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On Identifying and Locating-Dominating Codes in the Infinite King Grid

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Turku, November 2012 Mikko Pelto

List of original publications

This thesis is based on the following seven journal papers and manuscripts:

I. Mikko Pelto: A definition of uniqueness for optimal identifying and covering codes in infinite lattices, submitted 2012.

II. Mikko Pelto: New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid, Cryptography and Communications, Vol. 2, No. 1, 41–47 (2010).

III. Mikko Pelto: On locating-dominating codes in the infinite king grid, Ars Combinatoria, in press.

IV. Mikko Pelto: On $(r, \leq 2)$-locating-dominating codes in the infinite king grid, Advances in Mathematics of Communications, Vol. 6, No. 1, 27-–38 (2012).

V. Mikko Pelto: Optimal $(r, \leq 3)$-locating-dominating codes in the infinite king grid, submitted 2010.

VI. Mikko Pelto: On locating-dominating codes for locating large numbers of vertices in the infinite king grid, Australasian Journal of Combinatorics, Vol. 50, 127–139 (2011).

VII. Mikko Pelto: Optimal identifying codes in the infinite 3-dimensional king grid, submitted 2011.

Contents

Part I

Summary

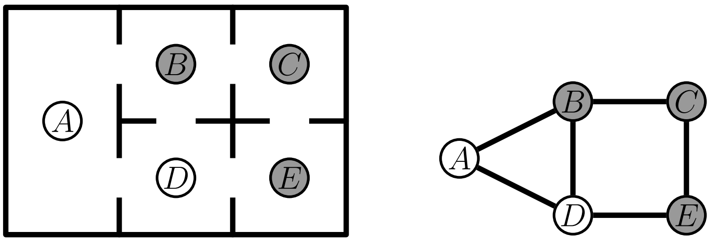
1 Background

Identifying codes, locating-dominating codes and related topics have been studied extensively during the last 15 years. The concept of locating-dominating codes was introduced already in the late of 1980s by Peter J. Slater [30, 31]. Identifying codes were introduced in the late of 1990s by Mark G. Karpovsky, Krishnendu Chakrabarty and Lev B. Levitin [17]. Both of these classes of codes can be applied finding faults in the sensor network.

Slater presented the following motivation for locating-dominating codes: We wish to perform a safeguard analysis of a facility using sensor networks. Assume that there is a facility which contains a lot of rooms. We want to detect motion in a facility and we can put some detectors against thieves, fire, etc. to the rooms of the facility. Assume that one sensor detects motion in the rooms where the sensor itself is or in the nearby rooms. Two rooms can for example be nearby if there is a doorway between the rooms. The sensor sends the symbol 2, if it detects motion in the room where the sensor is situated; symbol 1, if there is motion in some adjacent room, but not in the room where the sensor is; and symbol 0, otherwise. Now, we want to know, whether there is motion in some room. Moreover if there is, we are interested in knowing in which room there is motion — assuming that there is motion in at most one room — based on the reports which sensors sent symbol 2, which symbol 1, and which symbol 0.

In the case of identifying codes, we assume that a sensor sends symbol 1 instead of 2 also in the case when it detects motion in the room where it is. Moreover, cases when there can be motion in more than one room at the same time are studied, when we discuss $(r, \leq l)$-identifying codes and $(r, \leq l)$-locating-dominating codes.

Karpovsky, Chakrabarty and Levitin presented another motivation to such codes. They study fault diagnosis in multiprocessor systems. Here, we have a multiprocessor system. Some processors are chosen to perform the task of testing if the processor itself is faulty or if there is a faulty processor within distance r. Again, each of these processors sends a symbol 2, 1 or 0 depending on whether the processor itself, or some nearby processor is faulty, or the processor itself and all the adjacent processors are functioning properly.



(a) A floormap (b) The floormap as a graph

Figure 1: The identifying sets, i.e., the sets of sensors which detect motion if there is motion in a given room are distinct and non-empty: $I_1(A) = \{B\}$, $I_1(B) = \{B, C\}$, $I_1(C) = \{B, C, E\}$, $I_1(D) = \{B, E\}$ and $I_1(E) = \{C, E\}$.

In graph theoretic terms, we call rooms or processors to vertices and two neighbouring rooms or processors are adjacent vertices, i.e., they are connected by an edge. A chosen processor or room which contains a detector is called a codeword. In the future, we do not discuss a safeguard analysis or a fault diagnosis since we can simply discuss problems using mathematic terms and the answers can be applied to any corresponding practical problem.

An essential question about identifying codes is how many detectors we need. Indeed, every detector costs money. Therefore, the more detectors, the more expensive our safeguard analysis. Another interesting question is how the minimum number of detectors have to be placed so that they form an identifying or a locating-dominating code. One more interesting question is in how many different ways these detectors can be placed. All these question are studied in this thesis.

Example 1. Assume that we have a facility which contains five rooms A–E and whose floor map is as in Figure 1(a). Assume first that there is a sensor in each of the three rooms B, C and E. Suppose also that the sensors can only either observe motion or not, i.e., we have the case of identifying codes.

Now, if there is motion in room A, then only the sensor in room B sends an alarm. Otherwise, if there is motion in some room B–E, then the sensor in room C or E always alarms. Therefore, we know that there is motion in room A if the sensor in room B sends an alarm, but the other two sensors do not send an alarm. In the same way, we see that in all cases the combination of alarming sensors is unique. In particular, if there is motion in any room, then at least one sensor always give an alarm. Therefore, we can know whether there is motion in a facility on the whole. And moreover, we can identify the room where is motion, if there is motion at most in one room.

Next, we define the same problem using mathematical terms.

2 Preliminaries

We consider an undirected connected *graph* $G = (V, E)$ where V is the vertex set and E the edge set, i.e., a set of non-ordered pairs. A *path* from a vertex u to another vertex v is a sequence of different vertices such that u is the first and v the last element of this sequence and the edge $\{w, w'\}$ belongs to the edge set E for every two consecutive vertices w and w'. The *(graphical) distance* between two vertices u and v is denoted by $d(u, v)$ and it means the number of edges on any shortest path from u to v. Furthermore, the ball with center v and radius r is the set

$$B_r(v) = \{u \in V : d(u, v) \leq r\}.$$

This set is also called the *r-neighbourhood* of v. Define also the symmetric difference of two sets A and B as

$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$

Now, any set $C \subseteq V$ is called a *code* and any vertex of C is called a *codeword*. The *identifying set* of v with respect to C is defined by

$$I_r(v, C) = I_r(v) = C \cap B_r(v) = \{c \in C : d(c, v) \leq r\}.$$

Here, the distance $d(c, v)$ denotes the distance between c and v in $G = (V, E)$ (not in the subgraph induced by C). The set

$$I_r(F, C) = I_r(F) = C \cap B_r(F) = \bigcup_{v \in F} C \cap B_r(v)$$

is called the identifying set of a vertex set F.

A code is called an *r-covering code* if all identifying sets $I_r(v)$ are non-empty. Furthermore, an r-covering code is called an *r-identifying code* if all identifying sets are also distinct, i.e., $I_r(u) \neq I_r(v)$ or equivalently $I_r(u)\Delta I_r(v) \neq \emptyset$ for every two different vertices $u \in V$ and $v \in V$. On the other hand, an r-covering codes is called an r-locating-dominating code if the identifying sets are distinct for all non-codewords, i.e., $I_r(u) \neq I_r(v)$ for every two different vertices $u \in V \setminus C$ and $v \in V \setminus C$.

Example 2. The floormap of Example 1 is shown as a graph in Figure 1(b). We can now see that the set $\{B, C, E\}$ is a 1-identifying code. Indeed, the identifying sets, which are given in the caption, are non-empty and distinct.

On the other hand, $\{A, E\}$ is a 1-locating-dominating code. Indeed, then $I_r(B) = \{A\}$, $I_r(C) = \{E\}$ and $I_r(D) = \{A, E\}$, and for locating-dominating codes, it is enough that identifying sets are distinct for non-codewords. Notice that this is not a 1-identifying code.

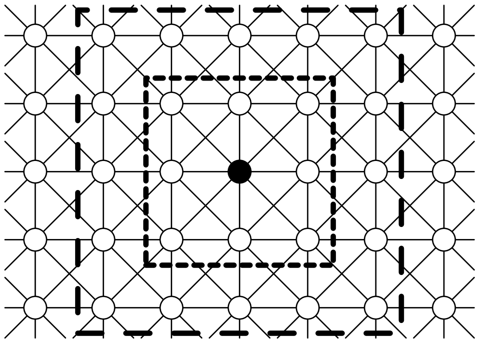


Figure 2: A part of the infinite king grid. The vertices within distance one and two from the black dot are surrounded by the dashed lines.

Furthermore, we define $(r, \leq l)$-identifying codes. A code is called an $(r, \leq l)$*-identifying code* if the identifying sets $I_r(F)$ are distinct for all such sets $F \subseteq V$ with cardinality at most l. Similarly, a code is called an $(r, \leq l)$-*locating-dominating code of type B*, i.e., an *r-LDB code* if the identifying sets $I_r(F)$ are distinct for all such sets $F \subseteq V \setminus C$ of non-codewords with cardinality at most l. Moreover, a code C is called an $(r, \leq l)$*-locating-dominating code of type A*, i.e., an *r-LDA code* if identifying sets $I_r(F_1)$ and $I_r(F_2)$ are distinct, i.e., $I_r(F_1) \Delta I_r(F_2) \neq \emptyset$ for all sets $F_1 \subseteq V$ and $F_2 \subseteq V$ where $F_1 \cap C = F_2 \cap C$, $|F_1| \leq l$, $|F_2| \leq l$ and $F_1 \neq F_2$.

In particular, the definition of $(r, \leq 1)$-identifying codes is equivalent to the definition of r-identifying codes. Also, the definitions of $(r, \leq 1)$-LDA codes, $(r, \leq 1)$-LDB codes and r-locating-dominating codes are equivalent. Moreover, an $(r, \leq l)$-identifying code is automatically an $(r, \leq l)$-LDA code and an $(r, \leq l)$-LDA code is always an $(r, \leq l)$-LDB code. Furthermore, an $(r, \leq l_1)$-identifying code (and also $(r, \leq l_1)$-LDA code and $(r, \leq l_1)$-LDB code) is automatically an $(r, \leq l_2)$-identifying code (and $(r, \leq l_2)$-LDA code and $(r, \leq l_2)$-LDB code, respectively) if $l_2 \leq l_1$.

In this thesis, we study identifying codes in the so-called *infinite king grid*. This grid is defined by the vertex set $V = Z^2$ and the edge set

$$E = \{\{u = (u_x, u_y), v = (v_x, v_y)\} : |u_x - v_x| \leq 1, |u_y - v_y| \leq 1, u \neq v\}.$$

A part of this grid is given in Figure 2. Thus, two different vertices are neighbours if the Euclidean distance between them is at most $\sqrt{2}$. In other words, the graph can also be defined with the help of the king in chess. Indeed, the distance between two vertices is the minimum number of moves a king needs in the infinite empty chessboard to move from one vertex to the other.

In this thesis, we also study identifying codes in the *n-dimensional infinite king grid*. The vertex set of this graph is $V = \mathbb{Z}^n$ and the edge set is

$$E = \{\{u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n)\} : |u_i - v_i| \leq 1 \text{ for all } i,\ u \neq v\}.$$

Thus, the 2-dimensional infinite king grid is the same as the infinite king grid.

In the n-dimensional infinite king grid, we define the *density* of the code C as

$$D(C) = \limsup_{k \to \infty} \frac{|C \cap Q_k|}{|Q_k|},$$

where $Q_k = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : |x_i| \leq k \text{ for all } i\}$, i.e., $Q_k = B_k((0, \ldots, 0))$. The main question of identifying codes is what is the minimum density of codewords for any identifying codes in given graph. In particular, an r-identifying code is called an *optimal* r-identifying if the density of the code is minimum. Similarly, we define optimal r-locating-dominating codes, optimal covering codes, etc.

Example 3. Consider again Figure 1. We see that the identifying code $\{B, C, E\}$ is optimal. First, it is easy to see that

$$I_r(v) \Delta I_r(u) = (B_r(v) \Delta B_r(u)) \cap C$$

for all graphs. Therefore, $B_1(A) \Delta B_1(B) = \{C\}$ has to contain a codeword. Thus, C is a codeword in every 1-identifying code. Similarly, because $B_1(A) \Delta B_1(D) = \{E\}$, then E is always a codeword. Finally, $I_r(A)$ must be non-empty. Therefore, A, B or D has to be a codeword. Thus, every 1-identifying code contains at least three codewords.

3 The structure of this thesis

This thesis consists of seven journal papers and manuscripts. The papers are concerned with identifying, locating-dominating, and covering codes in the infinite king grid.

Identifying codes, locating-dominating codes and related topics have been studied in many finite and infinite graphs. Hamming spaces, cycles and paths are examples of finite graphs where such codes have been studied. In addition to the infinite king grid, the most studied infinite graphs are the square and triangular lattices and the hexagonal mesh. We know many lower and upper bounds for r-identifying, r-locating-dominating codes, and so on. Over 200 papers on such codes and related topics have been published. Such papers are listed in the web bibliography [35]. However, it has not been previously studied how many different codes exist in infinite lattices.

In the first paper, we discuss how we can count the number of optimal identifying and covering codes in infinite lattices in a sensible way. Obviously, two codes are different if there is an element which belongs to one, but not the other. However, this definition leads to an uninteresting answer since in most of the interesting cases, there are infinitely many different optimal codes in the infinite king grid. Therefore, we choose another way to count the number of optimal codes.

We give a definition for *completely different codes* and study basic properties of this definition. In particular, we show that optimal r-identifying codes are unique according to our definition in the infinite king grid when $r \leq 3$. Moreover we see that there are many completely different optimal r-covering codes and r-identifying codes, when $r \geq 4$. In particular, we shall observe that there are infinitely many completely different r-covering codes, but the number of completely different optimal r-identifying codes is finite. Although we study completely different codes only for covering and identifying codes, this definition can also be applied for many other type of codes such as locating-dominating codes.

In the infinite king grid, the optimal bounds for the density of $(r, \leq l)$-identifying codes have been known for about ten years except for $(1, \leq 2)$- and $(2, \leq 2)$-identifying codes. Such codes are studied for example in [7–9, 12]. In the second paper, we give the best known constructions for $(2, \leq 2)$-identifying codes. The density of the code is $\frac{2}{7}$. Moreover, we prove the lower bound $\frac{5}{12}$ for $(1, \leq 2)$-identifying codes. However, a more recent and better lower bound for $(1, \leq 2)$-identifying codes has been proved in [11]. The summary of the known lower and upper bounds for $(r, \leq l)$-identifying codes is given in Table 1. Here, the upper bound means that there exists an identifying code with that density and the lower bound means that density of every identifying code is at least the value given in the table.

Locating-dominating codes have been studied clearly less than identifying codes in the infinite king grid. The optimal density for $(1, \leq 1)$-locating-dominating codes is known by [15], but the other cases have not been studied in the infinite king grid. However, in many other graphs such codes have been studied. For example, such codes were introduced and studied in Hamming spaces in [16].

In the next four papers III–VI, we discuss $(r, \leq l)$-locating-dominating codes of types A and B in the infinite king grid. The first of these papers concerns $(r, \leq 1)$-locating-dominating codes, i.e., r-locating-dominating codes. We show that for all odd r, we can construct an r-locating-dominating code the density of which is smaller than the density of the optimal r-identifying code. This is an interesting property since the definitions of these codes are quite similar, in particular for the cases when r is large. In the other three papers, we study $(r, \leq l)$-locating-dominating codes, when $l = 2$, $l = 3$ and $l \geq 4$, respectively. The known bounds for the density of optimal LDA codes and LDB codes are given in Tables 2 and 3, respectively.

In the final paper, we study r-identifying codes in the king grid with more than two dimensions. Previously, identifying codes have not been studied in the n-dimensional king grid. It is nevertheless an interesting graph. Identifying codes have already been studied in the n-dimensional square gird in [32] and hypercubes in [17], for example. We give general lower and upper bounds for r-identifying codes in the n-dimensional king grid. These bounds are very near to each other if r is much greater than n. In particular, we construct an r-identifying code with density $\frac{1}{8r^2}$ in the 3-dimensional king grid and prove that this code is optimal for all $r \geq 15$.

	$r = 1$		$r = 2$		$r \geq 3$	
$l = 1$	$\frac{2}{9}$	[8, 9]	$\frac{1}{8}$	[7, 8]	$\frac{1}{4r}$	[7]
$l = 2$	$\frac{47}{111} \leq D \leq \frac{3}{7}$	[11, 12]	$\frac{31}{120} \leq D \leq \frac{2}{7}$	[II, 12]	$\frac{1}{4}$	[12]
$l \geq 3$	do not exist	[12]	do not exist	[12]	do not exist	[12]

Table 1: The known lower and upper bounds for the density of $(r, \leq l)$-identifying codes in the king grid.

	$r = 1$		$r = 2$		$r \geq 3$		
$l = 1$	$\frac{1}{5}$	[15]	$\frac{1}{10} \leq D \leq \frac{1}{8}$	[III, 8]	$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ $\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$	if $2 \mid r$ if $2 \nmid r$	[III, 7]
$l = 2$	$\frac{1}{3}$	[IV]	$\frac{1}{5} \leq D \leq \frac{1}{4}$	[IV]	$\frac{1}{6} \leq D \leq \frac{r+1}{6r+3}$ $\frac{1}{6} \leq D \leq \frac{2r+3}{12r+6}$	if $r \equiv 0, 2, 5 \pmod 6$ if $r \equiv 1, 3, 4 \pmod 6$	[IV]
$l \geq 3$	1	[VI]	1	[VI]	1		[VI]

Table 2: The known lower and upper bounds for the density of $(r, \leq l)$-LDA codes in the king grid.

	$r=1$	$r=2$	$r \geq 3$
$l=1$	$\frac{1}{5}$ [15]	$\frac{1}{10} \leq D \leq \frac{1}{8}$ [III, 8]	$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ if $2 \mid r$ $\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$ if $2 \nmid r$ [III, 7]
$l=2$	$\frac{1}{3}$ [IV]	$\frac{1}{6} \leq D \leq \frac{1}{4}$ [IV]	$\frac{1}{6} \leq D \leq \frac{r+1}{6r+3}$ if $r \equiv 0,2,5 \ (6)$ $\frac{1}{6} \leq D \leq \frac{2r+3}{12r+6}$ if $r \equiv 1,3,4 \ (6)$ [IV]
$l=3$	$\frac{3}{5}$ [V]	$\frac{2}{3}$ [V]	$\frac{r}{r+1}$ [V]
$4 \leq l \leq 4r$	$\frac{2}{3}$ [VI]	$\frac{2}{3} \leq D \leq \frac{4}{5}$ [V,VI]	$\frac{r}{r+1} \leq D \leq \frac{2r}{2r+1}$ [V,VI]
$l > 4r$	$\frac{2}{3}$ [VI]	$\frac{4}{5}$ [VI]	$\frac{2r}{2r+1}$ [VI]

Table 3: The known lower and upper bounds for the density of $(r, \leq l)$-LDB codes in the king grid.

	$n=3$		$n \geq 4$	
r	lower bound	upper bound	lower bound	upper bound
1	$\frac{1}{12}$	$\frac{7}{72}$	$\frac{3}{4} \cdot \frac{1}{3^{n-1}}$	$\frac{2^n-1}{2^n} \cdot \frac{1}{3^{n-1}}$
2	$\frac{1}{40}$	$\frac{1}{32}$	$\frac{5}{8} \cdot \frac{1}{5^{n-1}}$	$\frac{2^n-1}{2^n} \cdot \frac{1}{5^{n-1}}$
3	$\frac{1}{84}$	$\frac{1}{72}$	$\frac{7}{12} \cdot \frac{1}{7^{n-1}}$	$\begin{cases} \frac{3}{6^n} & \text{if } n \leq 5 \\ \frac{2^n-1}{2^n} \cdot \frac{1}{7^{n-1}} & \text{if } n \geq 6 \end{cases}$
$4, \dots, 14$	$\frac{1}{8r^2+\frac{32}{3}}$	$\frac{1}{8r^2}$	$\frac{1}{(2r+1)^{n-3}(8r^2+\frac{32}{3})}$	$\min\{\frac{r}{(2r)^n}, \frac{2^n-1}{2\cdot(4r+2)^{n-1}}\}$
≥ 15	$\frac{1}{8r^2}$	$\frac{1}{8r^2}$	$\frac{1}{8r^2\cdot(2r+1)^{n-3}}$	$\min\{\frac{r}{(2r)^n}, \frac{2^n-1}{2\cdot(4r+2)^{n-1}}\}$

Table 4: The known bounds for identifying codes in the infinite n-dimensional king grid. All the values are in the paper [VII].

