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Almost holomorphic Poincaré series corresponding to products of harmonic Siegel–Maass forms

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Abstract

We investigate Poincaré series, where we average products of terms of Fourier series of real-analytic Siegel modular forms. There are some (trivial) special cases for which the products of terms of Fourier series of elliptic modular forms and harmonic Maass forms are almost holomorphic, in which case the corresponding Poincaré series are almost holomorphic as well. In general, this is not the case. The main point of this paper is the study of Siegel–Poincaré series of degree 2 attached to products of terms of Fourier series of harmonic Siegel–Maass forms and holomorphic Siegel modular forms. We establish conditions on the convergence and nonvanishing of such Siegel–Poincaré series. We surprisingly discover that these Poincaré series are almost holomorphic Siegel modular forms, although the product of terms of Fourier series of harmonic Siegel–Maass forms and holomorphic Siegel modular forms (in contrast to the elliptic case) is not almost holomorphic. Our proof employs tools from representation theory. In particular, we determine some constituents of the tensor product of Harish-Chandra modules with walls.

Keywords: Almost holomorphic modular forms, Siegel modular forms, Harish-Chandra modules

Mathematics Subject Classification: Primary 11F46; Secondary 11F30, 11F37, 11F70

Modular forms have a rich history with many beautiful applications in different sciences. Restricting our attention to cusp forms, we can distinguish two classes of modular forms, which behave drastically differently: Maass cusp forms and almost holomorphic modular forms. Almost holomorphic forms modular forms were introduced independently by Shimura [31] and Kaneko and Zagier [20]. Shimura initiated their study, because of their special arithmetic properties. Kaneko and Zagier's paper was stimulated by Dijkgraaf [7], who conjectured a relation between almost holomorphic modular forms and numbers of topologically inequivalent branched covers of an elliptic curve. Research on the latter aspect has been particularly successful, establishing connections to Kac–Moody algebras and related representation theoretic objects [4,9].

Poincaré series play a major role in the theory of automorphic forms (for example, see [5,8,10] among many others). It is not difficult to construct almost holomorphic Poincaré series by observing that almost holomorphic modular forms vanish under a power of the

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lowering operator $L := -2iy^2\partial_{\bar{\tau}}$, where throughout $\tau = x+iy \in \mathbb{H}$. Specifically, the elliptic Poincaré series $\sum_{\gamma}(y^{-d}e^{2\pi in\tau})\big|_k \gamma$ converges for $n, d \in \mathbb{Z}$, $n > 0$, $k > 2$, $0 \leq d < \frac{k}{2} - 1$, and it is almost holomorphic, since L is equivariant with respect to the usual slash action $|_k$ and $L^{d+1}(y^{-d}e^{2\pi in\tau}) = 0$. Slightly more general, if $s \in \mathbb{C}$, then Selberg's [29] Poincaré series $\sum_{\gamma}\left((y^s)\cdot(e^{2\pi in\tau})\right)\big|_k \gamma$ corresponds to a product of terms of Fourier series. The factor y^s is the 0-th term of the Fourier series of a (weight 0) Eisenstein series, and the factor $e^{2\pi in\tau}$ is a term of Fourier series of a (weight k) holomorphic modular form. In general, if the product of two terms of Fourier series is not almost holomorphic, then the associated Poincaré series is not almost holomorphic either, provided that it is nonzero. We discuss two examples in more detail in Sect. 1.2.

Siegel modular forms impact many different areas of mathematics: Algebraic and arithmetic geometry, invariant theory, representation theory, quantum theory, and conformal field theory, for example. Almost holomorphic Siegel modular forms were recently classified in [13,26], and they play an important role in the context of mirror symmetry in [13].

The purpose of this paper is to construct almost holomorphic Siegel–Poincaré series of degree 2, where we average products of terms of Fourier series of (harmonic) Siegel–Maass forms, but where these products are not almost holomorphic themselves. More specifically, in (3.3) we define the Siegel–Poincaré series

$$\mathbb{P}^{(2)}_{k,\ell;T,T'}(Z) = \sum_{M}\left(\Psi_k(T;Z)\cdot\Phi_\ell(T';Z)\right)\big|_{k+\ell} M,$$

where k, ℓ are positive even integers, T, T' are positive definite and symmetric 2×2 matrices, and where $\Psi_k(T;Z)$ and $\Phi_\ell(T';Z)$ are the T-th and T'-th terms of Fourier series of a weight k harmonic Siegel–Maass form (as in [6,27]) and weight ℓ holomorphic Siegel modular form, respectively. We apply an estimate of [30] to prove that $\mathbb{P}^{(2)}_{k,\ell;T,T'}$ converges for all ℓ large enough. Moreover, we employ a result of [17] to show that $\mathbb{P}^{(2)}_{k,\ell;T,T'}$ does not vanish identically for all ℓ large enough. The following theorem is our main result.

Theorem I *Assume the generalized Ramanujan conjecture for* GL_4. *The function* $\Psi_k(T;Z)\cdot\Phi_\ell(T';Z)$ *is not almost holomorphic. If* $\ell \geq 6 + 2b - k$, *where* $b > 0$ *is defined in* (3.4), *then* $\mathbb{P}^{(2)}_{k,\ell;T,T'}$ *converges and is almost holomorphic.*

Remark (1) Poincaré series are the starting point for Kuznetsov-type [18] trace formulas. It would be interesting to determine the Fourier series coefficients and the spectral decomposition of $\mathbb{P}^{(2)}_{k,\ell;T,T'}$ to discover a novel Kuznetsov-type trace formula for Siegel modular forms.

(2) It is possible to define $\mathbb{P}^{(2)}_{k,\ell;T,T'}$ for indefinite T. One can show that these Poincaré series converge for sufficiently large ℓ and that they are almost holomorphic. However, unfolding the Petersson scalar product against an arbitrary almost holomorphic Siegel modular form yields that they vanish identically.

The paper is organized as follows: In Sect. 1, we illustrate how almost holomorphic Poincaré series arise in the setting of elliptic modular forms. In Sect. 2, we review real-analytic Siegel modular forms. In particular, we recall the notions of almost holomorphic Siegel modular forms and harmonic Siegel–Maass forms. In Sect. 3, we define the Siegel–Poincaré series $\mathbb{P}^{(2)}_{k,\ell;T,T'}$, and we determine conditions on its convergence and nonvanishing. In Sect. 4, we prove Theorem I using tools from real representation theory: The Poincaré series $\mathbb{P}^{(2)}_{k,\ell;T,T'}$ yields a cuspidal automorphic representation for PGSp_n.

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The $(\mathfrak{g}, K)$-module attached to its component at the infinite place embeds into the tensor product of two Harish-Chandra modules generated by Ψ_k and Φ_ℓ. This puts severe restrictions on $(\mathfrak{g}, K)$-modules arising in our context. Specifically, we employ Muić's [25] study of decompositions of generalized principal series to control Harish-Chandra parameters of $(\mathfrak{g}, K)$-modules that have "walls" in their K-type support. In addition, $\mathbb{P}_{k,\ell;T,T'}^{(2)}$ gives rise to a scalar K-type. Consequently, we can invoke Arthur's [2] endoscopic classification of representations of $\mathrm{Sp}_2(\mathbb{R})$ to narrow down possibilities to holomorphic (limits of) discrete series.

1 Poincaré series for elliptic modular forms

We consider cuspidal Poincaré series in the case of elliptic modular forms to demonstrate possible phenomena and to explain available tools to investigate such series. From the introduction, recall the almost holomorphic Poincaré series attached to the function

$$\phi_{k[d]}(n; \tau) := y^{-d} e^{2\pi i n\tau}, \quad n > 0. \tag{1.1}$$

Note that $\phi_{k[d]}$ does not depend on k. Nevertheless, we also include it in the notation to indicate that it is a typical term of a weight k almost holomorphic modular form. If $d = 0$, then we write $\phi_k(n; \tau) := \phi_{k[0]}(n; \tau)$.

