that correspond to desirable properties of $\Box$ and $\Diamond$. While some of these questions are considered and answered in [110, 111], there are still very many interesting open questions in this area of many-valued modal logics.

Fragments of Many-Valued First-Order Logics

As mentioned before, one of the most important reasons why propositional modal logics grew so popular is that they provide an excellent compromise between expressivity and computability. While propositional logics are usually decidable but not very expressive and first-order logics are very expressive but usually undecidable, modal logic are fairly expressive and decidable in very many cases.

In the classical setting, it is well-known that modal propositional logics represent certain fragments of first-order logic. For example, the classical modal logic $\mathsf{S5}$ corresponds to the *one-variable fragment* of first order logic, a fact that we reflected in Chapter 5 in the setting of order-based logics. The classical modal logic K, on the other hand, embeds into the two-variable fragment of first-order logic (cf. [59]) by translating box- and diamond formulas as follows:

$$\pi(\Box\varphi) \;=\; (\forall y)(Rxy \to \pi(\varphi)(y)) \qquad \text{and} \qquad \pi(\Diamond\varphi) \;=\; (\exists y)(Rxy \wedge \pi(\varphi)(y))$$

As it was known that (satisfiability in) the *two-variable fragment* of first-order logic is decidable and NEXPTIME-complete (see e.g. [1]), it was held that it is the inclusion of modal logics in the two-variable fragment that makes modal logics so robustly decidable.

It was later argued, however, that this robustness in decidability rather stems from the tree-model property (cf. [119]), which is a powerful tool when it comes to designing efficient algorithms (see [108]).

Furthermore, van Benthem's Characterization Theorem tells us that the modal logic K corresponds to the *bisimulation invariant fragment* of first-order logic (see [13, 14]). A *bisimulation* is a binary relation connecting elements of two first-order structures (or two Kripke models) where the same unary predicates are satisfied (where the same propositional variables are true), and certain back-and-forth conditions concerning a binary relation on the structures (the accessibility relation) are fulfilled.

Moreover, it was shown that modal logics embed into the so-called *guarded fragment* of first-order logic (see [1]), where quantification is only allowed when it is guarded by some relation-symbol in much the same way as the translations above. The guarded fragment enjoys nice computational properties, e.g. it has the finite model property and a "tree-model-like" property, and is decidable and 2EXPTIME-complete (see [65]). Noting that the guarded fragment is not restricted to a certain number of variables, the satisfaction of these computational properties contrasts with the fact that the finite-variable fragments of first-order logic are undecidable for a number of variables ≥ 3 (see e.g. [1]).

With these useful connections between modal logics and fragments of first-order logic in mind, it is natural to ask how such connections can been drawn in the setting of many-valued logics and which fragments of many-valued first-order logics are decidable (see question (8) in the introduction).

In [67], Hájek embeds many-valued crisp $\mathsf{S5}$ logics based on continuous t-norms into the monadic fragments of their first-order counterparts, where the *monadic fragments* restricts the first-order language to unary predicate symbols and no function symbols (but allows arbitrarily many variables). While the monadic fragment is decidable in the classical setting, it is undecidable for first-order Łukasiewicz, product, and Gödel logic (see e.g. [73]). On the other hand, the many-valued crisp $\mathsf{S5}$ logics over complete MTL-algebras correspond to the one-variable fragments of their first-order counterparts, for which decidability of the validity problem is known for the standard Łukasiewicz algebra **Ł** (see [67]) and now also for the standard Gödel algebra **G** (and other order-based algebras; see Chapter 5). Decidability issues for the one-variable fragment of the first-order product logic remain open, however.