Bibliography

[1] Auger, D., Charon, I., Hudry, O., Lobstein A.: Watching systems in graphs: An extension of identifying codes, Discrete Applied Mathematics, in press.

[2] Auger, D., Honkala, I.: Watching systems in the king grid, Graphs and Combinatorics, in press.

[3] Ben-Haim, Y., Gravier, S., Lobstein, A., Moncel, J.: General bounds for identifying codes in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 8(1), R39 (2001).

[4] Ben-Haim, Y., Litsyn, S.: Exact minimum density of codes identifying vertices in the square grid, SIAM Journal on Discrete Mathematics, Vol. 19, 69–82 (2005).

[5] Bertrand, N., Charon, I., Hudry, O., Lobstein A.: Identifying and locating-dominating codes on chains and cycles, European Journal of Combinatorics, Vol. 25, 969–987 (2004).

[6] Charon, I., Honkala, I., Hudry, O., Lobstein, A.: General bounds for identifying codes in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 8(1), R39 (2001).

[7] Charon, I., Honkala, I., Hudry, O., Lobstein A.: The minimum density of an identifying code in the king lattice, Discrete Mathematics, Vol. 276, 95–109 (2004).

[8] Charon, I., Hudry, O. and Lobstein, A.: Identifying codes with small radius in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 9(1), R11 (2002).

[9] Cohen, G., Honkala, I., Lobstein, A., Zémor, G.: On codes identifying vertices in the two-dimensional square lattice with diagonals, IEEE Transactions on Computers, Vol. 50, 174–176 (2001).

[10] Day, R., Street, A. P.: Sequential Binary Arrays. I. The Square Grid. Journal of Combinatorial Theory, Ser. A, Vol. 32(1), 35–52 (1982).

[11] Foucaud, F., Laihonen, T., Parreau, A.: An improved lower bound for $(1, \leq 2)$-identifying codes in the king grid, arXiv:1111.2477 (2011).

[12] Honkala, I., Laihonen, T.: Codes for identification in the king lattice, Graphs and Combinatorics, Vol. 19, 505–516 (2003).

[13] Honkala, I., Laihonen, T.: On a new class of identifying codes in graphs, Information Processing Letters, Vol. 102, 92–98 (2007).

[14] Honkala, I., Laihonen, T.: On identifying codes in the triangular and square grids, SIAM Journal on Computing, Vol. 33, 304–312 (2004).

[15] Honkala, I., Laihonen, T.: On locating-dominating sets in infinite grids, European Journal of Combinatorics, Vol. 27, 218–227 (2006).

[16] Honkala, I., Laihonen, T., Ranto, S: On locating-dominating codes in binary Hamming spaces, Discrete Mathematics and Theoretical Computer Science, Vol. 6, 265–282, (2004).

[17] Karpovsky, M. G., Chakrabarty K., Levitin, L. B.: On a new class of codes for identifying vertices in graphs, IEEE Transactions on Information Theory, Vol. IT-44, 599–611 (1998).

[18] Martin, R., Stanton, B.: Lower bounds for identifying codes in some infinite grids, Electronic Journal of Combinatorics, Vol. 17(1), R122 (2010).

[19] Obata, K.: On identifying codes in lattices, Preprint, December 2003. http://www.kenjioba.net/images/Identifying.pdf (20.2.2012)

[20] Pelto, M.: A definition of uniqueness for optimal identifying and covering codes in infinite lattices, submitted.

[21] Pelto, M.: New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid, Cryptography and Communications, Vol. 2, 41–47 (2010).

[22] Pelto, M.: On locating-dominating codes for locating large numbers of vertices in the infinite king grid, Australasian Journal of Combinatorics, Vol. 50, 127–139 (2011).

[23] Pelto, M.: On locating-dominating codes in the infinite king grid, Ars Combinatoria, in press.

[24] Pelto, M.: On $(r, \leq 2)$-locating-dominating codes in the infinite king grid, Advances in Mathematics of Communications, Vol. 6, 27-–38 (2012).

[25] Pelto, M.: Optimal identifying codes in the infinite 3-dimensional king grid, submitted.

[26] Pelto, M.: Optimal $(r, \leq 3)$-locating-dominating codes in the infinite king grid, submitted.

[27] Ranto, S.: Identifying and locating-dominating codes in binary Hamming spaces, Ph. D. Thesis, University of Turku, Finland, 95 pages, (2007).

[28] Salomaa, A.: Jewels of Formal Language Theory. Computer Science Press, Rockville, MD, 1981, p. 6.

[29] Seo, S. J., Slater, P. J.: Open neighborhood locating dominating sets, Australasian Journal of Combinatorics, Vol. 46, 109–119 (2010).

[30] Slater, P. J.: Domination and location in acyclic graphs, Networks, Vol. 17, 55–64 (1987).

[31] Slater, P. J.: Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences, Vol. 22, 445–455 (1988).

[32] Stanton, B.: Vertex identifying codes for the n-dimensional lattice, Australasian Journal of Combinatorics, Vol. 53, 299– (2012).

[33] Street, A. P., Day, R.: Sequential binary arrays. II. Further results on the square grid. Combinatorial mathematics, IX (Brisbane, 1981), 392—418, Lecture Notes in Math., 952, Springer, Berlin-New York, 1982.

[34] http://www.infres.enst.fr/˜lobstein/biblio.html

[35] http://www.infres.enst.fr/˜lobstein/bibLOCDOMetID.html

Part II

Original publications

Publication I

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A definition of uniqueness for optimal identifying and covering codes in infinite lattices

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Abstract

A subset of vertices in a graph G is called an r-covering code if the sets $I_r(v) = \{c \in C \mid d(c,v) \leq r\}$ are non-empty for all vertices v. Furthermore, an r-covering code is called an r-identifying code if the sets $I_r(v)$ are also distinct for all vertices. In this paper we study codes in the infinite lattice, where the vertex set is $V = \mathbb{Z}^2$. In this lattice, an interesting question is what is the smallest density of the r-covering or r-identifying code. The density of the code C is defined by

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|},$$

where $Q_n = \{(x,y) : -n \leq x \leq n, -n \leq y \leq n\}$. We also study the question how many completely different codes the density of which is minimum exist. In this paper we give the following definition: Two codes C_a and C_b are called *completely different* if there exists $n \in \mathbb{N}_+$ such that $\alpha(C_a) \cap Q_n \neq \beta(C_b) \cap Q_n$ for all isometries α and β. In particular, we study the number of r-identifying and r-covering codes in the so-called infinite king grid.

Key words

identifying code, covering code, lattice, density, discrete geometry

1 Introduction

Identifying and covering codes are defined in a given graph $G = (V,E)$. For all vertices $v \in V$, the set $B_r(v) = \{u \in V \mid d(u,v) \leq r\}$ is called the *r-neighbourhood* of v. Moreover, vertices at distance one from v is called neighbours of v.

Any set C of vertices V is called a *code* and a vertex in the code is called a codeword. A code C is called an *r-covering code* if the *identifying sets*

$$I_r(v) = C \cap B_r(v)$$

are non-empty for all vertices $v \in V$. Furthermore, an r-covering code C is called an *r-identifying code* if the sets $I_r(v)$ are also distinct for all $v \in V$.

Covering codes have been studied for a long time and identifying codes were introduced in [10] in the 1990s. However, locating-dominating codes, which are a class of codes which are very closely related to identifying codes, were already introduced in the 1980s by Slater [16]. A motivation for such codes is a safeguard analysis of a facility using sensor networks [16]. Assume that we want to detect motion in a facility and we can put some detectors against thieves, fire, etc. to the rooms of the facility. Assume also that a detector gives an alarm if it detects motion in the room where the detector itself is or in the nearby rooms. Now, we want to place the detectors in such a way that some detector always alarms if there is motion in the facility. Moreover in the case of identifying codes, we also want to know which room the motion is in based on only the knowledge which detectors observed the motion. Now, if we place the detectors such a way that they form an r-covering or r-identifying code, then this is possible.

Covering and identifying codes have been studied in many different types of graphs, for example in Hamming spaces, paths, cycles and infinite lattices. A lot of papers on such codes and closely related topics can be found in the web pages [11, 12].

Consider first a finite graph. An interesting question about covering and identifying codes is how many vertices, at least, such codes must contain. Codes with the minimum numbers of codewords are called *optimal codes*. Another interesting question is how many different optimal codes exist.

Next, we consider infinite graphs. Now, the corresponding questions have to be formulated differently since an r-identifying code, for example, contains infinite number of codewords if the graph even has any r-identifying code. Therefore, we consider the density of a code instead of the number of codewords.

For lattices where the vertex set V is $\mathbb{Z}^2$, we define that the *density* of $C \subseteq V$ is

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|},$$

where $Q_n = \{(x, y) : -n \leq x \leq n, -n \leq y \leq n\}$. Moreover, a code is called optimal if the density is minimal (for the problem considered).

However, the second question, how many different codes exist, has not previously been studied in infinite lattices. In this paper, we discuss which kind of codes are different enough. We shall give a definition for completely different codes and study how many completely different r-identifying and r-covering codes there are in the infinite *king grid* where the vertex set is $V = \mathbb{Z}^2$ and the edge set

$$E = \{\{u = (u_x, u_y), v = (v_x, v_y)\} : |u_x - v_x| \leq 1,\ |u_y - v_y| \leq 1,\ u \neq v\}.$$

Such codes have been studied in [4–6] and we shall utilize some ideas of these proofs here. In other infinite lattices, identifying codes have for example been studied in [2, 3, 5, 7]. Some related topics — like locating-dominating codes, $(r, \leq l)$-identifying codes and watching systems — have been studied in the infinite king grid in [1, 9, 13, 14] for example.

2 A definition for completely different codes

Now, we restrict our attention to the graphs whose vertex set is $\mathbb{Z}^2$. However, the same idea can be used in many other infinite graphs. Define first an isometry in $\mathbb{Z}^2$.

Definition 1. A mapping $\alpha : \mathbb{Z}^2 \to \mathbb{Z}^2$ is called an isometry in $\mathbb{Z}^2$ if it maps vertices to vertices bijectively and preserves the edges and the Euclidean distances between all vertices. The isometries consist of translations, reflections, rotations and glide reflections.

First, we note that for two codes C and C' we have $C \neq C'$, if and only if the *symmetric difference* between C and C' is non-empty, i.e.,

$$C \Delta C' = (C \setminus C') \cup (C' \setminus C) \neq \emptyset.$$

However, two codes C and C' may be different although they are essentially very similar.

Example 2. We study 1-identifying codes in the infinite king grid. In [5, 6], one has shown that C_{opt} is an optimal r-identifying code. The code is defined in Theorem 15 in page 8 and a part of the code is in Figure 1(g). We notice that all vertices $(x, x) \in \mathbb{Z}^2$ are non-codewords in C_{opt}. However, we can define $C_{opt}^x = C_{opt} \cup \{(x, x)\}$. We observe that C_{opt}^x is also an optimal 1-identifying codes for all integers x since $C_{opt} \subset C_{opt}^x$ and

$$D(C_{opt}^x) = D(C_{opt}) + D(C_{opt}^x \setminus C_{opt}) = D(C_{opt}) + D(\{(x, x)\}) = D(C_{opt}).$$

Therefore, we see that the number of r-identifying codes is infinite.

Similarly, we can find infinite many optimal codes in all cases when the density of optimal codes is less than 1. Thus, we need a stronger definition, when we count the number of r-identifying or r-covering codes in a sensible way.

An idea to count the number of optimal codes can, for instance, be that two codes C and C' are different enough if these density of the symmetric difference of the codes is $D(C \Delta C') \neq 0$. Alternatively, the definition may be that two optimal codes are not different enough if one is the subset of the other code. However, both of these definitions would again be lead to the same problem — the optimal codes would be infinite number.

Instead, we do the following.

Definition 3. Two codes C_a and C_b are called *completely different* if there exists a positive integer n such that

$$\alpha(C_a) \cap Q_n \neq \beta(C_b) \cap Q_n$$

for all isometries α and β.

Moreover, if the code C is an optimal identifying (or covering) code and no optimal identifying (or covering, resp.) code is completely different from C, then C is called the *unique* optimal identifying (or covering, resp.) code.

This definition gives an interesting way to count the number of optimal codes. Indeed, for any optimal code in the given graph, we can often find a property with the help of which we can define how many codes exist in the given graph according to our definition. Such property can for example be that the density of certain vertices is zero. Then, we know that there is an arbitrary large square which does not contain such vertices. Indeed, if all squares with size $n \times n$ contain such a vertex, then the density of such vertices is at least $\frac{1}{n^2} > 0$.

3 Basic properties

Now, we assume our proof that two vertices can be neighbours only if the Euclidean distance between these vertices is at most $\sqrt{2}$. The following propositions also hold if the maximum Euclidean distance between two vertices is bounded above.

Proposition 4. Codes does not partition into equivalence classes based on the definition for completely different codes.

Proof. We give a counter example. Assume that codes C_1 and C_2 are completely different identifying codes. Then we can nevertheless define a code, say C_3, such that $C_3 = C_1$ for $x > r$ and $C_3 = C_2$ for $x < -r$ and $C_3 = V$ for $-r \leq x \leq r$. Now, C_3 is completely different neither with C_1 nor with C_2, but C_1 and C_2 are completely different. Therefore the relation is not transitive. □

Similarly, given a finite or countable infinite number of pairwise completely different optimal identifying codes, we can always define an optimal identifying code which is not completely different from any of them. The code can be defined, for example, by dividing the vertices into sectors about the origin.

Proposition 5. No r-identifying code is completely different from all optimal r-identifying codes, if there is an optimal code.

Proof. Let C be an optimal identifying code and C' be any identifying code in the infinite king grid. Then, the code

$$C'' = \{(a, b) \in C : a \leq (|b| + 4r)^2\} \cup \{(a, b) \in C' : a \geq b^2\}$$

is optimal but not completely different from C'. Certainly, it is nor completely different from C. Here, the density of vertices in $\{(a, b) : a < b^2\}$ is 0, then these vertices do not imply the density of the code and the other vertices alone denote the density. Therefore, the density of these code is optimal. However, the area in $\{(a, b) : a > (|b| + 4r)^2\}$ contains an arbitrary large square which is identical with C'. Moreover, the code contains so-called border zone, i.e., the zone where the vertex is a codeword if it is a codeword in C or C'. Now, the border zone must be enough large, i.e., if $B_r(u)$ and $B_r(v)$ intersect for two vertices u and v, then $B_r(u)$ and $B_r(v)$ are not on different side on the border zone (even partly). Therefore, two arbitrary vertices u and v are r-separated since C and C' are r-identifying codes. □

Proposition 6. Assume that there is an r-identifying code with density $D < 1$ and $D < D' \leq 1$. Then there are infinite many completely different r-identifying codes with density D'.

Proof. We prove the claim only to rational numbers $D' = \frac{t}{s} < 1$. The proof for the other cases are quite similar.

First, we choose a number l s.t. l is divisible by s and $l > \frac{4r+4}{D'-D} \geq \frac{4r+4}{D'-D}(1-D)$. Now, we choose a $(l - 2r - 2) \times (l - 2r - 2)$-sized square S the density of which is at most D in the case of the code C. This is possible since if the density of all areas with this size were more than D, then the density of C would also be more than D.

Now, we form an $l \times l$-sized tile where all vertices are codewords in the border the thickness of which is $r+1$ vertices and inside of the border form S. Then, the number of codewords in the tile is at most

$$\begin{aligned} d^* &= (l-2(r+1))^2 D + 4l(r+1) - 4(r+1)^2 \\ &= l^2 D - 4l(r+1)D + 4(r+1)^2 D + 4l(r+1) - 4(r+1)^2 \\ &= l^2 \left(D + \frac{4r+4}{l}(1-D) - \frac{4(r+1)^2}{l^2}(1-D) \right) \\ &< l^2 \left(D + (D'-D) - 0 \right) = l^2 D' \end{aligned}$$

Now, we can make a tiling of these $l \times l$-sized tiles and the density of which is $\frac{d*}{l^2} < \frac{l^2 D'}{l^2} = D'$. Then, we can change some tiles to $l \times l$-sized tiles whose all vertices are codewords s.t. the density of the whole code is D'. This can be make, for example, to choose t consecutive row in every $\frac{l^2 - d^*}{l^2 D' - d^*} t$ rows. For every $t \in \mathbb{N}$, we get a different periodic code with density D'. We note late that the claim follows from this. See Definition 7 and Proposition 9. □

By Propositions 5 and 6, the question, how many completely different codes exist, is sensible only if we restrict to optimal codes.

Definition 7. A code C is called *periodic* if there is a constant $t \in \mathbb{Z} \setminus \{0\}$ such that $(a \pm t, b)$ and $(a, b \pm t)$ are codewords if (a, b) is a codeword.

Remark 8. We could equivalently define that a code is periodic if there are two non-parallel integer vectors $\mathbf{s} = (s_x, s_y)$ and $\mathbf{r} = (r_x, r_y)$ s.t. a vertex v is a codeword if and only if $v + \mathbf{s}$ is a codeword if and only if $v + \mathbf{r}$ is a codeword. Definition 7 follows when we choose vectors $r_x \mathbf{s} - s_x \mathbf{r} = (0, r_x s_y - r_y s_x)$ and $s_y \mathbf{r} - r_y \mathbf{s} = (r_x s_y - r_y s_x, 0)$. Here, $r_x s_y - r_y s_x \neq 0$ since $\mathbf{s}$ and $\mathbf{r}$ are non-parallel.

Proposition 9. Two different periodic codes are automatically completely different.

Periodic codes are interesting since in almost all graphs where the optimal bound for the density is known, we can show that no optimal identifying code is completely different from all optimal periodic codes. We might even restrict only to the periodic codes our study. However, it is an open problem whether an optimal r-identifying code always exists in the well-studied lattices – square, triangular and hexagonal lattices. In fact, Obata [15] has constructed a lattice where there is an optimal non-periodic identifying (or covering) code, but no optimal periodic codes. Nevertheless, the 1-neighbourhoods of vertices in this graph are not even symmetric for all vertices. Instead, for 1-covering codes, there is also infinite number of optimal covering codes in the king or hexagonal grid and a part of these are completely different for all optimal periodic codes.

Therefore, it is better to use Definition 3 rather than try only to find all the optimal periodic codes.

Next, we study r-identifying and r-covering codes in the infinite king grid. In this grid, the optimal density of r-identifying and r-covering codes is known for all positive integers r. We shall see that there is an unique optimal r-identifying code when r is 1, 2 or 3. Furthermore, we shall see that the number of completely different optimal r-identifying codes is more than one, but finite, for $r \geq 4$. Moreover, there are infinite many completely different optimal r-covering codes for all r.

4 Proof for uniqueness of 1-identifying codes

In this section, we prove that no two optimal 1-identifying codes are not completely different. The proof is based on the proof of the lower bound on the density in [6].

Assume that C is a 1-identifying code. First, we define

$$\begin{aligned} L_i &= \{v \in V : |B_1(v) \cap C| = i|\}, \\ L_{\geq i} &= \{v \in V : |B_1(v) \cap C| \geq i\}, \\ C_i &= \{c \in C : |B_1(c) \cap L_{\geq 3}| = i\}, \\ C_{\geq i} &= \{c \in C : |B_1(c) \cap L_{\geq 3}| \geq i\} \end{aligned}$$

as in [6] and

$$C_3^* = \{c \in C_3 : B_1(c) \cap L_{\geq 4} = \emptyset\}.$$

The following three lemmas are also from [6].