To define elliptic Poincaré series, set $\Gamma_\infty^{(1)} := \left\{ \pm \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right), b \in \mathbb{Z} \right\}$, and recall the elliptic slash action $(f|_k \gamma)(\tau) := (c\tau + d)^{-k} f(\frac{a\tau+b}{c\tau+d})$ for $f : \mathbb{H} \to \mathbb{C}$, $k \in \mathbb{Z}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z})$. Products of almost holomorphic functions are almost holomorphic. Hence, it is trivial that the Poincaré series

$$P_{k[d],\ell;n,m}^{(1)}(\tau) := \sum_{\gamma \in \Gamma_\infty^{(1)} \backslash \mathrm{SL}_2(\mathbb{Z})} \left(\phi_{k[d]}(n; \tau) \, \phi_\ell(m; \tau) \right) \big|_{k+\ell} \gamma \tag{1.2}$$

is almost holomorphic, provided that it converges, which is the case if $k + \ell > 2 + 2d$. In the next two sections, we will demonstrate that analogous Poincaré series are, in general, not almost holomorphic.

1.1 Poincaré series that are not almost holomorphic

Consider a typical term of the nonholomorphic part of a weight k harmonic weak Maass form:

$$\widetilde{\psi}_k(n; \tau) := \Gamma(1 - k, 4\pi |n| y) \, e^{2\pi i n\tau}, \quad k \le 0, \, n < 0, \tag{1.3}$$

where Γ is the usual incomplete gamma function. Note that $\widetilde{\psi}_k(n; \cdot)$ decays rapidly toward infinity, but the Poincaré series $\sum_\gamma \widetilde{\psi}_k(n; \tau) \big|_k \gamma$ does not converge, due to its behavior as $y \to 0$. For the remainder of this section, let $\ell \in \mathbb{Z}$ such that $k + \ell > 2$, in which case the Poincaré series

$$\widetilde{\mathbb{P}}_{k,\ell;n,m}^{(1)}(\tau) := \sum_{\gamma \in \Gamma_\infty^{(1)} \backslash \mathrm{SL}_2(\mathbb{Z})} \left(\widetilde{\psi}_k(n; \tau) \, \phi_\ell(m; \tau) \right) \big|_{k+\ell} \gamma \tag{1.4}$$

converges.

Our main tool in this paper—see Sect. 4—is the theory of Harish-Chandra modules (i.e., real representation theory). We will apply it to the case of elliptic modular forms in Sect. 1.4. Guided by the emphasis of real representation theory on linear differential operators, we consider the lowering operator L instead of Bruinier and Funke's [3] ξ-operator. The ξ-operator and L map smooth functions on $\mathbb{H}$ to smooth functions on $\mathbb{H}$,

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and are defined by

$$\xi_k(f) := 2iy^k \overline{\partial_{\overline{\tau}} f} \quad \text{and} \quad L := L_k := -2iy^2 \partial_{\overline{\tau}},$$

where $\partial_{\overline{\tau}} f = \partial f/\partial \overline{\tau}$ and $\partial_\tau f = \partial f/\partial \tau$. The space $\mathbb{M}_k^{(1)}$ of harmonic Maass forms is mapped by ξ_k to $M_{2-k}^{(1)}$, the space of holomorphic modular forms. Equivalently, it is mapped by L_k to $y^{2-k}\overline{M_{2-k}^{(1)}}$, which is the space of antiholomorphic modular forms of antiholomorphic weight $2 - k$ normalized by y^{2-k} to holomorphic weight $k - 2$. Terms of Fourier series are related by the formula $L_k\left(\widetilde{\psi}_k(n;\tau)\right) = (4\pi|n|)^{1-k}\widetilde{\phi}_{k-2}(-n;\tau)$ with

$$\widetilde{\phi}_k(n;\tau) := y^{-k}\overline{\phi_{-k}(n;\tau)} = y^{-k}e^{-2\pi i n\overline{\tau}}, \quad k < 0, \, n < 0. \tag{1.5}$$

The image of $\widetilde{\mathbb{P}}_{k,\ell;n,m}^{(1)}$ under mL_k is therefore a scalar multiple of

$$\widetilde{P}_{k-2,\ell;n,m}^{(1)}(\tau) := \sum_{\gamma \in \Gamma_\infty^{(1)} \backslash \mathrm{SL}_2(\mathbb{Z})} \left(\widetilde{\phi}_{k-2}(n;\tau)\,\phi_\ell(m;\tau)\right)\big|_{k-2+\ell}\gamma. \tag{1.6}$$

For simplicity, we focus on $|m| > |n|$. In the next section, we find that *neither* $\widetilde{P}_{k,\ell;n,m}^{(1)}$ nor $\widetilde{\mathbb{P}}_{k,\ell;n,m}^{(1)}$ is almost holomorphic.

1.2 Spectral decomposition of Poincaré series

It is an important and nontrivial task to find the spectral decomposition of an automorphic form. The spectrum of the weight k hyperbolic Laplace operator on $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}, |_k)$ consists of four different contributions: (1) Eisenstein series $E_{k,s}$ with spectral parameter $s = 1/2 + it$, $t \in \mathbb{R}$. (2) The residual spectrum, which arises from unary theta series. (3) Almost holomorphic and almost antiholomorphic cusp forms. (4) Proper Maass cusp forms. The Poincaré series that we consider are all cuspidal, and so Eisenstein series do not occur in their spectral decomposition. Furthermore, the residual spectrum occurs only for half-integral weight, and therefore, it does not contribute to the weights that we treat. Consequently, Parseval's equation implies that if f is one of the Poincaré series, then

$$f = \sum_j \langle g_j, f \rangle g_j + \sum_j \langle u_j, f \rangle u_j$$

where g_j runs through a (finite) complete orthonormal set of almost holomorphic and antiholomorphic modular forms, and u_j runs through an (infinite) complete orthonormal set of proper Maass cusp forms.

We now show that $\widetilde{P}_{k,\ell;n,m}^{(1)}$ in (1.6) and $\widetilde{\mathbb{P}}_{k,\ell;n,m}^{(1)}$ in (1.4) are not almost holomorphic. More precisely, we show that both have spectral expansions with infinite support. The relation

$$L_k\left(\widetilde{\mathbb{P}}_{k,\ell;n,m}^{(1)}(\tau)\right) = (4\pi|n|)^{1-k}\,\widetilde{P}_{k-2,\ell;n,m}^{(1)}(\tau)$$

allows us to examine only $\widetilde{P}_{k,\ell;n,m}^{(1)}$.