Concerning decidability issues for the two-variable fragments or the guarded fragments of many-valued first-order logics, very little is known. While it is clear that the standard translation between a many-valued modal logic with K-like modalities and its first-order counterpart works in the same fashion as in the classical setting, there is only very limited literature on how the connections between many-valued modal logics and fragments of many-valued first-order logics might be exploited to obtain more insights into either of these topics. One of the first studies in this direction, providing an initial step towards a van Benthem-style characterization theorem, is Michel Marti and George Metcalfe's [92], where the Hennessy-Milner property (relating modal equivalence and bisimulations) is studied for the many-valued modal logics based on (image-finite) $\mathbb{K}^{\mathsf{c}}(\mathbf{A})$-models, where $\mathbf{A}$ is a complete MTL-chain, and some classification results are obtained.

It is therefore an interesting question whether our approach to order-based modal logics, i.e. restricting truth values of box- and diamond-formulas, extends to order-based first-order logics, where the truth values of quantified formulas are restricted. If it does, it might be useful in dealing with certain decidability issues of fragments of order-based first-order logics, e.g. the two-variable fragment or the guarded fragment. The question whether such an extension of our approach is possible and how it might be achieved will have to be left open here.

Bibliography

[1] H. Andréka, I. Németi and J. van Benthem. 'Modal Languages and Bounded Fragments of Predicate Logic'. In: *Journal of Philosophical Logic* 27.3 (1998), pp. 217–274.

[2] S. Arora and B. Barak. *Computational Complexity: A Modern Approach.* Cambridge University Press, 2009.

[3] A. Avron. 'A Constructive Analysis of RM'. In: *Journal of Symbolic Logic* 52.4 (1987), pp. 939–951.

[4] F. Baader, D. Calvanese, D. L. McGuinness, D. Nardi and P. F. Patel-Schneider, eds. *The Description Logic Handbook: Theory, Implementation, and Applications.* New York, NY, USA: Cambridge University Press, 2003.

[5] M. Baaz, P. Hájek, J. Krajíček and D. Švejda. 'Embedding Logics into Product Logic'. In: *Studia Logica* 61.1 (1998), pp. 35–47.

[6] M. Baaz, P. Hájek, F. Montagna and H. Veith. 'Complexity of t-tautologies'. In: *Annals of Pure and Applied Logic* 113.1 (2001), pp. 3–11.

[7] M. Baaz, A. Ciabattoni and C. G. Fermüller. 'Monadic Fragments of Gödel Logics: Decidability and Undecidability Results'. In: *Proceedings of LPAR 2007, Yerevan, Armenia.* Springer LNCS 4790, 2007, pp. 77–91.

[8] M. Baaz, N. Preining and R. Zach. 'First-order Gödel logics'. In: *Annals of Pure and Applied Logic* 147.1–2 (2007), pp. 23–47.

[9] M. Baaz and R. Zach. 'Compact Propositional Gödel Logics'. In: *28th International Symposium on Multiple-Valued Logic. May 1998, Fukuoka, Japan. Proceedings.* Los Alamitos: IEEE Computer Society Press, 1998, pp. 108–113.

[10] A. Beckmann, M. Goldstern and N. Preining. 'Continuous Fraïssé Conjecture'. In: *Order* 25.4 (2008), pp. 281–298.

[11] L. Běhounek and P. Cintula. 'Fuzzy Class Theory'. In: *Fuzzy Sets and Systems* 154.1 (2005), pp. 34–55.

[12] N. D. Belnap. ‘A Useful Four-Valued Logic’. In: *Modern Uses of Multiple-Valued Logic*. Ed. by J. M. Dunn and G. Epstein. D. Reidel, 1977.

[13] J. v. Benthem. ‘Modal Correspondence Theory’. PhD thesis. Mathematical Institute, University of Amsterdam, 1976.

[14] J. v. Benthem. *Modal Logic and Classical Logic*. Napoli: Bibliopolis, 1985.

[15] P. Blackburn, M. d. Rijke and Y. Venema. *Modal logic*. Cambridge University Press, 2001.

[16] W. J. Blok and C. J. Van Alten. ‘The finite embeddability property for residuated lattices, pocrims and BCK-algebras’. In: *Algebra Universalis* 48.3 (2002), pp. 253–271.

