ON TWO-SIDED CONTROLS OF A LINEAR DIFFUSION

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Pekka Matomäki
Turku, 17.9.2013

”In mathematics you don't understand things. You just get used to them.”
— J. von Neumann —

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LIST OF ORIGINAL RESEARCH PAPERS

(1) Matomäki, Pekka (2013) Optimal timing in a combined investment and exit problem, preprint.

(2) Matomäki, Pekka (2012) On solvability of a two-sided singular control problem, *Mathematical Methods of Operations Research*, vol. 76 (3), 239 – 271.

(3) Lempa, Jukka — Matomäki, Pekka (2013) A Dynkin game with asymmetric information, *Stochastics*, vol. 85 (5), 763 – 788.

(4) Alvarez, Luis H.R. — Matomäki, Pekka (2014) Optimal stopping of the maximum process, *Journal of Applied Probability*, To appear vol. 51 (3).

Part I

INTRODUCTION

1 INTRODUCTORY NOTES

In applied mathematics one always tries to construct a model that is complex enough to mimic the phenomenon under consideration and simple enough for analysis. A complex yet general model might not provide any concrete results due to its challenging nature. On the other hand, a very simple model is perhaps easy to solve but does not represent the real world whatsoever. So the choice of the model is practically always a trade off between generality and tractability.

In this thesis we use a one-dimensional, or linear, diffusion instead of, say, a multidimensional Markov process or Lévy process. Linear diffusions often arise naturally in various many situations and they can be used for modelling a wide range of phenomena from population dynamics to the path of a space ship. In addition, they can often be used to approximate more wildly behaving processes, such as multidimensional diffusions or Lévy processes, or at least they can offer insights on the behaviour of these more complex processes. While forming a large and applicable class of processes, at the same time linear diffusions are friendly and approachable. The main reasons for this are the facts that they are continuous, Markovian and one-dimensional, and therefore one can use a wide variety of tools to study them. Additionally, almost all interesting linear diffusions constitute a solution to a stochastic differential equation, allowing us to pick up additional techniques from Itô calculus and benefit from its intuitivity. All in all, although linear diffusions are general processes, they are surprisingly often simple enough to offer explicit solutions to problems and to allow one to analyse the nature of the problem quite deeply. One drawback of linear diffusions is that due to continuity they cannot always represent downside risks, e.g. in financial markets, natural resources, etc., as well as one would desire.

In this thesis, we will tour around the field of stochastic control theory. The tour will necessarily be incomplete, but it illuminates the great variety of problems encountered in the world of controlling linear diffusions. We will study four different optimal control problems, namely an investment problem, a two-player stochastic game (a Dynkin game), a singular control problem and a stopping problem involving a maximum process. In all these problems the optimal control turns out to be a two-sided control — at least under some conditions. In fact, one of our main tasks is to find such conditions.

One-sided stochastic control problems are widely studied and their solutions can often be determined explicitly. On the other hand, with two-sided control problems we often find it difficult to even prove the existence and uniqueness of an optimal solution due to the increase in free parameters. Thus, although the main lines of the analysis follow those in one-sided problems, we have to adopt new techniques to deal with the new obstacles.

In Figure 1 we see the research subjects of the thesis and their relations with each other. An ordinary optimal stopping problem constitutes the base problem and the investment problem is a special case of it, while the three other subjects can be seen as its generalisations. Moreover, the values of a Dynkin game and a two-sided singular control problem are closely related, as we shall see in Chapter 6.

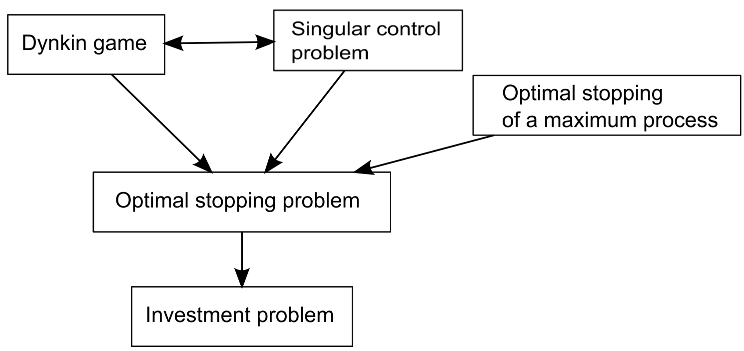

Figure 1: Research subjects of the thesis and their relations.

The content of the thesis is divided into two parts: the introductory part and the research part. In the introductory part we will first present an overview of the basic theory of linear diffusions. Then we will introduce ordinary optimal stopping problems and present solution methods for them. Finally we will introduce more general control problems; namely Dynkin games and singular control problems, and state solution methods for them. Throughout this first part of the thesis we will concentrate mainly on the techniques and theory needed in the research articles, but the tour also includes bypaths that open new perspectives to the considered subjects. The introduction is followed by the research part, which consists of four independent research papers.

2 STOCHASTIC PROCESS

2.1 General remarks on stochastic processes

We shall consider behaviour only on continuous time, and to that end, let the interval $[0,\infty)$ be our time set. Moreover, let $\{\Omega,\mathscr{F},\mathbb{P}\}$ be a probability space. A collection of σ-algebras $\{\mathscr{F}_t\}$, $t \in [0\infty)$, on the space $\{\Omega,\mathscr{F}\}$ is called a filtration if

$$\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F} \quad \text{for every } s \leq t.$$

The collection $(\Omega,\mathscr{F},\{\mathscr{F}_t\},\mathbb{P})$ is called a filtered probability space, and it is said to satisfy the usual conditions, if

- $\mathscr{F}$ is $\mathbb{P}$-complete (i.e. if there exist $A \subset \Omega$ and $A_1, A_2 \in \mathscr{F}$ such that $A_1 \subset A \subset A_2$ and $\mathbb{P}(A_1) = \mathbb{P}(A_2)$, then $A \in \mathscr{F}$);

- $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$;

- $\{\mathscr{F}_t\}$ is right-continuous (i.e. $\mathscr{F}_s = \mathscr{F}_{s+} := \bigcap_{t>s} \mathscr{F}_t$).

In this thesis we always assume the filtered probability space to satisfy the usual conditions.

Consider a random variable $X_t(\omega)$ defined on a filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\},\mathbb{P})$ and taking values in the state space $(\mathscr{I},\mathscr{B})$, where $\mathscr{I}$ is a non-empty (topological) space and $\mathscr{B}$ is the Borel σ-algebra on $\mathscr{I}$. A collection $X := \{X_t : \Omega \to \mathscr{I} \mid t \in [0,\infty)\}$ of such random variables is called a *stochastic process*. For a fixed scenario $\omega \in \Omega$ the mapping $t \to X_t(\omega)$ is said to be a *sample path* or a *trajectory* of the process X.

We say that X is *adapted* to the filtration $\{\mathscr{F}_t\}$ if X_t is $\mathscr{F}_t$-measurable for every t. The *natural filtration* $\{\sigma\{X_s \mid 0 \leq s \leq t\}\}$ is defined to be the smallest filtration to which X is adapted and it can be interpreted as the information generated by the history of the process X up to the date t. We denote by $\mathbb{P}_x$ the probability measure $\mathbb{P}$ conditioned on the initial state $X_0 = x$ and by $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$.

We define a random variable $\tau : \Omega \to [0,\infty]$ to be a *stopping time* (with respect to $\mathscr{F}$) if $\{\tau \leq t\} \in \mathscr{F}_t$ for every $t \in [0,\infty)$. Thus a stopping time is a random variable, which depends only on the history of the process X. Furthermore, if τ is a stopping time, then the *stopping time σ-algebra* is

$\mathscr{F}_\tau := \{F \in \mathscr{F} \mid F \cap \{\tau \le t\} \in \mathscr{F}_t \text{ for all } t \ge 0\}$, i.e. $\mathscr{F}_\tau$ is simply the history up to a stopping time τ.

Example 2.1.
- Examples of stopping times are
  - the first exit time from an open set $C \subset \mathscr{I}$: $\tau_C = \inf\{t \ge 0 \mid X_t \notin C\}$;
  - the first arrival time to a closed set $G \subset \mathscr{I}$: $\tau^G = \inf\{t \ge 0 \mid X_t \in G\}$;
  - the first hitting time to a point $z \in \mathscr{I}$: $\tau_z = \inf\{t \ge 0 \mid X_t = z\}$;
  - every deterministic time $t \in [0, \infty]$.
- The last exit time from a set $G \subset \mathscr{I}$, i.e. $\sup\{t \ge 0 \mid X_t \in G\}$, is not a stopping time, since it depends on the future evolution of the process.

A stopping time ζ is said to be a *life time*, or a *terminal time*, of a process X if the process is terminated at that time. We understand the termination of the process in the following way: We attach an additional *cemetery state* $\partial \notin \mathscr{I}$ to the state space and let $(\mathscr{I}^\partial, \mathscr{B}^\partial) = (\mathscr{I} \cup \{\partial\}, \sigma\{\mathscr{B}, \{\partial\}\})$ be this new state space and its Borel σ-algebra. Within this state space we understand the process X to be killed at the time ζ and immediately sent to the cemetery state where it stays for the rest of the time (see Dynkin 1965, Subsection 3.1). That is, the process with life time ζ evolves as

$$\begin{cases} X_t, & t < \zeta \\ \partial, & t \ge \zeta. \end{cases}$$

We extend an arbitrary function $f : \mathscr{I} \to \mathbb{R}$ to the enlarged state space $\mathscr{I}^\partial$ by defining $f(\partial) = 0$. In the sequel, we assume the cemetery state to be attached to the state space if needed, so that $\mathbb{P}_x(X_t \in \mathscr{I}) = 1$ for all $t \in [0, \infty)$ (i.e. the process is assumed to be *conservative*).

Example 2.2. Typical life times are

- $\zeta = \infty$ (so called *non-terminating* process);
- $\zeta = \inf\{t \ge 0 \mid X_t \notin C\}$ for some open set $C \subset \mathscr{I}$ (killed at the first exit time);
- $\zeta \sim \mathrm{Exp}(r)$ for some $r > 0$ (killed at an exponential rate);

- $\zeta = T$ for some deterministic time $T \in (0, \infty)$ (fixed finite time horizon, e.g. a maturity of an option).

Let us lastly define a few concepts for stochastic processes that will be used throughout the thesis. Firstly, a process X_t is *continuous* or *sample-continuous* if the sample paths $X_t(\omega)$ are continuous in t for $\mathbb{P}$-almost all ω. Secondly, a process is said to be *regular*, if every state can be reached from any other state, i.e. if $\mathbb{P}_x(X_t = y \text{ for some } t > 0) > 0$ for all $x, y \in \mathscr{I}$. Lastly, a process is said to be *time homogeneous* if the future evolution of the process does not depend on the current time, i.e. for all $B \in \mathscr{B}$ and $t, h > 0$ we have $\mathbb{P}_x(X_{t+h} \in B \mid X_t) = \mathbb{P}_{X_t}(X_h \in B)$.

From now on, we only consider regular, continuous, time-homogeneous processes unless otherwise stated.

2.2 Markov property

Consider a filtered probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P}\}$ and a stochastic process X_t on it, taking values on a state space $(\mathscr{I}, \mathscr{B})$.

Definition 2.3. Let $f : \mathscr{I} \to \mathbb{R}$ be a bounded and measurable function and let $B \in \mathscr{B}$.

(A) A process X_t is said to be a (time-homogeneous) *Markov process*, if for all $t, h > 0$

$$\mathbb{E}_x\{f(X_{t+h}) \mid \mathscr{F}_t\} = \mathbb{E}_x\{f(X_{t+h}) \mid X_t\} = \mathbb{E}_{X_t}\{f(X_h)\}, \quad \text{or equivalently}$$
$$\mathbb{P}_x(X_{t+h} \in B \mid \mathscr{F}_t) = \mathbb{P}_x(X_{t+h} \in B \mid X_t) = \mathbb{P}_{X_t}(X_h \in B) \quad \text{for all } B \in \mathscr{B}.$$

(B) A process X_t is said to be a (time-homogeneous) *strong Markov process* if the property above holds for all $h > 0$ and all finite $\mathscr{F}$-stopping times τ.

Roughly speaking, the Markov process is memoryless on meaning that the future evolution of the process depends only on its current state, not on how it got there. Another way to characterise the Markov property is to say that given the present state X_t, the past $\mathscr{F}_t$ and the future $\sigma\{X_s \mid s \geq t\}$ are independent. Moreover, since all deterministic times are stopping times, obviously a strong Markov process is also a Markov process, but the contrary is not true in general.

Example 2.4. • A one-dimensional Brownian motion[1] on $\mathbb{R}$ is a strong Markov process.

- Let W_t be a one-dimensional Brownian motion. Its *maximum process* S_t, defined by $S_t = \sup_{0\le s\le t}\{W_s\}$, is not a Markov process.

- Let us introduce an example of a Markov process which is not strong Markov. Let W_t be a one-dimensional Brownian motion and define X_t by

$$X_t = \begin{cases} W_t, & X_0 \neq 0 \\ 0, & X_0 = 0. \end{cases}$$

On can then show that X_t is a Markov process, but that it does not satisfy the strong Markov property for hitting times to zero (see p. 161 in Wentzell 1981).

An important feature in the general theory of Markov processes is the transition function, which enables one to interpret the expectation $\mathbb{E}_x\{f(X_t)\}$ as a semi-group operator, operating on $f(x)$; see e.g. Dynkin 1965, Chapter 1 in Blumenthal and Getoor 1968, Chapter 1 in Borodin and Salminen 2002 or Section II.4 in Peskir and Shiryaev 2006. However, the main goal of this thesis is not to deepen the theory of Markov processes, but to study optimal controls with respect to linear diffusions, a well-behaving class of Markov processes. Hence we shall not directly need the additional properties of a Markov structure that the transition functions can provide and so we will not discuss it further. An interested reader can consult the references above.

2.3 Martingale property

Another useful and important property for stochastic processes for applications are the martingales. In this section we consider a filtered probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}\}_t, \mathbb{P}\}$ and on it, a process X_t which almost surely has sample paths that are right continuous with left limits (so called *cadlag* processes).

Definition 2.5. Let X_t be adapted to the filtration $\{\mathscr{F}_t\}$ and and let it satisfy $\mathbb{E}_x\{|X_t|\} < \infty$ for all $t > 0$.

[1] One-dimensional Brownian motion W_t is a process that satisfies (i) $W_0 = 0$, (ii) path of W_t is almost surely continuous, and (iii) W_t has independent increments with distribution $W_t - W_s \sim N(0, t-s)$ for $0 \le s \le t$. See Chapter I of Rogers and Williams 2000a or Chapter IV in Borodin and Salminen 2002 for basic properties of Brownian motion.

(A) The process X_t is a *martingale*, if

$$\mathbb{E}_x\{X_{t+h} \mid \mathscr{F}_t\} = X_t \quad \text{for all } t,h > 0 \text{ and } x \in \mathscr{I}. \tag{1}$$

(B) *Supermartingale* is defined similarly, except that ”$=$” in (1) is replaced by ”$\leq$”.

(C) *Submartingale* is defined similarly, except that ”$=$” in (1) is replaced by ”$\geq$”.

The theory of martingales is wide and deep. For instance, there are powerful convergence theorems and strong inequalities for supermartingales (see e.g. Rogers and Williams 2000a). There are also weaker versions of the martingale-property:

Definition 2.6. (A) The process X_t is a *local martingale* if X_0 is $\mathscr{F}_0$-measurable and there exists an increasing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\tau_n = \infty$ almost surely, and for each $n \in \mathbb{N}$ the stopped process

$$X_{t\wedge\tau_n} - X_0 = \begin{cases} X_t - X_0, & t < \tau_n \\ X_{\tau_n} - X_0, & t \geq \tau_n \end{cases}$$

is a martingale.

(B) The process X_t is a *semi-martingale*, if $X_t = X_0 + M_t + N_t$, where M_t is a local martingale and N_t a finite variation process[2] and $M_0 = N_0 = 0$.

Example 2.7. • To get some idea how large the class of semi-martingales is, notice that all Lévy processes[3] are semi-martingales (see Theorem II.9 in Protter 2004).

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[2]A process N_t, which almost surely has sample paths that are right continuous with left limits, is a *finite variation process* if its variation over any finite time interval $[0,t)$ is finite, i.e

$$\sup \sum_{i=1}^{n} |N_{s_i} - N_{s_{i-1}}| < \infty,$$

where the supremum is taken over all partitions $0 = s_0 < s_1 < \cdots < s_n = t$. Informally we can say that a finite variation process fluctuates only moderately.

[3]A process L_t is a *Levy process*, if (i) $L_t - L_s$ is independent of $\mathscr{F}_s$ for all $0 \leq s < t < \infty$ (independent increments), (ii) $L_t - L_s \sim L_{t-s}$ for all $0 \leq s < t < \infty$ (stationary increments), and (iii) $\lim_{t\to s} L_t = L_s$ with probability 1 (continuous in probability). See e.g. Kyprianou 2006 for a comprehensive treatment of Lévy processes.

- A Brownian motion W_t on $\mathbb{R}$ is a martingale. Moreover, $M_t := W_t^2 - t$ is also a martingale (cf. Theorem I.27 in Protter 2004): since W_t is a martingale, we have $\mathbb{E}\{W_t W_s \mid \mathscr{F}_s\} = W_s^2$ for all $s < t$, so that

$$\begin{aligned}\mathbb{E}\{M_t - M_s \mid \mathscr{F}_s\} &= \mathbb{E}_x\left\{W_t^2 - 2W_tW_s + W_s^2 - (t-s) \mid \mathscr{F}_s\right\} \\ &= \mathbb{E}\left\{(W_s - W_t)^2\right\} - (t-s) = 0.\end{aligned}$$

- Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and X_t a martingale process on $\mathbb{R}$. Then, by Jensen inequality (i.e. $f(\mathbb{E}_x\{X_t\}) \leq \mathbb{E}_x\{f(X_t)\}$), a process $f(X_t)$ is a submartingale. Similarly, a concave function of a martingale is a supermartingale.

- Using Fatou's lemma one can show that a positive local martingale is always a supermartingale.

The theory of martingales is meaningful in this thesis since optimal stopping problems can be approached using martingale theory — the value function of an optimal stopping problem is the smallest supermartingale dominating the reward function. Another important issue for us is that the theory of Itô calculus (cf. Section 3.9 below), one of the most practical concepts in stochastics, is essentially based on the martingale properties of a Brownian motion. For a thorough treatment of local martingales, semi-martingales, and Itô calculus, see Rogers and Williams 2000b and Protter 2004.

3 LINEAR DIFFUSIONS

3.1 The definition of a linear diffusion

For the rest of the monograph, we shall consider only certain stochastic processes, namely one-dimensional, or linear, diffusions on an interval $\mathscr{I} \subset \mathbb{R}$ of the form

$$\mathscr{I} = (\alpha,\beta), \quad \mathscr{I} = [\alpha,\beta), \quad \mathscr{I} = (\alpha,\beta], \quad \text{or } \mathscr{I} = [\alpha,\beta],$$

for some endpoints $-\infty \leq \alpha < \beta \leq \infty$, depending on whether the process can hit the boundaries or not. For a thorough discussion of diffusions, consult Itô and McKean 1974, Chapter II in Borodin and Salminen 2002, or Chapter V in Rogers and Williams 2000b.

Definition 3.1. A sample-continuous stochastic process X_t on $\mathscr{I}$ is a (regular and time-homogeneous) *linear diffusion*, if it is a regular, time-homogeneous, strong Markov process.

Notice that in its widest definition, linear diffusions are neither regular nor time-homogeneous. However, in this thesis we only consider regular time-homogeneous processes. Therefore, from now on we understand linear diffusions to be regular and time-homogeneous unless otherwise stated.

Without any further restrictions, the definition is often too wide for applications (cf. Sections V.1–2 in Rogers and Williams 2000b), since it allows processes to behave in an unruly manner (e.g. there are diffusions that are not semi-martingales, see Example 3.24). Therefore, our next task is to find a more practical class of diffusions, and for this task an infinitesimal generator of a diffusion will be handy.

Definition 3.2. Let X_t be a diffusion on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ evolving on $\mathscr{I}$. The *infinitesimal generator* $\mathscr{A}$ of X is defined by

$$\mathscr{A} f := \lim_{t \to 0} \frac{\mathbb{E}_x\{f(X_t)\} - f(x)}{t},$$

where $f : \mathscr{I} \to \mathbb{R}$ is such that the limit exists for all $x \in \mathscr{I}$.

Intuitively, we can interpret $\mathscr{A} f(x)$ to be an expected growth rate of the process $f(X_t)$ at the point $f(x)$.

The generator has a computationally useful close connection to (partial) differential equations. This connection becomes visible after noticing (see Section V.5 in Dynkin 1965) that, for $f \in C^2(\mathscr{I})$, the infinitesimal generator is in fact a linear second order differential operator

$$\mathscr{A} f(x) = \frac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x) - c(x) f(x)$$

for some functions σ, μ, and c. Here $\sigma(x)$, $c(x) \geq 0$, and so the operator is elliptic. Now our aim is to create diffusions with given properties, and thus we would like to know when this connection can be used in the reverse direction: For what kinds of functions σ, μ, and c is the second order differential equation $\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - c(x)$ an infinitesimal generator of a diffusion? The following theorem answers to this question and allows us to build diffusions easily from elliptic differential equations. Before stating the theorem, let us define an *explosion time* to be the minimum of terminal time and the first hitting time of the process to its boundaries, i.e. $\inf\{t \geq 0 \mid X_t \in \{\alpha, \beta, \partial\}\}$.

Theorem 3.3. *Let $\sigma : \mathscr{I} \to \mathbb{R}_+$, $\mu : \mathscr{I} \to \mathbb{R}$, and $c : \mathscr{I} \to \mathbb{R}_+$ be continuous functions. Assume that they satisfy the conditions*

- *$\sigma(x) > 0$ and $c(x) \geq 0$ for all $x \in \mathscr{I}$; and*
- *for all $x \in \mathscr{I}$ there exists $\varepsilon > 0$ such that $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(s)|}{\sigma^2(s)} ds < \infty$.*

Then there exists a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ and a diffusion process X_t on it with state space $\mathscr{I}$ such that up to an explosion time the infinitesimal generator of X_t is

$$\mathscr{A} f(x) = \frac{1}{2}\sigma^2(x) f''(x) + \mu(x)u'(x) - c(x) f(x) \quad \textit{for } f \in C^2.$$

Furthermore, this diffusion process is unique in law[4].

This theorem is deep and its proof requires the concept of stochastic differential equations; the assumptions guarantee a (weak) unique solution to a certain stochastic differential equation (we talk more about this in Section 3.9 below). However, we observe that the theory defines a unique diffusion process

[4]Assume that μ, σ and c satisfy the conditions of Theorem 3.3. Then a process is unique in law, if whenever two processes X_t and Y_t are such that their infinitesimal generators $\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - c(x)$ coincide, they have the same distribution as processes.

up to an termination time ζ only on interior points of $\mathscr{I}$. Hence, in order to make the diffusion process unique everywhere, we need to define its behaviour at the boundaries and this is done in Section 3.4 below.

The analysis in this thesis extensively utilises the differential aspect of the infinitesimal generator, and for this reason we shall not study arbitrary general diffusions. Rather, we will restrict our scope to the diffusions which can be derived using Theorem 3.3 above. Although we exclude a proportion of diffusions from the study by doing so, in practice this proportion is so small that the choice is justifiable.

So from now on, unless otherwise stated, we assume that the *infinitesimal parameters* σ, μ, and c are given and satisfy the conditions of Theorem 3.3, and the linear diffusion $X := \{X_t : \Omega \to \mathscr{I} \mid t \in [0,\infty]\}$ is given by Theorem 3.3 above. The results in this section (as well as in the sections to come) hold true for more general diffusions than this. However, the formulation of the results with this definition is adequate for us. The parameters σ, μ and c are called, respectively, the *infinitesimal variance*, the *infinitesimal mean* and the *infinitesimal killing rate* of X, since often

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t}\mathbb{E}_x\left\{(X_t - x)^2\right\} &= \sigma^2(x), \\ \lim_{t\to 0} \frac{1}{t}\mathbb{E}_x\{X_t - x\} &= \mu(x), \\ \text{and} \quad \lim_{t\to 0} \frac{1}{t}\left(\mathbb{P}_x(\zeta > t) - 1\right) &= -c(x). \end{aligned}$$

Example 3.4. Examples of diffusions are

- standard Brownian motion (or Wiener process) for which $\mathscr{A} f(x) = \frac{1}{2} f''(x)$ and $\mathscr{I} = \mathbb{R}$;

- geometric Brownian motion with killing at an exponential rate r, and in this case $\mathscr{A} f(x) = \frac{1}{2}\sigma^2 x^2 f''(x) + \mu x f'(x) - r f(x)$ for some $\mu \in \mathbb{R}$ and $\sigma, r > 0$ and $\mathscr{I} = \mathbb{R}_+$;

- a mean reverting diffusion with killing at an exponential rate r. In this case $\mathscr{A} u(x) = \frac{1}{2}\sigma^2 x^2 f''(x) + \mu x(1-\gamma x) f'(x) - r f(x)$ for some $\mu \in \mathbb{R}$ and γ, $\sigma > 0$ and $\mathscr{I} = \mathbb{R}_+$. Here $1/\gamma$ (a state at which the infinitesimal drift term disappears) is known as the carrying capacity in biological applications. As γ approaches zero, the mean reverting diffusion approaches geometric Brownian motion.

Let us end this section with an important result showing the strong bonds between a diffusion and its infinitesimal generator (for the proof, see e.g. Theorem 7.4.1 in Øksendal 2007).

Theorem 3.5 (Dynkin's formula). *Let X_t be a diffusion and let $f \in C^2(\mathscr{I})$. Moreover, let τ be a stopping time for which $\mathbb{E}_x\{\tau\} < \infty$. Then*

$$\mathbb{E}_x\{f(X_\tau)\} = f(x) + \mathbb{E}_x\left\{\int_0^\tau \mathscr{A} f(X_s)ds\right\}.$$

3.2 Killing a diffusion

Especially in financial applications, the problem setting often involves discounting. In this section, we shall demonstrate that an expectation with continuous discounting is, in fact, nothing but an expectation of a killed diffusion.

Let X be a diffusion process with infinitesimal generator $\mathscr{A}$ and let $u : \mathscr{I} \to \mathbb{R}_+$ be a continuous function. Furthermore, define a continuous, non-negative process

$$\gamma_t := e^{-\int_0^t u(X_s)ds}$$

and a new diffusion process $\tilde{X}$ through the generator

$$\tilde{\mathscr{A}} = \mathscr{A} - u(x).$$

Let g be a continuous function, τ a stopping time, and $\tilde{\zeta}$ the terminal time of the process $\tilde{X}$. Then we have

$$\begin{aligned} \tilde{\mathbb{E}}_x\left\{g(\tilde{X}_t)\mathbb{1}_{\{t<\tilde{\zeta}\}}\right\} &= \mathbb{E}_x\{\gamma_t g(X_t)\} \\ \text{and} \quad \tilde{\mathbb{E}}_x\left\{\int_0^{\tau\wedge\tilde{\zeta}} g(\tilde{X}_s)ds\right\} &= \mathbb{E}_x\left\{\int_0^\tau \gamma_s g(X_s)ds\right\}, \end{aligned} \tag{2}$$

where $\tilde{\mathbb{E}}$ is an expectation taken with respect to the diffusion $\tilde{X}$. Especially, since γ_t is $\mathscr{F}_t$-measurable, taking $g(x) = \mathbb{1}_{\mathscr{I}}(x)$ gives

$$\gamma_t = \mathbb{E}_x\{\gamma_t \mathbb{1}_{\mathscr{I}}(X_t) \mid \mathscr{F}_t\} = \tilde{\mathbb{E}}_x\left\{\mathbb{1}_{\mathscr{I}}(\tilde{X}_t)\mathbb{1}_{\{t<\tilde{\zeta}\}} \mid \mathscr{F}_t\right\} = \tilde{\mathbb{P}}\left(\tilde{\zeta} > t \mid \mathscr{F}_t\right), \tag{3}$$

and so the discounting γ_t can be interpreted as the conditional probability that the trajectory of $\tilde{X}_t$ does not terminate before time t. For a more exact and thorough exposition, see Section X.4 in Dynkin 1965 or Chapter III in Blumenthal and Getoor 1968.

Example 3.6 (Continuous discounting as exponential killing)**.** Let X be a non-terminating diffusion (i.e. life time $\zeta = \infty$) associated with the generator $\mathscr{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ and introduce a constant killing $u(t) \equiv r > 0$ so that the generator of the killed diffusion $\tilde{X}$ is $\tilde{\mathscr{A}} = \mathscr{A} - r$. Now $\gamma_t = e^{-rt}$ is independent of X_t and hence, using (3), we get

$$\tilde{\mathbb{P}}_x\left(\tilde{\zeta} \leq t\right) = 1 - e^{-rt},$$

where the right hand side is a cumulative density function of the distribution $\mathrm{Exp}(r)$. Therefore, we see that the diffusion $\tilde{X}$ is killed at an exponential rate and we can interpret $\tilde{X}$ as

$$\tilde{X} = \begin{cases} X_t, & t < \zeta_r \\ \partial, & t \geq \zeta_r, \end{cases}$$

where $\zeta_r \sim$ Exp(r). Moreover, applying (2), we see that for any continuous function $g(x)$

$$\mathbb{E}_x\left\{e^{-rt}g(X_t)\right\} = \tilde{\mathbb{E}}_x\left\{g(\tilde{X}_t)\right\},$$

where we understand $g(\partial) = 0$. In other words, the expected payoff with continuous discounting can be written as an expectation of an undiscounted payoff with respect to a diffusion killed at an exponential rate.

3.3 Basic characteristics of a diffusion

3.3.1 Introducing the characteristics

Every diffusion has three basic characteristics: scale function S, speed measure m and killing measure k. These can be defined via the infinitesimal parameters σ, μ and c, for all $x \in \mathscr{I}$, as

$$S'(x) = e^{-B(x)}, \quad m(x) = \frac{2}{\sigma^2(x)}e^{B(x)}, \quad \text{and } k(x) = \frac{2}{\sigma^2(x)}c(x)e^{B(x)}, \tag{4}$$

where $B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}dy$. We can see m and k either as functions or measures; in the latter case we understand $m\{(a,b)\} = \int_a^b m(dz)$, and analogously $k\{(a,b)\} = \int_a^b k(dz)$, for all intervals $(a,b) \subset \mathscr{I}$.

The scale function $S : \mathscr{I} \to \mathbb{R}$ is increasing, and because we assumed that μ and σ satisfy the integrability condition of Theorem 3.3, it is also twice continuously differentiable. Moreover, S is determined uniquely up to additive and multiplicative constants[5] and it satisfies the equation $\frac{1}{2}\sigma^2(x)S''(x) + \mu(x)S'(x) = 0$. It is also closely connected to the hitting time distribution. Indeed, if $k((a,b)) = 0$ for an interval $(a,b) \subset \mathscr{I}$, then for $x \in (a,b)$

$$\mathbb{P}_x(\tau_a < \tau_b) = 1 - \mathbb{P}_x(\tau_b < \tau_a) = \frac{S(b) - S(x)}{S(b) - S(a)},$$

where $\tau_y = \inf\{t \geq 0 \mid X_t = y\}$ denotes the first hitting time of X to the state y. In other words, the scale function rescales the state space so that the hitting probabilities become proportional to actual distances. We say that the diffusion is in *natural scale* if $S(x) = x$. Furthermore, we see a connection to martingales, as the scaled diffusion $S(X_t)$ is a local martingale (see Corollary V.46.15 in Rogers and Williams 2000b).

Example 3.7. A standard Brownian motion is in natural scale. Moreover, for each diffusion X_t, the scaled diffusion $S(X_t)$ is in natural scale on the state space $S(\mathscr{I})$.

The speed measure $m : \mathscr{I} \to \mathbb{R}_+$ satisfies the adjoint equation $\frac{1}{2}\frac{d^2}{dx^2}(\sigma^2(x)m(x)) + \frac{d}{dx}(\mu(x)m(x)) = 0$, whenever σ and μ are smooth enough. Furthermore, m measures, in some sense, the speed of the process — in the regions where m is large, the diffusion moves slowly (cf. II.16 in Borodin and Salminen 2002). For each finite interval $(a,b) \subset \mathscr{I}$, the measure $m\{(a,b)\}$ is finite, but m does not necessary have a finite density over $\mathscr{I}$. However, if this density is finite, then the normalized function $\eta(x) := m'(x)/m(\mathscr{I})$ constitutes a probability density function defining, for the process X_t, a *stationary distribution*. This distribution can be used to calculate the average stationary behaviour of a process in the following way (see p. 37 in Borodin and Salminen 2002).

Lemma 3.8. *For every Borel-measurable bounded function $f : \mathscr{I} \to \mathbb{R}$ one*

[5] If S is a scale function, then also $c_1S + c_2$ is for $c_1 > 0$ and $c_2 \in \mathbb{R}$.

has

$$\lim_{t\to\infty} \mathbb{E}_x\{f(Z_t)\} = \int_{\mathscr{I}} f(y)\eta(y)dy.$$

3.3.2 Creating diffusions from the characteristics

It should be stressed that the characteristics are diffusion related functions and can be used without any references to the infinitesimal parameters. To illustrate this fact, let X_t be a general linear diffusion in widest sense of Definition 3.1 (i.e. it is not necessarily time-homogeneous or regular) on a state space $(\mathscr{I},\mathscr{B})$. Then we can use hitting time probabilities and the mean exit time from an interval $x \in (a,b) \subset \mathscr{I}$

$$\begin{aligned} p_{ab}(x) &= \mathbb{P}_x(\tau_a < \tau_b) \\ p_{ba}(x) &= \mathbb{P}_x(\tau_a > \tau_b) \\ e_{ab}(x) &= \mathbb{E}_x\{\tau_a \wedge \tau_b \wedge \zeta\} \end{aligned}$$

to introduce the characteristics according to the formulas

$$\begin{aligned} S_{ab}(dx) &= p_{ab}(x)p_{ba}(dx) - p_{ba}(x)p_{ab}(dx) \\ k_{ab}(dx) &= \frac{p'^{+}_{ab}(dx)}{p_{ab}(x)} = \frac{p'^{+}_{ba}(dx)}{p_{ba}(x)} \\ m_{ab}(dx) &= e_{ab}(x)k_{ab}(dx) - e'^{+}_{ab}(dx), \end{aligned}$$

where f'^{+} is a right derivative (see e.g. Section 4.1 in Itô and McKean 1974). Furthermore, this relation can be reversed, which gives rise to another way of creating diffusions. For the proof of the following theorem, see e.g. Section 5.6 in Itô and McKean 1974.

Theorem 3.9. *Let $S : \mathscr{I} \to \mathbb{R}$ be a strictly increasing function, and let m and k be non-negative measures on $\mathscr{B}$ such that $k\{(a,b)\}, m\{(a,b)\} < \infty$ for all $\alpha < a < b < \beta$. Then there exists a linear diffusion such that its basic characteristics are S, m, and k.*

Using this theorem, one can easily construct diffusions. Nevertheless, in this thesis we favour the construction of a diffusion from the infinitesimal parameters σ, μ, and c. First, by doing so we get well-behaving diffusions whose characteristics are absolutely continuous. Second, our construction method coincides better with the formal calculation of stochastic differential equations (cf. Section 3.9).

For a more thorough discussion about basic characteristics, consult e.g. Chapter II in Borodin and Salminen 2002, Chapters 4 and 5 in Itô and McKean 1974, or Chapter IV in Bass 1998.

3.4 Boundary classification

We noticed in Theorem 3.3 that given our assumptions, a diffusion is uniquely determined only up to an explosion time $\zeta \wedge \tau_\alpha \wedge \tau_\beta$, where τ_y is the first hitting time to the state y. Thus, in order to determine a diffusion uniquely, we need to specify what happens when the diffusion hits the boundaries α and β. In this section, we study the boundary behaviour of linear diffusions and divide the boundaries into four different classes based on basic characteristics. We shall analyse only the lower boundary α, as the properties at β are defined in a completely analogous way.

Let $\alpha < z < \beta$ and define

$$\Sigma(z) := \int_\alpha^z (m(\eta,z)+k(\eta,z))\,S(d\eta)$$
$$\text{and} \quad N(z) := \int_\alpha^z (S(z)-S(\eta))\,(m(d\eta)+k(d\eta))\,.$$

The function Σ measures, roughly speaking, the time it takes to reach the lower boundary α starting from z. On the other hand $N(z)$ measures, again roughly speaking, the time it takes to reach an interior point z starting from the boundary α (see page 231 in Karlin and Taylor 1981). In the boundary classification we are only interested in whether the measures Σ and N are finite or not:

Definition 3.10. Let $\alpha < z < \beta$. Then the lower boundary α is called

- *natural* if $\Sigma(z) = \infty$ and $N(z) = \infty$;
- *exit*, or exit-not-entrance, if $\Sigma(z) < \infty$ and $N(z) = \infty$;
- *entrance*, or entrance-not-exit, if $\Sigma(z) = \infty$ and $N(z) < \infty$;
- *regular*, or non-singular, if $\Sigma(z) < \infty$ and $N(z) < \infty$.

The names are intuitive: If α is an entrance, then the process can be started from α, from where it quickly moves to the interior and never comes back. Analogously the process can exit the state space from the exit boundary, but not start from it. A natural boundary is a boundary that is neither exit nor entrance, and a regular boundary is both exit and entrance.

A diffusion can be, a priori, started from a regular boundary, and it can reach a regular boundary in finite time. However, it turns out that the measures Σ and N are not enough to define a regular boundary uniquely. Thus we next classify the boundary behaviour of a regular boundary.

Definition 3.11. A regular boundary α is called

- *reflecting*, if $m(\{\alpha\}) = k(\{\alpha\}) = 0$,
- *killing*, if $m(\{\alpha\}) \neq \infty$ and $k(\{\alpha\}) = \infty$,
- *sticky*, if $\infty > m(\{\alpha\}) > 0$ and $k(\{\alpha\}) = 0$,
- *elastic*, if $m(\{\alpha\}) = 0$ and $k(\{\alpha\}) > 0$,
- *absorbing*, if $m(\{\alpha\}) = \infty$ and $k(\{\alpha\}) \geq 0$.

The two most often appearing regular boundaries are the reflecting and killing ones. This is due to the fact that when the basic characteristics are absolutely continuous with respect to the Lebesgue measure, as they are for example in this thesis, then these two are the only possible regular boundaries (see II.9 in Borodin and Salminen 2002).

Once again, the names of the boundaries are intuitive. A diffusion spends no time at a reflecting boundary (i.e. $\mathbb{P}_x(\text{Leb}(s \leq t \mid X_s = \alpha) = 0) = 1$) and does not die in it (i.e. $\mathbb{P}_x(X_{\zeta-} = \alpha) = 0$). At a killing boundary, the diffusion is immediately killed and sent to the cemetery state ∂. In practice, the behaviour of a diffusion at an exit-not-entrance boundary and at a killing boundary does not differ from one another — In both cases the diffusion is terminated immediately after hitting the boundary and never comes back. A process spends a positive amount of time (i.e. $\mathbb{P}_x(\text{Leb}(s \leq t \mid X_s = \alpha) > 0) > 0$) at a sticky boundary. At an elastic boundary, a process does not spend any time, and it can be killed there (i.e. $\mathbb{P}_x(X_{\zeta-} = \alpha) > 0$). Lastly, a diffusion gets stuck in an absorbing boundary, i.e. $\mathbb{P}_\alpha(X_t = \alpha) = 1$ for all $t < \zeta$.

Furthermore, a boundary α is said to be *attainable*, if it can be reached in a finite time. A boundary that is not attainable is *unattainable*. Hence exit and regular boundaries are attainable, while entrance and natural boundaries are unattainable.

For a more complete discussion on the boundary behaviour of diffusions, see pp. 15–21 in Borodin and Salminen 2002 and Section 15.6 in Karlin and Taylor 1981.

Example 3.12 (Killing a diffusion at the first exit time)**.** Let X be a diffusion on (α,β) with natural boundaries and let $C := (a,b)$, $\alpha < a < b < \beta$, be an open interval and let $x \in C$. Consider the diffusion $\tilde{X}$, which evolves as X until it hits the boundary of C where it is killed (killed at the first exit time from the set C, cf. Example 2.2). Then, the boundaries a and b of the diffusion $\tilde{X}$ on its state space C are regular and killing. The killing measure $\tilde{k}$ of $\tilde{X}$ is $k+\hat{k}$, where k is the killing measure for X and $\hat{k}$ is the killing measure on boundaries, being zero on C and infinite at the boundaries a and b.

3.5 Maximum processes

The *maximum process*, or the *running maximum*, S_t of a diffusion X_t is defined by

$$S_t = \sup_{s \le t} \{X_s\}\,.$$

Since S_t alone is not Markovian, we have to add an additional dimension and keep also record of X_t to get the two-dimensional process (X_t, S_t), which again is Markovian. Although an extra dimension often complicates, or even handicaps, the analysis of the studied problem, this is not the case here. This is due to the good nature of the maximum process: it is continuous even for a more general processes than just diffusions (e.g. for spectrally negative jump processes, see Subsection 2.6.2 in Kyprianou 2006), it is increasing, and, most significantly, it is constant most of the time since $X_t < S_t$ almost always. Indeed, between the hitting times of X_t to its maximum, the two-dimensional process (X_t, S_t) acts as a one-dimensional process (X_t, s) for some $s \in \mathscr{I}$.

In a similar manner, one can also define an *infimum process* $\inf_{s\le t}\{X_s\}$, and everything said above holds true with the obvious changes.

3.6 Fundamental solutions

Let X_t be a diffusion with an infinitesimal generator $\mathscr{A}$ and let $r > 0$ be a constant discounting rate (or a killing rate, cf. Example 3.6). The infinitesimal generator $\mathscr{A} - r$ grasps the information about the associated diffusion $\tilde{X}_t$ killed at the exponential rate r, and, as a differential operator, converts it to the language of pure analysis and ordinary differential equations. Furthermore, from the theory of differential equations we know that the second order differential equation $(\mathscr{A}-r)u(x)=0$ has two independent non-negative solutions ψ and φ. We can require ψ to be increasing and φ to be decreasing, in which case they are uniquely determined up to multiplicative constants. Since $\mathscr{A}-r$ describes the behaviour of the diffusion, understandably the fundamental solutions ψ and φ also carry lot of information about $\tilde{X}_t$ and thus also about X_t. One of the most important facts is the following hitting time distribution (or Laplace transform) result (see II.10 in Borodin and Salminen 2002 and Lemma 3.3 in Lamberton and Zervos 2013).

Proposition 3.13. *Let* $\tau_y = \inf\{t \geq 0 \mid X_t = y\}$.

(A) Then for all $r > 0$

$$\mathbb{E}_x\left\{e^{-r\tau_y}\right\} = \begin{cases} \frac{\psi(x)}{\psi(y)}, & x \leq y; \\ \frac{\varphi(x)}{\varphi(y)}, & x > y. \end{cases}$$

(B) More generally, for $x \in (a,b) \subset \mathscr{I}$*, one has*

$$\mathbb{E}_x\left\{e^{-r\tau_b}\mathbb{1}_{\{\tau_b<\tau_a\}}\right\} = \frac{\psi(x)\varphi(b)-\psi(b)\varphi(x)}{\psi(a)\varphi(b)-\psi(b)\varphi(a)}$$

$$\text{and}\quad \mathbb{E}_x\left\{e^{-r\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}\right\} = \frac{\psi(a)\varphi(x)-\psi(x)\varphi(a)}{\psi(a)\varphi(b)-\psi(b)\varphi(a)}.$$

The fundamental solutions also carry information about the boundary behaviour of the diffusion (see p. 19 in Borodin and Salminen 2002). We consider here only the lower boundary α, while analogous properties hold at β with the roles of φ and ψ interchanged. If α is a regular boundary, then the boundary condition for ψ depends on whether $\alpha \in \mathscr{I}$ or not:

- if $\alpha \in \mathscr{I}$, then $r\psi(\alpha)m(\{\alpha\}) = \frac{\psi'(\alpha)}{S'(\alpha)} - \psi(\alpha)k(\{\alpha\})$. Especially if α is reflecting, then $\frac{\psi'(\alpha)}{S'(\alpha)} = 0$;
- if $\alpha \notin \mathscr{I}$, then $\psi(\alpha+) = 0$ (killing boundary).

On the other hand, if α is not regular, then we have the following properties at α:

- if α is entrance,

$$\psi(\alpha+) > 0, \quad \frac{\psi'(\alpha+)}{S'(\alpha+)} = 0, \quad \varphi(\alpha+) = +\infty, \quad \frac{\varphi'(\alpha+)}{S'(\alpha+)} > -\infty;$$

- if α is exit,

$$\psi(\alpha+) = 0, \quad \frac{\psi'(\alpha+)}{S'(\alpha+)} > 0, \quad \varphi(\alpha+) < +\infty, \quad \frac{\varphi'(\alpha+)}{S'(\alpha+)} = -\infty;$$

- if α is natural,

$$\psi(\alpha+) = 0, \quad \frac{\psi'(\alpha+)}{S'(\alpha+)} = 0, \quad \varphi(\alpha+) = +\infty, \quad \frac{\varphi'(\alpha+)}{S'(\alpha+)} = -\infty.$$

Example 3.14. Let X be a diffusion with natural boundaries and let $C := (a,b)$, $\alpha < a < b < \beta$, be an open interval and let $x \in C$. Denote by $\psi(x)$ and $\varphi(x)$ the fundamental solutions associated with the diffusion X, killed at the rate $r > 0$.

- *(Killed at first exit time.)* Consider a diffusion $\tilde{X}$, which evolves as X until it hits the boundary of C where it is killed (killed at the first exit time from the set C). Now the fundamental solutions associated with $\tilde{X}$ can be chosen to be $\tilde{\psi}(x) = \psi(x)\varphi(a) - \varphi(x)\psi(a)$ and $\tilde{\varphi}(x) = \varphi(x)\psi(b) - \psi(x)\varphi(b)$.

- *(Reflected at the boundaries.)* Consider a diffusion $\hat{X}$, which evolves as X on C, and that the boundaries a and b are reflecting (i.e. let us define $\hat{m}(\{a\}) = \hat{m}(\{b\}) = \hat{k}(\{a\}) = \hat{k}(\{b\}) = 0$). Then the fundamental solutions associated with $\hat{X}$ can be chosen to be $\hat{\psi}(x) = -\psi(x)\varphi'(a) + \varphi(x)\psi'(a)$ and $\hat{\varphi}(x) = \varphi(x)\psi'(b) - \psi(x)\varphi'(b)$.

It should be mentioned that the fundamental solutions ψ and φ can be defined also in a more general context when the discounting $r : \mathscr{I} \to \mathbb{R}_+$ is a measurable function that is uniformly bounded away from zero (cf. Lamberton and Zervos 2013).

3.7 Fundamental solutions, the resolvent operator and decompositions

Fundamental solutions and the scale derivative

Since $\mathscr{A} - r$ is an ordinary differential equation, we can calculate its (constant) Wronskian determinant B (see e.g. p. 116 in Ince 1956)

$$B := \frac{\psi'(x)\varphi(x) - \psi(x)\varphi'(x)}{S'(x)}.$$

Thus, we see that the density of the scale S' is closely connected with ψ and φ. Indeed, each function in the triplet S', ψ, and φ can be expressed as a functional of the other two; using the theory of differential equations one can deduce the decompositions

$$\psi(x) = C_1\varphi(x)\int^x \frac{S'(y)}{\varphi^2(y)}dy, \quad \text{and} \quad \varphi(x) = C_2\psi(x)\int_x \frac{S'(y)}{\psi^2(y)}dy,$$

for some constants $C_1, C_2 \in \mathbb{R}_+$ (cf. p. 122 in Ince 1956).

The resolvent operator and its decomposition

Denote by $\mathscr{L}^1(\mathscr{I})$ the class of measurable functions $f : \mathscr{I} \to \mathbb{R}$ satisfying the integrability condition

$$\mathbb{E}_x\left\{\int_0^\infty e^{-rt}|f(X_t)|dt\right\} < \infty \quad \text{for all } x \in \mathscr{I}.$$

Definition 3.15. Define the *resolvent* (or *potential*, if one prefers the potential theoretic approach) operator R_r by

$$(R_r f)(x) := \mathbb{E}_x\left\{\int_0^\infty e^{-rt} f(X_t)dt\right\} \quad \text{for all } f \in \mathscr{L}^1(\mathscr{I}).$$

It is known (see e.g. Øksendal 2000 or p. 29 in Borodin and Salminen 2002) that the resolvent operator can be written as an integral decomposition:

$$(R_r f)(x) = B^{-1}\varphi(x)\int_\alpha^x \psi(y)f(y)m'(y)dy + B^{-1}\psi(x)\int_x^\beta \varphi(y)f(y)m'(y)dy.$$

This decomposition is computationally very useful (see e.g. the proof of Lemma 5.2(B) in Article II). More theory about resolvents can be found e.g. from Blumenthal and Getoor 1968.

Integral decompositions

Next, we introduce two further integral decompositions (see Corollary 3.2 in Alvarez 2004), which are closely connected to the Martin boundary theory. These will be applied several times in the thesis.

Lemma 3.16. *Assume that* $f \in C^2(\mathscr{I})$ *and that* $(\mathscr{A} - r)f(x) \in \mathscr{L}^1(\mathscr{I})$.

(A) Assume further that $\lim_{x \to \alpha} |f(x)| < \infty$. *Then*

$$\frac{f'(x)\psi(x)}{S'(x)} - \frac{f(x)\psi'(x)}{S'(x)} = \int_\alpha^x \psi(y)(\mathscr{A} - r)f(y)m'(y)dy - \delta,$$

where $\delta = 0$, *if* α *is unattainable, and* $\delta = B\frac{f(\alpha)}{\varphi(\alpha)}$ *otherwise.*

(B) Assume further that $\lim_{x \to \beta} \frac{f(x)}{\psi(x)} = 0$. *Then*

$$\frac{f'(x)\varphi(x)}{S'(x)} - \frac{f(x)\varphi'(x)}{S'(x)} = -\int_x^\beta \varphi(y)(\mathscr{A} - r)f(y)m'(y)dy.$$

3.8 Excessive and superharmonic functions

The fundamental concepts of r-harmonicity and r-excessivity will prove to be the key ingredients in the characterisation of the value of an optimal stopping problem. In this section we assume X_t to be a general regular linear diffusion in the sense of Definition 3.1.

Definition 3.17. (A) A lower semicontinuous[6] function $h : \mathscr{I} \to \mathbb{R}$ is r-*harmonic* with respect to X_t, if it is bounded from below and

$$\mathbb{E}_x \left\{ e^{-r\tau} h(X_\tau) \right\} = h(x) \tag{5}$$

for all stopping times τ and $x \in \mathscr{I}$.

(B) An r-*superharmonic* function is defined similarly, except that ”=” in (5) is replaced by ”$\leq$”.

(C) An r-*subharmonic* function is defined similarly, except that ”=” in (5) is replaced by ”$\geq$”.

[6] A function f is *lower semicontinuous* at x_0 if for every $\varepsilon > 0$ there exists a neighbourhood U such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in U$.

As is noticed on p. 16 in Dynkin 1965 vol II, if a Borel-measurable function h is bounded from below and satisfies condition $\mathbb{E}_x\{e^{-r\tau}h(X_\tau)\} \leq h(x)$ for all stopping times τ, then it is upper semicontinuous. Thus an r-superharmonic function is both lower and upper semicontinuous and consequently continuous.

Definition 3.18. A lower semicontinuous function $h : \mathscr{I} \to \mathbb{R}_+$ is *r-excessive* with respect to X_t, if

a) $\lim_{t\to 0}\mathbb{E}_x\{e^{-rt}h(X_t)\} = h(x)$ for all $x \in \mathscr{I}$; and

b) $\mathbb{E}_x\{e^{-rt}h(X_t)\} \leq h(x)$ for all $x \in \mathscr{I}$.

Observe that if h is continuous and X is a regular linear diffusion, then item a) in the definition holds. Furthermore, since an r-superharmonic function is continuous, we see at once that any non-negative r-superharmonic function must be r-excessive. However, for a nice enough process, also the converse is true; In Dynkin 1965 (p. 16 in vol II) it is remarked that in the class of quasi-left continuous[7], right continuous strong Markov processes any r-excessive function is also an r-superharmonic function. Further, we see a link to martingales, as for all $h \in \mathscr{L}^1$ the process $e^{-rt}h(X_t)$ is a (super/sub)martingale whenever h is r-(super/sub)harmonic with respect to X_t.

The fundamental solutions are crucial here, since they can be viewed as *minimal r-harmonic functions*: They span the set of all r-harmonic mappings so that every r-harmonic mapping is of the form $c_1\psi(x) + c_2\varphi(x)$ for some c_1, $c_2 \in \mathbb{R}$.

It is not convenient to determine directly from the definition if a given function is r-excessive or not. Hence the following proposition gives two more applicable characterisations, where the first one is a direct consequence of Dynkin's formula (Theorem 3.5) and the second one is Theorem 12.4.B in Dynkin 1965.

Proposition 3.19. *Let X be a regular linear diffusion in a sense of Definition 3.1.*

(A) Let $h : \mathscr{I} \to \mathbb{R}_+$ be twice continuously differentiable. If $(\mathscr{A} - r)h(x) \leq 0$ for all $x \in \mathscr{I}$, then h is r-excessive.

---

[7] A Markov process X_t is said to be *quasi-left continuous*, if for any sequence of stopping times $\tau_1, \tau_2, \ldots$ for which $\lim_{n\to\infty}\tau_n = \tau$ we have $\lim_{n\to\infty}X_{\tau_n} = X_\tau$, for all $x \in \mathscr{I}$.

(B) *Let $h : \mathscr{I} \to \mathbb{R}_+$ be a continuous function. Then h is r-excessive with respect to X_t if it satisfies the following condition. For any $\tau_{(a,b)} = \inf\{t \geq 0 \mid X_t \notin (a,b)\}$, where $\tau_{(a,b)}$ is the first exit time from an arbitrary open interval $(a,b) \subset \mathscr{I}$ whose compact closure is in $\mathscr{I}$, one has $\mathbb{E}_x \left\{ e^{-r\tau_{(a,b)}} h(X_{\tau_{(a,b)}}) \right\} \leq h(x)$, for all $x \in \mathscr{I}$.*

If the inequalities in the proposition are replaced by equalities, we get conditions for r-harmonicity.

3.9 Itô diffusion

The so called *Itô diffusions*, which are solutions to certain stochastic differential equations with respect to Brownian motion, are particularly important diffusions. In fact, we have used Itô diffusions all along without underlining this fact — the conditions in Theorem 3.3 guarantee the existence of a weak solution to a certain stochastic differential equation with a Brownian motion as the source of randomness, as we shall see shortly. For a thorough treatment of stochastic differential equations, consult Karatzas and Shreve 1988, Rogers and Williams 2000b, Protter 2004, or Øksendal 2007.

To formally define an Itô diffusion, let W_t be a one-dimensional Brownian motion. Then for a given drift $\mu : \mathscr{I} \to \mathbb{R}$ and volatility $\sigma : \mathscr{I} \to \mathbb{R}_+$, the dynamics of an Itô diffusion X_t can be represented in a stochastic integral form

$$X_t = X_0 + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

provided that the stochastic differential dW_t is rigorously defined. Stochastic differential calculus, developed by Itô in the 40's (and independently by Wolfgang Doeblin, see Bru and Yor 2002), describes how to define such differentials, but we will not go into details. It is adequate for us to know that one can rewrite the equation above as a *stochastic differential equation*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x. \tag{6}$$

We are interested in two kinds of solutions to this stochastic differential equation, and we also present two concepts for the uniqueness of the solution (see Chapter V in Karatzas and Shreve 1988).

Definition 3.20. (A) The equation (6) has a *weak solution*, if there exists a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ carrying a Brownian motion W_t and a stochastic process X_t, both adapted to $\mathscr{F}_t$, such that the pair (W_t, X_t) satisfies (6) and

$$\int_0^t \left\{ |\mu(X_t)| + |\sigma^2(X_t)| \right\} dt < \infty \tag{7}$$

up to the (possible) explosion time $\zeta \wedge \tau_\alpha \wedge \tau_\beta$.

(B) The equation (6) has a *strong solution*, if for any given filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ carrying a Brownian motion W_t, there exists a process X_t adapted to $\mathscr{F}_t$ satisfying (6) and (7) up to the explosion time $\zeta \wedge \tau_\alpha \wedge \tau_\beta$.

(C) The solution to the equation (6) is *unique in law* or *weakly unique*, if whenever two processes X_t and Y_t are solutions to (6), possibly on different filtered probability spaces, they have the same distribution as processes, i.e. for every Borel set $A \in \mathscr{B}$ we have $\mathbb{P}(\omega \mid t \to X_t(\omega) \in A) = \mathbb{Q}(\omega \mid t \to Y_t(\omega) \in A)$, where $\mathbb{P}$ and $\mathbb{Q}$ are the probability measures for X and Y respectively.

(D) The solution to the equation (6) is *pathwise unique*, if whenever the processes X_t and Y_t are solutions to (6) defined on the same filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$, they satisfy $\mathbb{P}(|X_t - Y_t| = 0 \quad \text{for all } t \geq 0) = 1$.

We have the following sufficient conditions for the existence and uniqueness of a weak and strong solution. The result 3.21(A) can be found e.g in Chapter V in Karatzas and Shreve 1988 and the result 3.21(B) in Section 5.2 in Øksendal 2007.

Theorem 3.21. *Let $\mu : \mathscr{I} \to \mathbb{R}$ and $\sigma : \mathscr{I} \to \mathbb{R}_+$ be Borel-measurable functions.*

(A) Assume that for all $x \in \mathscr{I}$ there exists $\varepsilon > 0$ such that the conditions $\sigma(x) > 0$ and $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(y)|}{\sigma^2(y)} dy < \infty$ are satisfied. Then there is a weak solution to the stochastic differential equation (6) *up to an explosion time and this solution is unique in law.*

(B) Assume that for all $x, y \in \mathscr{I}$ the Lipschitz condition $|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| < C|x - y|$ is satisfied for some constant $C > 0$. Then there

exists a strong solution to the stochastic differential equation (6) *up to an explosion time and this solution is pathwise unique.*

The solutions are defined up to an explosion time, and we can apply e.g. Feller's test for explosions (see e.g. Theorem 5.5.29 in Karatzas and Shreve 1988) to determine whether a solution to (6) hits either of the boundaries of $\mathscr{I}$ in finite time with a positive probability.

Example 3.22.

- *Geometric Brownian motion.* Choose $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ for some constants $\mu, \sigma > 0$. This clearly satisfies the conditions of Theorem 3.21(B) and thus there exists a pathwise unique, strong solution X_t to the equation $dX_t = \mu X_t dt + \sigma X_t dW_t$.