To exhibit the spectral decomposition, suppose that u is a Maass cusp form with spectral parameter s with $Re(s) > \frac{1}{2}$. Without loss of generality, we can raise its weight to weight $k + \ell$, such that its spectral parameter with respect to the weight $k + \ell$ Laplace operator is $s - (k + \ell)/2$. The Fourier series expansion of u has the form

$$u(\tau) = \sum_{n \in \mathbb{Z}\backslash\{0\}} c(u; n)\,(4\pi|n|y)^{-\frac{k+\ell}{2}}\,W_{\mathrm{sgn}(n)\frac{k+\ell}{2}, s-\frac{1}{2}}(4\pi|n|y)e^{2\pi i nx},$$

where $W_{\mu,\nu}$ denotes the usual W-Whittaker function. We unfold the Petersson scalar product of the Poincaré series $\widetilde{P}_{k,\ell;n,m}^{(1)}$ against u, and obtain by 7.621.3 of [11] that $\langle u, \widetilde{P}_{k,\ell;n,m}^{(1)} \rangle$ equals

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$$c(u; m - n) \int_0^\infty y^{-k} e^{-2\pi(m-n)y} \left(4\pi (m - n)y\right)^{-\frac{k+\ell}{2}} W_{\frac{k+\ell}{2}, s - \frac{1}{2}}(4\pi(m - n)y) \frac{\mathrm{d}y}{y^{2-(k+\ell)}}$$

$$= c(u; m - n) \, (4\pi(m-n))^{1-l} \, \frac{\Gamma\left(\frac{\ell-k}{2} + s - 1\right) \Gamma\left(\frac{\ell-k}{2} - s\right)}{\Gamma(-k)}$$

$$\times {}_2\mathrm{F}_1 \left(\frac{\ell-k}{2} + s - 1, s - \frac{k+\ell}{2}, -k; 0\right).$$

The hypergeometric series evaluated at 0 equals 1. The gamma factors do not vanish, and all Maass cusp forms with $c(u; m - n) \neq 0$ contribute to the spectral expansion of f. By Theorem 1.2 of [12], there are infinitely many such Maass cusp forms.

1.3 Almost holomorphic Poincaré series

In this section, we suggest another Poincaré series that is almost holomorphic. In light of what we discuss in Sect. 1.4, it is natural to replace $\widetilde{\psi}_k$ by

$$\psi_k(n; \tau) := y^{-k} \, \Gamma(1 + k, 4\pi n y) \, e^{2\pi i n \overline{\tau}}, \quad k \geq 0, \quad n > 0. \tag{1.7}$$

It is a term of the Fourier series of an "anti-harmonic" Maass form in $y^{-k} \overline{\mathbb{M}}^{(1)}_{-k}$, which is a space of functions that are mapped by the raising operator

$$R_k := 2i\partial_\tau + k y^{-1} \tag{1.8}$$

to holomorphic modular forms of weight $k + 2$.

Define the Poincaré series

$$\mathbb{P}^{(1)}_{k,\ell;n,m}(\tau) := \sum_{\gamma \in \Gamma^{(1)}_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \left(\psi_k(n; \tau) \, \phi_\ell(m; \tau)\right)\big|_{k+\ell} \gamma, \tag{1.9}$$

which converges if $\ell - k > 2$. Observe that

$$\psi_k(n; \tau) \, \phi_\ell(m; \tau) = y^{-k} \Gamma(1 + k, 4\pi n y) e^{2\pi i n \overline{\tau}} e^{2\pi i m \tau} = p(y^{-1}) e^{2\pi i (n+m)\tau},$$

where p is a polynomial of degree k. Applying the lowering operator $k+1$ times annihilates this product, i.e., $\mathbb{P}^{(1)}_{k,\ell;n,m}$ is almost holomorphic.

Finally, it is easy to see that $P^{(1)}_{k[d],\ell;n,m}$ [defined in (1.2)] is in the kernel of L^{d+1}, i.e., it is also almost holomorphic.

1.4 Harish-Chandra modules

The spectral decomposition of an automorphic form implies a decomposition of the associated Harish-Chandra module. Vice versa, one can deduce from the decomposition of the Harish-Chandra module attached to an automorphic form which parts of the spectrum contribute to its spectral expansion. In particular, it is possible to infer from the Harish-Chandra module alone whether an automorphic form is almost holomorphic. In this section, we give some details and examples. A precise statement in the case of automorphic forms for $\mathrm{Sp}_2(\mathbb{R})$ is presented in Sect. 4.

We reconsider the spectral decomposition of the Poincaré series $P^{(1)}_{k[d],\ell;n,m}$ and $\mathbb{P}^{(1)}_{k,\ell;n,n'}$ in light of Harish-Chandra modules. The Poincaré series in (1.4) cannot be completely analyzed, and their $(\mathfrak{g}, K)$-modules are not Harish-Chandra modules. An excellent treatment of Harish-Chandra modules can, for example, be found in [15,32]. To accommodate the classically inclined reader, we suggest a rather simple schematic way of thinking of Harish-Chandra modules in terms of lowering and raising operators, but for simplicity we suppress details on the correspondence of Harish-Chandra modules and the $\mathbb{C}[L, R]$-modules that we employ in this section. We will be more precise in Sect. 4, when we discuss the case of $\mathrm{Sp}_2(\mathbb{R})$.

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Given an even weight k and a function f, we study the action of $\mathbb{C}[L, R]$ where L and R are abstract lowering operators on the pair (f, k). The action is precisely given by $L(f, k) = (L_k(f), k - 2)$ and $R(f, k) = (R_k(f), k + 2)$, where L_k and R_k are the usual lowering and raising operators, as before given by

$$L_k := -2iy^2 \partial_{\overline{\tau}} \quad \text{and} \quad R_k := 2i\partial_\tau + ky^{-1}.$$

By viewing the second component of (f, k) as a grading, it makes sense to speak of the graded module $\mathbb{C}[L, R]f$, and study for which k its k-th graded component is nonzero. We are further interested in the action of the lowering and raising operators on the graded components of $\mathbb{C}[L, R](f, k)$. We illustrate the typical behavior on the function y^s with $s \in \mathbb{C}$. We clearly have $L_k(y^s) = sy^{s+1}$ and $R_k(y^s) = (s + k)y^{s-1}$. If $s = 0$ or $s = -k$, then y^s vanishes under L_k or R_k, respectively. Rewriting this in terms of pairs (f, k), we find that $L(y^s, 0) = (sy^{s+1}, -2)$ and $R(y^s, 0) = (sy^{s-1}, 2)$.

We now exhibit the behavior of terms of the Fourier series in (1.1), (1.3), (1.5), and (1.7) under lowering and raising operators. For any n, we have $L(\phi_k(n; \tau), k) = 0$, and no power of R annihilates $(\phi_k(n; \tau), k)$. The situation for $\psi_k(n; \tau), k)$ is similar. We have $L^{k+1}(\psi_k(n; \tau), k) = (0, -k - 2)$. In addition, $LR(\psi_k(n; \tau), k) = 0$. The behavior of $\widetilde{\phi}_k$ and $\widetilde{\psi}$ is analogous.

We now introduce a schematic way to describe these modules. Every graded component which is nonzero corresponds to a filled circle. Graded components that are zero correspond to (small) circles. Both are placed on a line to emphasize the $2\mathbb{Z}$-grading. We further encircle one dot that corresponds to the graded component, which we are focusing on. For example, in the case of $(y^{-d}, 0)$ we would encircle the 0-th graded component, and when considering (ϕ_k, k) we encircle the k-th one. Finally, we separate $\mathbb{C}[R, L]$-submodules by vertical lines decorated with an arrow. It means that applying the lowering operator (if the arrow points rightwards) or the raising operator (if the arrow points leftwards) applied to this graded component equals zero. In addition, we insert vertical dashed lines to indicate the relation to walls in so-called principal series representations. Here are diagrams for $\phi_{k[d]}$, ψ_k, $\widetilde{\phi}_k$, and $\widetilde{\psi}_k$.
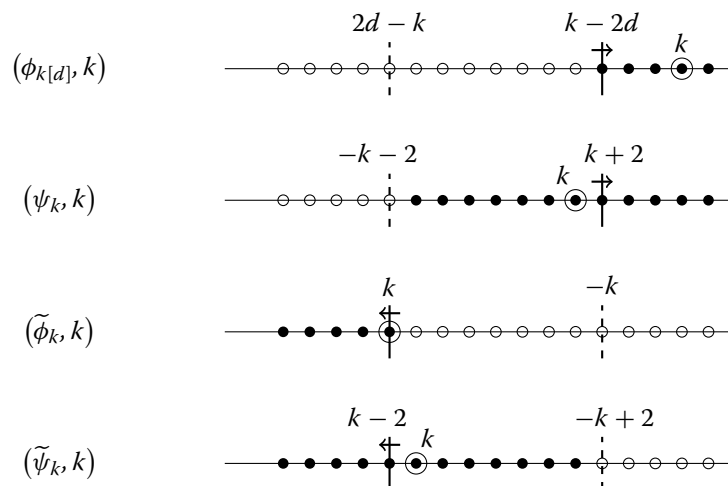
We now come back to the spectral decomposition of Poincaré series. We illustrate our approach in the first case $P_{k[d],\ell;n,m}$. It is central for the more general discussion in Sect. 4 to decompose the action of lowering operators and raising operators into two parts

according to the following analogue to the Leibniz product rule.