[17] W. Blok and D. Pigozzi. *Algebraizable Logics*. Memoirs of the American Mathematical Society, vol. 396, Providence, RI, 1986.

[18] F. Bobillo, M. Cerami, F. Esteva, A. García-Cerdaña, R. Peñaloza and U. Straccia. ‘Fuzzy Description Logics’. In: *Handbook of Mathematical Fuzzy Logic - Volume 3*. Ed. by P. Cintula, C. G. Fermüller and C. Noguera. Vol. 58. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2015, pp. 1105–1181.

[19] F. Bobillo, M. Delgado, J. Gómez-Romero and U. Straccia. ‘Fuzzy Description Logics Under Gödel Semantics’. In: *International Journal of Approximate Reasoning* 50.3 (2009), pp. 494–514.

[20] G. S. Boolos. *The Logic of Provability*. Cambridge University Press, 1993.

[21] F. Bou, F. Esteva, L. Godo and R. O. Rodríguez. ‘On the minimum many-valued logic over a finite residuated lattice’. In: *Journal of Logic and Computation* 21.5 (2011), pp. 739–790.

[22] F. Bou, F. Esteva and L. Godo. ‘Exploring a Syntactic Notion of Modal Many-Valued Logics’. In: *Mathware & Soft Computing* 15 (2008), pp. 175–188.

[23] R. Brafman, J.-C. Latombe, Y. Moses and Y. Shoham. ‘Knowledge as a tool in motion planning under uncertainty’. In: *Theoretical Aspects of Reasoning about Knowledge: Proc. Fifth Conference*. Ed. by R. Fagin. 1994, pp. 208–224.

[24] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. The Millenium Edition (2012). Vol. 78. Graduate Texts in Mathematics. Springer, 1981. URL: `https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`.

[25] C. C. Chang. ‘Algebraic Analysis of Many-Valued Logics’. In: *Transactions of the American Mathematical Society* 88 (1958), pp. 467–490.

[26] X. Caicedo, G. Metcalfe, R. O. Rodríguez and J. Rogger. 'A Finite Model Property for Gödel Modal Logics'. In: *Proceedings of WoLLIC 2013, Darmstadt, Germany.* Springer LNCS 8071, 2013, pp. 226–237.

[27] X. Caicedo, G. Metcalfe, R. O. Rodríguez and J. Rogger. 'Decidability of Order-Based Modal Logics'. In: *Journal of Computer and System Sciences* (to appear).

[28] X. Caicedo and R. O. Rodríguez. 'Standard Gödel Modal Logics'. In: *Studia Logica* 94.2 (2010), pp. 189–214.

[29] X. Caicedo and R. O. Rodríguez. 'Bi-modal Gödel logic over [0,1]-valued Kripke frames'. en. In: *Journal of Logic and Computation* 25.1 (2015), pp. 37–55.

[30] J. M. V. Castilho, M. A. Casanova and A. L. Furtado. 'A temporal framework for database specification'. In: *Proc. 8th Int. Conf. on Very Large Data Bases.* 1982, pp. 280–291.

[31] M. Cerami, F. Esteva and F. Bou. 'Decidability of a Description Logic over Infinite-Valued Product Logic'. In: *Proceedings of the Twelfth International Conference on the Principles of Knowledge Representation and Reasoning (KR 2010).* 2010, pp. 203–213.

[32] B. F. Chellas. *Modal Logic: An Introduction.* en. Cambridge University Press, 1980.

[33] A. Ciabattoni, F. Esteva and L. Godo. 'T-norm based logics with n-contraction'. In: *Neural Network World* 12.5 (2002), pp. 441–453.

[34] R. Cignoli, F. Esteva, L. Godo and A. Torrens. 'Basic fuzzy logic is the logic of continuous t-norms and their residua'. In: *Soft Computing* 4.2 (2000), pp. 106–112.

