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Michal Szabados

An Algebraic Approach to Nivat's Conjecture

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An algebraic approach to Nivat's conjecture

Algebrallinen lähestymistapa Nivat'n konjektuuriin

Michal Szabados

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University of Turku
Department of Mathematics and Statistics
FI-20014 Turku – Åbo

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Supervisor

Prof. Jarkko Kari
University of Turku
Finland

Reviewers

Dr. Svetlana Puzynina
St Petersburg State University
Russia

Prof. Luca Zamboni
Université Claude Bernard Lyon 1
France

Opponent

Prof. Bryna Kra
Northwestern University
USA

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Abstract

This thesis introduces a new, algebraic method to study multidimensional configurations, also sometimes called words, which have low pattern complexity. This is the setting of several open problems, most notably Nivat's conjecture, which is a generalization of Morse-Hedlund theorem to two dimensions, and the periodic tiling problem by Lagarias and Wang.

We represent configurations as formal power series over d variables where d is the dimension. This allows us to study the ideal of polynomial annihilators of the series. In the two-dimensional case we give a detailed description of the ideal, which can be applied to obtain partial results on the aforementioned combinatorial problems.

In particular, we show that configurations of low complexity can be decomposed into sums of periodic configurations. In the two-dimensional case, one such decomposition can be described in terms of the annihilator ideal. We apply this knowledge to obtain the main result of this thesis – an asymptotic version of Nivat's conjecture. We also prove Nivat's conjecture for configurations which are sums of two periodic ones, and as a corollary reprove the main result of Cyr and Kra from [CK15].

Tiivistelmä suomeksi

Tässä väitöskirjassa esitetään uusi, algebrallinen lähestymistapa moniulotteisiin, matalan kompleksisuuden konfiguraatioihin. Näistä konfiguraatioista, joita moniulotteisiksi sanoiksikin kutsutaan, on esitetty useita avoimia ongelmia. Tärkeimpinä näistä ovat Nivat'n konjektuuri, joka on Morsen-Hedlundin lauseen kaksiulotteinen yleistys, sekä Lagariaksen ja Wangin jaksollinen tiilitysongelma.

Väitöskirjan lähestymistavassa d-ulotteiset konfiguraatiot esitetään d:n muuttujan formaaleina potenssisarjoina. Tämä mahdollistaa konfiguraation polynomiannihilaattoreiden ihanteen tutkimisen. Väitöskirjassa selvitetään kaksiulotteisessa tapauksessa ihanteen rakenne tarkasti. Tätä hyödyntämällä saadaan uusia, osittaisia tuloksia koskien edellä mainittuja kombinatorisia ongelmia.

Tarkemmin sanottuna väitöskirjassa todistetaan, että matalan kompleksisuuden konfiguraatiot voidaan hajottaa jaksollisten konfiguraatioiden summaksi. Kaksiulotteisessa tapauksessa eräs tällainen hajotelma saadaan annihilaattori-ihanteesta. Tämän avulla todistetaan asymptoottinen versio Nivat'n konjektuurista. Lisäksi osoitetaan Nivat'n konjektuuri oikeaksi konfiguraatioille, jotka ovat kahden jaksollisen konfiguraation summia, ja tämän seurauksena saadaan uusi todistus Cyrin ja Kran artikkelin [CK15] päätulokselle.

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List of original publications

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Chapter 1

Introduction

Nivat's conjecture

In July 2017 it was exactly twenty years since the ICALP 1997 conference in Italy, where Maurice Nivat formulated a problem which until today is one of the most sought-after open problems in symbolic dynamics. Although it occurs in literature also without specifically mentioning Nivat [BV00, ST00, ST02], over the years it became known as *Nivat's conjecture*.

The conjecture is inspired by a much older combinatorial result by Morse and Hedlund. In their paper *Symbolic Dynamics* [MH38], besides giving formal foundations to a newly emerging mathematical field, they formulated the following theorem. Let an *alphabet* be any finite set $\mathcal{A}$, its elements are called *symbols*. A *bi-infinite word* w is an element of $\mathcal{A}^{\mathbb{Z}}$, that is, a two-way infinite sequence of symbols. A *subword* of w of length n is a finite sequence of n consecutive symbols occurring in w. The number of distinct subwords of w of length n, denoted by $P_w(n)$, is the *complexity function* of w.

Theorem (Morse-Hedlund)**.** *Let $w \in \mathcal{A}^{\mathbb{Z}}$ be a bi-infinite word. Then w is periodic if and only if $P_w(n) \leq n$ for some positive integer n.*

The theorem demonstrates an interesting phenomenon – a local restriction on structure, in this case a condition on complexity, implies a restriction on the global structure, in this case periodicity. Nivat's conjecture is a generalization of Morse-Hedlund theorem to two dimensions. To formulate it, we must answer three questions: What is a generalization of a word, what does it mean to be periodic, and what is the complexity function.

We start with a generalization of a word. Let d be a positive integer, the dimension. A *d-dimensional symbolic configuration* c is an element of $\mathcal{A}^{\mathbb{Z}^d}$, that is, a map assigning a symbol to every vertex of the lattice $\mathbb{Z}^d$. For a vector $\boldsymbol{v} \in \mathbb{Z}^d$, the symbol at position $\boldsymbol{v}$ is denoted $c(\boldsymbol{v})$, or also $c_{\boldsymbol{v}}$. For $\boldsymbol{u} \in \mathbb{Z}^d$, we say that c is *$\boldsymbol{u}$-periodic* if $c_{\boldsymbol{v}} = c_{\boldsymbol{v}+\boldsymbol{u}}$ holds for all $\boldsymbol{v} \in \mathbb{Z}^d$, and c is *periodic* if it is $\boldsymbol{u}$-periodic for some $\boldsymbol{u} \neq 0$.

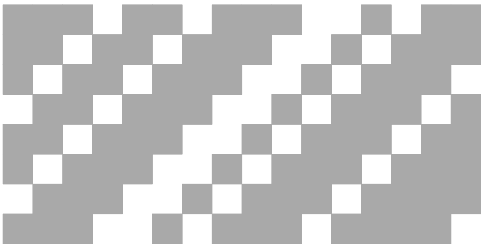

Figure 1.1: A piece of a periodic configuration having $P_c(m,n) = 2^{m+n-1}$ for all positive integers m, n.

For a finite domain $D \subseteq \mathbb{Z}^d$, the elements of $\mathcal{A}^D$ are *D-patterns*. For a fixed D, we denote by $c_{\boldsymbol{v}+D}$ the D-pattern in c in position $\boldsymbol{v}$, that is, the map $\boldsymbol{u} \mapsto c_{\boldsymbol{v}+\boldsymbol{u}}$ for all $\boldsymbol{u} \in D$. The number of distinct D-patterns in c is the *D-pattern complexity* $P_c(D)$ of c. In two dimensions, for an $m \times n$ rectangle $D = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ we denote $P_c(m,n) = P_c(D)$. In other words, $P_c(m,n)$ counts the number of distinct $m \times n$ block patterns occurring in c.

Now we have all the definitions needed to state the conjecture:

Conjecture (Nivat)**.** *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional configuration. If there exist positive integers m, n such that $P_c(m,n) \le mn$, then c is periodic.*

Unlike Morse-Hedlund theorem, the statement is not an equivalence. The reverse implication is indeed false: Let $c \in \{0,1\}^{\mathbb{Z}^2}$ be a two-dimensional configuration which in the first row observes a sequence containing every possible one-dimensional pattern (e.g. concatenation of binary expansions of all positive integers), and which is periodic with period $(1,1)$. See Figure 1.1 for an illustration. Such a configuration is periodic and satisfies $P_c(m,n) = 2^{m+n-1} > mn$ for all positive integers m, n.

It would be natural to analogously generalize Morse-Hedlund theorem to any dimension, however that is also not true [ST00]. A simple counterexample consisting of two non-intersecting perpendicular lines can be given for $d = 3$: For an integer $n \ge 3$, let $c(i,0,0) = c(0,i,n) = 1$ for all $i \in \mathbb{Z}$, and let $c(i,j,k) = 0$ otherwise, as in Figure 1.2. Then for D equal to the $n \times n \times n$ cube, $P_c(D) = 2n^2 + 1$ since the cube can either cover zeros only, or be pierced by exactly one of the 1-lines in n^2 positions. We have $P_c(D) = 2n^2 + 1 < n^3 = |D|$, but the configuration is not periodic.

If true, Nivat's conjecture is tight, as there exist non-periodic configurations which satisfy $P_c(m,n) = mn + 1$ for each m, n. Cassaigne [Cas99] gave a classification of all of them, the simplest example is a configuration having $c_{(0,0)} = 1$ and $c_{\boldsymbol{v}} = 0$ for $\boldsymbol{v} \ne (0,0)$.

Let us summarize past progress towards resolving the conjecture. There have been a number of results which show that $P_c(m,n) \le \alpha mn$ implies

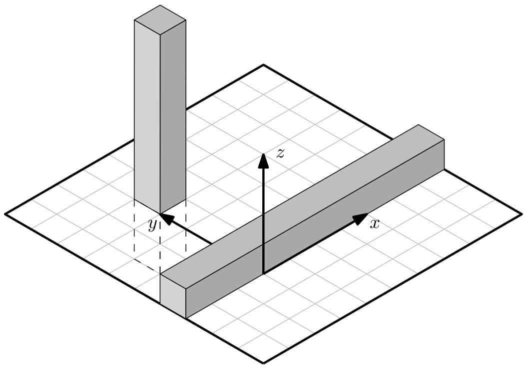

Figure 1.2: A non-periodic configuration of low complexity. If D is the $4 \times 4 \times 4$ cube, then $P(D) = 2 \cdot 4^2 + 1 = 33$.

periodicity for some constant $\alpha \in \mathbb{R}$. Epifanio, Koskas and Mignosi [EKM03] gave a proof for $\alpha = 1/144$, which was improved to $\alpha = 1/16$ by Quas and Zamboni [QZ04]. The best known constant $\alpha = 1/2$ is due to Cyr and Kra [CK15]. In another direction, there have been results proving Nivat's conjecture for short rectangles. Sander and Tijdeman [ST02] proved that having $P_c(2, n) \leq 2n$ for some n implies periodicity. Cyr and Kra [CK16] extended this condition to $P_c(3, n) \leq 3n$.

These results were obtained mostly by combinatorial analysis. Notably, the approach of Cyr and Kra uses symbolic dynamics, however the final arguments are still combinatorial. This to some extent can not be avoided as the conjecture is combinatorial in its nature. We introduce a new method to attack the conjecture, an algebraic method involving polynomials.

Periodic tiling problem

Before presenting our results let us introduce another open problem, the *periodic tiling problem* by Lagarias and Wang [LW96]. With a fixed dimension d, let a *tile* be an arbitrary finite set $T \subset \mathbb{Z}^d$. We say that T *tiles* $\mathbb{Z}^d$ if there exists $C \subset \mathbb{Z}^d$ such that

$$T \oplus C = \mathbb{Z}^d. \tag{1.1}$$

Informally, a tile is a collection of not necessarily adjacent unit hypercubes and the tilings considered are only by translations, no rotations are allowed. A set C satisfying (1.1) is a *co-tiler*. A tile T tiles $\mathbb{Z}^d$ *periodically* if there exists a co-tiler C which is translation invariant with respect to a non-zero vector.

Conjecture (Periodic tiling problem)**.** *If a tile tiles $\mathbb{Z}^d$, then it tiles it also periodically.*

The conjecture is easily seen to be true for $d = 1$ where, in fact, any tiling is periodic. Only recently, Bhattacharya [Bha16] has demonstrated that the conjecture holds also for $d = 2$ by a remarkable proof using ergodic theory. For other dimensions the conjecture remains open. It is known that it holds if $|T| = 4$ or if the size of T is a prime number [Sze98]. Moreover, in the latter case only periodic tilings exist. We give a short proof of this fact in Example 3.1.1 using the polynomial method.

It is natural to interpret a co-tiler C as a configuration having $c_{\boldsymbol{v}} = 1$ if $\boldsymbol{v} \in C$ and $c_{\boldsymbol{v}} = 0$ otherwise. The tiling condition (1.1) is then equivalent to saying that every $(-T)$-pattern in c contains exactly one coefficient 1. Szegedy [Sze98] showed that co-tilers of T and $-T$ coincide, therefore also every T-pattern contains exactly one 1. In particular, $P_c(T) = |T|$. We shall see that this condition, or more precisely $P_c(T) \leq |T|$, is the common denominator of the periodic tiling problem and Nivat's conjecture that is of our interest.

The polynomial method and low complexity configurations

We study symbolic configurations that satisfy the *low complexity condition*

$$P_c(D) \leq |D| \tag{1.2}$$

for some finite $D \subset \mathbb{Z}^d$. This is the case for $D = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ in the case of Nivat's conjecture and $D = T$ for periodic tiling problem.

When the symbols in $\mathcal{A}$ are chosen to be integers, a d-dimensional symbolic configuration can be identified with a formal power series with integer coefficients in d variables $x_1, \ldots, x_d$:

$$\sum_{\boldsymbol{v}=(v_1,\ldots,v_d)\in\mathbb{Z}^d} c_{\boldsymbol{v}} x_1^{v_1} \cdots x_d^{v_d}$$

We call this algebraic object a *configuration*. More precisely, we allow the coefficients to be complex numbers, and call a configuration *finitary integral* if they come from a finite subset of integers. The definitions are given in section 3.1.

Configurations are our main objects of study. If f is a Laurent polynomial, multiplication of a configuration c by f is well defined. We say that f is an *annihilator* of c if $fc = 0$. We further define $\text{Ann}(c)$ to be the ideal of all annihilators of c. For configurations satisfying the low complexity condition (1.2), $\text{Ann}(c)$ is always a non-trivial ideal (Lemma 3.1.2).

Having a polynomial ideal allows us to use methods of algebraic geometry. In particular, we use Hilbert's Nullstellensatz to obtain our first result, a decomposition theorem:

Theorem (Corollary 3.4.4)**.** *Let c be a low complexity configuration. Then c can be written as a sum of finitely many periodic configurations.*

There are examples of configurations which satisfy conditions of the theorem but are non-periodic. Therefore, in order to attack Nivat's conjecture, we need to use additional conditions from its statement. In Chapter 4 we focus on two-dimensional configurations. We prove:

Theorem (Theorem 4.1.1)**.** *Let c be a two-dimensional finitary integral configuration. Then* $\text{Ann}(c)$ *is a radical ideal.*

This result is followed by an analysis of the structure of $\text{Ann}(c)$, which allows us to formulate a more explicit version of decomposition theorem for two-dimensional configurations (Corollary 4.2.1). As a result of this analysis, we define $\text{ord}(c)$ as a number involved in the description of $\text{Ann}(c)$, and which is also the minimal number of periodic components c can be decomposed into. In particular, non-periodic configurations have $\text{ord}(c) \geq 2$.

Our main result is an asymptotic version of Nivat's conjecture. It is best understood when compared with Nivat's conjecture stated in the contrapositive direction:

Conjecture (Nivat)**.** *Let c be a non-periodic two-dimensional symbolic configuration. Then $P_c(m, n) > mn$ holds for all pairs of positive integers (m, n).*

Theorem (Theorem 5.5.4)**.** *Let c be a non-periodic two-dimensional symbolic configuration. Then $P_c(m, n) > mn$ holds for all but finitely many pairs of positive integers (m, n).*

The proof goes by analysis of hypothetical counterexamples to Nivat's conjecture. In Chapter 5 we connect the algebraic structure of $\text{Ann}(c)$ with complexity of the configuration. We end up with a proof which works except for one special case – a configuration with $\text{ord}(c) = 2$ which is a sum of vertically and horizontally periodic configuration.

Although it can be handled by combinatorial analysis [KS16], in Chapter 6 we prove a more general result which covers this case. We combine ideas of Cyr and Kra with the polynomial method. We give a short introduction to symbolic dynamics and define *balanced sets*, an essential tool of their approach. We prove:

Theorem (Theorem 6.5.1)**.** *Let c be a two-dimensional finitary integral configuration with* $\text{ord}(c) = 2$*. Then $P_c(m, n) > mn$ for all pairs of positive integers (m, n).*

That concludes the proof of our main theorem. As a corollary, we obtain the following result of Cyr and Kra:

Theorem (Cyr and Kra, Theorem 6.5.4)**.** *Let c be a non-periodic two-dimensional configuration. Then $P_c(m,n) > mn/2$ for all pairs of positive integers (m,n).*

The thesis is concluded with a short note on the decomposition theorem. We show an alternative proof of it which makes use of ultrafilers. Moreover, this proof gives an explicit description of the components. In particular, resulting components are *bounded*, i.e. with coefficients from a closed interval of reals:

Theorem (Corollary 7.3.3)**.** *Let c be a low complexity configuration. Then c can be written as a sum of finitely many bounded periodic configurations.*

The final Chapter 8 gives a summary of open problems.

Chapter 2

Preliminaries

In this chapter we establish notation and review a few algebraic topics which are covered in bachelor's and master's programmes in mathematics. Topics related to algebraic geometry can be found in [CLO92]. Readers are invited to skip these sections if they are familiar to them. We add a section which extends the theory to Laurent polynomials.

2.1 Basic notation

We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ integers, rationals, reals and complex numbers respectively. The symbols $\mathbb{N}$ or $\mathbb{Z}^+$ stand for positive integers, to include also zero we use $\mathbb{N}_0$ or $\mathbb{Z}_0^+$. Further we denote $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The set inclusion symbols "$\subset$" and "$\supset$" also admit set equality.

Let R be a commutative ring, we will usually consider integer or complex numbers. Denote by $R[x_1, \ldots, x_d]$ the set of polynomials over R in d variables. We adopt the usual simplified notation: for a d-tuple of non-negative integers $\boldsymbol{v} = (v_1, \ldots, v_d)$ set $X^{\boldsymbol{v}} = x_1^{v_1} \ldots x_d^{v_d}$, then we write

$$R[X] = R[x_1, \ldots, x_d]$$

and a general polynomial $f \in R[X]$ can be expressed as $f = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$, where $a_{\boldsymbol{v}} \in R$ and the sum ranges over finitely many d-tuples of non-negative integers $\boldsymbol{v}$. If we allow $\boldsymbol{v}$ to contain also negative integers we obtain *Laurent polynomials*, which are denoted by $R[X^{\pm 1}]$. Finally, by relaxing the requirement to have only finitely many $a_{\boldsymbol{v}} \neq 0$ we get *formal power series*:

$$R[[X^{\pm 1}]] = \left\{ \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}} \mid \boldsymbol{v} \in \mathbb{Z}^d,\ a_{\boldsymbol{v}} \in R \right\}.$$

Note that we allow infinitely many negative exponents in formal power series.

Example 2.1.1. Fix $d = 2$. Then $f(X) = X^{(1,1)} - 3X^{(1,0)} = x_1 x_2 - 3x_1$ is a polynomial and $g(X) = X^{(0,-1)} f(X) = x_1 - 3x_1 x_2^{-1}$ is a Laurent polynomial.

Every polynomial is a Laurent polynomial, every Laurent polynomial is a formal power series.

2.2 Commutative rings

Let us recall a few definitions from commutative algebra. Let R be a commutative ring. An *ideal* A of R, denoted $A \leq R$, is a subring of R which is closed under multiplication by elements from R. For $g_1, \ldots, g_n \in R$ denote by $\langle g_1, \ldots, g_n \rangle$ the ideal generated by g_i. An ideal A is *principal* or *one-generated* if $A = \langle g \rangle$ for some $g \in R$. A ring is *principal ideal domain*, or PID, if its every ideal is principal.

Let A, B be ideals of R. As usual, define $A+B = \{\, a+b \mid a \in A, b \in B \,\}$ and $AB = \langle\, ab \mid a \in A, b \in B \,\rangle$, they are ideals of R as well. Ideals A and B are said to be *comaximal* if $A + B = R$, or equivalently if $1 \in A + B$. The following fact is well-known:

Lemma 2.2.1. *If $A_1, \ldots, A_n \leq R$ are pairwise comaximal ideals, then $\bigcap A_i = \prod A_i$.*

An ideal $A \leq R$ is *prime* if $ab \in A, a, b \in R$ implies $a \in A$ or $b \in A$. An ideal $A \leq R$ is *radical* if $a^n \in A$ implies $a \in A$. Clearly, that happens if and only if $A = \sqrt{A}$ where

$$\sqrt{A} = \left\{\, a \in R \;\middle|\; \exists n : a^n \in A \,\right\}.$$

An invertible element of an integral domain R is called a *unit*, a non-zero element which cannot be written as a product of two non-unit elements is *irreducible*. Two elements are *associated* if they differ by a unit factor. An integral domain R is a *unique factorization domain* (UFD), if every non-zero non-unit element can be written as a product of irreducible elements. In that case the factorization is unique up to the order and association of factors. In a UFD any two elements have a greatest common divisor.

A commutative ring R is *noetherian*, or satisfies *ascending chain condition*, if every strictly ascending chain of its ideals is finite. A commutative ring R is noetherian iff every ideal of R is finitely generated. Trivially, every principal ideal domain is noetherian.

2.3 Polynomial ideals

The following facts are well-known.

Lemma 2.3.1. *Let R be a ring.*

- *If R is noetherian then $R[X]$ is noetherian.* (Hilbert's basis theorem.)

- *If R is a UFD then $R[X]$ is a UFD.*
- *If R is a field then $R[X]$ is a PID.*

Corollary 2.3.2. *Ideals in $\mathbb{Z}[X]$ and $\mathbb{C}[X]$ are finitely generated. Moreover $\mathbb{Z}[x]$ and $\mathbb{C}[x]$ are PID.*

Proof. $\mathbb{Z}[x]$ is a PID since it has division with remainder, the rest follows directly from Lemma 2.3.1. □

Hilbert's Nullstellensatz

The relation between polynomial ideals and varieties is of central interest in algebraic geometry. Let R be a commutative ring. Fix $d \in \mathbb{N}$ and for $I \leq R[X]$ and $V \subset R^d$ define:

$$\mathbf{V}(I) = \left\{ x \in R^d \,\middle|\, \forall f \in I \colon f(x) = 0 \right\}$$
$$\mathbf{I}(V) = \left\{ f \in R[X] \,\middle|\, \forall x \in V \colon f(x) = 0 \right\}$$

$\mathbf{V}(I)$ is the *algebraic variety* defined by I. It is the set of all common roots of all the polynomials in I. $\mathbf{I}(V)$ is on the other hand the set of all polynomials, for which every element of V is a root. It is easy to verify that $\mathbf{I}(V)$ is an ideal, in fact a radical ideal. The famous Hilbert's Nullstellensatz relates these two operators.

Theorem 2.3.3 (Hilbert's Nullstellensatz)**.** *Let R be an algebraically closed field and I an ideal of $R[X]$. Then*

$$\mathbf{IV}(I) = \sqrt{I}.$$

Radical ideals

It will be useful for us to describe structure of radical ideals in the special case when $R = \mathbb{C}$ and $d = 1, 2$. The following theorem is valid for any $d \in \mathbb{N}$:

Theorem 2.3.4 (Minimal decomposition)**.** *Let R be an algebraically closed field. Every radical ideal $A \leq R[X]$ can be uniquely written as a finite intersection of prime ideals $A = P_1 \cap \cdots \cap P_k$ where $P_i \not\subset P_j$ for $i \neq j$.*

Proof. See e.g. [CLO92] Chapter 4, §6, Theorem 5. □

To simplify notation, we use $\mathbb{C}[x]$ and $\mathbb{C}[x, y]$ in the place of $\mathbb{C}[x_1]$ and $\mathbb{C}[x_1, x_2]$. The case when $d = 1$ is easy since $\mathbb{C}[x]$ is a PID.