$$
\begin{aligned}
&L_{k+\ell}\left(\phi_{k[d]}(n;\tau)\cdot\phi_\ell(m;\tau)\big|_{k+\ell}\,\gamma\right)\\
&= L_{k+\ell}\left(\phi_{k[d]}(n;\tau)\big|_k\,\gamma\cdot\phi_\ell(m;\tau)\big|_\ell\,\gamma\right)\\
&= L_k\left(\phi_{k[d]}(n;\tau)\big|_k\,\gamma\,\right)\cdot\phi_\ell(m;\tau)\big|_\ell\,\gamma\,+\,\phi_{k[d]}(n;\tau)\big|_k\,\gamma\cdot L_\ell\left(\phi_\ell(m;\tau)\big|_\ell\,\gamma\right).
\end{aligned}
$$

An analogous formula holds for the raising operator. Applying both formulas iteratively yields that

$$
\begin{aligned}
\mathbb{C}[L,R]\left(P_{k[d],\ell;n,m},\,k+\ell\right) &\subseteq \mathbb{C}[L,R]\left(\phi_{k[d]}(n;\cdot)\cdot\phi_\ell(m;\cdot),\,k+\ell\right)\\
&\subseteq \left(\mathbb{C}[L,R]\left(\phi_{k[d]}(n;\cdot),\,k\right)\right)\otimes\left(\mathbb{C}[L,R]\left(\phi_\ell(m;\cdot),\,\ell\right)\right).
\end{aligned}
$$

The tensor product of (f,k) and (g,ℓ) is defined as $(fg,k+\ell)$, which is in accordance with the product $f|_k\,\gamma\cdot g|_\ell\,\gamma=(fg)|_{k+\ell}\,\gamma$.

With this machinery, we can now read off an upper bound for the support of the graded module $\mathbb{C}[L,R]\left(P_{k[d],\ell;n,m},\,k+\ell\right)$. We have already given diagrams for the left and right tensor component that correspond to it. All nonzero graded components are 1-dimensional, and we find that the weights of the nonzero graded components in the tensor product are at least $k-2d+\ell$.

As a last step, we use the classification of Harish-Chandra modules for $\mathrm{SL}_2(\mathbb{R})$: (Limits of) holomorphic discrete series are the only Harish-Chandra admissible modules which have a lowest weight. Therefore, the Harish-Chandra module attached to $P_{k[d],\ell;n,m}$ is a (finite) direct sum of (limits of) holomorphic discrete series. Graded components of such discrete series correspond to almost holomorphic modular forms, and this proves again what we have already observed in Sect. 1.3: The Poincaré series $P_{k[d],\ell;n,m}$ decomposes as a finite sum of almost holomorphic modular forms.

The inconclusive cases To analyze the Poincaré series in (1.4) and (1.6), we have to consider the tensor product of the $\mathbb{C}[L,R]$-modules generated by $\widetilde{\phi}_k$ and ϕ_ℓ, and by $\widetilde{\psi}_k$ and ϕ_ℓ, respectively. These tensor products are supported on all weights and all weight spaces are infinite dimensional. Thus, we cannot deduce anything definite. Nevertheless, this at least suggests that infinitely many Maass cusp forms appear in the spectral decomposition, which is what we verified directly in Sect. 1.2.

2 Real-analytic Siegel modular forms

We start by introducing necessary notation to define Siegel modular forms. Let $I=I^{(2)}$ be the 2×2 identity matrix, $\Gamma:=\Gamma^{(2)}:=\mathrm{Sp}_2(\mathbb{Z})$ be the symplectic group of degree 2 over $\mathbb{Z}$, $\mathbb{H}^{(2)}$ be the Siegel upper half space of degree 2, and let $Z=X+iY\in\mathbb{H}^{(2)}$ be a typical variable. If $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma$ and $Z\in\mathbb{H}^{(2)}$, then

$$
M\bullet Z:=(AZ+B)(CZ+D)^{-1}.
$$

Furthermore, if $F:\mathbb{H}^{(2)}\to\mathbb{C}$ and if $k\in\mathbb{Z}$, then

$$
\left(F\big|_k\,M\right)(Z):=\det(CZ+D)^{-k}\,F(M\bullet Z) \tag{2.1}
$$

for all $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma$.

Definition 2.1 A holomorphic (degree 2) Siegel modular form of weight k on Γ is a holomorphic function $F:\mathbb{H}^{(2)}\to\mathbb{C}$ such that, for all $M\in\Gamma$, $F\big|_k\,M=F$.

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2.1 Almost holomorphic Siegel modular forms

Almost holomorphic Siegel modular forms were introduced by Shimura [31]. In degree 2, they were classified in [13,26]. Write $Z = (z_{ij})$ and set $\partial_{\overline{Z}} := \left(\frac{1}{2}(1 + \delta_{ij})\partial_{\overline{z_{ij}}}\right)$ to define the lowering operator

$$L := L^{(2)} := Y^{\,t}\left(Y\partial_{\overline{Z}}\right). \tag{2.2}$$

By the d-th power of L, we mean its d-th tensor power.

Definition 2.2 An almost holomorphic (degree 2) Siegel modular form of weight k and depth d on Γ is a real-analytic function $F : \mathbb{H}^{(2)} \to \mathbb{C}$ satisfying the following conditions:

(1) For all $M \in \Gamma, F\big|_k M = F$.
(2) We have that $L^{d+1}(F) = 0$.

2.2 Harmonic Siegel–Maass forms

In [6], we introduced a certain space of harmonic (skew) Siegel–Maass forms, and we proved a connection of this space to the space of harmonic skew-Maass–Jacobi forms. In particular, we answered a question of Kohnen [16] on how skew-holomorphic Jacobi forms are related to real-analytic Siegel modular forms. We now introduce the (α, β)-slash action and the corresponding matrix-valued Laplace operator to recall the definition of harmonic (skew) Siegel–Maass forms in [6].

If $F : \mathbb{H}^{(2)} \to \mathbb{C}$, and if $\alpha, \beta \in \mathbb{C}$ such that $\alpha - \beta \in \mathbb{Z}$, then

$$\left(F\big|_{(\alpha,\beta)} M\right)(Z) := \det(CZ + D)^{-\alpha} \det(C\overline{Z} + D)^{-\beta} F(M \bullet Z) \tag{2.3}$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. If $\beta = 0$, we have $\left(F\big|_\alpha M\right)(Z) = \left(F\big|_{(\alpha,\beta)} M\right)(Z)$. As before, write $Z = (z_{ij})$ and set $\partial_Z := \left(\frac{1}{2}(1 + \delta_{ij})\partial_{z_{ij}}\right)$ to define the Laplace operator

$$\Omega_{\alpha,\beta} := -4Y^{\,t}\left(Y\partial_{\overline{Z}}\right)\partial_Z - 2i\beta Y\partial_Z + 2i\alpha Y\partial_{\overline{Z}}, \tag{2.4}$$

which is equivariant with respect to the action in (2.3) (see [24] for details). For the remainder, assume that κ is an odd integer such that $\kappa \notin \{1, 3\}$.