[35] P. Cintula. 'Weakly Implicative (Fuzzy) Logics I: Basic Properties'. In: *Archive for Mathematical Logic* 45.6 (2006), pp. 673–704.

[36] P. Cintula. 'From Fuzzy Logic to Fuzzy Mathematics'. PhD thesis. Czech Technical University in Prague, 2004.

[37] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna and C. Noguera. 'Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies'. In: *Annals of Pure and Applied Logic* 160.1 (2009), pp. 53–81.

[38] P. Cintula, C. G. Fermüller and C. Noguera, eds. *Handbook of Mathematical Fuzzy Logic - Volume 3.* Vol. 58. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2015. URL: http://www.mathfuzzlog.org/index.php/Handbook_of_Mathematical_Fuzzy_Logic.

[39] P. Cintula, P. Hájek and C. Noguera, eds. *Handbook of Mathematical Fuzzy Logic - Volume 1*. Vol. 37. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2011. URL: `http://www.mathfuzzlog.org/index.php/Handbook_of_Mathematical_Fuzzy_Logic`.

[40] P. Cintula, P. Hájek and C. Noguera, eds. *Handbook of Mathematical Fuzzy Logic - Volume 2*. Vol. 38. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2011. URL: `http://www.mathfuzzlog.org/index.php/Handbook_of_Mathematical_Fuzzy_Logic`.

[41] P. Cintula and C. Noguera. 'Implicational (semilinear) logics I: a new hierarchy'. In: *Archive for Mathematical Logic* 49.4 (2010), pp. 417–446.

[42] P. Cintula and C. Noguera. 'A General Framework for Mathematical Fuzzy Logic'. In: *Handbook of Mathematical Fuzzy Logic - Volume 1*. Ed. by P. Cintula, P. Hájek and C. Noguera. Vol. 37. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2011, pp. 103–207.

[43] P. Cintula, C. Noguera and J. Rogger. 'From Kripke to Neighborhood Semantics for Modal Fuzzy Logics'. In: *Information Processing and Management of Uncertainty in Knowledge Based Systems, IPMU 2016*. Ed. by J. Carvalho, M.-J. Lesot, U. Kaymak, S. Vieira, B. Bouchon-Meunier and R. Yager. (to appear).

[44] E. M. Clarke, E. A. Emerson and A. P. Sistla. 'Automatic verification of finite-state concurrent system using temporal logic specifications'. In: *ACM Transactrions on Programming Languages and Systems* 8.2 (1986), pp. 244–263.

[45] D. v. Dalen. 'Intuitionistic Logic'. In: *Handbook of Philosophical Logic*. Ed. by D. M. Gabbay and F. Guenthner. Vol. 5. Dordrecht: Springer Netherlands, 2002, pp. 1–114.

[46] A. Di Nola and I. Leuştean. 'Łukasiewicz Logic and MV-Algebras'. In: *Handbook of Mathematical Fuzzy Logic - Volume 2*. Ed. by P. Cintula, P. Hájek and C. Noguera. Vol. 38. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2011, pp. 496–583.

[47] D. Diaconescu and G. Georgescu. 'Tense Operators on MV-Algebras and Łukasiewicz-Moisil Algebras'. In: *Fundamenta Informaticae* 81.4 (2007), pp. 379–408.

[48] D. Diaconescu, G. Metcalfe and L. Schnüriger. 'Axiomatizing a Real-Valued Modal Logic'. In: *Proceedings of AiML 2016*. To appear.

[49] H. v. Ditmarsch, W. v. d. Hoek and B. Kooi. *Dynamic Epistemic Logic*. Vol. 337. Synthese Library. Springer Netherlands, 2008.

[50] M. Dummett. 'A Propositional Calculus with Denumerable Matrix'. In: *The Journal of Symbolic Logic* 24.2 (1959), pp. 97–106.