- *No strong solution.* Clearly a strong solution is always also a weak solution, but the converse does not hold: Take $\mu(x) \equiv 0$ and $\sigma(x) = \text{sgn}(x)$. Then there exists a weak solution, but not a strong one, see e.g. Example 5.3.2 in Øksendal 2007. For counterexamples with continuous $\sigma : \mathbb{R} \to \mathbb{R}_+$ with $0 < \delta < \sigma(x) < K$, see Barlow 1982.

We can now conclude that Theorem 3.21(A) together with the analysis of killed diffusions at Section 3.2 justifies Theorem 3.3, which characterises a diffusion through the infinitesimal parameters σ, μ, and c. To make sure that the weak solution is actually a diffusion it is enough to notice that it is continuous (e.g. Theorem 3.2.5 in Øksendal 2007) and a strong Markov process (e.g. Theorem V.21.1. in Rogers and Williams 2000b). Notice that in Theorem 3.3 we required that μ and σ are continuous, rather than just measurable, in order to make the basic characteristics of the diffusion absolutely continuous.

Now we can interpret a linear diffusion also as a solution to a stochastic differential equation. This enables us to combine the classical theory of linear diffusions with the theory of stochastic differential equations. As a result, we get a large tool kit for analysing stochastic problems. One of the most famous results in stochastic calculus is the following Itô's formula (e.g. Theorem 4.1.2 in Øksendal 2007), which can be useful, for example, when one tries to verify that a guessed stopping rule is indeed the optimal one for an optimal stopping problem.

Theorem 3.23. *Let X_t be a (weak) solution to* (6)*, and let* $g(t,x) : [0,\infty) \times \mathscr{I} \to$

$\mathbb{R}$ be $C^{1,2}$-function. Then $g(t,X_t)$ satisfies the stochastic differential equation

$$dg(t,X_t) = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\sigma^2(X_t)dt.$$

Another important issue is the fact that a stochastic integral $\int_0^t \sigma(X_s)dW_s$, when it exists, is a local martingale (e.g. Theorem IV.30.7 in Rogers and Williams 2000b), and consequently every solution to a stochastic differential equation of the type (6) is a semi-martingale. Moreover, if $\mathbb{E}_x\left\{\int_0^t \sigma^2(X_s)ds\right\} < \infty$ for all $t \geq 0$, then the integral $\int_0^t \sigma(X_s)dW_s$ is in fact a martingale (Corollary 3.2.6 in Øksendal 2007).

We conclude the section with an example that shows there are diffusions, in the sense of Definition 3.1, which are not Itô diffusions.

Example 3.24. *A diffusion that is not an Itô diffusion.* Let W_t be a one-dimensional Brownian motion and define $X_t := \sqrt{|W_t|}$. Then X_t is a diffusion in the sense of Definition 3.1, since $|W_t|$ is a diffusion and $x \to \sqrt{x}$ is a continuous bijection on $\mathbb{R}_+$. However, it is proven in Yor 1978 (see also Theorem 71 in Protter 2004 and II.5 in Borodin and Salminen 2002) that, unlike every Itô diffusion, X_t is not a semi-martingale.

4 OPTIMAL STOPPING

From now on, we assume that X is the non-terminating diffusion process on the interval $\mathscr{I} \subset \mathbb{R}$ given by Theorem 3.3. In other words, we assume that $c(x) \equiv 0$ for all $x \in \mathscr{I}$ and that the infinitesimal parameters $\mu : \mathscr{I} \to \mathbb{R}$, $\sigma : \mathscr{I} \to \mathbb{R}_+$ are continuous and that for all $x \in \mathscr{I}$ there exists $\varepsilon > 0$ such that the conditions $\sigma(x) > 0$ and $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(s)|}{\sigma^2(s)} ds < \infty$ hold.

4.1 What does ”optimal stopping” mean?

Since the path of a stochastic process is different at every observation, we need to clarify what we actually mean by ”optimal stopping”. Usually, as in this thesis, it means maximising the expectation of a given payoff over all stopping times. To be more specific, let $g : \mathscr{I} \to \mathbb{R}$ be a known payoff. Then we understand an ”optimal stopping problem” as the maximisation of the expected net present value

$$V(x) = \sup_{\tau} \mathbb{E}_x \left\{ e^{-r\tau} g(X_\tau) \right\}, \tag{8}$$

where the supremum is taken over all stopping times. It is worth noticing that, unless otherwise stated, we only consider problems with an infinite time horizon (i.e. there is no upper boundary for stopping times).

There are also other stopping criteria such as the variance criterion (see e.g. Pedersen 2011), where we choose τ so that the variance $\mathrm{Var}(g(X_\tau))$ gets maximised or minimised, sometimes with respect to other constraints; e.g. minimising the variance of a portfolio, so that it expectedly gives at least some certain profit. More generally we could try to maximise $U\left(\mathbb{E}_x \left\{g(X_\tau)\right\}\right)$, where $U(x)$ is any non-linear function. Although variance criterion is important in some applications, for instance in portfolio optimisation problems (see e.g. Zhou and Li 2000), these kinds of criteria are not studied nearly as much as ”normal” stopping criterion (8). One aspect that makes these variance criterion problems more complicated is the fact that the nonlinear term from the variance prevents one from using dynamic programming and the smooth fit principle. As a result, the value function cannot be characterised directly using the greatly developed optimal stopping machinery and each problem needs to be handled separately.

The payoff can also include a probability function. For example, we may want to maximise the function $\mathbb{P}_x(f(X_\tau) > b) - \mathbb{E}_x\{e^{-r\tau}g(X_\tau)\}$, where b is some exogenously determined level, and the expectation measures the cost of waiting more. However, since probabilities can be written as expectations, these problems can often be re-formulated as (8), see e.g. Chapter 21 in Peskir and Shiryaev 2006 and Theorem 7 in Pedersen 2005.

From now on by ”optimal stopping problem” we refer to a problem of the type (8) stated for a non-terminating linear diffusion. These kinds of problems are the simplest stochastic control problems, where the only allowed control is a straightforward and crude ”stop now” -control. However, at the same time these control problems are very important, as they have many practical and theoretical applications. For example analysing optimal exit from a market (see e.g. Alvarez 1998) and determining an optimal investment rule (see Dixit and Pindyck 1994) can be stated as optimal stopping problems. Other examples include deriving sharp inequalities arising in stochastic analysis, the quickest detection of a changed drift, option pricing and optimal prediction problems (see Chapters V, VI, VII, and VIII in Peskir and Shiryaev 2006, respectively, and the references therein).

For a brief introduction to the optimal stopping theory see Zabczyk 1979, and for a more comprehensive treatment consult Shiryaev 1978 and Peskir and Shiryaev 2006.

4.2 Solving optimal stopping problems

4.2.1 A procedure to reach the solution

A corner stone of the optimal stopping theory for diffusion processes is the following verification result, which dates back to Dynkin 1963. It unambiguously characterises the value and the optimal stopping time of the problem (8) (see e.g. Theorem 3.1 in Shiryaev 1978, or Theorem 2.7 together with Remark 2.10 in Peskir and Shiryaev 2006). In fact, this holds for an even larger class of Markov processes than just linear diffusions, but the following form is adequate for us.

Theorem 4.1 (Verification theorem for an optimal stopping problem)**.** *Let $g(x) : \mathscr{I} \to \mathbb{R}$ be an upper semicontinuous function such that $\mathbb{E}_x\left\{\sup_{t\geq 0}\{e^{-rt}g(X_t)\}\right\} < \infty$ for all $x \in \mathscr{I}$.*

(A) *The solution* $V(x)$ *to* (8) *is the smallest* r*-excessive majorant of the function* $g(x)$.

(B) *Define the continuation region* $C := \{x \in \mathscr{I} \mid g(x) < V(x)\}$*, and let* $\tau^* = \inf\{t \geq 0 \mid X_t \notin C\}$ *be the first exit time from* C*. If* $\mathbb{P}_x(\tau^* < \infty) = 1$ *for all* $x \in \mathscr{I}$*, then* τ^* *is the optimal stopping time for the problem* (8).

Besides being r-excessive, and consequently r-superharmonic everywhere, one can show that the value function $V(x)$ is r-harmonic on the continuation region C (see e.g. Lemma 4.2 in Salminen 1985). Theorem 4.1 also suggests that the state space $\mathscr{I}$ has a partition $\{C,S\}$, where C is the continuation region and $S = \mathscr{I} \setminus C$ is the stopping region (the optimal strategy is to allow the diffusion to evolve as long as it stays in C, and it is stopped immediately when it enters S, hence the names).

Also, this theorem allows us to utilise the theory of r-excessive functions in a potential theoretic way, and the Martin integral representation theory within it, to analyse the problem. We use this approach in this thesis; see Salminen 1984, 1985 for an excellent exposition on Martin boundary theory for linear diffusions, its relation with optimal stopping, and on how the representing measure of an r-excessive function can be characterised explicitly by relying on fundamental solutions and the scale function.

There are many other ways to approach an optimal stopping problem and some of these are surveyd in Subsection 4.2.3 below. Many methods, including the one used in this thesis, could be categorized as ”guess and verify” -methods, the previous theorem being of great help in the verification phase. In these methods, one typically first constructs (by ad hoc methods) some necessary first order optimality conditions. In the second step, one searches for a solution to these optimality conditions, after which the optimality of the proposed solution is validated by applying a verification theorem. In this thesis, we develop the necessary conditions and search for a solution to them with the help of fundamental solutions and the theory of linear diffusions. We adopt the name *fluctuation theory approach* to this method, as it was used in a slightly different situation in Kyprianou and Pistorius 2003. The normal procedure is as follows.

Step 1 Guess the nature of the stopping rule (e.g. one- or two-sided stopping boundary).

Step 2 Formulate the value with respect to the guessed stopping rule using the Laplace transform of the hitting time(s). Usually the value has as many free parameters as there are stopping boundaries in the guessed stopping rule.

Step 3 Derive the necessary first-order optimality conditions for the value with respect to the guessed stopping rule. This is often the crucial step, and one often needs additional assumptions in order to prove the existence and/or uniqueness of these optimality conditions.

Step 4 Verify that the guessed stopping rule, satisfying the necessary conditions from Step 3, indeed is the optimal solution (e.g. using Theorem 4.1).

Using these steps, we have a constructive method to reach the optimal exercise strategy without heavy differentiability preconditions. Interestingly, the celebrated *principle of smooth fit* (i.e. that the value function is continuously differentiable across the stopping boundary, cf. Subsection 4.2.3) is here often a consequence of optimality. We can also use the above mentioned procedure in cases where the payoff is not everywhere differentiable and, therefore, where the principle of smooth fit is not always satisfied. Moreover, we will see that this procedure can be applied to other stochastic control problems as well.

Despite the fact that optimal stopping problems for continuous time stochastic processes are well studied, it is often hard to find explicit solutions to these problems since the set of admissible strategies is very large and might involve rather exotic strategies. However, using the procedure introduced above we can often find necessary conditions under which a simple barrier strategy is the optimal one. Furthermore, this simpler barrier strategy ”stop as soon as X_t crosses barrier(s) $y_i \in \mathscr{I}$” often enables us to write the value function in a (quasi)-explicit form. This, in turn, is very helpful when, for example, studying comparative static properties of the value and the optimal strategy (see e.g. Article II and III). For a discussion on the benefits of barrier strategies, see also Section 3.5 in Rakkolainen 2009

4.2.2 A concrete example

Let us consider a situation where an investor has the opportunity to invest a sunk cost $k > 0$ at any time t to a project which then gives a return described by the diffusion X_t on $\mathbb{R}_+$. At any time the investor has the following two options: Either she invests now or postpones the decision to invest into the future. More specifically, the expected present value of this investment problem is

$$\Pi(x,t) = \mathbb{E}_x \left\{ e^{-rt}(X_t - k)^+ \right\},$$

where $(x-k)^+ = \max\{x-k, 0\}$. A rational investor naturally wants to choose the best time to make the investment and thus wants to find the stopping time τ^* such that

$$W^*(x) := \Pi(x,\tau^*) = \sup_{\tau} \Pi(x,\tau). \tag{9}$$

This problem is known as the pricing of an American call option with infinite time horizon, and for the case of geometric Brownian motion it was first treated by Samuelson 1965 and rigorously solved by McKean 1965 (for a discussion on American options see Section 25 of Peskir and Shiryaev 2006 and references therein). Here we solve the problem utilising the procedure introduced in the previous subsection.

Step 1 We judge that it will not be worthwhile to invest if the value of the underlying diffusion is small, while for sufficiently large values we shall exercise the investment opportunity. Hence we guess that the optimal stopping rule should be a one-sided threshold rule ”stop above a certain threshold” and we consider stopping times $\tau_y = \inf\{t \geq 0 \mid X_t \geq y\}$ where $y > k$.

Step 2 Fix $y > k$. Then

$$\begin{aligned}
\Pi(x,\tau_y) =& \mathbb{E}_x \left\{ e^{-r\tau_y}(X_{\tau_y} - k)^+ \right\} \\
&= \begin{cases} x-k, & x \geq y \quad \text{(stop immediately)} \\ \mathbb{E}_x \{e^{-r\tau_y}\} (y-k), & x < y \quad \text{(wait until } X_t \text{ hits } y) \end{cases} \\
&= \begin{cases} x-k, & x \geq y \\ \frac{y-k}{\psi(y)}\psi(x), & x < y, \end{cases}
\end{aligned}$$

where in the last equality we used the Laplace transform of the hitting time from Proposition 3.13. We see that $\Pi(x,\tau_y)$ has y as a free parameter.

Step 3 For y^* to be the optimal stopping boundary, it must maximise $\frac{y-k}{\psi(y)}$ and hence satisfy the first order necessary condition $\frac{\partial \Pi(x,\tau_y)}{\partial y}\Big|_{y=y^*-} = 0$. After the differentiation, the necessary condition reads as

$$\psi(y^*) - \psi'(y^*)(y^* - k) = 0. \tag{10}$$

If this equation has a unique solution, it must maximise (9) in the class of all one-sided threshold rules. Here we need additional assumptions to guarantee the unique existence of the solution to (10)[8].

Step 4 Firstly τ_{y^*} is an admissible stopping strategy and so $\Pi(x,\tau_{y^*}) \leq W^*(x)$. Secondly, to get the opposite inequality, it is enough to show that $\Pi(x,\tau_{y^*})$ is an r-excessive majorant for $(x-k)^+$ and then utilise the verification theorem 4.1.

Above we also see the link to the Martin boundary theory; The representative measure ν^{W^*} of the r-excessive value function W^* is given by (see Proposition 3.3 in Salminen 1985)

$$\begin{aligned} \nu^{W^*}\{(0,x)\} &= \frac{W^*(x)\psi'(x)}{S'(x)} - \frac{\psi(x)W^{*\prime}(x)}{S'(x)} \\ \text{and} \quad \nu^{W^*}\{(x,\infty)\} &= \frac{W^{*\prime}(x)\varphi(x)}{S'(x)} - \frac{\varphi'(x)W^*(x)}{S'(x)}. \end{aligned}$$

Furthermore, we can write the functional in (10) as

$$S'(x)\left(\frac{\psi(x)}{S'(x)} - \frac{\psi'(x)(x-k)}{S'(x)}\right).$$

Now, the function in parentheses is in fact the representing measure $\nu^{W^*}\{(0,x)\}$ for $x \in [y^*,\infty)$ and we see that it does not charge the optimal continuation region $(0,y^*)$ (cf. Section 4 in Salminen 1985).

To see this procedure in use in more complicated situations, see the articles of this thesis as well as e.g. Alvarez 2003 (a one-sided stopping rule), Lempa 2010 (a two-sided stopping rule), Alvarez and Lempa 2008 (an impulse and a singular control problem) and Alvarez and Rakkolainen 2009 (a spectrally negative Levy case).

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[8]A sufficient condition is, for example, that there exists $\tilde{x} > 0$ such that $(\mathscr{A} - r)(x-k) \gtreqqless 0$ for all $x \lesseqgtr \tilde{x}$ (cf. Theorem 6 in Alvarez 2003).

4.2.3 Other approaches to finding the solution

We only use the above mentioned potential theoretical method in this thesis, but we shall next briefly discuss some other approaches to optimal stopping problems. Figure 2 describes the relationships between the different approaches.

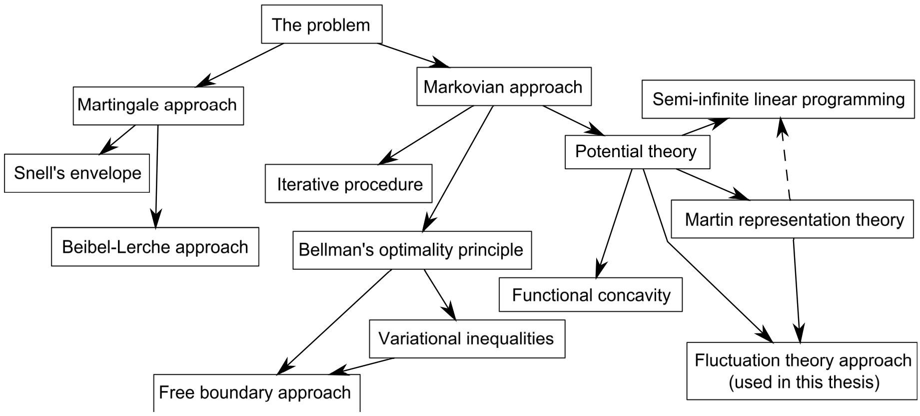

Figure 2: The presented approaches.

Functional concavity

Dayanik and Karatzas 2003 (further developed into a more general context in Dayanik 2008) presented another potential theoretic approach based on works by Dynkin 1965 and Dynkin and Yushkevich 1969. In this *functional concavity* technique it is shown that essentially, every optimal stopping problem can be transformed into an undiscounted stopping problem for a Brownian motion. Moreover, the value function in the Brownian motion case can be characterised as the smallest non-negative concave majorant of the (transformed) payoff function. The major benefit in this approach is that one does not need any prior guess about the optimal stopping region, instead the transformed problem is essentially solved by inspection. Functional concavity is also closely related to excessivity, which can be seen after realising that the set of non-negative concave functions coincides with the set of excessive functions under Brownian motion.

The free boundary approach

A very effective and widely used technique to solve optimal stopping problems is the *free boundary approach*, which highlights the partition $\{C,S\}$ of the state space into continuation and stopping regions. The free boundary approach works well in linear as well as in multidimensional cases and is especially powerful with regards to concrete examples. Here, one first guesses the form of the partition and then transforms the original problem (8) into a free boundary problem (a Dirichlet problem):

$$\begin{cases} (\mathscr{A}-r)V(x)=0, & x\in C \\ V(x)=g(x), & x\in \partial C \\ V(x)>g(x), & x\in C, \end{cases}$$

where both the value $V(x)$ and the continuation region C are unknown. Notice that in this Dirichlet problem in order to determine the region C, it is enough to determine its boundary ∂C (hence the terminology). However, in order to determine the boundary, one needs to apply non-trivial boundary conditions, and the so called *principle of smooth fit* is suitable in most cases. This principle says that the first derivatives of the value function and the payoff function agree at the optimal stopping boundary ∂C, i.e. one can add a boundary condition

$$V'(x)=g'(x), \quad x\in \partial C$$

to the Dirichlet problem above. Lastly, after solving the free boundary problem, one needs to verify the correctness of the initial guess using a verification theorem. For a good introduction to the subject and some examples, see Pedersen 2005. For a thorough discussion consult Peskir and Shiryaev 2006. We also further illustrate the free boundary approach at the end of this subsection.

The *variational inequalities* approach is closely related to the free boundary approach. Here one gathers a collection of (in)equalities that simultaneously serve as a free boundary problem and sufficient conditions for optimality, see e.g. Theorem 10.4.1 in Øksendal 2007. If one can find a function satisfying these variational inequalities, it is inevitably the unique solution. However, the variational inequalities require strong differential preconditions, and in many cases the value function is not smooth enough for applying the inequalities. For these cases, there is fortunately a technique based on the so called *vis-*

cosity solutions, first introduced in Crandall and Lions 1983. Basically these solutions are generalised solutions to a partial differential equation; see e.g. Øksendal and Reikvam 1998 for their use in optimal stopping problems.

Compared to the free boundary approach, the method of variational inequalities is typically more difficult to use with concrete examples, where we would like to find explicit solutions. However, it is powerful for finding sufficient conditions for the existence of a solution.

One very recent general method using the variational inequality technique was developed by Lamberton and Zervos 2013. They avoid the heavy differentiability assumptions by showing that a function $F : \mathscr{I} \to \mathbb{R}$ is r-excessive if and only if it is the difference of two convex functions[9] and $-(\frac{1}{2}\sigma^2 F'' + \mu F' - rF)$ is a positive measure. Using this characterisation of r-excessivity, rather than the direct characterisation based on the infinitesimal generator, they find necessary and sufficient conditions for the existence of a value function relying on the arguments of variational inequalities. Johnson 2012 demonstrates how this general theory can be applied to obtain explicit solutions to optimal stopping problems.

Semi-infinite linear programming approach

Another way to find the smallest r-excessive majorant is to transform the initial optimal stopping problem into a *semi-infinite linear program*, which in the linear diffusion case is taken over the coefficients of the minimal r-harmonic functions. In short, in the linear diffusion case one fixes the initial state $x_0 \in \mathscr{I}$ and the value $V(x_0)$ is a solution to the problem

$$\begin{aligned} \min_{c_1,\, c_2} \quad & c_1\psi(x_0) + c_2\varphi(x_0) \\ \text{s.t.} \quad & c_1\psi(x) + c_2\varphi(x) \geq g(x) \quad \text{for all } x \in \mathscr{I} \\ & c_1, c_2 \geq 0. \end{aligned}$$

See Helmes and Stockbridge 2010 for an analytical approach and Christensen 2012 for a numerical treatment.

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[9]A function $F : \mathscr{I} \to \mathbb{R}$ is a difference of two convex function if and only if its left-hand side derivative F'_- exists and its second distributional derivative is a measure. See Bačák and Borwein 2011 for a survey on difference of two convex functions.

General methods

Being a dynamic programming problem, an optimal stopping problem can also be approached using *Bellman's optimality principle*, which in this case can be written as *Hamilton-Jacobi-Bellman variational inequality*

$$\max\{(\mathscr{A}-r)V(x), g(x)-V(x)\} = 0 \quad \text{for all } x \in \mathscr{I}. \tag{11}$$

Here, the first component represents the continuation option, while the second component represents the fact that the process can always be stopped immediately. This is a very general way to characterise the value function, and in fact we see that the free boundary problem above can be derived from this by applying the partition of the state space into continuation and stopping regions. For more information on the Hamilton-Jacobi-Bellman inequality and its relation to stopping problems, see e.g. Dixit and Pindyck 1994.

The constructive but non-explicit *iterative procedure* -method, presented e.g. in Shiryaev 1978, is a general approach that requires no differentiability whatsoever. In this approach, the time is discretised after which the easier discrete-time results can be applied and so the value function $V(x)$ can be characterised as a limiting value

$$V(x) = \lim_{n\to\infty} \lim_{N\to\infty} Q_n^N g(x),$$
$$\text{where} \quad Q_n g(x) := \max\left\{g(x), \mathbb{E}_x\left\{e^{-r\frac{1}{2^n}} g(X_{\frac{1}{2^n}})\right\}\right\}$$

and Q_n^N is the Nth power of Q_n. Although this works in very general structures, unfortunately the iteration typically converges very slowly, and thus this approach usually provides only existence results.

A recent approach by Christensen et al. 2012 is also worth mentioning here. There the value function and the optimal stopping threshold are found in a very general setting utilising an expectation of a maximum process killed at an exponential time.

All the approaches we have considered so far apply analytical tools from the theory of Markov processes and can thus be classified as *Markovian approaches*. In these approaches the stochastic problem is often reduced to a pure analytical problem (e.g. solving Dirichlet problem) without any reference to any probabilistic construction. So, peculiarly, the value and the optimal stopping rule are found as a solution to a deterministic problem although the problem itself was initially stochastic in nature.

Martingale methods

Besides the Markovian approach, the other main approach is the *martingale approach*; We notice that an r-superharmonic function with respect to a diffusion X additionally constitutes a supermartingale $e^{-rt}f(X_t)$, and hence we could use probabilistic tools provided by the martingale theory. This approach was initiated by Snell 1952, and it relies on the fact that the value function constitutes the minimal supermartingale dominating the payoff function.

If we define the payoff process by $Y_t = e^{-rt}g(X_t)$, then the problem can be solved via *Snell's envelope*

$$S_t := \operatorname*{ess\,sup}_{t\leq\tau} \mathbb{E}\left\{Y_\tau \mid \mathscr{F}_t\right\}.$$

It can be shown that S_t coincides with the value process of the optimal stopping problem (8) (see Section 2.1 in Peskir and Shiryaev 2006).

The Beibel-Lerche method (see Beibel and Lerche 1997; Lerche and Urusov 2007), which can be viewed as optimal stopping via measure transformation, is a more recent method based on the martingale theory. In this method, we are interested in finding a function $h(x)$ and a positive martingale M_t such that $M_0 = 1$ and

$$\mathbb{E}_x^{\mathbb{P}}\left\{e^{-rt}g(X_t)\right\} = \mathbb{E}_x^{\mathbb{Q}}\left\{h(X_t)\right\}, \quad \text{where} \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = M_t.$$

If one can show that $h(x)$ attains a unique maximum value at some point x^*, then $\mathbb{E}_x^{\mathbb{P}}\{e^{-r\tau}g(X_\tau)\} \leq h(x^*)$ for all stopping times τ. The equality is attained with $\tau_{x^*} = \inf\{t \geq 0 \mid X_t = x^*\}$, which consequently is the optimal stopping rule. This method is especially powerful at the continuation region of a considered problem. Unfortunately, it does not always work on the stopping region.

A similar method, adding a touch from the potential theory, has been utilised recently in Christensen and Irle 2011, where they also proved an interesting characterisation of the stopping region: a point x is in the stopping region if and only if there exists an r-harmonic mapping h such that

$x = \operatorname{argmax}\left\{\frac{g}{h}\right\}$.

Help from time-, space-, or measure changes

There are cases where the considered problem is not solvable using any of the methods listed above. In these cases, one can try to transform the difficult problem into an easier one by making a time-, space-, or measure change. These methods are discussed in Sections 10–12 in Peskir and Shiryaev 2006, see also Barndorff-Nielsen and Shiryaev 2010.

4.2.4 Different methods in practice

In a way, categorizing methods and building fences between them is artificial, since at the heart they all tell the same story from a slightly different perspective.

To give a few explicit examples: Let us consider the Beibel-Lerche method applied to our concrete problem (9) of pricing an American call option. We can now choose $M_t = e^{-rt}\psi(X_t)$ as the positive martingale so that $h(x) = (x-k)^+/\psi(x)$ is the sought function. It follows that, for $x \leq x^*$, we can show the optimal stopping threshold to be $x^* = \operatorname{argmax}\{x - k/\psi(x)\}$, which is the same one we got from Step 3 at Subsection 4.2.2, but this time the justification comes from measure theoretical techniques and martingale properties rather than from analytic techniques.

For another example, let us solve the same problem (9) using the free boundary approach. Let us again guess that the stopping rule is of the type $\tau_y = \inf\{t \geq 0 \mid X_t \geq y\}$. The free boundary problem for the unknown value function V and the unknown optimal threshold point y^* can now be written as (cf. Section 25.1 in Peskir and Shiryaev 2006)

$$(\mathscr{A} - r)V(x) = 0, \qquad x > y^*; \tag{12a}$$

$$V(x) = (x-k)^+, \qquad x = y^*; \tag{12b}$$

$$V'(x) = 1, \qquad x = y^* \quad \text{(principle of smooth fit)}; \tag{12c}$$

$$V(x) > (x-k)^+, \qquad x > y^*; \tag{12d}$$

$$V(x) = (x-k)^+, \qquad 0 < x < y^*. \tag{12e}$$

It follows from (12a) that $V(x) = c_1\psi(x) + c_2\varphi(x)$, and by (12e) and (12b) we get that $c_2 \equiv 0$ and $c_1 = c_1(y) = \frac{y-k}{\psi(y)}$. The optimal threshold y^* is found by applying the principle of smooth fit (12c): y^* satisfies $\psi'(y^*)\frac{y^*-k}{\psi(y^*)} = 1$, which is the same condition as (10) at Step 3 in our fluctuation theory approach in Subsection 4.2.2. Finally, we must ensure that this guessed candidate for the value function and the optimal stopping time are indeed correct using an appropriate verification theorem.

After solving the concrete problem (9) by applying three different methods, we see that irrespective of the chosen method the ratio $(x-k)/\psi(x)$ and its maximum point play central roles. In other words, although each method bases its justification on different routes and theories, at the explicit level they behave more or less in a similar way and lead to the analysis of the same functionals. For a deeper analysis considering the relations between martingale and Markovian approaches (Beibel-Lerche vs. free boundary), see Gapeev and Lerche 2011.

4.3 Finite horizon

4.3.1 Fixed time horizon

Introducing a fixed, finite time horizon makes the optimal stopping problem (8) inherently two-dimensional, as one needs to also record time so that the studied process is (t, X_t). Moreover, the infinitesimal generator of the process (t, X_t) has an extra term $\partial/\partial t$ making the differential operator not only two-dimensional but also analytically more difficult to handle. For example, by adding a fixed finite time horizon (or maturity) $T < \infty$ to the problem (9) of stopping optimally an American call option we end up with the problem

$$\begin{aligned} V(t,x) &= \sup_{0 \le \tau \le T-t} \mathbb{E}_{(t,x)} \left\{ e^{-r\tau}(X_{t+\tau} - k)^+ \right\} \\ &= \sup_{\tau} \mathbb{E}_{(t,x)} \left\{ e^{-r\tau}(X_{t+\tau} - k)^+ \mathbb{1}_{\{\tau \le T-t\}} \right\}, \end{aligned}$$

where $X_t = x$ under $\mathbb{P}_{(t,x)}$ and τ is a stopping time.

In these kinds of problems the value $V(t,x)$ is typically two-dimensional, Moreover, the optimal stopping boundary is often a moving boundary $y^*(t)$ which depends on time, instead of being a fixed exercise threshold as is so often the case with an infinite time horizon. Unfortunately, the solutions can rarely be attained explicitly, and thus with concrete problems one must search for the solutions numerically. All in all, introducing a finite time horizon often means a considerable increase in the complexity of a problem. For an overview of a fixed finite time horizon, see Peskir and Shiryaev 2006.

4.3.2 Stochastic time horizon

Surprisingly, the stochastic time horizon case is not as complex as the fixed finite time horizon problems, at least when the horizon is exponentially distributed. For the results of this subsection, see Chakrabarty and Guo 2007, where the effect of random time horizon on optimal stopping has been studied extensively.

Let $T \sim \mathrm{Exp}(\lambda)$, $\lambda > 0$, be an exponentially distributed time horizon of the problem, independent of the diffusion X_t. Furthermore, denote by $\mathscr{T}$ the set of all $\mathscr{F}$-stopping times and, with a slight abuse of notation let $\hat{\mathscr{T}} = \mathscr{T} \cup \{T\}$ (rigorously $\hat{\mathscr{T}}$ should be defined through the enlarged filtration).

We assume that the decision maker is always aware of the existence of the terminating event, and we consider two problems: In the first problem the terminal time T of the terminating event is not observable, and cannot be used as a stopping time. In the second problem the terminal time T is observable and decision maker can use it as a stopping time. These problems are, respectively,

$$\begin{aligned} V_1(x) &= \sup_{\tau \in \hat{\mathscr{T}}} \mathbb{E}_x \left\{ e^{-r\tau} g(X_\tau) \mathbb{1}_{\{\tau < T\}} \right\} \\ \text{and} \quad V_2(x) &= \sup_{\tau \in \hat{\mathscr{T}}} \mathbb{E}_x \left\{ e^{-r\tau} g(X_\tau) \mathbb{1}_{\{\tau \leq T\}} \right\}, \end{aligned}$$

and they can be simplified into equivalent infinite time horizon problems (cf. Theorem 1 and 3 in Chakrabarty and Guo 2007)

$$\begin{aligned} V_1(x) &= \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left\{ e^{-(r+\lambda)\tau} g(X_\tau) \right\} \\ \text{and} \quad V_2(x) &= \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left\{ \lambda \int_0^\tau e^{-(r+\lambda)s} g^+(X_s) ds + e^{-(r+\lambda)\tau} g(X_\tau) \right\}. \end{aligned}$$

Thanks to this equivalence between random and infinite time horizons, first noticed in a portfolio optimization setting by Cass and Yaari 1967 and Merton 1971, it is easier to analyse random time horizon problems than fixed finite time horizon problems. Actually, this equivalence can also be used for pricing options in the fixed finite time horizon case (originating in Carr 1998 and further extended in e.g. Kyprianou and Pistorius 2003). In this Canadisation method one first chooses a random time horizon whose expected length equals the fixed maturity under investigation, secondly calculates a value for this new problem and finally lets the variance of the random horizon approach zero while maintaining the mean unchanged.

4.4 Optimal stopping of maximum processes

Let S_t be the supremum process of X_t and let us consider the problem

$$V(x,s) = \sup_{\tau} \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, S_\tau) \right\}, \tag{13}$$

where the exercise payoff $f(x,s)$ is assumed to be sufficiently smooth, decreasing in x, and increasing in s. Two of the most well-known examples of the problem (13) are the Russian option for which $f(x,s) = s$ (see e.g. Shepp and Shiryaev 1993; Lerche and Urusov 2007) and the American Lookback option with a floating strike for which $f(x,s) = s - x$ (see e.g. Conze and Viswanathan 1991). While both of these are path-dependent options, the latter can also be interpreted as a measure of the risk for a stock (see Douady et al. 2000; Magdon-Ismail et al. 2004).

We see at once that, as a two-dimensional problem, this is genuinely a more complex problem than the original (8). However, problem (13) is partly transformed into a linear problem after recalling that between the hitting times of X_t to its supremum, the two-dimensional process (X_t, S_t) behaves as a one-dimensional process (X_t, s) for some s. This method of conditioning a two-dimensional problem into a linear one is the main observation needed for solving (13).

Optimal stopping problems of the type (13) are typically solved by free boundary approach (see Peskir 1998 or Section 13 in Peskir and Shiryaev 2006) and the Beibel-Lerche method is also applicable in this situation (see Lerche and Urusov 2007). In Article IV of this thesis, we will introduce another approach, which is based on the discretisation of the maximum process. For a more detailed discussion cocerning the free boundary approach and the discretisation method, see Article IV and also Subsection 7.4 (summary of Article IV).

Since the minimum process is defined similarly, we can naturally solve problems involving the minimum process in the same way, and even problems where both the maximum and the minimum processes are present (see Peskir 2010).

5 DYNKIN GAMES

5.1 Introduction

A Dynkin game, originating from Dynkin 1969, is a stochastic zero-sum game involving two players who both try to stop the same underlying process optimally with respect to their own payoff functions. An additional challenge to this stopping problem comes from the fact that the process can be stopped only once and hence the players have to take into account the strategies of their opponent.

Let X_t be a linear diffusion, adapted to the filtration $\mathscr{F}$, and denote by $\mathscr{T}$ the set of all $\mathscr{F}$-stopping times. Furthermore, let g_i, for $i = 1,2,3$, be continuous mappings satisfying the inequalities $g_1(x) \leq g_2(x) \leq g_3(x)$.

Consider two players, the sup-player and the inf-player, who both choose a stopping rule, say τ and γ, respectively. The game terminates as soon as either one of the players decide to stop, that is at $\tau \wedge \gamma$ and at that time the inf-player pays to the sup-player the amount

$$g_1(X_\tau)\mathbb{1}_{\{\tau<\gamma\}} + g_2(X_\gamma)\mathbb{1}_{\{\tau>\gamma\}} + g_3(X_\gamma)\mathbb{1}_{\{\tau=\gamma\}}.$$

The expected present value of this Dynkin game is

$$\Pi(x;\tau,\gamma) := \mathbb{E}_x\left\{e^{-r(\tau\wedge\gamma)}\left(g_1(X_\tau)\mathbb{1}_{\{\tau<\gamma\}} + g_2(X_\gamma)\mathbb{1}_{\{\tau>\gamma\}} + g_3(X_\gamma)\mathbb{1}_{\{\tau=\gamma\}}\right)\right\},$$

and understandably the sup-player wants to maximise it while the inf-player tries to minimise it. The lower- and upper- values $\underline{V}$ and $\overline{V}$ of this *Dynkin game* are defined through

$$\underline{V}(x) := \sup_{\tau\in\mathscr{T}}\inf_{\gamma\in\mathscr{T}}\Pi(x;\tau,\gamma) \leq \overline{V}(x) := \inf_{\gamma\in\mathscr{T}}\sup_{\tau\in\mathscr{T}}\Pi(x;\tau,\gamma). \tag{14}$$

We say that the game has a value $V(x)$, if $\underline{V}(x) = \overline{V}(x) =: V(x)$, i.e. it has a Stackelberg equilibrium. Furthermore, a pair of stopping times (τ^*,γ^*) forms a saddle-point solution, or a Nash solution, for the game, if the condition

$$\Pi(x;\tau,\gamma^*) \leq \Pi(x;\tau^*,\gamma^*) \leq \Pi(x;\tau^*,\gamma) \tag{15}$$

is satisfied for all stopping times τ,γ (concepts of Stackelberg and Nash equilibriums are explained in detail in almost any game theory textbook, e.g. Fudenberg and Tirole 1991). It is worth noticing that the existence of a Nash equilibrium implies the existence of the Stackelberg equilibrium but the converse does not hold in general. However, from a study addressing this problem in a general Markovian setting (Ekström and Peskir 2008) one gets the following result

Theorem 5.1. *Let X_t be a strong Markov process. If X_t is right-continuous and left-continuous over stopping times, then both Nash and Stackelberg equilibriums exist and they are equivalent.*

One interpretation for the game is the following. Suppose that the issuer (inf-player in this case) has sold an American option with a payoff g_1 to the the holder (sup-player), but has left herself a right to cancel the option with an extra cost $g_2 \geq g_1$. This variant, called an Israeli option, was introduced by Kifer 2000 and further explicit solutions for some options were calculated in Kyprianou 2004.

Another way to interpret a Dynkin game is to think the sup-player as a risk averse decision-maker in an optimal stopping problem with a stochastic time horizon. Being a risk averse, she assumes that while she tries to maximise her payoff, the market plays against her by choosing the time horizon which minimises her payoff.

5.2 Solution methods

A Dynkin game can be seen as a generalised stopping problem.[10] As such, although a Dynkin game includes game theoretical elements, such as Nash and Stackelberg equilibriums, the solution can nonetheless be attained via optimal stopping methods.

In the ordinary optimal stopping problem (8) the verification theorem characterised the value as the smallest r-superharmonic majorant for the payoff function (see Theorem 4.1). We have a similar kind of verification theorem for a Dynkin game, but now the value is characterised as a mix of the smallest r-superharmonic majorant and the largest r-subharmonic minorant. The theorem holds for a class of Markov processes larger than just diffusions (see Theorem 2.1 in Peskir 2008), but the following form is adequate for our linear diffusion case.

[10] If we formally set $g_2 = \infty$, then the inf-player never stops and (14) reduces to the ordinary optimal stopping problem (8).

Theorem 5.2 (Verification theorem for a Dynkin game)**.** *Let X_t and g_i, for $i = 1,2,3$ be as in Section 5.1. Moreover, let $\hat{V}$ be the smallest r-superharmonic function lying between g_1 and g_2 and let $\check{V}$ be the largest r-subharmonic function lying between g_1 and g_2. Then*

(A) The Nash equilibrium (15) *holds if and only if $\hat{V} = \check{V} =: V$.*

(B) If item (A) *holds, then the pair*

$$\tau^* = \inf\{t \geq 0 \mid V(X_t) = g_1(X_t)\} \text{ and } \quad \gamma^* = \inf\{t \geq 0 \mid V(X_t) = g_2(X_t)\}$$

forms the saddle point solution and V is the value of the game.

According to Theorem 5.1, the Nash equilibrium exists and is equivalent to the value of the game in the linear diffusion case. Therefore Theorem 5.2 can be used as a verification theorem to characterise the value.

To find a candidate for the saddle point solution to a Dynkin game, we shall utilise a fluctuation theory approach similar to the one in Subsection 4.2.1 (for this approach in Dynkin game setting see e.g. Alvarez 2008, 2010 and Article III).

Step 1 Consider stopping policies which can be characterised as the first exit time from an open subinterval of the state space.

Step 2 Calculate, with the help of the Laplace transform of the hitting times, the value of the game for this policy. Usually the value has both boundary points of the subinterval as free parameters.

Step 3 Derive the first-order necessary conditions for the saddle point equilibrium value in the considered class of stopping policies.

Step 4 Verify that the stopping policy which satisfies the first order necessary conditions from Step 3 indeed constitutes the saddle point solution (e.g. using the verification theorem 5.2(A) above).

The fluctuation theory approach chosen here is by no means the only way to approach Dynkin games. Friedman 1973 studied stochastic zero-sum stopping games and their values using variational inequalities. Bensoussan and Friedman 1977 investigated stochastic stopping games in very general setting both in nonzero-sum as well as in zero-sum case using quasi-variational inequalities. The functional concavity together with r-excessivity have also been used to produce the optimal solution, e.g. in Ekström 2006 and Ekström and Villeneuve 2006. These methods, also familiar from the optimal stopping scene, can be classified as direct techniques. In contrast to these approaches, with Dynkin games it is also possible to use an indirect approach which is not as common in ordinary optimal stopping problems. In the indirect approaches the Dynkin game is shown to be equivalent to another problem and the latter one is then solved. This approach has been utilised for example in Boetius 2005, where the author characterises the value of the saddle point equilibrium as the derivative of the value function of a singular control problem[11] (for this connection, see also Subsection 6.5).

[11] In Guo and Tomecek 2008a,b the singular control problems are solved by showing one-to-one correspondence between a singular control and a switching problem. Hence, in principle, one could leap twice and also use an associated switching problem to solve a Dynkin game.

6 SINGULAR CONTROL PROBLEMS

6.1 Introducing the problem

In the previous chapter we saw that an optimal stopping problem is a special case of a Dynkin game. We shall learn that it can also be seen as a special case of an optimal control problem, where the only allowed control is the primitive control ”take the money and run”. To be able to study more refined problems like controlling a path of a space ship (see e.g. Jacka 2002), dividend payments problem (Asmussen and Taksar 1997) or rational harvesting (Lande et al. 1995), we need to include more refined controls.

Assume now that the state space is $\mathbb{R}_+$ with natural boundaries and that the diffusion without controls behaves as a non-terminating Itô diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where μ and σ satisfy the conditions of Theorem 3.3. We assume that the controller can, at any time, control the course of the process both downwards and upwards. An admissible control policy is defined as a pair of processes (D_t, U_t) such that both processes are non-negative, non-decreasing, right-continuous, and $\mathscr{F}^X$-adapted. Consequently any admissible control has finite variation. For an admissible control (D_t, U_t), we define the associated controlled process by

$$Z_t = X_t - D_t + U_t,$$

where D_t represents the cumulative downward control and U_t the cumulative upward control. For example, in a timber harvesting problem, D_t represents harvesting while U_t can be interpreted as replanting.

Now the problem under investigation is

$$V(x) = \sup_{(D,U)} \mathbb{E}_x \left\{ \int_0^\zeta e^{-rs} f(Z_s)ds + \int_0^\zeta e^{-rs} g_1(Z_s)dD_s - \int_0^\zeta e^{-rs} g_2(Z_s)dU_s \right\}, \tag{16}$$

where ζ is the first exit time from the state space, $f : \mathbb{R}_+ \to \mathbb{R}$ is the revenue function, $g_1(x)$ is the gain from downward control and $g_2(x) > g_1(x)$ is the cost of upward control and the supremum is taken over all admissible controls.

6.2 On admissible controls

An arbitrary admissible control C_t can be written as $C_t = C_t^c + \sum_{0\le s\le t} \Delta C_s$, where C_t^c is continuous and $\Delta C_s = C_s - C_{s-}$ is the size of the jump at time s. In this thesis, we are only interested in so called *singular* or *reflecting controls*. In these controls the jump part is absent, with the exception of a possible jump at time zero, and the control measure dC is singular with respect to Lebesgue measure dt as a function of time[12], whence the name. The easiest way to create a singular control is to divide the state space into two separate regions, an *action region* A and an *inaction region* $N = \mathscr{I} \setminus A$, and apply the following rule. From N, the exit of the process is prevented by reflecting at the boundary ∂N to an appropriate direction. From A, the process is immediately moved into the boundary ∂N. For an illustration, see Figure 3, and for a more detailed introduction see e.g. Chapters 2, 5 and 6 in Harrison 1985 and Article II.

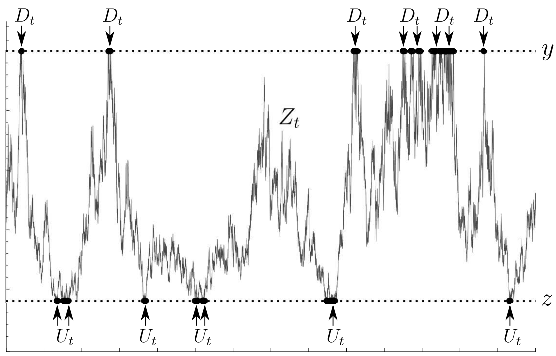

Figure 3: An illustration of a two-sided singular control policy at work, here $A = (0,z] \cup [y,\infty)$ and $N = (z,y)$.

The so-called *impulse controls*, where the continuous part is abolished, have also been widely studied. Impulse control policies are often described by (possibly finite) sequences of the form $\{(\tau_k, \xi_k)\}$, where τ_k prescribes the intervention time (an admissible stopping time) and ξ_k the corresponding impulse (size of the jump). However, in cases where there are no fixed transaction costs present, such as problem (16), the optimal control rarely has a jump structure.

[12] A measure μ is *singular* with respect to a measure λ, if there exist two disjoint sets A and B whose union is the whole space such that, for every measurable set E, $A \cap E$ and $B \cap E$ are measurable and $\mu(A \cap E) = \lambda(B \cap E) = 0$. That is, the sets for which λ does not vanish are the ones for which μ does, see p. 126 in Halmos 1950.

It is worth pointing out that for an arbitrary stopping time τ the stopping policy ”stop at time τ” can be interpreted as a degenerate impulse control policy, where the sequence $\{(\tau_k, \xi_k)\}$ has the pair $(\tau, X_{\tau-})$ as its only element. When obeying this impulse control policy one sends the state variable into the cemetery state at the first intervention time. Hence we see that optimal stopping problems are essentially special cases of (impulse) control problems.

Interestingly, from the impulse control point of view, we can heuristically treat singular control as a limiting impulse control, when the impulse approaches zero. Let us consider a downward impulse control which is activated at the instance when Z_t hits a level $y \in \mathbb{R}_+$ (i.e. $\tau_k = \inf\{t \geq \tau_{k-1} \mid Z_t = y\}$) and let the size of each downward impulse be $0 < \xi < y$, so that the impulse control is the sequence $\{(\tau_k, \xi)\}$. As $\xi \to 0$, we can heuristically see that the impulse control tends to the singular control reflected at the barrier y (i.e. the action and inaction regions are $A = (0, y)$ and $N = [y, \infty)$).

6.3 Control problem types

The problem (16) is called a *singular control problem* because the problem setting allows the control to be singular (and often the optimal control actually turns out to be singular).

Let us assume for a moment that there are fixed transaction costs in problem (16), i.e. we must pay a fixed cost every time a control is activated. Due to continuity, a singular control policy could easily lead to infinite costs and thus such control is not a reasonable choice anymore. Consequently, in such case the controller should use fixed-sized controls only at discrete times. Therefore, in these kind of problems an impulse control policy is often the optimal one. Accordingly, the problems with fixed transaction costs are called *impulse control problems*. Although impulse control problems can be approached in the same way as singular control problems, they are usually more difficult to handle as there are more free parameters; in a singular control policy the only free parameter is the intervention time, while in an impulse control policy one needs to determine also the size of the intervention. An interested reader finds more information on impulse control problems, in e.g. Bensoussan and Lions 1984, which is the seminal textbook on the subject. See also Alvarez 2004, Alvarez and Lempa 2008, and a survey on impulse control applications in finance by Korn 1999.

Besides singular, and impulse control problems, there exists a great variety of other stochastic control problems, e.g. ergodic control problems (i.e. optimizing a long period stationary behaviour), robust control problems (optimizing a risk measure of a controlled process) and stochastic targeting problems (targeting a controlled process as close as possible to an observed process at a termination time). For literature on these problems, consult e.g. Pham 2005 and references therein.

From now on we shall concentrate solely on singular controls.

6.4 Solution methods for singular control problems

The underlying Markovian structure still enables us to apply the familiar procedure of fluctuation theory approach from the optimal stopping scene — guess, apply diffusion tools and verify.

The verification theorem for control problems mainly relies on variational arguments and Itô's formula. However, depending on the problem type and technical assumptions, its formulation might slightly vary from problem to problem. For our singular control problem it can be stated in the following way (cf. Theorem 4.4. in Article II, see also Chapter 6 in Harrison 1985, Shreve et al. 1984 and Alvarez 1999).

Theorem 6.1 (Verification theorem for a singular control problem)**.** *Let V^* be the solution to* (16) *and let F be a function satisfying the conditions*

(i) $F \in C^2$*;*

(ii) $(\mathscr{A} - r)F(x) + f(x) \leq 0$ *for all* $x \in \mathscr{I}$*;*

(iii) $g_1(x) \leq F'(x) \leq g_2(x)$.

Then $F(x) \geq V^*(x)$.

6.4.1 A concrete example

In this subsection we will demonstrate the solution procedure with a concrete example.

Let us consider the simplest optimal dividend payment problem (cf. Subsection 3.2 in Alvarez and Virtanen 2006)

$$K(x) = \sup_D \mathbb{E}_x \{ \int_0^\zeta e^{-rs} dD_s \},$$

where there is only a downward control, and the controlled process is $Z_t = X_t - D_t$. Let us apply the four step procedure to this problem.

Step 1 Since there is no exercise cost of using control, we can guess that the optimal control is singular. Let us consider the simplest non-trivial singular control available. Namely, a control reflecting downwards at the boundary $y \in \mathbb{R}_+$. Now, the action and inaction regions are $A = [y, \infty)$ and $N = (0, y)$.

Step 2 For a fixed $y \in \mathbb{R}_+$ it can be shown (see e.g. Lemma 3.1 and the discussion below it in Article II) that with the chosen control the associated value is

$$K(x,y) = \begin{cases} x - y + \dfrac{\psi(y)}{\psi'(y)} & x \geq y \\ \dfrac{\psi(x)}{\psi'(y)} & x < y. \end{cases}$$

Step 3 For y^* to be the optimal reflecting barrier, it must satisfy the first order necessary condition $\frac{\partial K(x,y)}{\partial y}\Big|_{y=y^*-} = 0$. After the differentiation, this becomes $\psi''(y^*) = 0$. Here we need some additional assumptions to guarantee the unique existence of y^*[13].

Step 4 To confirm that y^*, which satisfies the necessary condition from Step 3, leads to the optimal control, one can show that $K(x, y^*)$ satisfies the conditions of Theorem 6.1 (under certain sufficient assumptions).

Specifically, we see that in the procedure above the necessary optimal condition forces, in a natural way, the value function to be twice continuously differentiable.

6.4.2 Other methods

Since singular stochastic control problems are dynamic programming problems, Bellman's optimality principle can be applied to them. This principle allows one to characterise the value function via Hamilton-Jacobi-Bellman variational inequalities, which in this case take the form (see e.g. Pham 2005 and

[13] Sufficient conditions are, for example, that there is a unique threshold $y^* \in \mathbb{R}_+$ such that $\mu(x) - rx$ is increasing on $(0, y^*)$ and decreasing on (y^*, ∞) and that $\lim_{x \to 0} \mu(x) \leq 0$ (cf. Lemma 3.1. in Alvarez and Virtanen 2006).

Weerasinghe 2005)

$$\max\{(\mathscr{A}-r)F(x)+f(x), f(x)-F'(x)\}=0, \quad \text{for all } x\in\mathscr{I}.$$

By splitting the state space into action and stopping regions, the Hamilton-Jacobi-Bellman inequalities give rise to an associated free boundary problem, similar to optimal stopping problems. In this case the non-trivial boundary condition turns out to be, not the C^1-smooth fit continuity as in optimal stopping problems, but C^2-smooth fit continuity.

It is also possible to rely on probabilistic methods. For example in Karatzas and Shreve 1984, 1985 and Karatzas and Wang 2001 the existence of the optimal control was proved by showing that the optimizing sequence of the considered problem converges to an admissible control using probabilistic reasoning and a weak compactness argument.

The approaches above can be classified as direct techniques. In contrast to these, in an indirect approach the control problem is shown to be equivalent with another problem and the latter one is then solved. The standard equivalence is the connection between singular control problems and optimal stopping problems or Dynkin games, see Subsection 6.5 below. This has been utilised e.g. in Karatzas and Wang 2001. Another indirect approach has been introduced in Guo and Tomecek 2008a,b, where the authors reveal a one-to-one correspondence between singular control problems and switching problems. They use this relation in a general multidimensional setting to find an integral representation for the value function and sufficient conditions for the existence of an optimal control.

Besides the Markovian methods above, Snell's envelope, based on martingale methods, has also been used successfully, e.g. in Bank 2005.

6.5 A link between singular control problems and stopping problems

The close connection between a one-sided singular control problem (i.e. either downward or upward control is allowed, but not both) and an optimal stopping problem was already present in the seminal paper by Bather and Chernoff 1966, and later studies have shown it to hold in general (see Karatzas and Shreve 1984, 1985, and Benth and Reikvam 2004). It is known from this literature that for every one-sided singular control problem there exists an associated optimal stopping problem such that the derivative of the value of the one-sided singular control problem is the value of the associated optimal stopping problem.

This connection can also be generalised to concern two-sided singular control problems (i.e. both downward and upward controls are present). Interestingly, for every two-sided singular control problem there exists an associated *Dynkin game*, such that the derivative of the two-sided singular control problem constitutes the value of the associated Dynkin game is (see Karatzas and Wang 2001 and Boetius 2005).

This connection partially explains the C^2-smooth fit condition for the value of a control problem (cf. e.g. Bayraktar and Egami 2008). Since the value of a stopping problem is often C^1 and is a derivative of the value of a control problem, we can interpret the C^2-condition as an inherited condition from a smooth fit condition of a stopping problem.

7 SUMMARIES OF THE INCLUDED ARTICLES

In this chapter we briefly summarise the four studies included in the thesis. Unless otherwise stated, we assume throughout the chapter that the underlying dynamics X_t evolves on $\mathbb{R}_+$ according to a linear Itô diffusion. Further, we assume that it is a weak solution to the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are as in Theorem 3.3.

7.1 Article I: Optimal timing in a combined investment and exit problem

In the first paper, we consider a situation where a decision-maker receives a revenue π_1 based on an underlying Itô diffusion X_t on an interval $(\alpha, \beta) \subset \mathbb{R}$. She has the following two options: She can invest irreversibly into an improved technology resulting in a new revenue function π_2 under a new diffusion Y_t, or she can exit the market. Of these two available options, which one she should use and when? And if she decides to invest, when is the right time to exit afterwards? This problem setting is modelled by

$$\begin{aligned} V(x) &= \sup_{\tau} \mathbb{E}_x \left\{ \int_0^{\tau} e^{-rs}\pi_1(X_s)ds + e^{-r\tau}\left(V_2\left(\theta(X_\tau)\right) - k\right)^+ \right\} \\ &= (R_r^X\pi_1)(x) + \sup_{\tau} \mathbb{E}_x \left\{ e^{-r\tau}\left[\left(V_2\left(\theta(X_\tau)\right) - k\right)^+ - (R_r^X\pi_1)(X_\tau) \right] \right\}, \end{aligned} \tag{17}$$

where $V(x)$ is the expected maximum present value for the decision maker,

$$V_2(x) = \sup_{\tau_2} \mathbb{E}_x \left\{ \int_0^{\tau_2} e^{-rs}\pi_2(Y_s)ds \right\}$$

is the expected net present value after the possible investment, k is the fixed cost of the investment, and $\theta(x)$ is a twice differentiable and increasing boost function, which describes how much the investment improves productivity. The second line in (17) is attained by utilising the strong Markov property, and the resolvent operator R_r^X is taken with respect to the initial diffusion X_t.

We will show that, under some mild assumptions, if the investment is eventually profitable, i.e. if (denoting by R_r^Y the resolvent with respect to the diffusion Y_t)

$$\lim_{x\to\beta}\left(R_r^Y\pi_2\right)(\theta(x)) - k > \lim_{x\to\beta}\left(R_r^X\pi_1\right)(x), \tag{18}$$

then the resulting optimal rule is a two-sided threshold rule. In other words, the decision-maker exits if the process X_t goes below a lower threshold and invests if it exceeds an upper threshold. After the possible investment the decision-maker faces a normal exit problem (see e.g. Alvarez 1998), where she exits below a certain threshold.

In the proof we will utilise and refine a fixed point method originating from Lempa 2010 and Alvarez and Lempa 2008. We will also show that if the inequality in 18 is reversed, we end up with either a one- or a three-sided threshold rule. In the former case it is never optimal to invest, whereas in the latter case it is, interestingly, optimal to to invest only on a certain finite interval.

7.2 Article II: On solvability of a two-sided singular control problem

In the second paper we consider singular control problems where the optimal control is a two-sided singular control. As described in Chapter 6, we consider a controlled process

$$Z_t = X_t - D_t + U_t,$$

where D_t and U_t are downward and upward controls (defined in Chapter 6) and we study a singular control problem

$$V(x) = \sup_{D,U}\mathbb{E}_x\left\{\int_0^{\zeta_Z} e^{-rs}\pi(Z_s) + p\int_0^{\zeta_Z} dD_t - q\int_0^{\zeta_Z} dU_t\right\}. \tag{19}$$

Here $\pi : \mathbb{R}_+ \to \mathbb{R}$ is a non-decreasing revenue function satisfying suitable growth and smoothness conditions (given in Article II), $q > p$ are exogenously given constants, $\zeta_Z = \inf\{t \geq 0 \mid Z_t \notin \mathbb{R}_+\}$ denotes the first exit from $\mathbb{R}_+$, and the supremum is taken over all admissible controls.

The following quasi-concavity assumption is the main condition which enforces the solution to be a two-sided control.

For every $b \in [p,q]$, there exists $\tilde{x}_b \in \mathbb{R}_+$ such that $\frac{d}{dx}(\pi(x) + b(\mu(x) -$

$rx)) \gtreqless 0$ whenever $x \lesseqgtr \tilde{x}_b$.

In a cash flow management application, the function $\pi(x) + b(\mu(x) - rx)$ can be seen measuring the expected net return accrued from postponing the dividend payment into the future instead of paying out dividends instantaneously (cf. p. 708 in Alvarez and Lempa 2008).

More precisely, we will establish that under the above mentioned assumption, and some mild additional conditions, the unique optimal control in problem (19) is a two-sided singular control. Moreover, under the same conditions, we will see that the value function can be written in a (quasi-)explicit form. Since we can identify the value function and control boundaries explicitly, we are also able to investigate the comparative static properties of the solution, which is the main contribution of the article. We shall see that known results concerning one-sided controls (see e.g. Alvarez 2001) generalise in a natural way for two-sided controls: we will prove that the value is decreasing with respect to the volatility and cost parameter q and increasing with respect to the gain parameter p. We will also show that when volatility increases, the inactivity region expands.