Definition 2.3 A harmonic (skew) Siegel–Maass form of weight κ on Γ is a real-analytic function $F : \mathbb{H}^{(2)} \to \mathbb{C}$ satisfying the following conditions:

(1) For all $M \in \Gamma, F\big|_{\left(\frac{1}{2}, \kappa - \frac{1}{2}\right)} M = F$.
(2) We have that $\Omega_{\frac{1}{2}, \kappa - \frac{1}{2}}(F) = 0$.
(3) We have that $|F(Z)| \leq c\,\mathrm{tr}(Y)^a$ for some $a, c > 0$ as $\mathrm{tr}(Y) \to \infty$.

Let $\mathbb{M}_\kappa^{\mathrm{sk}}$ denote the space of such harmonic Siegel–Maass forms of weight κ.

Remark 2.4 In [6], we only focused on (skew) Siegel–Maass forms of type $\left(\frac{1}{2}, \kappa - \frac{1}{2}\right)$, in order to establish links to the spaces of skew-Maass–Jacobi forms (if $\kappa < 0$) and skew-holomorphic Jacobi forms (if $\kappa > 3$). Nevertheless, many results of [6] extend to "holomorphic weights," which are more natural from a representation theoretic perspective. In fact, if $F \in \mathbb{M}_\kappa^{\mathrm{sk}}$, then the form $\det(Y)^{\kappa - 1/2}F(Z)$ has weight $k := 1 - \kappa$, i.e., it is invariant under (2.1) with $k = 1 - \kappa$, where k is an even integer such that $k \notin \{0, -2\}$. For convenience, we set

$$\mathbb{M}_k := \mathbb{M}_k^{(2)} := \det(Y)^{\frac{1}{2} - k} \mathbb{M}_{1-k}^{\mathrm{sk}}. \tag{2.5}$$

Let $\Gamma_\infty := \Gamma_\infty^{(2)} := \left\{ \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right) \in \Gamma \right\}$. Recall Maass' [23,24] nonholomorphic Eisenstein series

$$E_{\alpha,\beta}(Z) := \sum_{M \in \Gamma_\infty \backslash \Gamma} 1 \big|_{(\alpha,\beta)} M \tag{2.6}$$

and also the Poincaré–Eisenstein series

$$P_{\kappa,s}(Z) := \sum_{M \in \Gamma_\infty \backslash \Gamma} \det(Y)^s \big|_{\left(\frac{1}{2}, \kappa - \frac{1}{2}\right)} M. \tag{2.7}$$

Then, $P_{\kappa,s} = \det(Y)^s E_{s+1/2, s+\kappa-1/2}$, and in [6], we stated the following fact.

Proposition 2.5 *If $s = 0$ ($\kappa > 3$) or $s = \frac{3}{2} - \kappa$ ($\kappa < 0$), then $P_{\kappa,s} \in \mathbb{M}_\kappa^{\mathrm{sk}}$. In other words, we have*

$$\det(Y)^{\frac{1}{2}-k} P_{1-k,0} \in \mathbb{M}_k \qquad \text{for } k < -2;$$

$$\det(Y)^{\frac{1}{2}-k} P_{1-k,\frac{1}{2}+k} \in \mathbb{M}_k \qquad \text{for } k > 1.$$

We end this section with a remark on terms of the Fourier series of harmonic Siegel–Maass forms.

Remark 2.6 In [6], we determined Fourier series expansions of harmonic Siegel–Maass forms: For any nondegenerate, symmetric, half-integral 2×2 matrix T there exist functions $\Psi_k(T; Z)$ such that the T-th term of the Fourier series of any $F \in \mathbb{M}_k$ is given by

$$c(F; T) \Psi_k(T; Z), \qquad \text{where } c(F; T) \in \mathbb{C}.$$

The ideas in the proof of Proposition 4.6 can be used to show that for $k < -2$, $\Psi_k(T; Z) = 0$ for positive definite T.

3 Real-analytic Siegel–Poincaré series

In this section, we define Poincaré series attached to products of terms of Fourier series of holomorphic Siegel modular forms and harmonic Siegel–Maass forms. Recall from Proposition 2.5 that $\det(Y)E_{1+k,1} \in \mathbb{M}_k$. We can use its Fourier series to define $\Psi_k(T; Z)$ in Remark 2.6.

If $Re(\alpha), Re(\beta) > \frac{1}{2}$, then (see [24]) the terms of the Fourier series of the Eisenstein series $E_{\alpha,\beta}$ are, up to scalar multiples, given by

$$h_{\alpha,\beta}(T; Y) e^{2\pi i \operatorname{tr}(TX)},$$

where

$$h_{\alpha,\beta}(T; Y) := \iint\limits_{U \pm T > 0} \det(U + T)^{\alpha - \frac{3}{2}} \det(U - T)^{\beta - \frac{3}{2}} e^{-2\pi \operatorname{tr}(YU)} \, dU. \tag{3.1}$$

For the remainder, let T be positive definite, which we denote by $T > 0$. Set

$$\Psi_k(T; Z) := \det(TY) h_{k+1,1}(T; Y) e^{2\pi i \operatorname{tr}(TX)} \quad \text{and} \quad \Phi_\ell(T; Z) := e^{2\pi i \operatorname{tr}(TZ)}, \tag{3.2}$$

and for $T' > 0$ and positive even integers k and ℓ define the Poincaré series

$$P_{k,\ell; T,T'}^{(2)}(Z) := \sum_{M \in \Delta \backslash \Gamma} \left(\Psi_k(T; Z) \cdot \Phi_\ell(T'; Z) \right) \big|_{k+\ell} M, \tag{3.3}$$

where Δ is the subgroup of Γ defined by

$$\Delta := \left\{ \left(\begin{smallmatrix} I & B \\ 0 & I \end{smallmatrix} \right), B \in \operatorname{Mat}_2(\mathbb{Z}), \, {}^t B = B \right\}.$$

Observe that $\Phi_k(T; Z + B) = \Phi_k(T; Z)$ and $\Psi_\ell(T'; Z + B) = \Psi_\ell(T'; Z)$, and one finds that $P_{k,\ell; T,T'}^{(2)}$ is well defined.

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3.1 Convergence

We determine the convergence of $P^{(2)}_{k,\ell;\,T,T'}(Z)$ by comparing it to the Poincaré series

$$P_{k',T}(Z) := \sum_{M \in \Delta \backslash \Gamma} e^{2\pi i \operatorname{tr}(TZ)}\big|_{k'} M,$$

which converges absolutely and uniformly on compact subsets of $\mathbb{H}^{(2)}$ for even integers $k' \geq 6$ (see, for example, Proposition 3 on page 85 of [14]).

Consider $\det(Y) h_{k+1,1}(T; Y)$, where $T > 0$ and $Y > 0$. Let $0 < \lambda_1 \leq \lambda_2$ be the eigenvalues of TY. Then $\det(TY) = \lambda_1 \lambda_2$, $\operatorname{tr}(TY) = \lambda_1 + \lambda_2$, and $\lambda_1 \geq \frac{\det(TY)}{\operatorname{tr}(TY)}$.