[51] J. M. Dunn. 'Intuitive Semantics for First-Degree Entailments and 'Coupled Trees''. In: *Philosophical Studies* 29.3 (1976), pp. 149–168.

[52] J. M. Dunn and G. Restall. 'Relevance Logic'. In: *Handbook of Philosophical Logic.* Ed. by D. M. Gabbay and F. Guenthner. Vol. 6. Dordrecht: Springer Netherlands, 2002, pp. 1–128.

[53] F. Esteva, J. Gispert, L. Godo and F. Montagna. 'On the Standard and Rational Completeness of some Axiomatic Extensions of the Monoidal T-norm Logic'. In: *Studia Logica* 71.2 (2002), pp. 199–226.

[54] F. Esteva and L. Godo. 'Monoidal t-norm based logic: towards a logic for left-continuous t-norms'. In: *Fuzzy Sets and Systems* 124 (2001), pp. 271–288.

[55] F. Esteva, L. Godo, P. Hájek and M. Navara. 'Residuated fuzzy logics with an involutive negation'. In: *Archive for Mathematical Logic* 39.2 (2000), pp. 103–124.

[56] G. Fischer Servi. 'Axiomatizations for some intuitionistic modal logics'. In: *Rendiconti del Seminario Matematico Uniersità e Politecnico di Torino* 42 (1984), pp. 179–194.

[57] M. Fitting. 'Many-valued modal logics'. In: *Fundamenta Informaticae* 15 (1992), pp. 235–254.

[58] M. Fitting. 'Many-valued modal logics II'. In: *Fundamenta Informaticae* 17 (1992), pp. 55–73.

[59] D. M. Gabbay. 'Aspects of Philosophical Logic: Some Logical Forays into Central Notions of Linguistics and Philosophy'. In: ed. by U. Mönnich. Dordrecht: Springer Netherlands, 1981. Chap. Expressive Functional Completeness in Tense Logic (Preliminary report), pp. 91–117.

[60] J. Garson. 'Modal Logic'. In: *The Stanford Encyclopedia of Philosophy.* Ed. by E. N. Zalta. Spring 2016. URL: `http://plato.stanford.edu/archives/spr2016/entries/logic-modal`.

[61] G. Gentzen. 'Untersuchungen über das logische Schließen. I & II'. In: *Mathematische Zeitschrift* 39.1 (1935), pp. 176–210, 405–431.

[62] K. Gödel. 'Zum intuitionistischen Aussagenkalkül'. In: *Anzeiger der Akademie der Wissenschaften in Wien* 69 (1932), pp. 65–66.

[63] L. Godo, P. Hájek and F. Esteva. 'A fuzzy modal logic for belief functions'. In: *Fundamenta Informaticae* 57.2–4 (2003), pp. 127–146.

[64] L. Godo and R. Rodríguez. 'A fuzzy modal logic for similarity reasoning'. In: *Fuzzy Logic and Soft Computing.* 1999, pp. 33–48.

[65] E. Grädel. 'On the Restraining Power of Guards'. In: *The Journal of Symbolic Logic* 64.4 (1999), pp. 1719–1742.

[66] P. Hájek, D. Harmancová, F. Esteva, P. Garcia and L. Godo. 'On Modal Logics for Qualitative Possibility in a Fuzzy Setting'. In: *Proceedings of UAI 1994*. 1994, pp. 278–285.

[67] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 1998.

[68] P. Hájek. 'Making fuzzy description logic more general'. In: *Fuzzy Sets and Systems* 154.1 (2005), pp. 1–15.

[69] P. Hájek. 'On witnessed models in fuzzy logic'. In: *Mathematical Logic Quarterly* 53.1 (2007), pp. 66–77.

[70] P. Hájek. 'On witnessed models in fuzzy logic II'. In: *Mathematical Logic Quarterly* 53.6 (2007), pp. 610–615.

[71] P. Hájek. 'On fuzzy modal logics S5(C)'. In: *Fuzzy Sets Syst.* 161.18 (2010), pp. 2389–2396.