Lemma 10. [6, Lemma 3] *Every codeword c belongs to $C_{\geq 2}$. Moreover, if $c \in C_2$, then the surrounding of c is as in the constellation*

$$\begin{array}{ccccc} & x & - & - & x \\ x & - & - & - & - \\ - & - & c & - & - \\ - & - & - & - & x \\ x & - & - & x & \end{array}$$

or the symmetric constellation which is obtained by rotation by 90 degrees. Here, x denotes a codeword and $-$ a non-codeword.

Lemma 11. [6, Lemma 2] *Every codeword which has another codeword within graphical distance 1 — i.e, in $B_1(c)$ — belongs to the set $C_{\geq 4}$.*

Remark 12. By Lemma 11, if $c \in C_3^*$, then c is the only codeword in $B_1(c)$.

Lemma 13. [6, Lemma 4] *Every codeword has at most two codewords that belong to C_2 within Euclidean distance $\sqrt{5}$.*

Now, we show the essential condition when the code can be optimal 1-identifying code in the infinite king grid.

Lemma 14. *If C is an optimal 1-identifying code, then $D(C \setminus C_3^*) = 0$.*

Proof. In [5], one has proved that there exists an identifying code with density $\frac{2}{9}$. Figure 1(g) illustrates such a code and this code is called C_{opt}. This code is defined precisely in Theorem 15.

Now, the proof of the lower bound in [6] can be reformulated using the share technique and voting (discharging).

Let all vertices have one vote initially. Then we transfer votes to codewords by two voting steps. In the first voting step, every vertex gives its vote to codewords in its 1-neighbourhood s.t. it gives equally many votes to all codewords in its 1-neighbourhood. For example, if a vertex has three codewords in its 1-neighbourhood, it gives $\frac{1}{3}$ of a vote to three codewords each. (Notice that one of these three codewords can be the vertex itself. Then it can also give votes from itself to itself.) Now, every codeword has

$$S(c) = \sum_{v \in B_1(c)} \frac{1}{|I_1(v)|}$$

votes. This value is the so-called *share* of c.

Now, $|I_1(v)| = i$ for all vertices $v \in L_i$. In particular, at most one vertex in $B_1(c)$ can belong to L_1. Indeed, $c \in I_1(v)$ for all $v \in B_1(c)$ and if there was two vertices $v_1 \in L_1 \cap B_1(c)$ and $v_2 \in L_1 \cap B_1(c)$, then $I_r(v_1) = \{c\} = I_r(v_2)$ against the assumption that C is an 1-identifying code. Thus, at least eight vertices in $B_1(c)$ belong to $L_{\geq 2}$. Moreover by Lemma 10, at least two of the vertices in $B_1(c)$ belongs to $L_{\geq 3}$. In particular, the number of the votes of c is now

$$\begin{cases} S(c) \leq 1 + 6 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} = 4\frac{2}{3} & \text{for all } c \in C_2 \\ S(c) \leq 1 + 5 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} = 4\frac{1}{2} & \text{for all } c \in C_3 \\ S(c) \leq 1 + 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} = 4\frac{1}{3} & \text{for all } c \in C_{\geq 4}. \end{cases}$$

Moreover, if the 1-neighbourhood of $c \in C_3$ contains a vertex which belongs to the set $L_{\geq 4}$ (i.e., $c \in C_3 \setminus C_3^*$), then c has at most

$$S(c) \leq 1 + 5 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + \frac{1}{4} = 4\frac{5}{12} < 4\frac{4}{9}$$

votes.

In the second step, every codeword in C_2 gives $\frac{1}{18}$ of a vote to each codeword at Euclidean distance $\sqrt{5}$. By Lemmas 10 and 11, every such codeword gives $\frac{1}{18}$ of a vote to four codewords which all belong to the set $C_{\geq 4}$. Therefore, every codeword in C_2 gives $4 \cdot \frac{1}{18} = \frac{2}{9}$ of a vote away. Then such codeword has at most $4\frac{2}{3} - \frac{2}{9} = 4\frac{4}{9}$ votes. On the other hand, only the codewords in $C_{\geq 4}$ can get more votes in the second voting step. However by Lemma 13, they get $\frac{1}{18}$ of a vote from at most two codewords. Thus, a codeword in $C_{\geq 4}$ has at most $4\frac{1}{3} + \frac{2}{18} = 4\frac{4}{9}$ votes finally.

Now, we have shown that every codeword in C_3^* has finally at most $4\frac{1}{2}$ votes and each other codeword has at most $4\frac{4}{9}$ votes. Moreover, the vertices in Q_n have $|Q_n|$ votes all in all initially. On the other hand, all these votes are in the codewords in Q_{n+3} finally. Indeed, a vertex can give votes only to codewords at graphical distance one in the first step and at graphical distance two in the second step. Therefore, we have

$$\begin{aligned} |Q_n| &\leq 4\frac{1}{2}|C_3^* \cap Q_{n+3}| + 4\frac{4}{9}|(C \setminus C_3^*) \cap Q_{n+3}| \\ &\leq 4\frac{1}{2}|C \cap Q_n| - \frac{1}{18}|(C \setminus C_3^*) \cap Q_n| + 4\frac{1}{2}|Q_{n+3} \setminus Q_n|. \end{aligned}$$

Now,

$$\begin{aligned} D(C) &= \limsup_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|} \\ &\geq \limsup_{n \to \infty} \left(\frac{2}{9} + \frac{1}{81} \frac{|(C \setminus C_3^*) \cap Q_n|}{|Q_n|} - \frac{|Q_{n+3} \setminus Q_n|}{|Q_n|} \right) \\ &= \frac{2}{9} + \frac{1}{81} D(C \setminus C_3^*). \end{aligned}$$

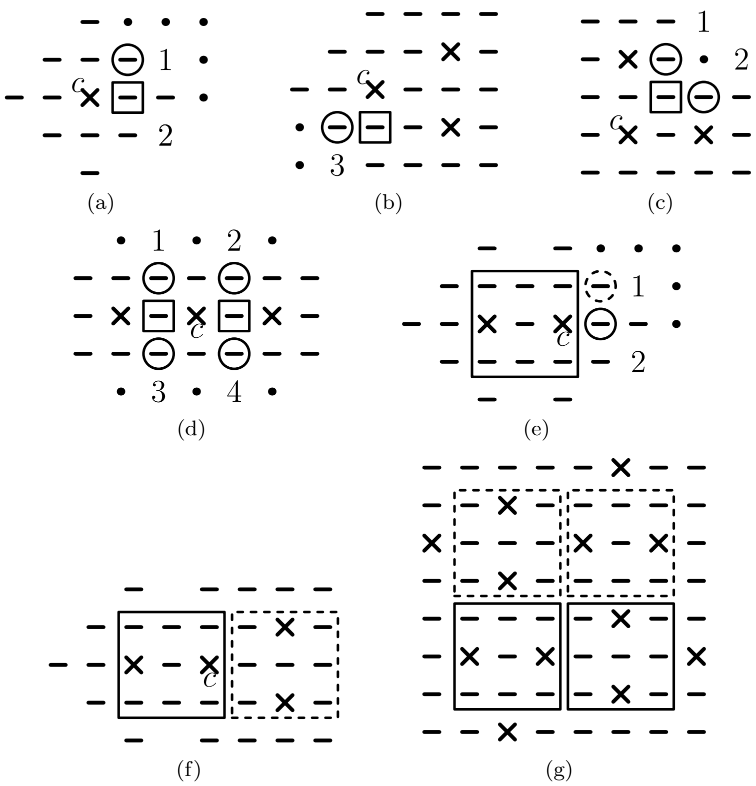


Figure 1: Some constellation for the proof of Theorem 15. The crosses are codewords and the lines are non-codewords as assumption. Also the numbers are codewords and the black dots are non-codewords. This is shown in the proof. Circles and squares are auxiliary marking for the proof.

Since the density of C is optimal, and the optimal density is $\frac{2}{9}$, then $D(C \setminus C_3^*) = 0$. ∎

Theorem 15. *The code*

$$C_{opt} = \left\{ (x, y) \;\middle|\; \begin{array}{llll} x + y \equiv \pm 1 & \pmod 6 & \text{if} \quad y \equiv 0 \pmod 3 & \text{or} \\ x + y \equiv \pm 2 & \pmod 6 & \text{if} \quad x \equiv 0 \pmod 3 & \end{array} \right\}.$$

is an unique optimal 1*-identifying code in the infinite king grid.*

Proof. In [5], one has proved that the density of C_{opt} is $\frac{2}{9}$. Moreover, the density of C_{opt} is optimal by the lower bound which has been proved in [6]. Figure 1(g) illustrates the code C_{opt}.

Assume next that C is any optimal 1-identifying code in the infinite king grid. Then $D(C) = D(C_{opt}) = \frac{2}{9}$ and furthermore $D(C \setminus C_3^*) = 0$ by Lemma 14. Therefore, for all n there exists a $(2n+1) \times (2n+1)$-square, which does not contain any codeword that does not belong to C_3^*.

Let α_n be an isometry s.t. $\alpha_n(C \setminus C_3^*) \cap Q_{n+3} = \emptyset$. Now, we show that every codeword c in $\alpha_n(C) \cap Q_n$ has exactly one codeword which is at Euclidean distance 2 from c. First, assume to the contrary that c has no codeword within

Euclidean distance 2 except c itself. See Figure 1(a). First, at least one of the vertices 1 and 2 has to be a codeword since c and the vertex surrounded by the square must be separated. Without loss of generality we can assume that the vertex 1 is a codeword. Now, the black dots are non-codewords because the vertex 1 belongs to C_3^* and none of such codewords has codeword neighbours by Remark 12. Furthermore, the vertex 2 has also to be a codeword. Indeed, vertices surrounded by the square and the circle have to be separated from each other. Moreover, the neighbours of the codeword 2 are non-codewords. See Figure 1(b). Now, the vertex 3 is a codeword since c and the vertex surrounded by the square in Figure 1(b) must be separated. Furthermore, the black dots are non-codewords. Now, the identifying sets of the vertices surrounded by the square and the circle in Figure 1(b) are the same, which is not possible since C is a 1-identifying code. Then we know that all codewords have another codeword within Euclidean distance 2.

Second, we assume to the contrary that there are at least two codewords c' and c'' at Euclidean distance 2 from c. Assume first that c, c' and c'' are not in the same line. See Figure 1(c). First, the black dot is a non-codeword since no neighbours of $c \in C_3^*$ belong to the set $L_{\geq 4}$. Now, vertices marked with numbers 1 and 2 must be codewords since the vertices marked with circles must be separated from c' and c''. Thus, these codewords nevertheless have a codeword neighbour, then they can not belong to C_3^*, which is a contradiction.

Next, we assume that c, c' and c'' are in the same line. See Figure 1(d). First, all black dots are non-codewords by the previous paragraph. Moreover, the vertices marked with the numbers 1–4 must be codewords since vertices surrounded by the circles must be separated from the vertices surrounded by the squares. Now, every vertex which has been surrounded with a circle belongs to the set $L_{\geq 3}$. Then $c \in C_{\geq 4}$ which is a contradiction.

Now, we are shown that every codeword $c \in \alpha_n(C) \cap Q_n$ has exactly one codeword at distance two (cf. Figure 1(e)). Then, we observe that every $c \in \alpha_n(C) \cap Q_n$ belongs to exactly one of such boxes which is illustrated in Figure 1(e) or its rotation. Again, we can observe in the same way as in Figure 1(a) that vertices marked with numbers in Figure 1(e) are codewords. Now, all neighbours of these numbers have to be non-codewords and then we have constellation in Figure 1(f). Now, nine vertices on the right-hand side of the box of c form a similar box which has been rotated by 90 degrees. By symmetry, also nine vertices above the right-hand side box must form such a box, and furthermore the nine vertices above the box containing c also form the similar box. And so on, we can see that $\alpha_n(C) \cap Q_n$ is unique except translation. □

5 Numbers of completely different optimal r-identifying codes when $r \geq 2$

In this section, we study r-identifying codes when $r \geq 2$. We use the similar idea with the help of which has been proved the lower bound $\frac{1}{4r}$ in [4] for r-identifying code when $r \geq 2$.

First, we define the set $K_r(v)$ in the same way as in [4]:

$$\begin{aligned} K_r((x,y)) &= \{(x+i, y+j) : i \in \{-r, r+1\}, -r \leq j \leq r+1\} \\ &\quad \cup \{(x+i, y+j) : -r \leq i \leq r+1, j \in \{-r, r+1\}\}. \end{aligned}$$

Now, the set $K_r((x,y))$ is the union of the pairwise symmetric differences of the sets $B_r((x,y))$, $B_r((x{+}1,y))$, $B_r((x,y{+}1))$ and $B_r((x{+}1,y{+}1))$. Furthermore, we call vertices $(x{-}r,y{-}r)$, $(x{-}r,y{+}r{+}1)$, $(x{+}r{+}1,y{-}r)$ and $(x{+}r{+}1,y{+}r{+}1)$ the *corners* of $K_r(v)$. The other vertices except corners form four *sides*.

Moreover, we give the name K^* to such sets which have exactly one codeword in the vertical sides and exactly one codeword in the horizontal sides and at most one codeword in its four corners. Furthermore, a vertex $v \in V$ belongs to the set V^* if $K_r(v) \in K^*$.

Lemma 16. *For all optimal r-identifying codes, the density $D(V \setminus V^*) = 0$, when $r \geq 2$.*

Proof. (Sketch) In [4], one proves that the sides in each $K_r(x,y)$ contain at least two codewords on average. This leads the lower bound $\frac{1}{4r}$ for r-identifying code in the infinite king grid when $r \geq 2$.

However, the proof can be reformulated using the voting method. Then, every codeword has $8r$ votes initially (since there are $8r$ vertices in sides in $K_r(x,y)$). Next, we transfer votes to the sets $K_r(x,y)$ using two voting steps:

Step 1: Every codeword gives one of its votes to each $K_r(x,y)$ which contains the codeword in its side. (Then a codeword gives exactly all its votes to the sets $K_r(x,y)$.)

Step 2a: A set $K_1 = K_r(x_1,y_1)$ gives $\frac{2}{2r+1}$ of a vote to another set $K_2 = K_r(x_2,y_2)$ if $x_1 = x_2$ and there are (at least) two codewords c and c' in the vertical sides of K_1 such that c is also a corner of K_2 and $c' \in K_2$. Notice that K_1 gives either $\frac{2}{2r+1}$ or 0 votes to K_2 in this step.

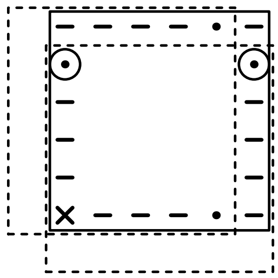
Step 2b: A set $K_1 = K_r(x_1,y_1)$ gives $\frac{2}{2r+1}$ of a vote to another set $K_2 = K_r(x_2,y_2)$ if $y_1 = y_2$ and there are (at least) two codewords c and c' in the horizontal sides of K_1 such that c is also a corner of K_2 and $c' \in K_2$. Notice that K_1 gives either $\frac{2}{2r+1}$ or 0 votes to K_2 in this step.

Now, we can show that every set $K_r(x,y)$ in K^* has at least two votes and the other sets $K_r(x,y)$ have at least $2\frac{1}{5}$ votes. Finally, we can furthermore transfer votes from $K_r(x,y)$ to the vertex (x,y). Thus, we can show that the vertices in Q_n have at least $2|Q_n| + \frac{|Q_n \setminus V^*|}{5}$ votes. Moreover, all these votes come from codewords in Q_{n+r+1} and these codewords have $8r|C \cap Q_{n+r+1}|$ votes in the beginning. Hence,

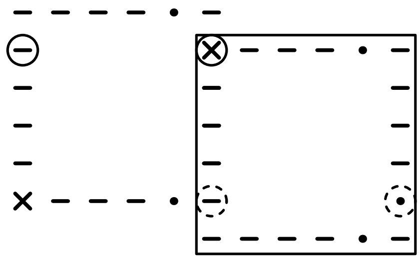
$$\begin{aligned} D(C) &= \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \\ &\geq \limsup_{n\to\infty} \left(\frac{1}{4r} + \frac{1}{40r}\frac{|Q_n \setminus V^*|}{|Q_n|} - \frac{|Q_{n+r+1} \setminus Q_n|}{|Q_n|} \right) \\ &= \frac{1}{4r} + \frac{1}{40r} D(V \setminus V^*). \end{aligned}$$

By [4], the density of optimal r-identifying code is $\frac{1}{4r}$ when $r \geq 2$. Thus, $D(V \setminus V^*) = 0$. □

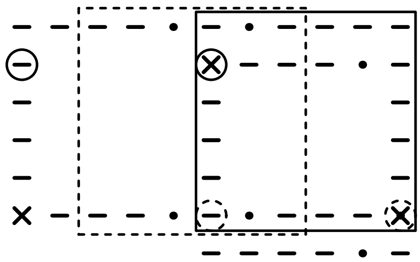
Lemma 17. *Every optimal r-identifying code in the infinite king grid contains an arbitrary large square which has period of size $2r$, i.e, which consists of tiles with size $2r \times 2r$. Moreover, every such tile contains exactly one codeword in exactly every two rows and every two columns.*



(a) The square with continuous line surrounds $K_r(x+r, y+r)$ and the squares with dashed lines surround $K_r(x+r-1, y+r)$ and $K_r(x+r, y+r-1)$.



(b) The square surrounds $K_r(x+3r+1, y+r-1)$.



(c) The square with continuous line surrounds $K_r(x+3r+1, y+r)$ and the square with dashed line surrounds $K_r(x+2r, y+r)$.

Figure 2: Some constellations for the proof of Lemma 17. The crosses are codewords and the lines are non-codewords. The cross in the lower left corner of each figure is the codeword (x, y).

Proof. Let C be an r-identifying code whose density is optimal, i.e., $D(C) = \frac{1}{4r}$. By Lemma 16, there exists, for all n, an $(n \times n)$-square which contains only the vertices that belong to the set V^*. Let α_n be an isometry and $C_n \subseteq V$ be a code s.t. $C_n = \alpha_n(C)$ and $\alpha_n(Q_{n+4r} \setminus V^*) = \emptyset$. Now, we assume that $(x, y) \in C_n$ is any codeword in Q_n. Next, we show that $(x, y+2r)$ is also a codeword.

First, $K_r(x+r, y+r)$ has a codeword in the lower left corner. See Figure 2(a). Now, the other corners of $K_r(x+r, y+r)$ must be non-codewords. Moreover, (x, y) belongs to the vertical side of $K_r(x+r, y+r-1)$, then other vertices in the vertical sides of $K_r(x+r, y+r-1)$ must be non-codewords. In the same way, the vertices in the horizontal sides of $K_r(x+r-1, y+r)$ except (x, y) must be non-codewords. Thus, only $(x+2r, y)$, $(x+2r, y+r+1)$, $(x, y+2r)$ and $(x+r+1, y+2r)$ can be codewords in the sides of $K_r(x+r, y+r)$. In particular, either $(x, y+2r)$ or $(x+2r+1, y+2r)$ is a codeword. See the circled dots in Figure 2(a).

Assume to the contrary that $(x+2r+1, y+2r) \in C_n$. See Figure 2(b). Now, $(x+2r+1, y+2r)$ is the upper left corner of $K_r(x+3r+1, y+r-1)$. Then we can decide in the same way as in the previous paragraph that exactly one of $(x+2r+1, y)$ and $(x+4r+2, y)$ (and exactly one of $(x+4r+1, y-1)$ and $(x+4r+1, y+2r)$, resp.) must be a codeword. However, we have already observed that $(x+2r+1, y)$ is a non-codeword, then $(x+4r+2, y) \in C_n$.

Now, $(x + 4r + 2, y)$ is the lower right corner of $K_r(x + 3r + 1, y + r)$. See Figure 2(c). Again, exactly one of $(x + 2r + 2, y)$ and $(x + 2r + 2, y + r + 1)$ has to be a codeword. However, $K_r(x + 2r, y + r)$ (bounded by the dashed lines in Figure 2(c)) now contains two codewords in the horizontal sides, which is a contradiction. Hence, $(x, y + 2r)$ must be a codeword for any codeword (x, y) in $C_n \cap Q_n$. By symmetry, we can observe in the similar way that $(x + 2r, y)$, $(x - 2r, y)$ and $(x, y - 2r)$ have to be codewords. This proves that the code contains an arbitrary large square which consists of tiles with size $2r \times 2r$.