Lemma 2.3.5.

- *Non-trivial prime ideals of $\mathbb{C}[x]$ are of the form $\langle \varphi \rangle$ where φ is an irreducible polynomial.*

- *Non-trivial radical ideals of $\mathbb{C}[x]$ are of the form $\langle \varphi_1 \cdots \varphi_n \rangle$ where φ_i are irreducible polynomials which are not associated to each other.*

Note that two polynomials are associated if they differ by a constant factor. Let us focus now on the case $d = 2$.

Lemma 2.3.6. *For a non-trivial prime ideal $P \leq \mathbb{C}[x, y]$ one of the following holds:*

- *P is a principal ideal generated by an irreducible polynomial, i.e. $P = \langle \varphi \rangle$ for some irreducible φ,*
- *or P is a maximal ideal, in which case $P = \langle x - \alpha, y - \beta \rangle$ for some $\alpha, \beta \in \mathbb{C}$.*

Proof. Follows by Proposition 1 in section 1.5 and Corollary 2 in section 1.6 of Fulton's book [Ful89]. □

Theorem 2.3.7. *Let $A \leq \mathbb{C}[x, y]$ be a non-trivial radical ideal. Then there are distinct principal ideals $R_1, \ldots, R_s$ generated by irreducible polynomials and distinct maximal ideals $M_1, \ldots, M_t$ such that $R_i \not\subset M_j$ and*

$$A = R_1 \cdots R_s \, M_1 \cdots M_t.$$

Moreover the ideals are determined uniquely and the ideals $R = R_1 \cdots R_s, M_1, \ldots, M_t$ are pairwise comaximal.

Proof. Apply Lemma 2.3.6 to Theorem 2.3.4 to obtain $A = R_1 \cap \cdots \cap R_s \cap M_1 \cap \cdots \cap M_t$ for R_i, M_j as in the statement. Observe that $\prod R_i = \bigcap R_i$ since R_i are generated by irreducible polynomials. The ideals $R, M_1, \ldots, M_t$ are pairwise comaximal since a maximal ideal is comaximal with any ideal not contained in it. Therefore $A = R \, M_1 \cdots M_t$ by Lemma 2.2.1. The uniqueness follows from uniqueness of minimal decomposition. □

2.4 Laurent polynomials

In this section we restate the theorems from the previous section in terms of Laurent polynomials. The proofs are rather technical, in all of them we reduce to polynomial version of the claim. A few auxiliary definitions are needed.

Definition 2.1.

- *For $I \leq \mathbb{C}[X^{\pm 1}]$ define $\lfloor I \rfloor = I \cap \mathbb{C}[X]$. Note that $f \in I$ iff there exists $\boldsymbol{v} \in \mathbb{Z}^d$ such that $X^{\boldsymbol{v}} f \in \lfloor I \rfloor$.*
- *For $J \leq \mathbb{C}[X]$ define $\langle J \rangle_\pm$ to be the ideal in $\mathbb{C}[X^{\pm 1}]$ generated by J.*

Hilbert's Nullstellensatz for Laurent polynomials

Recall notation $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and fix $d \in \mathbb{N}$. For $I \leq \mathbb{C}[X^{\pm 1}]$ and $V \subset (\mathbb{C}^*)^d$ define:

$$\mathbf{V}(I) = \left\{ x \in (\mathbb{C}^*)^d \;\middle|\; \forall f \in I \colon f(x) = 0 \right\}$$
$$\mathbf{I}(V) = \left\{ f \in \mathbb{C}[X^{\pm 1}] \;\middle|\; \forall x \in V \colon f(x) = 0 \right\}$$

Theorem 2.4.1 (Hilbert's Nullstellensatz for Laurent polynomials)**.** *Let I be an ideal of $\mathbb{C}[X^{\pm 1}]$. Then*

$$\mathbf{IV}(I) = \sqrt{I}.$$

Proof. We reduce to the polynomial case. To distinguish from polynomial operators, we temporarily use $\mathbf{V}_\pm$ and $\mathbf{I}_\pm$ to denote the operators on Laurent polynomials. Define $O = (\mathbb{C}^*)^d$ and let $J = \lfloor I \rfloor$. We prove three claims:

1. $\mathbf{V}_\pm(I) = \mathbf{V}(J) \cap O$
2. $\forall V \subset \mathbb{C}^d \colon \langle \mathbf{I}(V \cap O) \rangle_\pm = \langle \mathbf{I}(V) \rangle_\pm$
3. $\sqrt{I} = \langle \sqrt{J} \rangle_\pm$

Both inclusions in Claim 1 are easy. For the second claim, "$\supset$" follows from $V \cap O \subset V$. For "$\subset$" assume $f \in \mathbf{I}(V \cap O)$, then $x_1 \cdots x_n f \in \mathbf{I}(V)$ and therefore $f \in \langle \mathbf{I}(V) \rangle_\pm$. For Claim 3, "$\supset$" follows from $I \supset J$. For "$\subset$" assume $f \in \sqrt{I}$ and let n be such that $f^n \in I$. Choose $\boldsymbol{v} \in \mathbb{N}^d$ such that $X^{\boldsymbol{v}} f$ is a polynomial, then $X^{n\boldsymbol{v}} f^n \in J \Rightarrow X^{\boldsymbol{v}} f \in \sqrt{J} \Rightarrow f \in \langle \sqrt{J} \rangle_\pm$.

The theorem now follows:

$$\mathbf{I}_\pm \mathbf{V}_\pm(I) = \langle \mathbf{IV}_\pm(I) \rangle_\pm = \langle \mathbf{I}(\mathbf{V}(J) \cap O) \rangle_\pm = \langle \mathbf{IV}(J) \rangle_\pm = \langle \sqrt{J} \rangle_\pm = \sqrt{I}.$$

□

Radical ideals

Lemma 2.4.2. *Ideals in $\mathbb{C}[X^{\pm 1}]$ are finitely generated.*

Proof. For such an ideal I, $I = \langle \lfloor I \rfloor \rangle_\pm$ and $\lfloor I \rfloor$ is finitely generated. □

Lemma 2.4.3.

1. *If $P \leq \mathbb{C}[X^{\pm 1}]$ is prime, then $\lfloor P \rfloor \leq \mathbb{C}[X]$ is prime.*
2. *If $R \leq \mathbb{C}[X^{\pm 1}]$ is radical, then $\lfloor R \rfloor \leq \mathbb{C}[X]$ is radical.*

Proof.

1. Let $a, b \in \mathbb{C}[X]$ be such that $ab \in \lfloor P \rfloor$. Since P is prime, $a \in P$ or $b \in P$. Hence $a \in \lfloor P \rfloor$ or $b \in \lfloor P \rfloor$.

2. Let $f \in \mathbb{C}[X], n \in \mathbb{N}$ be such that $f^n \in \lfloor R \rfloor$. Since R is radical, $f \in R$ and hence $f \in \lfloor R \rfloor$.

□

Theorem 2.4.4 (Minimal decomposition)**.** *Every radical ideal $A \leq \mathbb{C}[X^{\pm 1}]$ can be uniquely written as a finite intersection of prime ideals $A = P_1 \cap \cdots \cap P_k$ where $P_i \not\subset P_j$ for $i \neq j$.*

Proof. By Lemma 2.4.3, $\lfloor A \rfloor$ is a radical ideal in $\mathbb{C}[x, y]$, let $\lfloor A \rfloor = Q_1 \cap \cdots \cap Q_k$ be its minimal decomposition as an intersection of prime ideals from Theorem 2.3.4. Let $P_i = \langle Q_i \rangle_\pm$.

Let us show that every P_i is a prime Laurent polynomial ideal. Fix i and assume $ab \in P_i$. There exist $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^d$ such that $X^{\boldsymbol{u}} a, X^{\boldsymbol{v}} b \in \mathbb{C}[x, y]$. Then $X^{\boldsymbol{u}} a X^{\boldsymbol{v}} b \in Q_i$. Since Q_i is a prime ideal, we have $X^{\boldsymbol{u}} a \in Q_i$ or $X^{\boldsymbol{v}} b \in Q_i$. Then $a \in P_i$ or $b \in P_i$, so P_i is prime.

Next we show $A = P_1 \cap \cdots \cap P_k$. We have that $A = \langle \lfloor A \rfloor \rangle_\pm = \langle Q_1 \cap \cdots \cap Q_k \rangle_\pm$. One inclusion is obtained easily by $\langle Q_1 \cap \cdots \cap Q_k \rangle_\pm \subset \langle P_1 \cap \cdots \cap P_k \rangle_\pm = P_1 \cap \cdots \cap P_k$. For the other one choose $f \in P_1 \cap \cdots \cap P_k$. There exists $\boldsymbol{v} \in \mathbb{N}^d$ such that $X^{\boldsymbol{v}} f$ is a polynomial. Therefore $\forall i$: $X^{\boldsymbol{v}} f \in Q_i$, or in other words $X^{\boldsymbol{v}} f \in Q_1 \cap \cdots \cap Q_k$, from which we have $f \in \langle Q_1 \cap \cdots \cap Q_k \rangle_\pm$. The two inclusions give $A = \langle Q_1 \cap \cdots \cap Q_k \rangle_\pm = P_1 \cap \cdots \cap P_k$.

For uniqueness assume there is another minimal decomposition $A = R_1 \cap \cdots \cap R_m$. Then $\lfloor A \rfloor = \lfloor R_1 \rfloor \cap \cdots \cap \lfloor R_m \rfloor$. By Lemma 2.4.3, $\lfloor R_i \rfloor$ are prime ideals. It is easy to verify that they satisfy the conditions of Theorem 2.3.4 so they have to be the same as $Q_1, \ldots, Q_k$ in some order. Since R_i are Laurent polynomial ideals generated by $\lfloor R_i \rfloor$, they are the same as $P_1, \ldots, P_k$. □

Note that invertible elements of $\mathbb{C}[X^{\pm 1}]$ are of the form $aX^{\boldsymbol{v}}$ where $a \in \mathbb{C}^*$ and $\boldsymbol{v} \in \mathbb{Z}^d$. Therefore a non-trivial Laurent polynomial ideal can not contain such a Laurent polynomial.

Lemma 2.4.5. *For a non-trivial prime ideal $P \leq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ one of the following holds:*

- *P is a principal ideal generated by an irreducible polynomial, i.e. $P = \langle \varphi \rangle$ for some irreducible φ,*

- *or P is maximal ideal, in which case $P = \langle x - \alpha, y - \beta \rangle$ for some $\alpha, \beta \in \mathbb{C}^*$.*

Proof. By Lemma 2.4.3, $\lfloor P \rfloor$ is a prime ideal in $\mathbb{C}[x, y]$. If we had $\lfloor P \rfloor = \mathbb{C}[x, y]$ then P would be trivial. By Lemma 2.3.6 there are two options for the form of the ideal.

Assume $\lfloor P \rfloor = \langle \varphi \rangle$ for $\varphi \in \mathbb{C}[x, y]$ irreducible. Note that φ is not of the form cx or cy for $c \in \mathbb{C}$ since P is a Laurent polynomial ideal. Therefore φ is also irreducible in $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ and $P = \langle \varphi \rangle$.

Assume $\lfloor P \rfloor = \langle x - \alpha, y - \beta \rangle$ for $\alpha, \beta \in \mathbb{C}$. The case $\alpha = 0$ or $\beta = 0$ is not possible since P is a Laurent polynomial ideal. We have $P = \langle x - \alpha, y - \beta \rangle$. □

Theorem 2.4.6. *Let $A \leq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be a non-trivial radical ideal. Then there are distinct principal ideals $R_1, \dots, R_s$ generated by irreducible polynomials and distinct maximal ideals $M_1, \dots, M_t$ such that $R_i \not\subset M_j$ and*

$$A = R_1 \cdots R_s \, M_1 \cdots M_t.$$

Moreover the ideals are determined uniquely and the ideals $R = R_1 \cdots R_s, M_1, \dots, M_t$ are pairwise comaximal.

Proof. The proof goes exactly as in Theorem 2.3.7, there is no need for reduction to the polynomial claim. Apply Lemma 2.4.5 to Theorem 2.4.4 to obtain $A = R_1 \cap \cdots \cap R_s \cap M_1 \cap \cdots \cap M_t$ for R_i, M_j as in the statement. Observe that $\prod R_i = \bigcap R_i$ since R_i are generated by irreducible polynomials. The ideals $R, M_1, \dots, M_t$ are pairwise comaximal since a maximal ideal is comaximal with any ideal not contained in it. Therefore $A = R \, M_1 \cdots M_t$ by Lemma 2.2.1. The uniqueness follows from uniqueness of minimal decomposition. □

Chapter 3

Decomposition theorem

We begin the exposition with definitions which are central to our approach. We define a *configuration* as a formal power series with complex coefficients, *annihilator* as a Laurent polynomial which annihilates a configuration by multiplication, and $\mathrm{Ann}(c)$ as the ideal of all annihilators of a configuration c.

We show that configurations satisfying the *low complexity condition* (1.2) have a non-trivial annihilator. Using Hilbert's Nullstellensatz, we prove that such configuration have also an annihilator of the form $(X^{\boldsymbol{v_1}} - 1) \cdots (X^{\boldsymbol{v_m}} - 1)$. More precisely, we first show in Lemma 3.2.3 that such a Laurent polynomial is in the radical of $\mathrm{Ann}(c)$, and then we define *line polynomials* to extend this result to $\mathrm{Ann}(c)$ in Theorem 3.3.3.

The main result of this chapter is what we call the Decomposition theorem (Theorem 3.4.1): Every low complexity configuration can be written as a sum of finitely many periodic configurations.

3.1 Configurations

Let d be a positive integer. Let us define a d-dimensional *configuration* to be any formal power series $c \in \mathbb{C}[[X^{\pm 1}]]$ and denote by $c_{\boldsymbol{v}}$ the coefficient of $X^{\boldsymbol{v}}$:

$$c = \sum_{\boldsymbol{v} \in \mathbb{Z}^d} c_{\boldsymbol{v}} X^{\boldsymbol{v}}$$

A configuration is *integral* if all coefficients $c_{\boldsymbol{v}}$ are integers, and it is *finitary* if there are only finitely many distinct coefficients $c_{\boldsymbol{v}}$.

Classically in symbolic dynamics configurations are understood as elements of $\mathcal{A}^{\mathbb{Z}^d}$. Because the actual names of the symbols in the alphabet $\mathcal{A}$ do not matter, they can be chosen to be integers. Then a symbolic configuration can be identified with a finitary integral configuration by simply setting the coefficient $c_{\boldsymbol{v}}$ to be the integer at position $\boldsymbol{v}$.

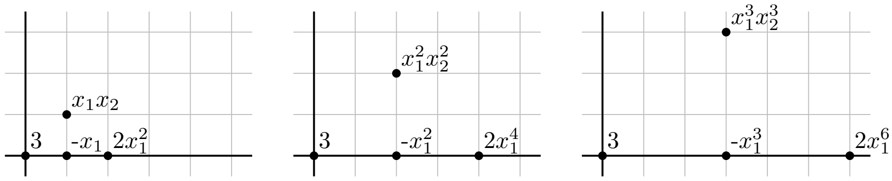

Figure 3.1: Plot of $f(X)$, $f(X^2)$ and $f(X^3)$ for the polynomial $f(X) = f(x_1, x_2) = 3 - x_1 + 2x_1^2 + x_1x_2$.

Multiplication of a formal power series by a Laurent polynomial is well defined and results again in formal power series. For example, $X^{\boldsymbol{v}}c$ is a translation of c by the vector $\boldsymbol{v}$. Another important example is that c is periodic if and only if there is a non-zero $\boldsymbol{v} \in \mathbb{Z}^d$ such that $(X^{\boldsymbol{v}} - 1)c = 0$. Here the right side is understood as the constant zero configuration.

For a Laurent polynomial $f(X) = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$ and a positive integer n define $f(X^n) = \sum a_{\boldsymbol{v}} X^{n\boldsymbol{v}}$. (See Figure 3.1.) The following example, and the proof of Lemma 3.2.2, use the well known fact that for any integral Laurent polynomial f and prime number p, we have $f^p(X) \equiv f(X^p) \pmod{p}$.

Example 3.1.1. The example concerns the periodic tiling problem. We provide a short proof of the fact – originally proved in [Sze98] – that if the size $p = |D|$ of tile D is a prime number then all co-tilers C are periodic. When the tile D is represented as the Laurent polynomial $f(X) = \sum_{\boldsymbol{v} \in D} X^{\boldsymbol{v}}$ and the co-tiler C as the power series $c(X) = \sum_{\boldsymbol{v} \in C} X^{\boldsymbol{v}}$, the tiling condition (1.1) states that $f(X)c(X) = \sum_{\boldsymbol{v} \in \mathbb{Z}^d} X^{\boldsymbol{v}}$. Multiplying both sides by $f^{p-1}(X)$, we get

$$f^p(X)c(X) = \sum_{\boldsymbol{v} \in \mathbb{Z}^d} p^{p-1} X^{\boldsymbol{v}} \equiv 0 \pmod{p}.$$

On the other hand, since p is a prime, $f^p(X) \equiv f(X^p) \pmod{p}$ so that

$$f(X^p)c(X) \equiv 0 \pmod{p}.$$

Let $\boldsymbol{v} \in D$ and $\boldsymbol{w} \in C$ be arbitrary. We have

$$0 \equiv [f(X^p)c(X)]_{\boldsymbol{w}+p\boldsymbol{v}} = \sum_{\boldsymbol{u} \in D} c(X)_{\boldsymbol{w}+p\boldsymbol{v}-p\boldsymbol{u}} \pmod{p}.$$

The last sum is a sum of p numbers, each 0 or 1, among which there is at least one 1 (corresponding to $\boldsymbol{u} = \boldsymbol{v}$). The only way for the sum to be divisible by p is by having each summand equal to 1. We have that $\boldsymbol{w} + p(\boldsymbol{v} - \boldsymbol{u})$ is in C for all $\boldsymbol{u}, \boldsymbol{v} \in D$ and $\boldsymbol{w} \in C$, which means that C is $p(\boldsymbol{v} - \boldsymbol{u})$-periodic for all $\boldsymbol{u}, \boldsymbol{v} \in D$. □

The next lemma grants us that for low complexity configurations there exists at least one Laurent polynomial that annihilates the configuration by formal multiplication.

Lemma 3.1.2. *Let c be a configuration and $D \subset \mathbb{Z}^d$ a finite domain such that $P_c(D) \leq |D|$. Then there exists a non-zero Laurent polynomial $f \in \mathbb{C}[X^{\pm 1}]$ such that $fc = 0$.*

Proof. Denote $D = \{\boldsymbol{u_1}, \ldots, \boldsymbol{u_n}\}$ and consider the set

$$\{ (1, c_{\boldsymbol{u_1}+\boldsymbol{v}}, \ldots, c_{\boldsymbol{u_n}+\boldsymbol{v}}) \mid \boldsymbol{v} \in \mathbb{Z}^d \}.$$

It is a set of complex vectors of dimension $n+1$, and because c has low complexity there is at most n of them. Therefore there exists a common non-zero orthogonal vector $(\overline{a_0}, \ldots, \overline{a_n})$. Let $g(X) = a_1 X^{-\boldsymbol{u_1}} + \cdots + a_n X^{-\boldsymbol{u_n}} \neq 0$, then the coefficient of gc at position $\boldsymbol{v}$ is

$$(gc)_{\boldsymbol{v}} = a_1 c_{\boldsymbol{u_1}+\boldsymbol{v}} + \cdots + a_n c_{\boldsymbol{u_n}+\boldsymbol{v}} = -a_0,$$

that is, gc is a constant configuration. Now it suffices to set $f = (X^{\boldsymbol{v}} - 1)g$ for arbitrary non-zero vector $\boldsymbol{v} \in \mathbb{Z}^d$. □

3.2 Annihilating Laurent polynomials

Let c be a configuration. We say that a Laurent polynomial f *annihilates* (or is an *annihilator* of) the configuration if $fc = 0$. Define

$$\mathrm{Ann}(c) = \{ f \in \mathbb{C}[X^{\pm 1}] \mid fc = 0 \}.$$

It is the set of all annihilators of c. Clearly it is an ideal of $\mathbb{C}[X^{\pm 1}]$. The zero polynomial annihilates every configuration; let us call the annihilator *non-trivial* if it is non-zero.

An easy, but useful observation is that if f is an annihilator, then any monomial multiple $X^{\boldsymbol{v}} f$ is also an annihilator. We shall use this fact without further reference.

There are good reasons why to study this ideal. Firstly, by Lemma 3.1.2, for low complexity configurations $\mathrm{Ann}(c)$ is non-trivial, which is the case of Nivat's conjecture and periodic tiling problem. Secondly, to prove that a configuration is periodic is equivalent to showing that $X^{\boldsymbol{v}} - 1$ annihilates c for some non-zero $\boldsymbol{v} \in \mathbb{Z}^d$.

We defined $\mathrm{Ann}(c)$ to consist of complex Laurent polynomials, so that we can later use Hilbert's Nullstellensatz directly, as it requires ideals over an algebraically closed field. We shall however occasionally work with integer coefficients when it is more convenient.

Convention. To simplify the exposition, from now on by *polynomial* we mean Laurent polynomial; if the classical meaning is needed we use the term *proper polynomial.*

In what follows we consider configurations which have an integral annihilator. Although it follows by a small modification of Lemma 3.1.2 that such an annihilator for integral configurations exists, a stronger statement holds:

Lemma 3.2.1. *Let c be an integral configuration. Then* $\mathrm{Ann}(c)$ *is generated by finitely many integral polynomials.*

Proof. We will show that $\mathrm{Ann}(c)$ is generated by integral polynomials, the claim then follows from Hilbert's Basis Theorem. Let $f \in \mathrm{Ann}(c)$ be arbitrary and denote

$$f(X) = \sum_{i=1}^{n} a_i X^{\boldsymbol{u_i}}.$$

Let V be a vector subspace of $\mathbb{C}^n$ defined by

$$V := \left\langle (c_{\boldsymbol{v}-\boldsymbol{u_1}}, \ldots, c_{\boldsymbol{v}-\boldsymbol{u_n}}) \mid \boldsymbol{v} \in \mathbb{Z}^d \right\rangle.$$

Then $fc = 0$ if and only if $(\overline{a_1}, \ldots, \overline{a_n}) \perp V$. All the vectors in V have integer coordinates, therefore the space $V^\perp$ has a basis consisting of rational, and therefore also integer vectors $\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(m)}$. Denote $\boldsymbol{b}^{(j)} = (b_1^{(j)}, \ldots, b_n^{(j)})$.

Consider integral polynomials $g^{(j)}(X) = \sum_{i=1}^{n} b_i^{(j)} X^{\boldsymbol{u_i}}$. Because $\overline{\boldsymbol{b}^{(j)}} = \boldsymbol{b}^{(j)} \perp V$ we have that $g^{(j)}$ is an integral annihilator of c. From construction the polynomial f is a linear combination of $g^{(1)}, \ldots, g^{(m)}$, which concludes the proof. □

Let us introduce additional notation: if $Z = (z_1, \ldots, z_d) \in (\mathbb{C}^*)^d$ is a complex vector, then it can be plugged into a polynomial. In particular, plugging into a monomial $X^{\boldsymbol{v}}$ results in $Z^{\boldsymbol{v}} = z_1^{v_1} \cdots z_d^{v_d}$. Recall that the notation $f(X^n)$ for positive integers n was defined in section 3.1.