[72] P. Hájek, L. Godo and F. Esteva. 'A complete many-valued logic with product-conjunction'. In: *Archive for Mathematical Logic* 35 (1996), pp. 191–208.

[73] P. Hájek, F. Montagna and C. Noguera. 'Arithmetical Complexity of First-Order Fuzzy Logics'. In: *Handbook of Mathematical Fuzzy Logic - Volume 2*. Ed. by P. Cintula, P. Hájek and C. Noguera. Vol. 38. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2011, pp. 853–908.

[74] Z. Haniková. 'Computational Complexity of Propositional Fuzzy Logics'. In: *Handbook of Mathematical Fuzzy Logic - Volume 2*. Ed. by P. Cintula, P. Hájek and C. Noguera. Vol. 38. Studies in Logic: Mathematical Logic and Foundations. London: College Publications, 2011, pp. 793–851.

[75] G. Hansoul and B. Teheux. 'Extending Łukasiewicz Logics with a Modality: Algebraic Approach to Relational Semantics'. In: *Studia Logica* 101.3 (2013), pp. 505–545.

[76] L. S. Hay. 'Axiomatization of the infinite-valued predicate calculus'. In: *Journal of Symbolic Logic* 28.1 (1963), pp. 77–86.

[77] U. Höhle. 'Commutative, residuated 1-monoids'. In: *Non-Classical Logics and their Applications to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory*. Ed. by U. Höhle and E. P. Klement. Dordrecht: Springer Netherlands, 1995, pp. 53–106.

[78] R. Iemhoff. 'Consequence relations and admissible rulese'. In: *Journal of Philosophical Logic* 45.3 (2016), pp. 284–315.

[79] S. Jenei and F. Montagna. 'A proof of standard completeness for Esteva and Godo's MTL logic'. In: *Studia Logica* 70.2 (2002), pp. 183–192.

[80] S. Kanger. *Provability in Logic*. Stockholm, Almqvist & Wiksell, 1957.

[81] R. Keefe. *Theories of Vagueness*. Cambridge University Press, 2000.

[82] S. C. Kleene. 'On Notation for Ordinal Numbers'. In: *The Journal of Symbolic Logic* 3.4 (1938), pp. 150–155.

[83] S. A. Kripke. 'Outline of a theory of truth'. In: *Journal of Philosophy* 72 (1975), pp. 690–716.

[84] S. A. Kripke. 'A Completeness Theorem in Modal Logic'. In: *Journal of Symbolic Logic* 24 (1959), pp. 1–14.

[85] S. A. Kripke. 'Semantical Analysis of Modal Logic (abstract)'. In: *Journal of Symbolic Logic* 24 (1959), pp. 323–324.

[86] S. A. Kripke. 'Semantical Analysis of Modal Logic I: Normal Modal Propositional Calculi'. In: *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 9 (1963), pp. 67–96.

[87] S. A. Kripke. 'Semantical Considerations on Modal Logic'. In: *Acta Philosophica Fennica* 16 (1963), pp. 83–94.

[88] R. E. Ladner. 'The Computational Complexity of Provability in Systems of Modal Propositional Logic'. In: *SIAM Journal on Computing* 6.3 (1977), pp. 467–480.

[89] J. Łukasiewicz. 'O logice trówartosciowej ("on three-valued logic")'. In: *Ruch filozoficzny* 5 (1920), pp. 170–171.

[90] J. Łukasiewicz. 'Zagadnienia prawdy ("the problems of truth")'. In: *Księga pamiątkowa XI zjazdu lekarzy i przyrodników polskich* (1922), pp. 84–85, 87.

[91] J. Łukasiewicz and A. Tarski. 'Untersuchungen über den Aussagenkalkül'. In: *Comp-tes Rendus de la Siciete des Sciences et des Lettres des Varsovie* 23 (1930), pp. 1–21.