The second claim – every two columns or rows contains one codeword – follows since the vertical sides of $K_r(x, y)$ contains $2r$ vertices in two different columns the distance of which is $2r + 1 \equiv 1 \pmod{2r}$. Now, if the column the x-coordinate of which is $0 \pmod{2r}$ contains a codeword, then the next column whose x-coordinate is $1 \pmod{2r}$ does not contain any codeword. Furthermore, the vertical sides of $K_r(x + 1, y)$ contains codewords in two columns whose x-coordinates are $1 \pmod{2r}$ and $2 \pmod{2r}$. The first does not contain a codeword as observed above, then the column with x-coordinate is $2 \pmod{2r}$ has to be contain a codeword. Now, we can decide the claim inductively for vertical sides and the proof for horizontal sides is equivalent. □

Let T_r be a $2r \times 2r$-sized tile which consists of exactly one codeword in every even row and every even column and no codewords in odd row and odd columns. Define a code C_{T_r} s.t. a vertex (x, y) is a codeword if $x \equiv a \pmod{2r}$ and $y \equiv b \pmod{2r}$ for any codeword $(a, b) \in T_r$.

Theorem 18. *No optimal r-identifying code in the infinite king grid is completely different from all codes C_{T_r}.*

Proof. The theorem follows from Lemma 17. □

In particular, C_{t_r} is an optimal r-identifying code with density $\frac{1}{4r}$ for every T_r when $r \geq 2$. Indeed, the identifying set of every vertex is non-empty. Moreover, the symmetric difference of the r-neighbourhoods of two vertices contains $2r + 1$ consecutive vertices in one even (and in one odd) column or $2r + 1$ consecutive vertices in one even (and in one odd) row. Then every such symmetric difference contains at least one codeword.

Now, the answer to the question *how many completely different optimal r-identifying codes are in the infinite king grid* is the same as the question *how many way it is possible to put a cross in the $r \times r$-array s.t. every row and every column contains exactly one cross when two arrays are the same if they are got by shift, rotation and reflection from each other.* Furthermore, this is an equivalent problem with the question how many non-equivalent sequential periodic binary arrays with the sequence $0^{r-1}1$ exists in the square grid. This problem is for instance studied in [8, 19].

The maximum number of completely different r-identifying codes in the infinite king grid for the small values of r is given in the Table 1. The exact value for such number is known when r is at most 29. Generally, it is known that the number is at least $\frac{r!}{8r^2}$ and it is also the approximate value for the large values of r.

r	1	2	3	4	5	6	7	8	9	10	11	12
N_r	1	1	1	2	4	10	28	127	686	4975	42529	420948

Table 1: The numbers of completely different r-identifying codes in the infinite king grid. The first value follows from Theorem 15 and the other values from [8, 18, 19].

6 On optimal covering codes

In this section, we show that the number of completely different optimal r-covering codes is infinite for all r in the infinite king grid. In particular, we also see that there is a non-periodic optimal r-covering code which is completely different from all optimal periodic r-covering codes.

Theorem 19. *Let y_x be a doubly-infinite sequence of integers in $[-r, r]$. Then the code*

$$C = \{(x, y) : x \equiv 0 \pmod{2r+1} \quad \text{and} \quad y \equiv y_{x/(2r+1)} \pmod{2r+1}\} \quad (1)$$

is an optimal r-covering code with density $\frac{1}{(2r+1)^2}$ in the infinite king grid.

Proof. Every vertex (x, y) in the infinite king grid can be written uniquely in the form $(x, y) = ((2r+1)k+s, y_k+(2r+1)l+t)$, where k, l, s and t are integers and $|s| \leq r$ and $|t| \leq r$. Now $((2r+1)k, (2r+1)l + y_k)$ is the only codeword in the r-neighbourhood of $(x, y) = ((2r+1)k + s, y_k + (2r+1)l + t)$. Therefore, every codeword has exactly one codeword in the r-neighbourhood, where the claim is following.

The optimality follows from the fact that the r-neighbourhood contains exactly $(2r+1)^2$ vertices for every vertex. Furthermore, each of these has to contain at least one codeword. □

Theorem 20. *No optimal r-covering code is completely different from all the codes in (1).*

Proof. Let C be an optimal r-covering code, i.e., $B_r(v)$ contains exactly one codeword for all vertices v in an arbitrary large square. First, if $(x+2r+1, y)$ and $(x, y+2r+1)$ are codewords for every codeword $(x, y) \in C$, then C corresponds to the code in the previous theorem when $y_x = \ldots, 0, 0, 0, \ldots$.

Thus, we can assume that $(x, y) \in C$ and $(x+2r+1, y) \notin C$ or $(x, y+2r+1) \notin C$. Furthermore, we can assume without loss of generality that $(x+2r+1, y) \notin C$. Moreover, we assume that every vertices in the large enough surround of (x, y) has exactly one codeword in its r-neighbourhood. Now, (x, y) does not cover $(x + r + 1, y)$ and only the vertices in $B = \{(x + 2r + 1, b) : b \in [-r, r]\}$ covers $(x + r + 1, y)$, but not vertices in $B_r(x, y)$. Therefore, at least one of the vertices in B has to be a codeword. Again, we can assume without loss of generality that $(x+2r+1, y+a) \in C$, where $0 < a \leq r$. Now, $(x+r, y+r+1)$ and $(x+r+1, y+a-r-1)$ can be covered by only the vertices $(x, y+2r+1)$ and $(x + 2r + 1, y + a - 2r - 1)$, respectively, such that none of the codeword is covered by two codewords. And so on, we can decide that $(x, y + (2r + 1)k)$ and $(x + 2r + 1, y + a + (2r + 1)k)$ are codewords for all $k \in \mathbb{Z}$.

Furthermore, $(x + 3r + 2, y)$ has to be covered by a codeword and only the vertices in $B_2 = \{(x + 2(2r + 1), b) : b \in [-r, r]\}$ can cover it such that no vertex is not covered by two codewords. When we are chosen, which vertex is a codeword in the set B_2, then vertices $(x + 3r + 2, a)$ can be covered only one way if all vertices are covered by exactly one codeword. In the same way, we see that exactly one of the vertices in $B_k = \{(x + k(2r + 1), b) : b \in [-r, r]\}$ is a codeword for all k. Again, the other codewords in these columns are determined uniquely. Thus, the code can be shown in the form of (1). □

Corollary 21. *There are infinite many completely different periodic r-covering codes in the infinite king grid.*

Proof. Let a_k be a family of sequences where

$$(a_k)_i = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{k} \\ 0, & \text{otherwise.} \end{cases}$$

Now, define code C_k with the help of a_k as equation (1). Then, C_k is optimal and periodic for all $k \in \mathbb{N}$. Moreover, they are pairwise completely different. □

Corollary 22. *There is a non-periodic optimal r-covering code which is completely different from all periodic optimal r-covering codes in the infinite king grid.*

Proof. Let f_k be a cube-free infinite sequence of 0 and 1. For example, Thue-Morse word is such a sequence [17]. Furthermore, let

$$C = \{(x, y) : x \equiv 0 \pmod{2r + 1} \quad \text{and} \quad y \equiv f_{|x/(2r+1)|} \pmod{2r + 1}\}$$

be an optimal non-periodic code and C' be an arbitrary optimal periodic code which is defined by the sequence a_k. Since C' is periodic, then so does a_k also. Let a_k have a period of length n. Now, no tile of size $6n \times 6n$ of C' can be equivalent with any $6n \times 6n$ sized tile of C. Indeed, otherwise there was subsequence S of $6n$ elements of f_k which is the same with some subsequence of C'. However without loss of generality we can assume that the indexes of $3n$ last (or first) elements of S are positive (or negative, respectively). However, the last (and first) $3n$ elements contain a periodic subsequence of n elements three times and so the subsequence and also f_k are not cube-free, which is a contradiction. □

References

[1] Auger, D., Honkala, I.: Watching Systems in the King Grid, Graphs and Combinatorics, in press.

[2] Ben-Haim, Y., Litsyn, S.: Exact minimum density of codes identifying vertices in the square grid, SIAM Journal on Discrete Mathematics, Vol. 19, 69–82 (2005).

[3] Charon, I., Honkala, I., Hudry, O., Lobstein, A.: General bounds for identifying codes in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 8(1), R39 (2001).

[4] Charon, I., Honkala, I., Hudry, O., Lobstein A.: The minimum density of an identifying code in the king lattice, Discrete Mathematics, Vol. 276, 95–109 (2004).

[5] Charon, I., Hudry, O., Lobstein, A.: Identifying codes with small radius in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 9(1), R11 (2002).

[6] Cohen, G., Honkala, I., Lobstein, A., Zémor, G.: On codes identifying vertices in the two-dimensional square lattice with diagonals, IEEE Transactions on Computers, Vol. 50, 174–176 (2001).

[7] Cranston, D. W., Yu, G.: A new lower bound on the density of vertex identifying codes for the infinite hexagonal grid, Electronic Journal of Combinatorics, Vol. 16(1), R113 (2009).

[8] Day, R., Street, A. P.: Sequential Binary Arrays. I. The Square Grid. Journal of Combinatorial Theory, Ser. A, Vol. 32(1), 35–52 (1982).

[9] Honkala, I., Laihonen, T.: Codes for identification in the king lattice, Graphs and Combinatorics, Vol. 19, 505–516 (2003).

[10] Karpovsky, M. G., Chakrabarty K., Levitin, L. B.: On a new class of codes for identifying vertices in graphs, IEEE Transactions on Information Theory, Vol. IT-44, 599–611 (1998).

[11] Lobstein, A.: Covering radius. Bibliography.
http://www.infres.enst.fr/~lobstein/biblio.html

[12] Lobstein, A.: Watching systems, identifying, locating-dominating and discriminating codes in graphs. Bibliography.
http://www.infres.enst.fr/~lobstein/bibLOCDOMetID.html

[13] Pelto, M.: New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid, Cryptography and Communications, Vol. 2, 41–47 (2010).

[14] Pelto, M.: On locating-dominating codes in the infinite king grid, Ars Combinatoria, in press.

[15] Obata, K.: On identifying codes in lattices, Preprint, December 2003. http://www.kenjioba.net/images/Identifying.pdf (20.2.2012)

[16] Slater, P. J.: Domination and location in acyclic graphs, Networks, Vol. 17, 55–64 (1987).

[17] Salomaa, A.: Jewels of Formal Language Theory. Computer Science Press, Rockville, MD (1981).

[18] Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences (OEIS). http://oeis.org/search?q=1%2C1%2C1%2C2%2C4%2C10%2C28%2C127

[19] Street, A. P., Day, R.: Sequential binary arrays. II. Further results on the square grid. Combinatorial mathematics, IX (Brisbane, 1981), 392—418, Lecture Notes in Math., 952, Springer, Berlin-New York, 1982.

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On locating-dominating codes in the infinite king grid

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Abstract

Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E. The ball $B_r(v)$ denotes the vertices within graphical distance r from v. A subset $C \subseteq V$ is called an r-locating-dominating code if the sets $I_r(v) = B_r(v) \cap C$ are distinct and non-empty for all $v \in V \setminus C$. A code C is an r-identifying code if the sets $I_r(v)$ are distinct and non-empty for all vertices $v \in V$. We study r-locating-dominating codes in the infinite king grid and in particular show that there is an r-locating-dominating code such that every r-identifying code has larger density. The infinite king grid is the graph with vertex set $\mathbb{Z}^2$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| \leq 1,\ |y_1 - y_2| \leq 1, (x_1, y_1) \neq (x_2, y_2)\}$.

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Denote by $d(u, v)$ the distance between two vertices u and v i.e. the number of edges on any shortest path from u to v. The ball with center v and radius r is

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

We call any $C \subseteq V$ a *code*. The vertices of C are called *codewords*. In particular, C is an *r-locating-dominating code* if the sets

$$I_r(v) = B_r(v) \cap C$$

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are non-empty and distinct for all non-codewords $v \in V \setminus C$. If the sets $I_r(v)$ are non-empty and distinct for all vertices $v \in V$, then C is an r-identifying code. In particular, an identifying code is always a locating-dominating code.

The symmetric difference of two sets A and B is denoted by

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

The r-locating-dominating code could also be defined by symmetric differences: code C is an r-locating-dominating code if and only if $I_r(v) \Delta I_r(u) \neq \emptyset$ and $I_r(v) \neq \emptyset$ for all non-codewords v and u. If $c \in I_r(v) \Delta I_r(u)$, we say that u and v are *separated* by c.

We study r-locating-dominating codes in the infinite king grid. The infinite king grid is the graph where $V = \mathbb{Z} \times \mathbb{Z}$ and two different vertices $u = (u_x, u_y)$ and $v = (v_x, v_y)$ are adjacent if $|u_x - v_x| \leq 1$ and $|u_y - v_y| \leq 1$. Thus vertices u and v are neighbours if the Euclidean distance between u and v is 1 or $\sqrt{2}$.

The density of $C \subseteq \mathbb{Z}^2$ is

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap B_n((0,0))|}{|B_n((0,0))|},$$

where $|C \cap B_n((0,0))|$ is the number of codewords in the ball $\{(x,y) \mid |x| \leq n, |y| \leq n\}$ and $|B_n((0,0))| = (2n+1)^2$ is the number of all vertices in the ball. We also denote $B_n((0,0)) = Q_n$. We search for the minimum density of locating-dominating codes for given r in the infinite king grid.

Locating-dominating codes were introduced in the late 1980s by Slater [17] and [18] and identifying codes in the late 1990s by Karpovsky, Chakrabarty and Levitin [10]. A motivation of such codes is a safeguard analysis of a facility using sensor networks [17] or a fault diagnosis of a multiprocessor system [10]. Assume that we have a multiprocessor system. Some processors are chosen to perform the task of testing if the processor itself is faulty or if there is a faulty processor within distance r. The chosen processor sends the symbol 2, if the processor itself is faulty; symbol 1, if it itself is not faulty, but there is a faulty processor within distance r; and symbol 0, otherwise. Finally, we get the reports from all the chosen processors and based on the reports alone we can perform the fault diagnosis. Here, processors are vertices and the chosen processors are codewords. If the chosen processors form an r-locating-dominating code, then we can locate the processor which is faulty if we assume that at most one of the processors is faulty. If the chosen processor sends symbol 1 instead of 2 also when the processor itself is faulty, then we can use an r-identifying code to locate the faulty processor.

r	Locating-dominating codes		ID codes	
1	$\frac{1}{5}$	[8]	$\frac{2}{9}$	[3],[4]
2	$\frac{1}{10} \leq D \leq \frac{1}{8}$	[3], Thm 2	$\frac{1}{8}$	[2],[3]
3	$\frac{1}{14} \leq D \leq \frac{2}{25}$	Thms 2 & 5	$\frac{1}{12}$	[2],[3]
≥ 4	$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ if $2 \mid r$ $\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$ if $2 \nmid r$	[2], Thms 2 & 5	$\frac{1}{4r}$	[2]

Table 1: The known lower and upper bounds for densities of locating-dominating and identifying codes (ID codes) in the infinite king grid.

Although the difference between the definitions of locating-dominating codes and identifying codes is quite small, we show in this paper that there exists an r-locating-dominating code with density D_r for any odd r such that there exist no r-identifying codes with the same or smaller density. Results when $r = 1$ have already been shown in [8]. We also prove two lower bounds for r-locating-dominating codes when $r > 1$. The proof of the better bound is long and quite similar to corresponding proof for r-identifying codes (Theorem 3 of [2]). Therefore, we only add some details to the proof and it is almost impossible to understand our proof if one does not know the proof of Theorem 3 of [2]. The complete proof would nevertheless be a duplicate in many respects of the proof for r-identifying codes, so it is reasonable to present only the differences in this paper. When reading our proof of Theorem 2, the reader should have a copy of [2] at hand. The proof of the weaker bound is short and easy to understand.

Furthermore, we observe in this paper that the bounds of r-locating-dominating codes when $r \geq 2$ are also valid for so-called *open neighbourhood r-locating-dominating codes* i.e. *r-OLD codes*. A code is an r-OLD code if sets $I_r(v) \setminus \{v\}$ are non-empty and distinct for all $v \in V$. OLD codes were considered in [9] and [16] and they can be used in fault diagnosis if the chosen processor sends symbol 0 if the processor itself is faulty. This corresponds for example to the case where the processor is unable to send an alarm if the processor itself is faulty.

Table 1 summarizes what is known about the density of r-locating-dominating codes and r-identifying codes in the infinite king grid. Here, the upper bound means that there exists a locating-dominating or an identifying code with that density and the lower bound means that density of every locating-dominating or identifying code is at least the value given in the table.

Locating-dominating codes, identifying codes and other closely related classes of codes in the infinite king grid and other graphs have also been studied in [1], [3]–[7] and [11]–[15]. See also the web bibliography [19].

2 Lower bounds

Theorem 1. *The density of an r-locating-dominating code is at least $\frac{1}{4r+4}$.*

Proof. In this proof, we use a standard technique for identifying codes. A more detailed presentation of the technique can be found in [1], for instance.

Let C be an r-locating-dominating code. Then

$$A_r(x,y) = (B_r(x,y) \Delta B_r(x,y+1)) \cup \{(x,y),(x,y+1)\}$$

contains at least one codeword for all (x,y). The claim follows from the fact $|A_r(x,y)| = 4r+4$. Indeed, a codeword can belong to only $4r+4$ such sets. □

Theorem 2. *The density of an r-locating-dominating code is at least $\frac{1}{4r+2}$.*

Proof. (Sketch) The claim is proved in [8], when $r=1$. Therefore we can assume that C is an r-locating-dominating code and $r \geq 2$. Next, we denote

$$\begin{aligned}
C_o(x,y) &= \{(x-r,y-r),(x-r,y+r+1),\\
&\quad (x+r+1,y-r),(x+r+1,y+r+1)\},\\
C_e(x,y) &= \{(x,y),(x,y+1),(x+1,y),(x+1,y+1)\},\\
S_v(x,y) &= \{(a,b) \mid a \in \{x-r,x+r+1\} \text{ and } y-r < b \leq y+r\},\\
S_h(x,y) &= \{(a,b) \mid x-r < a \leq x+r \text{ and } b \in \{y-r,y+r+1\}\},\\
L(x,y) &= C_o(x,y) \cup C_e(x,y) \cup S_v(x,y) \cup S_h(x,y).
\end{aligned}$$

We call C_o the set of *corners*, C_e the *center*, and S_v and S_h *vertical and horizontal sides* of set $L(x,y)$. Moreover, we observe that

$$L(x,y) = \bigcup_{u,v \in C_e(x,y)} (B_r(u) \ \Delta \ B_r(v)) \cup C_e(x,y).$$

Then $L(x,y)$ has to separate vertices in the center of $L(x,y)$.

Paper [2] shows that the sides of

$$\begin{aligned}
K(x,y) &:= K_r((x,y),(x+1,y),(x+1,y+1),(x,y+1))\\
&= \bigcup_{u,v \in C_e(x,y)} (I_r(u) \ \Delta \ I_r(v))
\end{aligned}$$

contain on average at least two codewords for all identifying codes. Now, we add a few rules to this proof and show that the sides and the center together contain at least two codewords on average for all locating-dominating codes.

If S_v contains codewords u and v, then these vertices separate the codewords of the center in the same way. Therefore, we say that v is *useless* for $L(x,y)$ if the y-coordinate of v is greater than the y-coordinate of u or if the y-coordinates of v and u are the same and the x-coordinate of v is greater than the x-coordinate of u. In the same way, if $u, v \in S_h(x,y)$, then v is useless for $L(x,y)$ if the x-coordinate of v is greater than the x-coordinate of u or if the x-coordinates of v and u are the same and the y-coordinate of v is greater than the y-coordinate of u. This is how useless codewords are defined for $K(x,y)$ in [2]. But in this paper, we give one more rule when a codeword is useless.