We shall also compare one-sided and two-sided singular control problems, and notice that the former ones are special cases of the latter ones. Moreover, we will show that in the two-sided case the controls are activated earlier.

7.3 Article III: A Dynkin game with asymmetric information

In the third paper, we consider an otherwise standard Dynkin game, defined in Chapter 5, except that we assume the time horizon to be stochastic with asymmetric information about it. To be more precise, we assume that there exists a terminating event at time $T \sim \text{Exp}(\lambda)$ which ends the game, and *only one* of the players observes the occurrence of the expiring time T. To define such a game, let $\hat{\mathscr{T}}$ be the set of all $\mathscr{F}$ stopping times augmented with T. We make a distinction between the cases where the sup-player (Game 1) or the inf-player (Game 2) can observe the time T. The respective values of the games are

$$
\begin{aligned}
V_1(x) &= \sup_{\tau \in \hat{\mathscr{T}}} \inf_{\gamma \in \mathscr{T}} \hat{\Pi}(x,\tau,\gamma) = \inf_{\gamma \in \mathscr{T}} \sup_{\tau \in \hat{\mathscr{T}}} \hat{\Pi}(x,\tau,\gamma),\\
V_2(x) &= \sup_{\tau \in \mathscr{T}} \inf_{\gamma \in \hat{\mathscr{T}}} \hat{\Pi}(x,\tau,\gamma) = \inf_{\gamma \in \hat{\mathscr{T}}} \sup_{\tau \in \mathscr{T}} \hat{\Pi}(x,\tau,\gamma),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\Pi}(x,\tau,\gamma) = \mathbb{E}_x \Big\{ e^{-r(\tau\wedge\gamma)} \big[& g_1(X_\tau)\mathbb{1}_{\{\tau<\gamma\}} \\
& + g_2\ (X_\gamma)\mathbb{1}_{\{\tau>\gamma\}} + g_3(X_\gamma)\mathbb{1}_{\{\tau=\gamma\}}\big]\ \mathbb{1}_{\{\tau\wedge\gamma\le T\}}\Big\}.
\end{aligned}
\tag{20}
$$

We will show that these asymmetric random time horizon games can be simplified to associated perpetual dividend paying games under the reference filtration $\mathscr{F}$ generated by the underlying diffusion X. More formally, we will show that the value functions of the games can be written as

$$
\begin{aligned}
V_1(x) &= \sup_{\tau \in \hat{\mathscr{T}}} \inf_{\gamma \in \mathscr{T}} \tilde{\Pi}_1(x,\tau,\gamma) = \inf_{\gamma \in \mathscr{T}} \sup_{\tau \in \hat{\mathscr{T}}} \tilde{\Pi}_1(x,\tau,\gamma),\\
V_2(x) &= \sup_{\tau \in \mathscr{T}} \inf_{\gamma \in \hat{\mathscr{T}}} \tilde{\Pi}_2(x,\tau,\gamma) = \inf_{\gamma \in \hat{\mathscr{T}}} \sup_{\tau \in \mathscr{T}} \tilde{\Pi}_2(x,\tau,\gamma),
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\Pi}_1(x,\tau,\gamma) = \mathbb{E}_x \Big\{ & \lambda \int_0^{\tau\wedge\gamma} e^{-rs} g_1^+(X_s)ds \\
& + e^{-(r+\lambda)(\tau\wedge\gamma)} \big[g_1(X_\tau)\mathbb{1}_{\{\tau<\gamma\}} + g_2(X_\gamma)\mathbb{1}_{\{\tau>\gamma\}} + g_3(X_\gamma)\mathbb{1}_{\{\tau=\gamma\}}\big]\Big\}\\
\tilde{\Pi}_2(x,\tau,\gamma) = \mathbb{E}_x \Big\{ & \lambda \int_0^{\tau\wedge\gamma} e^{-rs} g_2^-(X_s)ds \\
& + e^{-(r+\lambda)(\tau\wedge\gamma)} \big[g_1(X_\tau)\mathbb{1}_{\{\tau<\gamma\}} + g_2(X_\gamma)\mathbb{1}_{\{\tau>\gamma\}} + g_3(X_\gamma)\mathbb{1}_{\{\tau=\gamma\}}\big]\Big\},
\end{aligned}
$$

with $g_1^+ = \max\{g_1, 0\}$ and $g_2^- = \min\{g_2, 0\}$.

For $i = 1,2$, let z_i^* be the optimal exercise thresholds for inf-player in Game i and let y_i^* be the optimal exercise thresholds for the sup-player ($z_i^* < y_i^*$ always). We will show that, assuming that the games attain solutions, $V_1 \ge V_2$, $z_1^* \ge z_2^*$ and $y_1^* \ge y_2^*$ always. Verbally the last two conditions can be stated as *the more you know, the longer you wait*. Furthermore we will compare the random horizon games with the standard infinite horizon Dynkin game from Chapter 5 with (z^*, y^*) as the optimal exercise thresholds and V as the value of the game. We will see that if $g_2(x)$ is non-negative everywhere, then $V \ge V_1$,

$z^* \geq z_1^*$ and $y^* \geq y_1^*$. At the end of the paper we will also give an example where the value V of the infinite horizon game can, in fact, be the smallest of all presented games in case $g_2(x)$ attains also negative values.

We will also present limiting properties of Games 1 and 2, when the expected random time horizon $\mathbb{E}\{T\}$ goes to infinity or zero. In the case $\mathbb{E}\{T\} \to \infty$, there is expectedly no terminating event, and we retrieve an infinite horizon game. At the other end, when $\mathbb{E}\{T\} \to 0$, the games expectedly end immediately, and we get a solution where the games are either immediately or never stopped. Especially we will see that Games 1 and 2 are inseparable as $\mathbb{E}\{T\}$ approaches the limits infinity or zero. This is reasonable, since there is no advantage in seeing the expiring event if it does not happen or if both know that it happens immediately.

7.4 Article IV: Optimal stopping of the maximum process

In the fourth paper, we consider the optimal stopping problem

$$V(x,s) = \sup_{\tau} \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, S_\tau) \right\}, \tag{21}$$

where S_t is the maximum process and the exercise payoff $f(x,s)$ is assumed to be sufficiently smooth, decreasing in x, and increasing in s.

Typically, these kinds of problems are solved by applying the free boundary approach together with a non-trivial boundary condition called the *maximality principle*. This principle says that a certain non-linear differential equation attains a maximal solution $a^*(s)$ which stays below the diagonal (i.e. $a^*(s) < s$ for all $s \in \mathbb{R}_+$). Using the techniques from Peskir 1998 one can prove that under the maximality principle, the stopping rule $\tau^* = \inf\{t \geq 0 \mid X_t \leq a^*(S_t)\}$ provides an unique solution to (21).

The main contribution of the article is to demonstrate that the solution can be attained without the maximality principle. Let us briefly describe this concept. We first notice that, under mild conditions, the value $V(x,s)$ is finite, so that it can be written as

$$V(x,s) = \sup_{\tau} \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, s) \mathbb{1}_{\{\tau < \tau_s\}} + e^{-r\tau_s} V(s,s) \mathbb{1}_{\{\tau \geq \tau_s\}} \right\}, \tag{22}$$

where, for a given $s \in \mathbb{R}_+$, $V(s,s)$ is just a (still unknown) finite constant. For each given s the problem (22) is linear, and thus we can apply standard optimal stopping theory to show that there exists a unique optimal stopping rule that maximises the problem (22). This optimal stopping rule proves to be of the type

$$\tau^* = \inf\{t \geq 0 \mid X_t \leq a^*(s)\}, \tag{23}$$

where the values $a^*(s)$ are still unknown.

Next we discretise the maximum process, i.e. assume that the maximum process S_t can only attain values from a countable sequence. Then the discretised version of the problem (21) can be seen as a countable sequence of relatively easily solvable one-dimensional subproblems. Finally, as the sequence gets denser, the value of the discretised problem approaches the value (22) which is already known to have a unique solution. This discretisation approach is straightforward and easy for achieving an existence result as well as numerical results. Unfortunately it cannot provide explicit solutions.

To see why this problem is a two-sided control problem, observe that the problem (22) can be seen as a one-dimensional problem on the state space $(0,s]$, where the boundary s is killing and, once reached, leads to a terminal value $V(s,s)$. In other words, we can interpret (22) as a two-sided stopping problem, where the upper stopping threshold s is not a free parameter.

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Part II

THE RESEARCH PAPERS

Article I

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OPTIMAL TIMING IN A COMBINED INVESTMENT AND EXIT PROBLEM

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ABSTRACT

We study optimal timing in a combined investment and exit problem. We consider a situation where at any given time a company has the following three options: It can make an irreversible investment in order to obtain an improved technology resulting to a higher revenue flow, it can exit the market or it can postpone making the final decision. We prove the existence and uniqueness of an optimal strategy, which is a two-sided threshold rule: exit below one threshold and invest above another. We illustrate our results numerically with geometric Brownian motion.

keywords: irreversible investment, exit, optimal stopping, linear diffusion

AMS Classification: 60G40, 62L15, 60J60

1 Introduction

Consider a company operating in the presence of uncertainty and facing following options. Either the company invests irreversibly into an improved technology or machinery resulting in a higher profit flow, or the company exits the market. Of these two available options, which one the company should use and when? And if the company decides to exercise its opportunity to invest, when is the right time to exit afterwards?

Investment decisions are typically assumed to be irreversible; in most major investments capital is firm- or industry specific and thus investment expenditures cannot be recovered by using capital in a different firm or industry. Even if the investment is not firm- or industry specific, it is still often partly irreversible, since the resale value is frequently significantly below the purchase cost (see p. 8 in Dixit and Pindyck 1994). For more information about problems involving an irreversible decision under uncertainty see for example Dixit and Pindyck 1994, Alvarez and Stenbacka 2004, Chiarolla and Haussmann 2008 and the references therein. For a survey of literature of adopting a new technology, see Hoppe 2002.

Furthermore a company typically has a choice to exit the market and shut down the operation. And even in the case of individual investment projects, there are often possibilities of permanent exit or abandonment, in contrast to temporary exit or mothballing. Temporary exit is not a choice in situations where the capital disappears quickly after abandonment: mines flood, machines rust, brand recognition is lost and teams of skilled workers disband — all of these are example of definitive exit (Dixit and Pindyck 1994, p. 14). The literature on exit is extensive: For exit and entry problems see for example Dixit 1989, Dixit and Pindyck 1994, Zervos 2003, and Egami and Bayraktar 2010 and references therein. For a general linear Itô diffusion based exit studies, see Alvarez 1998 and Alvarez 2001.

In most studies concerning irreversible investment problems the analysis overlooks the embedded exiting option which is often a conceivable option. Whether it is an oil company considering opening a new oil field or a company pondering the possibility of entering a new market, there is always the back door possibility to exit the market irrevocably. On the other hand studies focusing on exit problems often neglect, with an exception of entry and exit or switching problems, the role of subsequent investment opportunities as a mechanism which potentially prolongs the operation of the company. In this paper, our aim is to combine these two viewpoints and examine the situation where both, the investment and exit options, are present. A company does not have to just wait for the optimal time to terminate production, it can operate actively to try to prevent such a situation. For example, the company could invest into an advertisement campaign to add the recognition of the product to improve sales, it could invest into an improved machinery to acquire a higher production rate or it can dismissal workers or hire more to increase the productivity.

The first one to study this question was Kwon in 2010 in Kwon 2010. He modelled it as an optimal stopping problem and found that the optimal stopping rule is a two-sided threshold rule: invest, if the profit flow is high enough and exit if it is low enough. Between these critical thresholds the company continues its operation with the incumbent technology. Furthermore, if the investment opportunity is excercised, Kwon discovered that it is subsequently optimal to exit when the profit flow falls below a certain threshold. He also analysed the sensitivity of the thresholds and observed that although previous studies (see for example Dixit 1992 and Alvarez 2003) suggest that it would be optimal to delay an irreversible action longer when uncertainty increases, in his model there were cases where this effect is reversed.

The pioneering work by Kwon in Kwon 2010 relies on Brownian motion with negative drift as the underlying diffusion modelling the profit flow. This approach overlooks some typically used diffusions like geometric Brownian motion or mean-reverting diffusions, not to speak of potentially more exotic profit flows. Another restriction is the assumption that the volatility of the profit flow is unaffected by the investment. It seems to be more reasonable to assume that technological change affects also the stochasticity of the profit flow. For example if uncertain production technology is upgraded, then the risk exposure of the company typically changes. Furthermore it is possible that mergers and acquisitions make companies more competently protected against random shocks, i.e. their volatility decreases, since they are more adaptable to react to different market fluctuations (see Thijssen 2008). Similar kinds of shifts in the stochasticity might also occur through information revelation (cf. Grenadier 1999) in which a company acquires better knowledge of the market after observing information revelation of other companies. All in all investments that affect the volatility capture a large spectrum of economic applications (see Alvarez and Stenbacka 2004).

These restrictions raise some questions: Can one be sure that the two-sided threshold rule indeed take place beyond the simple case of Brownian motion with negative drifts? Is the stopping rule still similar, if the volatility is considerably changed after the investment? The aim of this paper is to show that the answer to both of these questions is positive under certain mild assumptions. We generalise the results in Kwon 2010 to concern the above-mentioned widely used stochastic processes. To this end the problem is approached with a general linear Itô diffusion with different drifts and volatilities before and after the possible investment and with arbitrary increasing revenue functions.

Our study is, to some degree, also related to entry and exit studies. In classical entry and exit studies (see for example Dixit 1989 and Duckworth and Zervos 2001) the company follows a given fluctuating price process and is either ”active” and receives a positive revenue flow based on the price at the moment, or ”inactive” resulting to a zero or constant outcome. Furthermore, the company is allowed to make costly switchings between these two states. In the more general problem adjustments, often called switching problems, there are not necessary two, but many different possible states, each with different revenue flow and again the company can make costly switchings between these states. Moreover these problems have typically only one or another of the following properties: There exists an irrevocable exit state (for example Zervos 2003) or the switch changes the underlying stochasticity (for example Brekke and Øksendal 1994, Vath and Pham 2007, and Egami and Bayraktar 2010). Whereas in our model both of these are present; the exit state is a final absorbing state and the underlying stochastic diffusion changes when investing, or switching. We emphasise that although our model has a more complex starting point for a study here, we do not have the richness of reversible actions or many switching opportunities as is the case in the typical switching problem. Our study can merely be seen as an irreversible one-step switching problem with discretionary exit option. We have not given up the reversibility and arbitrary many switching possibilities for nothing though. Most switching studies find sufficient conditions for a general problem, but for the explicit solution the needed assumptions are tighten up. In this paper, by focusing on one switch, we find an unique explicit solution subject to relatively weak assumptions.

We will see that with our standing assumptions the optimal rule to invest or to exit is a two-sided threshold rule, which is a fairly expected result: It is optimal to invest, if the profit flow exceeds an optimal investment threshold, exit if it falls below another threshold and continue operation if the profit flow is between these thresholds. Due to the non-linearity and complexity of the solution, the sensitivity analysis is somewhat out of the scope of this study, and it is done only in a numerical example in §6.

The main results of this paper are the existence and uniqueness of a well defined two-sided threshold rule. There are several approaches for achieving this target, when underlying dynamics are continuous diffusions as here. One very common approach is the use of variational inequalities (see for example Brekke and Øksendal 1991, Øksendal 2007, Chapter 10). These are a set of sufficient inequalities, which characterise the optimal stopping strategy together with its value. This is a general approach in the sense that it can be used in multi-dimensional problems as well. However, in one-dimensional cases it has strong differentiability requirements. This is somewhat problematic, since the value does not need to be even differentiable (see for example Øksendal and Reikvam 1998). Moreover, there exists a rich classical theory of linear diffusions and its representation theorems, but the standard use of variational inequalities do not make full use of it.

Here we instead choose another approach, the use of r-excessive mappings and the classical theory of linear diffusions. Using this approach we get a constructive method to reach the optimal exercise strategy without heavy differentiability preconditions. In addition the theory of linear diffusions enables us to prove results for very general diffusions using argumentation with relatively light complexity: we will see that in the end our main problem reduces to obtaining a solution to a pair of non-linear equations. Furthermore one can choose whether to rest the analysis on functional concavity or r-excessive majorant -argument. The former reasoning is based on the fact that after a certain transformation the value function is the smallest non-negative concave majorant for the problem (see for example Dayanik and Karatzas 2003). Here we choose to follow the latter one, which counts on the fact that the optimal value function is minimal r-excessive majorant of the exercise payoff (see for example Salminen 1984, 1985, Alvarez 2003, 2004).

Nevertheless, regardless of the used approach, proving that the two-sided threshold rule is the optimal one in the class of all stopping times is not usually problematic. To prove that there exists a unique optimal two-sided threshold rule in the class of all two-sided threshold rules is naturally a more challenging task. The methodological significance in this paper is that for this purpose we refine a fixed point argument, a technique first developed by Lempa (2010) Lempa 2010, which is based on a work by Salminen (1985) Salminen 1985. Using this argument, one can directly verify the existence of unique two-sided thresholds. An advantage of the fixed point argument is that it simultaneously results into an algorithm for finding the optimal thresholds numerically as a limit of a converging sequence. In this way we not only prove that there exists a unique two-sided threshold rule, we also identify it. Moreover this argumentation might also proved to be useful in other situations, where one tries to find unique optimal thresholds. This said, it is worth stressing that the main ambition of the study is to solve the proposed investment-exit problem and this is done by fine tuning already existing methods.

The paper is organised as follows. The problem is represented in an exact form in §2. The needed assumptions and the necessary conditions for the optimal threshold rule are laid down in §3. In §4 we will prove that the proposed two-sided threshold rule is the optimal stopping rule. This is done with the help of minimal r-excessive mappings of the underlying diffusions. In brief section §5 we shall see that if the state after the investment is only a partial improvement, we will end up either one- or three-sided threshold rule. In §6 we will illustrate our results with explicit examples. We will see what the solution looks like with geometric Brownian motion and that Kwon's (Kwon 2010) results can be derived from the model of this paper. The study is concluded in §7.

2 The optimal stopping problem

2.1 The system

Denote the complete probability space satisfying the usual conditions by $(\Omega, \mathscr{F}_t, \mathbb{P})$ and let W_t be a standard one-dimensional (F_t)-Brownian motion. Assume that the state-space $\mathscr{I} = (\alpha, \beta)$ is open subset in $\mathbb{R}$ with natural boundaries and that the underlying dynamics defined on $(\Omega, \mathscr{F}_t, \mathbb{P})$ evolve on $\mathscr{I}$ according to regular linear Itô diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathscr{I},$$

where $\mu(x)$ and $\sigma(x)$ are the drift and volatility terms respectively. For simplicity we assume that $\sigma(x) > 0$ for all $x \in \mathscr{I}$. We assume further that $\mu(x)$ and $\sigma(x)$ satisfy the condition $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(s)|}{\sigma^2(s)} ds < \infty$ for all $x \in \mathscr{I}$ and $\varepsilon > 0$, so that above-mentioned Itô diffusion has an unique weak solution (see Section 5.5.B–C inKaratzas and Shreve 1988).

Define the basic characteristics of X_t, namely the scale function S and the speed measure m as

$$S'(x) = e^{-B(x)} \quad \text{and} \quad m'(x) = \frac{2}{\sigma^2(x)} e^{B(x)},$$

where $B(x) = \int^x \frac{2\mu(y)}{\sigma^2(y)} dy$. We assume that $\mu(x)$ and $\sigma(x)$ are such that the scale function S and the speed measure m are absolutely continuous with respect to the Lebesque measure, have smooth derivatives, and that the scale function S is twice continuously differentiable. For a characterisation on linear diffusion and its basic properties, see Chapter 2 in Borodin and Salminen 2002.

We denote the differential operator associated to the controlled diffusion X_t by

$$\mathscr{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

Furthermore we will denote by ψ and ϕ, respectively, the increasing and decreasing fundamental solution of the ordinary second-order linear differential equation $\mathscr{A}u = ru$, where $r > 0$ (for a characterisation and fundamental properties of ψ and ϕ, see pages 18–20 in Borodin and Salminen 2002). The assumed boundary classification of X_t implies the following limiting properties: $\lim_{x\nearrow\beta} \psi'(x)/S'(x) = \infty$ and $\lim_{x\searrow\alpha} \phi'(x)/S'(x) = -\infty$. Finally,

$$B = \frac{\psi'(x)\phi(x)}{S'(x)} - \frac{\phi'(x)\psi(x)}{S'(x)}$$

denotes the constant Wronskian determinant of the fundamental solutions.

The problem of this study can be mathematically seen as a recursive irreversible decision-making problem with two phases. At first we are in one phase with a certain diffusion and revenue function and we can at any time irreversibly either exit or switch to another phase with different diffusion and revenue function. Following the principle of dynamic programing, we study the problem backwards and start from the second phase:

2. phase: Solve the optimal stopping problem, a pure exit problem:

$$V_2(y) = \sup_{\tau_2} \mathrm{E}_y \left[\int_0^{\tau_2} e^{-rs}\pi_2(Y_s)ds \right]. \tag{1}$$

The underlying diffusion process, a profit flow Y_t after the possible investment, is given by the Itô equation

$$dY_t = \mu_2(Y_t)dt + \sigma_2(Y_t)dW_t, \quad Y_0 = y \in \mathscr{I},$$

where W_t denotes standard Brownian motion. The constant $r > 0$ is the discount rate and the revenue function $\pi_2(x) : \mathscr{I} \to \mathbb{R}$, which denotes the profit per time unit when the system is in the state x, is a continuous and increasing $\mathscr{L}^1$-function, meaning that $\int_0^\infty e^{-rt}|\pi_2(X_t)|dt < \infty$. In addition, we assume that there exists $x_2 \in \mathscr{I}$ such that $\pi_2(x) \gtreqless 0$ for all $x \gtreqless x_2$. Let ψ_2 and ϕ_2 denote the increasing and decreasing fundamental solutions of the differential equation

$$(\mathscr{A}_2 - r)u = \frac{1}{2}\sigma_2^2(y)\frac{d^2}{dy^2}u(y) + \mu_2(y)\frac{d}{dy}u(y) - ru(y) = 0.$$

We will substitute the value function V_2 to the first phase of the problem, which is the main problem to solve.

1. phase: Solve the optimal stopping problem

$$V_1(x) = \sup_{\tau_1} \mathbb{E}_x \left[\int_0^{\tau_1} e^{-rs}\pi_1(X_s)ds + e^{-r\tau_1}\left(V_2\left(\theta(X_{\tau_1})\right) - k\right)^+ \right]. \qquad (2)$$

The underlying diffusion process, a profit flow X_t, is given by the Itô equation

$$dX_t = \mu_1(X_t)dt + \sigma_1(X_t)dW_t, \quad X_0 = x \in \mathscr{I},$$

where W_t denotes standard Brownian motion. The parameter $k > 0$ is the investment cost which is assumed to be sunk. The boost function $\theta : \mathscr{I} \to \mathscr{I}$ is twice continuously differentiable, increasing and satisfies the inequality $\theta(x) \geq x$. The function θ describes how much investment improves the productivity of the second phase. The revenue function $\pi_1 : \mathscr{I} \to \mathbb{R}$, which denotes the profit per time unit in the first phase, is a continuous and increasing $\mathscr{L}^1$-function. In addition we assume that there exists $x_1 \in \mathscr{I}$ such that $\pi_1(x) \gtreqless 0$ for all $x \gtreqless x_1$. Let ψ_1 and ϕ_1 be the increasing and decreasing fundamental solutions of the differential equation

$$(\mathscr{A}_1 - r)u = \frac{1}{2}\sigma_1^2(x)\frac{d^2}{dx^2}u(x) + \mu_1(x)\frac{d}{dx}u(x) - ru(x) = 0.$$

2.2 Solving the phase 2 and rewriting the problem

Denote $(R_r^2\pi_2)(y) = \mathbb{E}_y\left[\int_0^\infty e^{-rt}\pi_2(Y_t)dt\right]$. This is the expected cumulative present value of the flow π_2, aka resolvent function of the function π_2, taken with respect to the diffusion of the second phase. Now, the solution V_2 to the problem (1) is known from Theorem 4B in Alvarez 2001 and it reads as follows.

Theorem 2.1. *Assume that π_2 is non-decreasing; that α, the lower boundary of the state space, is natural for the diffusion X; that $\lim_{x \searrow \alpha} \pi_2(x) < 0$; and that $\lim_{x \nearrow \beta} \pi_2(x) > 0$. Then $\bar{\tau}_2 = \inf\{t \geq 0 \mid X(t) \leq \bar{x}_2\}$ is the optimal stopping time and the value function is*

$$V_2(x) = \begin{cases} (R_r^2 \pi_2)(x) - (R_r^2 \pi_2)(\bar{x}_2) \dfrac{\phi_2(x)}{\phi_2(\bar{x}_2)}, & x > \bar{x}_2 \\ 0, & x \leq \bar{x}_2, \end{cases} \tag{3}$$

where $\bar{x}_2 = \operatorname{argmin}\left\{ \dfrac{(R_r^2 \pi_2)(x)}{\phi_2(x)} \right\}$ *is the unique optimal stopping boundary.*

This implies that $V_2\big(\theta(x)\big)$ is a non-decreasing and continuous and thus also $(V_2\big(\theta(x)\big) - k)^+$ is non-decreasing and continuous function. Moreover, there exists a unique $\hat{x} > 0$ such that $V_2\big(\theta(\hat{x})\big) = k$. In other words

$$\begin{aligned} (V_2\big(\theta(x)\big) - k)^+ &= 0, \quad \text{for all } x \leq \hat{x}, \\ (V_2\big(\theta(x)\big) - k)^+ &> 0, \quad \text{for all } x > \hat{x}. \end{aligned}$$

One brief word about the boost function $\theta(x)$. We assume that when the company decides to invest the amount k, it gets an additional boost while switching to a new phase. For example, if the investment option is to advertise the products of the company, the boost function θ could be $\theta(x) = x + \zeta$, where $\zeta > 0$. Now the function θ represents the (expected) demand boost after the advertising campaign.

Since Theorem 2.1 tells us all the necessary things about the behaviour of the value function of the second phase, we need to consider only the optimal behaviour of the first phase. The first thing is to rewrite our problem (2) to a more approachable form. By applying the strong Markov-property of the diffusions, we see that it can be written as (see also (1.13) in Lamberton and Zervos 2013)

$$V_1(x) = (R_r^1 \pi_1)(x) + \sup_{\tau_1} \mathbb{E}_x \left[e^{-r\tau_1} \left[\big(V_2\big(\theta(X_{\tau_1})\big) - k\big)^+ - (R_r^1 \pi_1)(X_{\tau_1}) \right] \right],$$

where $(R_r^1 \pi_1)(x) = \mathbb{E}_x \left[\int_0^\infty e^{-rt} \pi_1(X_t) dt \right]$ is the resolvent function of the function π_1 taken with respect to the diffusion of the first phase.

To ease the notations, let $\hat{g}(x) := (V_2(\theta(x)) - k)^+ - (R_r^1\pi_1)(x)$. We notice, that $\hat{g}$ is continuous for all $x \in \mathscr{I}$ and $\hat{g}'(\hat{x}+) > \hat{g}'(\hat{x}-)$. Moreover if $x < \hat{x}$, then $\hat{g}(x) = -(R_r^1\pi_1)(x)$ and thus $\hat{g}'(x) < 0$. By the help of $\hat{g}$, our problem can be rewritten as

$$V_1(x) = (R_r^1\pi_1)(x) + \sup_{\tau_1} \mathbb{E}_x \left[e^{-r\tau_1}\hat{g}(X_{\tau_1})\right]. \tag{4}$$

Now, we have the options to exit or invest. In this model the exit option simply means that the stopping time τ_1 in (4) is such that $\left(V_2\left(\theta(X_{\tau_1})\right) - k\right)^+ = 0$, which happens when X_{τ_1} is smaller than $\hat{x}$. Likewise when X_{τ_1} is greater than $\hat{x}$, we use the option to invest.

3 Preliminaries and notations

We shall solve the problem (4) in the next section, but before that, in this section, we need to lay down our standing assumptions and prove some auxiliary results.

3.1 Assumptions and definitions

First of all two notational remarks: for convenience we often use f_x for $f(x)$. Secondly for ease of notations, we write $(R_r f)(x) := (R_r^1 f)(x)$; $\mathscr{A} := \mathscr{A}_1$; $S(x) := S_1(x)$; $m(x) := m_1(x)$; $\psi(x) := \psi_1(x)$ and $\phi(x) := \phi_1(x)$.

Along the lines of Salminen (1985) Salminen 1985 (see also Alvarez 2004 and Lempa 2010) we define functions $I : \mathscr{I} \to \mathbb{R}$ and $J : \mathscr{I} \to \mathbb{R}$ as

$$\begin{aligned} I(x) &= \frac{\psi'(x)}{S'(x)}\hat{g}(x) - \frac{\hat{g}'(x)}{S'(x)}\psi(x) = -\frac{\psi^2(x)}{S'(x)}\frac{d}{dx}\left(\frac{\hat{g}}{\psi}\right)(x); \\ J(x) &= \frac{\hat{g}'(x)}{S'(x)}\phi(x) - \frac{\phi'(x)}{S'(x)}\hat{g}(x) = \frac{\phi^2(x)}{S'(x)}\frac{d}{dx}\left(\frac{\hat{g}}{\phi}\right)(x). \end{aligned} \tag{5}$$

Now for $x < \tilde{x}$, we have $\hat{g}(x) = -(R_r\pi_1)(x)$, and hence Corollary 3.2 in Alvarez 2004 says that for $x < \tilde{x}$ the functions I and J can be written in a useful integral form

$$
\begin{aligned}
I(x) &= \frac{-\psi'(x)}{S'(x)}(R_r\pi_1)(x) + \frac{(R_r\pi_1)'(x)}{S'(x)}\psi(x) = -\int_\alpha^x \psi(t)\pi_1(t)m'(t)dt;\\
J(x) &= \frac{\varphi'(x)}{S'(x)}(R_r\pi_1)(x) - \frac{(R_r\pi_1)'(x)}{S'(x)}\varphi(x) = -\int_x^\beta \phi(t)\pi_1(t)m'(t)dt,
\end{aligned}
\tag{6}
$$

if $\lim_{x\searrow\alpha} \frac{-(R_r\pi_1)(x)}{\phi(x)} = 0$ and $\lim_{x\nearrow\beta} \frac{-(R_r\pi_1)(x)}{\psi(x)} = 0$. That these limits hold in our case follow, in turn, straight from Proposition 4 in Johnson and Zervos 2007.

Further, we can calculate a pleasant connection between the derivatives of I and J: since the functions ϕ and ψ are the solutions of the differential equations $\mathscr{A}u = ru$, we find by straight derivation that

$$
J'(x) = \phi_x\big((\mathscr{A}-r)\hat{g}_x\big)m'_x = -\frac{\phi_x}{\psi_x}I'(x). \tag{7}
$$

In the proofs to come, we are interested when the functions $\frac{\hat{g}}{\phi}$ and $\frac{\hat{g}}{\psi}$ reaches their local maximum values. For that reason, we define two points

$$
\begin{aligned}
y^* &= \operatorname{argmax}\{\big(\frac{\hat{g}}{\phi}\big)(x) \mid x < \hat{x}\},\\
w^* &= \operatorname{argmax}\{\big(\frac{\hat{g}}{\psi}\big)(x) \mid x > \hat{x}\}.
\end{aligned}
\tag{8}
$$

In other words, points y^* and w^* are such that if $y^* < \hat{x}$, then $J(y^*) = 0$ and if $\hat{x} < w^*$, then $I(w^*) = 0$.

We study Problem (4) under the following assumptions.

Assumption 3.1. (i) Assume that there exists a point $x_0 \in [w^*, \beta)$ such that $(\mathscr{A}-r)\hat{g}_x > 0$ for all $x \in (w^*, x_0)$ and $(\mathscr{A}-r)\hat{g}_x < 0$ for all $x \in (x_0, \beta)$.

(ii) Assume that $\lim_{x\nearrow\beta}\big(R_r^2\pi_2\big)\big(\theta(x)\big) - k > \lim_{x\nearrow\beta}\big(R_r^1\pi_1\big)(x)$.

(iii) For $i = 1, 2$, assume that $\pi_i \in \mathscr{L}^1$ are continuous and increasing and that there exist $x_i \in \mathscr{I}$, such that $\pi_i(x) \gtreqless 0$ for all $x \gtreqless x_i$. In addition, assume that $\theta : \mathscr{I} \to \mathscr{I}$ is twice continuously differentiable, increasing and satisfies $\theta(x) \geq x$.

First of all, we make the following simple remark on the assumption (i), which shall be referred later on.

Lemma 3.2. *Let Assumption 3.1*(i) *hold. If $w^* > \hat{x}$, then $x_0 = w^*$.*

Proof. If $w^* > \hat{x}$ (or $I(w^*) = 0$) and $x_0 > w^*$, then it follows from equations (5) and (7) that $I(x)$ is negative, or $\left(\frac{\hat{g}}{\psi}\right)'(x)$ is positive, for all $x \in (w^*, x_0)$, which contradicts the definition of the point w^*. Thus x_0 could be greater than w^* only when $w^* = \hat{x}$. $\square$

From the application point of view, Assumption 3.1(i) is not too restricting: For $x > \hat{x}$, we have $\hat{g} = V_2(x) - k - \left(R_r^1 \pi_1\right)(x)$, and $V_2 = (R_r^2 \pi_2) - \frac{(R_r^2 \pi_2)(\bar{x}_2)}{\phi_2(\bar{x}_2)} \phi_2$ can be explicitly calculated from Theorem 2.1. Moreover $(R_r^i \pi_i)(x)$, for $i = 1, 2$, are twice differentiable in the classical sense as well as $\phi_2(x)$ and thus in the applications the validity of the differential assumption 3.1(i) can be typically checked. In this study we have taken as general approach as possible and therefore this condition can hardly be relaxed. It must be checked separately in each case, though some simpler verifiable special cases exist as we shall discuss below.

According to Assumption 3.1(ii) investment is eventually profitable: if x is large enough, then the total revenue flow after the investment with the sunk cost is greater than the total revenue flow if the investment option is not used. From this we can predict, that there ought to be an upper threshold, so that we always invest if profit flow surpasses this threshold. On the other hand we know from exit studies (for example Alvarez 2001) that typically the company should exit below a certain threshold. So intuitively it seems that if the considered profit flows and functions are nice enough, then the solution should be a two-sided threshold rule: invest above certain threshold and exit below another.

The differential operator $(\mathscr{A} - r)$ operating on $\hat{g}$ can be calculated and written in the following form:

$$\begin{aligned}(\mathscr{A} - r)\hat{g}_x = &\frac{1}{2}\left(V_2''\theta'^2 + V_2'^2\theta''\right)\left(\sigma_1^2(x) - \sigma_2^2(x)\right) \\ &+ V_2'\theta'\left(\mu_1(x) - \mu_2(x)\right) + \pi_1(x) - \pi_2\left(\theta(x)\right) + kr.\end{aligned}$$

From this formulation we see that if σ_1 is considerable greater than σ_2, then it might be that $(\mathscr{A} - r)\hat{g}_x > 0$ for all $x \in \mathscr{I}$ and Assumption 3.1(i) is not valid. In that case it is never optimal to invest, since high volatility of initial diffusion might lead to very high profit flow, so it encourages the company rather to wait.

From the representation of $(\mathscr{A} - r)\hat{g}$ above we also see a simpler verifiable conditions for Assumption 3.1(i). To this end assume that θ is convex, $\sigma_1 < \sigma_2$, $\mu_1 < \mu_2$, that $\sigma_1^2 - \sigma_2^2$, $\mu_1 - \mu_2$ and $\pi_1 - \pi_2 \circ \theta$ are non-increasing,

$\pi_1(\hat{x}) - \pi_2(\theta(\hat{x})) + kr < 0$, $\pi_1 < \pi_2 \circ \theta$, $\mu_2' < r$ and that π_2 and μ_2 are convex. Assume also that such differentiability conditions are satisfied that these assumptions make sense. Then it is possible to show that the last three conditions are sufficient for the convexity of V_2 ($\mu_2' < r$ implies convexity of ϕ_2 and the other two imply the convexity of $(R_r^2 \pi_2)$). Furthermore, by the equation above it follows that $(\mathscr{A} - r)\hat{g}_x$ is negative for all $x > \hat{x}$ and thus Assumption 3.1(i) is satisfied. Although this list is long, it nevertheless demonstrates that in some cases we can check the inequality in Assumption 3.1(i) just by looking the initial functions π_1, π_2, θ, σ_i and μ_i for $i = 1,2$. For example geometric Brownian motion with $\mu_2' < r$, $\sigma_1 < \sigma_2$, $\mu_1 < \mu_2$, $\pi_2 > \pi_1$ and with θ, π_1 and π_2 as a linear function falls into this class.

3.2 The necessary conditions

The aim of the subsection is to find the necessary conditions (equation (10) below) for the existence of an optimal two-sided stopping rule in the interval $\mathscr{I}$ for the considered problem (4). Later (in §4) we will show that this two-sided threshold rule exists uniquely and is the optimal one.

Now, we know from the general theory of optimal stopping (see for example Øksendal 2007, Chapter 10) that the continuation region of the optimal stopping problem $V(x) := V_1(x) - (R_r\pi_1)(x) = \sup_\tau \mathbb{E}_x[e^{-rt}\hat{g}(X_t)]$ is $\{x \in \mathscr{I} \mid V(x) > \hat{g}(x)\}$. In the case of two-sided stopping rule this means that the continuation region is a finite interval $(a^*, b^*) \subset \mathscr{I}$ such that the r-harmonic value function $V(x)$ satisfies on the interval (a^*, b^*) the Dirichlet problem $(\mathscr{A} - r)V(x) = 0$ subject to the boundary conditions $V(a^*) = \hat{g}(a^*)$ and $V(b^*) = \hat{g}(b^*)$.

The Dirichlet problem $(\mathscr{A} - r)u(x) = 0$, $x \in (a,b)$, with boundary conditions $u(a) = \hat{g}(a)$ and $u(b) = \hat{g}(b)$ for arbitrary $\alpha < a < y^* < x_0 < b < \beta$ (for the Dirichlet problem, see for example Øksendal 2007, Chapter 9) has a unique solution

$$F(x; a,b) = \mathbb{E}_x\left[e^{-r\tau_{(a,b)}}\hat{g}(X_{\tau_{(a,b)}})\right],$$

where $\tau_{(a,b)} = \inf\{t \geq 0 \mid X_t \notin (a,b)\}$. The function F is a value function constituted by a threshold stopping rule ”stop at time $\tau_{(a,b)}$” with free boundaries

a and b. Since $X_{\tau_{(a,b)}}$ is either a or b almost surely, we find that

$$\begin{aligned}
F(x;a,b) &= \mathbb{E}_x\left[e^{-r\tau_b} \mid \tau_b < \tau_a\right]\hat{g}(b) + \mathbb{E}_x\left[e^{-r\tau_a} \mid \tau_b > \tau_a\right]\hat{g}(a) \\
&= \frac{\phi_b\hat{g}_a - \hat{g}_b\phi_a}{\psi_a\phi_b - \phi_a\psi_b}\psi_x + \frac{\hat{g}_b\psi_a - \psi_b\hat{g}_a}{\psi_a\phi_b - \phi_a\psi_b}\phi_x \\
&=: h_1(a,b)\psi(x) + h_2(a,b)\phi(x) \qquad (9)
\end{aligned}$$

for all $x \in (a,b)$. Note that $\lim_{x\searrow a} F(x;a,b) = \hat{g}(a)$ and $\lim_{x\nearrow b} F(x;a,b) = \hat{g}(b)$. In other words the value-matching condition is satisfied on both boundaries a and b.

Above we have found an r-excessive value function $F(x;a,b)$ in arbitrary finite interval (a,b). The next theorem, which is essentially Theorem 4.7 in Salminen 1985, shows us the necessary condition for the points a and b to be the boundary points of the optimal continuation region (a^*,b^*).

Theorem 3.3. *Assume that* $\frac{\hat{g}'_x}{S'_x}$, $\frac{\psi'_x}{S'_x}$ *and* $\frac{\phi'_x}{S'_x}$ *exist in* $\mathscr{I} \setminus \{\hat{x}\}$. *Then the boundary points* a^* *and* b^* *satisfy*

$$\begin{cases} I(b^*) - I(a^*) = 0 \\ J(b^*) - J(a^*) = 0. \end{cases} \qquad (10)$$

We can verify by straight calculation that if the maximizing pair (a^*,b^*) exists, then the resulting function $F(x;a^*,b^*)$ satisfies the smooth pasting conditions $\lim_{x\nearrow b^*} \frac{\partial F}{\partial x}(x;a^*,b^*) = \hat{g}'(b^*)$ and $\lim_{x\searrow a^*} \frac{\partial F}{\partial x}(x;a^*,b^*) = \hat{g}'(a^*)$. Let us calculate the limit $x \nearrow b^*$:

$$\begin{aligned}
&\left(\frac{\partial F}{\partial x}(b^*-;a^*,b^*) - \hat{g}'(b^*)\right)(\psi_{a^*}\varphi_{b^*} - \psi_{b^*}\varphi_{a^*}) \\
&= \hat{g}_{b^*}\varphi'_{b^*}\psi_{a^*} - \hat{g}_{b^*}\varphi_{a^*}\psi'_{b^*} + \hat{g}_{a^*}(\psi'_{b^*}\varphi_{b^*} - \psi_{b^*}\varphi'_{b^*}) - \hat{g}'_{b^*}\psi_{a^*}\varphi_{b^*} + \hat{g}'_{b^*}\psi_{b^*}\varphi_{a^*} \\
&= -\psi_{a^*}J(b^*)S'_{b^*} - \varphi_{a^*}I(b^*)S'_{b^*} + \hat{g}_{a^*}BS'_{b^*} \\
&= (-\psi_{a^*}J(a^*) - \varphi_{a^*}I(a^*) + \hat{g}_{a^*}B\psi_{a^*}\varphi_{b^*} - \psi_{b^*}\varphi_{a^*})\frac{S'_{b^*}}{S'_{a^*}} \\
&= \left(\hat{g}'_{a^*}\left[\psi_{a^*}\varphi_{a^*} - \psi_{a^*}\varphi_{a^*}\right] + \hat{g}_{a^*}\left[\psi_{a^*}\varphi'_{a^*} - \psi'_{a^*}\varphi_{a^*} + \psi'_{a^*}\varphi_{a^*} - \varphi'_{a^*}\psi_{a^*}\right]\right)\frac{S'_{b^*}}{S'_{a^*}} \\
&= 0.
\end{aligned}$$

The limit $x \searrow a^*$ can be calculated similarly. So we see that the smooth pasting condition is a consequence of the optimality.

3.3 Auxiliary results

The last task in this section is to show some monotonicity properties and boundary behaviour of I and J. The results are gathered in the following lemma, which will be used when we prove the optimality in §4.

Lemma 3.4. *Let Assumption* 3.1 *hold. Then*

(A) $y^* < x_1$, *where* $x_1 = \pi_1^{-1}(0)$.

(B) the functions I and J are continuous in the domain $\mathscr{I} \setminus \{\hat{x}\}$. *In addition*

(i) The function J is monotonically decreasing in the domain $\mathscr{I} \setminus [y^*, x_0] = (\alpha, y^*) \cup (x_0, \beta)$. *Furthermore* $J(x) > 0$ *for all* $x < y^*$, $J(\alpha+) = \infty$, $J(x) < 0$ *for all* $x \in (y^*, \hat{x})$ *and* $J(\beta-) \in [0, \infty)$.

(ii) The function I is monotonically increasing in the domain $\mathscr{I} \setminus [y^*, x_0] = (\alpha, y^*) \cup (x_0, \beta)$. *Furthermore* $I(x) > 0$ *for* $x \in (\alpha, y^*)$, $I(\alpha+) = 0$ *and* $I(\beta-) = \infty$.

Proof. (A) Let $\hat{J}(x) = \frac{\phi^2(x)}{S'(x)}\left(\frac{-(R_r\pi_1)}{\phi}\right)'(x)$. (The function $\hat{J}$ is such that $\hat{J}|_{(\alpha,\hat{x})} = J|_{(\alpha,\hat{x})}$.) We have assumed that $\pi_1'(x) > 0$ and that there exist x_1, such that $\pi_1(x) \lesseqqgtr 0$ for all $x \lesseqqgtr x_1$. Thus using (7) we see that

$$\hat{J}'(x) = \phi_x \pi_1(x) m_x' \lesseqqgtr 0, \text{ when } x \lesseqqgtr x_1 \text{ and so } x_1 = \operatorname{argmin}\{\hat{J}(x) \mid x \in \mathscr{I}\}.$$

Furthermore $\hat{J}(x) = -\int_x^\beta \phi_t \pi_1(t) m_t' dt$ by (6), and since $\phi, m' > 0$ we see that for x large enough, $\hat{J}_x < 0$, which especially means that $\min \hat{J}(x) = \hat{J}(x_1) < 0$ and that $\hat{J}_x < 0$ for all $x > x_1$. By the assumptions on the boundaries $\hat{J}(\alpha+) = \infty$. This together with the derivative properties gives that there exists $y < x_1$, such that $\hat{J} \gtreqqless 0$ for all $x \lesseqqgtr y$. Now if $y < \hat{x}$, then $y^* = y$ and otherwise $y^* = \hat{x}- \leq y < x_1$.

(B) By Assumption 3.1(iii) the function $\hat{g}$ is differentiable in $\mathscr{I} \setminus \{\hat{x}\}$, and thus the functions I and J are continuous in $\mathscr{I} \setminus \{\hat{x}\}$. The desired monotonicity properties follow from (7) observing that since $y^* < x_1$ (part (1)), we have $(\mathscr{A} - r)\hat{g}_x = (\mathscr{A} - r)(-R_r\pi_1)(x) = \pi_1(x) < 0$ in the interval (α, y^*) and in the interval (x_0, β) we have $(\mathscr{A} - r)\hat{g}_x < 0$ by Assumption 3.1(i).

Combining (6) with the assumption on the lower boundary yields $I(\alpha+) = 0$ and $J(\alpha+) = \infty$. Moreover according to the proof of part (1) we have $J'(x) < 0$ for all $x < y^*$ and $J(y^*) \geq 0$. Combining these facts about J we conclude that $J(x) > 0$ for all $x \in (\alpha, y^*)$. If $y^* < \hat{x}$ then the proof of part (1) immediately implies that $J(x) < 0$ for all $x \in (y^*, \hat{x})$. The positiveness of I in (α, y^*) follows from observations $I(\alpha+) = 0$ and $I'(x) > 0$, for $x \in (\alpha, y^*)$.

In order to prove limiting properties of the function I in the boundary β, fix $s > x_0$. The mean value theorem for integrals implies that

$$\begin{aligned} I(s) &= I(x_0) + \int_{x_0}^{s} I'(t)dt = I(x_0) - \int_{x_0}^{s} \psi_t\big((\mathscr{A} - r)\hat{g}_t\big)m_t' dt \\ &= I(x_0) - \frac{(\mathscr{A} - r)\hat{g}(\eta)}{r}\Big(\frac{\psi'(s)}{S'(s)} - \frac{\psi'(x_0)}{S'(x_0)}\Big), \end{aligned}$$

where $\eta \in (x_0, s)$. Thus $(\mathscr{A} - r)\hat{g}(\eta) < 0$ by Assumption 3.1(i). The desired limit $\lim_{x \nearrow \beta} I(x) = \infty$ follows now from the limiting property $\lim_{x \nearrow \beta} \frac{\psi'(x)}{S'(x)} = \infty$.

To obtain the limiting property of the function J in the boundary β, we note that combining Assumption 3.1(ii) with the boundary behaviour of ϕ_2 yields $\lim_{x \nearrow \beta} \hat{g}(x) > 0$, which in turn gives $\lim_{x \nearrow \beta} \frac{\hat{g}(x)}{\phi(x)} = \infty$. We see from Assumption 3.1(i), that $J'(x) = \phi_x\big((\mathscr{A} - r)\hat{g}_x\big)m_x' < 0$ for all $x > x_0$. If, contrary to our claim, $J(\beta-) < 0$, there would be $v > \hat{x}$ such that $J(x) = \frac{\phi^2(x)}{S'(x)}\big(\frac{\hat{g}}{\phi}\big)'(x) < 0$ for all $x > v$. This is equivalent to saying that $\big(\frac{\hat{g}}{\phi}\big)'(x) < 0$ for all $x > v$. But that would mean that $\big(\frac{\hat{g}}{\phi}\big)(v) > \big(\frac{\hat{g}}{\phi}\big)(\beta-) = \infty$, a contradiction. Thus $\infty > J(\beta-) \geq 0$. □

4 Finding the solution

In the search of a solution to (10), we will need to get an even better grip on the functions I and J. The four points in the following definition will help us greatly on this task.

Definition 4.1. Define four auxiliary functions and points as

$$\begin{aligned} I_1 &= I|_{(\alpha,y^*)} & p_1 &= I_1^{-1}\big(I_2(x_0)\big) \\ I_2 &= I|_{(x_0,\beta)} & p_2 &= J_1^{-1}\big(J_2(x_0)\big) \\ J_1 &= J|_{(\alpha,y^*)} & q_1 &= I_2^{-1}\big(I_1(y^*)\big) \\ J_2 &= J|_{(x_0,\beta)} & q_2 &= J_2^{-1}\big(J_1(y^*)\big). \end{aligned}$$

If the point $J_2^{-1}\big(J_1(y^*)\big)$ does not exist, then we define $q_2 = \beta-$.

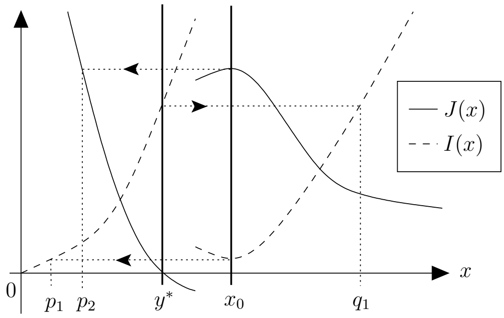

Figure 1: Definition of the points p_1, p_2, q_1 and q_2. In the figure $(\alpha,\beta) = (0,\infty)$, $x_0 > w^* = \hat{x}$, $y^* < \hat{x}$ and $q_2 = \infty$. We will see that under Assumption 3.1 we always have $p_2 > p_1$ and $q_2 > q_1$.

It will turn out that we can bound the examination to compact region: we will have $(a^*,b^*) \in [p_2,y^*] \times [x_0,q_1]$.

4.1 Existence and uniqueness of the pair (a^*,b^*)

Before proceeding to the main proposition about the existence of unique solution to (10), we need to make sure that the points in Definition 4.1 do exist and that they have certain ordering in $\mathscr{I}$. The following lemmata are proved in Appendix A.

Lemma 4.2. *Let Assumptions* 3.1 *hold. Then the points p_1, p_2, q_1 and q_2, which are defined in Definition* 4.1*, exist.*

Lemma 4.3. *Let p_1, p_2, q_1 and q_2 be as in Definition* 4.1 *and let Assumption* 3.1 *hold. Then $p_2 > p_1$ and $q_2 > q_1$.*

This ordering is vital for the fixed point argument in the main existence proposition, since we will look for the fixed point with respect to a function, which would be ill-defined, if this ordering did not hold. Now we are ready to prove our main result on solvability of the necessary condition (10).

Lemma 4.4. *Let Assumption* 3.1 *hold. Then the necessary conditions* (10) *have a unique solution* (a^*,b^*) *such that* $a^* \in (p_2,y^*)$ *and* $b^* \in (x_0,q_1)$.

Proof. We follow loosely the proof of Lemma 2.2 in Lempa 2010. Recall the definitions of I_i and J_i, for $i = 1,2$, from Definition 4.1 and define the function $K : [p_2,y^*) \to (p_2,y^*)$ by $K(x) = \left(J_1^{-1} \circ J_2 \circ I_2^{-1} \circ I_1\right)(x)$. By Lemma 3.4(B) we know that the functions I_i and J_i, $i = 1,2$, are monotonic in their domains (α,y^*) and (x_0,β) and thus we have

$$K'(x) = J_1^{-1\prime}(J_2(I_2^{-1}(I_1(x)))) \cdot J_2'(I_2^{-1}(I_1(x))) \cdot I_2^{-1\prime}(I_1(x)) \cdot I_1'(x) > 0$$

for all $x \in [p_2,y^*)$.

By invoking the inequality $q_2 > q_1$ (Lemma 4.3) and the monotonicity of the functions J_1 and J_2, we find that $p_2 = J_1^{-1}(J_2(x_0)) < J_1^{-1}(J_2(q_1)) < J_1^{-1}(J_2(q_2)) \le y^*$. Hence $K(y^*) = J_1^{-1}(J_2(q_1)) \in (p_2,y^*)$. Furthermore, we know that for some $x \in (x_0,q_1)$ we have $K(p_2) = J_1^{-1}(J_2(x))$. Therefore reasoning as above, we get $K(p_2) \in (p_2,y^*)$. Consequently, the function K is well defined and monotonically increasing in the interval $[p_2,y^*)$.

Let us create a sequence $a_n = K^n(p_2)(= (K \circ \cdots \circ K)(p_2))$. This sequence converges by induction: It is clear that $a_1 = K(p_2) > p_2$. Because K is an increasing function, we have $K\big(K(p_2)\big) > K(p_2)$. By induction $K^n(p_2) > K^{n-1}(p_2)$. Since the sequence a_n is increasing and bounded from above, it converges.

Since a_n converges, we can define $a^* = \lim_{n\to\infty} a_n$ and a^* is the fixed point of the function K. Defining $b^* = J_2^{-1}(J_1(a^*)) = I_2^{-1}(I_1(a^*))$, we get the pair (a^*,b^*) that satisfies the necessary conditions (10).

In order to prove the uniqueness, it suffices to establish that $K'(a^*) < 1$ for a given fixed point a^*. Utilizing the fixed point property $K(a^*) = a^*$ and the monotonicity properties of fundamental solutions ψ and ϕ, ordinary differentiation yields

$$K'(a^*) = \frac{\psi(a^*)}{\psi(b^*)}\frac{\phi(b^*)}{\phi(a^*)} < 1.$$

This means that whenever the curve $K(x)$ intersects the diagonal of $\mathbb{R}_+^2$, the intersection is from above. This observation completes the proof. □

In Lemma 4.4 we saw how the optimal threshold pair (a^*, b^*) can be found when it is identified as a fixed point. Analogous method was also used in Lemma 4.1 in Alvarez and Lempa 2008 on impulse control situation. In the optimal stopping problem the method was first used by Lempa in Lempa 2010, where Lempa had $y^* < \hat{x}$, $x_0 > \hat{x}$ and consequently he used the fixed point method in the case, where $p_1 = \alpha$ and $q_2 = \beta$. Here we generalise this to concern also cases where $y^* = \hat{x}$ and $x_0 = \hat{x}$, in which we needed the auxiliary points p_i and q_i, points which were not needed in Lempa's presentation.

Lemma 4.4 also shows how we can find the pair (a^*, b^*) numerically. First we identify the points y^*, x_0 and p_2. After that, we apply the function $K(x) = \left(J_1^{-1} \circ J_2 \circ I_2^{-1} \circ I_1\right)(x)$ to the point p_2 (actually any point in $[p_1, p_2]$ will do) and calculate $K^k(p_2)$, where we might for example set a stopping limit $\varepsilon > 0$ and stop at step k, when $|K^k(p_2) - K^{k-1}(p_2)| < \varepsilon$. After this we have $a^* \approx K^k(p_2)$ and $b^* \approx J_2^{-1}(J_1(a^*))$.

4.2 Proving the optimality

Now we are ready to represent our main theorem on the value and optimal stopping rule for the problem (4).

Theorem 4.5. *Let Assumption* 3.1 *hold. Then the optimal stopping time to the problem* (4) *with* $\hat{g}(x) = \left(V_2\big(\theta(x)\big) - k\right)^+ - (R_r\pi_1)(x)$ *is* $\tau^* = \inf\{t \geq 0 \mid X_t \notin (a^*, b^*)\}$ *and the value function is*

$$V_1(x) = (R_r\pi_1)(x) + \begin{cases} \hat{g}(x), & x \in (\alpha, a^*] \\ h_1(a^*, b^*)\psi(x) + h_2(a^*, b^*)\phi(x), & x \in (a^*, b^*) \\ \hat{g}(x), & x \in [b^*, \beta). \end{cases}$$

Here $a^* < y^*$ *and* $b^* > x_0$ *are the optimal stopping points found in Lemma* 4.4 *and the functions* h_1 *and* h_2 *are as defined in* (9).

Proof. Since the first term $(R_r\pi_1)(x)$ is independent of the stopping time, we need to consider only the problem

$$\sup_{\tau} \mathbb{E}\left[e^{-r\tau}\hat{g}(X_\tau)\right]. \tag{11}$$

Let the solution to (11) be $V^*(x)$, and let

$$V(x) = V_1(x) - (R_r\pi_1)(x)$$

Now $V(x) = \mathbb{E}_x\left[e^{-r\hat{\tau}}\hat{g}(X_{\hat{\tau}})\right]$, where $\hat{\tau} = \inf\{t \geq 0 \mid X_t \notin (a^*, b^*)\}$ is a stopping time. Since the supremum in (11) is taken over all stopping times, we observe that $V^*(x) \geq V(x)$.

The next step is to show that the inequality $V \geq V^*$ holds. Since the optimal value V^* is the smallest r-excessive majorant of $\hat{g}$, it is enough to show that V is an r-excessive majorant of $\hat{g}$. Firstly $V(x) \geq \hat{g}(x)$ for all $x \in (\alpha, \beta)$: In the domain $(\alpha, a^*] \cup [b^*, \beta)$ we have $V(x) = \hat{g}(x)$ and when $x \in (a^*, b^*)$, we have $V(x) = F(x, a^*, b^*) > F(x; a^*, x) = \hat{g}(x)$ by the optimality of the pair (a^*, b^*).