Lemma 3.2.2. *Let $c(X)$ be a finitary integral configuration and $f(X) \in$* $\mathrm{Ann}(c)$ *a non-zero integer polynomial. Then there exists an integer r such that for every positive integer n relatively prime to r we have $f(X^n) \in$* $\mathrm{Ann}(c)$*.*

Proof. Denote $f(X) = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$ and let $m \in \mathbb{N}$ be arbitrary. We prove that if $f(X^m)$ is an annihilator, then also $f(X^{pm})$ is an annihilator for every large enough prime p.

Let p be a prime. Since $f^p(X) \equiv f(X^p) \pmod{p}$ we especially have $f^p(X^m) \equiv f(X^{pm}) \pmod{p}$. We assume that $f(X^m)$ annihilates $c(X)$, therefore multiplying both sides by $c(X)$ results in

$$0 \equiv f(X^{pm})c(X) \pmod{p}.$$

The coefficients in $f(X^{pm})c(X)$ are bounded in absolute value by

$$s = c_{max} \sum |a_{\boldsymbol{v}}|,$$

where c_{max} is the maximum absolute value of coefficients in c. Note that the bound is independent of m. Therefore for any m, if $p > s$ we have $f(X^{pm})c(X) = 0$, which means $f(X^{pm}) \in \mathrm{Ann}(c)$.

To finish the proof, set $r = s!$. Now every n relatively prime to r is of the form $p_1 \cdots p_k$ where each p_i is a prime greater than s. Because $f(X)$ is an annihilator now it follows easily by induction that also $f(X^{p_1 \cdots p_k})$ is an annihilator. $\square$

Let us define the *support* of a polynomial $f = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$ as

$$\mathrm{supp}(f) = \{\, \boldsymbol{v} \in \mathbb{Z}^d \mid a_{\boldsymbol{v}} \neq 0 \,\}.$$

Lemma 3.2.3. *Let c be a finitary integral configuration and $f = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$ a non-trivial integer polynomial annihilator. Define*

$$g(X) = \prod_{\substack{\boldsymbol{v} \in \mathrm{supp}(f) \\ \boldsymbol{v} \neq \boldsymbol{v}_0}} (X^{r(\boldsymbol{v} - \boldsymbol{v}_0)} - 1)$$

where r is the integer from Lemma 3.2.2 and $\mathbf{v_0} \in \mathrm{supp}(f)$ arbitrary. Then $g(Z) = 0$ for any common root $Z \in (\mathbb{C}^*)^d$ of $\mathrm{Ann}(c)$, i.e. $g \in \mathbf{IV}(\mathrm{Ann}(c))$.

Proof. Fix Z. Let us define for $\alpha \in \mathbb{C}$

$$S_\alpha = \left\{ \boldsymbol{v} \in \mathrm{supp}(f) \;\middle|\; Z^{r\boldsymbol{v}} = \alpha \right\},$$
$$f_\alpha(X) = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} X^{\boldsymbol{v}}.$$

Because $\mathrm{supp}(f)$ is finite, there are only finitely many non-empty sets $S_{\alpha_1}, \ldots, S_{\alpha_m}$ and they form a partitioning of $\mathrm{supp}(f)$. In particular we have $f = f_{\alpha_1} + \cdots + f_{\alpha_m}$.

Numbers of the form $1 + ir$ are relatively prime to r for all non-negative integers i, therefore by Lemma 3.2.2, $f(X^{1+ir}) \in \mathrm{Ann}(c)$. Plugging in Z we obtain $f(Z^{1+ir}) = 0$. Now compute:

$$f_\alpha(Z^{1+ir}) = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} Z^{(1+ir)\boldsymbol{v}} = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} Z^{\boldsymbol{v}} \alpha^i = f_\alpha(Z)\alpha^i$$

Summing over $\alpha = \alpha_1, \ldots, \alpha_m$ gives

$$0 = f(Z^{1+ir}) = f_{\alpha_1}(Z)\alpha_1^i + \cdots + f_{\alpha_m}(Z)\alpha_m^i$$

Let us rewrite the last equation as a statement about orthogonality of two vectors in $\mathbb{C}^m$:

$$\left(\overline{f_{\alpha_1}(Z)}, \ldots, \overline{f_{\alpha_m}(Z)}\right) \perp (\alpha_1^i, \ldots, \alpha_m^i)$$

By Vandermode determinant, for $i \in \{0, \ldots, m-1\}$ the vectors on the right side span the whole $\mathbb{C}^m$. Therefore the left side must be the zero vector, and especially for α such that $\boldsymbol{v_0} \in S_\alpha$ we have

$$0 = f_\alpha(Z) = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} Z^{\boldsymbol{v}}.$$

Because Z does not have zero coordinates, each term on the right hand side is non-zero. But the sum is zero, therefore there are at least two vectors $\boldsymbol{v_0}, \boldsymbol{v} \in S_\alpha$. From the definition of S_α we have $Z^{r\boldsymbol{v}} = Z^{r\boldsymbol{v_0}} = \alpha$, so Z is a root of $X^{r(\boldsymbol{v}-\boldsymbol{v_0})} - 1$. □

3.3 Line polynomials

We say that a polynomial f is a *line polynomial* if its support contains at least two points and all the points lie on a single line. Let us call a vector $\boldsymbol{v} \in \mathbb{Z}^d$ *primitive* if its coordinates don't have a common non-trivial integer factor. Then every line polynomial can be expressed as

$$f(X) = X^{\boldsymbol{v}'}(a_n X^{n\boldsymbol{v}} + \cdots + a_1 X^{\boldsymbol{v}} + a_0)$$

for some $a_i \in \mathbb{C}$, $n \geq 1$, $a_n \neq 0 \neq a_0$, $\boldsymbol{v}', \boldsymbol{v} \in \mathbb{Z}^d$, where $\boldsymbol{v}$ is primitive. Moreover, the vector $\boldsymbol{v}$ is determined uniquely up to the sign. We define the *direction* of a line polynomial to be the vector space $\langle \boldsymbol{v} \rangle \subset \mathbb{Q}^d$.

Recall that an ideal $A \leq \mathbb{C}[X]$ is *radical* if $a^n \in A$ implies $a \in A$. The next lemma states that for one-dimensional configurations $\mathrm{Ann}(c)$ is radical.

Lemma 3.3.1. *Let $c \in \mathbb{C}[[x^{\pm 1}]]$ be a finitary one-dimensional configuration annihilated by f^m for a non-trivial polynomial f and $m \in \mathbb{N}$. Then it is also annihilated by f.*

Proof. The configuration c can be viewed as a sequence attaining only finitely many values, and f^m as a recurrence relation on it. Therefore c must be periodic, which means there is $n \in \mathbb{N}$ such that $x^n - 1 \in \mathrm{Ann}(c)$.

Then also $g = \gcd(x^n - 1, f^m) \in \mathrm{Ann}(c)$. Because g divides $x^n - 1$, it has only simple roots, and from $g \mid f^m$ we conclude $g \mid f$. Any multiple of g annihilates the sequence, hence also f does. □

Lemma 3.3.2. *Let c be a finitary configuration and $f_1, \ldots, f_k$ line polynomials such that $f_1^{m_1} \cdots f_k^{m_k}$ annihilates c. Then also $f_1 \cdots f_k$ annihilates it.*

Proof. We will show that if f is a line polynomial and f^m annihilates c, then also f annihilates c. Without loss of generality assume

$$f(X) = a_n X^{n\boldsymbol{v}} + \cdots + a_1 X^{\boldsymbol{v}} + a_0$$

for some $a_i \in \mathbb{C}$ and $\boldsymbol{v} \in \mathbb{Z}^d$. Define $g(t) = a_n t^n + \cdots + a_1 t + a_0 \in \mathbb{C}[t]$ so that $f^m(X) = g^m(X^{\boldsymbol{v}})$.

For any $\boldsymbol{u} \in \mathbb{Z}^d$ the sequence of coefficients $(c_{\boldsymbol{u}+i\boldsymbol{v}})_{i\in\mathbb{Z}}$ can be viewed as a one-dimensional configuration annihilated by g^m. By Lemma 3.3.1 it is also annihilated by g, therefore $g(X^{\boldsymbol{v}}) = f(X)$ annihilates c.

To finish the proof observe that $f_2^{m_2} \dots f_k^{m_k} c$ is a finitary configuration annihilated by $f_1^{m_1}$. Thus it is also annihilated by f_1 and $f_1 f_2^{m_2} \dots f_k^{m_k} c = 0$. The argument can be repeated for all f_i. □

Theorem 3.3.3. *Let c be a finitary integral configuration with a non-trivial annihilator. Then there exist vectors $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbb{Z}^d$ in pairwise distinct directions such that*

$$(X^{\boldsymbol{v_1}} - 1) \cdots (X^{\boldsymbol{v_m}} - 1) \in \mathrm{Ann}(c).$$

Proof. By Lemma 3.2.1, c has an integral annihilator. Therefore Lemma 3.2.3 applies, let $g \in \mathbf{IV}(\mathrm{Ann}(c))$ be as in its statement. By Hilbert's Nullstellensatz $g \in \sqrt{\mathrm{Ann}(c)}$, so there exists an integer m such that $g^m \in \mathrm{Ann}(c)$. Since g has only line polynomial factors of the form $X^{\boldsymbol{v}} - 1$, by Lemma 3.3.2, $g \in \mathrm{Ann}(c)$.

To finish the proof we have to guarantee that the vectors in exponents are in distinct pairwise directions. Observe that $(X^{a\boldsymbol{u}} - 1)(X^{b\boldsymbol{u}} - 1)$ divides $(X^{ab\boldsymbol{u}} - 1)^2$, and therefore any two factors in the same direction $(X^{a\boldsymbol{u}} - 1)(X^{b\boldsymbol{u}} - 1)$ can be by Lemma 3.3.2 replaced by a single factor $(X^{ab\boldsymbol{u}} - 1)$. □

Corollary 3.3.4. *Let c be a low complexity configuration. Then there exist vectors $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbb{Z}^d$ in pairwise distinct directions such that*

$$(X^{\boldsymbol{v_1}} - 1) \cdots (X^{\boldsymbol{v_m}} - 1) \in \mathrm{Ann}(c).$$

Proof. Since c is of low complexity it is also finitary. Let $\{a_1, \dots, a_n\}$ be the set of coefficients of c, then there exist configurations $c_1, \dots, c_n$ with coefficients in $\{0, 1\}$ such that $c = a_1 c_1 + \cdots + a_n c_n$. Moreover configurations c_i are also of low complexity, and by Lemma 3.1.2 each of them has a non-trivial annihilator.

By Theorem 3.3.3, for $i \in \{1, \dots, n\}$ there exist $m_i \in \mathbb{N}$ and non-zero vectors $\boldsymbol{v}_{i,j} \in \mathbb{Z}^2, j \in \{1, \dots, m_i\}$ such that

$$(X^{\boldsymbol{v_{i,1}}} - 1) \cdots (X^{\boldsymbol{v_{i,m_i}}} - 1) \in \mathrm{Ann}(c_i).$$

We can choose m and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ such that $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\} = \{\, \boldsymbol{v}_{i,j} \mid 1 \le i \le n, 1 \le j \le m_i \,\}$, then $(X^{\boldsymbol{v_1}} - 1) \cdots (X^{\boldsymbol{v_m}} - 1) \in \mathrm{Ann}(c)$. To have $\boldsymbol{v}_i$ in distinct directions we finish the proof the same way as in Theorem 3.3.3. □

3.4 Decomposition theorem

Multiplying a configuration by $(X^{\boldsymbol{v}}-1)$ can be seen as a *"difference operator"* on the configuration. Theorem 3.3.3 then says, that there is a sequence of difference operators which annihilates the configuration. We can reverse the process: let us start by a zero configuration and step by step *"integrate"* until we obtain the original configuration. This idea gives the Decomposition theorem:

Theorem 3.4.1 (Decomposition theorem)**.** *Let c be a finitary integral configuration with a non-trivial annihilator. Then there exist periodic integral configurations $c_1, \ldots, c_m$ such that $c = c_1 + \cdots + c_m$.*

The proof goes by a series of lemmas.

Lemma 3.4.2. *Let f, g be line polynomials in distinct directions and c a configuration annihilated by g. Then there exists a configuration c' such that $fc' = c$ and c' is also annihilated by g.*

Proof. Without loss of generality assume f, g are of the form

$$\begin{aligned} f(X) &= a_n X^{n\boldsymbol{u}} + \cdots + a_1 X^{\boldsymbol{u}} + a_0 \\ g(X) &= b_m X^{m\boldsymbol{v}} + \cdots + b_1 X^{\boldsymbol{v}} + b_0 \end{aligned}$$

for some vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^d$, $n, m \in \mathbb{N}$ and $a_i, b_i \in \mathbb{C}$ such that a_n, b_m, a_0, b_0 are all non-zero.

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent and the whole space $\mathbb{Z}^d$ is partitioned into two-dimensional sublattices (cosets) modulo $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$. Fix one such a sublattice Λ and a point $\boldsymbol{z} \in \Lambda$, then every point in the sublattice can be uniquely expressed as $\boldsymbol{z} + a\boldsymbol{u} + b\boldsymbol{v}$ for some $a, b \in \mathbb{Z}$. Denote $[a, b] = \boldsymbol{z} + a\boldsymbol{u} + b\boldsymbol{v}$.

The equation $fc' = c$ is satisfied if and only if

$$a_n c'_{[a-n,b]} + \cdots + a_1 c'_{[a-1,b]} + a_0 c'_{[a,b]} = c_{[a,b]} \tag{3.1}$$

holds for every $a, b \in \mathbb{Z}$ (on every sublattice Λ). This is a linear recurrence relation on the sequences $(c'_{[a,b]})_{a \in \mathbb{Z}}$. Let us define $c'_{[a,b]} = 0$ if $0 \leq a < n$, the rest of c' is then uniquely determined by the recurrence relation so that $fc' = c$ holds.

It remains to show that c' defined this way is annihilated by g. A simple computation shows that

$$f(gc') = g(fc') = gc = 0.$$

Therefore the configuration gc' satisfies a linear recurring relation defined by f on the sequences $\left((gc')_{[a,b]}\right)_{a \in \mathbb{Z}}$. Moreover we have $(gc')_{[a,b]} = 0$ for $0 \leq a < n$, from which it follows that gc' is zero everywhere. $\square$

Lemma 3.4.3. *Let $f_1, \ldots, f_m$ be line polynomials in pairwise distinct directions and c a configuration annihilated by their product $f_1 \cdots f_m$. Then there exist configurations $c_1, \ldots, c_m$ such that f_i annihilates c_i and*

$$c = c_1 + \cdots + c_m.$$

Proof. The proof goes by induction on m. For $m = 1$ there is nothing to prove, assume $m \geq 2$.

Since the configuration $f_m c$ is annihilated by $f_1 \cdots f_{m-1}$, by induction hypothesis we have

$$f_m c = b_1 + \cdots + b_{m-1}$$

where each b_i is annihilated by f_i for $1 \leq i < m$. Let c_i be such that $f_m c_i = b_i$ and c_i is annihilated by f_i, this is possible by Lemma 3.4.2. Then it suffices to set $c_m = c - c_1 - \cdots - c_{m-1}$; clearly $c = c_1 + \cdots + c_m$ and

$$f_m c_m = f_m(c - c_1 - \cdots - c_{m-1}) = 0.$$

□

Proof of Theorem 3.4.1. By Theorem 3.3.3 there is an annihilator of the form $(X^{\boldsymbol{v_1}} - 1) \cdots (X^{\boldsymbol{v_m}} - 1)$ where $(X^{\boldsymbol{v_i}} - 1)$ have distinct directions. Therefore by Lemma 3.4.3 there are $c_1, \ldots, c_m$ such that c is their sum and each c_i is periodic with the vector $\boldsymbol{v_i}$.

It remains to show that c_i can be integral. This follows from the fact that configurations in the proof of Lemma 3.4.2 are constructed by satisfying a recurrence relation (3.1), which for polynomials of the form $(X^{\boldsymbol{v_i}} - 1)$ has always integral solution. □

Corollary 3.4.4. *Let c be a low complexity configuration. Then there exist periodic configurations $c_1, \ldots, c_m$ such that $c = c_1 + \cdots + c_m$.*

Proof. The proof is identical to the first part of the proof of Theorem 3.4.1, using Corollary 3.3.4 instead of Theorem 3.3.3. □

Example 3.4.5. Recall the counterexample to the analogue of Nivat's conjecture in 3 dimensions from the introduction. It is the sum $c_1 + c_2$ where $c_1(i, 0, 0) = 1$ and $c_2(0, n, i) = 1$ for all $i \in \mathbb{Z}$, and all other entries are 0. Configurations c_1 and c_2 are $(1, 0, 0)$- and $(0, 0, 1)$-periodic, respectively, so that $(X^{(1,0,0)} - 1)(X^{(0,0,1)} - 1)$ annihilates $c = c_1 + c_2$. □

Example 3.4.6. The periodic configurations $c_1, \ldots, c_m$ in Theorem 3.4.1 may, for some configurations c, be necessarily non-finitary. Let $\alpha \in \mathbb{R}$ be irrational, and define three periodic two-dimensional configurations c_1, c_2 and c_3 by

$$c^{(1)}_{ij} = \lfloor j\alpha \rfloor, \qquad c^{(2)}_{ij} = \lfloor i\alpha \rfloor, \qquad c^{(3)}_{ij} = \lfloor (i + j)\alpha \rfloor.$$

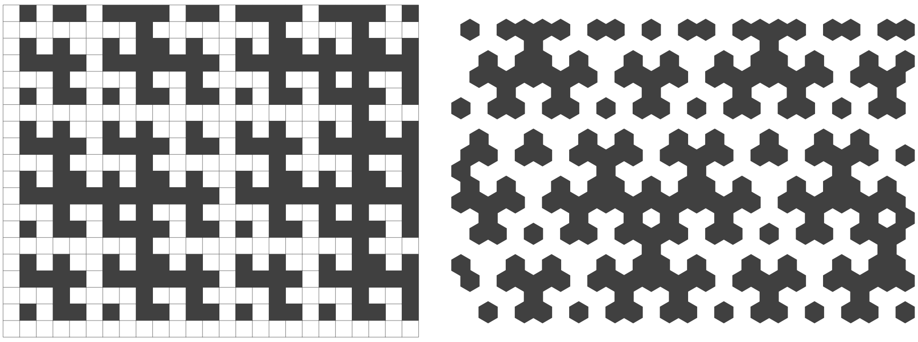

Figure 3.2: The configuration c from Example 3.4.6 when α is the golden ratio is shown on the left. On the right the configuration is skewed such that the three directions $\langle(1,0)\rangle$, $\langle(0,1)\rangle$ and $\langle(1,-1)\rangle$ became symmetrical, the bottom left corner is preserved.

Then $s = c_3 - c_1 - c_2$ is a finitary integral configuration over alphabet $\{0,1\}$, annihilated by the polynomial $(X^{(1,0)} - 1)(X^{(0,1)} - 1)(X^{(1,-1)} - 1)$, but it cannot be expressed as a sum of finitary periodic configurations (for a proof see [KS15a] or Example 4.3.4). Figure 3.2 illustrates the setup for α being the golden ratio. $\square$

Chapter 4

Two-dimensional configurations

4.1 Radicality of annihilator ideal

In the rest of the thesis we focus on two-dimensional configurations. We analyze $\mathrm{Ann}(c)$ using tools of algebraic geometry and provide a description of a polynomial ϕ which divides every annihilator. Moreover we show a theoretical result that $\mathrm{Ann}(c)$ is a radical ideal, which allows us to formulate a more explicit version of the decomposition theorem for two-dimensional configurations. We end the chapter by defining an important characteristic $\mathrm{ord}(c)$ of a configuration c as the minimal possible number of periodic configurations which sum to c.

To simplify the notation, we prefer to write $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ in the place of $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$. This section heavily uses algebraic structure of radical ideals in $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, see section 2.4 for a review of used claims.

Theorem 4.1.1. *Let c be a two-dimensional finitary integral configuration with a non-trivial annihilator. Then $\mathrm{Ann}(c)$ is a radical ideal. Moreover for every prime ideal P from the minimal decomposition of $\mathrm{Ann}(c)$ we have*

$$P = \langle x^a y^b - \omega \rangle \quad \textit{or} \quad P = \langle x - \omega_x, y - \omega_y \rangle$$

for $(a, b) \in \mathbb{Z}^2$ primitive vector and $\omega, \omega_x, \omega_y \in \mathbb{C}$ roots of unity.

Proof. Denote $A = \sqrt{\mathrm{Ann}(c)}$. Since c has a non-trivial annihilator, A is non-trivial. Let $A = P_1 \cap \cdots \cap P_k$ be its minimal decomposition.

Let P be one of P_i. Assume first that $P = \langle \varphi \rangle$ for an irreducible polynomial φ. By Theorem 3.3.3 there exist vectors $\boldsymbol{v_i}$ such that

$$(X^{\boldsymbol{v_1}} - 1) \cdots (X^{\boldsymbol{v_n}} - 1) \in A.$$

Since φ is an irreducible factor of this polynomial we have $\varphi \mid X^{\boldsymbol{v}} - 1$ for some $\boldsymbol{v}$. Let $\boldsymbol{v} = d\boldsymbol{w}$ for a primitive vector $\boldsymbol{w} = (a, b)$ and $d > 0$. Observe that

$$X^{\boldsymbol{v}} - 1 = X^{d\boldsymbol{w}} - 1 = (X^{\boldsymbol{w}} - \omega_1) \cdots (X^{\boldsymbol{w}} - \omega_d)$$

where $\omega_1, \dots, \omega_d$ are d-th roots of unity. Therefore φ is, up to a multiple by an invertible element, of the form

$$x^a y^b - \omega$$

for ω a root of unity. This classifies the case of principal ideals P.

Now assume that $P = \langle x - \alpha, y - \beta \rangle$ for some $\alpha, \beta \in \mathbb{C}^*$, without loss of generality let $P = P_1$. Choose $g \in \prod_{i=2}^{k} (P_i \setminus P_1)$ arbitrarily, then $g(x - \alpha) \in A$ and $g \notin A$. There exists $m \in \mathbb{N}$ such that $g^m (x - \alpha)^m \in \mathrm{Ann}(c)$, but $g^m \notin \mathrm{Ann}(c)$. In other words, $(x - \alpha)^m$ annihilates the non-zero finitary configuration $c' = g^m c$. By Lemma 3.3.2 also $x - \alpha$ annihilates c', and therefore for every $i, j \in \mathbb{Z}$

$$c'_{i,j} = c'_{0,j} \alpha^{-i}.$$

If α is not a root of unity then c' is not finitary, which is a contradiction. A similar argument applies to β.