Codeword $v \in C_e(x,y)$ is useless for $L(x,y)$ if other non-useless codewords in $L(x,y)$ separate vertices in the center of $L(x,y)$. When we define whether the codewords in the center are useless or not we go through then in the following order:

$$(x,y) \triangleleft (x,y+1) \triangleleft (x+1,y) \triangleleft (x+1,y+1).$$

If possible, we mark useless codewords for the associates of $L(x,y)$ in the same way as they are marked for the associates of $K(x,y)$ in paper [2]. However, it is possible, that the sides of $K(x,y)$ (or $L(x,y)$) do not contain any codeword for locating-dominating codes if $K(x,y) \in \mathcal{E}_2''$ i.e two opposite corners of $K(x,y)$ are codewords and the other two are not. Then at least one vertex in the center of $L(x,y)$ has to be a codeword. See Figure 1. In this case, we can not mark useless codewords as for identifying codes. Therefore we need new rules.

Assume that (x,y), $(x-r, y+r+1)$ and $(x+r+1, y-r)$ are codewords and there are no other codewords on the sides and in the corners (or else we can mark a useless codeword as in [2]). First, if the center also contains other codewords than (x,y), then all except one of the codewords in the center are useless and we can mark one of them for $L(x,y)$.

Second, we assume that (x,y) is the only codeword in the center of $L(x,y)$. Now,

$$\begin{aligned} &B_r((x-1,y)) \Delta B_r((x+1,y+1)) \\ &\subseteq S_v(x-1,y) \cup \{(x-r-1, y-r)\} \cup S_v(x,y) \cup S_h(x,y) \\ &\quad \cup \{(x-r, y-r), (x+r+1, y+r+1)\} \end{aligned} \tag{1}$$

contains at least one codeword or $(x-1,y)$ or $(x+1,y+1)$ is a codeword for all r-locating-dominating codes (cf. Figure 1). The last three sets of (1) and $\{(x+1,y+1)\}$ do not contain any codewords by our assumption. If

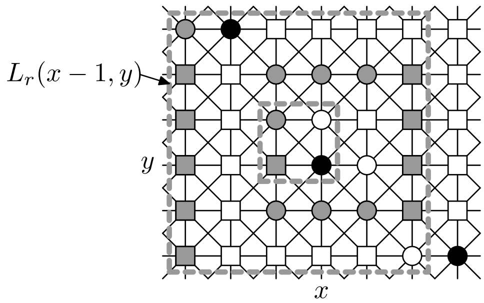

Figure 1: Black dots are codewords and white dots are non-codewords. These dots constitute $L_r(x,y)$. Gray dots can be codewords or non-codewords. The squares constitute the set $B_r((x-1,y))\Delta B_r((x+1,y+1))\cup\{(x-1,y),(x+1,y+1)\}$. Therefore, at least one of the squares must be a codeword.

$c \in S_v(x-1,y)$ is a codeword, then c and $(x-r, y+r+1) \in S_h(x-1,y)$ separates the codeword in $C_e(x-1,y)$ and so (x,y) is useless for $L(x-1,y)$ and it can be marked for $L(x-1,y)$.

If $S_v(x-1,y)$ does not contain codewords and $(x-r-1, y-r)$ is a codeword, then $(x-r, y+r+1)$ separates (x,y) from $(x-1,y+1)$ and $(x,y+1)$ and $(x-r-1,y-r)$ separates (x,y) from $(x-1,y)$. Thus (x,y) is useless for $L(x-1,y)$ and it can be marked for $L(x-1,y)$. Otherwise, $(x-1,y)$ must be a codeword, then it is useless and it is marked for $L(x-1,y)$, since $(x-r,y+r+1)$ and (x,y) separate $(x,y-1)$ from other vertices in the center of $L(x-1,y)$ and $(x-1,y)$ comes before (x,y) in our ordering.

The rotations of case $\mathcal{E}_2''$ are treated in the same way. In particular, we nevertheless mark useless codewords in the center only by left and right associates. Now, codewords can not be marked twice, because $r \geq 2$. Indeed, by the new rules we can mark codewords only from centers. Moreover, the useless codeword in the center of $L(x,y)$ can be marked only if there are no codewords in the sides of $L(x,y)$ or if there is exactly one codeword on the horizontal sides of $L(x,y)$ and it is at distance one from a corner.

Thus, equations (4) and (5) of [2] are also valid for r-locating-dominating codes when

$$\mathcal{S} = \{(L(x,y),c) \,|\, L(x,y) \in \mathcal{E}, c \in C \cap Q_n, c \text{ marked for } L(x,y)\},$$

but equation (6) is now

$$(8r+8)\cdot|C\cap Q_n| \geq 2|Q_n| + p_1 + p_2 + p_3 + 2p_4 + |\mathcal{S}| - 2hn$$

since $|L(x, y)| = 8r + 8$. Moreover,

$$p_1 + p_2 + p_3 + 2p_4 + |\mathcal{S}| \geq 4 \cdot |C \cap Q_n| - 10kn - 8n$$

as in [2]. Then

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n\to\infty} \left(\frac{1}{4r+2} - \frac{(10k + 2h + 8)n}{8r|Q_n|} \right) = \frac{1}{4r+2}.$$

□

Remark 3. An r-OLD code is also an r-locating-dominating code. Therefore the lower bounds for r-locating-dominating codes are also valid for r-OLD codes.

3 Upper bounds

Theorem 4. *There is an r-locating-dominating code with density $\frac{1}{4r}$ for all r.*

Proof. The code

$$C = \{ (x, y) \mid x \equiv y \pmod{2r},\ x \equiv 0 \pmod{2} \}$$

is an r-identifying code with density $\frac{1}{4r}$ for all r by [2]. The claim follows since r-identifying code is automatically an r-locating-dominating code. □

Theorem 5. *There exists an r-locating-dominating code with density $\frac{1}{4r+\frac{2}{r+1}}$ for all odd r.*

Proof. We show that the code

$$C = \left\{ \left(2k, (2r+2)k + 2rl + \left\lceil \frac{l}{r+1} \right\rceil \right) \middle| k, l \in \mathbb{Z} \right\}$$

is an r-locating-dominating code for all odd r. The density of C is

$$D(C) = \frac{r+1}{2(2r(r+1)+1)} = \frac{1}{4r + \frac{2}{r+1}}.$$

Indeed, when we look at any even-numbered column, then always $r + 1$ of any $(r + 1)2r + 1$ consecutive vertices belong to the code. Moreover, there are no codewords in the odd-numbered columns. A part of C when $r = 3$ is shown in Figure 2.

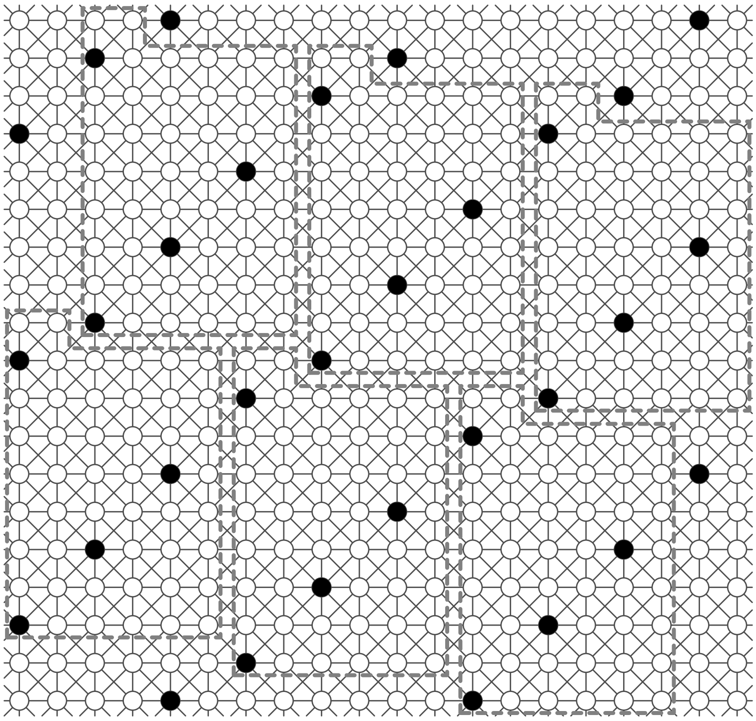

Figure 2: A 3-locating-dominating code. Black dots are codewords and white dots are non-codewords.

First, we make a small remark. Assume that $u = (x, y_u)$ and $v = (x, y_v)$ are two arbitrary vertices in the same column and $y_u < y_v$. If $I_r(u) \neq I_r(v)$ and $w = (x, y_w)$ is an arbitrary vertex where $y_w < y_u$ (or $y_w > y_v$), then also $I_r(w) \neq I_r(v)$ (or $I_r(w) \neq I_r(u)$, resp.). Indeed, $\Delta(I_r(u), I_r(v)) \subseteq \Delta(I_r(w), I_r(v))$ if $y_v - y_w \leq 2r$ or $I_r(w) \cap I_r(v) = \emptyset$, if $y_v - y_w > 2r$.

Now, we observe that at least one of $2r + 1$ consecutive vertices is a codeword in every even column. Then $I_r(v)$ is always non-empty and it contains a codeword from each even column that intersects $B_r(v)$. Thus, we see in which column v is. Indeed, if two vertices u and v are in different columns, then balls $B_r(u)$ and $B_r(v)$ can not intersect in exactly the same even columns.

Therefore, it is enough to show that $I_r(u) \neq I_r(v)$ for all non-codewords $u = (x, y_u)$ and $v = (x, y_v)$ in the same column.

Since the code consists of tiles with size $2r \times (2r + 2) \cup 2 \times 1$ (cf. Figure

separated vertices	codeword	x	t
$(x,-1+2t),\ (x,2t)$	$(-r-1+2t, r+2t)$	$0,\dots,-1+2t$	$1,\dots,\frac{r-1}{2}$
	$(r-1+2t,-r-1+2t)$	$2t,\dots,2r-1$	$0,\dots,\frac{r-1}{2}$
$(x,2t),\ (x,1+2t)$	$(-r-1+2t,-r+2t)$	$0,\dots,-1+2t$	$1,\dots,\frac{r-1}{2}$
	$(r+1+2t, r+1+2t)$	$1+2t,\dots,2r-1$	$0,\dots,\frac{r-1}{2}$
$(x,r+2t),\ (x,r+1+2t)$	$(2t,2t)$	$0,\dots,r+2t$	$0,\dots,\frac{r-1}{2}$
	$(2r+2+2t, 2r+1+2t)$	$r+2+2t,\dots,2r-1$	$0,\dots,\frac{r-3}{2}$
$(x,r-1+2t),\ (x,r+2t)$	$(2t,2r+2t)$	$0,\dots,r+2t$	$1,\dots,\frac{r-1}{2}$
	$(2r+2t,-1+2t)$	$r+2t,\dots,2r-1$	$1,\dots,\frac{r-1}{2}$
$(x,2r),\ (x,2r+1)$	$(r+1,3r+1)$	$1,\dots,2r-1$	
$(x,2r+1),\ (x,2r+2)$	$(-r+1,3r+2)$	$0,1$	

Table 2: The codeword is in the symmetric difference of the separated vertices. Here x and t are two integer parameters.

2), then it is enough to prove that each vertex (i,j) in the tile

$$\{(x,y) \mid 0 \le x < 2r, 0 \le y < 2r+2\} \cup \{(0,2r+2),(1,2r+2)\}$$

is separated from the vertex $(i,j-1)$. Table 2 shows that this is true except for the $r+1$ pairs

$$\{(2t,2t),(2t,2t+1)\}, \quad \text{for } t = 0,1,\dots,\frac{r-1}{2},$$

and

$$\{(r+2t+1,r+2t),(r+2t+1,r+2t+1)\}, \quad \text{for } t = 0,1,\dots,\frac{r-3}{2},$$

and

$$\{(0,2r),(0,2r+1)\}.$$

Moreover, exactly one of the vertices in each of these $r+1$ pairs is a codeword, and every codeword belongs to at most one of these pairs.

We now claim that a non-codeword (i,j) is separated from all other non-codewords in the same column. If $(i,j+1)$ is not in the code, then by the previous paragraph, (i,j) is separated from $(i,j+1)$ and therefore from all the non-codewords above (i,j), as we saw earlier. The same is true even if $(i,j+1) \in C$ unless $\{(i,j),(i,j+1)\}$ is one of the exceptional pairs listed above. But then $(i,j+2)$ is not in the code (by the structure of C) and $\{(i,j+1),(i,j+2)\}$ is not an exceptional pair (as each codeword is contained in at most one such pair). But then $(i, j+2)$ and $(i, j+1)$ are separated and by the argument proved earlier the same is true for $(i, j+2)$ and (i, j), and by referring to the same argument a second time, we see that (i, j) is separated from all non-codewords above it. This concludes the proof. $\square$

Remark 6. The codes in the proofs of Theorems 4 and 5 are also r-OLD codes, when $r \geq 2$. Indeed, $I_r(v) \setminus \{v\} = I_r(v)$ for any non-codeword v and by the previous proofs the sets $I_r(v)$ are distinct and non-empty for all non-codewords. Moreover, sets $I_r(c) \setminus \{c\}$ are non-empty for all codewords c. Then, the claim is true if $I_r(c) \setminus \{c\} \neq I_r(v) \setminus \{v\}$ for all $c \in C$ and for all $v \in V$.

Let $c = (a, b)$ be a codeword. Then either $c_1 = (a-2, b-2)$ or $c'_1 = (a-2, b-1)$ and either $c_2 = (a+2, b+2)$ or $c'_2 = (a+2, b+1)$ are also codewords for all codes in the proofs of Theorems 4 and 5. Now, $c_1, c'_1, c_2, c'_2 \in B_r(c)$. Then c_1 or c'_1 contains in $I_r(c) \setminus \{c\}$ and c_2 or c'_2 contains in $I_r(c) \setminus \{c\}$. Furthermore, if c_1 or c'_1 is in $B_r(v)$ and c_2 or c'_2 is in $B_r(v)$, then c also belongs to $B_r(v)$. Therefore, c is only vertex which has $c \notin I_r(v) \setminus \{v\}$, but $I_r(v) \setminus \{v\} \cap \{c_1, c'_1\} \neq \emptyset$ and $I_r(v) \setminus \{v\} \cap \{c_2, c'_2\} \neq \emptyset$.

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References

[1] I. Charon, I. Honkala, O. Hudry, A. Lobstein: General bounds for identifying codes in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 8(1), R39 (2001).

[2] I. Charon, I. Honkala, O. Hudry, A. Lobstein: The minimum density of an identifying code in the king lattice, Discrete Mathematics, Vol. 276, 95–109 (2004).

[3] I. Charon, O. Hudry, A. Lobstein: Identifying codes with small radius in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 9(1), R11 (2002).

[4] G. Cohen, I. Honkala, A. Lobstein, G. Zémor: On codes identifying vertices in the two-dimensional square lattice with diagonals, IEEE Transactions on Computers, Vol. 50, 174–176 (2001).

[5] I. Honkala, T. Laihonen: Codes for identification in the king lattice, Graphs and Combinatorics, Vol. 19, 505–516 (2003).

[6] I. Honkala, T. Laihonen: On a new class of identifying codes in graphs, Information Processing Letters, Vol. 102, 92–98 (2007).

[7] I. Honkala, T. Laihonen: On identifying codes in the triangular and square grids, SIAM Journal on Computing, Vol. 33, 304–312 (2004).

[8] I. Honkala, T. Laihonen: On locating-dominating sets in infinite grids, European Journal of Combinatorics, Vol. 27, 218–227 (2006).

[9] I. Honkala, T. Laihonen, S Ranto: On strongly identifying codes, Discrete Mathematics, Vol. 254, 191–205 (2002).

[10] M. G. Karpovsky, K. Chakrabarty, L. B. Levitin: On a new class of codes for identifying vertices in graphs, IEEE Transactions on Information Theory, Vol. IT-44, 599–611 (1998).

[11] M. Pelto: New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid, Cryptography and Communications, Vol. 2, 41–47 (2010).

[12] M. Pelto: On $(r, \leq 2)$-locating-dominating codes in the infinite king grid, submitted.

[13] M. Pelto: On locating-dominating codes for locating large numbers of vertices in the infinite king grid, Australasian Journal of Combinatorics, to appear.

[14] M. Pelto: Optimal $(r, \leq 3)$-locating-dominating codes in the infinite king grid, submitted.

[15] S. Ranto: Identifying and locating-dominating codes in binary Hamming spaces, Ph. D. Thesis, University of Turku, Finland, 95 pages, (2007).

[16] S. J. Seo, P. J. Slater: Open neighborhood locating dominating sets, Australasian Journal of Combinatorics, Vol. 46, 109–119 (2010).

[17] P. J. Slater: Domination and location in acyclic graphs, Networks, Vol. 17, 55–64 (1987).

[18] P. J. Slater: Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences, Vol. 22, 445–455 (1988).

[19] http://www.infres.enst.fr/˜lobstein/bibLOCDOMetID.html

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ON $(r, \leq 2)$-LOCATING-DOMINATING CODES IN THE INFINITE KING GRID

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Abstract. Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E. The ball $B_r(v)$ denotes the vertices within graphical distance r from v. Let $I_r(F) = \bigcup_{v \in F}(B_r(v) \cap C)$ be a set of codewords in the neighbourhoods of vertices $v \in F$. A subset $C \subseteq V$ is called an $(r, \leq l)$-locating-dominating code of type A if sets $I_r(F_1)$ and $I_r(F_2)$ are distinct for all subsets $F_1, F_2 \subseteq V$ where $F_1 \neq F_2$, $F_1 \cap C = F_2 \cap C$ and $|F_1|, |F_2| \leq l$. A subset $C \subseteq V$ is an $(r, \leq l)$-locating-dominating code of type B if the sets $I_r(F)$ are distinct for all subsets $F \subseteq V \setminus C$ with at most l vertices. We study $(r, \leq l)$-locating-dominating codes in the infinite king grid when $r \geq 1$ and $l = 2$. The infinite king grid is the graph with vertex set $\mathbb{Z}^2$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| \leq 1,\ |y_1 - y_2| \leq 1, (x_1, y_1) \neq (x_2, y_2)\}$.

1. Introduction

Let $G = (V, E)$ be an undirected graph where V is a vertex set and E an edge set. Denote by $d(u, v)$ the distance between two vertices u and v, i.e., the number of edges on any shortest path from u to v. The ball with center v and radius r is

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

We call any $C \subseteq V$ a *code*. The vertices of C are called *codewords*. We still define the set

$$I_r(F) = \left(\bigcup_{v \in F} B_r(v) \right) \cap C = B_r(F) \cap C.$$

Now, a code C is called an *$(r, \leq l)$-locating-dominating code of type A* if $I_r(F_1) \neq I_r(F_2)$ for all subsets $F_1 \subseteq V$ and $F_2 \subseteq V$ where $|F_1| \leq l$, $|F_2| \leq l$, $F_1 \neq F_2$ and $F_1 \cap C = F_2 \cap C$. Furthermore, C is an *$(r, \leq l)$-locating-dominating code of type B* if the sets $I_r(F)$ are distinct for all subsets $F \subseteq V \setminus C$ with at most l non-codewords. Moreover, we define an identifying code which is a closely related class of codes. A code C is an *$(r, \leq l)$-identifying code* if the sets $I_r(F)$ are distinct for all subsets $F \subseteq V$ with at most l vertices.

We call a locating-dominating code of type A (or type B) an *LDA code* (or an *LDB code*, resp.) for short. An identifying code is automatically an LDA code and

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an LDA code is automatically an LDB code. Moreover, if $l = 1$, then the definitions of LDA and LDB codes are equivalent.

We study codes in the infinite king grid. The infinite king grid is the graph where the vertex set $V = \mathbb{Z} \times \mathbb{Z}$ and two different vertices $u = (u_x, u_y)$ and $v = (v_x, v_y)$ are adjacent if $|u_x - v_x| \leq 1$ and $|u_y - v_y| \leq 1$. Thus vertices u and v are neighbours if the Euclidean distance between u and v is 1 or $\sqrt{2}$.