[92] M. Marti and G Metcalfe. 'A Hennessy-Milner Property for Many-Valued Modal Logics'. In: *Advances in Modal Logic* 10 (2014), pp. 407–420.

[93] P. McNamara. 'Deontic Logic'. In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. Winter 2014. URL: `http://plato.stanford.edu/archives/win2014/entries/logic-deontic`.

[94] G. Metcalfe and F. Montagna. 'Substructural Fuzzy Logics'. In: *The Journal of Symbolic Logic* 72.3 (2007), pp. 834–864.

[95] G. Metcalfe and N. Olivetti. 'Towards a Proof Theory of Gödel Modal Logics'. In: *Logical Methods in Computer Science* 7.2:10 (2011). Ed. by M. Giese, pp. 1–27.

[96] G. Metcalfe, N. Olivetti and D. Gabbay. *Proof Theory for Fuzzy Logics*. Vol. 36. Applied Logic Series. New York: Springer, 2009.

[97] R. Montague. 'Universal grammar'. en. In: *Theoria* 36.3 (1970), pp. 373–398.

[98] P. S. Mostert and A. L. Shields. 'On the structure of semigroups on a compact manifold with boundary'. In: *Annals of Mathematics* 65 (1957), pp. 117–143.

[99] D. Mundici. 'Satisfiability in many-valued sentential logic is NP-complete'. In: *Theoretical Computer Science* 52.1-2 (1987), pp. 145–153.

[100] H. Ono. 'The Finite Embeddability Property for MTL'. Unpublished note.

[101] H. Ono and Y. Komori. 'Logics without the contraction rule'. In: *Journal of Symbolic Logic* 50 (1985), pp. 169–201.

[102] H. Ono. 'Substructural Logics and Residuated Lattices — an Introduction'. In: *50 Years of Studia Logica*. Ed. by V. F. Hendricks and J. Malinowski. Vol. 21. Trends in Logic. Dordrecht: Springer Netherlands, 2003, pp. 193–228.

[103] E. Pacuit. *Neighborhood Semantics for Modal Logic: an Introduction.* ESSLLI 2007 Course Notes, 2007. URL: `http://ai.stanford.edu/~epacuit/classes/esslli/nbhdesslli.pdf`.

[104] C. H. Papadimitriou. *Computational Complexity.* Addison-Wesley Publishing Company, Inc, 1995.

[105] S. Peters and D. Westerståhl. *Quantifiers in Language and Logic.* Clarendon Press, 2006.

[106] G. Priest. 'Paraconsistent Logic'. In: *Handbook of Philosophical Logic*. Ed. by D. M. Gabbay and F. Guenthner. Vol. 6. Dordrecht: Springer Netherlands, 2002, pp. 287–393.

[107] G. Priest. *An Introduction to Non-Classical Logic: From If to Is.* Cambridge University Press, 2008.

[108] M. O. Rabin. 'Decidability of Second-Order Theories and Automata on Infinite Trees'. In: *Transactions of the American Mathematical Society* 141 (1969), pp. 1–35.

[109] G. Restall. *An Introduction to Substructural Logics.* London: Routledge, 1999.

[110] R. Rodríguez and L. Godo. 'Modal uncertainty logics with fuzzy neighborhood semantics'. In: *IJCAI-13 Workshop on Weighted Logics for Artificial Intelligence (WL4AI-2013).* 2013, pp. 79–86.

[111] R. Rodríguez and L. Godo. 'On the fuzzy modal logics of belief $KD45(\mathcal{A})$ and $Prob(Ł_n)$: axiomatization and neighbourhood semantics'. In: *IJCAI-15 Workshop on Weighted Logics for Artificial Intelligence (WL4AI-2015).* 2015, pp. 64–71.

[112] A. Rose and J. B. Rosser. ‘Fragments of many-valued statement calculi’. In: *Transactions of the American Mathematical Society* 87 (1958), pp. 1–53.