Shimura [30] studied a function $\omega(g, h; \alpha, \beta)$, which is closely related to (3.1). Specifically, if $g = 2\pi Y$, $h = T$, $\alpha = k + 1$, and $\beta = 1$, then

$$\omega(g, h; \alpha, \beta) := \frac{2^{-2-2k}}{\pi} \det(2\pi TY)^{\frac{1}{2}-k} \det(2\pi Y)^{k+\frac{1}{2}} \cdot h_{1+k,1}(T; Y).$$

Shimura established an estimate for ω, which implies that there exist constants $a, b > 0$ (depending on α and β, i.e., on k) such that

$$\begin{aligned}
\det(TY) &h_{k+1,1}(T; Y) \\
&\leq \det(TY)\, 2^{2k+2}\pi \det(2\pi TY)^{k-1/2} \det(2\pi Y)^{-k-1/2}\, a\, e^{-2\pi \operatorname{tr}(TY)} \left(1 + \lambda_1^{-b}\right) \\
&\leq c\, e^{-2\pi \operatorname{tr}(TY)} \left(1 + \left(\frac{\operatorname{tr}(TY)}{\det(TY)}\right)^b\right),
\end{aligned} \tag{3.4}$$

where $c := a\, 2^{1+2k} \det(T)^{k-b+1/2}$. Note that $\operatorname{tr}(TY)^b\, e^{-2\pi \operatorname{tr}(TY)} \leq d e^{-\pi \operatorname{tr}(TY)}$ for some constant $d > 0$. Thus,

$$\det(TY) h_{k+1,1}(T; Y) \ll \left(1 + \det(Y)^{-b}\right) e^{-\pi \operatorname{tr}(TY)}, \tag{3.5}$$

and one finds that $P^{(2)}_{k,\ell;\,T,T'}(Z)$ in (3.3) is dominated by

$$\sum_{M \in \Delta \backslash \Gamma} \left(\left(1 + \det(Y)^{-b}\right) e^{-\pi \operatorname{tr}(\widetilde{T}Y)}\right)\big|_{k+\ell} M \tag{3.6}$$

with $\widetilde{T} := T + T' > 0$. We conclude that $P^{(2)}_{k,\ell;\,T,T'}(Z)$ converges absolutely and uniformly on compact subsets of $\mathbb{H}^{(2)}$ if $\ell + k - 2b \geq 6$.

3.2 Nonvanishing

In this section, we show that $\lim_{\ell \to \infty} P^{(2)}_{k,\ell;\,T,T'}(iy_0 I) > 0$ for some $y_0 > 1$, which implies that $\mathbb{P}^{(2)}_{k,\ell;T,T'}$ does not vanish identically for all ℓ large enough. We apply the following lemma of Kowalski, Saha, and Tsimerman [17], where $U(y_0)$ stands for some neighborhood of $iy_0 I$ ($y_0 > 0$).

Lemma 3.1 ([17]) *There exists a real number $y_0 > 1$ such that for any $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ with $C \neq 0$ and for all $Z \in U(y_0)$, we have $|\det(CZ + D)| > 1$.*

Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. Suppose that $C \neq 0$. Choose $y_0 > 1$ as in Lemma 3.1 and consider $Z = iy_0 I$. Then $|\det(CZ + D)| > 1$, and if $T > 0$, then

$$\det(CZ + D)^{-k-\ell} \left(1 + |\det(CZ + D)|^{2b}\right) e^{2\pi i \operatorname{tr}(T\,M\bullet Z)} \xrightarrow{\ell \to \infty} 0.$$

Recall that (3.6) is a majorant of (3.3), which converges for all $\ell \geq 6 - k + 2b$, and we find that

$$\sum_{\substack{M = \left(\begin{smallmatrix} * & * \\ C & * \end{smallmatrix}\right) \in \Delta \backslash \Gamma \\ C \neq 0}} \left(\left(\Psi_k(T; Z) \cdot \Phi_\ell(T'; Z)\right)\right)\big|_{k+\ell} M \xrightarrow{\ell \to \infty} 0.$$

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Finally, suppose that $C = 0$. Then $D = {}^{t}A^{-1}$. Moreover, we factor out by Δ, and hence, we may (and do) assume that $B = 0$. If $M = \begin{pmatrix} A & 0 \\ 0 & {}^{t}A^{-1} \end{pmatrix}$ and again $Z = iy_0 I$, then

$$\det\left(\mathrm{Im}\left(M \bullet iy_0 I\right)\right) = \det\left(A\left(y_0 I\right){}^{t}A\right) = y_0^2 \det(A)^2 = y_0^2.$$

Thus,

$$\sum_{M = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \Delta \backslash \Gamma} \left(\left(\Psi_k(T; Z) \cdot \Phi_\ell(T'; Z)\right)\right)\Big|_{k+\ell} M$$

$$= y_0^2 \sum_{A \in \mathrm{GL}_2(\mathbb{Z})} \det(A)^{-k-\ell} h_{k+1,1}(T, y_0 I) e^{-2\pi y_0 \,\mathrm{tr}(\widetilde{T} A {}^{t}A)},$$

which is positive (k and ℓ are even), where again $\widetilde{T} := T + T' > 0$. In particular,

$$\lim_{\ell \to \infty} P_{k,\ell;T,T'}^{(2)}(iy_0 I) > 0.$$

4 (g, K)-modules generated by Poincaré series

Throughout this section, we focus on $(\mathfrak{g}, K)$-modules for $G := \mathrm{Sp}_2(\mathbb{R})$. We diverge from the classical notation $M \in \mathrm{Sp}_2(\mathbb{R})$ in favor of the representation theoretic notation $g \in G$. Recall the realization of the symplectic group as

$$\mathrm{Sp}_2(\mathbb{R}) = \left\{g \in \mathrm{Mat}_4(\mathbb{R}) : {}^{t}g J^{(2)} g = J^{(2)}\right\}, \quad J^{(2)} = \begin{pmatrix} 0 & -I^{(2)} \\ I^{(2)} & 0 \end{pmatrix},$$

with maximal compact subgroup

$$K = \left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} : A + iB \in \mathrm{U}_2(\mathbb{R})\right\} \cong \mathrm{U}_2(\mathbb{R}),$$

where $\mathrm{U}_2(\mathbb{R})$ are the $\mathbb{R}$-points of the unitary group U_2 attached to the quadratic extension $\mathbb{C}/\mathbb{R}$, which as an algebraic group is defined over $\mathbb{R}$.

Irreducible representations of $\mathrm{U}_2(\mathbb{R})$ are isomorphic to $\det^k \mathrm{sym}^\ell := \det^k \otimes \mathrm{sym}^\ell$ for some $k \in \mathbb{Z}$, $\ell \in \mathbb{Z}_{\geq 0}$. This is the classical way of denoting weights for Siegel modular forms. In the context of real-analytic representation theory, it is more common to parametrize irreducible K-representations by integers $a, b \in \mathbb{Z}$ subject to the condition that $a \geq b$. This notion stems from the action of the center of $\mathfrak{k}$ (the complexified Lie algebra of K) on a representation. Translation between the two conventions is straightforward: The pair (a, b) corresponds to $\det^b \mathrm{sym}^{b-a}$, while the weight $\det^k \mathrm{sym}^\ell$ corresponds to $(k + \ell, k)$.

A $(\mathfrak{g}, K)$-module is a simultaneous $\mathfrak{g}$ and K-module with compatibility relations imposed on them. A precise definition can be found in Section 3.3.1 of [32]. One invariant of $(\mathfrak{g}, K)$-modules is the set of nontrivial K-types (i.e., irreducible K-representations). A $(\mathfrak{g}, K)$-module ϖ viewed as a K-representation can be decomposed as a direct sum of irreducibles. We say that a K-type π_K occurs in ϖ if $\dim \mathrm{Hom}_K(\pi_K, \varpi) > 0$. If the multiplicity in ϖ of all π_K's is finite, then ϖ is called a Harish-Chandra module. The invariant that we will primarily encounter is the set of K-types that occur in a $(\mathfrak{g}, K)$-module.