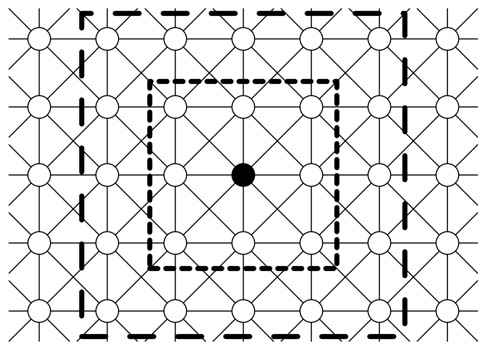

FIGURE 1. A part of the infinite king grid. The vertices within distance one and two from the black dot are surrounded by the dashed lines.

The density of $C \subseteq \mathbb{Z}^2$ is

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap B_n((0,0))|}{|B_n((0,0))|},$$

where $|C \cap B_n((0,0))|$ is the number of codewords in the ball $\{(x,y) \mid |x| \leq n, |y| \leq n\}$ and $|B_n((0,0))|$ is the number of all vertices in the ball. We also define $B_n((0,0)) = Q_n$. We search for the minimum density of locating-dominating codes of both types for given r and l in the infinite king grid.

Locating-dominating codes (of types A and B) were introduced in the late 1980s by Slater [15] and [16] and indentifying codes in the late 1990s by Karpovsky, Chakrabarty and Levitin [9] when $l = 1$. For general l LDB codes were introduced in [8] in the early 2000s. A motivation for such codes is a safeguard analysis of a facility [15] or a fault diagnosis of multiprocessor systems [9].

We study $(r, \leq 2)$-locating-dominating codes in the infinite king grid. Cases with $l \neq 2$ have been studied in papers [6, 11, 12, 13]. Identifying codes in the infinite king grid have been studied in [1, 2, 3, 4, 10]. More papers on locating-dominating and identifying codes in the infinite king grid and other graphs can be found in web bibliography [17].

In particular, we see that for large r the smallest densities of $(r, \leq 2)$-identifying codes are roughly $\frac{3}{2}$ times than the corresponding densities of $(r, \leq 2)$-LDA or $(r, \leq 2)$-LDB codes. Also, for small r the smallest densities of identifying codes are clearly larger than the densities of locating-dominating codes. Instead, the smallest densities of LDB codes when $l = 1$ are almost same as the smallest densities of identifying codes for large r, particularly. Also, the known bounds for the smallest possible densities of $(r, \leq 2)$-LDA codes are almost same as for $(r, \leq 2)$-LDB codes. See Table 1 which summarizes what is known about the density of $(r, \leq l)$-locating-dominating codes of types A and B and $(r, \leq l)$-identifying codes in the infinite

<table>
<tr><th>l</th><th>r</th><th>LDB codes</th><th>LDA codes</th><th>identifying codes</th></tr>
<tr><td rowspan="2">1</td><td>1</td><td colspan="2">$\frac{1}{5}$ [6]</td><td>$\frac{2}{9}$ [2, 3]</td></tr>
<tr><td>≥ 2</td><td colspan="2">$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ if $r \equiv 0 \pmod 2$
$\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$ if $r \equiv 1 \pmod 2$ [1, 12]</td><td>$\frac{1}{4r}$ [1]</td></tr>
<tr><td rowspan="3">2</td><td>1</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{5}{12} \leq D \leq \frac{3}{7}$ [4, 10]</td></tr>
<tr><td>2</td><td>$\frac{1}{6} \leq D \leq \frac{1}{4}$</td><td>$\frac{1}{5} \leq D \leq \frac{1}{4}$</td><td>$\frac{31}{120} \leq D \leq \frac{2}{7}$ [4, 10]</td></tr>
<tr><td>≥ 3</td><td colspan="2">$\frac{1}{6} \leq D \leq \frac{1}{6} + \frac{1}{12r+6}$ if $r \equiv 0, 2, 5 \pmod 6$
$\frac{1}{6} \leq D \leq \frac{1}{6} + \frac{2}{12r+6}$ if $r \equiv 1, 3, 4 \pmod 6$</td><td>$\frac{1}{4}$ [4]</td></tr>
</table>

TABLE 1. The known lower and upper bounds for the density of $(r, \leq l)$-LDA codes, $(r, \leq l)$-LDB codes, and $(r, \leq l)$-identifying codes when $l = 1$ or $l = 2$.

king grid when $l = 1$ or $l = 2$. The values of $(r, \leq 2)$-LDA and $(r, \leq 2)$-LDB codes are new and they are proved in this paper. The references of the other values is given in the table. The upper bound means that there exists such a code with that density and the lower bound means that density of every such code is at least the value given in the table.

2. LOWER BOUNDS

The next lemma gives a necessary but not sufficient condition for $(r, \leq 2)$-LDA and $(r, \leq 2)$-LDB codes.

Lemma 1. *Assume that C is an $(r, \leq 2)$-LDB code. Every translate (by integer vector) and rotation (by angle 0^o, 90^o, 180^o or 270^o about any vertex) of the set*

$$A_1 = \{(0,0), (0,1), (1,0), (1,1), (-r,-r), (-r, r+1), (r+1, -r), (r+1, r+1)\}$$

and of the set

$$A_2 = \{(0,-r), (0,-r+1), \ldots, (0,r); (r,0), (r+1,0)\}$$

contains at least one codeword of C.

Moreover, if C is an $(r, \leq 2)$-LDA code, then every translate and rotation of the set

$$A_3 = \{(0,-r), (0,-r+1), \ldots, (0,r); (r,0)\}$$

contains at least one codeword of C.

Proof. Assume first that C is an $(r, \leq 2)$-LDB code. First, if A_1 does not contain any codeword of C, then

$$(0,0), (1,1), (1,0), (0,1) \notin C$$

and

$$I_r((0,0),(1,1)) = I_r((1,0),(0,1)),$$

contradicting the fact that C is an $(r, \leq 2)$-LDB code. Second, if A_2 does not contain any codeword of C, then $(r+1, 0), (r, 0) \notin C$ and

$$I_r((r+1,0)) = I_r((r+1,0),(r,0))$$

which is again a contradiction.

Assume now that C is also an $(r, \leq 2)$-LDA code. If A_3 does not contain any codeword of C, then

$$I_r((r+1,0)) = I_r((r+1,0),(r,0))$$

and

$$\{(r+1,0)\} \cap C = \{(r+1,0),(r,0)\} \cap C.$$

By symmetry, every translate and rotation of the set A_1 or A_2 (and A_3, respectively) contains also a codeword in $(r, \leq 2)$-LDB codes (and $(r, \leq 2)$-LDA codes, respectively). □

In what follows, we also denote $A_1(x,y) = A_1 + (x,y)$. Furthermore, $A_2^{(k)}(x,y)$ (or $A_3^{(k)}(x,y)$) denotes the set A_2 (A_3, resp.) which has first been rotated by the angle $k \cdot 90^o$ counterclockwise about the origin and then translated by (x,y).

Lemma 2. *Let C be an $(r, \leq 2)$-LDA code. If the set*

$$A_4(x,y) = \{(x,y-r),\ldots,(x,y+r)\} \cup \{(x-r,y),\ldots,(x+r,y)\}$$

contains at most one codeword, then $(x,y) \in C$.

Proof. First, by Lemma 1 (about the sets $A_3^{(0)}(x,y)$), $A_4(x,y)$ contains at least one codeword. Then, we partition the set $A_4(x,y)$ into four branches ($\{(a,y) \mid a < x\}$, $\{(a,y) \mid a > x\}$, $\{(x,b) \mid b < x\}$ and $\{(x,b) \mid b > x\}$) and the center ($\{(x,y)\}$). Assume to the contrary that $A_4(x,y)$ contains only one codeword and that the center is a non-codeword. Without loss of generality we can further assume that $\{(a,y) | a < x\}$ is the only branch that contains a codeword. Now, the set $A_3^{(0)}(x,y)$ does not contain any codeword, which is a contradiction. □

Theorem 3. *The density of an $(r, \leq 2)$-LDA code is at least $\frac{1}{2r+1}$.*

Proof. (The idea of the proof has earlier been used in [7] Theorem 7.) Let C be an $(r, \leq 2)$-LDA code. Let now every codeword give two votes to itself and one vote to all vertices in the branches. Then a codeword gives $4r+2$ votes in all. On the other hand, every vertex gets at least two votes by Lemma 2. Hence the codewords in Q_n give in total $(4r+2)|C \cap Q_n|$ votes and every vertex in Q_{n-r} gets at least two votes from the codewords in Q_n. Then we have

$$(4r+2)|C \cap Q_n| \geq 2|Q_{n-r}| = 2(|Q_n| - 4r(2n-r+1))$$

and so the density of C is

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n\to\infty} \left(\frac{2}{4r+2} - \frac{8r(2n-r+1)}{(4r+2)|Q_n|} \right) = \frac{1}{2r+1}.$$

□

In particular, the previous theorem proves that the lower bounds for the densities of $(1, \leq 2)$-LDA and $(2, \leq 2)$-LDA codes are $\frac{1}{3}$ and $\frac{1}{5}$, respectively. Next, we show that the density of an $(r, \leq 2)$-LDB code is at least $\frac{1}{6}$ (Theorem 5). At the same time, we see that the density of an $(r, \leq 2)$-LDA code is at least $\frac{1}{6}$ since an $(r, \leq 2)$-LDA code is always an $(r, \leq 2)$-LDB code. This gives the better lower bound for $(r, \leq 2)$-LDA codes than Theorem 3 when $r \geq 3$. For LDB codes the lower bound $\frac{1}{6}$ is the best known lower bound also when $r = 2$.

Lemma 4. *Let C be an $(r, \leq 2)$-LDB code and*

$$A_5(x,y) = \{(x,y),(x,y+1),(x-r-1,y-r),(x-r-1,y+r+1),$$
$$(x+r,y-r),(x+r,y+r+1)\}.$$

If $A_5(x,y) \cap C = \emptyset$, then either there exists $t \in \mathbb{N}$ such that $A_5(x+t,y)$ contains at least two codewords and every set $A_5(x+i,y)$ contains a codeword for all $i = 1,\dots,t-1$ or every sets $A_5(x+i,y)$ contains exactly one codeword for all $i \in \mathbb{N}_+$.

Proof. By Lemma 1, $A_1(x,y) = A_5(x+1,y) \cup \{(x,y),(x,y+1)\}$ contains at least one codeword. Thus, $A_5(x+1,y)$ contains a codeword since $\{(x,y),(x,y+1)\} \subseteq A_5(x,y)$ does not contain any codeword by assumption.

Assume then that $A_5(x+i,y)$ contains exactly one codeword for all $i = 1,\dots,s-1$. Let k be an integer such that $(x+k,y+1) \notin C$ and $(x+k,y) \notin C$ and furthermore $(x+j,y) \in C$ or $(x+j,y+1) \in C$ for all $j = k+1,\dots,s-1$. Clearly, $k \geq 0$.

Assume first that $s-k > 2r$. Because $(x+k+r,y-r) \notin C$ or $(x+k+r,y+r+1) \notin C$, we can still assume without loss of generality that $(x+k+r,y-r) \notin C$. Now,

$$S = \{(x+k-r,y-r),\dots,(x+k+r,y-r)\} \cup \{(x+k,y),(x+k,y+1)\}$$

does not contain any codeword. Nevertheless, if C is an $(r, \leq 2)$-LDB code, S has to contain a codeword by Lemma 1.

Assume now that $s-k \leq 2r$. If $(x+s,y) \in C$ or $(x+s,y+1) \in C$, then $A_5(x+s,y) \cap C \neq \emptyset$. Assume then that neither of the two vertices is a codeword. Then, $\{(x+k,y),(x+s,y+1),(x+k,y+1),(x+s,y)\} \cap C = \emptyset$ and so the set

$$\begin{aligned} & I_r((x+k,y),(x+s,y+1)) \;\Delta\; I_r((x+k,y+1),(x+s,y)) \\ \subseteq\; & \{(x+k-r,y-r),\dots,(x+s-1-r,y-r)\} \\ & \cup \{(x+k-r,y+r),\dots,(x+s-1-r,y+r)\} \\ & \cup \{(x+k+1+r,y-r),\dots,(x+s+r,y-r)\} \\ & \cup \{(x+k+1+r,y+r),\dots,(x+s+r,y+r)\} \end{aligned}$$

has to contain at least one codeword. However, only the rightmost vertices of the four subsets can be codewords, by our assumptions. Thus $A_5(x+s,y) \cap C \neq \emptyset$. ☐

Theorem 5. *The density of an $(r, \leq 2)$-LDB code is at least $\frac{1}{6}$.*

Proof. Let $N(n)$ be the number of pairs (v,c) where $v \in \{(x,y) \mid -n+r < x \leq n-r \text{ and } -n+r \leq y < n-r\}$ and $c \in C \cap A_5(v)$. Then, $c \in Q_n$. Now, there are at least $2n-2r-1$ pairs for all $y \in [-n+r, n-r)$ and for given n by Lemma 4. Indeed, if there are two vertices (x_1,y) and (x_2,y) where $x_1 < x_2$ and $A_5(x_1,y) = A_5(x_2,y) = \emptyset$, then there has also to be a vertex (x_3,y) where $x_1 < x_3 < x_2$ and $|A_5(x_3,y)| \geq 2$. Thus, the number of pairs (v,c) is at least

$$\begin{aligned} N(n) &\geq (2n-2r-1)(2n-2r) \\ &= (2n+1)^2 - (2n+1)(4r+3) + 4r^2 + 6r + 2 \\ &= |Q_n| - (2n+1)(4r+3) + 4r^2 + 6r + 2. \end{aligned}$$

On the other hand, a codeword in Q_n can be the second element at most in six pairs. Then

$$6 \cdot |C \cap Q_n| \geq N(n) \geq |Q_n| - (2n+1)(4r+3) + 4r^2 + 6r + 2$$

and

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n\to\infty} \left(\frac{1}{6} - \frac{4r+3}{2n+1} + \frac{4r^2+6r+2}{(2n+1)^2} \right) = \frac{1}{6}.$$

□

We saw in Theorem 3 that the density of a $(1, \leq 2)$-LDA code is at least $\frac{1}{3}$. Now, we show that even the density of a $(1, \leq 2)$-LDB code is at least $\frac{1}{3}$.

Theorem 6. *The density of a $(1, \leq 2)$-LDB codes is at least $\frac{1}{3}$.*

Proof. First, we observe that every translate and rotation of the set

$$A_6 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(2,2)\}$$

contains at least one codeword. Otherwise, $I_r(\{(2,2)\}) = I_r(\{(1,1),(2,2)\})$.

Next, every codeword (x, y) gives one vote to itself and the vertices $(x, y-1)$ and $(x, y+1)$. We call this the *initial state.*

The vertices (x, y) without votes transfer one vote to itself from a nearby vertex by the following rules:

1. If the vertex $(x+1, y)$ has at least two votes, we transfer one vote from this vertex.
2. If the vertex $(x-1, y)$ is either a codeword with exactly two votes or a non-codeword with exactly one vote, then $(x+2, y)$ gives to (x, y) one vote.
3. Otherwise, we transfer one vote from $(x-1, y)$ to (x, y).

If we can transfer votes by more than one rule, we always use the rule with the smallest number.

Now, we show that after these rules have been applied every vertex has at least one vote.

First we show that a vertex (x, y) can not give votes both by the first and second rules. Indeed, if the vertex (x, y) gives votes by Rules 1 and 2, then vertices $(x-1, y)$ and $(x-2, y)$ do not have votes initially (i.e. $(x-1, y-1)$, $(x-1, y)$, $(x-1, y+1)$, $(x-2, y-1)$, $(x-2, y)$ and $(x-2, y+1)$ are non-codewords), but then $(x-3, y)$ must have three votes or else there is a set $A_2^{(2)}(x-1, y)$, $A_2^{(1)}(x-2, y-1)$ or $A_2^{(3)}(x-2, y+1)$ without codewords. Thus, $(x-2, y)$ does not get a vote from (x, y) by Rule 2. In particular, vertices can give at most two votes by Rules 1–3 and therefore vertices with three votes initially has at least one vote after they are applied.

Next, we assume that a vertex (x, y) with two votes gives one of them to the left by Rule 1 and so $(x-1, y)$ is a non-codeword. Now, if the vertex $(x+1, y)$ on the right also needs a vote (i.e. $(x+1, y-1)$, $(x+1, y)$ and $(x+1, y+1)$ are non-codewords), then (x, y) has to be a codeword or else there is a set $A_2^{(2)}(x+1, y)$ without codewords. Thus, $(x+1, y)$ gets a vote from the right by Rule 1 or 2 and therefore (x, y) does not give a vote by Rule 3. Hence, a codeword that gives a vote by Rule 1 has at least one vote after all voting rules have been applied.

Then we show that a vertex with exactly two votes initially can not give a vote by Rule 3 if it gives a vote by Rule 2. At the same time, we also observe that a vertex which gives votes by Rule 2 has at least two votes initially. Assume that vertex (x, y) gives a vote by Rule 2. Then $(x-2, y-1)$, $(x-2, y)$ and $(x-2, y+1)$ are non-codewords. Furthermore, either $(x-3, y-1)$ or $(x-3, y+1)$ is a codeword and the other is not. Without loss of generality, we assume $(x-3, y+1) \notin C$.

Now, $(x-1, y+1) \in C$ or else there is a set $A_2^{(3)}(x-2, y+1)$ without codewords. Moreover, $(x-1, y-1)$ and $(x-1, y)$ are not codewords because $(x-2, y)$ does not get a vote by Rule 1. Now, (x, y) and exactly one of the vertices of $(x, y-1)$ and $(x, y+1)$ must be in C or else there is a set $A_2^{(0)}(x-2, y)$ or $A_6 + (x-2, y-1)$ with no codewords. In particular, (x, y) has at least two votes initially. Thus, if $(x+1, y)$ needs a vote and (x, y) has exactly two votes initially, then $(x+1, y)$ gets a vote by Rule 1 or 2 on the right, but not from (x, y) by Rule 3. Now, we know that a vertex which gives a vote by Rule 2 has at least one vote after voting.

Finally, we have to show that the vertex which gives votes by Rule 3 has at least two votes initially. Let $(x+1, y)$ be a vertex that gets a vote from (x, y) by Rule 3. Then $(x+1, y-1)$, $(x+1, y)$ and $(x+1, y+1)$ are non-codewords. Moreover, $(x+2, y)$ has at most one vote or else $(x+1, y)$ would have got a vote by Rule 1. Now, at most one of the vertices $(x+2, y-1)$ and $(x+2, y+1)$ is a codeword. Assume that $(x+2, y+1)$ is not a codeword. Thus, $(x, y+1)$ has to be a codeword or else there is a set $A_2^{(3)}(x+1, y+1)$ without codewords. If (x, y) has only one vote, then $(x+1, y)$ gets a vote by Rule 2. Hence, (x, y) has at least two votes initially and at least one vote after voting.

Now the codewords in Q_n give $3 \cdot |C \cap Q_n|$ votes in all and every vertices in Q_{n-2} has got at least one vote from the codewords in Q_n. We therefore have

$$3 \cdot |C \cap Q_n| \geq |Q_{n-2}| = |Q_n| - 16n + 8$$

i.e.

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n \to \infty} \left(\frac{1}{3} - \frac{16n - 8}{3 \cdot |Q_n|} \right) = \frac{1}{3}.$$

□

3. Constructions

Theorem 7. *There exists*

- *a $(1, \leq 2)$-LDA code with density $\frac{1}{3}$ and*
- *a $(2, \leq 2)$-LDA code with density $\frac{1}{4}$.*

Proof. The code

$$C_1 = \{ (i, j) \mid i \equiv j \pmod 3 \}$$

is such a $(1, \leq 2)$-LDA code and

$$C_2 = \{ (i, j) \mid i - j \equiv 0 \text{ or } 3 \pmod 8 \}$$

is such a $(2, \leq 2)$-LDA code. See Figures 2 and 3. □

Theorem 8. *Assume that $r \geq 3$. There exists an $(r, \leq 2)$-locating-dominating code of type A with density*

$$\begin{cases} \frac{r+1}{6r+3} = \frac{1}{6} + \frac{1}{12r+6} & \text{if } r \equiv 0, 2, 5 \pmod 6, \\ \frac{2r+3}{12r+6} = \frac{1}{6} + \frac{1}{6r+3} & \text{if } r \equiv 1, 3, 4 \pmod 6. \end{cases}$$

In particular, for all $\varepsilon > 0$ there is an $(r, \leq 2)$-LDA code with density $\frac{1}{6} + \varepsilon$ for all large enough values of r.