To prove the radicality of $\mathrm{Ann}(c)$, observe that each P_i is generated by line polynomials. By Theorem 2.4.6 we have $A = P_1 \cdots P_k$, A has a finite set of generators $A = \langle g_1, \dots, g_k \rangle$ such that each g_i is a product of line polynomials. Then for each i there exists $m \in \mathbb{N}$ such that $g_i^m \in \mathrm{Ann}(c)$, and by Lemma 3.3.2 we have $g_i \in \mathrm{Ann}(c)$. $\mathrm{Ann}(c)$ contains a set of generators of its radical, and therefore it is a radical ideal. □

The proof of the radicality of $\mathrm{Ann}(c)$ relies on the decomposition of two-dimensional radical ideal into a product of primes. Although no analog of such statement is available in higher dimensions, we conjecture that $\mathrm{Ann}(c)$ is radical for higher dimensional finitary configurations as well.

To give a more explicit version of the decomposition theorem, let us study in greater detail how addition of configurations relates to their annihilators.

Lemma 4.1.2. *Let c be a configuration and $A_1, \dots, A_k$, $k \geq 2$ pairwise comaximal ideals such that* $\mathrm{Ann}(c) = A_1 \cap \cdots \cap A_k$. *Then there are uniquely determined configurations $c_1, \dots, c_k$ such that* $\mathrm{Ann}(c_i) = A_i$ *and* $c = c_1 + \cdots + c_k$.

Proof. Let us assume that none of A_i is the whole ideal, otherwise we can set $c_i = 0$ and exclude A_i from the list. Note that $\mathrm{Ann}(c) = A_1 \cdots A_k$. We use the following two easy to prove facts from commutative algebra. If A_i are pairwise comaximal then:

- (a) The ideals A_1 and $A_2 \cdots A_k$ are comaximal.
- (b) There exist $f_1, \dots, f_k$ such that $f_i \notin A_i$, $f_i \in \prod_{j \neq i} A_j$ and $f_1 + \cdots + f_k = 1$.

Let f_i be as in (b) and set $c_i = f_i c$. Then $c = c_1 + \cdots + c_k$. Let us show $A_1 \subset \mathrm{Ann}(c_1)$:

$$g \in A_1 \quad \Rightarrow \quad gf_1 \in A_1 \cdots A_k = \mathrm{Ann}(c) \quad \Rightarrow \quad g \in \mathrm{Ann}(f_1 c) = \mathrm{Ann}(c_1).$$

Next let us show $\mathrm{Ann}(c_1) \subset A_1$. Note that $(1 - f_1) = f_2 + \cdots + f_k \in A_1$ and compute:

$$g \in \mathrm{Ann}(c_1) \quad \Rightarrow \quad gf_1 \in \mathrm{Ann}(c) \subset A_1 \quad \Rightarrow \quad g = gf_1 + g(1 - f_1) \in A_1.$$

For the uniqueness assume $c = c_1' + \cdots + c_k'$ such that $c_1 \neq c_1'$ and $\mathrm{Ann}(c_i') = A_i$. By (a) let $f \in A_1$ and $g \in A_2 \cdots A_k$ be such that $f + g = 1$. Then

$$\begin{aligned} c_1 - c_1' &= f(c_1 - c_1') + g(c_1 - c_1') \\ &= f(c_1 - c_1') + g(-c_2 - \cdots - c_k + c_2' + \cdots + c_k') = 0. \end{aligned}$$

The argument can be repeated for all c_i. □

Theorem 4.1.3 (Two-dimensional decomposition theorem)**.** *Let c be as in Theorem 4.1.1 and $P_1 \cap \cdots \cap P_k$ be the minimal decomposition of* $\mathrm{Ann}(c)$*. Then there exist configurations $c_1, \ldots, c_k$ such that* $\mathrm{Ann}(c_i) = P_i$ *and* $c = c_1 + \cdots + c_k$.

Proof. Let $R_1, \ldots, R_s$, $M_1, \ldots, M_t$ be as in Theorem 2.4.6. By the same theorem, the ideals $R = \prod R_i, M_1, \ldots, M_t$ are pairwise comaximal, and by Lemma 4.1.2 there are configurations $c_R, c_{M_1}, \ldots, c_{M_t}$ annihilated by corresponding ideals such that $c = c_R + c_{M_1} + \cdots + c_{M_t}$.

By Theorem 4.1.1, $R_i = \langle \varphi_i \rangle$ for a line polynomial φ_i. These polynomials are in finitely many distinct directions m. Define $\phi_1, \ldots, \phi_m$ such that each ϕ_j is product of all φ_i in the same direction. Then, by Lemma 3.4.3, there are $c_{\phi_1}, \ldots, c_{\phi_m}$ annihilated by corresponding polynomials such that $c_R = c_{\phi_1} + \cdots + c_{\phi_m}$.

Moreover $\mathrm{Ann}(c_{\phi_i}) = \langle \phi_i \rangle$: if $f \in \mathrm{Ann}(c_{\phi_1})$, then $f\phi_2 \cdots \phi_m \in \mathrm{Ann}(c_R) = R$. The ideal R is one-generated, so $\phi_1 \cdots \phi_m \mid f\phi_2 \cdots \phi_m$ and therefore $f \in \langle \phi_1 \rangle$. Analogously for other ϕ_i.

For the final step define $S_1 \subset \{1, \ldots, s\}$ such that $\phi_1 = \prod_{i \in S_1} \varphi_i$. Since all φ_i for $i \in S_1$ have the same direction, by Theorem 4.1.1 they are all of the form $\varphi_i = x^a y^b - \omega_i$. Then $\langle \varphi_i \rangle = R_i$ for $i \in S_1$ are pairwise comaximal and by Lemma 4.1.2 there exist c_{R_i} annihilated by R_i such that $c_{\phi_1} = \sum_{i \in S_1} c_{R_i}$.

Analogously we can decompose each c_{ϕ_i}. To finish the proof observe that

$$c = c_{R_1} + \cdots + c_{R_s} + c_{M_1} + \cdots + c_{M_t}.$$

□

4.2 Decomposition of two-dimensional configurations

We say that a two-dimensional configuration is *doubly periodic* if there are two linearly independent vectors in which it is periodic. A configuration which is periodic but not doubly periodic is called *one-periodic.*

Corollary 4.2.1. *Let c be as in Theorem 4.1.1.*

(*a*) *There exist a non-negative integer m, line polynomials $\phi_1, \ldots, \phi_m$ in pairwise distinct directions, a polynomial $\phi := \phi_1 \cdots \phi_m$ and an ideal H which is an intersection of maximal ideals such that $\langle \phi \rangle$ and H are comaximal and*

$$\mathrm{Ann}(c) = \phi_1 \cdots \phi_m H = \phi H.$$

Moreover m and H are determined uniquely and $\phi, \phi_1, \ldots, \phi_m$ are determined uniquely up to an invertible factor and the order.

(*b*) *There exist configurations $c_\phi, c_H, c_1, \ldots, c_m$ such that*

$$c = c_1 + \cdots + c_m + c_H = c_\phi + c_H$$

where $\mathrm{Ann}(c_\phi) = \langle \phi \rangle$, $\mathrm{Ann}(c_H) = H$ and $\mathrm{Ann}(c_i) = \langle \phi_i \rangle$. Moreover c_ϕ and c_H are determined uniquely. Each c_i is one-periodic in the direction of ϕ_i, and c_H is doubly periodic.

Proof. Let us continue with the notation from the proof of Theorem 4.1.3.

(*a*) Let $H = \bigcap_{i=1}^{t} M_i$. Then $\phi, \phi_1, \ldots, \phi_m, H$ are as desired.

(*b*) Let $c_H = c_{M_1} + \cdots + c_{M_t}$, $c_\phi = c_R$ and $c_i = c_{\phi_i}$. The fact that $\mathrm{Ann}(c_H) = H$ follows by Lemma 4.3.1 introduced later and the uniqueness of c_ϕ and c_H follows by Lemma 4.1.2.

Let $\boldsymbol{v}$ be a primitive direction of the polynomial ϕ_1. There is $n \in \mathbb{N}$ such that each irreducible factor of ϕ_1 divides $X^{n\boldsymbol{v}} - 1$. Therefore this Laurent polynomial annihilates c_{ϕ_1} which means that c_{ϕ_1} has period $n\boldsymbol{v}$. If there was a period $\boldsymbol{u}$ in any other direction, then $\phi_1 \mid X^{\boldsymbol{u}} - 1$, which is impossible. Therefore c_{ϕ_1} is one-periodic, and so is any c_{ϕ_i}.

Denote $M_1 = \langle x - \omega_x, y - \omega_y \rangle$ and let $n \in \mathbb{N}$ be such that $\omega_x^n = 1$. Then c_{M_1} has a horizontal period n since $x^n - 1 \in M_1$. Similarly c_{M_1} has a vertical period. By a similar argument each c_{M_j} is doubly periodic. A finite sum c_H of doubly periodic configurations is also doubly periodic. □

Note. Theorem 4.1.3 and Corollary 4.2.1 assume that c is as in Theorem 4.1.1, i.e. that c is a finitary and integral configuration with a non-trivial annihilator. In all three propositions this assumption can be exchanged with a single assumption that c is a low complexity configuration. The latter condition implies that c is finitary and has a non-trivial annihilator (see

Corollary 3.3.4). The missing condition of c being integral is used in the proof of Theorem 4.1.1 when Theorem 3.3.3 is invoked. We can use Corollary 3.3.4 in that place instead. We omit a formal proof of these altered three proposition as we don't use them in the sequel.

Let us denote the number m from Corollary 4.2.1 by $\text{ord}(c)$. It is an important characteristic of the configuration which provides information about its periodicity.

Corollary 4.2.2. *Let c be as in Theorem 4.1.1. Then*

- $\text{ord}(c) = 0$ *if and only if c is doubly periodic,*
- $\text{ord}(c) = 1$ *if and only if c is one-periodic,*
- $\text{ord}(c) \geq 2$ *if and only if c is non-periodic.*

Proof. If $\text{ord}(c) = 0$ then $c = c_H$, which is doubly periodic. If $\text{ord}(c) = 1$ then $c = c_1 + c_H$ is a sum of one-periodic and doubly periodic configuration, which is one-periodic. If $\text{ord}(c) \geq 2$ then every annihilating polynomial is divisible by $\phi_1\phi_2$. Therefore $X^{\boldsymbol{v}} - 1$ cannot be an annihilator for any non-zero vector $\boldsymbol{v}$ and c is non-periodic. □

Corollary 4.2.3. *Let c be a two-dimensional configuration which can be written a sum of periodic configurations. If c is doubly periodic then* $\text{ord}(c) = 0$*, otherwise* $\text{ord}(c)$ *is the smallest possible number of periodic configurations which sum to c.*

Proof. The claim for doubly periodic c follows from Corollary 4.2.2. Otherwise, by Corollary 4.2.1(b), c can be written as a sum of $\text{ord}(c)$ periodic components, just add the doubly periodic component c_H to any other. If c could be written as a sum of $m < \text{ord}(c)$ periodic components, then c would be annihilated by $(X^{\boldsymbol{u_1}} - 1) \cdots (X^{\boldsymbol{u_m}} - 1)$ for some vectors $\boldsymbol{u_i}$. Since ϕ_i for $i \in \{1, \ldots, \text{ord}(c)\}$ are line polynomials in distinct directions, this polynomial is not divisible by at least one of them, which is a contradiction. □

Corollary 4.2.1 and Corollary 4.2.2 are powerful tools to analyze configurations from the structure of their annihilator ideals. The main improvement over the earlier decomposition theorem is that not only we know that c can be decomposed into a sum of periodic components, but also we can exactly describe the annihilator ideals of each component. Moreover each component is either one- or doubly periodic and the number of one-periodic components (in distinct directions) is unique and determines whether the original configuration is periodic or not.

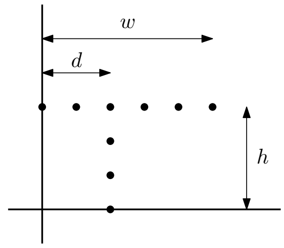

Figure 4.1: A T-shape with $w = 5$, $h = 3$ and $d = 2$. Convex hull of any non-collinear subset of its points is a triangle.

Example 4.2.4. Let us call $D \subset \mathbb{Z}^2$ a *T-shape* if it is of the form

$$D = \{0, \ldots, w\} \times \{h\} \cup \{d\} \times \{0, \ldots, h\}$$

for some $h, w, d \in \mathbb{N}$, $d \leq h$ (Figure 4.1). We show that if $P_c(D) \leq |D|$ for a T-shape D, then c is periodic. For a contradiction assume that the inequality holds and c is non-periodic.

We need a fact which is later proved in the next section as Lemma 5.1.3: The coefficients of c can be renamed such that if $P_c(D) \leq |D|$, then there is an annihilator polynomial with $\text{supp}(f) \subset -D$. Without loss of generality assume that the coefficients of c have been renamed and we have such an annihilator f.

By Corollary 4.2.2, $\text{ord}(c) \geq 2$, and in particular there are two line polynomials ϕ_1, ϕ_2 in distinct directions such that $\phi_1\phi_2$ divides any annihilator polynomial. The convex hull of $\text{supp}(\phi_1\phi_2)$ is a parallelogram, and therefore the convex hull of $\text{supp}(f)$ has two pairs of parallel sides because f is a polynomial multiple of $\phi_1\phi_2$. This is, however, impossible since $\text{supp}(f) \subset -D$ and convex hull of any non-collinear subset of points in $-D$ is a triangle.

Example 4.2.5. Let us prove a special case of the periodic tiling problem, originally proved by Szegedy [Sze98]. The claim says that if $D \subset \mathbb{Z}^2$, $|D| = 4$ tiles a plane, then there exists a periodic tiling by D.

Let C be any cotiler of D and let $c = \sum_{\boldsymbol{v} \in C} X^{\boldsymbol{v}}$ be a configuration encoding the cotiler. Let us assume that c is non-periodic, otherwise we are done. We have that $P_c(D) \leq 4$. Again we use Lemma 5.1.3 proved later which states that there is an annihilating polynomial f with support in $-D$ after renaming the coefficients of c.

Similarly as in the previous example, $\text{ord}(c) \geq 2$ and there exist ϕ_1, ϕ_2 line polynomials in distinct directions such that $\phi_1\phi_2 \mid f$. Convex hull of support of any polynomial multiple of $\phi_1\phi_2$ must have two sides parallel to the direction of ϕ_1 and two sides parallel to ϕ_2. Since $|\text{supp}(f)| \leq 4$, the only option is that $\text{supp}(f)$ consists of four points which form a parallelogram. Therefore $D = -\text{supp}(f)$.

To rephrase, we proved that if there exists an aperiodic tiling by a tile of size at most 4, the tile must consist of four points which form a parallelogram. Clearly, such a tile tiles the plane also periodically.

4.3 Annihilators of sums of configurations

Knowing a configuration and its annihilator, Theorem 4.1.3 gives a decomposition into a sum of configurations and provides their annihilators. We finish the section by giving a complementary claim: given configurations and their annihilators, we can describe the annihilator of their sum.

Lemma 4.3.1. *Let c_1, c_2 be configurations such that* $\mathrm{Ann}(c_1)$ *and* $\mathrm{Ann}(c_2)$ *are non-trivial radical ideals. Let $P_1, \dots, P_k$, $Q_1, \dots, Q_\ell$ be prime ideals such that*

$$\mathrm{Ann}(c_1) = \bigcap_{i=1}^{k} P_i \qquad \textit{and} \qquad \mathrm{Ann}(c_2) = \bigcap_{j=1}^{\ell} Q_j$$

are minimal decompositions. If $P_i \neq Q_j$ for all admissible i, j, then $\mathrm{Ann}(c_1 + c_2) = \mathrm{Ann}(c_1) \cap \mathrm{Ann}(c_2)$.

Proof. Denote $c = c_1 + c_2$, clearly $\mathrm{Ann}(c) \supset \mathrm{Ann}(c_1) \cap \mathrm{Ann}(c_2)$. To prove the other inclusion, for the contrary suppose there exists $f \in \mathrm{Ann}(c)$ such that $f \notin \mathrm{Ann}(c_1) \cap \mathrm{Ann}(c_2)$. Then f does not belong to at least one of the prime ideals. Without loss of generality assume $f \notin P_1$ and P_1 is minimal such ideal with respect to inclusion. In particular, we have $Q_j \nsubseteq P_1$ for every j.

Now choose any $g \in \prod_{j=1}^{\ell} (Q_j \setminus P_1)$, then we have $g \in \mathrm{Ann}(c_2) \setminus P_1$. Consider the polynomial fg. Since f annihilates c and g annihilates c_2, we have that fg annihilates $c - c_2 = c_1$. But $fg \notin P_1$, which is in contradiction with $\mathrm{Ann}(c_1) \subset P_1$. □

Corollary 4.3.2. *Let c_1, c_2 be two-dimensional finitary integral configurations having a non-trivial annihilator and* $k = \mathrm{ord}(c_1)$, $\ell = \mathrm{ord}(c_2)$ *such that*

$$\mathrm{Ann}(c_1) = \phi_1 \cdots \phi_k H_1 \qquad \textit{and} \qquad \mathrm{Ann}(c_2) = \psi_1 \cdots \psi_\ell H_2$$

where ϕ_i, ψ_j are line polynomials and H_1, H_2 intersections of maximal ideals as in Corollary 4.2.1. If ϕ_i and ψ_j have pairwise distinct directions, then $\mathrm{ord}(c_1 + c_2) = k + \ell$ *and there exists H an intersection of maximal ideals such that*

$$\mathrm{Ann}(c_1 + c_2) = \phi_1 \cdots \phi_k \psi_1 \cdots \psi_\ell H.$$

Example 4.3.3. Let us show that if c_1 and c_2 are two-dimensional finitary one-periodic configurations in distinct directions, then their sum is non-periodic.

By Corollary 4.2.2 we have $\mathrm{ord}(c_1) = \mathrm{ord}(c_2) = 1$, and therefore by Corollary 4.2.1 there are ϕ, ψ line polynomials such that $\mathrm{Ann}(c_1) = \phi H_1$ and $\mathrm{Ann}(c_2) = \psi H_2$ for some H_1, H_2 intersections of maximal ideals. Moreover ϕ and ψ have the same direction as is the unique direction of periodicity of c_1 and c_2 respectively. Therefore, by the previous lemma, $\mathrm{ord}(c_1 + c_2) = 2$ and therefore $c_1 + c_2$ is non-periodic by Corollary 4.2.2.

Example 4.3.4. Let us return to Example 3.4.6. As previously, define integral configurations $c^{(1)}, c^{(2)}, c^{(3)}$ by

$$c_{ij}^{(1)} = \lfloor j\alpha \rfloor, \qquad c_{ij}^{(2)} = \lfloor i\alpha \rfloor, \qquad c_{ij}^{(3)} = \lfloor (i+j)\alpha \rfloor.$$

we prove the following:

Claim. *Let $\alpha \in \mathbb{R}$ be irrational. The two-dimensional configuration s over the binary alphabet $\{0, 1\}$ defined by*

$$s = c^{(3)} - c^{(1)} - c^{(2)}$$

is a sum of three periodic integral configurations, but not a sum of finitely many finitary periodic configurations.

For a contradiction assume that $s = p^{(1)} + \cdots + p^{(m)}$ where each $p^{(i)}$ is finitary and periodic. By summing together all horizontally periodic components, without loss of generality we can assume $p^{(1)}$ to be periodic in horizontal direction (we allow possibly a zero configuration) and other $p^{(i)}$ each having a period in a non-horizontal direction. Let f_i for $i \geq 2$ be a line annihilator of $p^{(i)}$ in a non-horizontal direction.

We have $(c^{(1)} - p^{(1)}) + c^{(2)} + c^{(3)} - p^{(2)} - \cdots - p^{(m)} = 0$. Note that $c^{(1)}, c^{(2)}, c^{(3)}$ are annihilated respectively by $x-1$, $y-1$ and $x-y$. Multiply both sides by $(y-1)(x-y)f_2 \cdots f_m$ to obtain

$$g(y-1)(c^{(1)} - p^{(1)}) = 0$$

where $g = (x-y)f_2 \cdots f_m$. Let h be a horizontal line annihilator of $(c^{(1)} - p^{(1)})$. Since g and h don't have a common factor, the ideal $\langle g, h \rangle$ is zero-dimensional and in particular contains a vertical line polynomial g'. Then also

$$g'(y-1)(c^{(1)} - p^{(1)}) = 0,$$

which can be demonstrated by writing $g' = ag + bh$ for some polynomials a, b.

Consider the configuration $c' = (y-1)(c^{(1)} - p^{(1)})$, we have

$$\begin{aligned} c'_{ij} &= c^{(1)}_{i,j-1} - c^{(1)}_{i,j} + p^{(1)}_{i,j-1} - p^{(1)}_{i,j} \\ &= \lfloor (j-1)\alpha \rfloor - \lfloor j\alpha \rfloor + p^{(1)}_{i,j-1} - p^{(1)}_{i,j}. \end{aligned}$$

Because $\lfloor j\alpha \rfloor - \lfloor (j-1)\alpha \rfloor \in \{\lfloor \alpha \rfloor, \lfloor \alpha \rfloor + 1\}$ and $p^{(1)}$ is finitary, we have that c' is finitary as well. But g' is a vertical line annihilator of c', therefore c' is periodic with a vertical period. Denote by n the length of this period.

Let us focus on the zeroth column of c'. The sum of any n consecutive elements in it is constant, denote it by κ. Then summing the first $kn, k \in \mathbb{N}$ consecutive elements gives $k\kappa$:

$$\begin{aligned} k\kappa &= c'_{0,1} + \cdots + c'_{0,kn} \\ &= \lfloor 0 \rfloor - \lfloor \alpha \rfloor + p_1 - p_0 + \cdots + \lfloor (kn-1)\alpha \rfloor - \lfloor kn\alpha \rfloor + p_{kn} - p_{kn-1} \\ &= -\lfloor kn\alpha \rfloor + p_{kn} - p_0 \end{aligned} \tag{4.1}$$

where we denote $p_i = p^{(1)}_{0,i}$. Since $p^{(1)}$ is finitary, there are integers $k_1 < k_2$ such that $p_{k_1 n} = p_{k_2 n}$. Then $k_2\kappa - k_1\kappa = \lfloor k_1 n\alpha \rfloor - \lfloor k_2 n\alpha \rfloor$, and therefore κ is rational. Now divide both sides of Equation 4.1 by k and let $k \to \infty$:

$$\kappa = \lim_{k\to\infty} \frac{\lfloor kn\alpha \rfloor + p_{kn} - p_0}{k} = n\alpha.$$

We get a contradiction with irrationality of α.

□

Chapter 5

Approaching Nivat's conjecture

In this chapter we apply the facts we learned previously about annihilating polynomials and link them to the complexity of a configuration.

When going from a symbolic configuration to formal power series, we have to choose numerical representations of the symbols. We begin by showing that there is a particularly suitable choice, and we call such configurations *normalized*. Next, in order to attack Nivat's conjecture, we define a class of configurations called *counterexample candidates*. As the name suggests, these are potential counterexamples to the conjecture, and our goal is to prove that such configurations have high complexity.

To handle the complexity we need a suitable tool. We introduce *lines of blocks*, which are just sets of blocks $m \times n$ located on a common line in the configuration. We prove two complementary lemmas – the first one states that there are many disjoint lines of blocks, while the other gives a lower bound on the number of distinct blocks on a line. These combined result in a lower bound on the overall complexity.

Our main result is that if c is non-periodic then the condition $P_c(m, n) > mn$ is true for all but finitely many pairs m, n. In the proof we consider three different ranges of m and n:

Very thin blocks. If m or n is so small that the support of no annihilating polynomial fits in the $m \times n$ rectangle, then by a variation of Lemma 3.1.2 the configuration has complexity $P_c(m, n) > mn$.