The K-types in $(\mathfrak{g}, K)$-modules for $\mathrm{Sp}_2(\mathbb{R})$ can be displayed by a half-grid in 2 dimensions. A typical such grid looks as follows, where we have marked K-types that occur by a filled circle, and those that do not occur by an empty circle.

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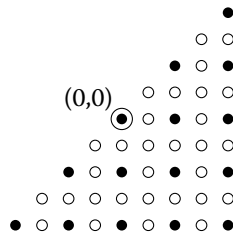
We say that a Harish-Chandra module ϖ has a vertical wall in the direction of $\leftarrow$ or $\rightarrow$, if there exists $a_0 \in \mathbb{Z}$ such that every K-type with highest weight (a, b) that occurs in ϖ satisfies $a \leq a_0$ or $a \geq a_0$, respectively. Horizontal walls in the direction of $\uparrow$ and $\downarrow$ can be defined analogously.
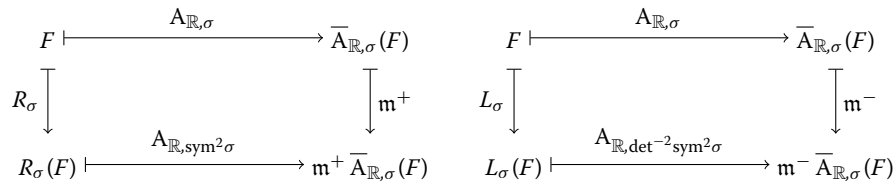
4.1 $(\mathfrak{g}, K)$-modules associated with modular forms

Section 2 of [33] gives an account of the connection between modular forms and Harish-Chandra modules. Weights are finite dimensional, holomorphic representations σ of $\mathrm{GL}_2(\mathbb{C})$. Their representation space is denoted by $V(\sigma)$. Given a weight σ and a smooth function $F : \mathbb{H}^{(2)} \to V(\sigma)$, we can attach a function $\mathrm{A}_{\mathbb{R},\sigma}(F) := \mathrm{A}_{\mathbb{R}}(F)$ on $G(\mathbb{R}) = \mathrm{Sp}_2(\mathbb{R})$:

$$\mathrm{A}_{\mathbb{R}}(F)(g) = \sigma^{-1}\left(j(g, iI^{(2)})\right) f(g \bullet iI^{(2)}), \qquad j(g, Z) = CZ + D. \tag{4.1}$$

From $\mathrm{A}_{\mathbb{R}}(F)$, one constructs the vector space $\overline{\mathrm{A}}_{\mathbb{R},\sigma}(F)(g)$ that is spanned by its coordinates. This space under right translation by K is isomorphic to the dual $\sigma^\vee$ of σ. The action of $\mathfrak{g}$ on this space generates a $(\mathfrak{g}, K)$-module that we denote by $\varpi(F)$. If F is an automorphic form or a term of the Fourier series of a modular form, then $\varpi(F)$ is a Harish-Chandra module.

Recall the compatibility of covariant differential operators acting on F and the $\mathfrak{g}$-action on $\varpi(F)$ that is stated and the end of Section 2.2 of [33] in terms of the following two commutative diagrams.

$$
\begin{array}{ccc}
F & \xrightarrow{\ \mathrm{A}_{\mathbb{R},\sigma}\ } & \overline{\mathrm{A}}_{\mathbb{R},\sigma}(F) \\
R_\sigma \big\uparrow & & \big\uparrow \mathfrak{m}^+ \\
R_\sigma(F) & \xrightarrow[\mathrm{A}_{\mathbb{R},\mathrm{sym}^2\sigma}]{} & \mathfrak{m}^+\,\overline{\mathrm{A}}_{\mathbb{R},\sigma}(F)
\end{array}
\qquad
\begin{array}{ccc}
F & \xrightarrow{\ \mathrm{A}_{\mathbb{R},\sigma}\ } & \overline{\mathrm{A}}_{\mathbb{R},\sigma}(F) \\
L_\sigma \big\uparrow & & \big\uparrow \mathfrak{m}^- \\
L_\sigma(F) & \xrightarrow[\mathrm{A}_{\mathbb{R},\det^{-2}\mathrm{sym}^2\sigma}]{} & \mathfrak{m}^-\,\overline{\mathrm{A}}_{\mathbb{R},\sigma}(F)
\end{array}
$$

This compatibility allows us to pass back and forth between the classical description of covariant differential operators acting on modular forms and the representation theoretic perspective.

Lemma 4.1 *Let $c : \mathbb{H}^{(2)} \to V(\sigma)$ be smooth function such that the Poincaré series*

$$P_c = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} c\big|_\sigma \gamma$$

is locally absolutely convergent. Then, there is an inclusion $\varpi(P_c) \hookrightarrow \varpi(c)$.

Proof This is an immediate consequence of viewing Poincaré series as intertwining maps from a suitable principal series to the automorphic spectrum. It can also be seen directly, by checking that

$$\mathrm{A}_{\mathbb{R}}(P_c) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{A}_{\mathbb{R}}(c) \circ \gamma.$$

$\square$

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Lemma 4.2 *Given two smooth functions $c_1 : \mathbb{H}^{(2)} \to V(\sigma_1)$ and $c_2 : \mathbb{H}^{(2)} \to V(\sigma_2)$, then there is an inclusion $\varpi(c_1 \cdot c_2) \hookrightarrow \varpi(c_1) \otimes \varpi(c_2)$.*

Proof This is a rephrasing of the Leibniz rule for differentials. □

4.2 The tensor products of $(\mathfrak{g}, K)$-modules

Proposition 4.3 *Let ϖ_1 and ϖ_2 be Harish-Chandra modules. If ϖ_1 has a vertical wall in the direction of $\to$ and ϖ_2 has a horizontal wall in the direction of $\uparrow$, then the tensor product $\varpi_1 \otimes \varpi_2$ has a vertical wall in the direction of $\to$. If ϖ_1 has a vertical wall in the direction of $\leftarrow$ and ϖ_2 has a horizontal wall in the direction of $\downarrow$, then the tensor product $\varpi_1 \otimes \varpi_2$ has a horizontal wall in the direction of $\downarrow$.*

Proof We prove the first case and leave the second one to the reader. Let a_0 and b_0' be such that $a \geq a_0$ if (a, b) occurs in ϖ_1 and $b' \geq b_0'$ if (a', b') occurs in ϖ_2. Let (a, b) and (a', b') be arbitrary K-types in ϖ_1 and ϖ_2. Then by the Clebsch-Gordan rules, their tensor product contains K-types of weight (a'', b'') with $a'' + b'' = a + a' + b + b'$ and $\max(a - b, a' - b') - \min(a - b, a' - b') \leq a'' - b'' \leq a - b + a' - b'$.

Adding these two, we find that

$$2a'' \geq a + a' + b' + b' + \max\left(a - b, a' - b'\right) - \min\left(a - b, a' - b'\right).$$

If $a - b \geq a' - b'$, then this equals $2(a + b') \geq 2(a_0 + b_0')$. Otherwise, it equals $2(a' + b)$ which is greater than $2(a + b') \geq 2(a_0 + b_0')$, because $a' - b' \geq a - b$. □

4.3 Harish-Chandra modules with walls

Proposition 4.4 *Assume the generalized Ramanujan conjecture for GL_4. Let ϖ be an irreducible, cuspidal, automorphic representation, with Harish-Chandra module ϖ_∞ at the infinite place. If ϖ_∞ has a vertical or horizontal wall, and if ϖ_∞ contains a scalar K-type, then ϖ_∞ is a holomorphic or antiholomorphic (limit of) discrete series.*

Proof We first show that the Harish-Chandra parameters (s_1, s_2) of ϖ_∞ are integral. The Langlands classification [15] exhausts irreducible Harish-Chandra modules as irreducible quotients of induced representations. We use [25] to determine their K-types. We adopt Muić's notation. Equations (9.3–9.5) of [25] allow us to focus on the induced representations

$$\begin{aligned}
&|\ |^p \mathrm{sgn}^p \times |\ |^t \mathrm{sgn}^t \rtimes 1, \qquad \delta\left(|\ |^s \mathrm{sgn}^\epsilon, k\right) \rtimes 1, \quad \zeta\left(|\ |^s \mathrm{sgn}^\epsilon, k\right) \rtimes 1 \\
&|\ |^s \mathrm{sgn}^\epsilon \rtimes X(p, \pm), \qquad |\ |^s \mathrm{sgn}^\epsilon \rtimes V_p
\end{aligned} \tag{4.2}$$

with nonintegral Harish-Chandra parameter. The first representation is irreducible by Lemma 9.1 of [25]. The remaining ones are irreducible by Theorem 12.1 of [25]. Hence, it suffices to check K-types of induced representations that occur in (4.2), which was done in Section 6 of [25]. This shows that $s_1, s_2 \in \mathbb{Z}$.