Let us then show that the function V is r-excessive. We see by straight calculation that $(\mathscr{A} - r)V(x) \leq 0$ for all $x \in \mathscr{I} \setminus \{a^*, b^*\}$: We have $J'(x) = \phi_x((\mathscr{A} - r)\hat{g}_x)m'_x < 0$ for all $x < a^*$ by Lemma 3.4(B). On (a^*, b^*) we have $(\mathscr{A} - r)V(x) = 0$, and lastly on (b^*, β) we have $(\mathscr{A} - r)V(x) \leq 0$ by Assumption 3.1(i). Moreover, by the definition of the functions V_2, π_1 and θ we know that the function $\hat{g}$ is twice continuously differentiable in the domain $\mathscr{I} \setminus \{\hat{x}\}$. Since $\hat{x} \in (a^*, b^*)$ and ψ and φ are twice continuously differentiable, we can deduce that $V(x)$ is twice continuously differentiable in $\mathscr{I} \setminus \{a^*, b^*\}$ and once continuously differentiable on $\mathscr{I}$. Moreover, it is clear that $|V''(a^*\pm)|, |V''(b^*\pm)| < \infty$. It follows (Theorem D.1 in Øksendal 2007) that there exist twice continuous functions f_j, $j = 1, 2, d, \ldots$, such that $f_j \to V$ uniformly on compact subsets of $\mathscr{I}$ and that $(\mathscr{A} - r)f_j \to (\mathscr{A} - r)V$ uniformly on compact subsets of $\mathscr{I} \setminus \{a^*, b^*\}$ as $j \to \infty$. Moreover, $\{(\mathscr{A} - r)f_j\}_{j=1}^{\infty}$ is locally bounded on $\mathscr{I}$. Applying Itô's theorem to the mapping $e^{-rt}f_j(x)$ and taking expectations, we have, for an arbitrary finite stopping time τ, that

$$\mathbb{E}_x\left\{e^{-r\tau}f_j(X_\tau)\right\} = f_j(x) + \mathbb{E}_x\left\{\int_0^\tau (\mathscr{A} - r)f_j(X_s)ds\right\}.$$

Letting $j \to \infty$, applying Fatou's theorem, and using the fact that $(\mathscr{A} - r)V(x) \leq 0$ for all $x \in \mathscr{I} \setminus \{a^*, b^*\}$ give

$$\mathbb{E}_x\left\{e^{-r\tau}V(X_\tau)\right\} \leq V(x)$$

proving the r-excessivity of $V(x)$. $\square$

Theorem 4.5 establish that the two-sided threshold rule ”stop at the time $\tau_{(a^*,b^*)}$” is the optimal stopping rule, among all admissible stopping rules. Following the optimal strategy, the stopping intervals $(\alpha, a^*]$ and $[b^*, \beta)$ are the exit and investment regions respectively. If we use the option to exit, we quit the market once and for all. If we use the investment option, our initial profit flow X_t changes to Y_t, with starting point $Y_0 = \theta X_{\tau^*}$, and we will exit when the new profit flow Y_t falls below the threshold $\bar{x}_2 = \text{argmin}\left\{\frac{(R_r^2\pi_2)(x)}{\phi_2(x)}\right\}$ (see Theorem 2.1).

Let us compare this strategy to the case where there is only a possibility to exit. Then the optimal strategy would be to exit below $\hat{a} = \text{argmax}\{\frac{-(R_r^1\pi_1)(x)}{\phi_1(x)}\}$ (cf. Theorem 2.1) and continue above. We recall that $y^* = \text{argmax}\{\frac{-(R_r^1\pi_1)(x)}{\phi_1(x)} \mid x < \hat{x}\}$ (Definition (8)) and so $\hat{a} \geq y^* > a^*$. In other words the investing opportunity brings more value for waiting and gaining more information before possible exit decision.

5 What if the second phase is only a partial improvement?

In this section we consider a case, where the second phase, that is the state after the investment, does not improve the first phase in all aspects. We will see that in this case, we will end up either to one- or three-sided threshold rule.

For an example what is meant by partial improvement, let us consider a car manufacturer. Assume that initially the factory produces quite cheap cars (revenue π_1) at rather fast pace (X_t). Further, let us assume that the manufacturer has an investment possibility to stop altogether manufacturing these low-cost cars and move into producing individual, almost handmade, luxury cars. Now, these luxury cars provide better outcome, meaning that $\pi_2 > \pi_1$, but they are slower to produce, and thus Y_t is expected to be smaller than X_t. The reverse case is also possible; that the company switches from manufacturing luxury cars to produce low-cost cars, so that the revenue function gets smaller and production rate higher.

To study the question attached with this kind of property we need to define a new point $z^* = \text{argmax}\{\left(\frac{\hat{g}}{\phi}\right)(x) \mid x > \hat{x}\}$. Hitherto the right boundary point of the state space, β, has been such a point. (It follows from Assumption 3.1(ii) that $\lim_{x\to\beta} \hat{g}(x) > 0$. Therefore $(\frac{\hat{g}}{\varphi})(\beta-) = \infty$ by the assumed boundary behaviour.) In the following we state our assumptions for this section.

Assumption 5.1. Assume Assumption 3.1(i) and (iii) to hold, with β replaced by z^*. Assume further that

(ii') $\lim_{x \nearrow \beta} \left(R_r^2 \pi_2\right)\left(\theta(x)\right) - k < \lim_{x \nearrow \beta} \left(R_r^1 \pi_1\right)(x)$.

Basically, the only difference to Assumption 3.1 is that the inequality in Assumption 3.1(ii) is reversed in (ii'). It means that in the long run, for x large enough, the investment is not profitable. Therefore one could predict that there ought to be an upper threshold, so that we do nothing if the profit flow is over that threshold.

The following theorem, which is the main result of the section, solves the problem (4) under Assumption 5.1. It turns out that location of the maximum point of $\frac{\hat{g}}{\phi}$ dictates whether the outcome is one- or three-sided threshold rule.

Theorem 5.2. *Let Assumption 5.1 hold.*

(A) Assume further that $\hat{x} > \operatorname{argmax}\{(\frac{\hat{g}}{\phi})(x)\}(= y^*)$. *Then, for the problem* (4), *the optimal stopping time is* $\tau = \inf\{t \geq 0 \mid X_t \leq y^*\}$, *the point* y^* *is the optimal stopping boundary and the optimal value reads as*

$$V_1(x) = \begin{cases} (R_r \pi_1)(x) + \dfrac{\hat{g}(y^*)}{\phi(y^*)}\phi(x), & x > y^* \\ 0, & x \leq y^*. \end{cases}$$

(B) Assume further that $\hat{x} < \operatorname{argmax}\{(\frac{\hat{g}}{\phi})(x)\}(= z^*)$. *Then, for the problem* (4), *the optimal stopping time is* $\tau^* = \inf\{t \geq 0 \mid X_t \in (\alpha, a^*] \cup [b^*, z^*]\}$ *and the optimal value reads as*

$$V_1(x) = (R_r \pi_1)(x) + \begin{cases} \hat{g}(x), & x \in (\alpha, a^*] \\ h_1(a^*, b^*)\psi(x) + h_2(a^*, b^*)\phi(x), & x \in (a^*, b^*) \\ \hat{g}(x), & x \in [b^*, z^*] \\ \frac{\hat{g}(z^*)}{\phi(z^*)}\phi(x), & x \in (z^*, \beta). \end{cases}$$

Here $a^* \in (\alpha, y^*)$ *and* $b^* \in (x_0, z^*)$ *are the optimal stopping boundaries found in Lemma* 4.4 *and the functions* h_1 *and* h_2 *are as defined in* (9).

Proof. (A) For the proof, see Theorem 3B in Alvarez 2001.

(B) Replacing the boundary point β by z^* in §§3–4, almost all results there are valid under Assumption 5.1, the only exceptions being that now

$0 = J(z^*) \neq J(\beta)$ and $I(z^*) \neq I(\beta)$ (the latter is not needed in this proof). Consequently, it is straightforward to go through the earlier proofs and conclude that the existence of an optimal continuation interval (a^*, b^*) on (α, z^*) and its uniqueness hold, and the claim follows as previously. □

According to previous theorem, there are two cases. If $(\frac{\hat{g}}{\varphi})(y^*) > (\frac{\hat{g}}{\varphi})(z^*)$, then we end up to the normal exit rule: exit below y^*, invest nowhere. The more interesting case, though, is when $(\frac{\hat{g}}{\varphi})(y^*) < (\frac{\hat{g}}{\varphi})(z^*)$. Then, in addition to the optimal continuation interval (a^*, b^*), there exists another continuation interval (z^*, β). In this case, the optimal stopping strategy is a *three*-sided threshold rule "stop at time $\tau_{(a^*,b^*,z^*)} = \inf\{t \geq 0 \mid X_t \in (\alpha, a^*] \cup [b^*, z^*]\}$". Here the interval $(\alpha, a^*]$ is the exit region and the finite interval $[b^*, z^*]$ is the investment region. The result is also in line with Theorem 5.2(A), which suggests that the interval $(\text{argmax}\{\frac{\hat{g}}{\varphi}\}, \beta)$ should be a continuation region.

A possible interpretation to our finding is that we may see the previous two-sided threshold rule as a special case of the three-sided threshold rule; if Assumption 3.1 hold, then $z^* = \beta-$, as mentioned above, and hence the upper continuation region (z^*, β) vanishes.

Economically the three-sided stopping region behaviour is interesting. If our profit flow is in the interval (a^*, b^*), we should wait until it surpasses b^* and then invest. However, if the profit flow is high enough (above z^*), it is again profitable to wait and invest only when the profit flow goes below z^*. This can be interpreted as an investment opportunity, which is profitable to do only when the company is, in some sense, doing badly, i.e. the profit flow is not too large.

Another interesting feature about Theorem 5.2 is that now we can also study the reverse investment opportunity. To this end, let X_t be again the initial profit flow and Y_t the profit flow after the possible investment and let Assumption 3.1 hold, so that there exists an optimal two-sided stopping rule. Consider then a reverse situation; that Y_t is the initial profit flow and X_t the profit flow after the possible investment. Then it is not too far-fetched to assume Assumption 3.1(i) &(iii) to hold. Hence we see that depending on the behaviour of $\frac{\hat{g}}{\varphi}$, it either might be optimal to sometimes reverse the investment (three- or two-sided rule) or not (one-sided rule). It follows that one could investigate more deeply the problem of costly reversible investment with optional exit utilizing Theorems 4.5 and 5.2. Unfortunately this is somewhat out of the scope of this study.

6 Examples

6.1 Geometric Brownian motion

To illustrate the main theorem (Theorem 4.5), let us consider an explicit example, where the diffusions are

$$X_t = \mu_1 X_t dt + \sigma_1 X_t dW_t, \quad X_0 = x > 0;$$
$$Y_t = \mu_2 Y_t dt + \sigma_2 Y_t dW_t.$$

Here W_t is a standard Brownian motion. Thus the diffusions are geometric Brownian motions and the state space $\mathscr{I} = (0,\infty)$. We assume that $r > \mu_2 > \mu_1$. Moreover, let the boost function be $\theta(x) = \zeta x$, where $\zeta \geq 1$ is a constant, and the revenue functions $\pi_1(x) = x - c_1$ and $\pi_2(x) = x - c_2$. With these choices our resolvent functions are $(R_r^i \pi_i)(x) = \frac{x}{r-\mu_i} - \frac{c_i}{r}$, for $i = 1, 2$.

In this case the fundamental solutions of the ordinary second order differential equation $(\mathscr{A}_1 - r)u = 0$ for the first phase are $\phi_1 = x^{\gamma_1^-}$ and $\psi_1 = x^{\gamma_1^+}$. Respectively the fundamental solutions of the differential equation $(\mathscr{A}_2 - r)u = 0$ for the second phase are $\phi_2 = x^{\gamma_2^-}$ and $\psi_2 = x^{\gamma_2^+}$. Here

$$\gamma_i^{\pm} = \frac{1}{\sigma_i^2}\left(\tfrac{1}{2}\sigma_i^2 - \mu_i \pm \sqrt{(\tfrac{1}{2}\sigma_i^2 - \mu_i)^2 + 2\sigma_i^2 r}\right)$$

are the solutions of the characteristic equation $\frac{1}{2}\sigma_i^2 \gamma_i(\gamma_i - 1) + \mu_i \gamma_i - r = 0$, for $i = 1, 2$. The functions ϕ_i are the decreasing solutions and ψ_i are the increasing ones.

6.1.1 Solving the problem

Now we can calculate that

$$\frac{(R_r^2 \pi_2)(x)}{\phi_2(x)} = \frac{x^{1-\gamma_2^-}}{r - \mu_2} - \frac{c_2 x^{-\gamma_2^-}}{r}.$$

This functional has a unique global minimum at the point $\bar{x}_2 = -\frac{\gamma_2^- c_2(r-\mu_2)}{r(1-\gamma_2^-)} < c_2$. Thus the solution to the second phase (Theorem 2.1) is

$$V_2(x) = \begin{cases} \frac{x}{r-\mu_2} - \frac{c_2}{r} + \frac{x^{\gamma_2^-}}{\bar{x}_2^{\gamma_2^-}} \cdot \frac{c_2}{r(1-\gamma_2^-)} & \text{when } x > \bar{x}_2, \\ 0 & \text{when } x \le \bar{x}_2. \end{cases}$$

The problem (4) is now $V_1(x) = (R_r^1\pi_1)(x) + \sup_{\tau_1} E_x\left[e^{-r\tau_1}\hat{g}(X_{\tau_1})\right]$, where

$$\hat{g}(x) = \left(\frac{\zeta x}{r-\mu_2} - \frac{c_2}{r} + \frac{x^{\gamma_2^-}}{\tilde{x}^{\gamma_2^-}} \cdot \frac{\zeta^{\gamma_2^-} c_2}{r(1-\gamma_2^-)} - k\right)^+ - \frac{x}{r-\mu_1} + \frac{c_1}{r}. \qquad (12)$$

With straight derivation we see that $V_2'(x) > 0$ and so there exists a unique $\hat{x} > 0$ such that $(V_2(\zeta x) - k)^+ > 0$ for all $x > \hat{x}$. Let us check that this set up will satisfy Assumption 3.1.

Firstly Assumption 3.1(i). For every $x > \hat{x}$ we have

$$\begin{aligned} (\mathscr{A}_1 - r)\hat{g}_x = \zeta^{\gamma_2^-} c_2 \frac{\frac{1}{2}\sigma_1^2\gamma_2^-(1-\gamma_2^-) + \mu_1\gamma_2^- - r}{\tilde{x}^{\gamma_2^-} r(1-\gamma_2^-)} x^{\gamma_2^-} \\ + x\left(1 - \frac{\zeta}{\frac{r-\mu_2}{r-\mu_1}}\right) + c_2 - c_1 + kr \\ =: a_1 x^{\gamma_2^-} + a_2 x + a_3. \end{aligned}$$

Here $a_1, a_3 \in \mathbb{R}$ and $a_2 < 0$. Since $\gamma_2^- < 0$, we see that this satisfy the required condition.

Secondly Assumption 3.1(ii). We see that

$$(R_r^2\pi_2)(\zeta x) - (R_r^1\pi_1)(x) - k \ge \frac{x(\mu_2-\mu_1)}{(r-\mu_2)(r-\mu_1)} + \frac{c_1-c_2}{r} - k > 0,$$

for all $x > \frac{(kr+c_2-c_1)(r-\mu_2)(r-\mu_1)}{r(\mu_2-\mu_1)} + k$. Thirdly our choices for functions π_1, π_2 and $\theta(x)$ satisfy Assumption 3.1(iii).

This set up satisfies the needed assumptions, and so by Theorem 4.5 the solution is two-sided threshold rule with optimal stopping pair (a^*, b^*). This pair exists uniquely (Lemma 4.4) and it is given by the pair of equations (10). The scale density of the geometric Brownian motion of the first phase reads as

$S_1'(x) = x^{-2\mu_1/\sigma_1^2}$, and so (10) can be written as

$$\begin{cases} b^{\gamma_1^+ + \frac{2\mu_1}{\sigma_1^2} - 1} L_b(\gamma_1^+) = -a^{\gamma_1^+ + \frac{2\mu_1}{\sigma_1^2} - 1} L_a(\gamma_1^+) \\ b^{\gamma_1^- + \frac{2\mu_1}{\sigma_1^2} - 1} L_b(\gamma_1^-) = -a^{\gamma_1^- + \frac{2\mu_1}{\sigma_1^2} - 1} L_a(\gamma_1^-), \end{cases}$$

where

$$L_b(y) = \frac{r(\zeta - 1) - \zeta\mu_1 + \mu_2}{(r - \mu_1)(r - \mu_2)} b(y-1) + y\left(\frac{c_1 - c_2}{r} - k\right) + \frac{c_2 b^{\gamma_2^-} \zeta^{\gamma_2^-} (\gamma_2^- - y)}{r\hat{x}^{\gamma_2^-} (\gamma_2^- - 1)};$$

$$L_a(y) = \frac{a(y-1)}{r - \mu_1} - \frac{c_1 y}{r}.$$

Unfortunately solving the optimal boundaries from these equations explicitly does not seem to be possible. So in the next subsection we make numerical illustrations.

6.1.2 Numerical results and the sensitivity analysis

Let us choose our parameters as $\mu_1 = 0.03$, $\mu_2 = 0.05$, $r = 0.08$, $c_1 = 2$, $c_2 = 3$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $k = 3$ and $\zeta = 1.4$. With these we get $(a^*, b^*) = (0.59, 1.54)$. In Figure 2(a) we see the value function of the problem (4) and in Figure 2(b) the function $(\mathscr{A}_1 - r)\hat{g}_x$.

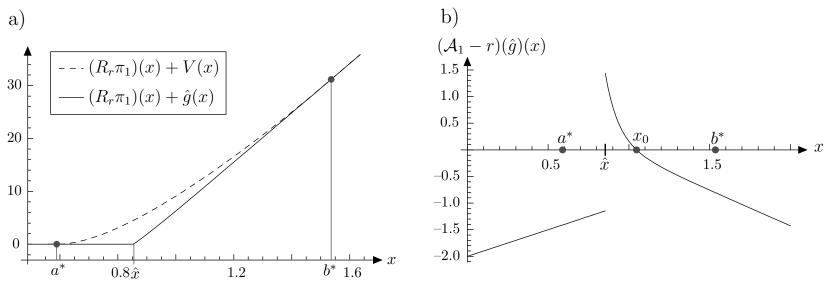

Figure 2: With made choices $y^* = \hat{x}-$, $w^* = \hat{x}+ < x_0$. (a) The solution of the problem is two-sided threshold rule. (b) The function $(\mathscr{A}_1 - r)\hat{g}(x)$.

We see that $(\mathscr{A}_1 - r)\hat{g}_x < 0$ when $x < \hat{x}$ or $x > x_0$ and $(\mathscr{A}_1 - r)\hat{g}_x > 0$, when $x \in (w^*, x_0) = (\hat{x}, x_0)$. So $\hat{g}$ behaves as we assume in Assumption 3.1.

In Figure 3 we see how the threshold alters, when we change parameters. We see expectedly that by increasing the boost effect of the investment (parameter ζ) the investment option becomes more attractive. In other words, the exercise threshold of the investment option becomes smaller (b^* diminishes) and the waiting for it to come becomes more attractive (exit threshold a^* diminishes).

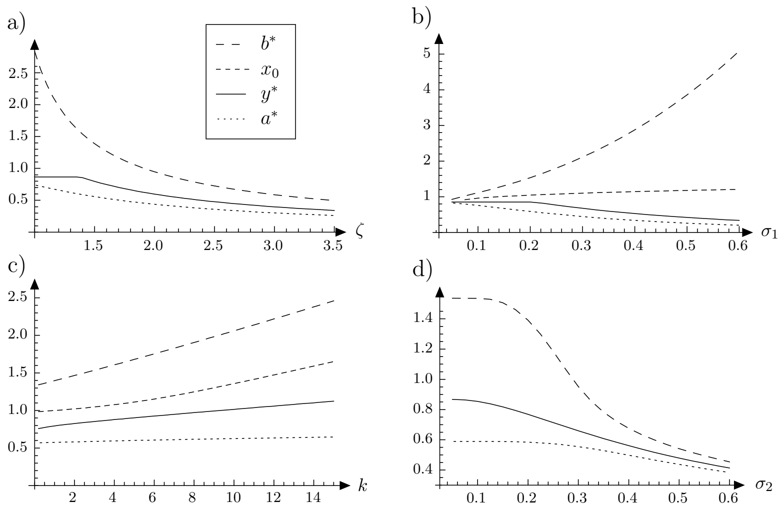

Figure 3: How the points a^*, b^* and y^* alters, when we change the parameter (a) ζ; (b) σ_1; (c) k; (d) σ_2. In figures (b) and (c) we also see the change of the point x_0.

Furthermore by increasing the volatility of the first phase (σ_1) we increase the value of the first phase and thus the investment option loses some of its attractiveness. And by increasing the sunk cost of the investment (k) we predictably make the investment opportunity less tempting and the exiting option more tempting.

Increasing the volatility of the second phase (σ_2), the value of the second phase comes greater (this is known form the previous works, see for example Dixit and Pindyck 1994). So it is sensible that we want to invest earlier (b^* diminishes), so that we may reach these greater values sooner. Also a^* diminishes, so it is more attractive to wait more information before exit.

6.2 Brownian motion with negative drift

In Kwon 2010 Kwon solves the problem (4) in the case, where the underlying diffusions were Brownian motions with negative drift. To be precise, in his paper Kwon studies a situation, where $\pi_1(x) = \pi_2(x) = x$; $\theta(x) = x + \zeta$, where $\zeta > 0$ is a constant; the diffusion of the first phase $dX_t = \mu dt + \sigma dW_t$, where $\mu < 0$ and $\sigma > 0$ are constants; the diffusion of the second phase $dY_t = (\mu + \delta)dt + \sigma dW_t$, where $\delta > 0$ is a constant such that $\mu + \delta < 0$ and the diffusion parameter σ is the same as in the first phase. The state space in this case is $\mathscr{I} = (-\infty, \infty)$. It is not difficult to show that the above mentioned setting satisfies Assumption 3.1, and consequently the theory presented in this paper can be seen as a generalisation of the results in Kwon 2010.

6.3 Same volatilities before and after the investment

In Kwon 2010 Kwon discovered that when the boost coefficient ζ is sufficiently large, at the case where underlying diffusions were Brownian motions with a drift with the same volatilities, then the investment threshold b^* decreases when σ increases. This is opposite to what normally happens when increasing the uncertainty of the future profit streams (see for example Alvarez 2003). However, we must remember that here we had the same volatility for both diffusions X_t and Y_t. We see from Figure 3(b) and (d) that if we increase the volatility of the first phase, then the investment threshold b^* increases, but if we increase the volatility of the second phase, then the investment threshold to decreases.

Suppose that the volatility is the same in both phases. There are cases when this is quite appropriate assumption. For example if one buys a new computers to an office, it is sensible to assume that they work as good as their precursors. Now, since the boost coefficient affects only the profit flow of the second phase, it could be that the adjustment made in the volatility affects more like changing the volatility of the second phase when the boost coefficient is sufficiently large. In other words, large boost coefficient emphasises the outcome of the second phase. This is illustrated in Figure 4 below with the geometric Brownian motion example from Subsection 6.1. We see that with small boost coefficient ζ, we get increasing investment boundary. More interesting is that with large boost coefficient the investment boundary is first decreasing and then increasing; a phenomenon that was not present in the case of Brownian motion with a drift.

Technically this can be explained as follows. The large boost coefficient emphasises the outcome of the second phase, and thus b^* is decreasing in σ,

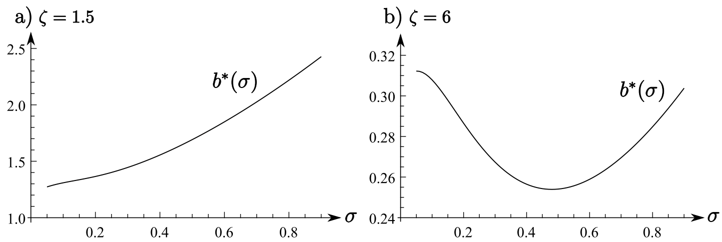

Figure 4: In figures we have the geometric Brownian motion situation of Subsection 6.1 with the same volatilities for both diffusions, i.e. $\sigma_1 = \sigma_2 =: \sigma$. a) With small values of ζ the investment boundary b^* is increasing in σ; b) With large values of ζ the boundary b^* is first decreasing and then increasing in σ. Other parameters: $r = 0.1$, $\mu_1 = 0.03$, $\mu_2 = 0.04$, $c_1 = 2.5$, $c_2 = 3$, $k = 1$.

as is predicted from Figure 3(d). But since the lower boundary of the state space, 0, is finite in the case of geometric Brownian motion, b^* cannot decrease but finitely amount. On the other hand increasing the volatility of the first phase increases b^*, and since the upper boundary is infinity, it can increase unbounded amount. It follows that with large boost, b^* first decreases, as the second phase is dominant, but after a while the first phase starts to dominate. In the case of Brownian motions, the lower boundary is unbounded, $-\infty$, and thus this kind of phenomenon did not happen. Economically this is even more interesting situation than σ-decreasing investment threshold. It means that as the risk grows, the investment opportunity is at first more preferable, but after some critical value, the investment starts to lose it attraction. So there lies some kind of trade off how much risk the investor is willing to tolerate.

Normally one would expect all the changes to be monotonic with respect to volatility, so this result reveals that the exit problem with embedded investment opportunity is highly non-linear resulting to surprising outcome.

7 Conclusion

In this paper we studied an irreversible decision-making problem, where the options either to invest or to exit were combined. We formulated the problem as two consecutive optimal stopping problems of a linear Itô diffusion. We started from relatively weak conditions and with the help of classical theory of linear diffusions and a fixed point argument we proved the existence and uniqueness of a well defined optimal solution. We saw that the solution is a two-sided threshold rule.

Given the novelty of the considered problem, there are still many interesting questions left for further research. Firstly it would be interesting to analyse the situation where there are not only one but several irreversible investment options, so that after making the investment, the company has always a new investment option to use. Then we would face a potentially infinite series of problems of the kind presented in this study. Secondly although many investments are irreversible, all of them are not. Thus one could also think how it would affect the result if the decision would be reversible, either partly or wholly. Combining these two enhancements, the problem would be a general switching problem with possibility to exit irrevocably, so that the analysis or techniques in this paper could be a stepping stone for a path to finding a solution to general switching problem. Thirdly in the present study the sensitivity analysis has been left to a minor role. It has been touched only via one numerical example. It could, however, be possible to do it explicitly and it would be interesting to know, how strongly the stopping thresholds are related to the diffusions and sunk cost. Fourthly, the investment opportunities could also have finite or stochastic time horizon, instead of infinite as here. It seems reasonable to predict that with finite time horizon the solution would still be two-sided threshold rule with lower investment threshold than in the infinite horizon case.

Acknowledgments

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A Proofs of lemmata 4.2 and 4.3

For clarity we denote

$$\begin{aligned}\Delta_I &= I(\hat{x}-) - I(\hat{x}+) = \frac{\psi(\hat{x})}{S'(\hat{x})}\big(\hat{g}'(\hat{x}+) - \hat{g}'(\hat{x}-)\big)(>0)\\ \Delta_J &= J(\hat{x}+) - J(\hat{x}-) = \frac{\phi(\hat{x})}{S'(\hat{x})}\big(\hat{g}'(\hat{x}+) - \hat{g}'(\hat{x}-)\big)(>0),\end{aligned} \tag{13}$$

so that Δ_I and Δ_J are the sizes of jumps of functions I and J at $\hat{x}$. Also during the proofs we will constantly use the fact, that if $w^* > \hat{x}$, then $x_0 = w^*$ (showed below Assumption 3.1) and the identity (7): $J'(x) = -\frac{\phi_x}{\psi_x} I'(x)$.

Proof of Lemma 4.2 We will prove that (a) $I(x_0) < I(y^*)$, and (b) $J(x_0) > J(y^*)$. Furthermore we prove that $I(\beta-) > I(y^*)$ and $J(\beta-) < J(y^*)$. Once these inequalities are proved, we know that the points p_1, p_2, q_1 and q_2 exist uniquely by invoking the results $I(\alpha+) = 0$, $J(\alpha+) = \infty$ and the monotonicity of the functions I and J (Lemma 3.4(B)). We will prove above mentioned inequalities separately.

Since $I(\beta-) = \infty$, the inequality $I(\beta-) > I(y^*)$ is always true. If $J(\beta-) > J(y^*)$, then we have $q_2 = \beta-$ by Definition 4.1. Thus we need to prove only (a) and (b).

(a): $\boldsymbol{I(x_0) < I(y^*)}$. Suppose first that $w^* > \hat{x}$. Then $x_0 = w^*$ and thus $I(x_0) = 0$. Moreover since $I(\alpha+) = 0$ and $I'(x) > 0$ for all $x < y^*$ (Lemma 3.4(B)), we have $I(y^*) > 0 = I(x_0)$.

Assume now that $w^* = \hat{x}+$. We can write $I(y^*) - I(x_0) = \big(I(y^*) - I(\hat{x}+)\big) + \big(I(\hat{x}+) - I(x_0)\big)$. Next we show that both summands are positive. Firstly $I(\hat{x}+) - I(x_0) > 0$ by the definition of the point x_0 and by the fact that $I'(x) < 0$ for all $x \in (\hat{x}+, x_0)$ (Assumption 3.1(i)).

In order to prove the positivity of $I(y^*) - I(\hat{x}+)$ we notice that since $J(y^*) \geq J(x^*-)$ (proof of Lemma 3.4(A)), we have

$$\begin{aligned}0 \leq J(y^*) - J(\hat{x}-) &= \int_{y^*}^{\hat{x}-} J'(x)dx = \int_{y^*}^{\hat{x}-} \big(-\frac{\phi_x}{\psi_x}\big) I'(x)dx\\ &= \big(-\frac{\phi_\eta}{\psi_\eta}\big)\big(I(\hat{x}-) - I(y^*)\big),\end{aligned}$$

for some $\eta \in [y^*, \hat{x})$. Hence $I(y^*) - I(\hat{x}+) = \Delta_I - (I(\hat{x}-) - I(y^*)) > 0$.

(b): $\boldsymbol{J(x_0) > J(y^*)}$. Suppose first that $y^* < \hat{x}$. Then $J(y^*) = 0$ and we know that $J'(x) = \phi_x((\mathscr{A} - r)\hat{g}_x)m'_x < 0$ for all $x > x_0$ by Assumption 3.1(i). These together with $J(\beta-) \geq 0$ (Lemma 3.4(B)) implies that $J(x_0) > 0 = J(y^*)$.

Assume now that $y^* = \hat{x}-$. Then $I(x_0) - I(\hat{x}+) < \Delta_I$: If $w^* > \hat{x}$, then $I(x_0) - I(\hat{x}+) = -I(\hat{x}+)$, otherwise $I(x_0) - I(\hat{x}+) \leq 0$. Thus we can write

$$\Delta_I = \frac{\psi(\hat{x})}{S'(\hat{x})}(\hat{g}'(\hat{x}+) - \hat{g}'(\hat{x}-)) > \int_{\hat{x}+}^{x_0} I'(x)dx = -\int_{\hat{x}+}^{x_0} \frac{\psi_x}{\phi_x} J'(x)dx$$
$$= -\frac{\psi(\xi)}{\phi(\xi)} \int_{\hat{x}+}^{x_0} J'(x)dx = -\frac{\psi(\xi)}{\phi(\xi)}\big(J(x_0) - J(\hat{x}+)\big),$$

for some $\xi \in (\hat{x}, x_0]$. Multiplying by $-\dfrac{\phi(\hat{x})}{\psi(\hat{x})}$ we get $K(J(x_0) - J(\hat{x}+)) > -\Delta_J$, where $K = \frac{\psi(\xi)}{\psi(\hat{x})}\frac{\phi(\hat{x})}{\phi(\xi)} \geq 1$. Therefore $J(x_0) - J(\hat{x}-) = \Delta_J + \left(\frac{k}{k}\right)\big(J(x_0) - J(\hat{x}+)\big) > \Delta_J - \frac{1}{k}\Delta_J \geq 0$.

Proof of Lemma 4.3 (1) Let us first prove the case $p_2 > p_1$. Suppose first that $w^* > \hat{x}$. Then $x_0 = w^*$ and $I(x_0) = 0$. Moreover since $I(\alpha+) = 0$ by Lemma 3.4(B), we can conclude that $p_1 = \alpha+$. Since $J(\alpha+) = \infty$ by Lemma 3.4(B), we have $p_2 > \alpha = p_1$.

In the remainder of this proof we assume that $w^* = \hat{x}+$, which implies that $I(\hat{x}+) \geq I(x_0)$ by the definition of the point x_0. By their definition (Definition 4.1), the points p_1 and p_2 satisfy $I(p_1) = I(x_0)$ and $J(p_2) = J(x_0)$. Adding additional terms in these equalities we see that p_1 and p_2 satisfy

$$\begin{aligned} \int_{p_1}^{\hat{x}-} I'_x dx &= \quad \Delta_I + I(\hat{x}+) - I(x_0); \\ \int_{p_2}^{\hat{x}-} J'_x dx &= -\Delta_J + J(\hat{x}+) - J(x_0). \end{aligned} \tag{14}$$

We now divide the proof into two cases according to the sign of $\int_{p_2}^{\hat{x}-} I'_x dx$.

1° $\int_{p_2}^{\hat{x}-} I'_x dx > 0$. Using the monotonicity of ϕ/ψ and the fact that $\int_{\hat{x}+}^{x_0} I'_x dx \leq 0$, we know that

$$\Delta_J = \frac{\phi(\hat{x})}{S'(\hat{x})}\big(\hat{g}'(\hat{x}+) - \hat{g}'(\hat{x}-)\big) = -\int_{p_2}^{\hat{x}-} J'_x dx - \int_{\hat{x}+}^{x_0} J'_x dx$$
$$= \int_{p_2}^{\hat{x}-} \frac{\phi(x)}{\psi(x)} I'_x dx + \int_{\hat{x}+}^{x_0} \frac{\phi(x)}{\psi(x)} I'_x dx > \frac{\phi(\hat{x})}{\psi(\hat{x})}\Big(\int_{p_2}^{\hat{x}-} I'_x dx + \int_{\hat{x}+}^{x_0} I'_x dx\Big).$$

Multiplying this expression by $\dfrac{\psi(\hat{x})}{\phi(\hat{x})}$ we get

$$\int_{p_2}^{\hat{x}-} I'_x dx + \int_{\hat{x}+}^{x_0} I'_x dx < \frac{\psi(\hat{x})}{S'(\hat{x})}\left(\hat{g}'(\hat{x}+) - \hat{g}'(\hat{x}-)\right) = \Delta_I = \int_{p_1}^{\hat{x}-} I'_x dx + \int_{\hat{x}+}^{x_0} I'_x dx.$$

In other words $I(p_2) > I(p_1)$. Since $I'(x) > 0$ for all $x < y^*$ (Lemma 3.4(B)), we have $p_2 > p_1$.

2° $\int_{p_2}^{\hat{x}-} I'_x dx \leq 0$. This condition means that $I(\hat{x}-) \leq I(p_2)$. However, it is true that $I(x_0) \leq I(\hat{x}+) < I(\hat{x}-)$ and so $I(x_0) < I(p_2)$. By the definition of the point p_1, we have $I(p_1) = I(x_0) < I(p_2)$. The desired result $p_2 > p_1$ follows now from the monotonicity of the function I in the interval (α, y).

(2) The proof of $q_2 > q_1$ is analogously to part (1). Now we just use the fact that $(\mathscr{A} - r)\hat{g}_x$ is negative for all $x \in (x_0, \beta)$ and that it is positive for all $x \in (w^*, x_0)$. In addition we need the facts that $\infty = I(\beta-) > I(y^*)$ and that $w^* < \beta$.

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ORIGINAL PAPER

On solvability of a two-sided singular control problem

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Abstract We study a two-sided singular control problem in a general linear diffusion setting and provide a set of conditions under which an optimal control exists uniquely and is of singular control type. Moreover, under these conditions the associated value function can be written in a quasi-explicit form. Furthermore, we investigate comparative static properties of the solution with respect to the volatility and control parameters. Lastly we illustrate the results with two explicit examples.

Keywords Singular stochastic control · Two-sided control · Linear diffusion

Mathematics Subject Classification (1991) 93E20 · 60J60

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t \mid t < \infty\}$ a right continuous, completed filtration. Consider the controlled process $Z_t = X_t + U_t - D_t$ where X_t is a general, linear time homogeneous Itô diffusion on $\mathbb{R}_+ := (0, \infty)$ and (U_t, D_t) is a pair of $\mathbb{F}$-adapted, non-decreasing cádlág processes on $\mathbb{R}_+$. We consider the one-dimensional two-sided singular, or reflecting, control problem

$$\sup_{(U_t, D_t)} \mathbb{E}_x \left\{ \int_0^{\zeta_Z} e^{-rs}\pi(Z_s)ds + p \int_0^{\zeta_Z} e^{-rs} dD_s - q \int_0^{\zeta_Z} e^{-rs} dU_s \right\},$$

where $\pi : \mathbb{R}_+ \to \mathbb{R}$ is a revenue function satisfying suitable conditions (given in Sect. 3), $r > 0$ and $q, p \in \mathbb{R}$, $q > p$, are exogenously given constants, $\zeta_Z = \inf\{t \geq$

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$0 \mid Z_t \notin \mathbb{R}_+\}$ denotes the first exit time from $\mathbb{R}_+$, and the supremum is taken over all admissible controls.

In this study we give sufficient conditions under which the above mentioned problem has a unique two-sided reflecting control as an optimal control. Moreover, under the same conditions, we see that the value function can be written in a (quasi-)explicit form. Further, since we can identify the value function and control boundaries explicitly, we are also able to investigate the comparative static properties of the value function with respect to the volatility and the control coefficients p and q.

Since the pioneering work by Bather and Chernoff (1966) appeared, singular stochastic control problems have been subjected to extensive investigation due to their applicability in various fields. These fields include for example a costly reversible investment problem, or an irreversible one, depending whether $U_t \equiv 0$ or not. In these problems the investor has a chance to purchase capital at price q and sell it with lower price $p < q$. In different specific forms the irreversible case is studied for example in Kobila (1993), Oksendal (2000), Chiarolla and Haussmann (2005) and the costly reversible case in Abel and Eberly (1996), Guo and Pham (2005), Alvarez (2011). Another example is an optimal dividend payments problem combined to obligative reinvestment (see Sethi and Taksar 2002; Paulsen 2008). The company pays dividends to the owners at rate p and on the other hand, the owners are obliged to reinvest if the value of the income process becomes too small. Without the reinvestment possibility, the dividend payments problem has been studied for example in Asmussen and Taksar (2006), Højgaard and Taksar (1999), Alvarez and Virtanen (2006). Further applications include, for example, rational harvesting (see e.g. Lande et al. 1995; Lungu and Oksendal 1997; Alvarez 2000; Alvarez and Koskela 2007), monotone fuel follower problem (Chow et al. 1985; Jacka 2002; Bank 2005), exchange rates (Mundaca and Oksendal 1998), inventory theory (Harrison and Taksar 1983) and controlling a dam (Faddy 1974).

Singular stochastic control problems can be approached in different ways. The one used also in this study is based on the theory of partial differential equations and on variational arguments. In this approach one typically first constructs (by ad hoc methods) a solution to some necessary (e.g. Hamilton–Jacobi–Bellman) conditions and then validates the optimality of the solution by a verification theorem (see Karatzas 1983; Shreve et al. 1984; Chow et al. 1985; Bayraktar 2008; Alvarez and Lempa 2008). Alternatively, it is also possible to rely on probabilistic methods. In Karatzas and Shreve (1984), Karatzas (1985a), and Karatzas and Wang (2001) the existence of an optimal control was proved by showing, leaning on a weak compactness argument, that the optimizing sequence of the considered problem converges to an admissible control. These two approaches could be classified as direct techniques, as the problem is approached straightforwardly. In contrast to this, in an indirect approach the control problem is showed to be equivalent with other type of problem and the latter one is then solved. For example in recent studies (Guo and Tomecek 2008a,b) the authors reveal one-to-one correspondence between a singular control and a switching problem. They then go on to use this relation in a general multidimensional case to find an integral representation for the value function and, moreover, sufficient conditions for the existence of an optimal control.

Although singular control problems have attained lots of attention in general, theory considering two-sided controls is not yet as vast as the theory of one-sided controls. There are some general existence results for a two-sided control problem, e.g. Shreve et al. (1984), Sethi and Taksar (2002), Guo and Tomecek (2008b), and Paulsen (2008), which provide sufficient conditions or verification theorems for the solution in a general diffusion setting. In this paper we also follow this path and give rather easily verifiable sufficient conditions for the optimality, but in addition we can also give a (quasi-) explicit form for the value function. To accomplish this task, we have chosen to combine some existing techniques (from Harrison 1985; Shreve et al. 1984; Alvarez 2008; Lempa 2010) in appropriate way with the classical theory of linear diffusions and r-excessive mappings.

More specifically, we formulate the problem in exact terms in Sect. 2, after which we derive necessary first order optimality conditions for the two-sided singular control in Sect. 3. In Sect. 4, we present our first result, leaning on techniques from Harrison (1985) and Shreve et al. (1984). We prove that if the derived necessary optimality conditions attain a solution, then under a set of weak assumptions this solution is unique and the associated reflecting control is the optimal one among all admissible controls. In Sect. 5 we will find sufficient assumptions under which the above mentioned first order optimality conditions obtain a solution, after which it follows from the first result that this solution must be unique. The solution to the optimality conditions is found by using a fixed point argument, originating from Alvarez and Lempa (2008), and Lempa (2010), which results directly into the verification of the existence of the optimal exercise thresholds. An advantage of this approach is that it simultaneously results into an algorithm for finding the optimal thresholds numerically as a limit of a converging sequence.

The most important results are presented in Sect. 6, where we consider the comparative static properties of the value function. Previously this kind of examination has been done with one-sided controls (e.g. Alvarez 2001), but the author is not aware of similar treatment concerning a general two-sided control problem. We show that the same set of sufficient assumptions as above guarantees that the value function is unambiguously decreasing with respect to the volatility. This in turn decelerates the usage of optimal controls by expanding the inactivity region where exerting the optimal policy is suboptimal. These findings are in line with the previous literature concerning one-sided policies, see e.g. Alvarez (2001). We also demonstrate the sensitiveness with respect to the control parameters, and in particular that the one-sided control problem can be attained as a special case of this two-sided problem when $p \to 0$ or $q \to \infty$. Lastly, we will illustrate our results with two explicit examples in Sect. 7.

2 Problem formulation

2.1 The underlying dynamics

Let $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ be a complete filtered probability space satisfying the usual conditions (see Borodin and Salminen 2002, p. 2). We assume that the regular linear diffusion process X_t is defined on $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ and evolves on $\mathbb{R}_+$ according to the dynamics described by the Itô stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \quad X_0 = x, \tag{1}$$

where W_t denotes a standard Brownian motion. We assume that both the drift coefficient $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and the volatility coefficient $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are once continuously differentiable and that $\sigma(x) > 0$ for all $x \in (0, \infty)$. These conditions are sufficient for the existence of a weak solution for the stochastic differential equation (1) (cf. (Karatzas and Shreve, 1988, Section 5.5.B–C)). Moreover, we assume that the boundary ∞ is unattainable (i.e. natural or entrance-not-exit) for the process X_t and that the boundary 0 can, in addition to being unattainable, be also attainable (i.e. exit or regular), and that whenever 0 is regular we assume that it is killing. Further, if 0 is attainable, we assume in addition that the condition $\mu(0+) \leq 0$ holds. It is also worth mentioning here that the assumption that the state space is $\mathbb{R}_+$ is for notational convenience.

We define the differential operator associated to the underlying diffusion process as

$$\mathcal{A} = \tfrac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

Let us denote, respectively, by ψ and φ the increasing and decreasing fundamental solution of the ordinary differential equation $(\mathcal{A} - r)u = 0$, where $r > 0$ is the discount coefficient (for a complete characterization and basic properties of these minimal r-excessive functions, see Borodin and Salminen 2002, pp. 18–20). We know that

$$BS'(x) = \psi'(x)\varphi(x) - \varphi'(x)\psi(x), \tag{2}$$

where B is the constant Wronskian of the fundamental solutions ψ and φ and

$$S'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right)$$

is the density of the scale function of X_t.

We denote by $\mathcal{L}^1$ the class of measurable mappings $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying the absolute integrability condition $\mathbb{E}_x \int_0^\infty e^{-rs}|f(X_s)|ds < \infty$. For all $f \in \mathcal{L}^1$ write

$$(R_r f)(x) = \mathbb{E}_x \int_0^\infty e^{-rs} f(X_s)ds$$

for the expected cumulative present value of a flow f. It is known from the literature on linear diffusion (e.g. Oksendal 2000, Proposition 4.3) that $(R_r f)(x)$ can be also re-expressed as

$$(R_r f)(x) = B^{-1}\varphi(x) \int_0^x \psi(y) f(y) m'(y) dy$$
$$+B^{-1}\psi(x) \int_x^\infty \varphi(y) f(y) m'(y) dy, \tag{3}$$

where $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the density of the speed measure of X_t.

2.2 The control and the problem

An admissible control policy is defined as a pair of processes (U_t, D_t) such that both processes are non-negative, non-decreasing, right-continuous, and $\{\mathcal{F}_t\}$-adapted. With admissible control (U_t, D_t), we define the associated controlled process $Z_t = X_t + U_t - D_t$. We associate a unit price p to the downward control D_t and a unit cost $-q$ to the upward control U_t. For example, in a timber harvesting example, D_t represents the cumulative harvest while U_t can be interpreted as the cumulative replanting. In capital theoretic or natural resource management applications of singular stochastic control, the unit price p is typically positive and the unit cost $-q$ is negative. However, there are cases where we may want to use negative values of p as well. For example if we consider controlling a boat in a stormy sea, with the controls as steering left and right, then it is sensible that both of these controls are costly, and so $p < 0$. So in order to grasp the most general aspect of the problem, we only assume $q > p$ without specifying their signs (the opposite inequality would lead easily to an infinite value function).

For an admissible control (U, D) our payoff function gets the form

$$H^{(U,D)}(x) = \mathbb{E}_x \left[\int_0^{\zeta_Z} e^{-rs}(\pi(Z_s)ds + pdD_s - qdU_s) \right], \tag{4}$$

where $\zeta_Z = \inf\{t \geq 0 : Z_t \notin \mathbb{R}_+\}$ denotes the first exit time of the controlled diffusion from its state space and $\pi : \mathbb{R}_+ \to \mathbb{R}$ captures the state dependent cash flow accrued from continuing operation, or it can be also interpreted as an utility function of the controller. Our objective is to solve the problem

$$V(x) = \sup_{(U,D)} H^{(U,D)}(x), \tag{5}$$

where the supremum is taken over all admissible policies (U_t, D_t). Our purpose is to delineate a set of fairly general assumptions under which there exists a well-defined and unique two-sided reflecting control policy for which the supremum (5) is attained.

3 Assumptions and preliminary results

3.1 Barrier policy and associated value function

For two arbitrary barriers z and y satisfying the inequality $0 < z < y < \infty$, we focus on barrier policies which maintain the state between these two barriers at all times. For given boundaries (z, y) we denote the exerted barrier policies, or reflecting controls, as U^z and D^y. If the initial state of the controlled process is between the boundaries, then the barrier policy (U^z, D^y) is obtained by assigning to the X_t the two-sided regulator so that U^z and D^y are continuous and increase only when $Z = z$ and $Z = y$, respectively. Thus, for $x \in (z, y)$, the controlled process evolves according to the diffusion X_t reflected at the boundaries z and y. If $x > y$, then we take $D_0^y = x - y$ resulting into an instantaneous gain $p(x - y)$ and apply the above mentioned regulator to $X - D_0^y$ from thereon. Similarly if $x < z$, we exert the policy $U_0^z = z - x$ resulting into the instantaneous cost $-q(z - x)$ and apply the regulator to $X + U_0^z$ from thereon. We shall see that the optimal control is of this class.

Next we shall write down the associated value function using the following application of Ito's lemma (cf. Harrison 1985, Corollary 5.2.4).

Lemma 3.1 *Let f be a twice continuously differentiable function. Fix $z < x < y$ and consider the barrier policy (U^z, D^y). Then*

$$f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-rs}\left[(r - \mathcal{A})f(Z_s)ds + f'(y)dD_s^y - f'(z)dU_s^z\right]\right].$$

Proof By (generalised) Ito's lemma

$$\begin{aligned} e^{-rt}f(Z_t) &= f(Z_0) + \int_0^t e^{-rs}df(Z_s) - r\int_0^t e^{-rs}f(Z_s)ds \\ &= f(x) + M_t + \int_0^t e^{-rs}\left[(\mathcal{A} - r)f(Z_s)ds - f'(y)dD_s^y + f'(z)dU_s^z\right], \end{aligned} \tag{6}$$

where $M_t = \int_0^t e^{-rs}\sigma(Z_s)f'(Z_s)dW_s$. Since $z < Z_s < y$ for all $s > 0$, we see that both $f(Z_s)$ and $f'(Z_s)$ are bounded and so $\lim_{t\to\infty} e^{-rt}f(Z_t) = 0$ and $\mathbb{E}_x\{M_t\} = 0$. Therefore the claim follows by taking expectation of both sides in (6) and letting $t \to \infty$. □

Fix barriers z and y, let π be an integrable and once continuously differentiable function, and let $H^{(z,y)}$ be the value function associated to the barrier policy (U^z, D^y). For $z < x < y$ we have, by definition,

$$H^{(z,y)}(x) = \mathbb{E}_x \left[\int_0^\infty e^{-rs} \left(\pi(Z_s)ds + pdD_s^y - qdU_s^z \right) \right]. \tag{7}$$

Consider now the function $f(x) = (R_r\pi)(x) + c_1\psi(x) + c_2\varphi(x)$, where $c_1 = c_1(z, y)$ and $c_2 = c_2(z, y)$ are such that $f'(z) = q$ and $f'(y) = p$. This is a twice continuously differentiable function and consequently, by the lemma above, for $x \in (z, y)$, we have

$$f(x) = \mathbb{E}_x \left[\int_0^\infty e^{-rs} \left[\pi(Z_s)ds + pdD_s^y - qdU_s^z \right] \right].$$

Comparing this to (7) we see that, for $x \in (z, y)$, we must have $H^{(z,y)}(x) = f(x)$. Furthermore, it is clear from the definition of barrier policy rule that for $x \geq y$ we have $H^{(z,y)}(x) = p(x-y) + H^{(z,y)}(y)$, and similarly for $x \leq z$ we have $H^{(z,y)}(x) = q(x-z) + H^{(z,y)}(z)$. Hence the proposed class of considered barrier policies (U^z, D^y) leads to the value function

$$H^{(z,y)}(x) = \begin{cases} p(x-y) + H^{(z,y)}(y) & x \geq y, \\ (R_r\pi)(x) + c_1(z, y)\psi(x) + c_2(z, y)\varphi(x) & z < x < y, \\ q(x-z) + H^{(z,y)}(z) & x \leq z, \end{cases} \tag{8}$$

where the z and y-dependent factors c_1 and c_2 are such that

$$\begin{cases} (R_r\pi)'(y) + c_1(z, y)\psi'(y) + c_2(z, y)\varphi'(y) = p, \\ (R_r\pi)'(z) + c_1(z, y)\psi'(z) + c_2(z, y)\varphi'(z) = q. \end{cases}$$

Notice that the value function $H^{(z,y)}$ is once continuously differentiable for all barriers $z < y$.

3.2 The first order optimality conditions

A necessary first order condition for a pair (z, y) to be optimal is that $\frac{dc_1}{dz} = \frac{dc_1}{dy} = 0 = \frac{dc_2}{dz} = \frac{dc_2}{dy}$. Carrying out the computations we see that these conditions are, in fact, equivalent to the smooth pasting requirement that the second derivative of $H^{(z,y)}$ vanishes at z and y, i.e. the requirement that $H^{(z,y)}$ is twice continuously differentiable everywhere. After performing the differentiations, our necessary optimality conditions for the two-sided threshold (z^*, y^*) can be written as

$$\begin{cases} J_q(z^*) - J_p(y^*) = 0, \\ I_q(z^*) - I_p(y^*) = 0, \end{cases} \tag{9}$$

where, for $b = p, q$,

$$J_b(x) := \frac{\big((R_r\pi)'(x) - b\big)\varphi''(x) - (R_r\pi)''(x)\varphi'(x)}{\psi'(x)\varphi''(x) - \varphi'(x)\psi''(x)}$$
$$\text{and} \quad I_b(x) := \frac{\big((R_r\pi)'(x) - b\big)\psi''(x) - (R_r\pi)''(x)\psi'(x)}{\psi'(x)\varphi''(x) - \varphi'(x)\psi''(x)}. \tag{10}$$

If the pair of equations (9) is solvable, then the factors c_1 and c_2 are $-J_q(z^*)$ and $I_q(z^*)$ respectively. Furthermore, provided that sufficient differentiability conditions hold, we get by straight differentiation, and using the harmonicity of $(R_r\pi)$, ψ and φ, that for $b = p, q$

$$J_b'(x) = \frac{\varphi'(x)\big(\pi'(x) + b(\mu'(x) - r)\big)}{rBS'(x)} = \frac{\varphi'(x)\rho_b'(x)}{rBS'(x)}$$
$$\text{and} \quad I_b'(x) = \frac{\psi'(x)\big(\pi'(x) + b(\mu'(x) - r)\big)}{rBS'(x)} = \frac{\psi'(x)\rho_b'(x)}{rBS'(x)}, \tag{11}$$

where $\rho_b(x) = \pi(x) + b(\mu(x) - rx)$.

3.3 Assumptions and auxiliary results

The assumptions presented here are needed to show that the solution is unique and of two-sided reflecting control type. So, throughout the study we will make the following assumptions.

Assumption 3.2 For $b \in [p, q]$, denote $\rho_b(x) := \pi(x) + b(\mu(x) - rx)$. Assume that

(i) $q > p$,
(ii) $\mu(x), \pi(x), \sigma(x) \in C^1(\mathbb{R}_+)$ and $\pi(x), \mu(x), x \in \mathcal{L}^1$,
(iii) $\mu'(x) < r$, and if 0 is attainable, then in addition $\mu(0+) \leq 0$ (these imply that ψ and φ are convex, see Lemma 3.3 below),
(iv) for every $b \in [p, q]$, there is $\tilde{x}_b \in \mathbb{R}_+$ such that $\frac{d}{dx}\rho_b(x) \gtreqqless 0$ whenever $x \lesseqqgtr \tilde{x}_b$.

Let us make a few remarks on Assumption 3.2. First the differentiability conditions for π in Assumption (ii) could be relaxed, but it would complicate matters without gaining any relevant extra insight.

Assumption (iii) seems a little restricting, but it is justified; in the opposite case ($\mu' > r$) we would easily end up to an infinite value function, implying an ill-posed problem setting. Moreover, often μ is assumed to be Lipschitz continuous, i.e. that for some $C > 0$ we have $\mu' < C$, and hence Assumption (iii) may be seen merely setting an upper bound for the Lipschitz constant. One could try to relax this assumption by assuming that $\mu' > r$ in some bounded subset of $\mathbb{R}_+$, but that would complicate the analysis and possibly lead to a peculiar behaviour (see e.g. Example 5.3 in Shreve et al. 1984).

The three first assumptions are more or less standard assumptions, setting no strict restrictions for the problem. It turns out that the last quasi-concavity assumption (iv), the only restraining assumption needed, is enough to ensure the uniqueness of a well-defined solution (cf. Proposition 4.2 and Theorem 4.4). The function $\rho_b(x)$ itself, for $b = p, q$, can be seen (cf. Alvarez and Lempa 2008) to measure the expected net return from postponing the dividend payments (or reinvestments, depending whether $b = p$ or q) into the future instead of paying out the dividends (or reinvesting) instantaneously.

We close this section by revealing vital monotonicity properties, which shall be used later on several times.

Lemma 3.3 *(A) Let Assumption 3.2 (iii) hold and assume that $x \in \mathcal{L}^1$. Then ψ and φ are convex functions.*

(B) Let Assumption 3.2 hold. Then

(1) for $b = p, q$, $\frac{d}{dx} J_b(x) \lesseqqgtr 0$, whenever $x \lesseqqgtr \tilde{x}_b$. In addition $J_p(x) > J_q(x)$ for all $x \in \mathbb{R}_+$.

(2) for $b = p, q$, $\frac{d}{dx} I_b(x) \gtreqqless 0$, whenever $x \lesseqqgtr \tilde{x}_b$. In addition $I_p(x) > I_q(x)$ for all $x \in \mathbb{R}_+$.

Proof See Appendix A.1. □

4 Uniqueness and optimality of the two-sided reflecting control

4.1 Uniqueness of (z^*, y^*)

Before proving the main proposition about the uniqueness of the solution of (9) we will show that we can restrict the examination to two disjoint sets on positive real line.

Lemma 4.1 *Let Assumption 3.2 hold. Assume further that the necessary condition (9) has a solution (z^*, y^*). Then $(z^*, y^*) \in (0, \tilde{x}_q) \times (\tilde{x}_p, \infty)$, where $\tilde{x}_q \leq \tilde{x}_p$ are as in Assumption 3.2(iv).*

Proof To see that the inequality $\tilde{x}_q \leq \tilde{x}_p$ holds, set $x < \tilde{x}_p$. Then

$$\rho_q'(x) = \pi'(x) + q(\mu'(x) - r) \leq \pi'(x) + p(\mu'(x) - r) = \rho_p'(x) \leq 0$$

by Assumption 3.2(iii) and (i). Thus by Assumption 3.2(iv) we must have $\tilde{x}_q \leq \tilde{x}_p$.

The rest of the proof follows that of Alvarez (2008, Theorem 4.3). For a fixed $y \in \mathbb{R}_+$, consider the functional

$$L_1^y(z) = J_q(z) - J_p(y).$$

By Lemma 3.3(B) we know that $L_1^y(y) < 0$ and that $L_1^y(z)$ is z-decreasing on $(0, \tilde{x}_q)$ and z-increasing on $(\tilde{x}_q, \infty)$. Thus, if there exists a root $z_y^* \in (0, y)$ satisfying the condition $L_1^y(z_y^*) = 0$, it has to be on the interval $(0, \tilde{x}_q)$.

Analogously, for a fixed $z \in \mathbb{R}_+$, consider the functional

$$L_2^z(y) = I_q(z) - I_p(y).$$

By Lemma 3.3(B) we know that $L_2^z(z) < 0$ and that $L_2^z(y)$ is y-decreasing on $(0, \tilde{x}_p)$ and y-decreasing on $(\tilde{x}_p, \infty)$. Thus, if there exists a root $y_z^* \in (z, \infty)$ satisfying the condition $L_2^z(y_z^*) = 0$, it has to be on the interval $(\tilde{x}_p, \infty)$. □

Previous lemma narrows the possible region for the optimal thresholds. We shall use this information in next proposition, which is our main result on the uniqueness of the solution to the necessary conditions (9).

Proposition 4.2 *Let Assumption 3.2 hold. Assume further that the necessary conditions (9) have a solution (z^*, y^*). Then the pair (z^*, y^*) is unique.*

Proof Define a function $K : (0, \tilde{x}_q] \to (0, \tilde{x}_q]$ by $K(x) = \big(\hat{J}_q^{-1} \circ \hat{J}_p \circ \hat{I}_p^{-1} \circ \hat{I}_q\big)(x)$, where $\hat{J}_q = J_q|_{(0,\tilde{x}_q]}$, $\hat{J}_p = J_p|_{[\tilde{x}_p,\infty)}$, $\hat{I}_q = I_q|_{(0,\tilde{x}_q]}$ and $\hat{I}_p = I_p|_{[\tilde{x}_p,\infty)}$.