Thin blocks. Consider fixed n, large enough so that the support of some annihilator fits inside a strip of height n. We show that there exists m_0 such that for all $m > m_0$ we have $P_c(m, n) > mn$. Analogously for a fixed m.

Fat blocks. We prove that there are constants m_0 and n_0 such that for $m > m_0$ and $n > n_0$ we have $P_c(m, n) > mn$.

These three ranges cover all but finitely many dimensions $m \times n$. Interestingly, a common approach works for all configurations except for the case of fat blocks when c is a sum of horizontally and vertically one-periodic configuration. This case is handled separately in Chapter 6.

5.1 Normalized configurations

There is a particularly suitable choice when representing a symbolic configuration as a formal power series. For a configuration c, define $\mathrm{Unif}(c)$ as the set of Laurent polynomials f such that fc is a constant configuration. We say that c is *normalized* if $\mathrm{Unif}(c) = \mathrm{Ann}(c)$, i.e. if the constant in the result of fc is always zero. Let us denote by $\mathbb{1}$ the constant one configuration.

Lemma 5.1.1. *Let c be a finitary configuration. Then there exists $a, b \in \mathbb{C}, a \neq 0$ such that $ac + b\mathbb{1}$ is normalized. Moreover if c is integral then $a, b \in \mathbb{Z}$.*

Proof. For $f \in \mathrm{Unif}(c)$ denote by $\kappa(f)$ the number such that $fc = \kappa(f)\mathbb{1}$ and by $\sigma(f)$ the sum of the coefficients of f. Then for $f, g \in \mathrm{Unif}(c)$:

$$\begin{aligned} &\sigma(f)\kappa(g)\mathbb{1} = fgc = gfc = \sigma(g)\kappa(f)\mathbb{1} \\ \Rightarrow \quad &g\big(\sigma(f)c - \kappa(f)\mathbb{1}\big) = \sigma(f)\kappa(g)\mathbb{1} - \kappa(f)\sigma(g)\mathbb{1} = 0. \end{aligned}$$

If there is f such that $\sigma(f) \neq 0$ we can choose $a = \sigma(f), b = -\kappa(f)$. Since $\mathrm{Unif}(c) = \mathrm{Unif}(ac+b\mathbb{1})$, for any g from this set we have $g(ac+b\mathbb{1}) = 0$ and we are done. Let us assume that for all $f \in \mathrm{Unif}(c)$ we have $\sigma(f) = 0$, we will show that then c is already normalized and therefore we can choose $a = 1, b = 0$.

For even k let C_k denote the hypercube $[-\frac{k}{2}, \frac{k}{2})^d \subset \mathbb{Z}^d$ of side k centered around the origin. Choose even $n \in \mathbb{N}$ such that $\mathrm{supp}(f) \subset C_n$ and consider arbitrary even integer $N > n$. Let us count the sum of coefficients of fc inside of C_N.

Since fc is a constant configuration the sum is surely $\kappa(f)N^d$. On the other hand, the coefficients of fc in C_N depend only on the coefficients of c in C_{N+n}. Each such coefficient $c_{\boldsymbol{v}}$ contributes to the sum by $\sigma(f)c_{\boldsymbol{v}}$, but we overcount in the region $C_{N+2n} \setminus C_N$ of fc, see Figure 5.1. This region is of size proportional to N^{d-1} and because c is finitary, the contribution to each position is bounded. Therefore

$$\kappa(f)N^d = \sum_{\boldsymbol{v} \in C_{N+n}} \sigma(f)c_v + O(N^{d-1}) = O(N^{d-1}).$$

Taking the limit $N \to \infty$ shows that $\kappa(f) = 0$. Therefore f is an annihilator and c is normalized.

For the "moreover" part we argue as in the proof of Lemma 3.2.1. Let $f = \sum a_i X^{\boldsymbol{u_i}}$, then

$$fc = a_0\mathbb{1} \quad \Leftrightarrow \quad (-\overline{a_0}, \overline{a_1}, \ldots, \overline{a_m}) \perp (1, c_{\boldsymbol{v}-\boldsymbol{u_1}}, \ldots, c_{\boldsymbol{v}-\boldsymbol{u_m}})$$

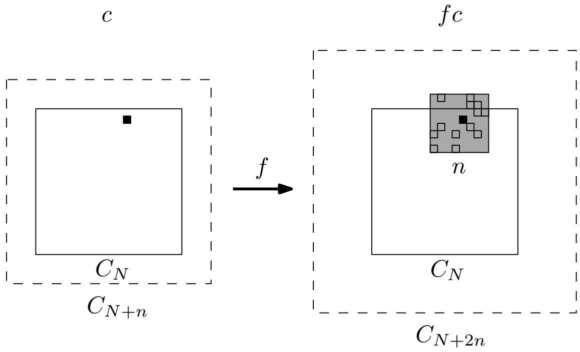

Figure 5.1: Proof of Lemma 5.1.1: Counting sum of coefficients of fc inside of C_N.

for all $\boldsymbol{v} \in \mathbb{Z}$. Thus all f form a vector space over $\mathbb{C}$ which has integral generators if c is integral. Therefore if there is f with $\sigma(f) \neq 0$, then there is also integral f' with $\sigma(f') \neq 0$. In that case necessarily $\sigma(f'), \kappa(f') \in \mathbb{Z}$. □

Corollary 5.1.2. *Either c is normalized, in which case $c + \kappa\mathbb{1}$ is normalized for all choices of $\kappa \in \mathbb{C}$, or there is unique $\kappa \in \mathbb{C}$ such that $c + \kappa\mathbb{1}$ is normalized.*

Proof. Follows from the proof of Lemma 5.1.1 by choosing $\kappa = b/a$. □

Note that the case when $\sigma(f) = 0$ for all f in the proof of the previous lemma can be handled easily for two-dimensional integral configurations. If the sum of coefficients of f is zero and fc is a constant configuration, then $f^2c = 0$. We proved that the ideal of annihilators is radical, so we can conclude $fc = 0$.

To link polynomials and complexity we use a variation of Lemma 3.1.2. Recall that for a finite shape $D \subset \mathbb{Z}^d$ we denote by $c_{\boldsymbol{v}+D}$ the pattern of shape D extracted from the position $\boldsymbol{v} \in \mathbb{Z}^d$. Formally we defined it as a function

$$\begin{aligned} c_{\boldsymbol{v}+D} : D &\to \mathbb{C} \\ \boldsymbol{d_i} &\mapsto c_{\boldsymbol{v}+\boldsymbol{d_i}}, \end{aligned}$$

and therefore it makes sense to talk about linear independence of patterns (over $\mathbb{C}$). If we denote $D = \{\boldsymbol{d_1}, \ldots, \boldsymbol{d_n}\}$, then this is the same as if we considered $c_{\boldsymbol{v}+D}$ to be the vector $(c_{\boldsymbol{v}+\boldsymbol{d_1}}, \ldots, c_{\boldsymbol{v}+\boldsymbol{d_n}}) \in \mathbb{C}^n$.

Let us say that a Laurent polynomial f *fits in* $S \subset \mathbb{Z}^d$ if a translate of $-\operatorname{supp}(f)$ is a subset of S. Here S can also be infinite, and usually will be a convex subset of $\mathbb{Z}^d$.

Lemma 5.1.3. *Let c be a configuration and $D \subset \mathbb{Z}^d$ a finite shape. Assume there is no annihilating Laurent polynomial f which fits in D. Then there* *are $|D|$ linearly independent patterns $c_{\boldsymbol{v}+D}$. Moreover if c is normalized then $P_c(D) > |D|$.*

Proof. Denote D as above and for contradiction assume the vectors $(c_{\boldsymbol{v}+\boldsymbol{d_1}}, \dots, c_{\boldsymbol{v}+\boldsymbol{d_n}}) \in \mathbb{C}^n$ span a space of dimension at most $n-1$. Then there exists a common orthogonal vector $(\overline{a_1}, \dots, \overline{a_n})$ and $f(X) = a_1 X^{-\boldsymbol{d_1}} + \dots + a_n X^{-\boldsymbol{d_n}}$ is an annihilating polynomial fitting in D.

For the second part for contradiction suppose $P_c(D) \leq n$, then the vectors $(1, c_{\boldsymbol{v}+\boldsymbol{d_1}}, \dots, c_{\boldsymbol{v}+\boldsymbol{d_n}}) \in \mathbb{C}^{n+1}$ span a space of dimension at most n. Let $(\overline{a_0}, \overline{a_1}, \dots, \overline{a_n})$ be their common orthogonal vector. Then f defined as previously has the property $fc = -a_0 \mathbb{1}$. If c is normalized then f is an annihilator. □

5.2 Counterexample candidates

We approach Nivat's conjecture by examining a potential counterexample to it. Let us recall the conjecture, in the contrapositive direction:

Conjecture (Nivat's conjecture)**.** *Let c be a non-periodic two-dimensional configuration. Then for all positive integers m, n we have $P_c(m, n) > mn$.*

If c is a counterexample, then it is surely a non-periodic two-dimensional configuration. It is finitary, since otherwise its complexity is not bounded. It also has to have an annihilator – otherwise by Lemma 3.1.2 for all m, n we have $P_c(m, n) > mn$. Moreover, without loss of generality, we can assume that c is integral. Let us make a formal definition:

Definition 5.1. *A configuration is a* counterexample candidate *if it is two-dimensional, non-periodic, finitary and integral configuration with an annihilator.*

Our goal is to show that any counterexample candidate c has a high complexity. In the proofs which follow we will frequently use the annihilator structure characterization from Corollary 4.2.1. Let us therefore define polynomials $\phi, \phi_1, \dots, \phi_{\mathrm{ord}(c)}$ and an ideal H such that

$$\mathrm{Ann}(c) = \phi H = \phi_1 \cdots \phi_{\mathrm{ord}(c)} H$$

as in the statement of Corollary 4.2.1. Note that since c is non-periodic we have $\mathrm{ord}(c) \geq 2$.

For a non-zero Laurent polynomial f let us define the *bounding box* of f to be the vector $\mathrm{box}(f) = (m, n)$ with m, n smallest integers such that f fits in a block $(m+1) \times (n+1)$. Equivalently,

$$\mathrm{box}(f) = (\max A - \min A,\ \max B - \min B)$$

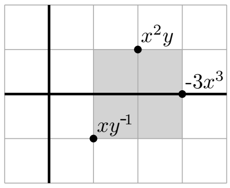

Figure 5.2: The bounding box of the polynomial $xy^{-1} + x^2y - 3x^3$ is $(2, 2)$.

where $A = \{ a \mid (a, b) \in \text{supp}(f) \}$ and $B = \{ b \mid (a, b) \in \text{supp}(f) \}$. Let us furthermore extend the definition to vectors: if $\boldsymbol{v} = (v_1, v_2)$ then define $\text{box}(\boldsymbol{v}) = (|v_1|, |v_2|)$.

Example 5.2.1. For example, $\text{box}(xy^{-1}+x^2y-3x^3) = (2, 2)$ and $\text{box}(X^{\boldsymbol{u}} - X^{\boldsymbol{v}}) = \text{box}(\boldsymbol{u} - \boldsymbol{v})$. If we plot the support of a polynomial as points in the plane, the bounding box are dimensions of the smallest rectangle which covers all of them, see Figure 5.2. Note however that a polynomial f never fits in $\text{box}(f)$.

With the framework that we just defined we get almost for free that counterexample candidates have high complexity for very thin rectangles:

Lemma 5.2.2 (Very thin blocks)**.** *Let c be a counterexample candidate and $(m_\phi, n_\phi) = \text{box}(\phi)$. If M, N are positive integers such that $M \leq m_\phi$ or $N \leq n_\phi$ then $P_c(M, N) > MN$.*

Proof. By Lemma 5.1.1 there exist $a, b \in \mathbb{Z}, a \neq 0$, such that $c' = ac + b\mathbb{1}$ is a finitary integral configuration which is normalized. Clearly $P_c(M, N) = P_{c'}(M, N)$. Let $\text{Ann}(c') = \phi' H'$. Since $\text{Ann}(ac) = \text{Ann}(c)$ and $\text{ord}(b\mathbb{1}) = 0$, by Corollary 4.3.2 we have $\phi' = \phi$.

Thus every annihilator of c' is a multiple of ϕ and therefore it cannot fit in an $M \times N$ rectangle. By Lemma 5.1.3 we have $P_{c'}(M, N) > MN$ which concludes the proof. $\square$

5.3 Disjoint lines of blocks

For a finite shape $D \subset \mathbb{Z}^2$ let us define a *line of D-patterns in direction* $\boldsymbol{v} \in \mathbb{Z}^2$, $\boldsymbol{v} \neq 0$ to be a set of the form

$$\mathcal{L} = \left\{ c_{\boldsymbol{u}+k\boldsymbol{v}+D} \;\middle|\; k \in \mathbb{Z} \right\}$$

for some vector $\boldsymbol{u} \in \mathbb{Z}^2$. Let $Lines_{\boldsymbol{v}}(D)$ be the set of all lines in the same direction, i.e.

$$Lines_{\boldsymbol{v}}(D) = \left\{ \{ c_{\boldsymbol{u}+k\boldsymbol{v}+D} \mid k \in \mathbb{Z} \} \;\middle|\; \boldsymbol{u} \in \mathbb{Z}^2 \right\}.$$

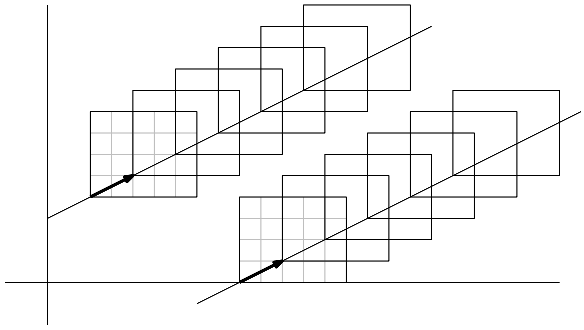

Figure 5.3: Two lines of blocks 5×4 in direction $(2,1)$. They are elements of $Lines_{(2,1)}(5,4)$.

Note that $Lines_{\boldsymbol{v}}(D)$ is a family of sets. In our usual setup the vector $\boldsymbol{v}$ will be primitive and as the shape D we will consider rectangular blocks $M \times N$. In that case we talk about *lines of $M \times N$ blocks in direction $\boldsymbol{v}$* and denote more conveniently by $Lines_{\boldsymbol{v}}(M,N)$. Figure 5.3 illustrates this definition.

Our strategy is to prove two complementary lemmas. The first one gives a lower bound on the number of pairwise disjoint sets in $Lines_{\boldsymbol{v}}(M,N)$ for a suitable choice of $\boldsymbol{v}, M, N$. The second one gives a lower bound for the number of blocks in any $\mathcal{L} \in Lines_{\boldsymbol{v}}(M,N)$. Combined, they give a lower bound on the complexity of the configuration.

We make use of the structure of the annihilator ideal $\mathrm{Ann}(c) = \phi H$. When talking about *minimal* polynomials, we mean minimal with respect to polynomial division. In polynomials in one variable, all ideals have (up to an invertible factor) unique minimal polynomial which generates the ideal. In our case the situation can be more complicated.

Clearly, minimal polynomials of $\mathrm{Ann}(c)$ are of the form ϕh where h is a minimal polynomial of H. Moreover, in that case $\mathrm{Ann}(hc) = \langle \phi \rangle$. Note that we cannot take any polynomial from H in the place of h – for example, $\phi h \in H$ but $\mathrm{Ann}(\phi h c) = \mathrm{Ann}(0) = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$.

We claim that H contains a line polynomial in arbitrary non-zero direction $\boldsymbol{v} \in \mathbb{Z}^2$ which is minimal. If $H = \mathbb{C}[X^{\pm 1}]$ this is trivially true. Otherwise let $Z_i \in \mathbb{C}^2$ be the roots of H, then $\prod_i (X^{\boldsymbol{v}} - Z_i^{\boldsymbol{v}}) \in H$ is a line polynomial in the direction $\boldsymbol{v}$. It suffices to choose a minimal polynomial from H which divides it.

Lemma 5.3.1. *Let f be a line Laurent polynomial and $\boldsymbol{v}$ a primitive vector in the direction of f. Let c be a configuration such that* $\mathrm{Ann}(c) = \langle f \rangle$. *Denote* $(m_f, n_f) = \mathrm{box}(f)$, $(m,n) = \mathrm{box}(\boldsymbol{v})$ *and let $M > m_f, N > n_f$ be positive integers. Then $Lines_{\boldsymbol{v}}(M,N)$ contains at least $(M - m_f)n + m(N - n_f)$ pairwise disjoint sets.*

Proof. Without loss of generality assume $\boldsymbol{v} = (m,n)$, otherwise a mirrored

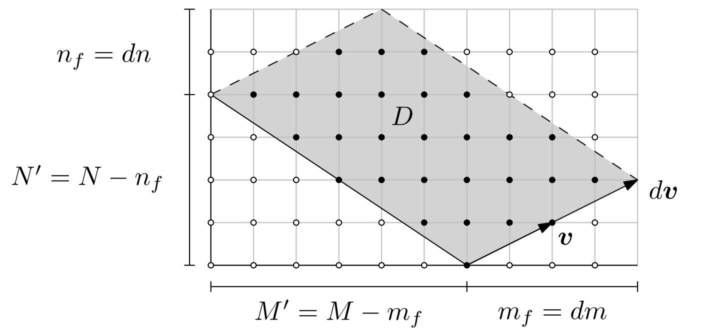

Figure 5.4: The shape D in Lemma 5.3.1. The marked points are elements of the $M \times N$ block, the filled ones belong to D.

or rotated configuration can be considered. There is an integer $d \in \mathbb{N}$ such that $(m_f, n_f) = (dm, dn) = d\boldsymbol{v}$. Denote $M' = M - m_f, N' = N - n_f$ and define

$$D = \left\{ (M', 0) + a(-M', N') + b(m_f, n_f) \;\middle|\; a, b \in [0, 1) \right\} \cap \mathbb{Z}^2.$$

The shape D is contained in an $M \times N$ block and $|D| = M'n_f + m_f N'$, see Figure 5.4. Moreover no multiple of f fits in D, thus by Lemma 5.1.3 there are at least $M'n_f + m_f N' = d(M'n + mN')$ linearly independent patterns $c_{\boldsymbol{v}+D}$.

Let $\mathcal{L}$ be a line of patterns from $Lines_{\boldsymbol{v}}(D)$. Then f gives a linear recurrence relation of degree d on the elements of $\mathcal{L}$. Therefore the vector space generated by the elements of $\mathcal{L}$ has dimension at most d. In particular, each line contains at most d of the $|D|$ linearly independent patterns $c_{\boldsymbol{v}+D}$. It follows that there are at least $M'n + mN'$ distinct lines in $Lines_{\boldsymbol{v}}(D)$.

We claim that if two lines are distinct then they are disjoint. Indeed, if a line contains a particular D-pattern, then f uniquely determines the next and the previous pattern on the line. Therefore the lines either contain exactly the same patterns or they are disjoint.

We proved that $Lines_{\boldsymbol{v}}(D)$ contains at least $M'n + mN'$ pairwise disjoint lines, therefore also $Lines_{\boldsymbol{v}}(M, N)$ does. □

Lemma 5.3.2. *Let c be a counterexample candidate, $f \in \operatorname{Ann}(c)$ be minimal and $\boldsymbol{v}$ be a primitive vector in the direction of ϕ_1. Denote $(m_f, n_f) = \operatorname{box}(f)$, $(m, n) = \operatorname{box}(\boldsymbol{v})$ and let $M > m_f$, $N > n_f$ be integers. Then $Lines_{\boldsymbol{v}}(M, N)$ contains at least $(M - m_f)n + m(N - n_f)$ disjoint sets.*

Proof. Let $c' = (f/\phi_1)c$, then c' is a one-periodic configuration with $\operatorname{Ann}(c') = \langle \phi_1 \rangle$. Denote $(m_1, n_1) = \operatorname{box}(\phi_1)$, then by Lemma 5.3.1, $Lines_{\boldsymbol{v}}(M - m_f + m_1, N - n_f + n_1)$ in c' contains at least $(M - m_f)n +$

$m(N-n_f)$ disjoint elements. An $M \times N$ block in c when multiplied by f/ϕ_1 determines an $(M - m_f + m_1) \times (M - n_f + n_1)$ block in c'. Therefore the lower bound applies also for $Lines_{\boldsymbol{v}}(M, N)$ in c. $\square$

5.4 Non-periodic stripes

Define a *stripe* to be a set of integer points between two parallel lines, i.e. a set of the form

$$\left\{\, \boldsymbol{w} + a\boldsymbol{u} + b\boldsymbol{v} \;\middle|\; a \in [0,1), b \in \mathbb{R} \,\right\} \cap \mathbb{Z}^2,$$

where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{Z}^2$ are arbitrary, $\boldsymbol{v} \neq 0$. The vector $\boldsymbol{w}$ specifies the position of the stripe, $\boldsymbol{u}$ determines its width and the stripe extends infinitely along $\boldsymbol{v}$. Let us call the vector space $\langle \boldsymbol{v} \rangle \subset \mathbb{Q}^2$ the *direction* of the stripe.

Lemma 5.4.1. *Let c be a counterexample candidate and $\boldsymbol{v} \in \mathbb{Z}^2$ a non-zero vector. Let S be an infinite stripe in the direction of $\boldsymbol{v}$ of maximal width such that ϕ does not fit in. Then c restricted to the stripe S is non-periodic in the direction of $\boldsymbol{v}$.*

Proof. Since $\mathrm{ord}(c) \geq 2$ there are at least two line polynomial factors of ϕ in different directions. Without loss of generality assume that $\boldsymbol{v}$ is distinct from the direction of ϕ_1.

Let $h \in H$ be a minimal line polynomial in the direction of $\boldsymbol{v}$. Then $f = \phi h$ is a minimal polynomial from $\mathrm{Ann}(c)$. Consider $c' = (f/\phi_1)c$. It is a one-periodic configuration in the direction of ϕ_1. Let S' be a narrower stripe in c' determined from S in c by the multiplication by f/ϕ_1. S' is of maximal width such that ϕ_1 does not fit in.

For a contradiction assume that c restricted to S is periodic in the direction of $\boldsymbol{v}$, then also c' restricted to S' is. Moreover S' determines the whole configuration c' – the annihilator ϕ_1 gives a linear recurrence relation on the coefficients of c' lying on lines in the direction of ϕ_1, and S' is wide enough so that every coefficient is determined. Therefore c' is periodic also in the direction of $\boldsymbol{v}$, which is in contradiction with one-periodicity of c'. $\square$

Lemma 5.4.2. *Let c be a counterexample candidate and $\boldsymbol{v} \in \mathbb{Z}^2$ a non-zero vector. Denote $(m_\phi, n_\phi) = \mathrm{box}(\phi)$, $(m, n) = \mathrm{box}(\boldsymbol{v})$ and let $M > m_\phi$, $N > n_\phi$ be integers. Let $\mathcal{L} \in Lines_{\boldsymbol{v}}(M, N)$ be arbitrary.*

(a) If $\boldsymbol{v}$ is neither horizontal nor vertical, then

$$|\mathcal{L}| \geq \min\left\{ \frac{M - m_\phi + 1}{m}, \frac{N - n_\phi + 1}{n} \right\}.$$

(b) Assume $\boldsymbol{v}$ is not horizontal. If $M \geq (N + n_\phi)\frac{m}{n} + m_\phi$ then

$$|\mathcal{L}| \geq \frac{N+1}{n}.$$

Proof. Without loss of generality assume $\boldsymbol{v} = (m, n)$, the other cases are mirrored or rotated. Also assume that there is a block in $\mathcal{L}$ with $(0, 0)$ as its bottom left corner. The proof is illustrated in Figure 5.5.