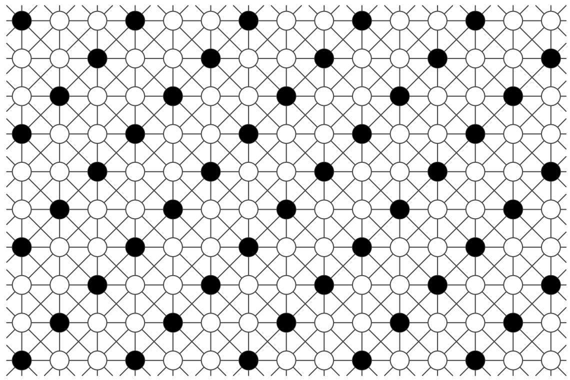

FIGURE 2. A $(1, \leq 2)$-LDA code.

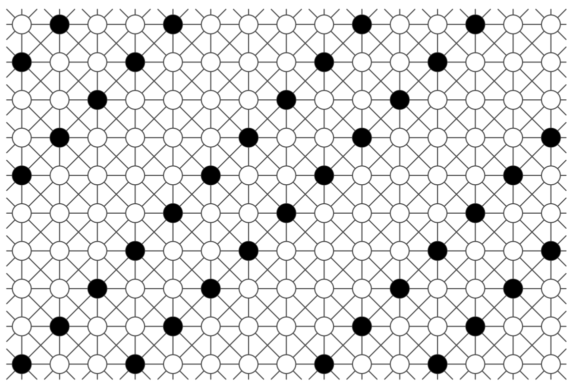

FIGURE 3. A $(2, \leq 2)$-LDA code.

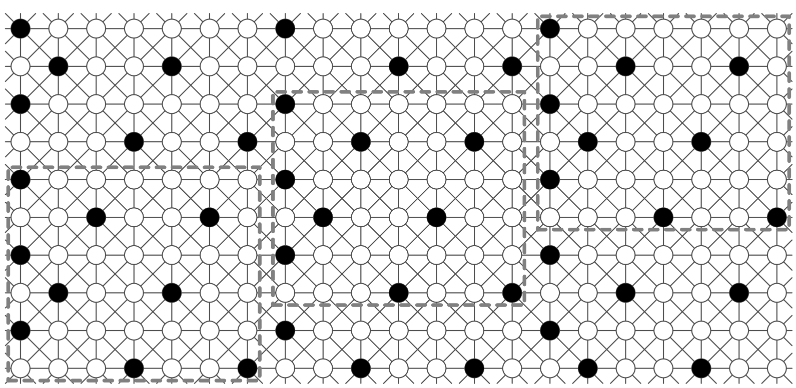

FIGURE 4. A $(3, \leq 2)$-LDA code.

Proof. Define

$$\begin{aligned} C_3 &= \{(i,j) \mid 2i - j \equiv 0 \pmod 6 \text{ and } i \not\equiv 0 \pmod{2r+1}\} \\ &\quad \cup \{(i,j) \mid i \equiv 0 \pmod{2r+1} \text{ and } j \equiv 1 \pmod 2\}, \end{aligned}$$

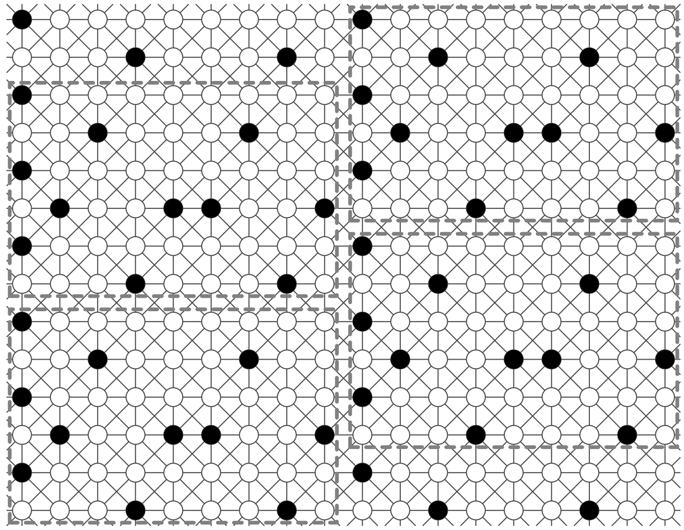

FIGURE 5. A $(4, \leq 2)$-LDA code.

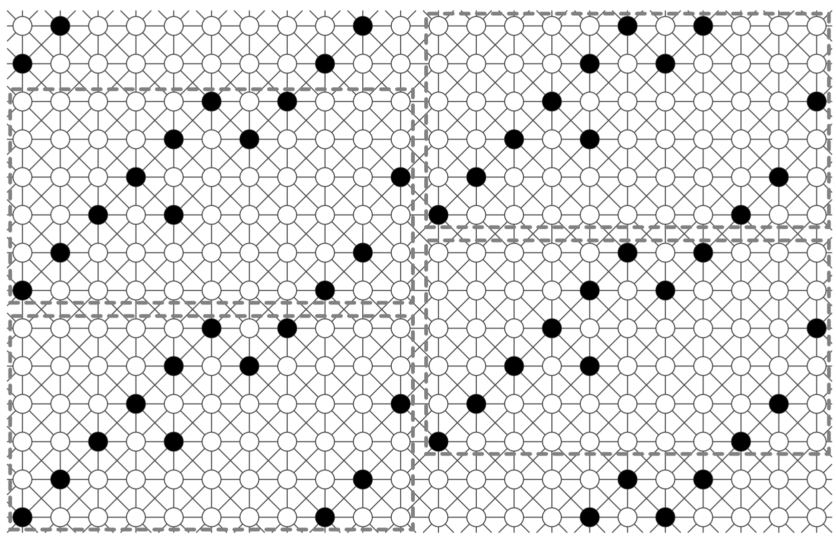

FIGURE 6. A $(5, \leq 2)$-LDA code.

$$C_4 = \left\{(i,j) \;\middle|\; 2i - j \equiv 2\left\lfloor \frac{i}{r+\frac{1}{2}} \right\rfloor \pmod 6 \quad \text{and} \quad i \not\equiv 0 \pmod{2r+1}\right\}$$
$$\cup \{(i,j) \mid i \equiv 0 \pmod{2r+1} \quad \text{and} \quad j \equiv 1 \pmod 2\},$$

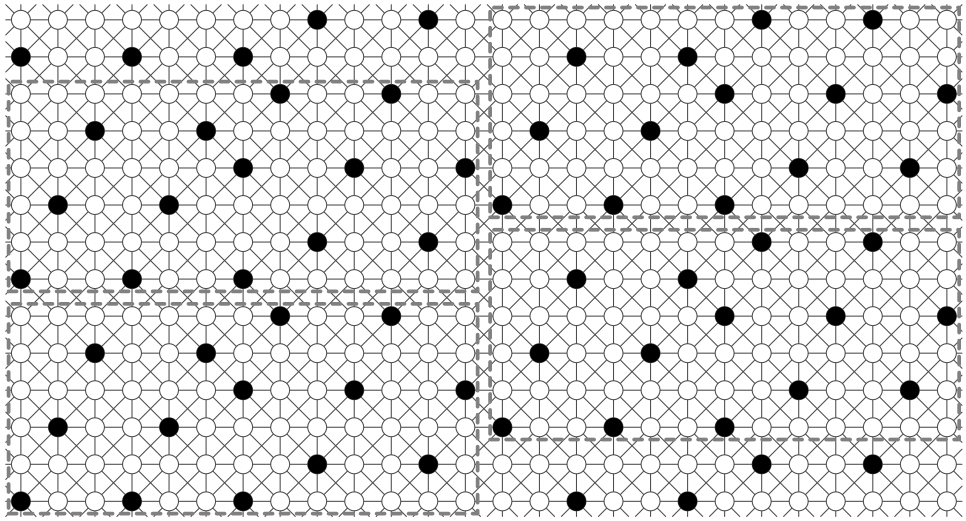

FIGURE 7. A $(6,\leq 2)$-LDA code.

$$\begin{aligned} C_5 \;=\; & \left\{ (i,j) \;\middle|\; i-j \equiv 3\left\lfloor \frac{i}{2r+1} \right\rfloor \pmod 6,\ i \equiv 0,\dots,r \pmod{2r+1} \right\} \\ & \cup \left\{ (i,j) \;\middle|\; i-j \equiv 1+3\left\lfloor \frac{i}{2r+1} \right\rfloor \pmod 6,\ i \equiv 2r \pmod{2r+1} \right\} \\ & \cup \left\{ (i,j) \;\middle|\; i-j \equiv 2+3\left\lfloor \frac{i}{2r+1} \right\rfloor \pmod 6, \right. \\ & \qquad\qquad \left. i \equiv r-1; r+1, r+2, \dots, 2r-1 \pmod{2r+1} \right\} \end{aligned}$$

and

$$\begin{aligned} C_6 \;=\; & \{\, (i,j) \;\mid\; 2i-j \equiv 0 \pmod 6, \quad i \equiv 0,1,\dots,r \pmod{2r+1} \,\} \\ & \cup \{\, (i,j) \;\mid\; 2i-j \equiv 3 \pmod 6, \quad i \equiv r,\dots,2r \pmod{2r+1} \,\}. \end{aligned}$$

We show that C_6, (C_3, C_4 and C_5, resp.) is an $(r,\leq 2)$-LDA code when $r \equiv 0 \pmod 6$ ($r \equiv 3 \pmod 6$, $r \equiv 1 \pmod 3$ and $r \equiv 2 \pmod 3$, resp.). Furthermore, we see that the densities of the codes are as claimed. Parts of the codes when $r = 3, 4, 5$ or 6 are shown in Figures 4, 5, 6 and 7, respectively.

First, we observe that every column has period six and at least one vertex of each period is a codeword. Moreover, at least one of $2r+1$ consecutive vertices is a codeword in every row. Consequently, if $B_r(u)$ contains at least one vertex from a certain row or column, then so does $I_r(u)$. We can therefore easily see, which row (or column) is the lowest (or highest, leftmost or rightmost) row (or column) that $B_r(U)$ has a non-empty intersection with.

Thus, we can draw a rectangle such that all vertices in the given set U are contained in the rectangle and the size of the rectangle is minimal. Then there is a vertex of U on every side of the rectangle. Because we assume that the size of U is at most two, then the vertices in U are in the opposite corners of the rectangle. Now, we have at most two choices for set U. If the vertices of the rectangle are in the same row or column, then there is only one choice for U and we can identify

U. Assume then that the opposite corners of the rectangle are in different rows and different columns.

Next, we observe that if some corner of the rectangle is a codeword or there is a codeword within distance r from exactly one of the corners, then we know which corners are in U. Furthermore, if we can draw a smaller rectangle inside the rectangle and if there is a codeword within distance r from exactly one of the corners for the smaller rectangle, then an analogous codeword exists for the original rectangle. Indeed, the same codeword (which is within r from exactly one corner of the smaller rectangle) is within distance r from exactly one corner of the larger rectangle or else the length or the height of the larger rectangle is at least $2r+2$ and so there are $2r+1$ consecutive vertices (in the same row or in the same column) within distance r from only one and the same corner and at least one of the vertices is a codeword.

Now, we show that every rectangle with the lower left corner in some six consecutive values of y satisfies the condition (given in the first sentence in the previous paragraph) when $x = 0, 1, \dots, 2r$ and r is divisible by 6. By symmetry, this proves the claim when 6 divides r and the other cases are proved in the same way. First, we see if $(x-r, y-r)$ is a codeword, then also $(x-r, y+r)$, $(x+r+1, y-r+2)$ and $(x+r+1, y+r+2)$ are codewords. Thus rectangles with (x, y) as the lower left corner (resp., (x, y) as the upper left corner, $(x+1, y+2)$ as the lower right corner and $(x+1, y+2)$ as the upper right corner) satisfy the condition, since the lower left (upper left, lower right and upper right, resp.) corner of the rectangle is the only corner within distance r from the codeword $(x-r, y-r)$ ($(x-r, y+r)$, $(x+r+1, y-r+2)$ and $(x+r+1, y+r+2)$, resp.). In particular, rectangles with lower left corner in the set $\{(x, y+b) \mid b = -1, 0, 1 \text{ and } 2\}$ satisfy the condition if $(x-r, y-r)$ is a codeword.

We first study rectancles with lower left corner in columns where $x = 0, 1, \dots, r-2$. Now, $(x, 2x)$ is a codeword then rectangles with $(x, 2x)$ as the lower left corner satisfy the condition. Let then $(x, 2x+1)$ be the lower left corner of the rectangle. If the height (or width, resp.) of the rectangle is at least two, then the vertex in the upper left (or lower right, resp.) corner is the only corner within graphical distance r from codeword $(x-r, 2x+r+3)$ (or $(x+r+2, 2x-r+1)$, resp.). Otherwise, the height and width are one and then the upper right corner $(x+1, 2x+2)$ is a codeword. Also, rectangles with lower left corner in the set $\{(x, 2x+b) \mid b = 2, 3, 4 \text{ or } 5\}$ satisfy the condition by the observation made in the previous paragraph because $(x-r, 2x+3-r)$ is a codeword.

In the same way, rectangles with lower left corner in columns where $x = r, r+1, \dots, 2r-1$ satisfy the condition. Indeed, rectangles with lower left corner in the set $\{(x, 2x+b) \mid b = -1, 0, 1, 2 \text{ or } 3\}$ satisfy the condition since $(x-r, 2x-r)$ and $(x, 2x+3)$ are codewords. Moreover, the rectangles with lower left corner in $(x, 2x-2)$ satisfy the condition because $(x+1, 2x-1)$, $(x-r, 2x+r)$ and $(x+r+2, 2x-2-r)$ are codewords.

Also, rectangles with lower left corner in $(2r, y)$, $y \in \mathbb{Z}$, satisfy the condition. Indeed, $(r, y) \in C$ for all y divisible by 3. Then, by our previous observation, every rectangle has a codeword that is within distance r from exactly one corner.

Finally, we assume that the lower left corner of the rectangle is in the set $\{(r-1, b) \mid b = 0, 1, \dots, 5\}$. Again, rectangles with lower left corner in the set $\{(r-1, 2(r-1)+b) \mid b = 0, 2, 3, 4 \text{ or } 5\}$ satisfy the condition since $(r-1, 2r-2)$ and $(-1, r+1)$ are codewords. Assume then that $(r-1, 2r-1)$ is the lower left corner. Now, if the width and height of the rectangle is one, the upper right corner is a codeword and the rectangle satisfies the condition. Otherwise, the width or height is at least two and the rectangles satisfy the condition since $(0, 3r)$ or $(-1, 3r+1)$ is a codeword within distance r only from the upper left corner.

Hence, C_6 is an $(r, \leq 2)$-LDA code when r is divisible by 6. □

REFERENCES

[1] I. Charon, I. Honkala, O. Hudry and A. Lobstein, *The minimum density of an identifying code in the king lattice*, Discrete Math., **276** (2004), 95–109.

[2] I. Charon, O. Hudry and A. Lobstein, *Identifying codes with small radius in some infinite regular graphs*, Electronic J. Combin., **9** (2002), R11.

[3] G. Cohen, I. Honkala, A. Lobstein and G. Zémor, *On codes identifying vertices in the two-dimensional square lattice with diagonals*, IEEE Trans. Comput., **50** (2001), 174–176.

[4] I. Honkala and T. Laihonen, *Codes for identification in the king lattice*, Graphs Combin., **19** (2003), 505–516.

[5] I. Honkala and T. Laihonen, *On identifying codes in the triangular and square grids*, SIAM J. Comput., **33** (2004), 304–312.

[6] I. Honkala and T. Laihonen, *On locating-dominating sets in infinite grids*, European J. Combin., **27** (2006), 218–227.

[7] I. Honkala and T. Laihonen, *On a new class of identifying codes in graphs*, Inform. Proc. Letters, **102** (2007), 92–98.

[8] I. Honkala, T. Laihonen and S. Ranto, *On locating-dominating codes in binary Hamming spaces*, Discrete Math. Theor. Comput. Sci., **6** (2004), 265–282,.

[9] M. G. Karpovsky, K. Chakrabarty and L. B. Levitin, *On a new class of codes for identifying vertices in graphs*, IEEE Trans. Informa. Theory, **IT-44** (1998), 599–611.

[10] M. Pelto, *New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid*, Crypt. Commun., **2** (2010), 41–47.

[11] M. Pelto, *On locating-dominating codes for locating large numbers of vertices in the infinite king grid*, Australas. J. Combin., **50** (2011), 127–139,.

[12] M. Pelto, *On locating-dominating codes in the infinite king grid*, Ars Combin., to appear.

[13] M. Pelto, *Optimal $(r, \leq 3)$-locating-dominating codes in the infinite king grid*, submitted.

[14] S. Ranto, "Identifying and Locating-Dominating Codes in Binary Hamming Spaces," Ph.D thesis, University of Turku, Finland, 2007.

[15] P. J. Slater, *Domination and location in acyclic graphs*, Networks, **17** (1987), 55–64.

[16] P. J. Slater, *Dominating and reference sets in a graph*, J. Math. Phys. Sci., **22** (1988), 445–455.

[17] `http://www.infres.enst.fr/~lobstein/bibLOCDOMetID.html`

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Publication V

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Optimal $(r, \leq 3)$-locating-dominating codes in the infinite king grid

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Abstract

Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E. The ball $B_r(v)$ denotes the vertices within graphical distance r from v. A subset $C \subseteq V$ is called an $(r, \leq l)$-locating-dominating code of type B if the sets $I_r(F) = \bigcup_{v \in F}(B_r(v) \cap C)$ are distinct for all subsets $F \subseteq V \setminus C$ with at most l vertices. We give examples of optimal $(r, \leq 3)$-locating-dominating codes of type B in the infinite king grid for all $r \in \mathbb{N}_+$ and prove optimality. The infinite king grid is the graph with vertex set $\mathbb{Z}^2$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| \leq 1,\ |y_1 - y_2| \leq 1\}$.

Keywords

locating-dominating code, king grid, graph, density

1 Introduction

Let $G = (V, E)$ be an undirected graph where V is the vertex set and E the edge set. The ball with center v and radius r is

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}$$

where $d(u, v)$ is the distance between two vertices u and v i.e. the number of edges on any shortest path from u to v.

We call any $C \subseteq V$ a *code.* The vertices of C are called *codewords.* In particular, C is an $(r, \leq l)$*-locating-dominating code of type* B or $(r, \leq l)$*-LDB code* for short if the *identifying sets*

$$I_r(F) = \left(\bigcup_{v \in F} B_r(v) \right) \cap C = B_r(F) \cap C$$

are distinct for all subsets $F \subseteq V \setminus C$ with at most l non-codewords. The set $B_r(F)$ is called the *r-neighbourhood* of F. The identifying set of F is always a subset of the r-neighbourhood of F.

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We study codes in the infinite king grid which has vertex set $V = \mathbb{Z}^2$ and edge set

$$E = \{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| \leq 1,\ |y_1 - y_2| \leq 1\}.$$

Therefore, vertices u and v are neighbours if the Euclidean distance between u and v is 1 or $\sqrt{2}$. The *density* of code C is defined as

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|}$$

where $Q_n = B_n((0,0)) = \{(x,y) \mid -n \leq x \leq n \text{ and } -n \leq y \leq n\}$. In particular, we say that $(r, \leq l)$-LDB code is *optimal* if there is no $(r, \leq l)$-LDB code with smaller density.