By Lemma 3.3(B) we know that the functions $\hat{J}_b$ and $\hat{I}_b$, for $b = p, q$, are monotonic in their domains $(0, \tilde{x}_q]$ and $[\tilde{x}_p, \infty)$ and therefore

$$K'(x) = \hat{J}_q^{-1\prime}(\hat{J}_p(\hat{I}_p^{-1}(\hat{I}_q(x)))) \cdot \hat{J}_p'(\hat{I}_p^{-1}(\hat{I}_q(x))) \cdot \hat{I}_p^{-1\prime}(\hat{I}_q(x)) \cdot \hat{I}_q'(x) > 0,$$

for all $x \in (0, \tilde{x}_q)$ and thus K is monotonically increasing.

Moreover, we see at once that if there exists a pair (z^*, y^*) satisfying the necessary conditions (9), then z^* must be a fixed point for K, that is $K(z^*) = z^*$. In order to prove the uniqueness, it suffices to establish that $K'(z^*) < 1$ for any given fixed point z^*. Utilizing the fixed point property $K(z^*) = z^*$ and the monotonicity properties of ψ' and φ' [Lemma 3.3(A)], ordinary differentiation yields

$$K'(z^*) = \frac{\psi'(z^*)}{\psi'(y^*)} \frac{\varphi'(y^*)}{\varphi'(z^*)} < 1.$$

This means that whenever the curve $K(x)$ intersects the diagonal of $\mathbb{R}_+^2$, the intersection is from above. This observation completes the proof. □

Thus, if the first order optimality conditions (9) attain a solution (z^*, y^*), it must be unique under Assumption 3.2. Next we shall concentrate on the optimality of the associated control (U^{z^*}, D^{y^*}).

4.2 Proving the optimality of the barrier policy

The two-sided barrier policy (z^*, y^*), which satisfy the pair of equations (9), leads to the value function [cf. (8)]

$$V(x) = \begin{cases} p(x - y^*) + V(y^*) & x \geq y^*, \\ (R_r\pi)(x) + c_1^*\psi(x) + c_2^*\varphi(x) & z^* < x < y^*, \\ q(x - z^*) + V(z^*) & x \leq z^*, \end{cases}$$

where $c_1^* = -J_q(z^*) = -J_p(y^*)$ and $c_2^* = I_q(z^*) = I_p(y^*)$ with I and J as in (10). Using the expressions $c_1^* = -J_p(y^*)$ and $c_2^* = I_p(y^*)$, applying the harmonicity of

$(R_r\pi)$, ψ and φ, and using the identity (2) we can calculate the limit in the boundary y^* to get

$$V(y^*-) = \frac{\left(p\frac{2\mu(y^*)}{\sigma^2(y^*)} - (R_r\pi)''(y^*) - \frac{2\mu(y^*)}{\sigma^2(y^*)}(R_r\pi)'(y^*) + \frac{2r}{\sigma^2(y^*)}(R_r\pi)(y^*)\right) S'(y^*)B}{\frac{2r}{\sigma^2(y^*)}S'(y^*)B}$$
$$= \frac{1}{r}\left[p\mu(y^*) + \pi(y^*)\right].$$

Similarly, using now the expressions $c_1^* = -J_q(z^*)$ and $c_2^* = I_q(z^*)$, we get

$$V(z^*+) = \frac{1}{r}\left[q\mu(z^*) + \pi(z^*)\right],$$

and so the value function can be written as

$$V(x) = \begin{cases} p(x-y^*) + \frac{1}{r}\left[p\mu(y^*) + \pi(y^*)\right] & x \geq y^*, \\ (R_r\pi)(x) + c_1^*\psi(x) + c_2^*\varphi(x) & z^* < x < y^*, \\ q(x-z^*) + \frac{1}{r}\left[q\mu(z^*) + \pi(z^*)\right] & x \leq z^*. \end{cases} \quad (12)$$

To prove that the two-sided barrier control (U^{z^*}, D^{y^*}) is the optimal control among all admissible controls and that $V(x)$ above is the optimal value function we shall need the following concavity result, which is a slight modification of Shreve et al. (1984, Lemma 4.2).

Lemma 4.3 *Let Assumption 3.2 hold, let (z^*, y^*) be a solution to (9) and let V be as in (12). Then*

(A) $V''(x) \leq 0$ for all $x \in (z^*, y^*)$.
(B) V is an increasing function.

Proof See Appendix A.2. □

Now we are ready to prove the main result about optimality of a reflecting control.

Theorem 4.4 *Let Assumption 3.2 hold and assume in addition that the necessary conditions (9) have a solution (z^*, y^*). Then the barrier policy (U^{z^*}, D^{y^*}) is the unique optimal policy to the problem (5) and the optimal value function $V(x)$ is as in (12).*

Proof Let V^* be the optimal value of the problem (5) and let V be as in (12). Since V is obtained with an admissible control $(U_t^{z^*}, D_t^{y^*})$, we know that $V^* \geq V$. The following properties will be proved to be sufficient for the opposite inequality:

(i) $V \in C^2$;
(ii) $(\mathcal{A} - r)V(x) + \pi(x) \leq 0$ for all $x \in \mathbb{R}_+$;
(iii) $p \leq V'(x) \leq q$ for all $x \in \mathbb{R}_+$.

Let us show that V satisfies these. Firstly the case (i) is valid, since (z^*, y^*) was chosen so that V is twice continuously differentiable. To show that (ii) hold, we get by straight differentiation that

$$(\mathcal{A}-r)V(x)+\pi(x)=\begin{cases}\rho_p(x)-\rho_p(y^*) & \text{if } x\geq y^*,\\ 0 & \text{if } x\in(z^*,y^*),\\ \rho_q(x)-\rho_q(z^*) & \text{if } x\leq z^*.\end{cases}$$

Here the first and the last expressions are non-positive due to Assumption 3.2(iv) and Lemma 4.1, and thus the case (ii) follows. The case (iii) is obtained as soon as we notice that combining the concavity of V from Lemma 4.3(A) with the fact that $V'(z^*+)=q>p=V'(y^*-)$ yields $p\leq V'(x)\leq q$ for $z^*\leq x\leq y^*$ and that $V'(x)=p$ for $x>y^*$ and $V'(x)=q$ for $x<z^*$.

To show that these three properties imply $V\geq V^*$, let (U_t,D_t) be an arbitrary admissible control, fix $T<\infty$ and define

$$U_t^c=U_t-\sum_{0<s\leq t}\Delta U_s \quad\text{and}\quad D_t^c=D_t-\sum_{0<s\leq t}\Delta D_s,$$

where $\Delta U_s=U_s-U_{s-}$ so that U_t^c and D_t^c are the continuous parts of U_t and D_t respectively. Letting $\tau_T=T\wedge\zeta_Z$, which is an almost surely finite stopping time, we apply generalised Ito's lemma to the function $e^{-r\tau_T}V(Z_{\tau_T})$ to get

$$\begin{aligned}E_x\left[e^{-r\tau_T}V(Z_{\tau_T})\right]&=V(x)+E_x\left[\int_0^{\tau_T}e^{-rs}(\mathcal{A}-r)V(Z_s)ds\right]\\&+E_x\left[\int_0^{\tau_T}e^{-rs}V'(Z_s)(dU_s^c-dD_s^c)\right]\\&+E_x\left[\sum_{0\leq s\leq\tau_T}e^{-rs}\Delta V(Z_s)\right],\end{aligned}$$

where $\Delta V(Z_s)=V(Z_s)-V(Z_{s-})$.

Let v be the value function corresponding the chosen control (U_t,D_t). Set

$$v_{\tau_T}(x)=E_x\left[\int_0^{\tau_T}e^{-rs}(\pi(Z_s)ds+p\,dD_s-q\,dU_s)+e^{-r\tau_T}V(Z_{\tau_T})\right]. \tag{13}$$

This is a compound policy, which follows the arbitrarily chosen policy (U_t,D_t) until time τ_T and thereafter applies the barrier policy (U^{z^*},D^{y^*}) with value function $V(x)$. Using the expression for $E_x\left[e^{-r\tau_T}V(Z_{\tau_T})\right]$ above and utilizing the three properties of the function V above we can calculate that

$$v_{\tau_T}(x)=V(x)+E_x\left[\int_0^{\tau_T}e^{-rs}\left((\mathcal{A}-r)V(Z_s)+\pi(Z_s)\right)ds\right]$$

$$
\begin{aligned}
&+E_x\left[\int_0^{\tau_T} e^{-rs}(V'(Z_s)-q)dU_s^c\right]\\
&+E_x\left[\int_0^{\tau_T} e^{-rs}(p-V'(Z_s))dD_s^c\right]+E_x\left[\sum_{0\le s\le\tau_T}\Delta V(Z_s)-q\Delta U_s+p\Delta D_s\right]\\
&\le V(x)+E_x\left[\sum_{0\le s\le\tau_T}\Delta V(Z_s)-q\Delta U_s+p\Delta D_s\right].
\end{aligned}
$$

Here the last sum is non-positive: assume that $\Delta U_s > 0$ and $\Delta D_s = 0$. Then $\Delta Z_s = \Delta U_s$ and

$$
\begin{aligned}
&\Delta V(Z_s)-q\Delta U_s+p\Delta D_s\\
&\quad = V(Z_s)-V(Z_s-\Delta U_s)-q\Delta U_s \le q\Delta U_s - q\Delta U_s = 0,
\end{aligned}
$$

where the inequality follows from the fact that $V'(x) \le q$ for all $x > 0$. Similar arguments apply to the case, where $\Delta U_s = 0$ and $\Delta D_s > 0$ as well as to the case $\Delta U_s > 0$ and $\Delta D_s > 0$. In every case $v_{\tau_T}(x) \le V(x)$. As $V(x)$ is bounded from below, $\lim_{T\to\infty} e^{-rT}V(Z_T) \ge 0$. Letting $T\to\infty$ in (13) we see that $v(x) \le v_{\tau_T}(x) \le V(x)$ for all admissible policies (U_t, D_t). Therefore also $V^* \le V$. Lastly, the uniqueness follows from Proposition 4.2.

The argument in the proof has been used for example in Harrison (1985, Chapter 6), where it is called a policy improvement logic. The theorem itself confirms that if we have already found a solution satisfying the first order optimality conditions (9), then fairly weak conditions ensure it to be unique and the corresponding control to be optimal for the problem (5) and the value function can be written explicitly as in (12). All in all, this is a pleasant result for the applications, since often if a solution to the necessary conditions (9) exists, it can be found numerically without too much difficulty.

Moreover we have seen in Lemma 4.3 that under Assumption 3.2 the marginal value $V'(x)$ is positive but diminishing everywhere. This generalises the known result from one-sided control, e.g. (Alvarez, 2001, Theorem 5), to two-sided ones.

A connection to the Dynkin game is also worth mentioning. There is a strong connection between one-sided singular control and optimal stopping, which is known already from the pioneering work (Bather and Chernoff 1966). It says that a derivative of the value function of a one-sided control problem constitutes the value function of an associated optimal stopping problem, see also Karatzas and Shreve (1984) and Karatzas (1985b) and Alvarez (2001). The two-sided control problem, like ours, is in turn known to have a similar connection with an associated two-player optimal stopping game known as a Dynkin game, see for example Karatzas and Wang (2001) and Boetius (2005).

5 Sufficient conditions for the solution

5.1 Assumptions and auxiliary results

Although one could try to find numerically the solution to the necessary conditions (9), we are nevertheless in a state of uncertainty whether there does exist a solution or not. To make things clearer, in this section we shall provide a set of sufficient conditions under which there exists a unique pair (z^*, y^*) satisfying the first order optimality conditions (9). These conditions are summarised in the following.

Assumption 5.1 Assume that Assumption 3.2 hold, that the boundaries 0 and ∞ are natural and in addition that for $b = p, q$

(v) $\rho_b(\infty) = -\infty$ and that $\rho_b'(0+) > 0$
(vi) $\lim_{x\downarrow 0} - \int_x^{\tilde{x}_b} \varphi'(t)/S'(t)dt = \infty$.

Basically all these additional assumptions aim to dictate the boundary behaviour of the auxiliary functions I and J, so that we can be sure they intersect each other. Of these assumptions, especially (vi) seems a bit bizarre and hard to verify, but it has a clear interpretation; the assumption that 0 is natural means that it is also not-entrance, implying that the scale derivative $-\varphi'(x)/S'(x)$ approaches infinity as x tends to zero. Now, Assumption (vi) requires the scale derivative to be even steeper at zero, namely that also the integral $-\int_x^{\tilde{x}_b} \varphi'(t)/S'(t)dt$ approaches infinity as x tends to zero. So loosely speaking one could say that Assumption (vi) makes zero even more forbidden entrance than the naturality assumption of the boundary. Since this assumption can be troublesome to verify, we shall give in Lemma 5.2 below two different conditions which imply the assumption. Before that we need to introduce the associated diffusion

$$d\hat{X}_t = (\mu(\hat{X}_t) + \sigma'(\hat{X}_t)\sigma(\hat{X}_t))dt + \sigma(\hat{X}_t)dW_t,$$

with killing rate $r - \mu'(x)$. (Its infinitesimal generator $\hat{\mathcal{A}} - \big(r - \mu'(x)\big) = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \big(\mu(x) + \sigma'(x)\sigma(x)\big)\frac{d}{dx} - \big(r - \mu'(x)\big)$ is got by differentiating the generator $\mathcal{A} - r$.)

Lemma 5.2 *Assume that either*

(A) $\mu'(x) < r$ *for all* $x \geq 0$, μ *is concave near zero, and the boundary* 0 *is not-entrance for the associated diffusion* $\hat{X}_t$*; or*
(B) $\psi'(0) = 0$ *and* $(R_r\mathrm{id})'(0+) > 0$.

Then $\lim_{x\downarrow 0} - \int_x^{\tilde{x}_b} \varphi'(t)/S'(t)dt = \infty$ *for* $b = p, q$.

Proof See Appendix A.3. □

In previous lemma the condition (A) can be checked from initial functions, while condition (B) can be convenient, if ψ and $(R_r\mathrm{id})$ can be calculated explicitly. Moving on, in the following lemma we see that I and J from (10) can be written in a tidy integral form.

Lemma 5.3 *Let Assumption 5.1 hold. Then, for $b = p, q$, the functions I and J from (10) can be written as*

$$\begin{cases} J_b(x) = -\frac{1}{B}\left(\int_x^\infty \varphi_t(\rho_b(x) - \rho_b(t))m_t' dt\right) \\ I_b(x) = \frac{1}{B}\left(\int_0^x \psi_t(\rho_b(x) - \rho_b(t))m_t' dt\right) \end{cases}$$

Proof See Appendix A.4 □

We have previously proved (Lemma 3.3) that these auxiliary functions satisfy certain monotonicity properties, which were adequate for the uniqueness of a solution. But for the existence we also need to know something about their boundary behaviour.

Lemma 5.4 *Let Assumption 5.1 hold. Then*

(A) for $b = p, q$, $J_b(0+) = \infty$ and $J_b(\infty) \leq 0$.
(B) for $b = p, q$, $I_b(0+) \geq 0$ and $I_b(\infty) < 0$.

Proof See Appendix A.5. □

5.2 Proving the existence of (z^*, y^*)

We already know from Lemma 4.1 that if there exists a pair (z^*, y^*) satisfying the condition (9), then it must be in the set $(0, \tilde{x}_q) \times (\tilde{x}_p, \infty)$. Now with stricter assumptions, we can shrink this acceptable set into a bounded set.

Lemma 5.5 *Let Assumption 5.1 hold. Assume further that the necessary conditions (9) have a solution (z^*, y^*). Then $(z^*, y^*) \in (x_q^J, \tilde{x}_q) \times (\tilde{x}_p, x_p^I)$, where $x_q^J, x_p^I \in \mathbb{R}_+$ are the unique interior points for which $J_q(x_q^J) = 0$ and $I_p(x_p^I) = 0$ and $\tilde{x}_q, \tilde{x}_p$ are as in Assumption 3.2(iv).*

Proof The proof follows that of Theorem 4.3 in Alvarez (2008). From Lemma 5.4 we get $J_b(0+) > 0$ and $J_b(\infty) \leq 0$ for $b = p, q$. Combining these facts with the monotonicity properties (Lemma 3.3) we see that there must exist a unique $x_b^J < \tilde{x}_b$ such that $J_b(x) \gtreqless 0$ for all $x \lesseqgtr x_b^J$. Especially we see that $J_b(x) < 0$ for all $x > \tilde{x}_b$. Analogously we see that there exists a unique $x_b^I > \tilde{x}_b$ such that $I_b(x) \gtreqless 0$ for all $x \lesseqgtr x_b^I$, and especially that $I_b(x) > 0$ for all $x < \tilde{x}_b$.

To prove the new lower boundary for z^*, we notice first that by Lemma 4.1 we have $y^* > \tilde{x}_p$, and thus, since z^* satisfies (9), using the sign results above we get $J_q(z^*) = J_p(y^*) < 0$. Moreover utilizing the sign results above once more we get $z^* > x_q^J$. The new upper boundary for y^* follows similarly.

So the possible region for optimal thresholds is narrowed to a compact region. This information is useful in next theorem, which is our main result on the solvability of the necessary conditions (9).

Theorem 5.6 *Let Assumption 5.1 hold. Then there exists a unique pair* (z^*, y^*) *satisfying the first order optimality conditions* (9).

Proof As in proof of Proposition 4.2, define a function $K : [x_q^J, \tilde{x}_q] \to [x_q^J, \tilde{x}_q]$ by $K(x) = \big(\hat{J}_q^{-1} \circ \hat{J}_p \circ \hat{I}_p^{-1} \circ \hat{I}_q\big)(x)$, where $\hat{J}_q = J_q|_{(0,\tilde{x}_q]}$, $\hat{J}_p = J_p|_{[\tilde{x}_p,\infty)}$, $\hat{I}_q = I_q|_{(0,\tilde{x}_q]}$ and $\hat{I}_p = I_p|_{[\tilde{x}_p,\infty)}$. As before, we notice that K is increasing. Notice that now the domain of K is different.

To ensure that K is well defined, we will show that the endpoints $x_q^J, \tilde{x}_q$ are mapped into the domain of K. Firstly $0 < x_q^J < \tilde{x}_q$, and so $I_q(x_q^J) > 0$. Since $I_p(x) > I_q(x)$ for all $x \in \mathbb{R}_+$, there exists a point $s_1 \in (\tilde{x}_p, x_p^I)$ such that $I_p(s_1) = I_q(x_q^J)$. Moreover, $J_p(s_1) < 0$ and since $J_p(x) > J_q(x)$ for all $x \in \mathbb{R}_+$, there exists a point $s_2 \in (x_q^J, \tilde{x}_q)$ such that $J_p(s_1) = J_q(s_2)$, so especially $K(x_q^J) = s_2 \in (x_q^J, \tilde{x}_q)$. For the upper endpoint, since $I_p(x) > I_q(x)$ for all $x \in \mathbb{R}_+$, we know that there exists $t_1 \in (\tilde{x}_p, x_p^I)$ such that $I_p(t_1) = I_q(\tilde{x}_q)$. Reasoning as above, we get that there exists $t_2 \in (x_q^J, \tilde{x}_q)$ such that $J_p(t_1) = J_q(t_2)$ so in particular $K(\tilde{x}_q) = t_2 \in (x_q^J, \tilde{x}_q)$ and K is well defined.

Let us define a sequence $z_n = K^n(x_q^J)(= (K \circ \cdots \circ K)(x_q^J))$. This sequence converges by induction: It is clear that $z_1 = K(z_0) > z_0$. Because K is an increasing function, we have $K\big(K(z_0)\big) > K(z_0)$. By induction $K^n(z_0) > K^{n-1}(z_0)$. Since the sequence z_n is increasing and bounded from above, it converges.

Writing $z^* = \lim_{n\to\infty} z_n$, we see that z^* is the fixed point of the function K. Defining $y^* = \hat{J}_p^{-1}(\hat{J}_q(z^*))(= \hat{I}_p^{-1}(\hat{I}_q(z^*)))$, we get a pair (z^*, y^*) that satisfies the necessary conditions (9). The uniqueness of such a pair follows directly from Proposition 4.2. □

In the previous theorem we saw that under Assumption 5.1 the unique pair (z^*, y^*) satisfying (9) always exists. Furthermore we saw how it can be found when it is identified as a fixed point. Analogous fixed point argument is used also in Alvarez and Lempa (2008) in an impulse control situation and in Lempa (2010) in a traditional optimal stopping situation. Theorem 5.6 also shows how we can find the pair (z^*, y^*) numerically. First we identify the point x_q^J. After that, we apply the function $K(x) = \big(\hat{J}_q^{-1} \circ \hat{J}_p \circ \hat{I}_p^{-1} \circ \hat{I}_q\big)(x)$ to that point (actually any point in $(0, x_q^J]$ will do) and calculate $K^k(x_q^J)$, where we might for example set a stopping limit $\varepsilon > 0$ and stop at the first step step k, for which $|K^k(x_q^J) - K^{k-1}(x_q^J)| < \varepsilon$. After this we have $z^* \approx K^k(x_q^J)$ and $y^* \approx J_2^{-1}(J_1(K^k(x_q^J)))$.

6 Comparative analysis

Let us next study the sensitiveness of the value function and the optimal barriers, firstly and most importantly with respect to the volatility, and secondly with respect to the control parameters. We shall also compare the differences between the solutions of two-sided and one-sided control problems.

6.1 Volatility sensitiveness

Our main results on the effect of the increased volatility are summarised in the following.

Theorem 6.1 *Let Assumption 3.2 hold and let (z^*, y^*) be a solution to (9). Then*

(A) $V(x)$ is non-increasing in σ.
(B) if we assume further that the inequalities concerning ρ_b' in Assumption 3.2(iv) are strict, the inactivity region (z^*, y^*) widens as σ increases.

Proof Let $\hat{\sigma}(x) \geq \sigma(x)$ for all $x \geq 0$ and let $\hat{\mathcal{A}} = \frac{1}{2}\hat{\sigma}^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ be the infinitesimal generator, $\hat{V}$ the optimal value function and $(\hat{z}^*, \hat{y}^*)$ the optimal inactivity region with respect to the volatility $\hat{\sigma}$.

(A) We have

$$(\hat{\mathcal{A}} - r)V(x) + \pi(x) = \begin{cases} \rho_p(x) - \rho_p(y^*) \leq 0 & \text{if } x \geq y^* \\ \frac{1}{2}\left(\hat{\sigma}^2(x) - \sigma^2(x)\right) V''(x) \leq 0 & \text{if } x \in (z^*, y^*) \\ \rho_q(x) - \rho_q(z^*) \leq 0 & \text{if } x \leq z^*, \end{cases}$$

the first and the last expressions being non-positive due to Assumption 3.2(iv) and the middle expression due to the concavity of V (Lemma 4.3). Hence V satisfies the property (ii) in the proof of Theorem 4.4 with respect to $\hat{\sigma}$, while the properties (i) and (iii) can be handled as previously. Therefore analysis similar to that in the proof of Theorem 4.4 shows that $V \geq \hat{V}$.

(B) Let us first prove the ordering for the lower boundaries. Suppose, contrary to our claim, that $z^* < \hat{z}^*$. Now from the value function expression (12) we see that

$$V(z^*) = \frac{1}{r}\rho_q(z^*) + qz^* < \frac{1}{r}\rho_q(\hat{z}^*) + qz^* = \hat{V}(z^*) - q(z^* - \hat{z}^*) + q(z^* - \hat{z}^*) = \hat{V}(z^*),$$

where the inequality follow from strict inequality in Assumption 3.2(iv). This contradicts the fact $V \geq \hat{V}$ (item (A)). The same reasoning applies also to the case $y^* < \hat{y}^*$. □

According to our theorem, increased volatility affects negatively both the optimal policy and its value. Put differently, our theorem shows that increased volatility expands the inactivity region and postpones the usage of singular policies by decreasing the marginal value of the optimal policy. This result generalises previous findings based on one-sided policies (e.g. Theorem 6 in Alvarez 2001) to a two-sided setting.

6.2 Comparing the two-sided and one-sided solutions

It is also of interest to study the relationship between two-sided and one-sided controls. Obviously, since not using a control is an admissible control, the optimal value function is greater in the two-sided case. But are the reflected barriers from these two problems ordered consistently, and if so, how? To this end let Assumption 5.1 hold and let (z^*, y^*) be the optimal reflecting barriers in two-sided control problem.

Consider first the case where the dynamics are controlled only downwards, so that $Z = X - D$. In that case the value reads as $\sup_D \mathbb{E}_x \int_0^{\zeta Z} e^{-rs}\big(\pi(Z_s)ds + pdD_s\big)$. Under Assumption 5.1 this one-sided control problem is known to have solution (actually, weaker assumptions are sufficient, see Lemma 3.4 in Alvarez and Lempa 2008) and the optimal control is reflecting control with the reflecting barrier at x_p^I (the unique point for which $I_p(x_p^I) = 0$, cf. Lemma 5.5), and we know from Lemma 5.5 that $y^* < x_p^I$. So, in the harvest example, in the absence of a replanting opportunity we harvest later.

Similarly, consider the case where the dynamics are controlled only upwards, so that $Z = X + U$. In this case the value reads as $\sup_U \mathbb{E}_x \int_0^{\zeta Z} e^{-rs}\big(\pi(Z_s)ds - qdU_s\big)$. Going through the reasoning in Alvarez and Lempa (2008), one could verify that under Assumption 5.1 this one-sided control problem has a solution, where the optimal control is a reflecting control with the reflecting barrier at x_q^J (the unique point for which $J_q(x_q^J) = 0$, cf. Lemma 5.5), and from Lemma 5.5 we know that $z^* > x_q^J$. Now, in a dividend payments problem with obligative reinvestment example, in the absence of dividend payments we reinvest later.

6.3 Sensitiveness on control parameters

Next we shall consider the sensitiveness with respect to the control parameters p and q in the following two propositions.

Proposition 6.2 *Let Assumption 5.1 hold. Then*

(A) $V(x)$ is p-increasing and q-decreasing.
(B) the inactivity region (z^*, y^*) shrinks as p increases and widens as q increases.

Proof Fix $p_1 < p_2 (< q)$ and let $V_i(x) := V(x; p_i)$ and (z_i^*, y_i^*) be the value function and optimal reflecting barriers, respectively, with respect to p_i.

(A) We see that

$$\begin{aligned} V_1(x) &= \mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(Z_t)dt + p_1 dD_t^{y_1^*} - qdU_t^{z_1^*})\right] \\ &\leq \mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(Z_t)dt + p_2 dD_t^{y_1^*} - qdU_t^{z_1^*})\right] \\ &\leq \sup_{(D,U)} \mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(Z_t)dt + p_2 dD_t - qdU_t)\right] = V_2(x). \end{aligned}$$

Proving that $V(x; q_2) \leq V(x; q_1)$ for all $q_2 > q_1$ is analogous.

(B) Let us first study the sensitiveness with respect to p. Fix again $p_1 < p_2 (< q)$ and let (z_i^*, y_i^*) be the optimal reflecting barriers with respect to p_i. Furthermore let $K_i(x) = \left(\hat{J}_q^{-1} \circ \hat{J}_{p_i} \circ \hat{I}_{p_i}^{-1} \circ \hat{I}_q\right)(x)$, for $i = 1, 2$, be as in Theorem 5.6. Since $\psi'', \varphi'' > 0$ by Lemma 3.3(A) we can use the expression (10) to obtain inequalities $I_{p_2}(x) < I_{p_1}(x)$ and $J_{p_2}(x) < J_{p_1}(x)$. Combining these with the monotonicity properties of $\hat{J}$ and $\hat{I}$ yields

$$\begin{aligned}
&\hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*)) > \hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*)) \\
&\quad \Longrightarrow \hat{J}_{p_1}(\hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*))) > \hat{J}_{p_1}(\hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*))) > \hat{J}_{p_2}(\hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*))) \\
&\quad \Longrightarrow \hat{J}_q^{-1}(\hat{J}_{p_1}(\hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*)))) < \hat{J}_q^{-1}(\hat{J}_{p_2}(\hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*)))),
\end{aligned}$$

where all the inequalities are strict. In other words $K_2(z_1^*) > K_1(z_1^*) = z_1^*$. Now proceeding as in the proof of Theorem 5.6, we can deduce that as a limit of an increasing sequence $z_2^* = \lim_{n\to\infty} K_2^n(z_1^*)\big(= (K_2 \circ \cdots \circ K_2)(z_1^*)\big) > z_1^*$. Moreover by the monotonicity of functions $\hat{I}$ we get

$$y_1^* = \hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*)) > \hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*)) > \hat{I}_{p_2}^{-1}(\hat{I}_q(z_2^*)) = y_2^*.$$

Let us then consider the sensitiveness with respect to q. Same arguments as above with slight changes applies to this case. Fix $q_2 > q_1$ and let (z_i^*, y_i^*) be the optimal continuation region with respect to q_i. Now we need to define functions H_i : $[\tilde{x}_p, x_p^I) \to [\tilde{x}_p, x_p^I)$ (for the definitions of $\tilde{x}_p$ and x_p^I see Lemma 5.5) as $H_i(y) = (\hat{I}_p^{-1} \circ \hat{I}_{q_i} \circ \hat{J}_{q_i}^{-1} \circ \hat{J}_p)(y)$, for $i = 1, 2$, so that $H_1(y_1^*) = y_1^*$. Reasoning as above we can deduce that $H_2(y_1^*) > y_1^*$. Now $H_2^n(y_1^*)$ is an bounded increasing sequence and therefore $y_2^* = \lim_{n\to\infty} H_2^n(y_1^*) > y_1^*$. Lastly by monotonicity of functions $\hat{J}$ we get

$$z_1^* = \hat{J}_{q_1}^{-1}(\hat{J}_p(y_1^*)) > \hat{J}_{q_2}^{-1}(\hat{J}_p(y_1^*)) > \hat{J}_{q_2}^{-1}(\hat{J}_p(y_2^*)) = z_2^*.$$

□

Proposition 6.2 verifies intuitively clear facts: increasing the income (p) from using an upper barrier, the value is understandably also increasing and the controller is encouraged to use the controls, thus the inactivity region is narrowing. The contrary is true when the cost q of using control at the lower barrier is increased.

Subsequent questions are the limiting properties, which are considered in the following.

Proposition 6.3 *Let Assumption 5.1 hold. Then*

(A) $z^* \searrow 0$ *and* $y^* \nearrow x_p^I$ *as* $q \nearrow \infty$*;*
(B) if in addition π *is increasing, we have* $z^* \searrow x_q^J$ *and* $y^* \nearrow \infty$ *as* $p \searrow 0$
(C) the inactivity region (z^*, y^*) *shrinks arbitrary small as* $q - p \searrow 0$*. Moreover* $\tilde{x}_p - \tilde{x}_q \searrow 0$ *and* $z^* \nearrow \tilde{x}_q$, $y^* \searrow \tilde{x}_q$ *and the value function approaches, from below, a function* $q(x - \tilde{x}_q) + \frac{1}{r}\left(q\mu(\tilde{x}_q) + \pi(\tilde{x}_q)\right)$.

(D) if in addition π is increasing, we have $(z^*, y^*) \searrow (0, 0)$ as $q, p \to \infty$ and $(z^*, y^*) \nearrow (\infty, \infty)$ as $q, p \to 0$.

Proof (A) Let $q_1 < q_2$. Then

$$\rho'_{q_1}(x) = \pi'(x) + q_1(\mu'(x) - r) > \pi'(x) + q_2(\mu'(x) - r) = \rho'_{q_2}(x).$$

Therefore we can deduce that $\tilde{x}_{q_1} > \tilde{x}_{q_2}$. Now $\rho'_\infty(x) = -\infty$ for all $x > 0$, and so $\lim_{q\to\infty} \tilde{x}_q = 0$. By Lemma 4.1 we know that $z^* < \tilde{x}_q$, and therefore we can conclude that $z^* \to 0$ as $q \to \infty$.

Since zero was assumed to be natural, the process never reaches the state 0, and it follows that $U_t^{z^*} = U_t^0 \equiv 0$ as $q \to \infty$. And so, when $q \to \infty$, the problem reduces to

$$\sup_D \mathbb{E}_x \int_0^{\zeta_Z} e^{-rs} (\pi(Z_s)ds + p dD_s) .$$

But this is the one-sided control problem introduced in Sect. 6.2, and its optimal policy is known to be $D_t^{x_p^I}$ (see Lemma 3.4 in Alvarez and Lempa (2008)), and so $\lim_{q\to\infty} y^* = x_p^I$. Moreover, from Lemma 5.5 we know that $y^* < x_p^I$ for all q, and so the convergence must be from below.

(B) Let $0 < p_1 < p_2$. Then

$$\rho'_{p_2}(x) = \pi'(x) + p_2(\mu'(x) - r) < \pi'(x) + p_1(\mu'(x) - r) = \rho'_{p_1}(x).$$

Therefore we can deduce that $\tilde{x}_{p_1} > \tilde{x}_{p_2}$, and this holds for all $\pi(x)$. Now if π is increasing, then $\rho'_0(x) = \pi'(x) \geq 0$ for all $x > 0$, and so $\lim_{p\to 0} \tilde{x}_p = \infty$. And since $y^* > \tilde{x}_p$ (by Lemma 5.5), the rest of the reasoning is similar to the one in (A).

(C) First of all, Proposition 6.2(B) implies that the inactivity region (z^*, y^*) shrinks as $q - p \searrow 0$. Moreover, above we saw that $\tilde{x}_q$ is decreasing in q and $\tilde{x}_p$ is increasing in p. Furthermore, since $\rho_b(x)$ is b-continuous, it is clear that as $q - p \searrow 0$, we get in fact $\tilde{x}_p - \tilde{x}_q \searrow 0$ ($\tilde{x}_p \geq \tilde{x}_q$ always by Lemma 4.1).

Without lost of generality, we from now on fix q and let p approach q. For all $p < q$ we know from Theorem 5.6 that there exist $z^*(p) < \tilde{x}_q$ and $y^*(p) > \tilde{x}_q$. Further, $z^*(p)$ is p-increasing and $y^*(p)$ is p-decreasing by Proposition 6.2(B). Moreover from the proof of Proposition 6.2(B) we see that $z^*(p)$ is p-continuous, since the functions I_b, J_b, I_b^{-1}, J_b^{-1}, for $b = p, q$, are. Similarly also $y^*(p)$ is p-continuous.

It follows that there exist $Z^* = \lim_{p\nearrow q} z^*(p)$ and $Y^* = \lim_{p\nearrow q} y^*(p)$, which satisfy the fixed point properties in the proof of Theorem 5.6 at the limit $p \nearrow q$; i.e. properties $K(Z^*) = Z^*$, $Y^* = (I_q^{-1}|_{[\tilde{x}_q,\infty)} \circ I_q|_{[x_q^J,\tilde{x}_q]})(Z^*)$ and $K'(Z^*) < 1$. But now since the pair $(\tilde{x}_q, \tilde{x}_q)$ also satisfies these properties at the limit $p \nearrow q$, and the fixed point is unique, we must have $(Z^*, Y^*) = (\tilde{x}_q, \tilde{x}_q)$.

The value $V(x)$ is p-increasing by Proposition 6.2(A). Moreover, from the value function expression (12), we see that since $z^*, y^* \to \tilde{x}_q$ as $p \to q$, we have limit $\lim_{p\to q} V(x) = q(x - \tilde{x}_q) + \frac{1}{r}\left(q\mu(\tilde{x}_q) + \pi(\tilde{x}_q)\right)$ for $x \geq y^*$ and $x \leq z^*$. And since $z^* - y^* \to 0$, as $p \to q$, this expression holds everywhere.

(D) Consider first the case $q, p \to \infty$. We have already shown that $z^* \searrow 0$ as $q \to \infty$, so we are left to prove that $y^* \searrow 0$ as $p \to \infty$. Since $y^* < x_p^I$ (by Lemma 5.5), it is sufficient to show that $\lim_{p\to\infty} x_p^I = 0$. Now

$$I_p(x) = \frac{1}{B}\int_0^x \psi_t(\rho_p(x) - \rho_p(t))m_t' dt,$$

and since $\lim_{p\to\infty} \tilde{x}_p = 0$, we know that $\rho_p(x) - \rho_p(t) < 0$ for all $t < x$ as $p \to \infty$. Hence $I_p(x) < 0$ for all $x > 0$ at the limit $p \to \infty$. Consequently $x_p^I \to 0$.

Let us then turn to the case $q, p \to 0$. We already know that $\lim_{p\to 0} y^* = \infty$, and thus it remains to prove that $\lim_{q\to 0} z^* = \infty$. Since $z^* > x_q^J$ (by Lemma 5.5), it is sufficient to show that $\lim_{q\to 0} x_q^J = \infty$. Now

$$J_q(x) = -\frac{1}{B}\int_x^\infty \varphi_t(\rho_q(x) - \rho_q(t))m_t' dt,$$

and since $\lim_{q\to 0} \tilde{x}_q \to \infty$, we know that $\rho_q(x) - \rho_q(t) < 0$ for all $t > x$ as $q \to 0$. Hence $J_q(x) > 0$ for all $x > 0$ at the limit $q \to 0$, and consequently $x_p^J \to \infty$.

In Proposition 6.3(A)–(B) we see that at the limits $q \to \infty$ and $p \to 0$ we get the solutions of the associated one-sided control problems (cf. Sect. 6.2), so that the theory presented in this paper can be seen as a natural generalisation of the one-sided problem. Moreover, we see that the upper boundary x_p^I is approached from below and the lower boundary x_q^J from above. It is also worth stressing that in Proposition 6.3(B) the requirement that π is increasing is necessary; we shall see an example in Sect. 7.2 where a concave revenue function π enables the upper threshold y^* to be finite even with negative values of p.

From case (C) we see that as p and q approach each other, the inactivity region (z^*, y^*) becomes arbitrarily small. Noteworthy is that, although technically at the limit $p \nearrow q$ we get reflecting barriers $(z^*, y^*) = (\tilde{x}_q, \tilde{x}_q)$, the corresponding pair of controls $(U^{\tilde{x}_q}, D^{\tilde{x}_q})$ are no longer admissible policies.

In the last case (D) we see that when both control parameters are set to the same limit, either 0 or ∞, we, respectively, either raise both of the thresholds z^* and y^* up toward infinity, or lower them down toward zero. Noteworthy is that in the limit neither the control U^∞ nor D^0 are admissible, since they usher the diffusion to the state ∞ or 0, respectively, which are not in the state space.

6.4 Stationary distribution

The controlled process $Z_t = X_t + U_t^{z^*} - D_t^{y^*}$ is well defined on the finite interval $[z^*, y^*]$, and so it follows that $M := m(z^*, y^*) = \int_{z^*}^{y^*} m'(u)du < \infty$. Moreover, since the boundaries of the controlled process are reflecting, we can define a stationary probability distribution for controlled process Z_t as $\eta(x) := m'(x)/M$. Now, for every Borel-measurable bounded function $f : [z^*, y^*] \to \mathbb{R}$ we have (see Borodin and Salminen 2002, p. 37)

$$\lim_{t\to\infty} \mathbb{E}_x [f(Z_t)] = \int_{x^*}^{y^*} f(u)\eta(u)du.$$

7 Examples

7.1 Geometric Brownian motion

To illustrate our results explicitly, assume that the underlying uncontrolled diffusion evolves as geometric Brownian motion, i.e.

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t),$$

where $\sigma \in \mathbb{R}_+$, $\mu \in (-\infty, r)$ are exogenously given constants. Furthermore, assume that the revenue flow is $\pi(x) = x^a - c$, with $a \in (0, 1)$ and $c \in \mathbb{R}$, so that

$$(R_r\pi)(x) = \frac{x^a}{r + \frac{1}{2}\sigma^2\left(a - a^2\right) - a\mu} - \frac{c}{r}.$$

It is worth mentioning that with linear payoff function ($a = 1$), there would not emerge a two-sided reflecting barrier as an optimal rule due to invalidity of Assumption 3.2(iv). Furthermore let us still assume that $q > p$.

With geometric Brownian motion our fundamental solutions of the ordinary differential equation $(\mathcal{A} - r)u = 0$ are $\psi(x) = x^{\gamma^+}$ and $\varphi(x) = x^{\gamma^-}$, where

$$\gamma^\pm = \frac{1}{\sigma^2}\left(\frac{1}{2}\sigma^2 - \mu \pm \sqrt{(\frac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2 r}\right) \tag{14}$$

are the solutions of the characteristic equation $\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \mu\gamma - r = 0$. Especially we see that $\gamma^+ > 1$ since $\mu < r$.

7.1.1 Solution to the problem

Let us check that this setup satisfies Assumption 5.1. Now the boundaries are natural and Assumption (i) is already assumed to hold, and clearly conditions in (ii) are satisfied. Furthermore, we assumed $\mu < r$ and so (iii) holds. By straight differentiation $\rho_b'(x) = ax^{a-1} + b(\mu - r)$, which satisfies assumption (iv) since $0 < a < 1$. Furthermore $\rho_b'(0+) = \infty$ and $\rho_b(\infty) = -\infty$, thus (v) is valid. Lastly $\psi'(0) = 0$ since $\gamma^+ > 1$, and $(R_r\text{id})(x) = x/(r - \mu)$ so that $(R_r\text{id})'(0) > 0$ and therefore we can conclude by Lemma 5.2 that also Assumption (vi) is valid.

Hence the results from Sect. 5 can be applied, so especially the optimal solution to (5) is a two-sided reflected control. The optimal reflecting barriers (z^*, y^*) are the

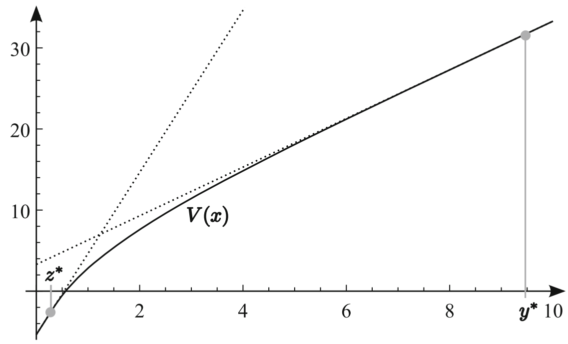

Fig. 1 Optimal value function for a control problem, *dashed lines* are tangents at the points z^* and y^*

unique solution to the necessary conditions (9), which can now be written as

$$\begin{cases} z^{-\gamma^-}\left[2az^a(a-\gamma^-)+Aqz(\gamma^- -1)\right]=y^{-\gamma^+}\left[2ay^a(a-\gamma^-)+Apy(\gamma^- -1)\right] \\ z^{-\gamma^-}\left[2az^a(a-\gamma^+)+Aqz(\gamma^+ -1)\right]=y^{-\gamma^-}\left[2ay^a(a-\gamma^+)+Apy(\gamma^+ -1)\right], \end{cases}$$

where $A = 2r + a(\sigma^2(1-a) - 2\mu)$. Unfortunately this seems impossible to solve explicitly, but we shall illustrate the optimal barriers numerically below.

With optimal barriers, the value function gets the form

$$V(x) = \begin{cases} p(x-y^*)+\frac{1}{r}\left[p\mu y^* + y^{*a} - b)\right] & x \geq y^*, \\ \frac{x^a}{r+\frac{1}{2}\sigma^2(a-a^2)-a\mu} - \frac{c}{r} - J_q(z^*)x^{\gamma^+} + I_q(z^*)x^{\gamma^-} & z^* < x < y^*, \\ q(x-z^*)+\frac{1}{r}\left[q\mu z^* + z^{*a} - b\right] & x \leq z^*, \end{cases}$$

where J_q and I_q are as in (10).

7.1.2 Numerical illustration

Let us illustrate numerically the results under the parameter configuration $\mu = 0.05$, $\sigma = 0.2$ $r = 0.08$, $a = 1/3$, $c = 1$, $p = 3$ and $q = 10$. With these choices $(z^*, y^*) \approx (0.28, 9.45)$, and the value function is drawn in Fig. 1. As was shown in Lemma 4.3, $V(x)$ is concave.

In Fig. 2 we see how the thresholds are altered, when we change parameter values. By increasing a we increase the payoff function π (for $x > 1$), so that it is sensible that the upper barrier y^* increases. As was proved in Theorem 6.1, higher volatility (σ) leads to a wider inactivity region. Moreover the impact of a change in p and q affects as proved in Propositions 6.2 and 6.3 (now $(x_q^J, \tilde{x}_q, x_p^I) \approx (0.26, 1.17, 9.453)$). What is not seen from those propositions though, is the exceptional rapid widening of the interval (z^*, y^*) with respect to q, when q is near p: With $p = q = 3$, we have $z^* = y^*$, but already with $q = 3.02$, we have $y^* - z^* \approx 3.0$ and with $q = 3.1$, $y^* - z^* \approx 4.8$. Consequently q reaches its upper barrier x_p^I rather quickly. On the other hand, a change in p does not affect the boundaries so strongly. This suggests that the optimal policy is more sensitive with respect to changes in costs than in revenues.

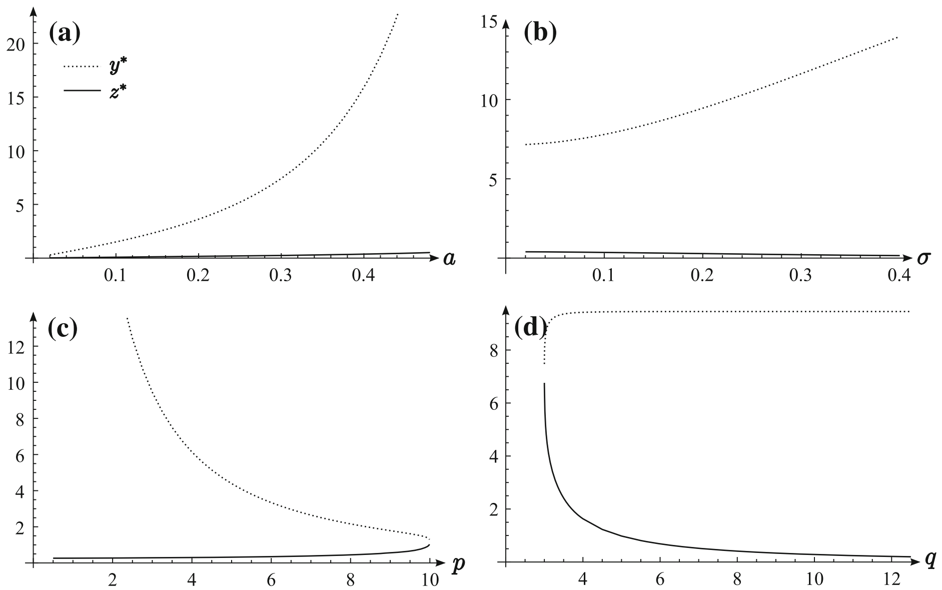

Fig. 2 Sensitivity of the inactivity region with respect to the parameters **a** a; **b** σ; **c** p; **d** q

Furthermore now $m'(x) = \frac{2}{\sigma^2} x^{2\left(\frac{\mu}{\sigma^2}-1\right)}$. Thus, if $\mu \neq \frac{1}{2}\sigma^2$, the stationary probability distribution is

$$\eta(x) = \frac{2\mu - \sigma^2}{\sigma^2 \left(y^{*\frac{2\mu}{\sigma^2}-1} - z^{*\frac{2\mu}{\sigma^2}-1} \right)} x^{2\left(\frac{\mu}{\sigma^2}-1\right)}.$$

Using Sect. 6.4, we can calculate that, with the chosen numerical values, $\lim_{t\to\infty} \mathbb{E}[Z_t] = 5.70$ (the midpoint of the interval (z^*, y^*) is 4.9), and that the variance of the long run stationary state is $\lim_{t\to\infty} \mathrm{Var}(Z_t) = 6.00$. Moreover, choosing $A = [6.4, y^*]$ (the upper third of the interval $[z^*, y^*]$), we get $\lim_{t\to\infty} \mathbb{E}[\mathbf{1}_A(Z_t)] = \lim_{t\to\infty} \mathbb{P}(Z_t \in A) = 0.45$. All this advocates that, in the long run, the controlled process spends more time near the upper threshold y^* than near the lower threshold z^*.

7.2 Mean reverting diffusion

As a slightly more challenging setting, consider that without a control the underlying diffusion X_t follows a mean reverting diffusion:

$$dX_t = \mu X_t(1 - \beta X_t)dt + \sigma X_t dW_t, \quad X_0 = x,$$

where $\mu > 0$ is exogenous constant and $\beta > 0$ is the degree of the mean-reversion and $\sigma > 0$ is the volatility coefficient. In this subsection we shall demonstrate a case, where the "gain" p from downward control can also be negative. An example, where this kind of behaviour might arouse is the following.

Let us consider a house owner who wants to control the inside temperature of her home and dislikes both cold and hot temperature, so that her temperature dependent utility function, represented by π, is a concave function. The house owner can naturally control the temperature of her home either by heating or cooling, by paying a fixed cost q and p for it, respectively. Since both heating and cooling are costly operations, we must have $q > 0 > p$.

To carry on to a more specific analysis, fix $q > 0 > p$ and the utility function $\pi(x) = -x^2 + ax$, where $a > 0$ is an exogenously given constant. Let us next check that this structure satisfies Assumption 5.1(i)–(vi). We notice that Assumption (i) is already assumed and that the smoothness conditions in Assumption (ii) are valid. To see that the integrability Assumption (ii) holds, observe first that by Itô's Lemma

$$X_t^2 = x^2 + \int_0^t 2\left(\sigma^2 + \mu\,(1 - \beta X_s)\right) X_s^2 ds + \int_0^t 2\sigma X_s^2 dW_s.$$

It is now straightforward to show that

$$2\left(\sigma^2 + \mu\,(1 - \beta X_s)\right) X_s^2 \leq \frac{2(\sigma^2 + \mu)}{3\mu\beta}, \quad \text{and thus} \quad \mathbb{E}_x[X_t^2] \leq x^2 + \frac{2(\sigma^2 + \mu)t}{3\mu\beta}.$$

Thus, it follows that $\mathbb{E}_x \int_0^\infty e^{-rt} X_t^2 dt = \int_0^\infty e^{-rt} \mathbb{E}_x \left[X_t^2\right] dt < \infty$, and consequently $\pi, \mu(x), x \in \mathcal{L}^1$ and assumption (ii) holds.

By straight calculations, Assumption (iii)–(v) hold under the sufficient conditions $\mu < r, q < \frac{a}{r-\mu}$ and $p > -\frac{1}{\mu\beta}$. Finally, since Assumption (iii) is valid and the drift $\mu x(1 - \beta x)$ is concave, the last Assumption (vi) follows from Lemma 5.2 if 0 is non-entrance for the associated diffusion $\hat{X}_t$, which in this case is

$$d\hat{X}_t = (\mu(1 - \beta\hat{X}_t) + \sigma^2)\hat{X}_t dt + \sigma\hat{X}_t dW_t$$

and we observe that 0 is non-entrance for it.

It follows that under the above mentioned conditions, the results from Sect. 5 can be applied. Unfortunately, due to complicated nature of ψ and φ in this case (see Section 6.5 in Dayanik and Karatzas 2003), we cannot solve explicitly any results, but an illustrative numerical solution is seen in Fig. 3.

Furthermore, in this case the speed density is

$$m'(x) = \frac{2}{\sigma^2} x^{2\left(\frac{\mu}{\sigma^2} - 1\right)} e^{-\frac{2\mu\beta}{\sigma^2} x},$$

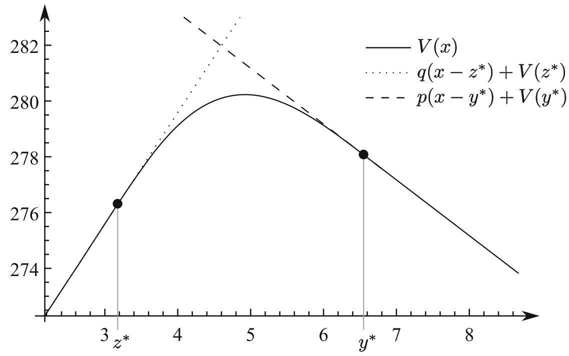

Fig. 3 A numerical illustration of the solution to (5) with the mean reverting set up, introduced above, with the parameter configuration $\mu = 0.04$, $\beta = 0.05$, $\sigma = 0.3$, $r = 0.08$, $a = 10$, $q = 4$ and $p = -2$

and thus the stable stationary distribution on (z^*, y^*) is

$$\eta(x) = \frac{x^{2\left(\frac{\mu}{\sigma^2}-1\right)} e^{-\frac{2\mu\beta}{\sigma^2}x}}{\left(\frac{\sigma^2}{2\mu\beta}\right)^{\frac{2\mu}{\sigma^2}-1}\left(\Gamma(\frac{2\mu}{\sigma^2}-1, \frac{2\mu\beta}{\sigma^2}z^*) - \Gamma(\frac{2\mu}{\sigma^2}-1, \frac{2\mu\beta}{\sigma^2}y^*)\right)},$$

where $\Gamma(s, x) = \int_x^\infty t^{s-a}e^{-t}dt$ is the upper incomplete gamma function.

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Appendix A: Omitted proofs

Firstly, we introduce the following general integral representation result (Corollary 3.2 in Alvarez 2004), which will be referred later on.

Lemma 7.1 *A Assume that* $f \in C^2(\mathbb{R}_+)$, *that* $\lim_{x\to 0+}|f(x)| < \infty$ *and that* $(\mathcal{A} - r)f(x) \in \mathcal{L}^1$. *Then*

$$\frac{f'(x)\psi(x)}{S'(x)} - \frac{\psi'(x)f(x)}{S'(x)} = \int_0^x \psi(t)\big((\mathcal{A} - r)f\big)(t)m'(t)dt - \delta,$$

where $\delta = 0$ *if* 0 *is unattainable and* $\delta = Bf(0)/\varphi(0)$ *if* 0 *is attainable for* X_t.

B Assume that $f \in C^2(\mathbb{R}_+)$, *that* $\lim_{x\to\infty} f(x)/\psi(x) = 0$, *and that* $(\mathcal{A}-r)f(x) \in \mathcal{L}^1$. *Then*

$$\frac{f'(x)\varphi(x)}{S'(x)} - \frac{\varphi'(x)f(x)}{S'(x)} = -\int_x^\infty \varphi(t)\big((\mathcal{A}-r)f\big)(t)m'(t)dt$$

Proof of Lemma 3.3

(A) This follows directly from Corollary 1 in Alvarez (2003), if the so called transversality condition $\lim_{t\to\infty} \mathbb{E}_x\left[e^{-rt}X_t; t < \tau_0\right] = 0$ holds. Here $\tau_0 = \inf\{t \geq 0 \mid X_t \notin \mathbb{R}+\}$. But we assumed that $x \in \mathcal{L}^1$, meaning that $\mathbb{E}_x\left[\int_0^\infty e^{-rt}X_t dt\right] < \infty$, and so the transversality condition must hold.

(B) The derivative properties follow from the derivative form (11) by using Assumption 3.2 (iv) together with the facts $\varphi' < 0$ and $\psi' > 0$. Furthermore, from straight calculation we get

$$J_p(x) - J_q(x) = \frac{(q-p)\varphi''(x)\sigma^2(x)}{2rBS'(x)},$$

which is positive due to the fact $q > p$ [Assumption 3.2(i)] and convexity of φ (item (A) of this lemma). Similarly, from straight calculation we get $I_p(x) - I_q(x) = \frac{(q-p)\psi''(x)\sigma^2(x)}{2rBS'(x)}$, which is positive due to the fact $q > p$ [Assumption 3.2(i)] and convexity of ψ (item (A) of this lemma). □

Proof of Lemma 4.3

(A) The function V satisfies the differential equation $(\mathcal{A}-r)V + \pi(x) = 0$ on the interval (z^*, y^*). Differentiating this we obtain

$$\frac{1}{2}\sigma^2(x)V'''(x) = (r - \mu'(x))V'(x) - \big(\mu(x) + \sigma(x)\sigma'(x)\big)\, V''(x) - \pi'(x).$$

We begin by proving the claim in the case $\mu(x) + \sigma(x)\sigma'(x) \equiv 0$. Since the necessary conditions (9) hold, V is twice continuously differentiable, and $V''(z^*) = V''(y^*) = 0$ and $V'(z^*) = q > p = V'(y^*)$, so

$$\begin{aligned} &\frac{1}{2}\sigma^2(z^*)V'''(z^*) = (r - \mu'(z^*))V'(z^*) - \pi'(z^*) = -\rho_q'(z^*) < 0 \\ \text{and} \quad &\frac{1}{2}\sigma^2(y^*)V'''(y^*) = (r - \mu'(y^*))V'(y^*) - \pi'(y^*) = -\rho_p'(y^*) > 0, \end{aligned}$$

where the inequalities follow from the facts that $z^* < \tilde{x}_q$ and $y^* > \tilde{x}_p$ (Lemma 4.1). Therefore $V''(x) \leq 0$ for all x in the neighbourhoods of z^* and y^*. Let $\bar{y} = \sup\{y \in (z^*, y^*) \mid V'''(x) < 0 \text{ for all } z^* < x < y\}$. Then, since $V''(x) < 0$ for all $x < \bar{y}$, we have $V'(x) < q$ for all $x < \bar{y}$. Further, since for all $x \leq \tilde{x}_q$ and

$b < q$ we have $0 < \rho_q'(x) = (\mu'(x) - r)q + \pi'(x) < (\mu'(x) - r)b + \pi'(x)$, and $V'''(x) = -\frac{2}{\sigma^2(x)}\rho_{V'(x)}'(x)$, we must have $\bar{y} > \tilde{x}_q$.

If $V'' \leq 0$ for all $x \in (z^*, y^*)$, then the lemma is proved. So consider for a moment, contrary to our claim, that there exists at least one point for which $V'' > 0$ and let $w_1 < y^*$ be the supremum of such points and let $w_2 < w_1$ be the supremum of the points for which V'' intersects x-axis from below. In other words $V''(w_2) = 0$, $V'''(w_2) > 0$ and $V''(w_1) = 0$, $V'''(w_1) < 0$ and $V'(w_1) > p$. In fact we also have $V'(w_1) \leq q$; If this would not be true, we would have $0 < -\frac{1}{2}\sigma^2(w_1)V'''(w_1) = \rho_{V'(w_1)}'(w_1) < \rho_q'(w_1)$, which contradicts Assumption 3.2(iv), since above we have shown that $\tilde{x}_q < \bar{y} < w_1$.

Since $V''(x) \geq 0$ for all $w_2 < x < w_1$, we have $V'(w_2) < V'(w_1)$. Thus we can calculate that $0 > -\frac{1}{2}\sigma^2(w_2)V'''(w_2) = \rho_{V'(w_2)}(w_2) > \rho_{V'(w_1)}(w_2)$, but above we chose w_1 so that $0 < -\frac{1}{2}\sigma^2(w_1)V'''(w_1) = \rho_{V'(w_1)}'(w_1)$. Since $V'(w_1) \in (p, q)$, this contradicts Assumption 3.2(iv), since $w_2 < w_1$. Therefore we must have $V''(x) \leq 0$ for all $x \in (z^*, y^*)$.