(a) Consider the stripe

$$S_1 = \left\{ (0, n_\phi) + a(m_\phi, -n_\phi) + b\boldsymbol{v} \;\middle|\; a \in [0, 1), b \in \mathbb{R} \right\} \cap \mathbb{Z}^2.$$

Since (m_ϕ, n_ϕ) is the bounding box of ϕ, the stripe S from Lemma 5.4.1 fits in S_1. Therefore S_1 is non-periodic in the direction of $\boldsymbol{v}$, and in particular there exists a "fiber" $f = \{ \boldsymbol{u} + k\boldsymbol{v} \mid k \in \mathbb{Z} \}$ inside of the stripe on which c spells a non-periodic sequence.

Each block from $\mathcal{L}$ contains the same number of consecutive points from a fixed fiber in S_1, let $p(f)$ be this number for f. Clearly, one of the two fibers on the boundaries of S_1 lower bounds this quantity. Therefore, by computing the number of points on the boundary fibers,

$$p(f) \geq \min\left\{ \left\lfloor \frac{M - m_\phi}{m} \right\rfloor, \left\lfloor \frac{N - n_\phi}{n} \right\rfloor \right\}.$$

Now by Morse-Hedlund theorem there are at least $p(f)+1$ distinct blocks in $\mathcal{L}$. The proof is finished by verifying that $\lfloor p/q \rfloor + 1 \geq (p+1)/q$ for $p, q \in \mathbb{N}$.

(b) Consider the stripe

$$S_2 = \left\{ a(m_\phi, -n_\phi) + b\boldsymbol{v} \;\middle|\; a \in [0, 1), b \in \mathbb{R} \right\} \cap \mathbb{Z}^2.$$

As in the part (a), it contains a non-periodic fiber. Moreover, if the condition on M is satisfied, then the boundary of S_2 intersects every block in $\mathcal{L}$ on the top edge. Therefore $\lfloor N/n \rfloor$ lower bounds the number of points from any fiber of S_2 contained in a block in $\mathcal{L}$. The rest follows as in (a). □

5.5 Asymptotic Nivat's conjecture

Let us combine the above lemmas to get a lower bound on the complexity of a counterexample candidate.

Lemma 5.5.1 (Thin blocks)**.** *Let c be a counterexample candidate and $(m_\phi, n_\phi) = \mathrm{box}(\phi)$. Fix an integer $N > n_\phi$. Then there exists M_0 such that if $M > M_0$ then $P_c(M, N) > MN$.*

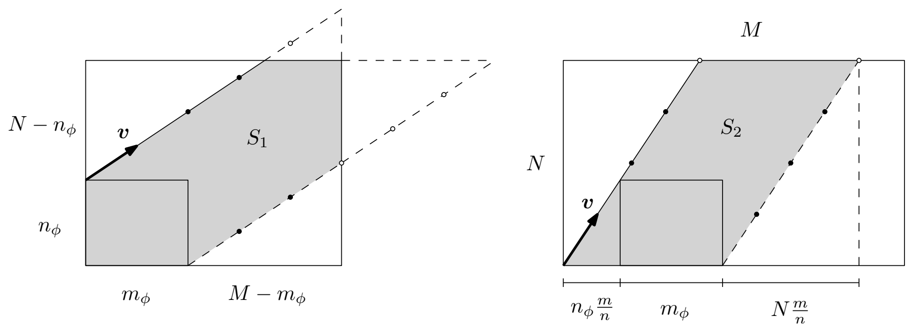

Figure 5.5: The stripes S_1 and S_2 from the proof of Lemma 5.4.2.

Proof. Since $\operatorname{ord}(c) \geq 2$ we can without loss of generality assume that the direction of ϕ_1 is not horizontal. Let $\boldsymbol{v}$ be a primitive vector in that direction and denote $(m,n) = \operatorname{box}(\boldsymbol{v})$.

Let $h \in H$ be a horizontal line polynomial and let $f = \phi h$, $(m_f, n_f) = \operatorname{box}(f)$. Clearly $n_f = n_\phi$. Assume $M \geq (N+n_\phi)\frac{m}{n} + m_\phi$. Then by Lemma 5.3.2 and Lemma 5.4.2(b) for $M > m_f, N > n_f$ we have

$$\begin{aligned} P_c(M,N) &= \left| \bigcup Lines_{\boldsymbol{v}}(M,N) \right| \\ &\geq \big((M-m_f)n + m(N-n_f)\big)\frac{N+1}{n} \\ &\geq (M-m_f)(N+1) = MN + M - m_f(N+1). \end{aligned}$$

The proof is finished by choosing $M_0 = \max\left\{m_f(N+1),\ (N+n_\phi)\frac{m}{n} + m_\phi\right\}$. $\square$

Lemma 5.5.2 (Fat blocks I). *Let c be a counterexample candidate and let $\boldsymbol{v}$ be the direction of ϕ_1. If $\boldsymbol{v}$ is neither horizontal nor vertical, then there exist positive integers M_0, N_0 such that for $M > M_0$ and $N > N_0$ holds $P_c(M,N) > MN$.*

Proof. Let $f \in \operatorname{Ann}(c)$ be minimal and denote $(m,n) = \operatorname{box}(\boldsymbol{v})$, $(m_\phi, n_\phi) = \operatorname{box}(\phi)$, $(m_f, n_f) = \operatorname{box}(f)$. Assume $M > m_f$, $N > n_f$ and let $\alpha = \frac{m}{n}$. We consider three ranges of M. The proof is illustrated in Figure 5.6.

(a) Assume $(N+n_\phi)\alpha + m_\phi \leq M$. This condition is equivalent to the one in Lemma 5.4.2(b), therefore by combining with Lemma 5.3.2

$$\begin{aligned} P_c(M,N) &\geq \big((M-m_f)n + m(N-n_f)\big)\frac{N+1}{n} \\ &= (M-m_f)(N+1) + (N-n_f)(N+1)\frac{m}{n} \\ &= MN + M + \Theta(N^2). \end{aligned}$$

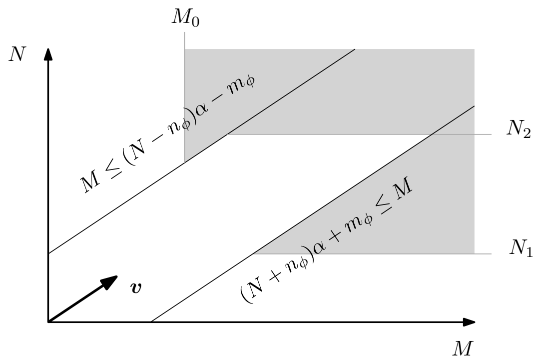

Figure 5.6: Three different ranges for M from the proof of Lemma 5.5.2.

Therefore there exist an integer N_1 such that for $N > N_1$ the complexity is at least MN.

(b) Assume $(N - n_\phi)\alpha - m_\phi < M < (N + n_\phi)\alpha + m_\phi$. Then $M = \Theta(N)$. Now combine Lemma 5.3.2 and Lemma 5.4.2(a):

$$\begin{aligned} P_c(M, N) &> \big((M - m_f)n + m(N - n_f)\big) \min\left\{\frac{M - m_\phi}{m}, \frac{N - n_\phi}{n}\right\} \\ &\geq \big((M - m_f)n + m(N - n_f)\big) \min\left\{\frac{M - m_f}{m}, \frac{N - n_f}{n}\right\} \\ &= (M - m_f)(N - n_f) + \min\left\{(M - m_f)^2 \frac{n}{m}, (N - n_f)^2 \frac{m}{n}\right\} \\ &= (M - m_f)(N - n_f) + \Theta(N^2) \\ &= MN + \Theta(N^2). \end{aligned}$$

Therefore there is an integer N_2 such that for $N > N_2$ the complexity exceeds MN.

(c) Assume $M \leq (N - n_\phi)\alpha - m_\phi$. This is equivalent to the condition in Lemma 5.4.2(b) when the roles of horizontal and vertical direction are exchanged. Therefore, similarly as in (a), there exists M_0 such that for $M > M_0$ the complexity is at least MN. The whole proof is finished by choosing $N_0 = \max\{N_1, N_2\}$. □

Now we are just a step away from our main theorem. Suppose we knew that Lemma 5.5.2 holds also when there are only horizontal and vertical ϕ_i components:

Lemma 5.5.3 (Fat blocks II)**.** *Let c be a counterexample candidate,* $\mathrm{ord}(c) = 2$ *and the directions of ϕ_1, ϕ_2 are horizontal and vertical, respectively. Then there exist positive integers M_0, N_0 such that for $M > M_0$ and $N > N_0$ holds $P_c(M, N) > MN$.*

This is the case when c is a sum of horizontally one-periodic and vertically one-periodic configurations. We postpone the proof of Lemma 5.5.3 to the next chapter. Assuming the lemma is valid, we can finally give a proof of our main theorem.

Theorem 5.5.4 (The main result)**.** *Let c be a two-dimensional non-periodic configuration. Then $P_c(M,N) > MN$ holds for all but finitely many choices $M, N \in \mathbb{N}$.*

Proof. By the discussion preceding Definition 5.1, it is enough to consider counterexample candidates c. Note that either at least one of ϕ_i is neither horizontal nor vertical, or $\mathrm{ord}(c) = 2$ and the directions of ϕ_1, ϕ_2 are horizontal and vertical in some order. In either case, by Lemma 5.5.2 or Lemma 5.5.3, there are M_0, N_0 such that for $M > M_0, N > N_0$ we have $P_c(M,N) > MN$.

Let $(m_\phi, n_\phi) = \mathrm{box}(\phi)$ and assume $n_\phi < N \leq N_0$. By Lemma 5.5.1 for each such N all but finitely many M satisfy $P_c(M,N) > MN$. Therefore for the whole range $n_\phi < N \leq N_0$ the condition can be violated only finitely many times. The situation for $m_\phi < M \leq M_0$ is symmetric.

Finally, if $M \leq m_\phi$ or $N \leq n_\phi$ the complexity is greater than MN by Lemma 5.2.2. This concludes the proof. □

Corollary 5.5.5. *If c is a two-dimensional configuration such that $P_c(M,N) \leq MN$ holds for infinitely many pairs $M, N \in \mathbb{N}$, then c is periodic.*

Chapter 6

Sums of two periodic configurations

To finish the proof of the asymptotic version of Nivat's conjecture (Theorem 5.5.4) it remains to handle the case when a configuration is a sum of horizontally and vertically periodic configuration. In the paper [KS16] we have shown how to do it in a rather technical combinatorial way. Here we take a different approach.

We revisit the method of Van Cyr and Bryna Kra [CK15, CK16]. They approach Nivat's conjecture from the point of view of symbolic dynamics. They use a refined version of the classical notion of expansiveness of a subshift, a so called *one-sided non-expansiveness*. A key definition of theirs is that of a *balanced set* – it is a shape $D \subset \mathbb{Z}^2$ which satisfies a particular condition on the complexity $P_c(D)$. (Note that this notion is different from balancedness usual in combinatorics on words.) The crucial tool they developed is a combinatorial lemma which links one-sided non-expansiveness and balanced sets to periodicity of a configuration. However, in order to obtain the main result of the paper from the lemma it still takes a rather lengthy technical analysis.

In this chapter we combine the algebraic method with ideas of Cyr and Kra. We start the exposition with a very basic introduction to the topic of symbolic dynamics. In section 6.1 we define a subshift, in section 6.2 we fix some geometric terminology, and in section 6.3 we give definitions of non-expansiveness and one-sided non-expansiveness of a subshift.

In section 6.4 we introduce a simplified version of a balanced set and prove Lemma 6.4.3 which connects balanced sets with periodicity using the ideas of Cyr and Kra. We use the lemma together with decomposition theorem to prove the following, from which our main result follows:

Theorem (Theorem 6.5.1)**.** *Let c be a counterexample candidate with* $\mathrm{ord}(c) = 2$. *Then $P_c(m, n) > mn$ for all $m, n \in \mathbb{N}$.*

As a corollary, we obtain an alternative proof of Theorem 1.2 of [CK15], the main result of their paper:

Theorem (Cyr, Kra)**.** *Let c be a configuration satisfying $P_c(m, n) \leq mn/2$ for some $m, n \in \mathbb{N}$. Then c is periodic.*

Results in this chapter have been published in [Sza18].

6.1 Symbolic dynamics and subshifts

Let us recall basic facts from symbolic dynamics, for a comprehensive reference and proofs see [Kůr03].

Let $\mathcal{A}$ be a finite set, we call it an *alphabet*. Let $d \in \mathbb{N}$ be a fixed dimension. A *symbolic configuration* is a mapping $c\colon \mathbb{Z}^d \to \mathcal{A}$. The set of all symbolic configurations is denoted $\mathcal{A}^{\mathbb{Z}^d}$. To simplify notation, for $\boldsymbol{v} \in \mathbb{Z}^d$ we denote $c_{\boldsymbol{v}} = c(\boldsymbol{v})$.

For a vector $\boldsymbol{u} \in \mathbb{Z}^d$ we define the *shift* operator $\tau_{\boldsymbol{u}}\colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ by

$$(\tau_{\boldsymbol{u}}(c))_{\boldsymbol{v}} = c_{\boldsymbol{v}-\boldsymbol{u}}.$$

If we interpret c as assignment of "colors" from $\mathcal{A}$ to the grid $\mathbb{Z}^d$, then $\tau_{\boldsymbol{u}}(c)$ is the coloring c translated in the direction of vector $\boldsymbol{u}$.

For purposes of symbolic dynamics it does not matter what are the actual elements of $\mathcal{A}$, let us assume they are complex numbers. Then a symbolic configuration can be naturally identified with power series $\sum_{\boldsymbol{v}} c_{\boldsymbol{v}} X^{\boldsymbol{v}}$, which is our notion of finitary configuration from before. Note that the shift operator $\tau_{\boldsymbol{u}}$ corresponds to multiplication by $X^{\boldsymbol{u}}$. We will identify the two concepts from now on.

Symbolic dynamics studies $\mathcal{A}^{\mathbb{Z}^d}$ as a topological space. Let us first make $\mathcal{A}$ a topological space by endowing it with the discrete topology. Then $\mathcal{A}^{\mathbb{Z}^d}$ is considered to be a topological space with the product topology.

Open sets in this topology are for example sets of the following form. Let $D \subset \mathbb{Z}^d$ be finite and $p\colon D \to \mathcal{A}$ arbitrary. Then

$$Cyl(p) := \left\{ c \in \mathcal{A}^{\mathbb{Z}^d} \;\middle|\; \forall \boldsymbol{v} \in D\colon c_{\boldsymbol{v}} = p_{\boldsymbol{v}} \right\}$$

is an open set, also called a *cylinder*. In fact, the collection of cylinders $Cyl(p)$ for all possible p forms a base of the topology on $\mathcal{A}^{\mathbb{Z}^d}$. We leave this fact without a proof.

The set $\mathcal{A}^{\mathbb{Z}^d}$ is called the *full shift*. A subset $X \subset \mathcal{A}^{Z^d}$ is called a *subshift* if it is a topologically closed set which is invariant under all shifts $\tau_{\boldsymbol{u}}$:

$$\forall \boldsymbol{u} \in \mathbb{Z}^d\colon c \in X \Rightarrow \tau_{\boldsymbol{u}}(c) \in X.$$

Subshifts are the central objects of study in symbolic dynamics.

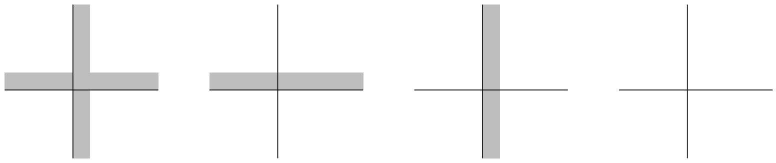

Figure 6.1: Four types of configurations in the orbit closure X_c from Example 6.1.1. The gray color corresponds to value 1, white is 0.

Let c be a configuration. We denote by X_c the *orbit closure* of c, that is, the smallest subshift which contains c. It can be shown that c contains exactly those configurations c' whose finite patterns are among the finite patterns of c. In particular, for any $c' \in X_c$ and a finite domain D we have $P_{c'}(D) \leq P_c(D)$.

Example 6.1.1. Let us give an example of taking orbit closure. Let $c \in \{0,1\}^{\mathbb{Z}^2}$ be such that $c_{ij} = 1$ if $i = 0$ or $j = 0$, and $c_{ij} = 0$ otherwise. When pictured, the configuration c consists of a large cross with its center at $(0,0)$. The orbit closure X_c then consist of four types of configurations: a cross, a horizontal line, a vertical line and all zero configurations, with all possible translations, see Figure 6.1. It is easy to see that any pattern which occurs in them also occurs in c, and not difficult to prove that those are all such configurations.

Example 6.1.2. One way to obtain a subshift is to specify a set of forbidden patterns. Let $D \subset \mathbb{Z}^d$ be finite and let $p_1, \ldots, p_k \in \mathcal{A}^D$ be patterns. We define $X_{p_1,\ldots,p_k}$ to be the set of configurations which do not contain any of the patterns p_i. A subshift that can be defined this way is called a *subshift of finite type*.

An example in one dimension would be the *golden mean subshift*. Let $\mathcal{A} = \{0,1\}$, then the subshift X_{11} consists of all sequences $c \in \mathcal{A}^{\mathbb{Z}}$ which do not contain "11" as a subpattern. An example sequence in this subshift would be a bidirectional version of the Fibonacci word $\ldots 0100101001001010010 10 \ldots$, or a sequence consisting of a single one $\ldots 0001000 \ldots$.

6.2 Geometric notation and terminology

In the sequel we will be concerned with the geometry of $\mathbb{Z}^2$. Let us establish some notation and terminology.

We view $\mathbb{Z}^2$ as a subset of the vector space $\mathbb{Q}^2$. A *direction* is an equivalence class of $\mathbb{Q}^2 \setminus \{(0,0)\}$ modulo the equivalence relation $u \sim v$ iff $u = \lambda v$ for some $\lambda > 0$. By a slight abuse of notation, we identify a non-zero vector $\boldsymbol{u} \in \mathbb{Z}^2$ with the direction $\boldsymbol{u}\mathbb{Q}^+$.

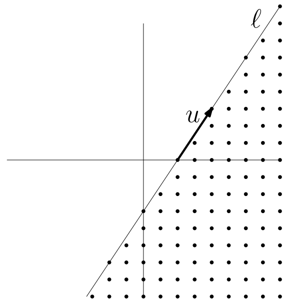

Figure 6.2: Marked points belong to the half-plane H_ℓ determined by the directed line ℓ in direction $\boldsymbol{u} = (2, 3)$.

Let $\boldsymbol{u} \in \mathbb{Z}^2$ be non-zero. An (undirected) *line* in $\mathbb{Z}^2$ is a set of the form

$$\{\, \boldsymbol{v} + q\boldsymbol{u} \mid q \in \mathbb{Q} \,\} \cap \mathbb{Z}^2$$

for some $\boldsymbol{v} \in \mathbb{Z}^2$. We call both $\boldsymbol{u}$ and $-\boldsymbol{u}$ a *direction* of the line. We define a *directed line* to be a line augmented with one of the two possible directions. A pair of directed lines in opposite directions are called *antiparallels*.

Let ℓ be a directed line in direction $\boldsymbol{u}$ going through $\boldsymbol{v} \in \mathbb{Z}^2$. The *half-plane* determined by ℓ is defined by

$$H_\ell = \big\{\, \boldsymbol{v} + \boldsymbol{w} \;\big|\; \boldsymbol{w} \in \mathbb{Z}^2, w_1 u_2 - u_1 w_2 \geq 0 \,\big\}.$$

With the usual choice of coordinates it is the half-plane "on the right" from the line, see Figure 6.2. Let $H_{\boldsymbol{u}}$ denote the half-plane determined by the directed line in direction $\boldsymbol{u}$ going through the origin.

We say that a non-empty $D \subset \mathbb{Z}^2$ is *convex* if D can be written as an intersection of half-planes. *Convex hull* of D, denoted $Conv(D)$, is the smallest convex set containing D. Assume ℓ is a directed line in direction $\boldsymbol{u}$ such that $D \subset H_\ell$ and $\ell \cap D$ is non-empty. If $|\ell \cap D| > 1$ we call it the *edge* of D in direction $\boldsymbol{u}$, otherwise we call it the *vertex* of D in direction $\boldsymbol{u}$. Note that a vertex is a vertex for many directions, but an edge has a unique direction (as long as D is not contained in a line). See Figure 6.3 for an example.

Stripes

Let us amend the definition of a stripe from section 5.4 by augmenting it with one of the two possible directions. Let $\boldsymbol{u}$ be a direction and ℓ, ℓ' two directed lines in direction $\boldsymbol{u}$. If

$$S = H_\ell \setminus H_{\ell'}$$

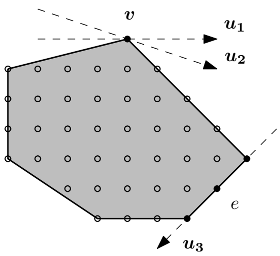

Figure 6.3: A convex set. The point $\boldsymbol{v}$ is a vertex of the set for both directions $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. The set of three marked points e is the edge in direction $\boldsymbol{u_3}$.

is non-empty, then S is called a *stripe* in direction $\boldsymbol{u}$. Let $S^\circ = S \setminus \ell$ be the *interior* of S.

Suppose $D \subset \mathbb{Z}^2$ is finite. Then a stripe S *envelopes* D if D fits in S, but does not fit in S°. We say that S envelopes a vector $\boldsymbol{v}$ or a polynomial f if S envelopes $\{0, \boldsymbol{v}\}$ or $\operatorname{supp}(f)$ respectively.

6.3 Non-expansiveness and one-sided non-expansiveness

It can be verified that the topology on $\mathcal{A}^{\mathbb{Z}^d}$ is compact, and also metrizable for example with the metric

$$d(c,e) = \begin{cases} 0 \text{ if } c = e, \\ 2^{-\min\{|\boldsymbol{v}| \,:\, c_{\boldsymbol{v}} \neq e_{\boldsymbol{v}}\}} \text{ otherwise.} \end{cases}$$

Note that shift operators $\tau_{\boldsymbol{u}}$ are continuous maps on $\mathcal{A}^{\mathbb{Z}^d}$. Expansiveness can be defined in general for a continuous action on a compact metric space, the definition is however too general for our purposes. We give a definition specific to the case of $\mathcal{A}^{\mathbb{Z}^2}$.

Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be a subshift and $\boldsymbol{u}$ a direction. Then $\boldsymbol{u}$ is an *expansive direction* for X if there exists a stripe S in direction $\boldsymbol{u}$ such that

$$\forall c, e \in X: \quad c\restriction_S = e\restriction_S \;\Rightarrow\; c = e.$$

Informally speaking, $\boldsymbol{u}$ is an expansive direction for X if a configuration in X is uniquely determined by its coefficients in a wide enough stripe in direction $\boldsymbol{u}$.