Assume that ϖ_∞ is tempered. Among the (limits of) discrete series, only the holomorphic ones contain scalar K-types. To show that no other tempered representation can occur, observe that tempered representations that are not (limits of) discrete series are fully induced from discrete series attached to the Levi factor of a parabolic subgroup by Knapp and Zuckerman [19]. Section 9 of [25] lists the nontempered constituents of inductions of discrete series, and this reduces us to the Harish-Chandra parameter $(0, 0)$.

By Corollary 5.2 of [25], the induced representation $1 \times 1 \rtimes 1$ is irreducible. Lemma 6.1 reveals that it has no walls. The principal series

$$\mathrm{sgn} \times \mathrm{sgn}^\epsilon \rtimes 1 \cong \mathrm{sgn}^\epsilon \times \mathrm{sgn} \rtimes 1 \cong \mathrm{sgn}^\epsilon \rtimes X(0,+) \oplus \mathrm{sgn}^\epsilon \rtimes X(0,-)$$

contains K-types (k', k') for odd k' only.

Assume that ϖ_∞ is nontempered. Then by the generalized Ramanujan conjecture for GL_4 and by Arthur's endoscopic classification [2], the only nontempered contributions to the automorphic spectrum are lifts of (1) Soudry type, (2) Saito–Kurokawa type, (3) Howe–Piatetski–Shapiro type, or (4) one-dimensional type. For a detailed explanation, see [1]. Local components at the infinite places can be determined via the local Langlands correspondence for reductive groups over the reals, which was established in [21]. Sections 1 and 2 of [28] summarize both results briefly. For Soudry type lifts, we apply Lemma 6.1 of [25] to discover that no scalar K-types occur. The holomorphic Saito–Kurokawa lift does contain scalar K-types. At the infinite place, it is a holomorphic discrete series. A Saito–Kurokawa lift with integral Harish-Chandra parameter that is not a holomorphic discrete series contains no scalar K-type by Lemmas 6.1 and 9.2 of [25]. The Howe–Piatetski–Shapiro type corresponds to the Langlands quotient of the Borel subgroup. In the case of integral Harish-Chandra parameters, it contains no scalar K-type by, again, Lemma 6.1 and 9.2 of [25]. This also applies to the one-dimensional type. This establishes the claim. □

4.4 Proof of the main theorem
Before we investigate Poincaré series, we recall two results about their Fourier series:

Proposition 4.5 *Let* $c : \mathbb{H}^{(2)} \to V(\sigma)$ *be a holomorphic function. Then* $\varpi(c)$ *has a horizontal wall in the direction of* $\uparrow$. *Moreover,* $\varpi(\Psi_k(T;Z))$ *has a vertical wall in the direction of* $\rightarrow$, *where* $\Psi_k(T;Z)$ *is defined in* (3.2).

Proof The first statement is classical and follows from the description of holomorphic discrete series. The second one is a direct consequence of Proposition 4.1 of [33], or alternatively can be extracted from [22]. □

Proposition 4.6 *There is no* $d \in \mathbb{Z}_{\geq 0}$ *such that*

$$L^d \left(\Psi_k(T; \cdot) \Phi_\ell(T'; \cdot) \right)$$

vanishes.

Proof Since $L(\Phi_\ell(T; \cdot) = 0$, we have

$$L^d \left(\Psi_k(T; \cdot) \Phi_\ell(T'; \cdot) \right) = L^d \left(\Psi_k(T; \cdot) \right) \cdot \Phi_\ell(T'; \cdot).$$

Using the equivariance

$$\Psi_k({}^t U T U; Z) = \Psi_k(T; {}^t U Z U), \quad U \in \mathrm{GL}_2(\mathbb{R})$$

we can focus on the case that $\Psi_k(T; \cdot)$ is a nonvanishing term of the Fourier series of an Eisenstein series. We then obtain an embedding of the Harish-Chandra module ϖ_k generated by that Eisenstein series into the generalized Whittaker model $W_{k,T}$ associated with $\Psi_k(T; \cdot)$ (which, in fact, is a Bessel model). Using the decomposition series of principal series given in [22], we find that the K-types $(k + 2a, k)$ occur in ϖ_k for all $a \geq 0$.

Moreover, all K-types in ϖ_k occur with multiplicity at most 1. Let π be the projection of the weight (i.e., $\mathrm{GL}_2(\mathbb{C})$-representation) $\det^k(\mathrm{sym}^2)^d$ to $\det^k \mathrm{sym}^{2d}$. Our argument shows that

$$\pi\left(L^d\left(\Psi_k(T;\,\cdot\,)\right)\right)$$

generates the K-type $(k+2d,k)$ of the image of ϖ_k in $W_{k,T}$. In particular, it does not vanish. □

Proof of Theorem I By Sect. 3, the Poincaré series $P^{(2)}_{k,\ell;T,T'}$ is a real-analytic cusp form. We have to show that it vanishes under some tensor power of the lowering operator L. By the connection of modular forms and $(\mathfrak{g},K)$-modules, elaborated on in Sect. 4.1, it suffices to show that $\varpi\left(P^{(2)}_{k,\ell;T,T'}\right)$ is a finite sum of holomorphic (limits of) discrete series.

Cuspidality implies that $\varpi\left(P^{(2)}_{k,\ell;T,T'}\right)$ is a direct sum of irreducibles. The theorem is proved if we show that any of its irreducible subquotients is a holomorphic (limit of) discrete series.

Lemma 4.1 asserts that we can restrict on the codomain of

$$\varpi\left(P^{(2)}_{k,\ell;T,T'}\right) \hookrightarrow \varpi\left(\Psi_k(T;Z)\cdot\Phi_\ell(T';Z)\right),$$

and Lemma 4.2 allows us to further reduce our considerations to

$$\varpi\left(\Psi_k(T;Z)\cdot\Phi_\ell(T';Z)\right)\hookrightarrow \varpi_{k,\ell;T,T'} = \varpi\left(\Psi_k(T;Z)\right)\otimes\varpi\left(\Phi_\ell(T';Z)\right),$$

where $\Phi_\ell(T';Z)$ is viewed as a function from $\mathbb{H}^{(2)}$ to $V(\det^\ell)$.

Proposition 4.5 guarantees that the first tensor factor has a horizontal wall, and the second one has a vertical wall. Consequently, we can apply Proposition 4.3. It implies that $\varpi_{k,\ell;T,T'}$ has a vertical wall in the direction of $\rightarrow$. If it occurs in $\varpi\left(P^{(2)}_{k,\ell;T,T'}\right)$, then it contains a scalar K-type. Hence, it is a holomorphic discrete series by Proposition 4.4. This completes the proof. □

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