We now turn to the case $\delta(x) := \mu(x) + \sigma(x)\sigma'(x) \not\equiv 0$. Let us introduce a change of variable $f(x) = \int_0^x \exp(\int_0^u \delta(v)dv)du$ and define a function $l'(y) = (V' \circ f^{-1})(y)$. Then by straight derivation

$$\frac{1}{2}(\sigma^2 \circ f^{-1})(y)l'''(y) = \frac{\big(r - (\mu' \circ f^{-1})(y)\big)\, l'(y) - (\pi' \circ f^{-1})(y)}{(f' \circ f^{-1})^2(y)}.$$

Since $l''(f(x)) = V''(x)/f'(x)$, we see that $l''(f(x))$ has the same sign as $V''(x)$ and thus the claimed property of V follows from that of l.

(B) From (12) we see that $V'(x) > 0$ for all $x \leq z^*$ and $x \geq y^*$. Since, by item (A) $V''(x) \leq 0$ in between, we must also have $V'(x) > 0$ for $x \in (z^*, y^*)$. □

Proof of Lemma 5.2

(A) Let $\tilde{y}_b \in (0, \tilde{x}_b)$ be such that $\mu(x)$ is concave for all $0 < x \leq \tilde{y}_b$ and let $x < \tilde{y}_b$. We can write

$$-\int_x^{\tilde{x}_b} \frac{\varphi'(t)}{S'(t)}dt = -\int_x^{\tilde{y}_b} \frac{\varphi'(t)}{S'(t)}dt - \int_{\tilde{y}_b}^{\tilde{x}_b} \frac{\varphi'(t)}{S'(t)}dt.$$

Here the latter integral in the right-hand side is finite, so we need to show that former one tends to infinity when x tends to zero. To that end let us inspect more closely the associated diffusion $\hat{X}_t$. Straight calculation shows the density of the scale function and the density of the speed measure to be $\hat{S}'(x) = S'(x)/\sigma^2(x)$ and $\hat{m}'(x) = 2/S'(x)$. Moreover, by convexity of φ [Lemma 3.3(A)], we can verify the decreasing fundamental solution to be $\hat{\varphi}(x) = -\varphi'(x)$. Utilizing these together with the concavity of μ allows us to write

$$-\int_x^{\tilde{y}_b} \frac{\varphi'(v)}{S'(v)}dv = \frac{1}{2}\int_x^{\tilde{y}_b} \frac{\mu'(v)-r}{\mu'(v)-r}\hat{\varphi}(v)\hat{m}'(v)dv$$
$$< \frac{1}{2(\mu'(0)-r)}\int_x^{\tilde{y}_b} (\mu'(v)-r)\hat{\varphi}(v)\hat{m}'(v)dv.$$

We can now use Lemma 7.1(B) for the diffusion $\hat{X}_t$ to obtain

$$\frac{1}{2(\mu'(0)-r)}\int_x^{\tilde{y}_b} (\mu'(v)-r)\hat{\varphi}(v)\hat{m}'(v)dv = \frac{1}{2(\mu'(0)-r)}\left(\frac{\hat{\varphi}'(x)}{\hat{S}'(x)} - \frac{\hat{\varphi}'(\tilde{y}_b)}{\hat{S}'(\tilde{y}_b)}\right).$$

Assumed boundary behaviour for $\hat{X}_t$ at 0 and the fact that $\mu'(0) - r < 0$ guarantee that this approach to infinity as x approach to zero, which was desired.

(B) Derivating (3) we get

$$B(R_r\mathrm{id})'(x) = \varphi'(x)\int_0^x \psi(t)tm'(t)dt + \psi'(x)\int_x^\infty \varphi(t)tm'(t)dt.$$

We know that $\lim_{x\downarrow 0}\varphi'(x)\int_0^x \psi(t)tm'(t)dt \leq 0$, so we must have $\lim_{x\downarrow 0}\psi'(x)\int_x^\infty \varphi(t)tm'(t)dt > 0$, for otherwise $(R_r\mathrm{id})'(0)$ cannot be positive. But $\psi'(0+) = 0$, so $\lim_{x\downarrow 0}\int_x^\infty \varphi(t)tm'(t)dt = \infty$. The proof is completed by showing that this integral is smaller than the claimed one. To see this, apply Fubini's Theorem:

$$\int_x^\infty \varphi(t)tm'(t)dt = \int_{t=x}^\infty \int_{v=0}^t \varphi(t)m'(t)dtdv \leq \lim_{u\to 0}\int_{v=u}^\infty \int_{t=v}^\infty \varphi(t)m'(t)dtdv$$
$$= \lim_{u\to 0} -\frac{1}{r}\int_u^\infty \frac{\varphi'(v)}{S'(v)}dv,$$

where the last equality follows from Lemma 7.1 (B). Now, since

$$\int_u^\infty \frac{\varphi'(v)}{S'(v)}dv = \int_u^{\tilde{x}_b} \frac{\varphi'(v)}{S'(v)}dv + \int_{\tilde{x}_b}^\infty \frac{\varphi'(v)}{S'(v)}dv$$

and the last integral in the right hand side is finite, this completes the proof. □

Proof of Lemma 5.3

Let us first prove the integral form for the function $J_b(x)$. Since φ satisfies the differential equation $(\mathcal{A}-r)\varphi=0$ and $(R_r\pi)(x)$ satisfies $(\mathcal{A}-r)(R_r\pi)=-\pi$, we can write J_b from (10) as

$$J_b(x)=\frac{1}{rBS'(x)}\left[\frac{1}{2}b\sigma^2(x)\varphi''(x)-\pi(x)\varphi'(x)+r\big((R_r\pi)(x)\varphi'(x)-(R_r\pi)'(x)\varphi(x)\big)\right]$$
$$=\frac{1}{Br}\left[\frac{1}{2}b\sigma^2(x)\frac{\varphi''(x)}{S'(x)}+r\int_x^\infty\varphi(t)\big(\pi(x)-\pi(t)\big)m'(t)dt\right],$$

where the integral representation follows from Lemma 7.1. Now to cope with the first term observe that since $(\mathcal{A}-r)\varphi=0$, we can write

$$\frac{1}{2}\sigma^2(x)\frac{\varphi''_x}{S'_x}=r\frac{\varphi_x-x\varphi'_x}{S'}-(\mu_x-rx)\frac{\varphi'_x}{S'_x}=r\frac{\varphi_x-x\varphi'_x}{S'}+(\mu_x-rx)r\int_x^\infty\varphi_t m'_t dt$$

so that we need an integral form to $\frac{\varphi_x-x\varphi'_x}{S'}$. But choosing $f(x)=x$ in Lemma 7.1, we get $\frac{\varphi_x-x\varphi'_x}{S'}=-\int_x^\infty\varphi_t(\mu_t-rt)m'_t dt$. Combining all these forms together gives the desired integral representation for $J_b(x)$. The proof for $I_b(x)$ is similar. □

Proof of Lemma 5.4

(A) To calculate the value at the upper boundary, let $x>\tilde{x}_b$, for $b=p,q$. Using the integral representation from Lemma 5.3 we can calculate that

$$\lim_{x\to\infty}J_b(x)=\lim_{x\to\infty}$$
$$-\frac{1}{B}\left(\int_x^\infty\varphi_t(\rho_b(x)-\rho_b(t))m'_t dt\right)\le\lim_{x\to\infty}-\frac{1}{B}\left(\int_x^\infty\varphi_t(\rho_b(t)-\rho_b(t))m'_t dt\right)=0,$$

where the inequality follows from Assumption 3.2(iv).

To calculate the value at the lower boundary, let $\tilde{y}_b\in(0,\tilde{x}_b)$ and $\varepsilon>0$ be such that $\rho'_b(x)>\varepsilon$ for all $0\le x\le\tilde{y}_b$. This is possible for some constant ε since $\rho'_b(0+)>0$ [Assumption 5.1(v)]. Let $x<\tilde{y}_b$ and apply Fubini's Theorem, Lemma 7.1(B) and inequality $\rho'_b(x)>\varepsilon$ to get

$$J_b(x)=\frac{1}{B}\int_{t=x}^\infty\int_{v=x}^t\varphi(t)m'(t)\rho'_b(v)dvdt=\frac{1}{B}\int_{v=x}^\infty\int_{t=v}^\infty\varphi(t)m'(t)\rho'_b(v)dtdv$$

$$= -\frac{1}{Br}\int_{x}^{\infty}\frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv = -\frac{1}{Br}\int_{x}^{\tilde{y}_b}\frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv - \frac{1}{Br}\int_{\tilde{y}_b}^{\infty}\frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv$$

$$> -\frac{\varepsilon}{Br}\int_{x}^{\tilde{y}_b}\frac{\varphi'(v)}{S'(v)}dv - \frac{1}{Br}\int_{\tilde{y}_b}^{\infty}\frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv.$$

Here the last integral term is finite and $\lim_{x\downarrow 0} -\int_x^{\tilde{y}_b}\frac{\varphi'(v)}{S'(v)}dv = \infty$ by Assumption 5.1(vi), so $J_b(0+) = \infty$.

(B) To calculate the value at the lower boundary, let $x < \tilde{x}_b$, for $b = p, q$. Using the integral representation from Lemma 5.3 we can calculate that

$$\lim_{x\to 0} I_b(x) = \lim_{x\to 0}\frac{1}{B}\left(\int_0^x \psi_t(\rho_b(x) - \rho_b(t))m_t'dt\right)$$

$$\geq \lim_{x\to 0}\frac{1}{B}\left(\int_0^x \psi_t(\rho_b(t) - \rho_b(t))m_t'dt\right) = 0,$$

where the inequality follows from Assumption 3.2(iv).

To calculate the value at the upper boundary, let $x > \tilde{x}_b$. We can write

$$I_b(x) = \frac{1}{B}\int_0^{\tilde{x}_b}\psi_t\big(\rho_b(x) - \rho_b(t)\big)m_t'dt + \frac{1}{B}\int_{\tilde{x}_b}^{x}\psi_t\big(\rho_b(x) - \rho_b(t)\big)m_t'dt$$

$$= \frac{1}{Br}\big(\rho_b(x) - \rho_b(\eta)\big)\frac{\psi'(\tilde{x}_b)}{S'(\tilde{x}_b)} + \frac{1}{B}\int_{\tilde{x}_b}^{x}\psi_t\big(\rho_b(x) - \rho_b(t)\big)m_t'dt,$$

for some $\eta \in (0, \tilde{x}_b)$ by mean value theorem for integrals. The last term in the last row is always negative, since $\rho_b(x) < \rho_b(t)$ for all $t > x > \tilde{x}_b$ and the first term in the last row tends to minus infinity as x tends to infinity since by Assumption 5.1(v) $\rho_b(\infty) = -\infty$. Hence $I_b(\infty) = -\infty$. □

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Article III

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A Dynkin game with asymmetric information

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We study a Dynkin game with asymmetric information. The game has a random expiry time, which is exponentially distributed and independent of the underlying process. The players have asymmetric information on the expiry time, namely only one of the players is able to observe its occurrence. We propose a set of conditions under which we solve the saddle point equilibrium and study the implications of the information asymmetry. Results are illustrated with an explicit example.

Keywords: Dynkin game; Nash equilibrium; Israeli option; default risk; linear diffusion; resolvent operator

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1. Introduction

Dynkin games are game variants of optimal stopping problems, for the seminal study see [12]. Such a game has two players, 'buyer' and 'issuer', and both of them can stop the underlying process prior to the terminal time. In this paper, we study the following formulation of the game. First, we assume that the underlying process X is a time homogenous diffusion; we will elaborate the assumptions on X in the next section. At the initial time $t = 0$, the players choose their own stopping times τ (buyer) and γ (issuer) and at the time of the first exercise, i.e. at $\tau \wedge \gamma$, the issuer pays the buyer the amount

$$g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}} + g_3(X_\gamma)1_{\{\tau=\gamma\}}; \tag{1.1}$$

we will pose assumptions on the pay-off functions g_i in the next section. An interpretation of this is that, at any stopping time γ, the issuer can cancel the buyer's right to exercise, but she has to pay the cost $g_2(X_\gamma)$ to do so. Now, it is the buyer's (issuer's) objective to choose the stopping time τ (γ) such that the expected present value of the exercise pay-off

$$\Pi(x, \tau, \gamma) = \mathbf{E}_x\left\{e^{-r(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}} + g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]\right\} \tag{1.2}$$

is maximized (minimized). Here, $r > 0$ is the constant rate of discounting.

The objective of this paper is to study a version of this game with random time horizon, the infinite horizon game given by the expression (1.2) being already analysed comprehensively, e.g. in [2] and [14]. To introduce the random time horizon, we assume

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that, in addition to the diffusion X, there is also an independent Poisson process N defined on the underlying probability space. Furthermore, we assume that the game expires at the first jump time of the Poisson process, in other words we assume that the game has an exponentially distributed random time horizon. The existence of the terminating event and its rate is assumed to be known to the players, while the information of it is asymmetric: we assume that the occurrence of the expiring event is observable *only to one* of the players. Here, the information asymmetry has an interpretation as inside information. Indeed, the player who observes the default taking place has more information than is commonly available on the market and can be considered as an insider. We make a distinction between the cases when either buyer (Game 1) or issuer (Game 2) observes the jump of the Poisson process and study both of these cases separately.

Optimal stopping games are relevant for financial applications. For instance, a game variant of an American option, where the issuer has the possibility to terminate the contract early by making a payment, is introduced in [20] by Kifer. He shows that the pricing and hedging of this contract reduces to solving a saddle point of an associated Dynkin game and coins the contract as a game or an Israeli option. Explicit solutions for some perpetual Israeli options are computed in [21]. Furthermore, a characterization of the value function of a perpetual game option in terms of excessive functions is provided in [13] for general one-dimensional diffusion dynamics along with a further discussion on explicitly solvable games. For the pricing theory of Israeli options in a general semimartingale framework, see e.g. [18] and [4]. There is also a branch of literature that studies convertible bonds (or more general contingent claims) in terms of defaultable Israeli options, see e.g. [19] and the series of papers including [4] and [5] for more recent references. A convertible bond is a derivative security which can be converted into a given number of stocks by the holder and cancelled for a charge by the issuer. Thus, the pricing of such contract has a natural game-theoretic character. Our study touches this branch, since our random time horizon can be naturally interpreted as a default time of the game. In addition, a Dynkin game with random time horizon can be regarded also as a Canadized version of a finite horizon Israeli option. Canadization is a method for pricing options with finite maturity introduced originally in [9] and further extended in, for example, [21] and [22]. The concept of Canadization was extended in [7] to handle stochastic control problems.

In economic applications of game theory, asymmetric information is an important concept and, as we mentioned, our specification is compatible with the notion of inside information. Obviously, this is not the only way one can formulate asymmetric information in a game. For example, in [8] the game is set up such that neither of the players know the true pay-off but they have only partial information on it whereas, in contrast to our game, the time horizon is deterministic and known to both players. So in general the information sets of the players are separate. However, if the pay-off matrix (g_{ij}) defined in Section 1 in [8] is reduced to a vector, be it row or column, then one of the players will have full information as she knows the true pay-off. In this case, there is an inclusion of the information sets and the interpretation of inside information applies.

Our approach to the problem is built on Markovian approach to Dynkin games. There is a substantial literature in this area highlighting various parts of the theory. For instance, studies [2] and [3] are concerned with deriving explicit characterization for the value and saddle point equilibrium using classical theory of diffusions and standard nonlinear programming techniques. A generalized concavity approach is used in [13] and [14] to produce the optimal solution via the theory of excessive functions. In [15] and [25], the authors study equilibrium properties of Dynkin games under very general Markovian set-up. Our set-up and approach is closely related to [2] and can be regarded as a partial extension of it. We start our analysis by first deriving partly heuristically a free boundary problem which gives us a candidate for the solution. To set up the free boundary problem, we assume that the optimal continuation region is an interval with compact closure with constant thresholds. Given the time homogeneity of the diffusion X and the fact that the discount rate r and the jump rate of N are constants, this is indeed a reasonable assumption.

We derive necessary and sufficient conditions for the existence of a unique Nash equilibrium for Games 1 and 2 under which the value functions can be expressed in a (quasi-)explicit form. These values admit a decomposition on continuation region into terminal pay-off and early exercise premium. We also carry out a comparison of the solutions showing that whenever Games 1 and 2 have a saddle point solution, the value of Game 1 dominates the value of Game 2. Furthermore, we show that if the pay-off g_2 is non-negative, the value of the infinite horizon game dominates both the value of Games 1 and 2. Interestingly, we find that if g_2 admits also negative values, then the value of the infinite horizon game can even be the smallest of the three. We discuss also the symmetric information case where the expiring event is not observable to either of the players – denote this as Game 3. In this case, we find that the value is in between the values of Game 1 and Game 2. We also show that the optimal continuation regions of Games 1–3 are related in a way that can be described as follows: If you are able to observe the terminating event, you will wait longer – *The more you know, the longer you wait*.

The reminder of the paper is organized as follows. In Section 2, we set up the underlying dynamics and introduce the Dynkin games. In Sections 3 and 4, we study the solvability of the games and discuss some implications of the information asymmetry. In Section 5, we compare the optimal solutions of the games and study limiting behaviour of the solutions. In Section 6, we illustrate the main results of the study with an explicit example.

2. The games

2.1. Underlying dynamics

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, with $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, be a complete filtered probability space satisfying the usual conditions, see [6], p. 2. In addition, let W be a Wiener process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$. We assume that the state process X is a regular linear diffusion defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, evolving on $\mathbf{R}_+$, and given as the solution of the Itô equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{2.1}$$

where the coefficients $\mu : \mathbf{R}_+ \to \mathbf{R}$ and $\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ are assumed to be sufficiently smooth to guarantee the existence of a unique (weak) solution of (2.1), see [6], pp. 46–47. In line with the most economical and financial applications, we assume that X does not die inside the state space $\mathbf{R}_+$, i.e., that killing of X is possible only at the boundaries 0 and ∞. Therefore, the boundaries 0 and ∞ are either natural, entrance, exit or regular. In the case a boundary is regular, it is assumed to be killing, see [6], pp. 18–20, for a characterization of the boundary behaviour of diffusions. The assumption that the state space is $\mathbf{R}_+$ is done for reasons of notational convenience. In fact, we could assume that the state space is any interval $\mathcal{I}$ in $\mathbf{R}$ and all our subsequent analysis would hold with obvious modifications. Denote as $\mathscr{A} = (1/2)\sigma^2(x)(d^2/dx^2) + \mu(x)(d/dx)$ the differential operator associated with the process X. For notational convenience we denote $\mathcal{G}_\beta = \mathscr{A} - \beta$ for a given constant $\beta > 0$.

For any given constant $\beta > 0$, we denote as $\mathcal{L}_1^\beta$ the class of real-valued measurable functions f on $\mathbf{R}_+$ satisfying the condition

$$\mathbf{E}_x\left\{\int_0^\zeta e^{-\beta t}|f(X_t)|dt\right\} < \infty,$$

where $\zeta := \inf\{t > 0 : X_t \notin \mathbf{R}_+\}$ denotes the lifetime of X. In addition, for any given constant $\beta > 0$, we denote, respectively, as ψ_β and φ_β the increasing and the decreasing solution of the ordinary second-order linear differential equation $\mathcal{G}_\beta u(x) = 0$ defined on the domain of the characteristic operator of X – for the characterization and fundamental properties of the minimal β-excessive functions ψ_β and φ_β, see [6], pp. 18–20. Denote as $B_\beta = \psi'_\beta(x)\varphi_\beta(x)/S'(x) - \varphi'_\beta(x)\psi_\beta(x)/S'(x)$ the Wronskian determinant, where

$$S'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right)$$

denotes the density of the scale function of X, see [6], p. 19. We remark that the value of the Wronskian does not depend on the initial state x but on the constant β. For a function $f \in \mathcal{L}_1^\beta$, the *resolvent* $R_\beta f : \mathbf{R}_+ \to \mathbf{R}$ is defined as

$$(R_\beta f)(x) = \mathbf{E}_x\left\{\int_0^\zeta e^{-\beta t} f(X_t)dt\right\}, \tag{2.2}$$

for all $x \in \mathbf{R}_+$. The resolvent R_β and the solutions ψ_β and φ_β are connected in a computationally very useful way. Indeed, we know from the literature, see [6], pp. 17–20 and p. 29, that for a given $f \in \mathcal{L}_1^\beta$ the resolvent $R_\beta f$ can be expressed as

$$(R_\beta f)(x) = B_\beta^{-1}\varphi_\beta(x)\int_0^x \psi_\beta(y)f(y)m'(y)dy + B_\beta^{-1}\psi_\beta(x)\int_x^\infty \varphi_\beta(y)f(y)m'(y)dy,$$

for all $x \in \mathbf{R}_+$, where $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the speed density of X.

To close the subsection, we denote as N a Poisson process with intensity $\lambda > 0$, and assume that N is independent of the underlying X. Now, the first jump time T of N is an exponentially distributed random time with mean $1/\lambda$. Denote as $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_t\}_{t\geq 0}$ the enlarged filtration defined as $\hat{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\{T \leq s\} : s \leq t)$. In other words, the filtration $\hat{\mathbb{F}}$ carries the information of the evolution of underlying X and the first jump of the Poisson process N. We denote as $\mathcal{T}_0$ the set of all $\mathbb{F}$-stopping times and as $\mathcal{T}_1$ the set $\mathcal{T}_0$ augmented with T, i.e., the set of all $\hat{\mathbb{F}}$-stopping times.

2.2. The games

Dynkin game is an optimal stopping game between two players, 'buyer' and 'issuer'. In contrast to classical optimal stopping problems, also the issuer can now exercise. Recall now the definition of the expected present value of the exercise pay-off from (1.2). We make the following standing assumptions for the pay-offs g_i.

Assumption 2.1. We assume that the pay-offs $g_i : \mathbf{R}_+ \to \mathbf{R}$, $i = 1, 2, 3$, are continuous and non-decreasing functions satisfying the ordering $g_1 \leq g_3 \leq g_2$ and that g_1 is bounded from below. Furthermore, we assume that $g_1 \in \mathcal{L}_1^r$ and $g_i \in C^1(\mathbf{R}_+) \cap C^2(\mathbf{R}_+\backslash D)$, where the set D is finite and $|g''_i(y\pm)| < \infty$ for all $y \in D$.

We make some remarks on Assumptions 2.1. First, the integrability condition is posed to guarantee that the resolvent $\lambda(R_{r+\lambda}g_1)$ is well defined for all $\lambda > 0$. This assumption is integral for our study as the resolvent gives us a necessary tool to handle the random time horizon. Second, our smoothness assumptions are stricter than those in [14] where only continuity of the pay-offs is required initially. However, in [14], Section 4, where the authors study the saddle point property (which is the main focus of our study), they assume that $g_i \in C^1(\mathbf{R}_+ \backslash D) \cap C^2(\mathbf{R}_+ \backslash D)$. A potential approach to reduce our smoothness assumptions to this case could the convolution approximation method introduced in [1]. This analysis is, however, out of the scope of our study.

In order to propose a value and notions of equilibrium for the considered games, define first the lower and upper values $\underline{V}$ and $\bar{V}$ as

$$\underline{V}(x) = \sup_{\tau \in \mathcal{T}} \inf_{\gamma \in \mathcal{T}} \Pi(x, \tau, \gamma), \quad \bar{V}(x) = \inf_{\gamma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \Pi(x, \tau, \gamma), \tag{2.3}$$

where $\mathcal{T}$ is the class of admissible stopping times. Following [14], pp. 1578, we remark that $g_1 \leq \underline{V} \leq \bar{V} \leq g_2$. If, on the other hand, the values satisfy $\underline{V} \geq \bar{V}$, we say that the game has the value $V := \underline{V} = \bar{V}$, i.e. has a Stackelberg equilibrium. Moreover, if there exists stopping times τ^* and γ^* such that

$$\Pi(x, \tau, \gamma^*) \leq \Pi(x, \tau^*, \gamma^*) \leq \Pi(x, \tau^*, \gamma),$$

for all $x \in \mathbf{R}_+$, then the pair (τ^*, γ^*) constitutes a saddle point, i.e., a Nash equilibrium of the game. We remark that the existence of a saddle point implies the existence of the value but the converse does not hold in general – for a study addressing this problem in a general Markovian setting, see [15]. However, in our setting the underlying process is nice enough so that Stackelberg equilibrium is equivalent to Nash equilibrium.

The main objective of this paper is to study two Dynkin games which are associated via a certain type of information asymmetry. To make a precise statement, recall the Poisson process N from the previous section. At the initial time $t = 0$, the underlying X and exogenous N are both started. At the first jump time T, the game ends. Thus, the considered games have an exponentially distributed random time horizon which is independent of X. The information asymmetry is introduced as follows: we assume that the occurrence of the expiring event is observable *only to one* of the players. Let us formalize this setting first in the case when T is observable to the *buyer*; later this case will be referred to as Game 1. First, recall the definitions of the sets $\mathcal{T}_0$ and $\mathcal{T}_1$ from the previous subsection. At the start of the game, issuer chooses a stopping time from the set $\mathcal{T}_0$ and the buyer from the set $\mathcal{T}_1$. The expected present value Π_1 of the exercise pay-off is written as

$$\Pi_1(x, \tau, \gamma) = \mathbf{E}_x\left\{ e^{-r(\tau \wedge \gamma)} \left[g_1(X_\tau) 1_{\{\tau < \gamma\}} + g_2(X_\gamma) 1_{\{\tau > \gamma\}} + g_3(X_\gamma) 1_{\{\tau = \gamma\}} \right] 1_{\{\tau \wedge \gamma \leq T\}} \right\}, \tag{2.4}$$

and the upper and lower values are defined as

$$\underline{V}_1(x) = \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau, \gamma), \quad \bar{V}_1(x) = \inf_{\gamma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_1} \Pi_1(x, \tau, \gamma). \tag{2.5}$$

For Game 1, we denote the value function as V_1 and a saddle point equilibrium as $\left(\tau_1^*, \gamma_1^*\right)$.

The set-up of the second game, which will be referred to as Game 2, is completely analogous. For Game 2, we assume that the random time T is a stopping time to issuer. Similarly to Game 1, we define the expected present value Π_2 of the exercise pay-off as

$$\Pi_2(x, \tau, \gamma) = \mathbf{E}_x\left\{ e^{-r(\tau \wedge \gamma)} \left[g_1(X_\tau) 1_{\{\tau < \gamma\}} + g_2(X_\gamma) 1_{\{\tau > \gamma\}} + g_3(X_\gamma) 1_{\{\tau = \gamma\}} \right] 1_{\{\gamma \wedge \tau \leq T\}} \right\}, \tag{2.6}$$

and the upper and lower values are defined as

$$\underline{V}_2(x) = \sup_{\tau\in\mathcal{T}_0}\inf_{\gamma\in\mathcal{T}_1}\Pi_2(x,\tau,\gamma), \quad \bar{V}_2(x) = \inf_{\gamma\in\mathcal{T}_1}\sup_{\tau\in\mathcal{T}_0}\Pi_2(x,\tau,\gamma). \tag{2.7}$$

Analogously to Game 1, the value function of Game 2 is denoted as V_2 and a saddle point equilibrium as (τ_2^*, γ_2^*).

3. Game 1

3.1. Equivalent formulation of the game

First, we introduce some additional definitions and notations. Following [2] (see also [27]), define the operators L_ψ^β and L_φ^β for sufficiently smooth functions $f : \mathbf{R}_+ \to \mathbf{R}$ as

$$\begin{cases} \left(L_\psi^\beta f\right)(x) = \frac{f'(x)}{S'(x)}\psi_\beta(x) - \frac{\psi'_\beta(x)}{S'(x)}f(x), \\ \left(L_\varphi^\beta f\right)(x) = \frac{f'(x)}{S'(x)}\varphi_\beta(x) - \frac{\varphi'_\beta(x)}{S'(x)}f(x), \end{cases} \tag{3.1}$$

for a given constant $\beta > 0$. In order to simplify the upcoming notation, define the functions $\hat{g}_i : \mathbf{R}_+ \to \mathbf{R}$, $i = 1,2$, as

$$\begin{cases} \hat{g}_1(x) = g_1(x) - \lambda\left(R_{r+\lambda}g_1^+\right)(x), \\ \hat{g}_2(x) = g_2(x) - \lambda\left(R_{r+\lambda}g_1^+\right)(x), \end{cases} \tag{3.2}$$

where $g_1^+(x) = \max\{g_1(x), 0\}$. We remark that since we assumed $g_1 \le g_2$, also $\hat{g}_1 \le \hat{g}_2$.

In this subsection, we transform Game 1 into an adjusted perpetual game and study its solvability. To this end, we derive first a candidate G_1 for the optimal value function in a partly heuristic way – for a related study in a different context, see [16]. We start with the *ansatz* that the game has a saddle point equilibrium. Because the exponential distribution has memoryless property and the underlying dynamic structure is time homogeneous, we assume that the state space $\mathbf{R}_+$ is partitioned into continuation and action regions, where the continuation region $(z_1^*, y_1^*) \subset \mathbf{R}_+$ has compact closure. If $x \in (z_1^*, y_1^*)$, the players wait by definition. Now, in an infinitesimal time interval dt, the Poisson process jumps (expiring the exercise opportunities) with probability λdt. Because the buyer can exercise at time T, she will exercise at that time if and only if $g_1 \ge 0$; this yields the terminal payoff $g_1^+(x)$. On the other hand, with probability $1 - \lambda$dt the contract lives on yielding additional expected present value. Denote as G_1 the candidate for the value function. Formally, this suggests with a heuristic use of Dynkin's theorem, see e.g. [24], that

$$\begin{aligned} G_1(x) &= g_1^+(x)\lambda \mathrm{d}t + (1-\lambda \mathrm{d}t)\mathbf{E}_x\left[\mathrm{e}^{-r\mathrm{d}t}G_1(X_{dt})\right] \\ &= \lambda g_1^+(x)\mathrm{d}t + (1-\lambda\mathrm{d}t)[G_1(x) + \mathcal{G}_r G_1(x)\mathrm{d}t] \\ &= G_1(x) + \mathcal{G}_r G_1(x)\mathrm{d}t + \lambda\left(g_1^+(x) - G_1(x)\right)\mathrm{d}t, \end{aligned}$$

for all $x \in (z_1^*, y_1^*)$ under the intuition $\mathrm{d}t^2 = 0$. This yields the condition

$$\mathcal{G}_{r+\lambda}G_1(x) = -\lambda g_1^+(x), \tag{3.3}$$

for all $x \in (z_1^*, y_1^*)$ – for an analogous result, see Equation (10) in [9]. The solutions of equation (3.3) can be expressed as $G_1(x) = \lambda\left(R_{r+\lambda}g_1^+\right)(x) + c_1\psi_{r+\lambda}(x) + c_2\varphi_{r+\lambda}(x)$ for some positive constants c_1 and c_2. We assume that the candidate G_1 satisfies that the value-matching condition, i.e., is continuous over the boundary of (z_1^*, y_1^*). This condition can be expressed as

$$\begin{cases} \lambda\left(R_{r+\lambda}g_1^+\right)\left(z_1^*\right) + c_1\psi_{r+\lambda}\left(z_1^*\right) + c_2\varphi_{r+\lambda}\left(z_1^*\right) = g_2\left(z_1^*\right), \\ \lambda\left(R_{r+\lambda}g_1^+\right)\left(y_1^*\right) + c_1\psi_{r+\lambda}\left(y_1^*\right) + c_2\varphi_{r+\lambda}\left(y_1^*\right) = g_1\left(y_1^*\right). \end{cases}$$

Using the notation from (3.2), it is a matter of elementary algebra to show that

$$\begin{cases} c_1 = \dfrac{\varphi_{r+\lambda}\left(y_1^*\right)\hat{g}_2\left(z_1^*\right) - \varphi_{r+\lambda}\left(z_1^*\right)\hat{g}_1\left(y_1^*\right)}{\varphi_{r+\lambda}\left(y_1^*\right)\psi_{r+\lambda}\left(z_1^*\right) - \varphi_{r+\lambda}\left(z_1^*\right)\psi_{r+\lambda}\left(y_1^*\right)} =: h_1\left(z_1^*, y_1^*\right) \\ c_2 = \dfrac{\psi_{r+\lambda}\left(z_1^*\right)\hat{g}_1\left(y_1^*\right) - \psi_{r+\lambda}\left(y_1^*\right)\hat{g}_2\left(z_1^*\right)}{\varphi_{r+\lambda}\left(y_1^*\right)\psi_{r+\lambda}\left(z_1^*\right) - \varphi_{r+\lambda}\left(z_1^*\right)\psi_{r+\lambda}\left(y_1^*\right)} =: h_2\left(z_1^*, y_1^*\right). \end{cases} \tag{3.4}$$

To proceed, denote as $\tau_{(z_1^*, y_1^*)}$ the first exit time of X from the interval (z_1^*, y_1^*). We know from Theorem 13.11 in [11], that the function $x \mapsto \mathbf{E}_x\left[\mathrm{e}^{-(r+\lambda)\tau_{(z_1^*, y_1^*)}}\right]$ solves the boundary value problem $\mathcal{G}_{r+\lambda}u(x) = 0$ on (z_1^*, y_1^*) with boundary conditions $u(z_1^*) = u(y_1^*) = 1$. Using this, we find that

$$\mathbf{E}_x\left\{\mathrm{e}^{-(r+\lambda)\left(\tau_{y_1^*}\wedge\gamma_{z_1^*}\right)}1_{\{\tau_{y_1^*}<\gamma_{z_1^*}\}}\right\} = \frac{\varphi_{r+\lambda}(x)\psi_{r+\lambda}\left(z_1^*\right) - \varphi_{r+\lambda}\left(z_1^*\right)\psi_{r+\lambda}(x)}{\varphi_{r+\lambda}\left(y_1^*\right)\psi_{r+\lambda}\left(z_1^*\right) - \varphi_{r+\lambda}\left(z_1^*\right)\psi_{r+\lambda}\left(y_1^*\right)},$$

$$\mathbf{E}_x\left\{\mathrm{e}^{-(r+\lambda)\left(\tau_{y_1^*}\wedge\gamma_{z_1^*}\right)}1_{\{\tau_{y_1^*}>\gamma_{z_1^*}\}}\right\} = \frac{\varphi_{r+\lambda}\left(y_1^*\right)\psi_{r+\lambda}(x) - \varphi_{r+\lambda}(x)\psi_{r+\lambda}\left(y_1^*\right)}{\varphi_{r+\lambda}\left(y_1^*\right)\psi_{r+\lambda}\left(z_1^*\right) - \varphi_{r+\lambda}\left(z_1^*\right)\psi_{r+\lambda}\left(y_1^*\right)},$$

see also [23]. Consequently, the candidate G_1 can be rewritten as

$$\begin{aligned} G_1(x) = \lambda(R_{r+\lambda}g_1^+)(x) &+ \hat{g}_1\left(y_1^*\right)\mathbf{E}_x\left\{\mathrm{e}^{-(r+\lambda)(\tau_{y_1^*}\wedge\gamma_{z_1^*})}1_{\{\tau_{y_1^*}<\gamma_{z_1^*}\}}\right\} \\ &+ \hat{g}_2\left(z_1^*\right)\mathbf{E}_x\left\{\mathrm{e}^{-(r+\lambda)(\tau_{y_1^*}\wedge\gamma_{z_1^*})}1_{\{\tau_{y_1^*}>\gamma_{z_1^*}\}}\right\}, \end{aligned} \tag{3.5}$$

for all $x \in (z_1^*, y_1^*)$. Since the sample paths of X are (almost surely) continuous, an application of the strong Markov property of the underlying X yields

$$\begin{aligned} G_1(x) = \boldsymbol{E}_x\Bigg\{&\lambda\int_0^{\tau_{y_1^*}\wedge\gamma_{z_1^*}}\mathrm{e}^{-(r+\lambda)s}g_1^+(X_s)\mathrm{d}s \\ &+\mathrm{e}^{-(r+\lambda)(\tau_{y_1^*}\wedge\gamma_{z_1^*})}\left[g_1(X_{\tau_{y_1^*}})1_{\{\tau_{y_1^*}<\gamma_{z_1^*}\}} + g_2(X_{\gamma_{z_1^*}})1_{\{\tau_{y_1^*}>\gamma_{z_1^*}\}}\right]\Bigg\} \end{aligned} \tag{3.6}$$

for all $x \in \mathbf{R}_+$. This result indicates the form of the equivalent perpetual game. The next proposition confirms that this partly heuristic derivation gives the correct form of the adjusted perpetual problem. For a rigorous proof we though need an auxiliary lemma.

LEMMA 3.1. For $\tau \in \mathcal{T}_1$, there exists $\tau' \in \mathcal{T}_0$ such that $\tau \wedge T = \tau' \wedge T$ a.s.

Proof. See [26], Lemma, Section VI.3, p. 378. □

PROPOSITION 3.2. The upper and lower values for Game 1 can be rewritten as

$$\bar{V}_1(x) = \inf_{\gamma\in\mathcal{T}_0}\sup_{\tau\in\mathcal{T}_0}\tilde{\Pi}_1(x,\tau,\gamma), \quad \underline{V}_1(x) = \sup_{\tau\in\mathcal{T}_0}\inf_{\gamma\in\mathcal{T}_0}\tilde{\Pi}_1(x,\tau,\gamma),$$

where

$$\tilde{\Pi}_1(x,\tau,\gamma) = \mathbf{E}_x\left\{\lambda\int_0^{\tau\wedge\gamma} e^{-(r+\lambda)s} g_1^+(X_s)ds + e^{-(r+\lambda)(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}} + g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]\right\}$$

for all $x\in\mathbf{R}_+$.

Proof. Let $\hat{\mathcal{T}}_1$ denote the set containing $\hat{\mathbb{F}}$-stopping times satisfying $\tau\leq T$ for all ω. We know that for all $\tau\in\hat{\mathcal{T}}_1$, there is a $\tau'\in\mathcal{T}_1$ for which $\tau'=\tau\wedge T$. Because buyer's objective is to maximize the expected present value of the pay-off and she is aware that after the observable expiry time T the pay-off will be zero, we reason that

$$\begin{aligned}
V_1(x) &= \sup_{\tau\in\mathcal{T}_1}\inf_{\gamma\in\mathcal{T}_0}\mathbf{E}_x\{e^{-r(\tau\wedge\gamma)}[g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}} + g_3(X_\gamma)1_{\{\tau=\gamma\}}]1_{\{\tau\wedge\gamma\leq T\}}\}\\
&= \sup_{\tau\in\mathcal{T}_1}\inf_{\gamma\in\mathcal{T}_0}\mathbf{E}_x\{e^{-r(\tau\wedge\gamma)}\left[(g_1(X_\tau)1_{\{\tau<T\}} + g_1^+(X_T)1_{\{\tau\geq T\}})1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}}\right.\\
&\quad + g_3(X_\gamma)1_{\{\tau=\gamma\}}]1_{\{\tau\wedge\gamma\leq T\}}\}\\
&= \sup_{\hat{\tau}\in\hat{\mathcal{T}}_1}\inf_{\gamma\in\mathcal{T}_0}\mathbf{E}_x\{e^{-r(\hat{\tau}\wedge\gamma)}\left[(g_1(X_{\hat{\tau}})1_{\{\hat{\tau}<T\}} + g_1^+(X_T)1_{\{\hat{\tau}\geq T\}})1_{\{\hat{\tau}<\gamma\}} + g_2(X_\gamma)1_{\{\hat{\tau}>\gamma\}}\right.\\
&\quad + g_3(X_\gamma)1_{\{\hat{\tau}=\gamma\}}]1_{\{\hat{\tau}\wedge\gamma\leq T\}}\}\\
&= \sup_{\hat{\tau}\in\hat{\mathcal{T}}_1}\inf_{\gamma\in\mathcal{T}_0}\mathbf{E}_x\{e^{-r(\hat{\tau}\wedge\gamma)}\left[(g_1(X_{\hat{\tau}})1_{\{\hat{\tau}<T\}} + g_1^+(X_T)1_{\{\hat{\tau}\geq T\}})1_{\{\hat{\tau}<\gamma\}} + g_2(X_\gamma)1_{\{\hat{\tau}>\gamma\}}\right.\\
&\quad + g_3(X_\gamma)1_{\{\hat{\tau}=\gamma\}}]\}\\
&= \sup_{\tau\in\mathcal{T}_1}\inf_{\gamma\in\mathcal{T}_0}\mathrm{E}_x\{e^{-r((\tau\wedge T)\wedge\gamma)}\left[(g_1(X_\tau)1_{\{\tau<T\}} + g_1^+(X_T)1_{\{\tau\geq T\}})1_{\{\tau\wedge T<\gamma\}}\right.\\
&\quad + g_2(X_\gamma)1_{\{\tau\wedge T>\gamma\}} + g_3(X_\gamma)1_{\{\tau\wedge T=\gamma\}}]\}.
\end{aligned} \tag{3.7}$$

Now, it follows from Lemma 3.1 that the last expression is equivalent with the form

$$\sup_{\tau\in\mathcal{T}_0}\inf_{\gamma\in\mathcal{T}_0}\mathbf{E}_x\{e^{-r((\tau\wedge T)\wedge\gamma)}\left[\left(g_1(X_\tau)1_{\{\tau<T\}} + g_1^+(X_T)1_{\{\tau\geq T\}}\right)1_{\{\tau\wedge T<\gamma\}} + g_2(X_\gamma)1_{\{\tau\wedge T>\gamma\}} + g_3(X_\gamma)1_{\{\tau\wedge T=\gamma\}}\right]\}.$$

Finally, let $\tau,\gamma\in\mathcal{T}_0$. Since T is independent of X, we conclude that

$$\begin{aligned}
&\mathbf{E}_x\{e^{-r((\tau\wedge T)\wedge\gamma)}\left[\left(g_1(X_\tau)1_{\{\tau<T\}} + g_1^+(X_T)1_{\{\tau\geq T\}}\right)1_{\{\tau\wedge T<\gamma\}} + g_2(X_\gamma)1_{\{\tau\wedge T>\gamma\}}\right.\\
&\quad \left.+ g_3(X_\gamma)1_{\{\tau\wedge T=\gamma\}}\right](1_{\{\tau\geq T\}} + 1_{\{\tau<T\}})\}\\
&= \mathbf{E}_x\{e^{-rT}g_1^+(X_T)1_{\{\tau\wedge\gamma\geq T\}} + e^{-r(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}}\right.\\
&\quad \left.+ g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]1_{\{\tau\wedge\gamma<T\}}\}\\
&= \mathbf{E}_x\left\{\lambda\int_0^{\tau\wedge\gamma} e^{-(r+\lambda)s}g_1^+(X_s)ds + e^{-(r+\lambda)(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}}\right.\right.\\
&\quad \left.\left.+ g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]\right\},
\end{aligned}$$

for all $x \in \mathbf{R}_+$. This computation proves the claimed result for the lower value $\underline{V}_1$. The result for the upper value $\bar{V}_1$ is proved completely similarly. □

In Proposition 3.2, we showed that the random horizon game can be transformed into an equivalent adjusted perpetual game. In particular, the existence of the value function for Game 1 follows now from [14] even when the pay-offs are assumed only to be continuous. Moreover, we observe that the form of the value function (3.6) associated with constant threshold policy is consistent with Proposition 3.2. It is also worth mentioning that the buyer follows actually a stopping rule '*Stop at time* $\tau \wedge T$' which results into the pay-off $g_1(X_\tau)1_{\{\tau<T\}} + g_1^+(X_T)1_{\{\tau\geq T\}}$. This property was used in (3.7).

Put slightly different, Proposition 3.2 essentially shows that the value of the random time horizon game under the extended filtration $\hat{\mathbb{F}}$ can be determined via an associated perpetual dividend paying game under the reference filtration $\mathbb{F}$ generated by the underlying asset price X. The same idea appears in varying contexts in the series of papers by Bielecki et al. including [4] and [5]. In particular, Proposition 3.2 resembles Lemma 3.6. in [5] where asset prices follow general semimartingale processes and the default of game option is modelled using a hazard process. In our case, the hazard process would be the independent Poisson process N. We also refer to [10], where an analogous result is proved for optimal stopping problems with underlying diffusion dynamics.

3.2. Necessary conditions

Having the expression (3.6) at hand, we proceed with the derivation of necessary conditions. Define the function $Q_1 : \mathbf{R}_+^3 \to \mathbf{R}_+$ as

$$\begin{aligned} Q_1(x,z,y) &= \boldsymbol{E}_x\{\mathrm{e}^{-(r+\lambda)(\tau_y\wedge\gamma_z)}\big[\hat{g}_1(X_{\tau_y})1_{\{\tau_y<\gamma_z\}} + \hat{g}_2(X_{\gamma_z})1_{\{\tau_y>\gamma_z\}}\big]\} \\ &= h_1(z,y)\psi_{r+\lambda}(x) + h_2(z,y)\varphi_{r+\lambda}(x), \end{aligned}$$

recall the definition of the functions $\hat{g}_i$ and h_i, $i = 1,2$, from (3.2) and (3.4), respectively (cf. [2], expression (15)). We assume now that the thresholds z_1^* and y_1^* give rise to an extremal expression for Q_1 in the sense that for all fixed (initial) states x, the point (z_1^*, y_1^*) is a saddle point for the surface $(z,y) \mapsto Q_1(x,z,y)$. Using Lemma 4.1 in [2], we conclude that the thresholds z_1^* and y_1^* must then satisfy the conditions

$$\begin{cases} (L_\varphi^{r+\lambda}\hat{g}_2)(z_1^*) - (L_\varphi^{r+\lambda}\hat{g}_1)(y_1^*) = 0, \\ (L_\psi^{r+\lambda}\hat{g}_2)(z_1^*) - (L_\psi^{r+\lambda}\hat{g}_1)(y_1^*) = 0. \end{cases} \tag{3.8}$$

Define now the candidate

$$G_1(x) = \begin{cases} g_1(x), & x \geq y_1^*, \\ \lambda(R_{r+\lambda}g_1^+)(x) + Q_1(x, z_1^*, y_1^*), & x \in (z_1^*, y_1^*), \\ g_2(x), & x \leq z_1^*, \end{cases} \tag{3.9}$$

where z_1^* and y_1^* are given by (3.8). We point out that it follows from Lemma 4.1 in [2] that G_1 is continuously differentiable over the exercise boundaries.

It is interesting to note from (3.9) that the candidate G_1 admits a value decomposition on the continuation region. The resolvent term $\lambda(R_{r+\lambda}g_1^+)$ gives the expected present value of the terminal pay-off at the exponentially distributed independent random time T

(pay-off of a randomized European contingent claim) and the term Q_1 has the natural interpretation as the early exercise premium, cf. [9], p. 604. This decomposition is analogous to Equation (12) in [9], where a Canadian put option is considered in the classical Black–Scholes framework.

Having the candidate G_1 formulated, the next proposition contains our main result on the necessary conditions for the optimal solution for Game 1.

Proposition 3.3. Assume that there is a pair (z_1^*, y_1^*) satisfying conditions (3.8) and that there exist thresholds $\hat{x}_i$, $i = 1,2$, such that

$$\mathcal{G}_{r+\lambda}\hat{g}_i(x) \gtreqless 0, \text{ whenever } x \lesseqgtr \hat{x}_i. \tag{3.10}$$

Then the pair (z_1^*, y_1^*) is unique and $z_1^* < \hat{x}_2$ and $\hat{x}_1 < y_1^*$. Moreover the value of Game 1 reads as $V_1(x) = G_1(x)$ for all $x \in \boldsymbol{R}_+$, where G_1 is defined in (3.9).

Proof. We know from [2], Theorem 4.3, that under assumption (3.10) a pair satisfying (3.8) is necessary unique and that $z_1^* < \hat{x}_2$ and $\hat{x}_1 < y_1^*$. The proof that the value of the game reads as in (3.9) is similar to that of Theorem 4.3 in [2], the only difference being that g_i is replaced by $\hat{g}_i$, for $i = 1,2$. □

In Proposition 3.3, we showed that given the additional condition (3.10) a solution of the pair (3.8) is necessarily unique. From a practical point of view, this is a convenient result. Indeed, if we attempt to solve the pair (3.8) numerically for a particular example and our scheme converges to a solution, we can be sure that it is the unique optimal one. Condition (3.10) was needed in the proof of Proposition 3.3 to assure that functionals $L_\cdot^{r+\lambda}\hat{g}_i$ behave nicely enough for the uniqueness result to hold – remember that $(L_\cdot^{r+\lambda}\hat{g}_i)'(x) \propto \left(\mathcal{G}_{r+\lambda}\hat{g}_i\right)(x)$. We propose in the next lemma a set of sufficient conditions for the assumption (3.10).

Lemma 3.4. Assume that there are thresholds $\tilde{x}_i$, $i = 1,2$, such that

$$\mathcal{G}_r g_i(x) \gtreqless 0, \text{ whenever } x \lesseqgtr \tilde{x}_i.$$

In addition, assume that

- $g_1(x) \geq 0$ for all $x > 0$ *or* that $\mathcal{G}_r g_1$ is non-increasing, and
- $(g_1^+ - g_2)$ and $\mathcal{G}_r g_2$ are non-increasing.

Then condition (3.10) holds.

Proof. The result follows from the expressions $\mathcal{G}_{r+\lambda}\hat{g}_1 = \mathcal{G}_r g_1 + \lambda(g_1^+ - g_1)$, where $g_1^+ - g_1$ is non-increasing and $\mathcal{G}_{r+\lambda}\hat{g}_2 = \mathcal{G}_r g_2 + \lambda(g_1^+ - g_2)$. □

We note also from Proposition 3.3 that the stopping times $\tau_{y_1^*}$ and $\gamma_{z_1^*}$ do not tell the entire story about the optimal stopping rules. Indeed, the optimal stopping rule for the issuer is '*stop at time* $\gamma_{z_1^*} = \inf\{t \geq 0 | X_t \leq z_1^*\}$', but for the buyer optimal rule is '*stop at time* $\tau_{y_1^*} = \inf\{t \geq 0 | X_t \geq y_1^*\}$, *but if* $T < \tau_{y_1^*}$, *stop at time* T *whenever* $g_1(X_T) > 0$' so the optimal rule for the buyer is not a pure threshold rule. However, it is analogous to the exercise rule of a finite horizon American option. Consider, for example, American call option in classical Black–Scholes framework. Then it is well known that the optimal exercise boundary is given by a decreasing, concave curve in space-time truncated at the fixed terminal time. In our case, the buyer's optimal exercise boundary is of rectangular shape in space-time, but its length in time is random – see [9], p. 605, for the same observation for a Canadian put option.

While Proposition 3.3 catches a relatively large range of problems, our assumptions are not usually satisfied if exercise pay-offs have option characteristics – for example if $g_i(x) = (x - c_i)^+$, where $c_1 > c_2 > 0$. In the next result, we propose a set of necessary conditions for a class of simple option type problems.

Corollary 3.5. Assume that there exists $\bar{x}_i < \hat{x}_i$ so that $g_i(x) = 0$ on $(0, \bar{x}_i)$, $(\mathcal{G}_{r+\lambda}\hat{g}_i) > 0$ on $(\bar{x}_i, \hat{x}_i)$, and $(\mathcal{G}_{r+\lambda}\hat{g}_i) < 0$ on $(\hat{x}_i, \infty)$, $i = 1,2$. Assume also that the threshold

$$\hat{y}^*_{\bar{x}_2} = \operatorname*{argmax}_y \left\{ \frac{\hat{g}_1(y)}{\psi_{r+\lambda}(y) - \frac{\psi_{r+\lambda}(\bar{x}_2)}{\varphi_{r+\lambda}(\bar{x}_2)} \varphi_{r+\lambda}(y)} \right\}$$

exists. If there exists a pair $(z_1^*, y_1^*) \in (\bar{x}_2, \hat{x}_2) \times (\hat{x}_1, \hat{y}^*_{\bar{x}_2})$ satisfying the first-order conditions (3.8), then the conclusion of Proposition 3.3 is satisfied and the value of the game reads as in (3.9).

Proof. The result follows from Proposition 3.3 after noticing that $\hat{y}^*_{\bar{x}_2}$ is the corner solution to the lower equation of (3.8) (or its alternative formulation, see (20) in [2]). □

If there does not exist an internal solution, then the pair $(\bar{x}_2, \hat{y}^*_{\bar{x}_2})$ constitutes a corner solution, which is a saddle point solution and the solution reads as

$$V_1(x) = \begin{cases} g_1(x), & x \geq \hat{y}^*_{\bar{x}_2}, \\ \lambda\left(R_{r+\lambda} g_1^+\right)(x) + Q_1\left(x, \bar{x}_2, \hat{y}^*_{\bar{x}_2}\right), & x \in \left(\bar{x}_2, \hat{y}^*_{\bar{x}_2}\right), \\ 0, & x \leq \bar{x}_2. \end{cases}$$

As mentioned before, one possible way to generalize the result above to a more general class of option type pay-offs would be the use of convolution approximation method from [1], but this is out of the scope of this study.

3.3. *Sufficient conditions*

The main objective of this section is to propose a set of sufficient conditions for the solvability of the game. To this end, we prove first the following lemma.

Lemma 3.6. Let $b \in \boldsymbol{R}_+$. Then $\frac{\psi_r(x)}{\psi_r(b)} > \frac{\psi_{r+\lambda}(x)}{\psi_{r+\lambda}(b)}$ for all $x < b$ and the function $x \mapsto \frac{\psi_{r+\lambda}(x)}{\psi_r(x)}$ is monotonically increasing.

Proof. Let $x < b < \infty$. From [6], p. 18, we have $\boldsymbol{E}_x\{e^{-r\tau_b}\} = \frac{\psi_r(x)}{\psi_r(b)}$, where $\tau_b = \inf\{t \geq 0 \,|\, X_t = b\}$. Then

$$\frac{\psi_r(x)}{\psi_r(b)} = \boldsymbol{E}_x\{e^{-r\tau_b}\} > \boldsymbol{E}_x\{e^{-(r+\lambda)\tau_b}\} = \frac{\psi_{r+\lambda}(x)}{\psi_{r+\lambda}(b)}.$$

From this, we also see that $\frac{\psi_{r+\lambda}}{\psi_r}$ is monotonically increasing. □

The next theorem, which is the main result of this subsection, gives a set of conditions under which the optimal solution for Game 1 is given by (3.8) and (3.9).

THEOREM 3.7. Assume that the boundaries 0 and ∞ are natural for the underlying X, that condition (3.10) holds, and that for $i = 1,2$,

(1) $\mathcal{G}_r g_i, g_2 \in \mathcal{L}_1^r$,
(2) $\lim_{x\to\infty}\left|\frac{g_i(x)}{\psi_r(x)}\right| = 0$,
(3) $\mathcal{G}_r g_1(x) > \mathcal{G}_r g_2(x)$ for all $x \in \boldsymbol{R}_+ \setminus D$.

Then there exist a unique pair (z_1^*, y_1^*) satisfying the first-order conditions (3.8) and the value V_1 of Game 1 reads as in (3.9).

Proof. First, we find by coupling assumption (3) with the inequality $g_2 \geq g_1$ that

$$\begin{aligned}\left(\mathcal{G}_{r+\lambda}\hat{g}_1\right)(x) &= (\mathcal{G}_r g_1)(x) + \lambda(g_1^+(x) - g_1(x)) > (\mathcal{G}_r g_2)(x) + \lambda(g_1^+(x) - g_2(x))\\ &= \left(\mathcal{G}_{r+\lambda}\hat{g}_2\right)(x),\end{aligned} \tag{3.11}$$

for all $x \in \boldsymbol{R}_+ \setminus D$. Furthermore, since the functions $g_i \in \mathcal{L}_1^r$, assumption (1) implies that

$$\mathcal{G}_{r+\lambda}\hat{g}_i = \mathcal{G}_r g_i + \lambda(g_1^+ - g_i) \in \mathcal{L}_1^{r+\lambda}, \tag{3.12}$$

for $i = 1, 2$. Our next objective is to show that

$$\lim_{x\to\infty}(L_\varphi^{r+\lambda}\hat{g}_i)(x) = \lim_{x\to 0}(L_\psi^{r+\lambda}\hat{g}_i)(x) = 0. \tag{3.13}$$

To this end, let $b \in \boldsymbol{R}_+$. Since the function $\frac{\psi_r(x)}{\psi_{r+\lambda}(x)}$ is decreasing, see Lemma 3.6, we find

$$0 \leq \lim_{x\to\infty}\left|\frac{\hat{g}_i(x)}{\psi_{r+\lambda}(x)}\right| \leq \frac{\psi_r(b)}{\psi_{r+\lambda}(b)}\lim_{x\to\infty}\left|\frac{g_i(x) - \lambda(R_{r+\lambda}g_1^+)(x)}{\psi_r(x)}\right| = 0, \tag{3.14}$$

for $i = 1,2$. Here, the last inequality follows from assumption (2) and Proposition 4 from [17]. By coupling (3.14) with (2.1) and (3.12), we find that

$$(L_\varphi^{r+\lambda}\hat{g}_i)(x) = -\int_x^\infty \varphi_{r+\lambda}(y)(\mathcal{G}_{r+\lambda}\hat{g}_i)(y)m'(y)\mathrm{d}y \to 0, \text{ as } x \to \infty,$$

where the integral representation follows from [2], Corollary 3.2. In addition, since g_1 and g_2 are bounded from below, Corollary 3.2 from [2] implies that

$$(L_\psi^{r+\lambda}\hat{g}_i)(x) = \int_0^x \psi_{r+\lambda}(y)(\mathcal{G}_{r+\lambda}\hat{g}_i)(y)m'(y)\mathrm{d}y \to 0, \text{ as } x \to 0.$$

Thus, we have established condition (3.13). Now, conditions (3.10) and (3.11)–(3.13) guarantee that the claimed result follows from [2], Theorem 4.4. □

Theorem 3.7 states a set of conditions under which a unique pair (z_1^*, y_1^*) satisfying the first-order conditions (3.8) exists and under which the value of Game 1 can be written as (3.9). We remark that these conditions do not depend on the jump rate λ. Furthermore, we know from Lemma 3.4 that condition (3.10) can be substituted with a set of conditions that are also independent of λ. Thus, when using our results to check whether a particular example of Game 1 has a (unique) solution, the value of λ does not play any role.

4. Game 2

4.1 Equivalent formulation of the game

This section is devoted to the study of the solvability of Game 2. The analysis is completely analogous to Section 3. Again, we begin with the *ansatz* that the game has a saddle point equilibrium and that the continuation region $(z_2^*, y_2^*) \subset \boldsymbol{R}_+$ has compact closure. Now, because the terminal date T is observable to the issuer and she knows that after that time the buyer cannot exercise, it is clear that she will exercise at time T if and only if $g_2(X_T) < 0$. Thus, in an infinitesimal time interval $\mathrm{d}t$, the Poisson process jumps with probability $\lambda \mathrm{d}t$ leaving the buyer with pay-off $g_2^-(x) = \min\{g_2(x), 0\}$. With probability $1 - \lambda \mathrm{d}t$ there is no jump which results in additional expected present value. Analogously to Game 1, we deduce that the candidate G_2 must satisfy the condition $\mathcal{G}_{r+\lambda} G_2(x) = -\lambda g_2^-(x)$ for all $x \in (z_2^*, y_2^*)$ and, consequently, the candidate can be represented as

$$G_2(x) = \boldsymbol{E}_x\left\{\lambda \int_0^{\tau_{y_2^*} \wedge \gamma_{z_2^*}} \mathrm{e}^{-(r+\lambda)s} g_2^-(X_s)\mathrm{d}s + \mathrm{e}^{-(r+\lambda)(\tau_{y_2^*} \wedge \gamma_{z_2^*})}\left[g_1(X_{\tau_{y_2^*}})1_{\{\tau_{y_2^*} < \gamma_{z_2^*}\}} + g_2(X_{\gamma_{z_2^*}})1_{\{\tau_{y_2^*} > \gamma_{z_2^*}\}}\right]\right\} \tag{4.1}$$

for all $x \in \boldsymbol{R}_+$. As in Game 1, this form is the correct form of the value function for the associated perpetual game.