The following theorem links double periodicity of a configuration with expansiveness. It is a corollary of a theorem by Boyle and Lind [BL97].

Theorem 6.3.1. *Let c be a symbolic configuration. Then c is doubly periodic iff all directions are expansive for X_c.* □

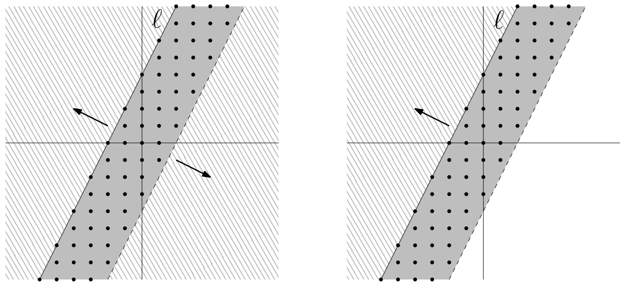

Figure 6.4: The figure on the left illustrates expansiveness – values of the configuration inside the stripe determine the whole configuration. On the right we see one-sided expansiveness in direction $(1, 2)$ – values in the half-plane H_ℓ, or equivalently in a wide enough stripe, determine the values in the half-plane $\mathbb{Z}^2 \setminus H_\ell$.

Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be a subshift and $\boldsymbol{u}$ a direction. Then $\boldsymbol{u}$ is a *one-sided expansive direction* for X if

$$\forall c, e \in X\colon \quad c{\restriction_{H_{\boldsymbol{u}}}} = e{\restriction_{H_{\boldsymbol{u}}}} \quad \Rightarrow \quad c = e.$$

Equivalently, $\boldsymbol{u}$ is a one-sided expansive direction for X if there exists a wide enough stripe S in direction $\boldsymbol{u}$ such that $\forall c, e \in X\colon c{\restriction_S} = e{\restriction_S} \Rightarrow c{\restriction_{H_{-\boldsymbol{u}}}} = e{\restriction_{H_{-\boldsymbol{u}}}}$. See Figure 6.4 for a comparison of the notion of expansiveness and one-sided expansiveness.

Example 6.3.2 (Ledrappier's subshift)**.** It is possible for a subshift to be one-sided expansive but non-expansive in the same direction. Consider a subshift $X \subset \{0, 1\}^{\mathbb{Z}^2}$ consisting of configurations c which satisfy $c_{ij} \equiv c_{i,j+1} + c_{i+1,j+1} \pmod 2$. Upper half-plane of a configuration determines the whole, since any single row determines the one below it. Therefore $(-1, 0)$ is a one-sided expansive direction for X. However, no stripe in direction $(-1, 0)$ determines a configuration from the subshift; for any row, there are always two possibilities for the row above it (they are complements of each other). Therefore any horizontal stripe can be extended to the upper half-plane in infinitely many ways.

We are primarily interested in non-expansive directions. In our setup, there are at most finitely many of them. This follows from existence of generating sets introduced by Cyr and Kra [CK15], here we show a proof using polynomials:

Lemma 6.3.3. *Let c be a finitary integral configuration with a non-trivial annihilator. Let $Ann(c) = \phi_1 \cdots \phi_m H$ as in Corollary 4.2.1. Then non-expansive directions of X_c are among directions of ϕ_i.*

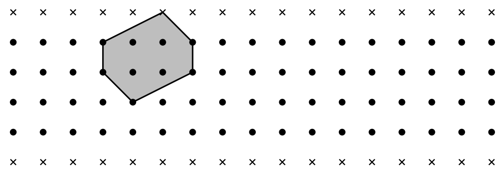

Figure 6.5: A stripe of height 4 (full dots) and its two adjacent lines (crosses). The convex hull of supp(f) is positioned such that one vertex lies on one of the lines, the rest lies in the stripe.

Proof. Let $h \in H$ be in the same direction as ϕ_1 and let $f = \phi_1 \cdots \phi_m h$. Denote $F = \text{supp}(f)$. Note that $Conv(F)$ has edges only in directions of ϕ_i.

Let $c' \in X_c$ be arbitrary. Since c' contains only patterns contained in c, f annihilates c'. Let $\boldsymbol{u}$ be any direction distinct from the directions of ϕ_i and S a wide enough stripe in direction $\boldsymbol{u}$ such that F fits in. We show that values in the stripe determine the values in the two nearest lines in direction $\boldsymbol{u}$ not contained in S. Then, by repeating the argument for a shifted stripe, all values can be determined, and therefore X_c is expansive in direction $\boldsymbol{u}$.

Since $Conv(F)$ has a vertex both in direction $\boldsymbol{u}$ and $-\boldsymbol{u}$, the set F can be translated such that one point lies on one of the closest lines in direction $\boldsymbol{u}$ not contained in S, and the rest is contained in S. The polynomial f gives a linear combination on the values in any translation of F. Therefore the values inside of S determine the value in the neighbouring line. By shifting F along the stripe the whole line is determined. See Figure 6.5. □

We conjecture also the converse:

Conjecture 6.1. *Let c be a finitary integral configuration with a non-trivial annihilator. Let $Ann(c) = \phi_1 \cdots \phi_m H$ as in Corollary 4.2.1. Then non-expansive directions of X_c are exactly the directions of ϕ_i.*

For later use it will be practical to define non-expansiveness explicitly. Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be a subshift and S a stripe in direction $\boldsymbol{u}$. We say that S is an *ambiguous stripe in direction $\boldsymbol{u}$* if there exist $c, e \in X$ such that

$$c\restriction_{S^\circ} = e\restriction_{S^\circ}, \quad \text{but} \quad c\restriction_S \neq e\restriction_S \,. \tag{6.1}$$

We say that $c \in X$ *contains* an ambiguous stripe S if there exists $e \in X$ satisfying (6.1). Informally, a stripe is ambiguous if its interior does not determine the inner boundary.

Definition 6.1. *Let $\boldsymbol{u}$ be a direction and $X \subset \mathcal{A}^{\mathbb{Z}^2}$ a subshift. Then $\boldsymbol{u}$ is* one-sided non-expansive direction *if there exists an ambiguous stripe in direction $\boldsymbol{u}$ of arbitrary width.*

We leave the proof that this is the converse of the earlier definition of one-sided expansiveness to the reader.

6.4 Balanced sets

Let c be a fixed symbolic configuration.

Definition 6.2. *Let $B \subset \mathbb{Z}^2$ be a finite and convex set, $\boldsymbol{u}$ a direction and E an edge or a vertex of B in direction $\boldsymbol{u}$. Then B is $\boldsymbol{u}$-balanced if:*

(i) $P_c(B) \leq |B|$

(ii) $P_c(B) < P_c(B \setminus E) + |E|$

(iii) Intersection of B with all lines in direction $\boldsymbol{u}$ is either empty or of size at least $|E| - 1$.

The three conditions of the definition can be interpreted as follows. The first one states that B is a low complexity shape, and therefore in particular there exists an annihilator such that its support fits in $-B$ (if c is normalized). The second condition limits the number of $(B \setminus E)$-patterns which do not extend uniquely to a B-pattern, there is strictly less than $|E|$ of them. The third condition is implied if the length of the edge in direction $\boldsymbol{u}$ is smaller or equal to the length of the edge in the opposite direction, as can be seen in the next proof.

Lemma 6.4.1. *Let c be such that $P_c(m, n) \leq mn$ holds for some $m, n \in \mathbb{N}$ and $\boldsymbol{u}$ a direction. Then there exists a $\boldsymbol{u}$-balanced or $(-\boldsymbol{u})$-balanced set. Moreover, if $\boldsymbol{u}$ is horizontal or vertical, then there exists a $\boldsymbol{u}$-balanced set.*

Proof. Let D be an $m \times n$ rectangle, we have $P_c(D) \leq |D|$. Let us define a sequence of convex shapes $D = D_0 \supset D_1 \supset \cdots \supset D_k = \emptyset$ such that $D_i \setminus D_{i+1}$ is the edge of D_i in direction $(-1)^i\boldsymbol{u}$. Informally, the sequence represents shaving off an edge (or a vertex) of the shape alternately in directions $\boldsymbol{u}$ and $-\boldsymbol{u}$. See Figure 6.6 for an illustration.

Consider the expression $P_c(D_i) - |D_i|$ as a function of i. For $i = 0$ its value is non-positive and for $i = k$ its value is 1. Let $i \in [0, k-1]$ be smallest such that $0 < P_c(D_{i+1}) - |D_{i+1}|$, then we have

$$P_c(D_i) - |D_i| \leq 0 < P_c(D_{i+1}) - |D_{i+1}|.$$

Denote $E = D_i \setminus D_{i+1}$, it is an edge or a vertex of D_i in direction $\boldsymbol{u}$ or $-\boldsymbol{u}$. Adding $|D_i|$ to the inequality and rewriting gives $P_c(D_i) \leq |D_i| < P_c(D_i \setminus E) + |E|$.

We show that $B = D_i$ is a balanced set by showing that *(iii)* of Definition 6.2 holds. Without loss of generality let the direction of E be $\boldsymbol{u}$. Then, by construction, the length of E is smaller or equal to the edge in direction $-\boldsymbol{u}$. In fact, if we consider the convex hull of B in $\mathbb{Q}^2$, any line in direction $\boldsymbol{u}$ intersects it in a line segment longer or equal to d, the length of the edge.

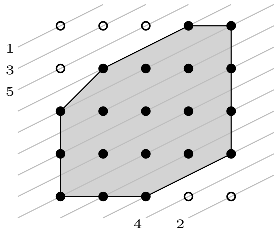

Figure 6.6: Shaving off edges or vertices of a 5×5 rectangle alternately in directions $(2,1)$ and $(-2,-1)$. Small numbers indicate the order in which the edges or vertices were removed.

Any line segment of length at least d in direction $\boldsymbol{u}$ intersects either none or at least $|E|-1$ integer points, and we are done.

If $\boldsymbol{u}$ is either horizontal or vertical, instead of alternating the direction of shaved off edges, we can always shave off the edge in direction $\boldsymbol{u}$. It will be always the shortest edge in direction $\boldsymbol{u}$, therefore verification of part *(iii)* goes through. □

Next we present Lemma 6.4.3 which connects non-expansiveness and balanced sets with periodicity, based on the method of Cyr and Kra. Periodicity in the proof first arises in a stripe from the use of the Morse–Hedlund theorem. This part of the proof follows Lemma 2.24 from [CK15]. The periodicity is then extended to the whole configuration by the following variant of Lemma 5.4.1:

Lemma 6.4.2. *Let c be a finitary integral two-dimensional configuration and S a stripe in direction $\boldsymbol{u}$ such that there exists a non-trivial annihilator which fits in S. If S° is periodic with a period in direction $\boldsymbol{u}$ then also c is periodic with a period in direction $\boldsymbol{u}$.*

Proof. Let $Ann(c) = \phi_1 \cdots \phi_m H = \phi H$ as in Corollary 4.2.1. Further let $h \in H$ be a line polynomial in direction $\boldsymbol{u}$ (for its existence see discussion before Lemma 5.3.1). Since ϕ divides the annihilator which fits in S, also ϕh is an annihilator which fits in S. If there is i such that ϕ_i is in direction $\boldsymbol{u}$, let $g = \phi_i$, otherwise let $g = 1$. Then the support of ϕ/g does not have an edge in direction $\boldsymbol{u}$ or $-\boldsymbol{u}$.

Let $\boldsymbol{v}$ be a period of S. Now consider $c' = (X^{\boldsymbol{v}} - 1)ghc$ and let S' be a stripe in c' determined by multiplication by the line polynomial $(X^{\boldsymbol{v}} - 1)gh$ from the values in S. S' has the same width as S and in particular ϕ/g fits in. Note that c' contains only zeros in $(S')^\circ$ and ϕ/g annihilates c'. Since ϕ/g does not have an edge in direction $\boldsymbol{u}$ or $-\boldsymbol{u}$ we have that $c' = 0$. In particular, c is annihilated by the line polynomial $(X^{\boldsymbol{v}} - 1)gh$, and therefore it is periodic in direction $\boldsymbol{u}$. □

Lemma 6.4.3. *Let c be a two-dimensional configuration and B a $\boldsymbol{u}$-balanced set. Assume that c contains an ambiguous stripe for X_c in direction $\boldsymbol{u}$ such that B fits in the stripe. Then c is periodic in direction $\boldsymbol{u}$.*

Proof. Let E be the edge or vertex of B in direction $\boldsymbol{u}$, denote S the stripe and let ℓ be the inner boundary of S in direction $\boldsymbol{u}$. Without loss of generality assume $B \subset S$ and $E \subset \ell$. Further assume that $\boldsymbol{u}$ is a primitive vector. Let $e \in X_c$ be such that Equation (6.1) holds.

Denote points in E consecutively by $e_1, \ldots, e_n$ (see Figure 6.7). Define a sequence $B = D_n \supset \cdots \supset D_1 \supset D_0 = B \setminus E$ by setting $D_{i-1} = D_i \setminus \{e_i\}$. Consider the values $P(D_i) - |D_i|$. Since B is a balanced set, by *(ii)* we have $P_c(D_n) - |D_n| < P_c(D_0) - |D_0|$, let $k \in [0, n-1]$ be such that

$$P_c(D_{k+1}) - |D_{k+1}| < P_c(D_k) - |D_k|.$$

Adding $|D_{k+1}|$ to both sides yields $P_c(D_{k+1}) < P_c(D_k) + 1$. On the other hand, $P_c(D_k) \leq P_c(D_{k+1})$ since $D_k \subset D_{k+1}$, and therefore we have $P_c(D_k) = P(D_{k+1})$. In other words, a D_k-pattern uniquely determines the value at position e_{k+1}.

We will show that $\forall i\colon c\restriction_{B+i\boldsymbol{u}} \neq e\restriction_{B+i\boldsymbol{u}}$. For the contrary, assume that there is j such that $c\restriction_{B+j\boldsymbol{u}} = e\restriction_{B+j\boldsymbol{u}}$, then also $c\restriction_{D_k+j\boldsymbol{u}} = e\restriction_{D_k+j\boldsymbol{u}}$. Using the property of D_k, we have $c\restriction_{e_{k+1}+j\boldsymbol{u}} = e\restriction_{e_{k+1}+j\boldsymbol{u}}$. Therefore $c\restriction_{D_k+(j+1)\boldsymbol{u}} = e\restriction_{D_k+(j+1)\boldsymbol{u}}$ and we can proceed by induction to show $c\restriction_{D_k+j'\boldsymbol{u}} = e\restriction_{D_k+j'\boldsymbol{u}}$ for all $j' > j$. Consequently, $c\restriction_{B+j'\boldsymbol{u}} = e\restriction_{B+j'\boldsymbol{u}}$ for all $j' > j$. Analogously, by constructing sets D_i by removing edge points from the other end, it can be shown that also $c\restriction_{B+j'\boldsymbol{u}} = e\restriction_{B+j'\boldsymbol{u}}$ for all $j' < j$. We proved $c\restriction_S = e\restriction_S$, which is a contradiction with ambiguity of S.

We have that all $(B \setminus E)$-patterns $c\restriction_{(B\setminus E)+i\boldsymbol{u}}$ have at least two possible extensions into a B-pattern. Part *(ii)* of Definition 6.2 implies that there are at most $|E| - 1$ such patterns. Let T be a thinner stripe in direction $\boldsymbol{u}$ defined by $T = \bigcup_{i\in\mathbb{Z}} (B \setminus E) + i\boldsymbol{u}$. Using part *(iii)*, values of c on every line $\lambda \subset T$ in direction $\boldsymbol{u}$ contain at most $|E| - 1$ distinct subsegments of length at least $|E| - 1$. By Morse–Hedlund theorem, the values on the line repeat periodically. Therefore $c\restriction_T$ is periodic in direction $\boldsymbol{u}$.

Since B is a balanced set, c is of low complexity and therefore finitary. To finish the proof, without loss of generality assume that c is integral and also normalized. Then by *(i)* of Definition 6.2 and Lemma 5.1.3 there is an annihilator which fits in $-B$. In particular, it fits in $T \cup \ell$. Since T is periodic in direction $\boldsymbol{u}$, by Lemma 6.4.2 also c is periodic in direction $\boldsymbol{u}$. □

6.5 Nivat's conjecture for $\mathrm{ord}(c) = 2$

Theorem 6.5.1. *Let c be a counterexample candidate such that $\mathrm{ord}(c) = 2$. Then $\forall m, n \in \mathbb{N}\colon P_c(m, n) > mn$.*

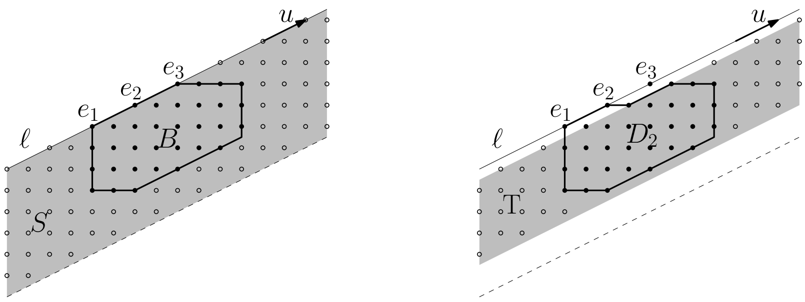

Figure 6.7: Illustration of the proof of Lemma 6.4.3.

Proof. For contradiction assume there exist $m, n \in \mathbb{N}$ such that $P_c(m,n) \le mn$. By Theorem 3.4.1 there exist periodic configurations c_1, c_2 such that $c = c_1 + c_2$. Denote by $\boldsymbol{u_1}, \boldsymbol{u_2}$ their respective vectors of periodicity and define a parallelogram

$$D = \left\{\, a\boldsymbol{u_1} + b\boldsymbol{u_2} \;\middle|\; a, b \in [0,1) \,\right\} \cap \mathbb{Z}^2.$$

We can choose $\boldsymbol{u_1}, \boldsymbol{u_2}$ large enough so that an $m \times n$ rectangle fits in. We can also assume that $\boldsymbol{u_2} \in H_{\boldsymbol{u_1}}$. Denote $D_j = D + j\boldsymbol{u_2}$ and define a sequence of stripes $S_j = \bigcup_{i \in \mathbb{Z}} D_j + i\boldsymbol{u_1}$. The setup is illustrated in Figure 6.8.

Assume that there are $j \neq j'$ such that $c\restriction_{D_j} = c\restriction_{D_{j'}}$. We claim that then $c\restriction_{S_j} = c\restriction_{S_{j'}}$. Note that $c' = (X^{(j-j')\boldsymbol{u_2}} - 1)c$ is annihilated by $X^{\boldsymbol{u_1}} - 1$ and $c'\restriction_{D_j}$ consists of zeros. Therefore c' is $\boldsymbol{u_1}$-periodic and $c'\restriction_{S_j}$ is zero, from which we have $c\restriction_{S_j} = c\restriction_{S_{j'}}$.

Since c is finitary there are only finitely many possible D-patterns, let N be an upper bound on their number. There are also finitely many stripe patterns $c\restriction_{S_j}$ since the pattern in S_j is determined by the pattern in D_j. Because c is not periodic, there exists $k \in \mathbb{Z}$ such that $c\restriction_{S_k} \neq c\restriction_{S_{k-N!}}$.

By Lemma 6.4.1, there is either a $\boldsymbol{u_1}$-balanced or $(-\boldsymbol{u_1})$-balanced set B, without loss of generality assume the former. Since c is non-periodic, by Lemma 6.4.3 there is no ambiguous stripe in c in direction $\boldsymbol{u_1}$ in which B fits. B fits in any stripe S_j, therefore values in any stripe S_j determine the values in the whole half-plane of the side on the inner boundary of S_j.

By pigeonhole principle, there are $j < j' \in [0, N]$ such that $c\restriction_{S_{k+j}} = c\restriction_{S_{k+j'}}$. The two stripes extend uniquely to the half-planes on the side of their inner boundary. Therefore the half-plane $H = \bigcup_{i \le j'} S_{k+i}$ has period $(j'-j)\boldsymbol{u_2}$. Since $j'-j$ divides $N!$ and $S_k, S_{k-N!} \subset H$, we have a contradiction with $c\restriction_{S_k} \neq c\restriction_{S_{k-N!}}$. □

Corollary 6.5.2. *Let c be a two-dimensional configuration satisfying $P_c(m,n) \le mn$ for some $m, n \in \mathbb{N}$. If c is a sum of two periodic configurations then it is periodic.*

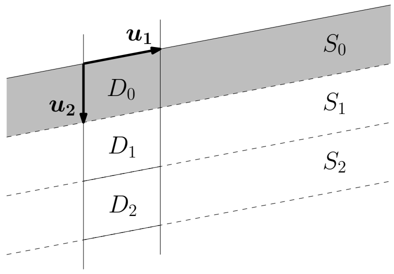

Figure 6.8: Proof of Theorem 6.5.1.

Corollary 6.5.3. *If a non-periodic two-dimensional configuration c is a sum of two periodic ones, then $P_c(m,n) \geq mn+1$ for all $m,n \in \mathbb{N}$.*

Theorem 6.5.1 is a special case of Lemma 5.5.3, and therefore this finishes the proof of asymptotic Nivat's conjecture Theorem 5.5.4. As a corollary, we also obtain the result of Cyr and Kra from [CK15]:

Theorem 6.5.4. *Let c be a configuration such that $P_c(m,n) \leq mn/2$ for some $m,n \in \mathbb{N}$. Then c is periodic.*

Proof. For a contradiction assume that c is a counterexample candidate. Let f be an annihilator of c which fits in an $m \times n$ rectangle. Using Corollary 4.2.1, we can write $f = \phi_1 \cdots \phi_m h$. If $\mathrm{ord}(c) \leq 2$, c is periodic by Theorem 6.5.1. Assume $\mathrm{ord}(c) \geq 3$, we will show that it leads to a contradiction.

Let $g = \phi_3 \cdots \phi_m$, denote $(m_g, n_g) = \mathrm{box}(g)$ and let $c' = gc$, see Figure 6.9. Note that an $(m - m_g) \times (n - n_g)$ block in c' is determined by multiplication by g from an $m \times n$ block in c. Therefore $P_c(m,n) \geq P_{c'}(m - m_g, n - n_g)$. Furthermore c' is a finitary configuration with $\mathrm{ord}(c') = 2$. We can apply Theorem 6.5.1 to get

$$P_c(m,n) \geq P_{c'}(m - m_g, n - n_g) > (m - m_g)(n - n_g).$$

Let $\boldsymbol{v}$ be an arbitrary vertex of the convex hull of $-\,\mathrm{supp}(g)$. Consider all translations of $-\,\mathrm{supp}(g)$ which are a subset of the rectangle $[m] \times [n]$, denote R the locus of $\boldsymbol{v}$ under these translations. There are $(m-m_g)(n-n_g)$ such translations, therefore the size of R is the same number.

Now let us define a shape $U = [m] \times [n] \setminus R$. It is a shape such that no polynomial multiple of g fits in $-U$. In particular no annihilator of c fits in $-U$, and thus by Lemma 5.1.3,

$$P_c(m,n) \geq P_c(U) > |U|.$$

Since either $(m - m_g)(n - n_g) = |R| \geq mn/2$ or $|U| \geq mn/2$, we have $P_c(m,n) > mn/2$, a contradiction. $\square$

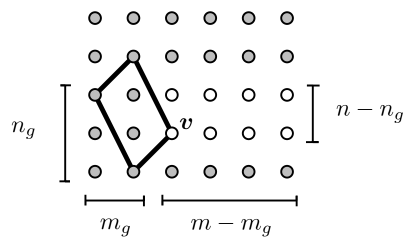

Figure 6.9: Proof of Theorem 6.5.4. The quadrilateral depicts the convex hull of $-\operatorname{supp}(g)$ for a polynomial g, positioned in the bottom left corner of an $m \times n$ block. The white points form the set R and the shaded points form the set U. We have $|U| \geq mn/2$ or $|R| \geq mn/2$.