[113] W. J. Savitch. ‘Relationships Between Nondeterministic and Deterministic Tape Complexities’. In: *Journal of Computer and System Sciences* 4.2 (1970), pp. 177–192.

[114] S. Schockaert, M. D. Cock and E. Kerre. ‘Spatial reasoning in a fuzzy region connection calculus’. In: *Artificial Intelligence* 173.2 (2009), pp. 258–298.

[115] D. Scott. ‘Advice on Modal Logic’. en. In: *Philosophical Problems in Logic*. Ed. by K. Lambert. Synthese Library 29. Springer Netherlands, 1970, pp. 143–173.

[116] S. Shapiro. *Vagueness in Context*. Oxford University Press, 2006.

[117] N. J. J. Smith. *Vagueness and Degrees of Truth*. Oxford University Press, 2009.

[118] G. Takeuti and T. Titani. ‘Intuitionistic fuzzy logic and intuitionistic fuzzy set theory’. In: *Journal of Symbolic Logic* 49.3 (1984), pp. 851–866.

[119] M. Vardi. ‘Why is modal logic so robustly decidable?’ In: *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* 31 (1997), pp. 149–184.

[120] Y. Venema. ‘Temporal Logic’. In: *The Blackwell Guide to Philosophical Logic*. Ed. by L. Goble. Blackwell Publishers, 2001, pp. 203–223.

[121] A. Vidal, F. Esteva and L. Godo. ‘On modal extensions of Product fuzzy logic’. In: *Journal of Logic and Computation* (2015). URL: `http://logcom.oxfordjournals.org/content/early/2015/08/10/logcom.exv046`.

[122] T. Williamson. *Vagueness*. London: Routledge, 1994.

[123] L. A. Zadeh. ‘Fuzzy sets’. In: *Information and Control* 8.3 (1965), pp. 338–353.

Erklärung

gemäss Art. 28 Abs. 2 RSL 05

Name/Vorname:	Rogger Jonas
Matrikelnummer:	06-125-959
Studiengang:	Mathematik
Art des Abschlusses:	Dissertation
Titel der Arbeit:	Decidability of Order-Based Modal Logics
LeiterIn der Arbeit:	Prof. Dr. George Metcalfe

Ich erkläre hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen benutzt habe. Alle Stellen, die wörtlich oder sinngemäss aus Quellen entnommen wurden, habe ich als solche gekennzeichnet. Mir ist bekannt, dass andernfalls der Senat gemäss Artikel 36 Absatz 1 Buchstabe r des Gesetzes vom 5. September 1996 über die Universität zum Entzug des auf Grund dieser Arbeit verliehenen Titels berechtigt ist. Ich gewähre hiermit Einsicht in diese Arbeit.

Bern, 05. Juli 2016 Jonas Rogger

Lebenslauf

Name:	Jonas Rogger
Heimatort:	Nebikon, LU
Geburtstag:	13. Dezember 1985

2012 – 2016	Doktorat in Mathematik an der Universität Bern Titel der Arbeit: “Decidability of Order-Based Modal Logics”
2015	Forschungsaufenthalt (5 Monate) an der Tschechischen Akademie der Wissenschaften (AV ČR) in Prag, Tschechische Republik
2010 – 2012	Masterstudium in Philosophie an der Universität Bern Titel der Arbeit: “Semantic Contextualism United” Minor: Mathematik
2011 – 2012	Forschungsaufenthalt (10 Monate) an der Universität von Oxford, England
2010	Erasmus-Mastersemester (6 Monate) an der Universität Helsinki, Finnland
2006 – 2010	Bachelorstudium in Philosophie an der Universität Bern Titel der Arbeit: “Neale’s Gödel’s Slingshot” Minor: Mathematik
2001 – 2005	Kantonsschule Wettingen
1997 – 2001	Bezirksschule Brugg
1992 – 1997	Primarschule Villigen