Proposition 4.1. The upper and lower values can for Game 2 be rewritten as

$$\bar{V}_2(x) = \inf_{\gamma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} \tilde{\Pi}_2(x, \tau, \gamma), \underline{V}_2(x) = \sup_{\tau \in \mathcal{T}_0} \inf_{\gamma \in \mathcal{T}_0} \tilde{\Pi}_2(x, \tau, \gamma),$$

where

$$\tilde{\Pi}_2(x, \tau, \gamma) = \boldsymbol{E}_x\left\{\lambda \int_0^{\tau \wedge \gamma} \mathrm{e}^{-(r+\lambda)s} g_2^-(X_s)\mathrm{d}s + \mathrm{e}^{-(r+\lambda)(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}} + g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]\right\}$$

for all $x \in \boldsymbol{R}_+$.

Proof. Completely similar to the proof of Proposition 3.2. □

Similarly to Game 1, we remark that the issuer follows now a stopping rule '*Stop at time* $\gamma \wedge T$' which results into the pay-off $g_2(X_\gamma)1_{\{\gamma<T\}} + g_2^-(X_T)1_{\{\gamma \geq T\}}$.

4.2. Necessary conditions

In order to simplify the notations, we denote

$$\begin{cases} \check{g}_1(x) = g_1(x) - \lambda\left(R_{r+\lambda} g_2^-\right)(x), \\ \check{g}_2(x) = g_2(x) - \lambda\left(R_{r+\lambda} g_2^-\right)(x). \end{cases} \tag{4.2}$$

Moreover define the function $Q_2 : \boldsymbol{R}_+^3 \to \boldsymbol{R}_+$ as

$$Q_2(x,z,y) = \boldsymbol{E}_x\left\{\mathrm{e}^{-(r+\lambda)(\tau_y\wedge\gamma_z)}\left[(\check{g}_1(X_{\tau_y})1_{\{\tau_y<\gamma_z\}} + \check{g}_2(X_{\gamma_z})1_{\{\tau_y>\gamma_z\}}\right]\right\}$$
$$= k_1(z,y)\psi_{r+\lambda}(x) + k_2(z,y)\varphi_{r+\lambda}(x),$$

where the functions $k_1 : \boldsymbol{R}_+^2 \to \boldsymbol{R}$ are defined as

$$\begin{cases} k_1(z,y) = \dfrac{\varphi_{r+\lambda}(y)\check{g}_2(z) - \varphi_{r+\lambda}(z)\check{g}_1(y)}{\varphi_{r+\lambda}(y)\psi_{r+\lambda}(z) - \varphi_{r+\lambda}(z)\psi_{r+\lambda}(y)}, \\ k_2(z,y) = \dfrac{\psi_{r+\lambda}(z)\check{g}_1(y) - \psi_{r+\lambda}(y)\check{g}_2(z)}{\varphi_{r+\lambda}(y)\psi_{r+\lambda}(z) - \varphi_{r+\lambda}(z)\psi_{r+\lambda}(y)}. \end{cases} \tag{4.3}$$

Analogously to Section 3, we assume that for every fixed x, the surface $(z,y) \to Q_2(x,z,y)$ has a unique saddle point (z_2^*, y_2^*), which does not depend on x. Then the first-order necessary conditions for this saddle point can be written as

$$\begin{cases} (L_\varphi^{r+\lambda}\check{g}_2)(z_2^*) - (L_\varphi^{r+\lambda}\check{g}_1)(y_2^*) = 0, \\ (L_\psi^{r+\lambda}\check{g}_2)(z_2^*) - (L_\psi^{r+\lambda}\check{g}_1)(y_2^*) = 0. \end{cases} \tag{4.4}$$

The next proposition contains our main result on the necessary conditions for the optimal solution for Game 2.

PROPOSITION 4.2. Assume that there is a pair (z_2^*, y_2^*) satisfying conditions (4.4) and that there are thresholds $\check{x}_i$, $i = 1,2$, such that

$$\mathcal{G}_{r+\lambda}\check{g}_i(x) \gtreqqless 0, \text{ whenever } x \lesseqqgtr \check{x}_i. \tag{4.5}$$

Then the pair (z_2^*, y_2^*) is unique and $z_2^* < \check{x}_2$ and $\check{x}_1 < y_2^*$. Furthermore the value V_2 of Game 2 reads as

$$V_2(x) = \begin{cases} g_1(x), & x \geq y_2^*, \\ \lambda(R_{r+\lambda}g_2^-)(x) + Q_2(x, z_2^*, y_2^*), & x \in (z_2^*, y_2^*), \\ g_2(x), & x \leq z_2^*, \end{cases} \tag{4.6}$$

where the functions k_i, $i = 1,2$, are defined in (4.3).

Proof. Completely analogous to the proof of Proposition 3.3. □

Similarly to Proposition 3.3, we posed in Proposition 4.2 the additional assumption (4.5) to assure that the functionals $L_\cdot^{r+\lambda}\check{g}_i$ behave well enough so that the uniqueness of the solution is guaranteed. In this case, as in Game 1, we propose sufficient conditions to (4.5) which do not depend on λ. These conditions are listed in the next lemma.

LEMMA 4.3. Assume that there are thresholds $\tilde{x}_i$, $i = 1,2$, such that $(\mathcal{G}_r g_i)(x) \gtreqqless 0$, whenever $x \lesseqqgtr \tilde{x}_i$. In addition, assume that

- $(\mathcal{G}_r g_1)$ and $g_2^- - g_1$ are non-increasing, and
- $(\mathcal{G}_r g_2)$ is non-increasing or $g_2 \leq 0$ for all $x > 0$.

Then condition (4.5) holds.

Proof. Similar to the proof of Lemma 3.4. □

Similar to Proposition 3.3, the stopping times $\tau_{y_2^*}$ and $\gamma_{z_2^*}$ do not tell the whole truth about the optimal stopping rules. The optimal stopping rule for the issuer is now '*stop at time* $\gamma_{z_2^*}$, *but if* $T < \gamma_{z_2^*}$ *and* $g_2(X_T) < 0$, *stop at time* T, *else do not stop*', whilst the optimal stopping rule for the buyer is '*stop at time* $\tau_{y_2^*} = \inf\{t \geq 0 | X_t \geq y_2^*\}$'. Analogously to Game 1, the value is decomposed in continuation region into the terminal pay-off $\lambda(R_{r+\lambda} g_2^-)$ and the early exercise premium $Q_2(x, z_2^*, y_2^*)$.

COROLLARY 4.4. Assume that there exists $\bar{x}_i < \check{x}_\mathrm{i}$ so that $g_i(x) = 0$ on $(0, \bar{x}_i)$, $(\mathcal{G}_{r+\lambda}\check{g}_i) > 0$ on $(\bar{x}_i, \check{x}_i)$, and $(\mathcal{G}_{r+\lambda}\check{g}_i) < 0$ on $(\check{x}_\mathrm{i}, \infty)$, $i = 1, 2$. Assume also that the threshold

$$\check{y}^*_{\bar{x}_2} = \underset{y}{\operatorname{argmax}} \left\{ \frac{\check{g}_1(y)}{\psi_{r+\lambda}(y) - \frac{\psi_{r+\lambda}(\bar{x}_2)}{\varphi_{r+\lambda}(\bar{x}_2)} \varphi_{r+\lambda}(y)} \right\}$$

exists. If there exists a pair $(z_2^*, y_2^*) \in (\bar{x}_2, \check{x}_2) \times (\check{x}_1, \check{y}^*_{\bar{x}_2})$ satisfying the first-order conditions (4.4), then the conclusion of Proposition 4.2 is satisfied and the value of the game reads as in (4.6).

Proof. Proof is similar to that of Corollary 3.5. □

If there does not exist an internal solution, then the pair $(\bar{x}_2, \check{y}^*_{\bar{x}_2})$ constitutes a corner solution, which is a saddle point solution and the solution reads as

$$V_2(x) = \begin{cases} g_1(x), & x \geq \check{y}^*_{\bar{x}_2}, \\ \lambda(R_{r+\lambda} g_2^-)(x) + Q_2(x, \bar{x}_2, \check{y}^*_{\bar{x}_2}), & x \in (\bar{x}_2, \check{y}^*_{\bar{x}_2}), \\ 0, & x \leq \bar{x}_2. \end{cases}$$

4.3. Sufficient conditions

The next theorem contains a set of sufficient conditions for the optimal solution for Game 2.

THEOREM 4.5. Assume that the boundaries 0 and ∞ are natural for the underlying X, that condition (4.5) hold, and that conditions 1–3 in Theorem 3.7 holds for $i = 1,2$. Then there exist a unique pair (z_2^*, y_2^*) satisfying the first-order conditions (4.4) and the value V_2 of Game 2 reads as in (4.6).

Proof. The proof is analogous to that of Theorem 3.7. □

Theorem 4.5 states sufficient conditions under which an optimal pair (z_2^*, y_2^*) uniquely exists and under which the value of Game 2 can be expressed as in (4.6). Using Lemma 4.3 condition (4.5) can be expressed independently of λ. Therefore, similarly to Game 1, we remark that for a particular example, the conditions of the theorem can be checked without any reference to the jump rate λ.

5. Comparison and asymptotics

In the previous sections, we studied the solvability of Games 1 and 2. In particular, we derived necessary and sufficient conditions for the solutions to be given by (3.9) and (4.6).

In this section, we study further the properties of these solutions. In particular, we are interested in finding orderings of the stopping thresholds and the value functions. Furthermore, we study the asymptotic behaviour of the optimal characteristics with respect to jump rate λ. To this end, we define two more Dynkin games. First of these is the infinite horizon Dynkin game, which is defined using (1.2) and (2.3) in the absence of terminating event taking place at time T. For a comprehensive analysis of this game, see [2]. Denote the value of this game as V and the optimal exercise thresholds as (z^*, y^*). The second additional game is the game with random time horizon in the case where the terminating event is not observable to either of the players – we refer to this game as Game 3. The upper and lower values of Game 3 are $\inf_{\gamma\in\mathcal{T}_0}\sup_{\tau\in\mathcal{T}_0}\tilde{\Pi}_3(x,\tau,\gamma)$ and $\sup_{\tau\in\mathcal{T}_0}\inf_{\gamma\in\mathcal{T}_0}\tilde{\Pi}_3(x,\tau,\gamma)$, respectively, where

$$\tilde{\Pi}_3(x,\tau,\gamma)=\boldsymbol{E}_x\left\{e^{-(r+\lambda)(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}}+g_2(X_\gamma)1_{\{\tau>\gamma\}}+g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]\right\}.$$

In fact, Game 3 is an infinite horizon game with discount rate $r+\lambda$. Hence, we know from [2] that under certain assumptions this game has a Nash equilibrium given by the unique thresholds (z_3^*, y_3^*). We denote the value of this game as V_3. It is worth pointing out that Proposition 3.2 implies that if the function g_1 is non-positive, the value of Game 1 coincides with the value of Game 3. Similarly, Proposition 4.1 implies that if the function g_2 is non-negative, the value of Game 2 coincides with the value of Game 3.

5.1. Ordering of the thresholds and the values

The following proposition is our main result on the ordering of optimal characteristics of the games.

PROPOSITION 5.1.

(A) Assume that Game 1, Game 2 and Game 3 have unique saddle point solutions. Then the following orderings hold
- $V_1(x)\geq V_3(x)\geq V_2(x)$ everywhere.
- $z_1^*\geq z_3^*\geq z_2^*$ and $y_1^*\geq y_3^*\geq y_2^*$ always.

(B) If in addition the infinite horizon game has a unique saddle point solution and g_2 is non-negative, then
- $V(x)\geq V_1(x)\geq V_3(x)\geq V_2(x)$ for all $x\in\boldsymbol{R}_+$.
- $z^*\geq z_1^*\geq z_3^*\geq z_2^*$ and $y^*\geq y_1^*\geq y_3^*\geq y_2^*$.

Proof.

(A) Let us first prove the orderings between Game 1 and Game 2. Recall the definitions of $\tilde{\Pi}_1(x,\tau,\gamma)$ and $\tilde{\Pi}_2(x,\tau,\gamma)$ from Propositions 3.2 and 4.1, respectively. Now

$$\begin{aligned}\tilde{\Pi}_1(x,\tau,\gamma)=\ &\boldsymbol{E}_x\left\{\lambda\int_0^{\tau\wedge\gamma}e^{-(r+\lambda)s}g_1^+(X_s)ds+e^{-(r+\lambda)(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}}\right.\right.\\&\left.\left.+g_2(X_\gamma)1_{\{\tau>\gamma\}}+g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]\right\}\\&\geq\boldsymbol{E}_x\left\{\lambda\int_0^{\tau\wedge\gamma}e^{-(r+\lambda)s}g_2^-(X_s)ds+e^{-(r+\lambda)(\tau\wedge\gamma)}\left[g_1(X_\tau)1_{\{\tau<\gamma\}}\right.\right.\\&\left.\left.+g_2(X_\gamma)1_{\{\tau>\gamma\}}+g_3(X_\gamma)1_{\{\tau=\gamma\}}\right]\right\}=\tilde{\Pi}_2(x,\tau,\gamma),\end{aligned}$$

for all $x \in \boldsymbol{R}_+$ and $\tau, \gamma \in \mathcal{T}_0$. Thus,

$$V_1(x) = \sup_{\tau \in \mathcal{T}_0} \inf_{\gamma \in \mathcal{T}_0} \tilde{\Pi}_1(x, \tau, \gamma) \geq \sup_{\tau \in \mathcal{T}_0} \inf_{\gamma \in \mathcal{T}_0} \tilde{\Pi}_2(x, \tau, \gamma) = V_2(x) \tag{5.1}$$

Suppose now, contrary to our claim, that $y_1^* < y_2^*$ and let $x \in (y_1^*, y_2^*)$ so that x is in the stopping region of Game 1, and in the continuation region of Game 2. Then $V_1(x) = g_1(x) < V_2(x)$, contrary to (5.1). The same reasoning applies to the case $z_2^* \leq z_1^*$. Next, recall the definition of $\tilde{\Pi}_3$ from beginning of the section. We see that $\tilde{\Pi}_1 \geq \tilde{\Pi}_3 \geq \tilde{\Pi}_2$ and using reasoning as above we find that $V_1 \geq V_3 \geq V_2$. The claimed inequalities for the thresholds follow as above.

(B) Let g_2 be non-negative and recall the definition of $\Pi(x, \tau, \gamma)$ from (1.2). We shall compare it to Π_1 from (2.4). We know that the value function satisfies $V(x) = \sup_{\tau \in \mathcal{T}_0} \inf_{\gamma \in \mathcal{T}_0} \Pi(x, \tau, \gamma)$ and similarly $V_1(x) = \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau, \gamma)$. To prove the claim, we first write

$$V_1(x) = \max\left\{ \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau \wedge T, \gamma); \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau \vee T, \gamma) \right\}. \tag{5.2}$$

Now, for the first term on the right-hand side of (5.2) we observe that

$$\sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau \wedge T, \gamma) = \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi(x, \tau \wedge T, \gamma). \tag{5.3}$$

For the second term, we find the following

$$\begin{aligned} &\sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau \vee T, \gamma) \\ &= \max\left\{ \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, T, \gamma); \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \infty, \gamma) \right\} = \max\left\{ \inf_{\gamma \in \mathcal{T}_0} \Pi(x, T, \gamma); \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \infty, \gamma) \right\} \\ &\leq \max\left\{ \inf_{\gamma \in \mathcal{T}_0} \Pi(x, T, \gamma); \inf_{\gamma \in \mathcal{T}_0} \Pi(x, \infty, \gamma) \right\} \leq \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi(x, \tau \vee T, \gamma). \end{aligned} \tag{5.4}$$

Here, the first inequality holds since g_2 is non-negative. Furthermore, the second inequality holds since the stopping times 0 and ∞ belong to $\mathcal{T}_1$. By substituting (5.3) and (5.4) into (5.2) we obtain

$$\begin{aligned} V_1(x) &\leq \max\left\{ \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi(x, \tau \wedge T, \gamma); \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi(x, \tau \vee T, \gamma) \right\} \\ &= \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi(x, \tau, \gamma) = V(x), \end{aligned} \tag{5.5}$$

where the last equality follows from the fact that $\mathcal{T}_1 = \mathcal{T}_0$ in the absence of terminating event.

Suppose, contrary to our claim, that $y^* < y_1^*$ and let $x \in (y^*, y_1^*)$, so that x is in the continuation region of stochastic time horizon case, and in the stopping region of infinite time horizon case. Then $V(x) = g_1(x) < V_1(x)$, contrary to (5.5). The same reasoning applies to the case $z_1^* \leq z^*$. □

Intuitively, item (A) of Proposition 5.1 is not surprising. Indeed, if the issuer has inside information about the terminating event, it will make the value of the game smaller as there is one additional stopping time in the set over which the issuer minimizes. Similarly, if the buyer has inside information about the terminating event, the value will be larger. In Game 3, the value is naturally in between these two extremes. Furthermore, the exercise thresholds are ordered as one could guess from orderings of the value functions, the principal idea being: *The more you know, the longer you wait*.

The item (B) is also intuitively quite clear. Since $g_2 \geq 0$, there is no risk of ending up on trajectory leading inevitably into negative pay-off. By coupling this with the fact that Game 1 will end in finite time almost surely, the ordering $V \geq V_1$ becomes evident as there is less time to maximize the pay-off which is bound to be non-negative. We stress here that the positiveness of g_2 is indeed required for the inequalities $V \geq V_1$, $z^* \geq z_1^*$ and $y^* \geq y_1^*$ to hold. We will give a numerical example at the end of Section 6 where these inequalities are reversed for a function g_2 that takes also negative values.

5.2. *Some asymptotics*

In this subsection, we study the limiting behaviour of the optimal characteristics of Games 1 and 2 when the jump rate λ tends to infinity as well as when it tends to zero. The next proposition is our main result on this matter.

PROPOSITION 5.2. Let $\bar{x}_i$ be the greatest point such that $g_i(\bar{x}_i) = 0$. The value functions V_i, $i = 1, 2$, and the corresponding optimal thresholds satisfy the limiting properties

$$\lim_{\lambda \to \infty} V_i(x) = V^\infty(x) := \begin{cases} g_1(x), & x \geq \bar{x}_1 \\ 0, & x \in (\bar{x}_2, \bar{x}_1) \\ g_2(x) & x \leq \bar{x}_2. \end{cases}$$

and

$$\lim_{\lambda \to 0} V_i(x) = V(x) \text{ and } \begin{cases} \lim_{\lambda \to 0} z_i^* = z^* \\ \lim_{\lambda \to 0} y_i^* = y^*. \end{cases}$$

Proof. We will prove the proposition only for Game 1; Game 2 is handled similarly. Let us first prove the case $\lambda \to \infty$. Recall from (2.4) and (2.5) that the value of the Game 1 reads as

$$V_1(x) = \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau, \gamma) = \inf_{\gamma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_1} \Pi_1(x, \tau, \gamma),$$

where $\Pi_1(x, \tau, \gamma) = \boldsymbol{E}_x\{e^{-r(\tau \wedge \gamma)}[g_1(X_\tau)1_{\{\tau<\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}} + g_3(X_\gamma)1_{\{\tau=\gamma\}}]1_{\{\tau \wedge \gamma \leq T\}}\}$. Letting $\lambda \to \infty$, we see that

$$\begin{aligned} \Pi_1(x, \tau, \gamma) &= 0, && \text{if } \tau, \gamma > 0 \\ \Pi_1(x, \tau, \gamma) &= g_1(x), && \text{if } \tau = 0 < \gamma \\ \Pi_1(x, \tau, \gamma) &= g_2(x), && \text{if } \tau > 0 = \gamma \\ \Pi_1(x, \tau, \gamma) &= g_3(x), && \text{if } \tau = 0 = \gamma. \end{aligned} \tag{5.6}$$

In light of these findings, let us show that the claimed function V^∞ is indeed a saddle point solution when λ approaches to infinity. There are three cases to be considered depending whether $x \leq \bar{x}_2$, $x \in (\bar{x}_2, \bar{x}_1)$ or $x \geq \bar{x}_1$. (Note that since $g_2 \geq g_1$, we always have $\bar{x}_2 \leq \bar{x}_1$).

Let $x \leq \bar{x}_2$. Now $g_1(x) \leq g_3(x) \leq g_2(x) \leq 0$ and so we can check straightforwardly, using (5.6), that $\sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau, \gamma) = g_2(x) = \inf_{\gamma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_1} \Pi_1(x, \tau, \gamma)$.

The same reasoning applies also to the cases $x \in (\bar{x}_2, \bar{x}_1)$ and $x \geq \bar{x}_1$, and the claimed limiting property follows.

Next, we turn our eyes on the case $\lambda \to 0$. Since $g_1^+ \in L_1^r$, we find that $\lambda(R_{r+\lambda} g_1^+)(x) \to 0$ as $\lambda \to 0$ for all $x \in \boldsymbol{R}_+$. Given this limiting property together with the definition of V_1 in (3.9), we find that the claimed limiting property holds. Finally, given the convergence result of value function V_1, the claimed convergence results hold also for the thresholds z_1^* and y_1^*. □

It is interesting to observe that the values of Game 1 and Game 2 are the same at the limit $\lambda \to 0$ and also at $\lambda \to \infty$. In the limit $\lambda \to 0$, this result is intuitively plausible: if the expected waiting time for the Poisson process to jump is infinite, the game will not expire unexpectedly, and as a result we get the solution of an infinite horizon game. Also the limit $\lambda \to \infty$ has a natural explanation: there is no advantage of observing the jump, since both players already know that the jump will occur at the time zero. While our asymptotic analysis is general in terms of the underlying dynamics and pay-off structure, it does not, unfortunately, answer the interesting question about the *rate* of this convergence.

6. Explicit example with geometric Brownian motion

We illustrate the main results of the study in this section with an explicit example. Let the underlying diffusion be a geometric Brownian motion, that is, let X be the solution of the Itô equation

$$dX_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \tag{6.1}$$

where W is a Wiener process. Furthermore, we assume $\mu \in \mathbf{R}_+$, $\sigma > 0$ and that $r > \mu$. Further let $g_1(x) = x - c_1$ and $g_2(x) = x - c_2$ and assume that $c_1 > c_2 > 0$, so that $g_2 > g_1$. Given this set-up, we find that $(R_{r+\lambda} g_i)(x) = x/(r + \lambda - \mu) - c_i/(r + \lambda)$. Also clearly $g_1, g_2 \in \mathcal{L}_1^r$.

In this case, the decreasing and increasing fundamental solutions of the ordinary second-order differential equation $(\mathscr{A} - \beta)u = 0$ are $\varphi_\beta(x) = x^{\gamma_1^\beta}$ and $\psi_\beta(x) = x^{\gamma_2^\beta}$ respectively. Here

$$\gamma_i^\beta = \frac{1}{\sigma^2}\left(\frac{1}{2}\sigma^2 - \mu + (-1)^i \sqrt{(\frac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2\beta}\right),$$

for $i = 1,2$, are the solutions of the characteristic equation $\frac{1}{2}\sigma^2\gamma_i(\gamma_i - 1) + \mu\gamma_i - r = 0$. Finally, the scale density reads as $S'(x) = x^{-2\mu/\sigma^2}$.

6.1. Game 1 has a solution

We know that $(R_{r+\lambda} g_1^+)(x)$ satisfies the differential equation $\frac{1}{2}\sigma^2 x^2 (R_{r+\lambda} g_1^+)''(x) + \mu x (R_{r+\lambda} g_1^+)'(x) - (r + \lambda)(R_{r+\lambda} g_1^+)(x) = -g_1^+(x)$. Therefore $(R_{r+\lambda} g_1^+)$ satisfies the following conditions:

$$(R_{r+\lambda} g_1^+)(x) = \begin{cases} a_1 \psi_{r+\lambda}(x) + a_2 \varphi_{r+\lambda}(x), & x \leq c_1 \\ a_3 \psi_{r+\lambda}(x) + a_4 \varphi_{r+\lambda}(x) + (R_{r+\lambda} g_1)(x), & x > c_1. \end{cases}$$

Since $(R_{r+\lambda} g_1^+)(0+) \neq \infty$, we must have $a_2 = 0$ and since $\lim_{x\to\infty}\big((R_{r+\lambda} g_1^+)(x) - (R_{r+\lambda} g_1)(x)\big) = 0+$, we must have $a_3 = 0$. Furthermore $(R_{r+\lambda} g_1^+)$ is continuous and differentiable. Thus, the coefficients a_1 and a_4 can be solved from conditions

$\lim_{x\to c_1+}(R_{r+\lambda}g_1^+)(x) = \lim_{x\to c_1-}(R_{r+\lambda}g_1^+)(x)$ and $\lim_{x\to c_1+}(R_{r+\lambda}g_1^+)'(x) = \lim_{x\to c_1-}(R_{r+\lambda}g_1^+)'(x)$. It is a matter of elementary calculation to show that

$$a_1 = \frac{1}{\psi_{r+\lambda}(c_1)}\left((R_{r+\lambda}g_1)(c_1) + \frac{(R_{r+\lambda}g_1)'(c_1)\psi_{r+\lambda}(c_1) - (R_{r+\lambda}g_1)(c_1)\psi'_{r+\lambda}(c_1)}{\varphi_{r+\lambda}(c_1)\psi'_{r+\lambda}(c_1) - \varphi'_{r+\lambda}(c_1)\psi_{r+\lambda}(c_1)}\varphi_{r+\lambda}(c_1)\right)$$

$$a_4 = \frac{(R_{r+\lambda}g_1)'(c_1)\psi_{r+\lambda}(c_1) - (R_{r+\lambda}g_1)(c_1)\psi'_{r+\lambda}(c_1)}{\varphi_{r+\lambda}(c_1)\psi'_{r+\lambda}(c_1) - \varphi'_{r+\lambda}(c_1)\psi_{r+\lambda}(c_1)}.$$

Next, we show that the presented set-up satisfies the sufficient conditions of Theorem 3.7. Since $\mathcal{G}_r g_i(x) = (\mu - r)x + rc_i$, for $i = 1,2$, we find that $\mathcal{G}_r g_i \in \mathcal{L}_1^r$, for $i = 1,2$. The assumption $c_1 > c_2$ implies that $\mathcal{G}_r g_1 > \mathcal{G}_r g_2$ – thus conditions (1) and (3) in Theorem (3.7) hold. Moreover, since we assumed $r > \mu$, we have that $\gamma_2^r > 1$, therefore $g_i(x)/\psi_r(x) = x^{1-\gamma_2^r} + c_i x^{-\gamma_2^r}$, for $i = 1,2$, satisfy condition (2) in Theorem 3.7. Finally, for condition (3.10) recall that $\hat{g}_i = g_i - \lambda(R_{r+\lambda}g_1^+)$. Thus, $\mathcal{G}_{r+\lambda}\hat{g}_i = \mathcal{G}_r g_i + \lambda(g_1^+ - g_i)$ and we get

$$\mathcal{G}_{r+\lambda}\hat{g}_1(x) = \begin{cases} (\mu - r - \lambda)x + (r+\lambda)c_1, & x < c_1 \\ (\mu - r)x + rc_1, & x \geq c_1, \end{cases}$$

$$\mathcal{G}_{r+\lambda}\hat{g}_2(x) = \begin{cases} (\mu - r - \lambda)x + (r+\lambda)c_2, & x < c_1 \\ (\mu - r)x + (r+\lambda)c_2 - \lambda c_1, & x \geq c_1. \end{cases}$$

From these expressions, we see that condition (3.10) holds and $\hat{x}_1 > c_1$.

It follows that we can apply Theorem 3.7 and, consequently, that there exists a unique pair (z_1^*, y_1^*) satisfying the necessary optimality conditions (3.8). If $z_1^* < c_1$, conditions (3.8) can be written as (to simplify notation, we write $\gamma_i := \gamma_i^{r+\lambda}$)

$$\begin{cases} \left(y^{\gamma_1}\lambda a_4(\gamma_1 - \gamma_2) + \frac{y(\gamma_2-1)(r-\mu)}{r+\lambda-\mu} - \frac{r\gamma_2 c_1}{r+\lambda}\right)y^{\gamma_2+\frac{2\mu}{\sigma^2}-1} = z^{\gamma_2+\frac{2\mu}{\sigma^2}-1}(z(\gamma_2-1) - \gamma_2 c_2) \\ \left(\frac{y(\gamma_1-1)(r-\mu)}{r+\lambda-\mu} - \frac{r\gamma_1 c_1}{r+\lambda}\right)y^{\gamma_1+\frac{2\mu}{\sigma^2}-1} = z^{\gamma_1+\frac{2\mu}{\sigma^2}-1}\left(z(\gamma_1-1) - \gamma_1 c_2 + z^{\gamma_2}\lambda a_1(\gamma_2-\gamma_1)\right). \end{cases}$$

If, on the other hand, $z_1^* \geq c_1$, conditions (3.8) take the form

$$\begin{cases} \left(y^{\gamma_1}\lambda a_4(\gamma_1 - \gamma_2) + \frac{y(\gamma_2-1)(r-\mu)}{r+\lambda-\mu} - \frac{r\gamma_2 c_1}{r+\lambda}\right)y^{\gamma_2+\frac{2\mu}{\sigma^2}-1} = z^{\gamma_2+\frac{2\mu}{\sigma^2}-1}\left(\frac{(\gamma_2-1)(r-\mu)z}{r+\lambda-\mu} + \frac{\gamma_2\lambda c_1}{r+\lambda} - \gamma_2 c_2 + z^{\gamma_1}(\gamma_1-\gamma_2)\lambda a_4\right) \\ \left(\frac{y(\gamma_1-1)(r-\mu)}{r+\lambda-\mu} - \frac{r\gamma_1 c_1}{r+\lambda}\right)y^{\gamma_1+\frac{2\mu}{\sigma^2}-1} = z^{\gamma_1+\frac{2\mu}{\sigma^2}-1}\left(\frac{z(\gamma_1-1)(r-\mu)}{r+\lambda-\mu} + \frac{\gamma_1\lambda c_1}{r+\lambda} - \gamma_1 c_2\right). \end{cases}$$

Now $y_1^* > \hat{x}_1 > c_1$ (see Proposition 3.3), but we do not know whether $z_1^* < c_1$ or the other way around. Therefore, we have two alternative formulation for (3.8). Nevertheless, only one of these has solution, since Theorem 3.7 guarantees the uniqueness of the solution. Furthermore at the point $z = c_1$, these two pairs of equations become the same. Unfortunately, solving the optimal boundaries from these equations explicitly does not seem to be possible. Therefore, we illustrate the results numerically. But before that, let us see through the solvability of Game 2.

6.2. Game 2 has a solution

Similarly to Game 1, we find that

$$(R_{r+\lambda}g_2^-)(x) = \begin{cases} a_5\psi_{r+\lambda}(x) + (R_{r+\lambda}g_2)(x), & x < c_2 \\ a_6\varphi_{r+\lambda}(x), & x \geq c_2, \end{cases}$$

where

$$a_5 = \frac{1}{\psi_{r+\lambda}(c_2)}\left(-(R_{r+\lambda}g_2)(c_2) + \frac{(R_{r+\lambda}g_2)'(c_2)\psi_{r+\lambda}(c_2) - (R_{r+\lambda}g_2)(c_2)\psi'_{r+\lambda}(c_2)}{\varphi'_{r+\lambda}(c_2)\psi_{r+\lambda}(c_2) - \varphi_{r+\lambda}(c_2)\psi'_{r+\lambda}(c_2)}\varphi_{r+\lambda}(c_2)\right)$$

$$a_6 = \frac{(R_{r+\lambda}g_2)'(c_2)\psi_{r+\lambda}(c_2) - (R_{r+\lambda}g_2)(c_2)\psi'_{r+\lambda}(c_2)}{\varphi'_{r+\lambda}(c_2)\psi_{r+\lambda}(c_2) - \varphi_{r+\lambda}(c_2)\psi'_{r+\lambda}(c_2)}.$$

In particular $a_5, a_6 < 0$.

Next, we verify that the sufficient conditions in Theorem 4.5 hold. We already showed with Game 1 that conditions (1)–(3) hold, so it suffices to check whether condition (4.5) holds. Recall that $\check{g}_i = g_i - \lambda(R_{r+\lambda}g_2^-)$ so that $\mathcal{G}_{r+\lambda}\check{g}_i = \mathcal{G}_r g_i + \lambda(g_2^- - g_i)$. Thus,

$$\mathcal{G}_{r+\lambda}\check{g}_1(x) = \begin{cases} (\mu - r)x + (r+\lambda)c_1 - \lambda c_2, & x < c_2 \\ (\mu - r - \lambda)x + (r+\lambda)c_1, & x \geq c_2, \end{cases}$$

$$\mathcal{G}_{r+\lambda}\check{g}_2(x) = \begin{cases} (\mu - r)x + rc_2, & x < c_2 \\ (\mu - r - \lambda)x + (r+\lambda)c_2, & x \geq c_2. \end{cases}$$

From these expressions, we see that condition (4.5) holds and $\check{x}_1 > c_2$.

Again, we can apply Theorem 4.5 and there exists a unique pair (z_2^*, y_2^*) which satisfies the necessary optimality condition (4.4). This time, if $z_2^* < c_2$, the condition can be written as (to simplify notation we write $\gamma_i := \gamma_i^{r+\lambda}$)

$$\begin{cases} \left(y^{\gamma_1}(\gamma_1 - \gamma_2)\lambda a_6 + y(\gamma_2 - 1) - \gamma_2 c_1\right)y^{\gamma_2 + \frac{2\mu}{\sigma^2} - 1} = z^{\gamma_2 + \frac{2\mu}{\sigma^2} - 1}\left(\frac{z(r-\mu)(1-\gamma_2)}{r+\lambda-\mu} - \frac{r\gamma_2 c_2}{r+\lambda}\right) \\ \left(y(\gamma_1 - 1) - \gamma_1 c_1\right)y^{\gamma_1 + \frac{2\mu}{\sigma^2} - 1} = z^{\gamma_1 + \frac{2\mu}{\sigma^2} - 1}\left(\frac{z(r-\mu)(\gamma_1 - 1)}{r+\lambda-\mu} - \frac{r\gamma_1 c_2}{r+\lambda} + z^{\gamma_2}(\gamma_2 - \gamma_1)\lambda a_5\right) \end{cases}$$

If, on the other hand, $z_2^* \geq c_2$, condition (4.4) takes the form

$$\begin{cases} \left(y^{\gamma_1}(\gamma_1 - \gamma_2)\lambda a_6 + y(\gamma_2 - 1) - \gamma_2 c_1\right)y^{\gamma_2 + \frac{2\mu}{\sigma^2} - 1} = z^{\gamma_2 + \frac{2\mu}{\sigma^2} - 1}\left(z(\gamma_2 - 1) - \gamma_2 c_2 + z^{\gamma_1}(\gamma_1 - \gamma_2)\lambda a_6\right) \\ \left(y(\gamma_1 - 1) - \gamma_1 c_1\right)y^{\gamma_1 + \frac{2\mu}{\sigma^2} - 1} = z^{\gamma_1 + \frac{2\mu}{\sigma^2} - 1}(z(\gamma_1 - 1) - \gamma_1 c_2). \end{cases}$$

Similarly to Game 1, we know that $y_2^* > \check{x}_1 > c_2$ (cf. Proposition 4.2), but we do not know whether $z_2^* < c_2$ or not. Therefore, we have two alternative formulations of (4.4), but only one of these has a solution. Again, solving the optimal boundaries from these equations explicitly does not seem to be possible and so we are prompted to do numerical illustrations.

6.3. Numerical illustration

To illustrate the optimal characteristics numerically, we fix the parameter configuration $\mu = 0.03, r = 0.08, \sigma = 0.35, c_1 = 3, c_2 = 2$ and $\lambda = 0.1$. Under this choice, the value functions for Game 1 and Game 2 are given in Figure 1(a),(b).

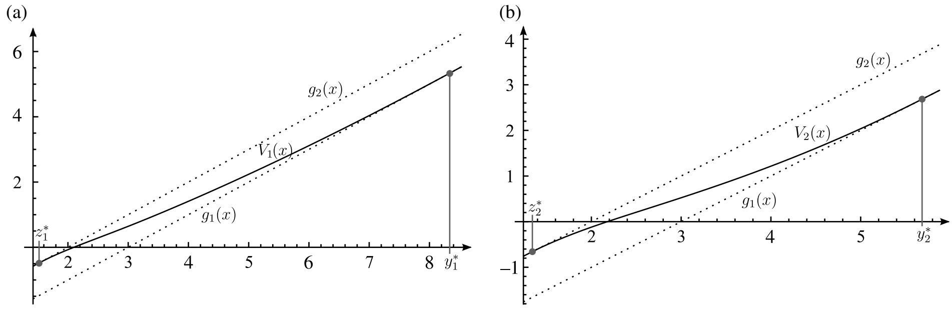

Figure 1. (a) The solution of Game 1; (b) The solution of Game 2. Now $(z_1^*, y_1^*) = (1.52, 8.34)$, whereas $(z_2^*, y_2^*) = (1.34, 5.68)$. For comparison in infinite horizon game $(z^*, y^*) = (1.60, 8.99)$ so that now $z_2^* < z_1^* < z^*$ and $y_2^* < y_1^* < y^*$.

The values V, V_1, V_2 and V_3 are compared graphically in Figure 2, recall the definition of V and V_3 from Section 5.

In line with Proposition 5.1, we observe that the inequalities $V \geq V_1 \geq V_3 \geq V_2$ hold in this case. We point out that $V \geq V_1$ in this case even though g_2 takes also negative values. The values V, V_1 and V_2 appear to differ quite significantly from each others, which indicates that the mere existence of the expiry time and the inside information on it can have substantial impact on the optimal exercise rule. For example, if $x = 4$ for the given parameters, we have $V(4) \approx 1.55$ and $V_1(4) \approx 1.41$ the difference being 0.14, so that $V(4)$ is about 10% greater. However, we observe that the value V_3 does not differ much from V_2. This means that in this example when the issuer has inside knowledge about Poisson clock (Game 2), she rarely takes advantage of this information. This, in turn, is because she exercises at the jump time T only if $g_2(X_T) < 0$. This happens rarely, since g_2 is usually positive.

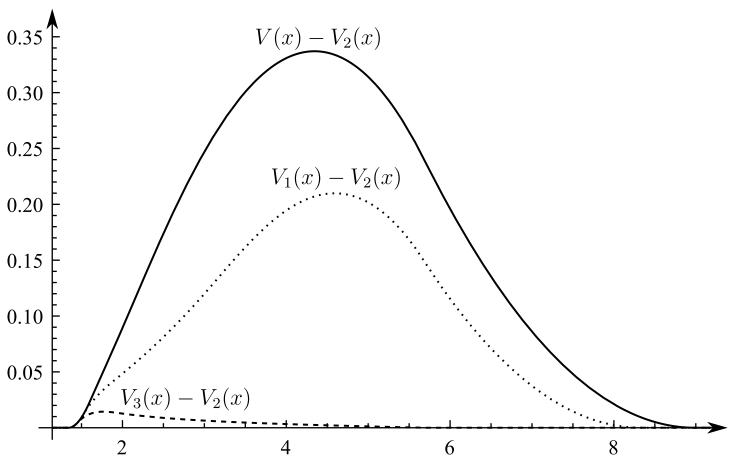

Figure 2. The differences $V - V_2$, $V_1 - V_2$ and $V_3 - V_2$.

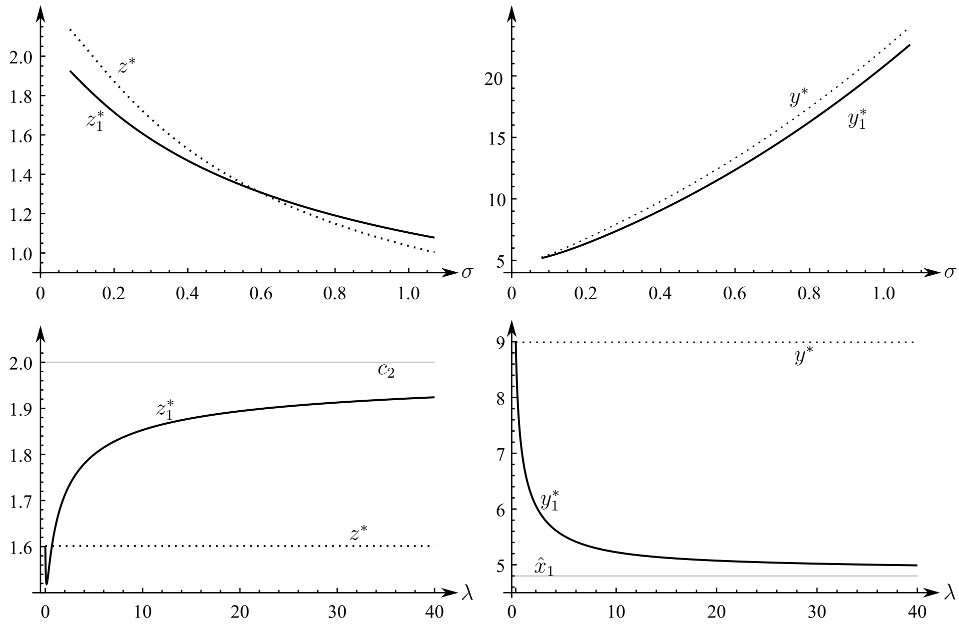

Figure 3. The changes of thresholds in Game 1 and in infinite horizon game, when changing σ and λ.

In Figure 3, we illustrate the sensitivities of the exercise thresholds with respect to parameters σ and λ in Game 1 and in the infinite horizon game. We notice that the order of the lower thresholds change as σ increases. This is possible, since g_2 takes also negative values (cf. Proposition 5.1). Moreover we see that as σ increases, the continuation region gets wider. This result is in line with the literature. Furthermore, we observe that the continuation region shrinks as λ increases which is again natural in the current example. In particular, the issuer lets her exercise threshold grow towards c_2 so that she could increase her chances of exercising with negative pay-off.

6.4. *Counterexample for inequalities $V_1 \leq V$, $z_1^* \leq z^*$ and $y_1^* \leq y^*$*

In Proposition 5.1, we prove that if $g_2 \geq 0$, then for the optimal stopping boundaries we have the inequalities $z_1^* \leq z^*$ and $y_1^* \leq y^*$ and for the values we have $V(x) \geq V_1(x)$. In this subsection, we show that if g_2 is allowed to be negative, then these inequalities are not necessary true, a hint of this can also be seen from Figure 3.

Let the underlying diffusion still be a geometric Brownian motion and the parameter configuration as $\mu = 0.03$, $\sigma = 1.0$, $\lambda = 0.1$ and $r = 0.08$. Furthermore, let $g_1 = \sqrt[3]{x} - 3$ and $g_2 = \sqrt{x} - 2$; in particular, $g_2 > g_1$. It is a straightforward task to check that there exist unique saddle point solutions for Game 1, Game 2 and infinite time horizon game and that the optimal thresholds read as $(z_1^*, y_1^*) \approx (0.56, 44.7)$; $(z_2^*, y_2^*) \approx (0.24, 39.4)$ and $(z^*, y^*) \approx (0.21, 30.0)$. Now contrary to Proposition 5.1(B), we have $z_1^* > z^*$ and $y_1^* > y^*$. Moreover, we have also $z_2^* > z^*$ and $y_2^* > y^*$. On the other hand, the boundaries of Game 2 are lower than the ones of Game 1, see Proposition 5.1(A). Moreover, we find that $V(x) \leq V_2(x) \leq V_1(x)$ which is illustrated in Figure 4.

It is interesting to observe that the value of a random time horizon game can dominate the value of an infinite horizon game. In fact, it can be that the infinite horizon game can

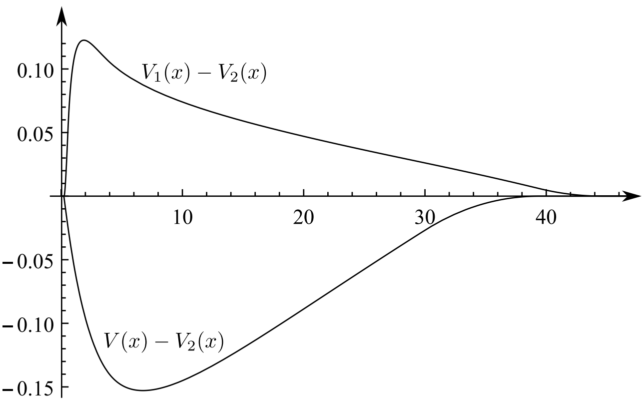

Figure 4. The differences $V - V_2$ and $V_1 - V_2$. We have $V \leq V_2 \leq V_1$ in contrast to Proposition 5.1.

have the smallest value of the games considered in this paper, which seems first rather counterintuitive. However, this is all due to the 'sufficient negativeness' of g_2. Indeed, as the game will end almost surely in finite time, the issuer's chances of exercising with a very negative payoff are reduced in comparison to the infinite horizon game.

6.5. Option type example

In this subsection, we examine an option type example, where the pay-offs g_i are not continuously differentiable. We illustrate our results only for Game 1, Games 2 and 3 being analogous. To begin with, set $c_1 > c_2 > 0$ and define $g_i = (x - c_i)^+$, so that the pay-offs are compatible with Corollary 3.5. Moreover, let the underlying diffusion follow the geometric Brownian motion (6.1) with $r > \mu > 0$. Under these choices we set up the optimal stopping game

$$V_1(x) = \sup_{\tau \in \mathcal{T}_1} \inf_{\gamma \in \mathcal{T}_0} \Pi_1(x, \tau, \gamma) = \inf_{\gamma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_1} \Pi_1(x, \tau, \gamma).$$

Since $\mathcal{G}_{r+\lambda}\hat{g}_i = \mathcal{G}_r g_i + \lambda(g_1 - g_i)$, it is a matter of straightforward calculus to show that

$$\mathcal{G}_{r+\lambda}\hat{g}_1(x) = \begin{cases} 0, & x \leq c_1 \\ -rx + \mu + rc_1, & x > c_1 \end{cases} \quad \text{and}$$

$$\mathcal{G}_{r+\lambda}\hat{g}_2(x) = \begin{cases} 0, & x \leq c_2 \\ -(r+\lambda)x + \mu + (r+\lambda)c_2, & c_1 > x > c_2 \\ -rx + \mu + rc_2 + \lambda(c_2 - c_1), & x \geq c_1. \end{cases}$$

We see immediately that since $r > \mu > 0$, there exists a unique $\hat{x}_1 > c_1$ satisfying (3.10). Moreover, since $\mathcal{G}_{r+\lambda}\hat{g}_2$ is decreasing for all $x > c_2$ and is continuous over c_1, there exist also a unique $\hat{x}_2 > c_2$ satisfying (3.10). Finally, to show the existence of $\hat{y}^*_{\hat{x}_2}$, we know that $\hat{g}_1(c_2) < 0$. It can be calculated that $\hat{\psi}_{c_2}(x) := \psi_{r+\lambda}(x) - \psi_{r+\lambda}(c_2)\varphi_{r+\lambda}(x)/\varphi_{r+\lambda}(c_2)$ is increasing for all $x > c_2$ and that it tends to zero as x tends to c_2. Thus $\hat{g}_1(c_2)/\hat{\psi}_{c_2}(c_2) = -\infty$, and it is increasing in a neighbourhood of c_2. On the other hand, it can be shown that there exists $K > c_2$ such that $\hat{g}_1(x)/\hat{\psi}_{c_2}(x)$ is

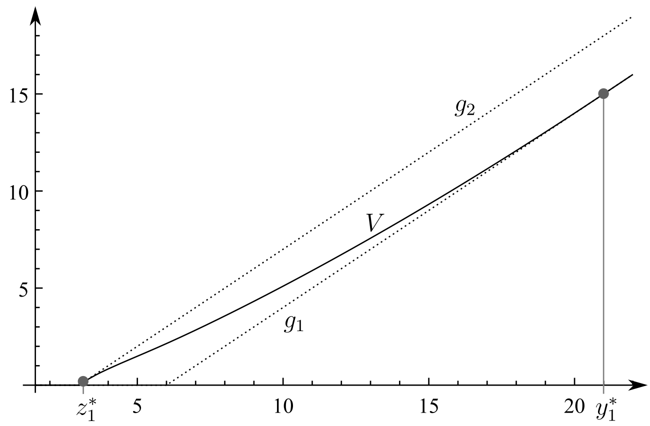

Figure 5. The solution of Game 1 with $c_1 = 6$, $c_2 = 3$, $\sigma = 0.2$, $\mu = 0.05$, $r = 0.08$ and $\lambda = 0.03$. Now the continuation region $(z_1^*, y_1^*) \approx (3.1, 21.00)$. With this parameter selection, the state $\hat{y}^*_{\bar{x}_2} \approx 21.01$.

decreasing for all $x > K$. Thus, we conclude that $\hat{g}_1(x)/\hat{\psi}_{c_2}(x)$ attains its maximum on the compact interval $[c_2, K]$.

It follows that the conditions of Corollary 3.5 are met, and we know that *if* there exists an internal solution (z_1^*, y_1^*) for the necessary conditions (3.8) (i.e. $z_1^*, y_1^* \notin \{c_1, c_2\}$), *then* it must be unique and the value of the game reads as in (3.9). If this is not the case, then the solution is a corner solution with the exercise boundaries c_2 and $\hat{y}^*_{\bar{x}_2}$. Finally, we illustrate the value function graphically for a particular parameter configuration in Figure 5.

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Note

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Article IV

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OPTIMAL STOPPING OF THE MAXIMUM PROCESS

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Abstract

We consider a class of optimal stopping problems involving both the running maximum as well as the prevailing state of a linear diffusion. Instead of tackling the problem directly via the standard free boundary approach, we take an alternative route and present a parameterized family of standard stopping problems of the underlying diffusion. We apply this family to delineate circumstances under which the original problem admits a unique well-defined solution. We then develop a discretized approach resulting into a numerical algorithm for solving the considered class of stopping problems. The algorithm is illustrated in two different cases for a GBM and a mean reverting diffusion.

Keywords: Optimal stopping, linear diffusions, maximum process

2010 Mathematics Subject Classification: Primary 60J60, 62L15

Secondary 60G40

1. Introduction

Let X_t be an Itô diffusion evolving on the state space $\mathbb{R}_+$ and denote as $S_t = \sup_{s\leq t}\{X_t\}$ its running supremum. In this paper our objective is to analyze and solve the infinite horizon optimal stopping problem

$$\sup_{\tau} \mathbb{E}_{(x,s)}\left\{e^{-r\tau}f(X_\tau,S_\tau)\right\}, \tag{1.1}$$

where the exercise payoff $f(x,s)$ is assumed to be decreasing in x, increasing in s, $r>0$ denotes the exogenously given constant discount rate, and τ is a stopping time.

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Two well-known examples belonging to this class of stopping problems are the Russian option for which $f(x, s) = s$ (see e.g. [27, 19]) and the American lookback option with a floating strike for which $f(x, s) = s - x$ (see e.g. [9, 19]). While both of these cases constitute perpetual path-dependent options, the latter problem has also an alternative interpretation as a measure of a risk for a stock (see [10, 20]).

Typically optimal stopping problems of the type (1.1) are solved by considering an associated free boundary problem (for a pioneering treatment, see [25]; for a comprehensive treatment of these problems see Chapter III and Section 13 in [26]). In [25] the considered stopping problem was of the form

$$\sup_{\tau} \mathbb{E}_{(x,s)} \left\{ F(S_\tau) - \int_0^\tau c(X_s) ds \right\},$$

where F is an increasing function, c a positive function and X_t Brownian motion. In that study, a powerful *maximality principle* was developed. According to that principle, the first-order differential equation characterizing the optimal exercise boundary admits a maximal solution which stays strictly below the diagonal in $\mathbb{R}^2_+$. It was then shown that the maximality principle is equivalent to the existence of a finite solution, and that the optimal stopping strategy can be characterized as the first time the process X_t falls below the maximal solution. More recently this technique has been further refined in [22] extending the original results of [25] to a more general setting. The optimal stopping of the running minimum within an optimal prediction of the ultimate minimum setting has recently been investigated and solved in [12] by relying on a free boundary approach. Further, the maximality principle has also been adapted to problems involving spectrally negative Lévy processes (see e.g. [18, 23], and the references therein).

In this paper, we address the optimal stopping problem (1.1) under a set of reasonable basic regularity and smoothness assumptions on exercise payoff and the underlying diffusion. Instead of relying on a free boundary approach, we take an alternative route and present a parameterized family of associated standard stopping problems which we solve explicitly by relying on ordinary optimization techniques. We subsequently apply our findings in deriving, independently of the free boundary problem, a set of sufficient conditions under which (1.1) indeed attains a finite solution. Our approach relies on the r-excessivity of the values of the associated stopping problems. In that way it avoids the immediate application of the smooth pasting and instantaneous reflection conditions even though especially the former of these conditions is to some extent embedded in the considered class of optimization problems.

Having established the existence of a solution for the considered class of stopping problems, we then develop a discretized approach which can be applied for determining the optimal policy and its value. In a finite horizon case of the problem, one can discretize time, leading to a familiar binomial tree framework similar to the well known CRR-model (see eg. [4, 16]). However, within an infinite time horizon setting this approach is no longer possible and a somewhat different discretization is required. As our study demonstrates, discretizing the state of the supremum process is an appropriate technique leading to a desired outcome. In the chosen discretization framework, the supremum process can only take values from an arithmetic sequence. Since the supremum process increases only at states where it coincides with the underlying diffusion, we notice that at any given date the underlying process has hit only finitely many times its discretized supremum. Between these hitting times, the two-dimensional process (X_t, S_t) behaves as one-dimensional. It then follows that the discretized problem can be seen as a countable sequence of relatively easily solvable one-dimensional subproblems. Since this sequence is shown to converge to the optimal solution under a set of typically satisfied conditions, our study complements the existing approaches by presenting a technique which does not require the analysis of the ordinary differential equation characterizing the optimal boundary. This discretization simultaneously results into an algorithm for finding the optimal threshold and value numerically as a limit of a converging sequence. In this way we do not only prove that there exists a unique threshold rule, we also identify it. For the sake of generality, we also consider an extension of the original problem (1.1) in a case where there are no monotonicity requirements for the exercise payoff $f(x, s)$. It turns out that our approach applies in that case as well, leading to a convergent sequence approaching the solution.

In order to illustrate our findings explicitly, we solve the value and optimal exercise strategy of a lookback option with a floating strike for a general Itô diffusion. We also determine the value and optimal stopping strategy of a π-option ($f(x, s) = x^{\kappa} s^{\eta} - K$, with $\kappa, \eta, K \geq 0$) introduced in [14]. The efficiency of the developed discrete algorithm is then illustrated for these two option models under two different dynamic specifications for the underlying diffusion process. All in all, our examples seem to indicate that the discretization method can be successfully used to solve a great variety of different stopping problems involving a running supremum process, the primary restrictive factor being an s-Hölder-continuity of f.

It is at this point worth mentioning that there are also other approaches that avoid the use of the free boundary conditions and the maximality principle. In [5] an alternative technique based on a measure transform was introduced. This technique, known as the Beibel-Lerche approach, has been successfully applied in the solution of some optimal stopping problems of the running supremum of a geometric Brownian motion (see [19]). Another alternative approach was developed in [15]. Instead of analyzing the free boundary problem subject to appropriate boundary conditions, [15] computes directly the expected value of stopping strategies defined with respect to a suitable class of boundaries and then chooses the optimal one by relying on arguments familiar form the calculus of variations.

The contents of this study are as follows. The problem and the basic assumptions are represented in an exact form in Section 2. In Section 3, we then prove the existence of a solution to (1.1) by solving a parameterized family of associated stopping problems. We show in Section 4 that the optimal value and stopping boundary can be found also by using the discretization method. Our findings are then illustrated numerically in Section 5 and 6.

2. The Optimal Stopping Problem

Let $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ be a complete filtered probability space satisfying the usual conditions (see p. 2 in [7]). Let X_t be a regular linear diffusion defined on $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ and evolving on $\mathbb{R}_+$ according to the dynamics described by the Itô differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

Here W_t denotes the standard Brownian motion and both the drift term $\mu : \mathbb{R}_+ \to \mathbb{R}$ and volatility term $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are assumed to be sufficiently smooth for guaranteeing the existence and uniqueness of a (weak) solution for the above stochastic differential equation (for example, if the conditions of Theorem 5.15 in [17] are met). Especially we assume that $\sigma(x) > 0$ for $x \in \mathbb{R}_+$ in order to avoid interior singularities. We also assume that the boundaries of the state space are natural for the process X_t. Furthermore, given the underlying diffusion X_t, we denote as

$$S_t = \max\{s, \sup_{0<u\leq t} \{X_u\}\}, \quad S_0 = s \geq x$$

the supremum up to date t of the underlying diffusion. The time $t = 0$ can be interpreted as the time when the considered optimal stopping problem arises, e.g. as the time when the lookback option is issued. In this light s can be seen as the historical supremum of X, reached before the stopping problem aroused, explaining the case $s > x$.

As usually, we define the differential operator associated with the underlying diffusion as

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

and denote as $\mathcal{G}_r := \mathcal{A} - r$ the differential operator associated with the underlying diffusion killed at the constant rate r. Given these differential operators, we denote as ψ and φ the increasing and the decreasing fundamental solutions of the ordinary differential equation $(\mathcal{G}_r u)(x) = 0$, respectively. As is well-known from the literature on linear diffusions, $BL'(x) = \psi'(x)\varphi(x) - \varphi'(x)\psi(x)$, where B is the constant Wronskian of the fundamental solutions and

$$L'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right)$$

denotes the density of the scale function of X_t. Moreover $m'(x) = 2/(\sigma^2(x)L'(x))$ denotes the density of the speed measure of X_t. For a complete characterization of the basic characteristics of a linear diffusion and the associated fundamental solutions, see Chapter 2 in [7].