Chapter 7

Bounded decomposition using ultrafilters

In section 3.4 we proved that any finitary configuration with a non-trivial annihilator can be written as a sum of periodic integral configurations. These configurations, however, could be unbounded, as demonstrated by Example 4.3.4. Moreover, the individual components could not be easily obtained from the original configuration.

In this short chapter we show an alternative way how to obtain the decomposition theorem. We are concerned with *bounded configurations* – configurations with coefficients from a real interval $[a, b]$. Unlike in Theorem 3.4.1, we give a recipe how to construct the periodic components. The method is, vaguely speaking, by averaging values along lines in the configuration. The resulting components however will not be integral anymore, their values will be reals from the interval $[a, b]$. To simplify notation, for vectors $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m}$ let us write $\partial_{\boldsymbol{u_1} \cdots \boldsymbol{u_m}} = (X^{\boldsymbol{u_1}} - 1) \cdots (X^{\boldsymbol{u_m}} - 1)$.

Theorem. *Let $c \in [a, b]^{\mathbb{Z}^d}$ be a d-dimensional configuration with an annihilator of the form $\partial_{\boldsymbol{u_1} \cdots \boldsymbol{u_m}}$ for some non-zero vectors $\boldsymbol{u_i}$. Then there exist periodic configurations $c_1, \ldots, c_n \in [a, b]^{\mathbb{Z}^d}$ such that $c = c_1 + \cdots + c_n$.*

Explicit description of the components is given in the statement of Theorem 7.3.2 later in this chapter. For the two-dimensional case the decomposition is particularly simple, as described in Corollary 7.3.4.

It was first indicated in the paper of Bhattacharya [Bha16] that such a bounded decomposition is possible. The paper was however concerned with two-dimensional configurations only and the set-up there was more involved – configurations were represented by a measurable function on a suitable subshift which was endowed with an ergodic measure, and the result was obtained by an application of ergodic theorem and spectral theorem for unitary operators.

Our proof makes use of ultrafilters. We use them as a tool to define a shift-invariant mean (average) of a sequence of numbers, section 7.1 introduces the theory needed. Equipped with this powerful tool, the proof of the theorem is quite straightforward. We present it in section 7.3.

7.1 Ultrafilters and ultralimit

Let us give a brief overview of the theory of ultrafilters, for proofs see e.g. [Kru]. Let X be an infinite set, for us usually $\mathbb{N}$. An (ultra)filter is a consistent choice of "large" subsets of X:

Definition 7.1. *A* filter *on X is $\mathfrak{F} \subset \mathcal{P}(X)$ such that*

1. $X \in \mathfrak{F}, \emptyset \notin \mathfrak{F}$ *(X is large, $\emptyset$ is not)*
2. $A \in \mathfrak{F}, A \subset B \Rightarrow B \in \mathfrak{F}$ *(supersets of large sets are large)*
3. $A, B \in \mathfrak{F} \Rightarrow A \cap B \in \mathfrak{F}$ *(finite intersections of large sets are large)*

Definition 7.2. *An* ultrafilter *is a filter that for any $A \subset X$ satisfies*

4. $A \in \mathfrak{F}$ *or* $(X \setminus A) \in \mathfrak{F}$ *(either a set or its complement is large)*

Ultrafilters are filters which are maximal with respect to the order by inclusions. The following is a consequence of (but not equivalent to) Zorn's lemma:

Theorem 7.1.1 (Ultrafilter lemma)**.** *Every filter is contained in an ultrafilter.*

An ultrafilter is *principal* if it is generated by one point, i.e. if it is of the form $\mathfrak{U}_x = \{ A \subset X \mid x \in A \}$ for some $x \in X$. Let us define the *cofinite filter* $\mathfrak{F}_{cof}$ to consist of all cofinite subsets of X, it is easy to check that it is a filter. By ultrafilter lemma, $\mathfrak{F}_{cof}$ is contained in some ultrafilter $\mathfrak{U}$. Such an ultrafilter is then necessarily non-principal. In fact, $\mathfrak{F}_{cof}$ is contained in any non-principal ultrafilter.

Definition 7.3. *Let T be a topological space, $f\colon X \to T$ a map and $\mathfrak{F}$ a filter on X. Let us define the* limit *of f with respect to $\mathfrak{F}$ to be such a point $y \in T$ which satisfies*

$$\forall U \in B(y)\colon f^{-1}(U) \in \mathfrak{F},$$

where $B(y)$ denotes the set of all open neighbourhoods of y. If such a point y exists we denote $\lim_{x \to \mathfrak{F}} f(x) = y$. If $\mathfrak{F}$ is an ultrafilter, we call it an ultralimit.

Lemma 7.1.2. *Let T be a compact Hausdorff space, $f\colon X \to T$ a map and $\mathfrak{U}$ an ultrafilter on X. Then $\lim_{x \to \mathfrak{U}} f(x)$ exists and is unique.*

We apply the theory for $X = \mathbb{N}$ and $T = [a, b]$ a non-empty real interval. Let us fix a non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ and let $s : \mathbb{N} \to [a, b]$ be a sequence, we denote more conveniently $s_n = s(n)$. Then $\lim_{n\to\mathfrak{U}} s_n$ is such a number $y \in [a, b]$ that satisfies

$$\forall U \in B(y)\colon \{\, n \mid s_n \in U \,\} \in \mathfrak{U}.$$

It can be shown that this is equivalent to

$$\forall \varepsilon > 0\colon \{n\colon |s_n - y| < \varepsilon\} \in \mathfrak{U}.$$

Note that $\lim_{n\to\mathfrak{U}} s_n$ always exists since $[a, b]$ is compact and Hausdorff. Further note that if we exchange $\mathfrak{U}$ with the cofinite filter $\mathfrak{F}_{cof}$ in the previous equation, we get the definition of the ordinary limit. $\mathfrak{F}_{cof}$ is contained in every non-principal ultrafilter, therefore if the ordinary limit exists, it agrees with the ultralimit.

Example 7.1.3. Let $(s_n)_{n\in\mathbb{N}} = 1, 0, 1, 0, \ldots$ be the sequence of alternating ones and zeros. Clearly it does not converge in the usual sense. However, for an ultrafilter $\mathfrak{U}$, either $\lim_{n\to\mathfrak{U}} s_n = 0$ or $\lim_{n\to\mathfrak{U}} s_n = 1$, depending on whether the set of even numbers, or its complement belongs to $\mathfrak{U}$.

7.2 Shift invariant means

Let $\mathfrak{U}$ be a fixed non-principal ultrafilter on $\mathbb{N}$ and $s \in [a, b]^{\mathbb{N}}$ a sequence. Let us define the *mean* of s to be

$$\mu(s) = \lim_{n\to\mathfrak{U}} \frac{1}{n} \sum_{i=1}^{n} s_i.$$

Lemma 7.2.1. *Mean is a linear shift invariant operator on bounded real sequences, i.e. for $s \in [a, b]^{\mathbb{N}}, t \in [a', b']^{\mathbb{N}}$:*

1. *If $\alpha, \beta \in \mathbb{R}$ then $\mu(\alpha s + \beta t) = \alpha\mu(s) + \beta\mu(t)$.*
2. *If $\exists k \in \mathbb{N}$ such that $t_n = s_{n+k}$, then $\mu(s) = \mu(t)$.*

Proof.

1. Ultralimit is a linear operator on bounded real sequences. The proof follows in a straightforward fashion from the definition of mean.
2. Compute:

$$\mu(s) - \mu(t) = \mu(s - t) = \lim_{n\to\mathfrak{U}} \frac{1}{n} \left(\sum_{i=1}^{k} s_i - \sum_{i=n-k+1}^{n} t_i \right).$$

Since s and t are bounded, the limit

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{k} s_i - \sum_{i=n-k+1}^{n} t_i \right)$$

exists and is zero. Therefore also the ultralimit exists and is zero.

□

Let $a, b \in \mathbb{R}$ and $c \in [a, b]^{\mathbb{Z}^d}$ be a configuration, we call such a configuration *bounded*. Let $\boldsymbol{u} \in \mathbb{Z}^d$ be a non-zero vector. Let us define the configuration $c^{\boldsymbol{u}}$ by

$$(c^{\boldsymbol{u}})_{\boldsymbol{v}} = \mu((c_{\boldsymbol{v}+n\boldsymbol{u}})_{n\in\mathbb{N}}) = \lim_{n\to\mathfrak{U}} \frac{1}{n} \sum_{i=1}^{n} c_{\boldsymbol{v}+i\boldsymbol{u}}.$$

Informally, the coefficient of $c^{\boldsymbol{u}}$ at position $\boldsymbol{v}$ is the mean of values of c lying on a discrete line starting at $\boldsymbol{v}$ and extending in direction $\boldsymbol{u}$ with jumps of length $\boldsymbol{u}$. Let us adopt a notational convention $c^{\boldsymbol{u_1}\boldsymbol{u_2}} = (c^{\boldsymbol{u_1}})^{\boldsymbol{u_2}}$.

Lemma 7.2.2. *Let c be a bounded d-dimensional configuration and $\boldsymbol{u} \in \mathbb{Z}^d$ non-zero. Then:*

1. *$c^{\boldsymbol{u}}$ is $\boldsymbol{u}$-periodic.*
2. *If c is $\boldsymbol{u}$-periodic then $c = c^{\boldsymbol{u}}$.*
3. *$c^{\boldsymbol{u}\boldsymbol{u}} = c^{\boldsymbol{u}}$.*

Proof. Part 1 is a direct consequence of shift invariance of mean. If c is $\boldsymbol{u}$-periodic, then $(c^{\boldsymbol{u}})_{\boldsymbol{v}}$ is a mean of a constant sequence of which every term equals $c_{\boldsymbol{v}}$, which shows part 2. Part 3 follows from 1 and 2. □

Lemma 7.2.3. *Let c be a bounded d-dimensional configuration and $\boldsymbol{u_1}, \ldots, \boldsymbol{u_k} \in \mathbb{Z}^d$ non-zero vectors. Then $\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}$ are periods of the configuration $c^{\boldsymbol{u_1}\cdots\boldsymbol{u_k}}$.*

Proof. The proof goes by induction, the case $k = 1$ is covered in Lemma 7.2.2. Let us assume $k > 1$ and that $c' = c^{\boldsymbol{u_1}\cdots\boldsymbol{u_{k-1}}}$ has periods $\boldsymbol{u_1}, \ldots, \boldsymbol{u_{k-1}}$. Since the operator $e \mapsto e^{\boldsymbol{u_k}}$ from definition commutes with translations, also $(c')^{\boldsymbol{u_k}}$ has periods $\boldsymbol{u_1}, \ldots, \boldsymbol{u_{k-1}}$. It has also period $\boldsymbol{u_k}$ by Lemma 7.2.2. □

Note that in general $c^{\boldsymbol{u}\boldsymbol{v}} \neq c^{\boldsymbol{v}\boldsymbol{u}}$. Let $\boldsymbol{u} = (1, 0)$, $\boldsymbol{v} = (0, 1)$ and c be a configuration defined by $c_{(i,j)} = 0$ if $i \geq j$ and $c_{(i,j)} = 1$ otherwise. Then $c^{\boldsymbol{u}} = c^{\boldsymbol{u}\boldsymbol{v}}$ is a constant 0 configuration, whereas $c^{\boldsymbol{v}} = c^{\boldsymbol{v}\boldsymbol{u}}$ is a constant 1 configuration. See Figure 7.1.

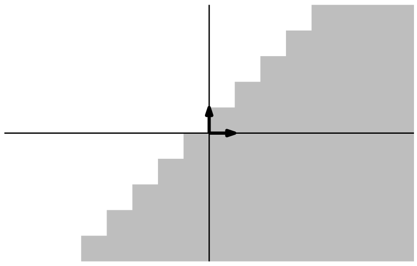

Figure 7.1: A configuration c for which $c^{(1,0)(0,1)} \neq c^{(1,0)(0,1)}$. White color denotes the coefficient 1 and dark the coefficient 0.

7.3 Bounded decomposition

Lemma 7.3.1. *Let c be a bounded d-dimensional configuration, $\boldsymbol{u} \in \mathbb{Z}^d$ a non-zero vector and $g \in \mathbb{R}[X^{\pm 1}]$ a Laurent polynomial. If $\partial_{\boldsymbol{u}} g$ annihilates c, then g annihilates $c - c^{\boldsymbol{u}}$.*

Proof. We repeatedly use linearity of mean and Lemma 7.2.2. The configuration $g(c - c^{\boldsymbol{u}})$ is $\boldsymbol{u}$-periodic since both gc and $gc^{\boldsymbol{u}}$ are. Then $g(c - c^{\boldsymbol{u}}) = (g(c - c^{\boldsymbol{u}}))^{\boldsymbol{u}} = g(c - c^{\boldsymbol{u}})^{\boldsymbol{u}} = g(c^{\boldsymbol{u}} - c^{\boldsymbol{u}\boldsymbol{u}}) = 0$. □

Theorem 7.3.2. *Let c be a bounded d-dimensional configuration and $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m} \in \mathbb{Z}^d$ non-zero vectors such that $\partial_{\boldsymbol{u_1} \cdots \boldsymbol{u_m}} c = 0$. Then*

$$c = \sum_{1 \le i \le m} c^{\boldsymbol{u_i}} - \sum_{1 \le i < j \le m} c^{\boldsymbol{u_i}\boldsymbol{u_j}} + \sum_{1 \le i < j < k \le m} c^{\boldsymbol{u_i}\boldsymbol{u_j}\boldsymbol{u_k}} - \cdots \pm c^{\boldsymbol{u_1} \cdots \boldsymbol{u_m}}. \quad (7.1)$$

Proof. For $\boldsymbol{u} \in \mathbb{Z}^d$ let us define an operator on bounded configurations by

$$(1 - \cdot^{\boldsymbol{u}}) \colon e \mapsto e - e^{\boldsymbol{u}}.$$

Since $\partial_{\boldsymbol{u_m} \cdots \boldsymbol{u_1}} c = 0$, inductive application of Lemma 7.3.1 yields

$$(1 - \cdot^{\boldsymbol{u_m}}) \cdots (1 - \cdot^{\boldsymbol{u_1}}) c = 0.$$

It is straightforward to verify that this expression evaluates to (7.1). □

Corollary 7.3.3. *Let c be a low complexity configuration. Then c can be written as a sum of finitely many bounded periodic configurations.*

Proof. Since c is of low complexity, it is finitary and therefore bounded. By Corollary 3.3.4 there are vectors $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m}$ such that $\partial_{\boldsymbol{u_1} \cdots \boldsymbol{u_m}} c = 0$. By Theorem 7.3.2, Equation (7.1) holds. Since the right-hand side of it consists of bounded periodic configurations we are done. □

Corollary 7.3.4. *Let c be a two-dimensional finitary integral configuration with a non-zero annihilator. Then there exist $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m} \in \mathbb{Z}^d$ in distinct directions and a doubly periodic configuration c^0 such that*

$$c = c^0 + c^{\boldsymbol{u_1}} + \cdots + c^{\boldsymbol{u_m}}.$$

Proof. By Theorem 3.3.3 there exist $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m} \in \mathbb{Z}^2$ in distinct directions such that $\partial_{\boldsymbol{u_1} \cdots \boldsymbol{u_m}} c = 0$. Define

$$c^0 = -\sum_{1 \leq i < j \leq m} c^{\boldsymbol{u_i}\boldsymbol{u_j}} + \sum_{1 \leq i < j < k \leq m} c^{\boldsymbol{u_i}\boldsymbol{u_j}\boldsymbol{u_k}} - \cdots \pm c^{\boldsymbol{u_1} \ldots \boldsymbol{u_m}}.$$

Then, by Theorem 7.3.2, $c = c^0 + c^{\boldsymbol{u_1}} + \cdots + c^{\boldsymbol{u_m}}$. It remains to show that c^0 is doubly periodic. Since vectors $\boldsymbol{u_i}$ are in distinct directions, by Lemma 7.2.3 every summand of c^0 is doubly periodic, and we are done. □

Example 7.3.5. Recall the configuration from Example 3.4.6 defined by

$$s_{ij} = \lfloor (i+j)\alpha \rfloor - \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor,$$

it is a binary configuration annihilated by $\partial_{\boldsymbol{u_1}\boldsymbol{u_2}\boldsymbol{u_3}}$ with $\boldsymbol{u_1} = (1,0)$, $\boldsymbol{u_2} = (0,1)$ and $\boldsymbol{u_3} = (-1,1)$. We can apply Corollary 7.3.4 to get $s = s^0 + s^{\boldsymbol{u_1}} + s^{\boldsymbol{u_2}} + s^{\boldsymbol{u_3}}$ where

$$\begin{aligned}
s^0_{ij} &= -1 \\
s^{\boldsymbol{u_1}}_{ij} &= j\alpha - \lfloor j\alpha \rfloor \\
s^{\boldsymbol{u_2}}_{ij} &= i\alpha - \lfloor i\alpha \rfloor \\
s^{\boldsymbol{u_3}}_{ij} &= 1 - (i+j)\alpha + \lfloor (i+j)\alpha \rfloor.
\end{aligned}$$

Note that all the components are bounded, and moreover coefficients of $s^{\boldsymbol{u_1}}, s^{\boldsymbol{u_2}}, s^{\boldsymbol{u_3}}$ are from the interval $[0,1]$. Also note that none of them is finitary. Indeed, in Example 4.3.4 we proved that s can not be written as a sum of finitely many finitary periodic components.

Let us demonstrate the derivation for $s^{\boldsymbol{u_1}}$. Note that

$$\lim_{n \to \mathfrak{U}} \frac{\lfloor (n+k)\alpha \rfloor}{n} = \lim_{n \to \infty} \frac{\lfloor (n+k)\alpha \rfloor}{n} = \alpha$$

holds for any $k \in \mathbb{Z}$. The result now follows by a computation:

$$\begin{aligned}
s_{ij}^{\boldsymbol{u_1}} &= \lim_{n\to\mathfrak{U}} \frac{1}{n}\sum_{k=1}^{n} s_{i+k,j} = \lim_{n\to\mathfrak{U}} \frac{1}{n}\sum_{k=1}^{n} \big(\lfloor (k+i+j)\alpha\rfloor - \lfloor j\alpha\rfloor - \lfloor (k+i)\alpha\rfloor\big)\\
&= -\lfloor j\alpha\rfloor + \lim_{n\to\mathfrak{U}} \frac{1}{n}\sum_{k=1}^{n} \big(\lfloor (k+i+j)\alpha\rfloor - \lfloor (k+i)\alpha\rfloor\big)\\
&= -\lfloor j\alpha\rfloor + \lim_{n\to\mathfrak{U}} \frac{1}{n}\left(\sum_{k=n-j+1}^{n} \lfloor (k+i+j)\alpha\rfloor - \sum_{k=1}^{j}\lfloor (k+i)\alpha\rfloor\right)\\
&= -\lfloor j\alpha\rfloor + \lim_{n\to\mathfrak{U}} \frac{1}{n}\sum_{\ell=1}^{j} \lfloor (n+i+\ell)\alpha\rfloor\\
&= -\lfloor j\alpha\rfloor + \sum_{\ell=1}^{j} \lim_{n\to\mathfrak{U}} \frac{\lfloor (n+i+\ell)\alpha\rfloor}{n}\\
&= j\alpha - \lfloor j\alpha\rfloor,
\end{aligned}$$

where if $j \leq 0$, we define $\sum_{\ell=1}^{j} = -\sum_{\ell=j}^{-1}$.

Chapter 8

Summary of open problems

Let us recall Nivat's conjecture, the main motivation for contents of this thesis:

Conjecture (Nivat)**.** *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional symbolic configuration such that $P_c(m,n) \leq mn$ for some $m, n \in \mathbb{N}$. Then c is periodic.*

Despite our efforts, it still remains open. We propose two weaker versions of its statement that we believe would shed more light onto the topic:

Conjecture 8.1. *There exists a real number $\frac{1}{2} < \alpha \leq 1$ such that if $c \in \mathcal{A}^{\mathbb{Z}^2}$ is a two-dimensional symbolic configuration satisfying $P_c(m,n) \leq \alpha mn$ for some $m, n \in \mathbb{N}$, then c is periodic.*

Conjecture 8.2. *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional symbolic configuration such that $P_c(m,n) \leq mn$ for some $m, n \in \mathbb{N}$. Then there exists a periodic configuration in X_c.*

Interestingly enough, for configurations satisfying $c_H = 0$ with notation from Corollary 4.2.1, Conjecture 8.1 holds with $\alpha = 2/3$. Indeed, the unique doubly periodic component c_H in the decomposition of c provides an obstacle when extending our proof to arbitrary c. We omit the proof, note however that the technique is identical to the proof of Theorem 6.5.4, with the use of Corollary 2(c) of [KS15b].

In Chapter 4 we proved that $\mathrm{Ann}(c)$ is a radical ideal for two-dimensional configurations. We conjecture that the same is the case for higher dimensions:

Conjecture 8.3. *Let c be a configuration. Then* $\mathrm{Ann}(c)$ *is a radical ideal.*

In the same chapter we define $\mathrm{ord}(c)$, which is the smallest possible number of periodic configurations which sum to c. Another direction of weakening Nivat's conjecture is to prove it for configurations with given ord. In

Chapter 6 we prove it for $\mathrm{ord}(c) = 2$. For $\mathrm{ord}(c) = 3$ we can prove Conjecture 8.2, i.e. that there is a periodic point in $\mathrm{ord}(c)$. We also think that proving Nivat's conjecture in this case is possible, however, the technique we used closely follows Cyr and Kra's proof from [CK15]. Since it is very technical and does not bring any major new ideas, we decided to omit it. Nevertheless, it would be interesting to see a nice proof of the fact.

Related to the subshift X_c and the decomposition of c from Chapter 4, we formulated Conjecture 6.1. Let us rephrase it in simpler terms:

Conjecture 8.4. *Let c be a two-dimensional configuration which can be written as a sum of periodic configurations and which is not doubly periodic. Let $c = c_1 + \cdots + c_m$ where m is smallest possible and c_i is a configuration with period $\boldsymbol{u_i}$. Then non-expansive directions of X_c are exactly the directions $\boldsymbol{u_i}$.*

Providing a proof of this conjecture would be interesting by itself since its statement does not involve any concepts defined in this thesis.

Let us finish by mentioning the periodic tiling problem, which is another long-term open problem to which the polynomial method can be applied:

Conjecture (Lagarias, Wang). *If $T \subset \mathbb{Z}^d$ is a tile which tiles $\mathbb{Z}^d$ by translations, there exists also periodic tiling of $\mathbb{Z}^d$ by T.*

After a recent remarkable proof for $d = 2$ [Bha16], the conjecture remains open for $d \geq 3$.

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Turku Centre *for* Computer Science

http://www.tucs.fi
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University of Turku
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- Institute of Information Systems Science

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Faculty of Science and Engineering
- Computer Engineering
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