Given the underlying diffusion and its running maximum, our objective is to analyze and solve the infinite horizon optimal stopping problem

$$V(x,s) = \sup_{\tau} \mathbb{E}_{(x,s)}\left\{e^{-r\tau} f(X_\tau, S_\tau)\right\} \tag{1.1}$$

under the following standing assumptions:

Assumption 2.1. *We assume that the exercise payoff* $f : \mathbb{R}_+^2 \mapsto \mathbb{R}$ *is* x*-non-increasing,* s*-increasing and satisfies the following conditions*

(a) $f(x,s) \in C^{2,1}(\mathbb{R}_+^2)$ *for all* $0 < x \leq s < \infty$ *and* $0 < f(0+,s) < \infty$ *for all* $s > 0$*;*

(b) for a given $s > 0$*, there exists* $\tilde{x}_s \in (0, s]$ *such that* $(\mathcal{G}_r f)(x,s) \lesseqqgtr 0$ *for all* $x \gtreqqless \tilde{x}_s$ *and that* $(\mathcal{G}_r f)(0+,s) < 0$*.*

3. Associated Stopping Problem

3.1. The auxiliary problem and its solution

Instead of tackling the considered two dimensional optimal stopping problem directly via variational inequalities, we now take an alternative approach and consider first an associated parameterized family of one dimensional stopping problems of the underlying linear diffusion process. To that end, let $Q(s)$ be a (finite) nonnegative continuous function satisfying the inequality $Q(s) \geq f(s,s)$ for all $s \in \mathbb{R}_+$. Our first aim is to solve, for $x \leq s$, the auxiliary problem

$$V^Q(x,s) = \sup_{\tau} \mathbb{E}_x \left\{ e^{-r\tau} f(X_\tau, s) \mathbb{1}_{\{\tau < \gamma_s\}} + e^{-r\gamma_s} Q(s) \mathbb{1}_{\{\tau \geq \gamma_s\}} \right\}, \tag{3.1}$$

where $\gamma_s = \inf\{t \geq 0 \mid X_t = s\}$. This problem can be seen as a one-dimensional problem on the state space $(0, s]$, where the boundary s is killing and once reached, it leads to a terminal value $Q(s)$. In what follows, we will show that the set $\{V^Q\}$ generates a family of r-excessive majorants for the payoff f, from which we can later choose the specific V^Q constituting the solution to the original problem (1.1). It is worth pointing out that an approach based on first exit times from open intervals has also been utilized in [13].

To attain our objective, denote by

$$\hat{\psi}(x) = \varphi(y)\psi(x) - \psi(y)\varphi(x)$$

the increasing and by

$$\hat{\varphi}(x) = \varphi(x)\psi(s) - \psi(x)\varphi(s)$$

the decreasing minimal r-excessive mappings for X killed at the boundaries y and s, $y < s$ (cf. pp. 18–20 in [7]). Moreover, for the sake of notational simplicity we also define for any twice continuously differentiable r-harmonic function u and sufficiently smooth function g the functional $(\mathcal{L}_u g)$ as

$$(\mathcal{L}_u g)(x,s) = \frac{g'_x(x,s)}{L'(x)}u(x) - \frac{u'(x)}{L'(x)}g(x,s). \tag{3.2}$$

Especially, we notice that differentiating (3.2) with respect to the current state x yields

$$(\mathcal{L}_u g)'_x(x,s) = (\mathcal{G}_r g)(x,s)u(x)m'(x)$$

due to the assumed r-harmonicity of the function $u(x)$.

We now restrict our analysis to ordinary first passage time type stopping rules $\tau_y = \inf\{t \geq 0 \mid X_t \leq y\}$ and consider for a given upper boundary $s \in \mathbb{R}_+$ and initial state $x \in [y,s]$ the functional

$$\begin{aligned} v(y,x,s) &= \mathbb{E}_{(x,s)}\left\{e^{-r\tau_y} f(X_{\tau_y},s)\mathbb{1}_{\{\tau_y<\gamma_s\}} + e^{-r\gamma_s}Q(s)\mathbb{1}_{\{\tau_y\geq\gamma_s\}}\right\} \\ &= \frac{\hat{\varphi}(x)}{\hat{\varphi}(y)}f(y,s) + \frac{\hat{\psi}(x)}{\hat{\psi}(s)}Q(s). \end{aligned} \tag{3.3}$$

Having stated the associated valuation (3.3), we will show that *there exists a unique threshold* $a_s^Q = a(s,Q) \in (0,s]$ *maximizing the functional* $v(y,x,s)$ *as a function of the boundary* y. Moreover, we will prove that *the associated stopping rule* $\tau_{a_s^Q}$ *constitute the optimal one to the auxiliary problem* (3.1). We first observe by differentiating the functional $v(y,x,s)$ with respect to y that

$$\frac{\partial v(y,x,s)}{\partial y} = \frac{\hat{\varphi}(x)L'(y)}{\hat{\varphi}^2(y)}\left\{(\mathcal{L}_{\hat{\varphi}} f)(y,s) - BQ(s)\right\}.$$

Consequently, we find that a maximizing threshold exists provided that the difference in the brackets changes sign from positive to negative only once on the state space $(0,s]$. This result is established in the following auxiliary lemma.

Lemma 3.1. *There exists a unique maximizing threshold* $a_s^Q \in (0,\tilde{x}_s]$ *satisfying the ordinary first order condition* $(\mathcal{L}_{\hat{\varphi}} f)(a_s^Q,s) = BQ(s)$.

Proof. Consider the functional $H(x,s) = (\mathcal{L}_{\hat{\varphi}} f)(x,s) - BQ(s)$. We first notice that $\lim_{x\uparrow s} H(x,s) \leq 0$ demonstrating that $H(x,s)$ is non-positive at the upper boundary s. On the other hand, since $H'_x(x,s) = (\mathcal{G}_r f)(x,s)\hat{\varphi}(x)m'(x)$, the functional $H(x,s)$ can be re-expressed as

$$H(x,s) = B(f(s,s) - Q(s)) - \int_x^s (\mathcal{G}_r f)(t,s)\hat{\varphi}(t)m'(t)dt$$

showing that $H(x,s) < 0$ for all $x \in (\tilde{x}_s, s]$ by Assumption 2.1(b). Moreover, for $x < x_1 < \tilde{x}_s$, applying the mean value theorem for integrals, we get

$$\begin{aligned} H(x,s) &= H(x_1,s) - \int_x^{x_1} (\mathcal{G}_r f)(t,s)\hat{\varphi}(t)m'(t)dt \\ &= H(x_1,s) - \frac{1}{r}(\mathcal{G}_r f)(\xi,s)\left\{\frac{\hat{\varphi}'(x_1)}{L'(x_1)} - \frac{\hat{\varphi}'(x)}{L'(x)}\right\}, \end{aligned}$$

where $\xi \in (x, x_1)$. The assumed boundary behavior together with Assumption 2.1(b) guarantees that $H(0+,s) = \infty$. Combining this observation with the continuity and monotonicity of the functional $H(x,s)$ then completes the proof of the existence and uniqueness of the maximizing boundary $a_s^Q \in (0, \tilde{x}_s)$.

Having demonstrated that there is a unique boundary maximizing the functional (3.3), we are now in position to prove the following:

Theorem 3.1. *Let Assumption 2.1 hold. Then, for a given s, $\tau_{a_s^Q} = \inf\{t \geq 0 \mid X_t \leq a_s^Q\}$ is the optimal stopping time for the problem* (3.1) *and the value is*

$$V^Q(x,s) = \begin{cases} v(a_s^Q, x, s) & x \in (a_s^Q, s], \\ f(x,s) & x \in (0, a_s^Q]. \end{cases} \tag{3.4}$$

Moreover, if $Q(s)$ is differentiable, then

$$\lim_{x \uparrow s} \frac{\partial V^Q(x,s)}{\partial s} = \left\{Q'(s)B - (\mathcal{L}_\Phi f)(a_s^Q, s)\right\} \frac{\hat{\varphi}^2(a_s^Q)}{B}, \tag{3.5}$$

where $\Phi(x) = \varphi(x)\psi'(s) - \varphi'(s)\psi(x)$ denotes the minimal decreasing r-excessive function for the underlying diffusion reflected at s.

Proof. Let $V^Q(x,s)$ be the solution to (3.1) and denote by $J(x,s)$ the value given in (3.4). Obviously $J(x,s)$ is obtained by following an admissible stopping strategy and, therefore, $V^Q(x,s) \geq J(x,s)$. In order to prove the opposite inequality, we first observe that it is clear by construction that $J(x,s)$ is continuous on $(0,s]$ and that $J'_x(a_s^Q-, s) = f'_x(a_s^Q, s)$. Furthermore, since

$$\frac{\partial v(a_s^Q, x, s)}{\partial x} = \frac{\varphi(a_s^Q)Q(s) - \varphi(s)f(a_s^Q, s)}{\hat{\varphi}(a_s^Q)}\psi'(x) + \frac{\psi(s)f(a_s^Q, s) - \psi(a_s^Q)Q(s)}{\hat{\varphi}(a_s^Q)}\varphi'(x)$$

we find by letting $x \downarrow a_s^Q$ and invoking the optimality condition $(\mathcal{L}_{\hat{\varphi}} f)(a_s^Q, s) = BQ(s)$ that $J'_x(a_s^Q+, s) = f'_x(a_s^Q, s)$ proving the continuous differentiability of $J(x,s)$. Next, let $x \in (a_s^Q, s]$. Then $v(a_s^Q, x, s) \geq v(x-, x, s) = f(x, s)$ the inequality following from the optimality of a_s^Q. This shows that $J(x, s)$ is a continuously differentiable majorant of $f(x, s)$.

It remains to establish that $J(x, s)$ is r-excessive for the underlying diffusion X killed at s. To see that this is indeed the case, we first observe that $(\mathcal{G}_r J)(x, s) = 0$ on $(a_s^Q, s]$ and $(\mathcal{G}_r J)(x, s) = (\mathcal{G}_r f)(x, s) < 0$ on $(0, a_s^Q)$. The alleged result then follows from the inequality $|f''_{xx}(a_s^Q \pm, s)| < \infty$. We have thus established that $J(x, s)$ is an r-excessive majorant of $f(x, s)$. Since the optimal value V^Q is the smallest of such majorants, we conclude that $J \geq V^Q$.

Finally, differentiating the value $J(x, s)$ with respect to s and invoking the optimality condition $(\mathcal{L}_{\hat{\varphi}} f)(a_s^Q, s) = Q(s)B$ then yields (3.5).

Given the assumed differentiability of the exercise payoff, we notice by implicit differentiation that the sensitivity of the optimal threshold with respect to changes in the exogenous upper boundary s can be expressed as

$$a_s^{Q\prime} = \frac{BQ'(s) - (\mathcal{L}_\Phi f)(a_s^Q, s) - (\mathcal{L}_{\hat{\varphi}} f_s)(a_s^Q, s)}{(\mathcal{G}_r f)(a_s^Q, s)\hat{\varphi}(a_s^Q) m'(a_s^Q)}.$$

On the other hand, Theorem 3.1 guarantees that $V^Q(x, s)$ constitutes an excessive majorant of the exercise payoff as long as the inequality $Q(s) \geq f(s, s)$ is fulfilled. Combining these observations show that if $Q(s)$ is chosen so that also condition $(\mathcal{L}_\Phi f)(a_s^Q, s) = BQ'(s)$ is satisfied, then the value $V^Q(x, s)$ satisfies the instantaneous reflection condition $V_s^{Q\prime}(s-, s) = 0$ as well and the optimal exercise boundary satisfies the differential equation

$$a_s^{Q\prime} = \frac{1}{2}\sigma^2(a_s^Q)\frac{\hat{\varphi}'(a_s^Q) f_s'(a_s^Q, s) - \hat{\varphi}(a_s^Q) f''_{xs}(a_s^Q, s)}{(\mathcal{G}_r f)(a_s^Q, s)\hat{\varphi}(a_s^Q)}. \tag{3.6}$$

It's worth pointing out that utilizing the standard free boundary approach for solving the considered stopping problem results into the differential equation (3.6) as well (cf. Section 13 in [26]).

3.2. The solution to the main problem

Before proving our main existence theorem for (1.1), we first need to assure the finiteness of the value of the stopping problem.

Lemma 3.2. *Let Assumption 2.1 hold and assume that* $\int_0^\infty \mathbb{E}_{(x,s)}\left\{e^{-rt}f(0,S_t)\right\}dt < \infty$ *for all* $0 < x \leq s < \infty$. *Then the value function* (1.1) *is finite.*

Proof. Fix $0 < x \leq s < \infty$ and denote by $T_r \sim \mathrm{Exp}(r)$ an exponentially distributed random time, independent of W_t. Since f is continuous and (X_t, S_t) is a strong Markov process, it is known (see e.g. Proposition 2.1 in [11] and also Lemma 2.2 in [8]) that

$$u(x,s) := \mathbb{E}_{(x,s)}\left\{\sup_{0\leq t\leq T_r} f(0,S_t)\right\} = \mathbb{E}_{(x,s)}\left\{f(0,S_{T_r})\right\}$$

is r-excessive. Moreover, it is clear that

$$f(x,s) \leq f(0,s) = \mathbb{E}_{(x,s)}\left\{f(0,s)\right\} \leq \mathbb{E}_{(x,s)}\left\{f(0,S_{T_r})\right\} = u(x,s),$$

demonstrating that u dominates f. Since V constitutes the minimal r-excessive majorant of f, we notice that $V \leq u$.

Furthermore, by straight calculations

$$\begin{aligned} u(x,s) = \mathbb{E}_{(x,s)}\left\{f(0,S_{T_r})\right\} &= \mathbb{E}_{(x,s)}\left\{r\int_0^\infty e^{-rt}f(0,S_t)dt\right\} \\ &= r\int_0^\infty \mathbb{E}_{(x,s)}\left\{e^{-rt}f(0,S_t)\right\}dt. \end{aligned}$$

The last term on the right hand side of this equality is finite by assumption and thus $V(x,s) \leq u(x,s) < \infty$.

Having established the finiteness of the value of the optimal stopping strategy we are now in position to state our main theorem characterizing the value and optimal exercise policy of problem (1.1).

Theorem 3.2. *Let Assumption 2.1 hold and assume that* $\int_0^\infty \mathbb{E}_{(x,s)}\left\{e^{-rt}f(0,S_t)\right\}dt < \infty$. *Then there exists a unique function* $a_s^* \in (0, \tilde{x}_s)$, *such that* $\tau_{a_s^*} = \inf\{t \geq 0 \mid X_t \leq a_{S_t}^*\}$ *is the optimal stopping time for the considered problem* (1.1). *Moreover, there exists a unique* $Q(s)$ *for which the value* $V(x,s)$ *reads as in* (3.4).

Proof. For $x \leq s$, the problem (1.1) can be re-written as

$$\sup_\tau \mathbb{E}_{(x,s)}\left\{e^{-r\tau}f(X_\tau,s)\mathbb{1}_{\{\tau<\gamma_s\}} + e^{-r\gamma_s}\sup_\xi \mathbb{E}_{\{s,s\}}\left\{e^{-r\xi}f(X_\xi,S_\xi)\right\}\mathbb{1}_{\{\tau\geq\gamma_s\}}\right\},$$

where $\gamma_s = \inf\{t \geq 0 \mid X_t = s\}$, so that it is of the form of the auxiliary problem (3.1), with

$$Q(s) = \sup_{\xi} \mathbb{E}_{\{s,s\}} \left\{ e^{-r\xi} f(X_\xi, S_\xi) \right\}.$$

Obviously $Q(s) \geq f(s,s)$ and by Lemma 3.2 $Q(s) < \infty$ for all $s < \infty$. Therefore, the alleged claim now follows from Theorem 3.1.

Theorem 3.2 states a set of sufficient conditions under which the auxiliary stopping problem constitutes the value of the optimal stopping problem (1.1). The sensitivity of the value and optimal boundary with respect to changes in the volatility of the underlying diffusion are now summarized in the following.

Theorem 3.3. *Assume that the conditions of Theorem 3.2 are satisfied, that the difference* $\mu(x) - rx$ *is non-increasing, and that a transversality condition* $\lim_{t\to\infty} \mathbb{E}_x \{e^{-rt} X_t\} = 0$ *holds. Then, the value function is strictly convex as a function of the current state* x *on the continuation set* $(a_s^*, s]$ *and increased volatility increases the value* $V(x,s)$ *and decreases the optimal stopping boundary* a_s^*.

Proof. Fix $s < \infty$ and let $\sigma_1(x) \leq \sigma_2(x)$, for all x. For $i = 1, 2$, denote by V_i and a_i^* the value function and the optimal stopping boundary for the problem (1.1), respectively, with respect to σ_i. The assumed monotonicity of the difference $\mu(x) - rx$ together with the transversality condition guarantee that the fundamental solutions are convex (see Corollary 1 in [2]). Furthermore, the r-excessivity of the value $V(x,s)$ implies that it constitutes a positive affine transformation of the minimal solutions $\psi(x)$ and $\varphi(x)$ on the set (a_s^*, s) where it is r-harmonic and consequently it is convex there. Since the sign of the relationship between increased volatility and the value of an r-excessive mapping is positive on the set where it is r-harmonic (cf. Theorem 4 in [2]), we find that $V_1(x,s) \leq V_2(x,s)$. Suppose, contrary to our claim, that $a_1^* < a_2^*$, and let $x \in (a_1^*, a_2^*)$ so that x is in the continuation region with respect to V_1, and in the stopping region with respect to V_2. Then $V_2(x,s) = f(x,s) < V_1(x,s)$, which contradicts the inequality derived above.

Theorem 3.3 states a set of conditions under which increased volatility unambiguously increases the value of the optimal stopping policy and postpones exercise by lowering the optimal boundary.

3.3. A useful extension

It turns out that our existence result can be directly generalized to also cover a general class of continuous exercise payoffs satisfying a boundedness condition. To this end, we consider the problem (1.1) under the following weakened assumptions.

Assumption 3.1. *For each $s > 0$, let $\underline{x}_s \in [0, s]$ be the point at which $f(x, s)$ is maximized. Assume also that the exercise payoff $f : \mathbb{R}^2_+ \mapsto \mathbb{R}$ satisfies the following conditions*

(a) $f(x, s) \in C(\mathbb{R}^2_+)$ *for all* $0 < x \leq s < \infty$ *and* $0 < f(\underline{x}_s, s) < \infty$ *for all* $s > 0$.

(b) $\mathbb{E}_{(x,s)} \left\{ \sup_{0 \leq t \leq T_r} f(\underline{x}_{S_t}, S_t) \right\} dt < \infty$.

Under this assumption, we can again constitute the auxiliary problem (3.1) for a non-negative, continuous $Q(s)$ satisfying $Q(s) \geq f(s, s)$, and the following proposition holds. (Denote by τ_Q^* and τ^* the optimal stopping times for auxiliary problem (3.1) and (1.1), respectively.)

Proposition 3.1. *Let Assumption 3.1 hold. Then*

(A) the value $V(x, s)$ is finite;

(B) there exists a unique $Q(s)$ such that $V^Q(x, s) = V(x, s)$. Moreover, if $\tau_Q^* < \infty$ a.s., $\tau^* = \tau_Q^*$.

Proof. (A) The proof is completely analogous with the one of Lemma 3.2. (B) For each finite $Q(s)$ the linear auxiliary problem (3.1) has a solution by general existence results concerning linear diffusions (see e.g. [26]). Establishing the alleged claim is analogous with the proof of Theorem 3.2.

Interestingly, the unique existence of solution to a general problem involving a maximum process can be reduced to a search of a unique solution to a linear diffusion problem. However, with general assumptions we cannot, naturally, guarantee the shape of the stopping region. The following corollary presents an example how this existence result can be used (cf. Proposition 6.1).

Corollary 3.1. *Let Assumption 3.1 hold. Further, assume that for all $s > 0$ and $Q(s)$ there exists a unique stopping region $\mathfrak{S}_s^Q$ such that $\tau_Q^* = \left\{ t \geq 0 \mid X_t \in \mathfrak{S}_s^Q \right\}$ is* *the optimal stopping time for the auxiliary problem* (3.1). *Then for each* $s > 0$ *there exists a unique stopping region* $\mathfrak{S}_s \subset [0, s]$ *such that* $\tau^* = \{t \geq 0 \mid X_t \in \mathfrak{S}_{S_t}\}$, *where* $\mathfrak{S}_s = \mathfrak{S}_s^Q$ *for some* Q.

4. The discretization

Our objective is now to develop a sequence of optimal stopping problems by discretizing the state of the supremum process and to show that the sequence converges in the limit to the original stopping problem (1.1). To this end, we need the following two additional assumptions.

Assumption 4.1. *Assume, in addition to Assumption 2.1, that*

(c) $\int_0^\infty \mathbb{E}_{(x,s)} \{e^{-rt} f(0, S_t)\}\, dt < \infty$ *for all* $0 < x \leq s < \infty$;

(d) $f(x, s)$ *is s-Hölder-continuous, i.e. there exist* $M > 0$ *and* $0 < \alpha \leq 1$ *such that* $|f(x, s_1) - f(x, s_2)| \leq M|s_1 - s_2|^\alpha$ *for all* $s_1, s_2 \in \mathbb{R}_+$ *and* $x < \min\{s_1, s_2\}$.

First of all, let us prove an equivalent condition to Assumption 4.1(c).

Lemma 4.1. *Let Assumption 2.1 hold. Then* $\int_s^\infty \frac{f_s'(0,u)}{\psi(u)} du < \infty$ *for all* $s \in \mathbb{R}_+$, *if and only if Assumption 4.1*(c) *holds.*

Proof. Let $T \sim \mathrm{Exp}(r)$. From the proof of Lemma 3.2 we know that $\mathbb{E}_{(x,s)} \{f(0, S_T)\} = r \int_0^\infty \mathbb{E}_{(x,s)} \{e^{-rt} f(0, S_t)\}\, dt$.

Observe that (see p. 26 in [7]) for all $x < y$ we have $\mathbb{P}_x (S_T \leq y) = \mathbb{P}_x (\tau_y > T) = 1 - \psi(x)/\psi(y)$, where $\tau_y = \inf\{t \geq 0 \mid X_t \geq y\}$. Using this fact and Fubini's theorem, we can calculate

$$\begin{aligned} r \int_0^\infty \mathbb{E}_{(x,s)} \left\{e^{-rt} f(0, S_t)\right\} dt = \mathbb{E}_{(x,s)} \{f(0, S_T)\} &= \int_s^\infty f(0, y) d\mathbb{P}(S_T \leq y) \\ &= \int_s^\infty f(0, y) \frac{\psi(x)\psi'(y)}{\psi^2(y)} dy \\ &= f(0, s) + \psi(x) \int_s^\infty \frac{f_s'(0, u)}{\psi(u)} du, \end{aligned}$$

whence the claim follows.

4.1. Introducing the recursive algorithm

Fix a step size $z > 0$, number of steps $n \in \mathbb{N}$ and starting points $(X_0, S_0) = (x, s)$ with $0 < x < s$ and, for all s, fix a terminal payoff function $Q(s)$, which satisfies $Q(s) \geq f(s,s)$ and $\lim_{t\to\infty} \mathbb{E}_{(x,s)}\left\{e^{-rt}Q(S_t)\right\} = 0$. Denote $s_k := s + kz$, for $k \in \mathbb{N}$. From now on in this section we assume that the supremum process S_t can only take values s_k, $k = 0, 1, 2\ldots, n$ (for convenience we denote $s := s_0$). That is, as we start from (x, s), the supremum process S_t jumps to the state s_1 when the diffusion X_t reaches the point s and we "restart" (X_t, S_t) from the state (s, s_1). Again, when X_t reaches the new supremum value s_1, the process S_t jumps to s_2 and we again "restart" (X_t, S_t), now from (s_1, s_2). The discretization is graphically illustrated in Figure 1. It is worth mentioning that since S_t takes values from a finite arithmetic sequence, we know that at any time $t > 0$ there has been only finitely many jumps in the path of the discretized supremum. Furthermore, we consider s_n to be the highest possible level for X_t, and consequently for S_t. This means that when X_t reaches s_n, the process is stopped (killed) and we receive the terminal payoff $Q(s_n)$ at that state.

Having presented the discretized version of the running supremum of the underlying diffusion, we now apply the findings of our Theorem 3.1 and define recursively a sequence of continuously differentiable r-excessive values dominating the exercise payoff. To this end, we first define the terminal value of the sequence as $V_{n+1} \equiv Q(s_n)$. Given the terminal value V_{n+1}, we now define recursively for any index $1 \leq k \leq n$ the values V_k as $V_k := J(s_{k-1}, s_k)$,

$$J(x, s_k) = \sup_{\tau} \mathbb{E}_{(x,s_k)} \left\{ e^{-r\tau} f(X_\tau, s_k) \mathbb{1}_{\{\tau < \gamma_{s_k}\}} + e^{-r\gamma_{s_k}} V_{k+1} \mathbb{1}_{\{\tau \geq \gamma_{s_k}\}} \right\} \tag{4.1}$$

$$= \begin{cases} \frac{\varphi(x)\psi(s_k) - \psi(x)\varphi(s_k)}{\varphi(\hat{a}_{s_k})\psi(s_k) - \psi(\hat{a}_{s_k})\varphi(s_k)} f(\hat{a}_{s_k}, s_k) \\ \qquad + \frac{\psi(x)\varphi(\hat{a}_{s_k}) - \varphi(x)\psi(\hat{a}_{s_k})}{\psi(s_k)\varphi(\hat{a}_{s_k}) - \varphi(s_k)\psi(\hat{a}_{s_k})} V_{k+1} & x \in (\hat{a}_{s_k}, s_k] \\ f(x, s_k) & x \in (0, \hat{a}_{s_k}], \end{cases}$$

Here $\gamma_{s_k} = \inf\{t \geq 0 \mid X_t = s_k\}$, for $k = 1, \ldots, n$, denotes the first hitting time of X to the state s_k, and $\hat{a}_{s_k} \in (0, \tilde{x}_{s_k})$ constitutes the unique root of the ordinary first order condition $(\mathcal{L}_{\hat{\varphi}} f)(\hat{a}_{s_k}, s_k) = BV_{k+1}$ (cf. Theorem 3.1). Finally, the initial value is chosen as $V_0 = J(x, s)$. It is clear that these identities completely characterize the sequence of values $\{V_k\}_{k=0}^{n+1}$ and the sequence of optimal exercise boundaries $\{\hat{a}_{s_k}\}_{k=0}^{n}$.

Moreover, we also observe that for $z > 0$ and $n \in \mathbb{N}$ this discretized problem can be written in a compact form

$$
\begin{aligned}
&J(z,n,x,s) := \\
&:= \sup_{\tau} \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, s + z \sum_{k=1}^{n-1} \mathbb{1}_{\{\gamma_{s_k} < \tau\}} \right) \mathbb{1}_{\{\tau < \gamma_{s_n}\}} + e^{-r\gamma_{s_n}} Q(s_n) \mathbb{1}_{\{\tau \geq \gamma_{s_n}\}} \right\} \\
&= \sup_{\tau} \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, S_\tau) \mathbb{1}_{\{\tau < \gamma_{s_n}\}} + e^{-r\gamma_{s_n}} Q(s_n) \mathbb{1}_{\{\tau \geq \gamma_{s_n}\}} \right\},
\end{aligned}
\tag{4.2}
$$

where S_t denotes the discretized supremum process.

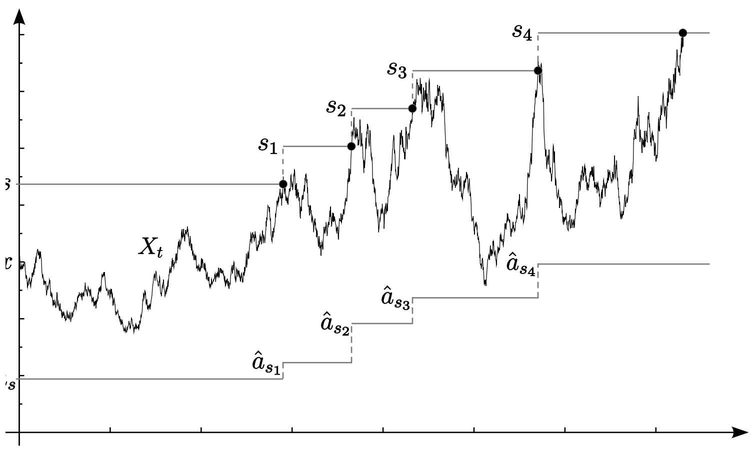

FIGURE 1: An illustrative example of how a sample path evolves in constructed discretized problem. Here $n = 4$ and $\hat{a}_{s_k}$ is the optimal stopping boundary at step k, for $k = 0, 1, 2, 3, 4$. We stop immediately after the diffusion hits either the lower boundary $\hat{a}_{s_k}$, when we receive a payoff $f(\hat{a}_k, s_k)$, or the maximum level s_4 implying payoff $Q(s_4)$ (exogenously given terminal function).

Lastly a notational remark. In this section, we shall denote by S_t a discretized supremum process, whereas the normal continuous one is denoted by S_t^0.

4.2. Proving that the algorithm works

Let us first establish that the limiting value function of the sequence does not depend on the choice of terminal value function $Q(s)$.

Lemma 4.2. *Let Assumption 4.1 hold and fix $s, z > 0$. Furthermore, let $Q(s) \geq f(s,s)$ be such that $\lim_{t\to\infty} \mathbb{E}_{(x,s)} \left\{ e^{-rt} Q(S_t^0) \right\} = 0$. Then, the limit $\lim_{n\to\infty} J(z,n,x,s)$ does not depend on the choice of $Q(s)$.*

Proof. Fix $n \in \mathbb{N}$ and $Q_1(s) > Q_2(s)$. For $i = 1, 2$, denote as J^i the value function associated to the terminal payoff $Q_i(s)$, and let τ_1 be the optimal stopping rule maximizing the dicsretised problem with $Q_1(s)$ as a terminal value (this exists by Theorem 3.1). Since $Q_1(s) > Q_2(s)$, we know that $J^1(z,n,x,s) - J^2(z,n,x,s) \geq 0$. On the other hand we can apply (4.2) to make an estimate

$$
\begin{aligned}
J^1(z,n,x,s) - J^2(z,n,x,s) = \\
&= \mathbb{E}_{(x,s)} \left\{ e^{-r\tau_1} f(X_{\tau_1}, S_{\tau_1}) \mathbb{1}_{\{\tau_1 < \gamma_{s_n}\}} + e^{-r\gamma_{s_n}} Q_1(s_n) \mathbb{1}_{\{\tau_1 \geq \gamma_{s_n}\}} \right\} \\
&\quad - \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, S_\tau) \mathbb{1}_{\{\tau < \gamma_{s_n}\}} + e^{-r\gamma_{s_n}} Q_2(s_n) \mathbb{1}_{\{\tau \geq \gamma_{s_n}\}} \right\} \\
&\leq \mathbb{E}_{(x,s)} \left\{ e^{-r\tau_1} f(X_{\tau_1}, S_{\tau_1}) \mathbb{1}_{\{\tau_1 < \gamma_{s_n}\}} + e^{-r\gamma_{s_n}} Q_1(s_n) \mathbb{1}_{\{\tau_1 \geq \gamma_{s_n}\}} \right\} \\
&- \mathbb{E}_{(x,s)} \left\{ e^{-r\tau_1} f(X_{\tau_1}, S_{\tau_1}) \mathbb{1}_{\{\tau_1 < \gamma_{s_n}\}} + e^{-r\gamma_{s_n}} Q_2(s_n) \mathbb{1}_{\{\tau_1 \geq \gamma_{s_n}\}} \right\} \\
&= \mathbb{E}_{(x,s)} \left\{ e^{-r\gamma_{s_n}} (Q_1(s_n) - Q_2(s_n)) \mathbb{1}_{\{\tau_1 \geq \gamma_{s_n}\}} \right\} \\
&\leq \mathbb{E}_{(x,s)} \left\{ e^{-r\gamma_{s_n}} (Q_1(s_n) - Q_2(s_n)) \right\}.
\end{aligned}
$$

Since $\lim_{n\to\infty} \mathbb{E}_{(x,s)} \{ e^{-r\gamma_{s_n}} Q_i(s_n) \} = 0$, for $i = 1, 2$, by assumption, we notice that the last term tends to zero as n approaches infinity.

According to Lemma 4.2 the algorithm results into the same value irrespective of the chosen terminal value $Q(s)$ as long as it satisfies the relatively weak conditions of our lemma. Hence, depending on the precise form of the exercise payoff and its behavior at the upper boundary s, natural choices for $Q(s)$ are, for example, additive forms $Q(s) = f(s,s) + a$, $a \geq 0$, or multiplicative forms $Q(s) = bf(s,s)$, $b \geq 1$.

It remains to establish that the sequence of optimal boundaries and value functions converge towards the corresponding ones of the original problem (1.1) as $n \to \infty$ and $z \to 0$. This property is established in our next theorem.

Theorem 4.1. *Let Assumption 4.1 hold.*

(a) Fix $z > 0$. Then the limit $J(z,x,s) := \lim_{n\to\infty} J(z,n,x,s)$ exists finitely. Furthermore $\lim_{z\to 0} J(z,x,s) = \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, S_\tau^0) \right\}$.

(b) Fix $s > 0$. Then $\hat{a}_s$ approaches the optimal stopping boundary a_s^* as $n \to \infty$ and $z \to 0$.

Proof. (a) Choose the terminal value function as $Q(s_k) = f(s_k, s_k)$. We see at once that this choice satisfies the conditions of Lemma 4.2 under Assumption 4.1. Moreover, with this choice, the value $J(z, n, x, s)$ constitutes an increasing sequence in n. To see this fix $N \in \mathbb{N}$ and let $\{V_k^N\}_{k=0}^N$ be a sequence with respect to the number of steps N. Then $V_N^N = Q(s_{N-1}) = f(s_{N-1}, s_{N-1})$. On the other hand, with the number of steps being $N+1$ we get $V_{N+1}^{N+1} = Q(s_N) = f(s_N, s_N)$ leading to

$$\begin{aligned} V_N^{N+1} &= \sup_\tau \mathbb{E}_{(s_{N-1}, s_N)} \left\{ e^{-r\tau} f(X_\tau, s_N) \mathbb{1}_{\{\tau < \gamma_{s_N}\}} + e^{-r\gamma_{s_N}} Q(s_N) \mathbb{1}_{\{\tau \geq \gamma_{s_N}\}} \right\} \\ &\geq f(s_{N-1}, s_N) \geq f(s_{N-1}, s_{N-1}) = V_N^N \end{aligned}$$

Consequently $V_k^{N+1} \geq V_k^N$ for all $k \leq N$, so that especially $V_0^{N+1} = J(z, N+1, x, s) \geq J(z, N, x, s) = V_0^N$.

Moreover, utilizing the expression (4.2) and applying the assumed s-Hölder-continuity we can make the following estimation for an arbitrary $n \in \mathbb{N}$

$$\begin{aligned} J(z, n, x, s) &\leq \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, S_\tau^0 + z\right) \mathbb{1}_{\{\tau < \gamma_{s_n}\}} + e^{-r\gamma_{s_n}} Q(s_n) \mathbb{1}_{\{\tau \geq \gamma_{s_n}\}} \right\} \\ &\leq \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, S_\tau^0 + z\right) \right\} + \mathbb{E}_{(x,s)} \left\{ e^{-r\gamma_{s_n}} Q(s_n) \right\} \\ &\leq \sup_\tau \left\{ \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, S_\tau^0\right) \right\} + \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} M z^\alpha \right\} \right\} + \mathbb{E}_{(x,s)} \left\{ e^{-r\gamma_{s_n}} Q(s_n) \right\} \\ &\leq \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, S_\tau^0\right) \right\} + M z^\alpha + \mathbb{E}_{(x,s)} \left\{ e^{-r\gamma_{s_n}} Q(s_n) \right\} < \infty, \end{aligned}$$

where the finiteness follows from Lemma 3.2. Since $J(z, n, x, s)$ is a bounded increasing sequence, it converges as $n \to \infty$. Since $\lim_{n\to\infty} \mathbb{E}_{(x,s)} \left\{ e^{-r\gamma_{s_n}} Q(s_n) \right\} = 0$, we get

$$J(z, x, s) = \lim_{n \to \infty} J(z, n, x, s) \leq \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, S_\tau^0\right) \right\} + M z^\alpha \tag{4.3}$$

On the other hand, utilizing again expression (4.2) we also obtain the inequality

$$\begin{aligned} J(z, x, s) &= \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, s + z \sum_{k=1}^\infty \mathbb{1}_{\{\gamma_{s_k} < \tau\}} \right) \right\} \\ &\geq \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, S_\tau^0 - z\right) \right\} \\ &\geq \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f\left(X_\tau, S_\tau^0\right) \right\} - M z^\alpha. \end{aligned}$$

Combining this with (4.3) we see that

$$\sup_\tau \mathbb{E}_{(x,s)} \left\{e^{-r\tau} f\left(X_\tau, S_\tau^0\right)\right\} - Mz^\alpha \leq J(z,x,s) \leq \sup_\tau \mathbb{E}_{(x,s)} \left\{e^{-r\tau} f\left(X_\tau, S_\tau^0\right)\right\} + Mz^\alpha, \tag{4.4}$$

so that by letting $z \to 0$ we get $J(z,x,s) \to \sup_\tau \mathbb{E}_{(x,s)} \left\{e^{-r\tau} f\left(X_\tau, S_\tau^0\right)\right\}$.

(b) The value function V of the original problem (1.1) can be written as (cf. Theorem 3.2)

$$V(x,s) = \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, s) \mathbb{1}_{\{\tau<\gamma_s\}} + e^{-r\gamma_s} \sup_\xi \mathbb{E}_{(s,s)} \left\{e^{-r\xi} f(X_\xi, S_\xi^0)\right\} \mathbb{1}_{\{\tau\geq\gamma_s\}} \right\}, \tag{4.5}$$

where τ and ξ are admissible stopping times, and the supremum is attained with $\tau_{a_s^*} = \inf\{t \geq 0 \mid X_t \leq a_s^*\}$, where $a_s^* \in (0, \tilde{x}_s)$ is the unique stopping boundary from Theorem 3.2.

On the other hand, the discretized problem can be written as

$$\begin{aligned} \lim_{z\to 0} J(z,x,s) &= \lim_{z\to 0} \sup_\tau \mathbb{E}_{(x,s)} \left\{e^{-r\tau} f(X_\tau, s)\mathbb{1}_{\{\tau<\gamma_s\}} + e^{-r\gamma_s} V_1 \mathbb{1}_{\{\tau\geq\gamma_s\}}\right\} \\ &= \sup_\tau \mathbb{E}_{(x,s)} \left\{e^{-r\tau} f(X_\tau, s)\mathbb{1}_{\{\tau<\gamma_s\}} + e^{-r\gamma_s} \mathbb{1}_{\{\tau\geq\gamma_s\}} \lim_{z\to 0} V_1\right\}, \end{aligned}$$

where the supremum is attained with $\tau_{\hat{a}_s}$, where $\hat{a}_s \in (0, \tilde{x}_s)$ is the unique stopping boundary. Now $J(z, s-, s) = V_1$ and according to part (a) $\lim_{z\to 0} J(z, s-, s) = \sup_\tau \mathbb{E}_{(s,s)} \left\{e^{-r\tau} f(X_\tau, S_\tau^0)\right\}$. Hence, we get the equality

$$\begin{aligned} &\lim_{z\to 0} J(z,x,s) = \\ &\quad = \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} f(X_\tau, s) \mathbb{1}_{\{\tau<\gamma_s\}} + e^{-r\gamma_s} \sup_\xi \mathbb{E}_{(s,s)} \left\{e^{-r\xi} f(X_\xi, S_\xi^0)\right\} \mathbb{1}_{\{\tau\geq\gamma_s\}} \right\}, \end{aligned}$$

which coincides with (4.5). It follows that we have $\hat{a}_s = a_s^*$.

Theorem 4.1 demonstrates that the developed algorithm indeed converges to the proposed limit. However, it does not characterize the speed of convergence to the limit as the discretization step becomes smaller. This subject is addressed in the following.

Corollary 4.1. *Let Assumption 4.1 hold. Then, the rate of convergence* $\lim_{z\to 0} J(z,x,s) = V(x,s)$ *is of order* $\mathcal{O}(z^\alpha)$.

Proof. From (4.4) we see straight that $J(z,x,s) = V(x,s) + \mathcal{O}(z^{\alpha})$.

Unfortunately, Corollary 4.1 characterizes the convergence of the algorithm only in terms of the denseness of the applied discretization and not in terms of the number of steps. In order to characterize that, we would have to be able to estimate the difference $|V(x,s) - J(z,n,x,s)|$, which is a highly process dependent quantity.

4.3. A useful extension

Let us present a discretization associated to the generalization introduced in Subsection 3.3. The proofs are analogous to those in Subsection 4.2, and are thus omitted.

Theorem 4.2. *Let Assumption 3.1 hold. In addition, assume that*

(c) $f(x,s)$ *is s-Hölder-continuous.*

(d) $Q(s) \geq f(s,s)$ *is such that* $\lim_{t\to\infty} \mathbb{E}_{(x,s)} \left\{ e^{-rt} Q(S_t^0) \right\} = 0$*;*

Then $\lim_{z\to\infty} \lim_{n\to\infty} J(z,n,x,s) = V(x,s)$*, where* J *is defined through* (4.1)*.*

Proposition 4.1. *Let the assumptions of Theorem 4.2 hold. In addition, assume that for all* $s > 0$ *and* Q *there exists a unique stopping region* $\mathfrak{S}_s^Q$ *such that* $\tau_Q^* = \left\{ t \geq 0 \mid X_t \in \mathfrak{S}_s^Q \right\}$ *provides the value for the auxiliary problem* (3.1)*. Then*

(a) $\lim_{z\to\infty} \lim_{n\to\infty} J(z,n,x,s) = V(x,s)$*;*

(b) $\lim_{z\to 0} \lim_{n\to\infty} \hat{\tau}_{\hat{\mathfrak{S}}_s} = \tau_s^*$*, where* $\hat{\tau}_{\hat{\mathfrak{S}}_s} = \inf\{t \geq 0 \mid X_t \in \hat{\mathfrak{S}}_s\}$ *and* $\hat{\mathfrak{S}}_s$ *is the stopping region for the discretized problem with a state* s*, and* τ_s^* *is the optimal stopping time for the problem* (1.1) *with a state* s*.*

Theorem 4.2 illustrates that under the stated assumptions the discretization approaches the value irrespectively on whether the value is attained with a finite stopping time or not. In addition, if we know that for all admissible $Q(s)$ the value of the auxiliary problem is attained with an admissible stopping time, then the stopping region "approaches" the stopping region of the initial problem as well. All in all, the generalization in Subsection 3.3 assures that the proof of the existence of a solution to problem (1.1) reduces to the proof of the existence of a solution for a linear problem

(3.1). On top of this, the results above guarantees that these solutions can be attained numerically.

5. Explicit Illustration: Perpetual Lookback with Floating Strike

In order to illustrate the algorithm developed in our paper, we now consider the valuation and optimal exercise of a perpetual lookback option with a floating strike. In that case the exercise payoff reads as $f(x,s) = (s - kx)$, where $k \in \mathbb{R}_+$ is a known exogenously given constant. Therefore, our objective is to analyze and solve the stopping problem (cf. [9, 10, 19, 20, 24])

$$V(x,s) = \sup_{\tau} \mathbb{E}_{(x,s)}[e^{-r\tau}(S_\tau - kX_\tau)]. \tag{5.1}$$

It is clear that by letting $k \downarrow 0$ the problem becomes the valuation of a perpetual Russian option (cf. [27]). As our general findings indicate, in this case we have the following result.

Proposition 5.1. *Assume that* $\int_0^\infty \mathbb{E}_{(x,s)}\{e^{-rt}S_t\}\,dt < \infty$*, that there is a single state* $\tilde{x}_s \in \mathbb{R}_+$ *so that* $k(rx - \mu(x)) \lesseqqgtr rs$ *for* $x \lesseqqgtr \tilde{x}_s$*, and that* $\lim_{x\downarrow 0}\mu(x) \geq 0$*. Then, the value function of the problem* (5.1) *reads as*

$$V_{a^*}(x,s) = \begin{cases} \dfrac{(s-ka_s^*)\psi'(a_s^*) + k\psi(a_s^*)}{BL'(a_s^*)}\varphi(x) \\ \qquad + \dfrac{(ka_s^* - s)\varphi'(a_s^*) - k\varphi(a_s^*)}{BL'(a_s^*)}\psi(x) & \text{if } x \in (a_s^*, s) \\ s - kx & \text{if } x \in (0, a_s^*], \end{cases}$$

where a_s^* *can be seen either as the limit boundary stated in Theorem 4.1 or, alternatively, as the solution of the ordinary differential equation*

$$a_s' = \frac{\hat{\varphi}'(a_s)\sigma^2(a_s)}{2\hat{\varphi}(a_s)\big(r(ka_s - s) - k\mu(a_s)\big)},$$

subject to the maximality principle. The optimal stopping time is $\tau^* = \inf\{t \geq 0 \mid X_t \leq a_{S_t}^*\}$.

5.1. Geometric Brownian motion example

Assume now that X_t evolves according to a geometric Brownian motion characterized by the stochastic differential equation $dX_t = \mu X_t dt + \sigma X_t dW_t$, where $\mu \in (-\infty, r)$

and $\sigma > 0$. In this case, the decreasing and increasing fundamental solutions read as $\varphi(x) = x^{\gamma_1}$ and $\psi(x) = x^{\gamma_2}$, where

$$\gamma_i = \frac{1}{\sigma^2}\left(\frac{1}{2}\sigma^2 - \mu + (-1)^i\sqrt{(\frac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2 r}\right) \tag{5.2}$$

are the solutions of the characteristic equation $\frac{1}{2}\sigma^2\gamma(\gamma-1) + \mu\gamma - r = 0$, for $i = 1, 2$. Notice that $\gamma_1 < 0$ and, since $\mu < r$, we have $\gamma_2 > 1$. Under this setting, the problem (5.1) can be solved explicitly (see [19, 24]):

Proposition 5.2. *When X_t is a geometric Brownian motion, the value of the perpetual lookback* (5.1) *is*

$$V^*(x,s) = \begin{cases} \frac{x/\beta}{\gamma_2-\gamma_1}\Big\{(\gamma_2 - k\beta(\gamma_2-1))\left(\frac{x}{\beta s}\right)^{\gamma_1-1} \\ \qquad - (\gamma_1 - k\beta(\gamma_1-1))\left(\frac{x}{\beta s}\right)^{\gamma_2-1}\Big\} & \text{if } \beta s < x \le s \\ s - kx & \text{if } 0 < x \le \beta s \end{cases}$$

and the optimal stopping time is given by $\tau^* = \inf\{t \ge 0 \mid X_t \le \beta S_t\}$, where β is the unique solution to the equation

$$\beta^{\gamma_2-\gamma_1} = \frac{(\gamma_2-1)(\gamma_1 - k\beta(\gamma_1-1))}{(\gamma_1-1)(\gamma_2 - k\beta(\gamma_2-1))}$$

The comparison between the exact and an approximate result are summarized in Table 1. We see from it that $\hat{a}$ is decreasing while $\hat{V}$ is increasing in n (as proof of Theorem 4.1 indicates) and that the computing time is linear. Another positive feature is that the algorithm simultaneously produces approximations for the optimal boundary a_s^* for other s's as well along the discretized supremum process.

In Table 2 we see that while the original approximation for a_{10}^* was very good, also other estimates for $a_{15}^*, \ldots, a_{50}^*$ are quite good, every single one being under half percent away from the exact value.

5.2. Mean reverting diffusion

To illustrate our findings in a somewhat more complicated setting, let $dX_t = \mu X_t(\theta - X_t)dt + \sigma X_t dW_t$, where $\mu, \theta, \sigma > 0$ are exogenously given constants. The fundamental solutions are now $\psi(x) = x^{\gamma_2}M(\gamma_2, 1+\gamma_2-\gamma_1, \frac{2\mu\theta}{\sigma^2}x)$ and $\varphi(x) = x^{\gamma_1}U(\gamma_1, 1+\gamma_1-\gamma_2, \frac{2\mu\theta}{\sigma^2}x)$, where $M : \mathbb{R}^+ \to \mathbb{R}_+$ and $U : \mathbb{R}_+ \to \mathbb{R}_+$ denote the confluent hypergeometric

n	$V^* - \hat{V}$	$\hat{a} - a^*$	Time
100	0.89	1.0	0.2sec
1 000	0.27	0.21	3sec
10 000	0.049	0.036	29sec
100 000	0.0020	0.0014	288sec

TABLE 1: The values for the model are $(\sigma, \mu, r, k) = (0.2, 0.05, 0.08, 1)$, $(x, s) = (7, 10)$, and for the approximation we chose $Q(s) = f(s, s) \equiv 0$, $z = 0.1$. The exact values are: $V(7, 10) = 4.03$ and $a^*_{10} = 5.34$.

s	a^*	$\hat{a} - a^*$	$\frac{\hat{a}-a^*}{a^*}$
15	8.0	0.0099	0.12%
20	10.7	0.021	0.19%
25	13.4	0.034	0.25%
30	16.0	0.048	0.30%
40	21.3	0.084	0.39%
50	26.7	0.13	0.47%

TABLE 2: A comparison of the exact values a^*_s with the approximate values.

functions of the first and second kind, respectively (cf. p. 504 in [1]), and γ_i, $i = 1, 2$ are as in (5.2). These functions are very difficult to handle analytically and, therefore, we analyze numerically the solution to (5.1) under the following parameter specifications: $\mu = 0.05$, $\theta = 0.1$, $\sigma = 0.15$, $r = 0.08$, $k = 1$.

Let us apply the algorithm. From table 3 we see that it has only a minor impact to the solution whether we choose the highest possible state for X_t to be $s_n = 75$ or $s_n = 200$. Therefore, the choice $s_n = 75$ is adequate for the estimation when $s \leq 10$. Moreover, since the $f(x, s)$ is now s-Lipschitz-continuous with Lipschitz constant 1, we see from Corollary 4.1 that we can quite surely say that $|J(x, s) - V(x, s)| < z$, for $s \leq 10$, where J is our approximative and V the (unknown) optimal value function. In table 4 we see the effect of changing the grid parameter z. The impact of increased volatility on the optimal boundary and the value are, in turn, illustrated in Figure 2.

Differences $|J(0.1, n_1, 9.9, 10) - J(0.1, n_2, 9.9, 10)|$

s_n	50	75	100	200
50	-	$4.3 \cdot 10^{-5}$	$4.3 \cdot 10^{-5}$	$4.3 \cdot 10^{-5}$
75		-	$6.6 \cdot 10^{-9}$	$6.6 \cdot 10^{-9}$
100			-	$5.2 \cdot 10^{-13}$
Time	50sec	118sec	234sec	598sec

TABLE 3: The grid $z = 0.1$ is fixed, and the differences $|J(0.1, n_1, 9.9, 10) - J(0.1, n_2, 9.9, 10)|$ are calculated, where n_i is such that the the highest state for X_t is s_{n_i}.

z	$J(2,3)$	$\hat{a}_3$	$\hat{a}_7$	$\hat{a}_{10}$	Time
0.1	1.000889	1.97771	4.57640	6.44145	95sec
0.01	1.000233	1.98858	4.58651	6.45046	958sec
0.005	1.000209	1.98917	4.58707	6.45096	1875sec

TABLE 4: The initial point $(x, s) = (2, 3)$ and the highest state $s_n = 75$ is fixed. We compare how the solution change as we change z (in each case n is chosen such that $s_n = 75$).

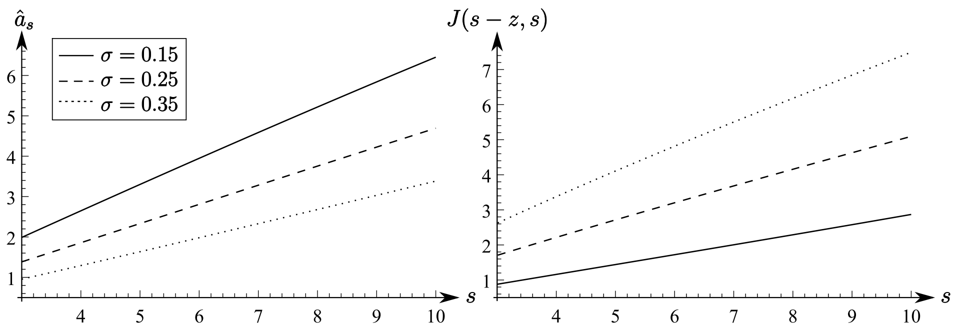

FIGURE 2: The stopping boundaries $\hat{a}_s$ and the values $J(s-z, s)$ are calculated for $s \in (3, 10)$, and $\sigma = 0.15, 0.25, 0.35$. We have chosen $z = 0.01$ and s_n to be 75 (with $\sigma = 0.15$), 100 (with $\sigma = 0.25$) and 150 (with $\sigma = 0.35$).

6. Explicit Illustration: π-option

In order to utilize our findings on the generalized case introduced in Subsections 3.3 and 4.3, we will now consider the valuation and optimal exercise of a π-option introduced in [14]. In that case the exercise payoff reads as $f(x,s) = x^\kappa s^\eta - K$, where $\kappa, \eta, K \geq 0$ are known exogenously given constant. That is, we plan to analyze and solve the stopping problem

$$V(x,s) = \sup_\tau \mathbb{E}_{(x,s)} \left\{ e^{-r\tau} \left(X_\tau^\kappa S_\tau^\eta - K \right) \right\}. \tag{6.1}$$

Let $Q(s) \geq f(s,s)$ be a continuous function and assume that $\lim_{t\to\infty} \mathbb{E}_{(x,s)} \{e^{-rt} Q(S_t)\} = 0$. Consider an auxiliary problem

$$V^Q(x,s) = \sup_\tau \mathbb{E}_x \left\{ e^{-r\tau} \left(X_\tau^\kappa S_\tau^\eta - K \right) \mathbb{1}_{\{\tau < \gamma_s\}} + e^{-r\gamma_s} Q(s) \mathbb{1}_{\{\tau \geq \gamma_s\}} \right\}. \tag{6.2}$$

Applying our tools, we get the following result (cf. Section 4 in [14]).

Proposition 6.1. *Assume that* $\int_0^\infty \mathbb{E}_x \{e^{-rt} f(S_t, S_t)\}\, dt < \infty$ *and that for each* $s > 0$ *there exists* $\tilde{x}_s \in (0, s]$ *such that* $(\mathcal{A} - r) f(x,s) \gtreqless 0$ *for all* $x \lesseqgtr \tilde{x}_s$*. Then, for each* $s > 0$*, the value for* (6.1) *is finite and the optimal stopping time is* $\tau_{\mathfrak{S}_s} = \inf\{t \geq 0 \mid X_t \in \mathfrak{S}_s\}$*, where* $\mathfrak{S}_s$ *is* s*-dependent and is either* $\emptyset$ *or of the form* $[u_s^*, y_s^*]$*, where* $0 < u_s^* \leq y_s^* < s$ *are uniquely determined.*

Proof. Let us apply Proposition 3.1 and Corollary 3.1 and let us show that for each $s > 0$ and $Q(s)$ the stopping region $\mathfrak{S}_s^Q$ of the auxiliary problem (6.2) is of the claimed form. Denote by C_s^Q the continuation region at a fixed state $s > 0$. Clearly $f(0,s) < 0$, for all $s > 0$, which implies that the region near the boundary 0 belongs to the continuation region. It follows from Corollary 4 in [21] (see also Theorem 2 in [6]) that $(0, \min\{u_s^*, s\})$ belongs to a continuation region, where $u_s^* = \operatorname{argmax}_{x\in\mathbb{R}_+}\{\frac{f(x,s)}{\psi(x)}\}$. Moreover, under our assumptions, u_s^* is unique (cf. Lemma 3.6 in [3]). If $u_s^* > s$, then $(0,s) \subset C^Q$ and $\mathfrak{S}_s^Q = \emptyset$. Assume now that s is such that $u_s^* < s$. We know by Dynkin's formula that $u_s^* > \tilde{x}_s$. Now, proceeding as in the proof of Lemma 3.1, we see that there exists a unique $y_s^* \in [u_s^*, s)$ maximizing $v(y,x,s)$ (see (3.3)) for all $x \in (y_s^*, s)$. Moreover, either the derivative $v_y'(y_s^*, x, s) = 0$ and $y_s^* \geq u_s^*$ or $v_y'(y_s^*, x, s) < 0$ and $y_s^* = u_s^*$. In the former case $\mathfrak{S}_s^Q = [u_s^*, y_s^*]$ and in the latter case $\mathfrak{S}_s^Q = \emptyset$. The optimality of the stopping time $\tau_{\mathfrak{S}_s^Q}$ for the auxiliary problem follows after noticing that the resulting value is a r-excessive majorant of the exercise payoff. The alleged results now follow from Proposition 3.1 and Corollary 3.1.

The stopping region $\mathfrak{S}_s$ and its dependence on s can be characterized more closely under more restricting assumptions. However, since our purpose is not to provide an exhaustive treatment of this subject, we will not go deeper into the analysis of π-option.

6.1. Numerical example

By Theorem 4.2 our discretization works for the π-option. In our numerical illustration we have chosen $\kappa = 0.9$, $\eta = 1$, $K = 9$, and $Q(s) = f(s,s)$. Although the numerics indicate that the algorithm converges also for $\eta > 1$, we were not able to prove the convergence in Theorem 4.1 without Hölder continuity.

6.1.1. *Geometric Brownian motion* Let the setting be as in Subsection 5.1. In [14] the valuation of π-option has been solved under the geometric Brownian motion, which gives us a baseline for our numerical approximations.

n	$V^* - \hat{V}$	$\hat{y} - y^*_{13}$	Time
400	7.50	0.6	0.7sec
4 000	2.5	0.18	4.7sec
40 000	0.70	0.050	47sec
400 000	0.07	0.0053	467sec

TABLE 5: The value of π-option for geometric Brownian motion. The values for the model are $(\sigma, \mu, r) = (0.2, 0.03, 0.1)$, $(x, s) = (10, 13)$, and $z = 0.25$. The exact values are: $V(10, 13) = 115.4$ and $y^*_{13} = 7.076$.

The table shows that in about 50 seconds, we were able to attain results that are within 1% error margin. Notice that $u^*_{13} := \text{argmax}\{\frac{f(x,13)}{\psi(x)}\} = 1.29$ is independent of $Q(s)$, z, and n and is always exact.

6.1.2. *Mean reverting diffusion* Let the setting be as in Subsection 5.2. Now, there is no known exact solution. As was the case earlier (Subsection 5.2), it has only a minor impact to the solution whether we choose the highest possible state for X_t to be $s_n = 70$ or $s_n = 200$. Therefore, the choice $s_n = 75$ is adequate for our estimation. The results are summarized in Table 6.

z	$\hat{V}(11,13)$	$\hat{y}_{13}$	$\hat{y}_{15}$	$\hat{y}_{20}$	Time
0.1	105.470	9.9854	11.8577	16.5847	38sec
0.01	105.342	10.0184	11.8919	16.6212	300sec
0.005	105.335	10.0202	11.8938	16.6233	600sec
0.0025	105.332	10.0211	11.8947	16.6243	1200sec

TABLE 6: The value of π-option in the case of mean reverting diffusion. The values for the model are $(\sigma, \gamma, \mu, r) = (0.2, 0.1, 0.03, 0.08)$, $(x, s) = (11, 13)$. The highest state $s_n = 75$ is fixed. We compare how the solution change as we change z (in each case n is chosen such that $s_n = 75$). Now $u^*_{13} = 1.40$.

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