Moreover, we say that vertices $u_1, u_2, \ldots, u_k$ are *consecutive* in the king grid if all of them have the same x-coordinate (or y-coordinate) and the y-coordinates (or x-coordinates, resp.) are consecutive. Finally, we define that $v_1, v_2, \ldots, v_k$ are k *successive non-codeword neighbours* of (x, y) if they are k successive non-codewords of cycle

$$(x+1, y), (x+1, y+1), (x, y+1), (x-1, y+1),$$
$$(x-1, y), (x-1, y-1), (x, y-1), (x+1, y-1).$$

Locating-dominating codes were introduced in the 1980s in [18, 19] when $l = 1$ and general case of LDB codes in [11]. A motivation of locating-dominating codes is a safeguard analysis of a facility using sensor networks [18] or a fault diagnosis of a multiprocessor system [12]. Assume that we have a multiprocessor system. Some processors are chosen to perform the task of testing if the processor itself is faulty or if there is a faulty processor within distance r. The chosen processor sends the symbol 2, if the processor itself is faulty; symbol 1, if it itself is not faulty, but there is a faulty processor within distance r; and symbol 0, otherwise. Finally, we get the reports from all the chosen processors and based on the reports alone we can perform some kind of a fault diagnosis.

Here, processors are vertices and the chosen processors are codewords. If the chosen processors form an $(r, \leq l)$-locating-dominating code, then we can locate processors which are faulty if we assume that there are at most l faulty processors. LDB codes can be used in two-step fault diagnosis: in the first step, we test if chosen processors are faulty and, in the second step, we test other processors. The so-called *locating-dominating codes of type A* or *LDA codes* for short can be used in one-step fault diagnosis.

LDB and LDA codes have also been studied in the infinite king grid in [10, 14, 15, 16]. Moreover locating-dominating codes and closely related classes of codes have been studied in the infinite king grid and many other graphs in [1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 17]. More papers on such codes can be found in [20]. In this paper, we study $(r, \leq 3)$-LDB codes and show that the density of optimal $(r, \leq 3)$-LDB codes is $\frac{3}{5}$ if $r = 1$ and $\frac{r}{r+1}$ if $r > 1$. Table 1 summarizes what is now known about the density of $(r, \leq l)$-LDB codes in the infinite king grid. Here, the upper bound means that there exists an LDB code with that density and the lower bound means that density of every LDB code is at least the value given in the table. If the table entry consists of a single value, then these lower and upper bounds coincide.

	$r=1$		$r=2$		$r \geq 3$	
$l=1$	$\frac{1}{5}$	[10]	$\frac{1}{10} \leq D \leq \frac{1}{8}$	[6, 14]	$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ if $2 \mid r$ $\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$ if $2 \nmid r$	[5, 14]
$l=2$	$\frac{1}{3}$	[15]	$\frac{1}{6} \leq D \leq \frac{1}{4}$	[15]	$\frac{1}{6} \leq D \leq \frac{r+1}{6r+3}$ if $r \equiv 0,2,5 \ (6)$ $\frac{1}{6} \leq D \leq \frac{2r+3}{12r+6}$ if $r \equiv 1,3,4 \ (6)$	[15]
$l=3$	$\frac{3}{5}$		$\frac{2}{3}$		$\frac{r}{r+1}$	
$4 \leq l \leq 4r$	$\frac{2}{3}$	[16]	$\frac{2}{3} \leq D \leq \frac{4}{5}$	[16]	$\frac{r}{r+1} \leq D \leq \frac{2r}{2r+1}$	[16]
$l > 4r$	$\frac{2}{3}$	[16]	$\frac{4}{5}$	[16]	$\frac{2r}{2r+1}$	[16]

Table 1: The known lower and upper bounds for the density of LDB codes.

2 Lower bounds

Lemma 1. *If C is an $(r, \leq 3)$-LDB code, then at most two of $2r+2$ consecutive vertices can be non-codewords.*

Proof. Assume to the contrary that (x,y), $(x+a,y)$ and $(x+b,y)$ (or (x,y), $(x,y+a)$ and $(x,y+b)$, resp.) where $0 < a < b < 2r+2$ are all non-codewords. Since

$$B_r(x+a,y) \subseteq B_r(\{(x,y),(x+b,y)\}),$$

we have

$$I_r\left(\{(x,y),(x+b,y)\}\right) = I_r\left(\{(x,y),(x+a,y),(x+b,y)\}\right),$$

which is a contradiction. □

Corollary 2. *The density of an $(r, \leq 3)$-LDB code is at least $\frac{r}{r+1}$.*

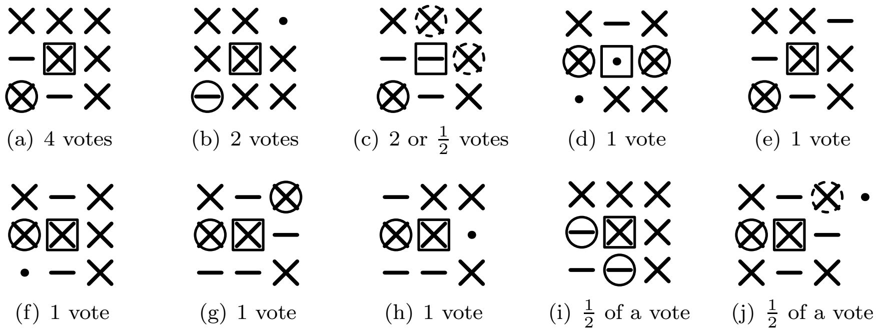
Theorem 3. *The density of a $(1, \leq 3)$-LDB code is at least $\frac{3}{5}$.*

Proof. Assume that C is a $(1, \leq 3)$-LDB code. We use in this proof a voting method. It can also be called a discharging method.

First, every codeword gives one vote to itself and three votes for all its neighbours. Then every codeword gives 25 votes in all. This is the initial state.

Next, vertices give votes to neighbours as in Figure 1. Also, vertices in rotations and reflections of these figures give votes in the same way. Figure 1 and its caption tell how many votes neighbours get. We shall show that after these voting rules every vertex has at least 15 votes. This proves the claim.

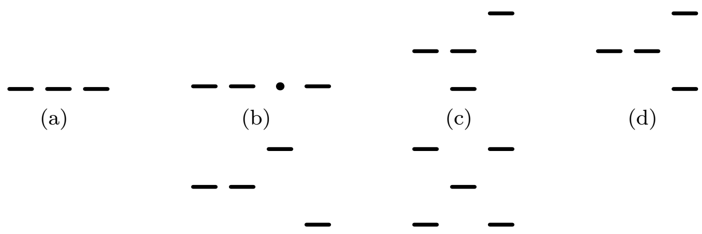
First, we observe that there are six forbidden patterns, which can not be in any $(1, \leq 3)$-LDB code; see Figure 2. Indeed, Lemma 1 explains the two first patterns. The highest non-codeword in pattern 2(c), 2(d) or 2(e) is the only vertex which is in the 1-neighbourhood of the set consisting of three other non-codewords, but not in the 1-neighbourhood of the set consisting of the lowest and the leftmost non-codewords. Therefore the identifying sets of these two non-codeword sets are the same. In the same way, the 1-neighbourhood of the lower left, upper right and the center non-codewords of constellation 2(f) contains



(a) 4 votes (b) 2 votes (c) 2 or $\frac{1}{2}$ votes (d) 1 vote (e) 1 vote

(f) 1 vote (g) 1 vote (h) 1 vote (i) $\frac{1}{2}$ of a vote (j) $\frac{1}{2}$ of a vote

Figure 1: The squared vertices give 4, 2, 1 or $\frac{1}{2}$ votes to all circled vertices. The non-codeword in constellation 1(c) gives two votes to the codeword surrounded by the solid circle and $\frac{1}{2}$ of a vote to both the codewords surrounded by the dashed circle. Moreover, the codeword surrounded dashed circle in constellation 1(j) gets $\frac{1}{2}$ of a vote only if the black dot is a non-codeword. The crosses are codewords and the lines are non-codewords. The black dots may be codewords or non-codewords.



(a) (b) (c) (d)

(e) (f)

Figure 2: Forbidden patterns

only two vertices (but no codewords) which are not the 1-neighbourhood of the lower left and the upper right non-codewords.

Step 1. We shall show that vertices which gives votes by rules 1(a)–1(j) have at least 15 votes finally. First, we observe that codewords can gives votes by only one of the rules. Furthermore, vertices except constellations 1(b) and 1(f) can give votes only one rotation or reflection. (In particular, the lower right corner in constellation 1(j) does not get votes from the squared codeword if the black dot is a non-codeword since the vertex to the rigth from the lower right corner must be a codeword or else there would be forbidden pattern 2(f).) 1(b) also gives votes its rotation if the black dot is a non-codeword and 1(f) gives a vote its reflection if the black dot is a codeword. In any case, vertices except 1(g) (and 1(h) if the black dot is a non-codeword) get enough votes initially, so they have at least 15 votes after all voting rules.

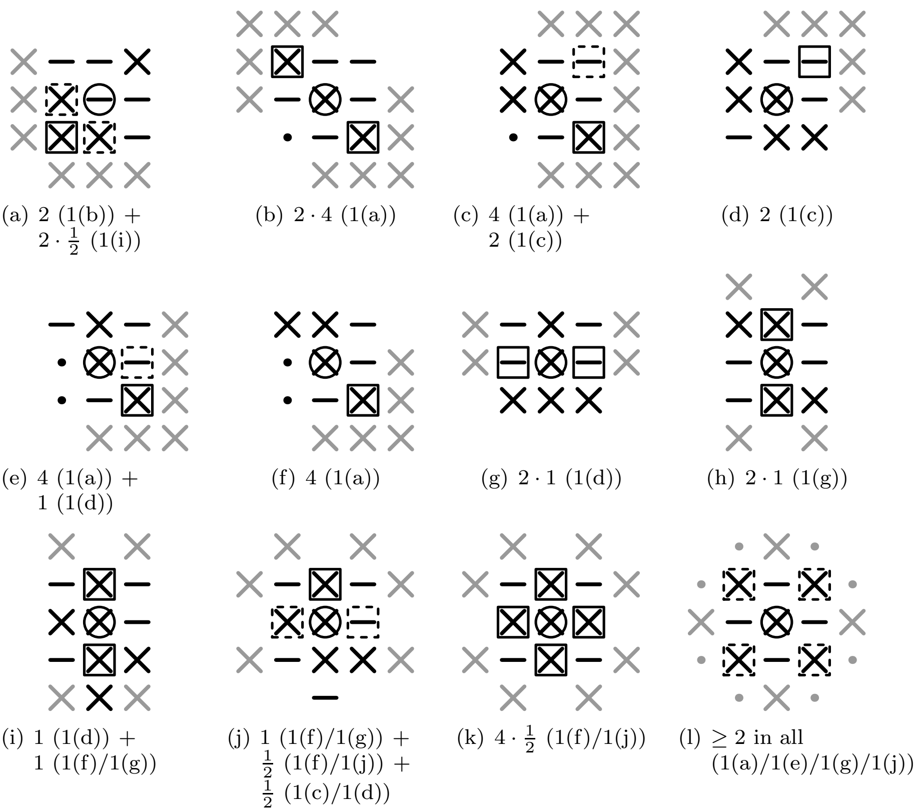
The squared codeword in 1(g) (or 1(h) if the black dot is a non-codeword, resp.) gets 13 votes initially ($4 \cdot 3$ votes from codeword neighbours and one vote from itself). Furthermore all the neighbours of the lower right corner which are not neighbours of the squared codeword must be codewords. Indeed, if any of these vertices were a non-codeword, then there would be a forbidden pattern 2(c), 2(d), 2(e), 2(b), or 2(d). Now, the lower left corner is of type 1(a) and it gives four votes to the squared codeword in 1(g), and therefore the squared codeword gets at least 17 votes and it gives two votes.

Step 2. Prove the claim for non-codewords. Let v be a non-codeword, say $v = (0, 0)$. First, v gets enough votes if it has at least five codeword neighbours. Indeed, then v gets at least 15 votes initially and, by Step 1, we can assume that v does not give votes by Figure 1. Assume that this is not the case, i.e., at least four of its neighbours are non-codewords. If all the vertices at Euclidean distance one were codewords, then all vertices at Euclidean distance $\sqrt{2}$ were non-codewords, and then there would be forbidden pattern 2(f). Thus, we can assume that $(1, 0)$ is a non-codeword. Now $(-1, 0)$ has to be a codeword since there can not be three consecutive non-codewords (forbidden pattern 2(a)).

Next, we observe that only one of $(1, 1)$ and $(1, -1)$ can be a non-codeword or else there are again three consecutive non-codewords. Also, only one of $(-1, 1)$ and $(-1, -1)$ can be a non-codeword since otherwise there is a forbidden pattern 2(d). Then, at least one of $(0, 1)$ and $(0, -1)$ has to be a non-codeword because at least four neighbours of $(0, 0)$ are non-codewords. Without loss of generality we can assume that $(0, 1)$ is a non-codeword. Now, $(0, -1)$ and $(-1, -1)$ are codewords or else there is forbidden pattern 2(a) or 2(c). Now, we see that the only way the non-codewords can be situated is that the other non-codewords are $(1, -1)$ and $(-1, 1)$. See Figure 3(a) (where v is the circled non-codeword).

Now, $(-2, 1)$, $(-2, 0)$, $(0, -2)$, and $(1, -2)$ all have to be codewords or else there would be a forbidden pattern 2(a) or 2(b). Also $(-2, -1)$ and $(-1, -2)$ must be codewords; otherwise, there is a forbidden pattern 2(e). We now see that $(-1, -1)$ gives two votes by 1(b) and $(-1, 0)$ and $(0, -1)$ give $\frac{1}{2}$ of a vote each to v by 1(i). Thus, v gets 12 votes initially and 3 votes by 1(b) and 1(i), therefore at least 15 votes in all.

Step 3. In this step we shall see that every codeword also has at least 15 votes after both voting steps. So assume that c is a codeword and it is situated in the origin. If c has at least five codeword neighbours, the claim is clear since, again by Step 1, we can assume that c does not give votes by Figure 1. So we assume that c has at most four codeword neighbours. The proof contains many



(a) 2 (1(b)) + $2 \cdot \frac{1}{2}$ (1(i))

(b) $2 \cdot 4$ (1(a))

(c) 4 (1(a)) + 2 (1(c))

(d) 2 (1(c))

(e) 4 (1(a)) + 1 (1(d))

(f) 4 (1(a))

(g) $2 \cdot 1$ (1(d))

(h) $2 \cdot 1$ (1(g))

(i) 1 (1(d)) + 1 (1(f)/1(g))

(j) 1 (1(f)/1(g)) + $\frac{1}{2}$ (1(f)/1(j)) + $\frac{1}{2}$ (1(c)/1(d))

(k) $4 \cdot \frac{1}{2}$ (1(f)/1(j))

(l) ≥ 2 in all (1(a)/1(e)/1(g)/1(j))

Figure 3: The circled codeword (except 3(a)) gets one vote from itself and three votes from all codeword neighbours. In addition, the circled vertex gets extra votes from squared vertices by rules in Figure 1. The amount of extra votes is given in the caption of subfigure. The vertices with continuous square give more extra votes than the vertices with dashed square.

cases; see Figure 3. We shall observe that grey crosses must be codewords or else there would be a forbidden pattern 2(a)–2(e).

First, we assume that there are three successive non-codeword neighbours of c. Without loss of generality, we can assume that these non-codewords are $(0,1)$, $(1,1)$, and $(1,0)$ (since three non-codewords can not be consecutive). Then $(-1,1)$ and $(1,-1)$ have to be codewords (to avoid forbidden pattern 2(a)). Now, we divide the case into three subcases depending on if $(-1,0)$ and $(0,-1)$ are codewords or non-codewords. See Figures 3(b)–3(d). In the final subcase, $(-1,-1)$ must be a non-codeword since we assume that the neighbourhood of c contains at most five codewords including codeword c.

Second, we assume that there are two (but not three) succesive non-codeword neighbours of c. Again, we can without loss of generality assume that $(1,0)$ and $(1,1)$ are non-codewords and $(0,1)$ (and also $(1,-1)$) is a codeword. First, we assume that also $(0,-1)$ is a non-codeword. Now there are two subcases depending on whether $(-1,1)$ is a codeword or not (Figures 3(e) and 3(f)). Observe that at least one of the black dots has to be a codeword or else there would be three successive non-codeword neighbours of c.

Next we assume that $(0,-1)$ is a codeword. Assume that also $(-1,0)$ is a non-codeword. Then there are two subcases depending on if $(-1,-1)$ or $(-1,1)$ is a codeword (Figures 3(g) and 3(h)). Indeed, exactly one of these has to be a codeword since there can not be three consecutive non-codewords and we have assumed that c has at most four codeword neighbours. Finally, we assume that $(-1,0)$ is a codeword. Then $(-1,-1)$ and $(-1,1)$ must be non-codewords or c has more than four codeword neighbours. However, we have two subcases depending on whether $(0,-2)$ is a codeword or not. See Figures 3(i) and 3(j).

Third, we assume that there are no two successive non-codeword neighbours of c. Now, there are only two ways how the non-codewords can be situated if c has at most four codeword neighbours. See Figures 3(k) and 3(l). The first of these two cases is clear. In Figure 3(l), at least one of the grey dots below (left, right and top, respectively) must be a codeword. Otherwise, there would be a forbidden pattern 2(f). Hence, at least four of the grey dots have to be codewords. On the other hand, if both grey dots $(1,2)$ and $(2,1)$ are codewords, then $(1,1)$ gives at least one vote to c by Figure 1(a) or 1(e). If exactly one of $(1,2)$ and $(2,1)$ is a codeword, then $(1,1)$ gives at least $\frac{1}{2}$ of a vote to c by Figure 1(g) or 1(j). Thus, the squared codewords give at least $\frac{1}{2}$ of a vote per grey dot which is a codeword, i.e., at least two votes in all.

Hence, we have shown that codewords in Q_n give $25 \cdot |C \cap Q_n|$ votes. Moreover, every vertex gets at least 15 votes from itself and neighbours. Then vertices in Q_{n-1} get at least $15 \cdot |Q_{n-1}|$ votes from Q_n in all. Thus we have

$$25 \cdot |C \cap Q_n| \geq 15 \cdot |Q_{n-1}| = 15 \cdot (|Q_n| - 8n)$$

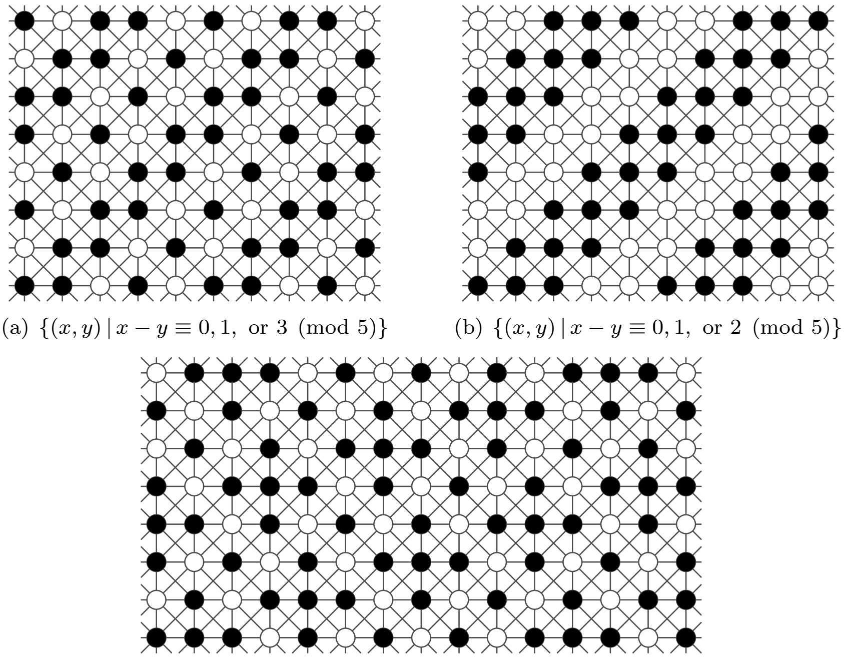
and so

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n\to\infty} \left(\frac{15}{25} - \frac{120n}{25 \cdot |Q_n|} \right) = \frac{3}{5}.$$

□

3 Constructions

In Figure 4 we give three different optimal $(1, \leq 3)$-LDB codes with density $\frac{3}{5}$.



(a) $\{(x, y) \mid x - y \equiv 0, 1, \text{ or } 3 \pmod 5\}$ (b) $\{(x, y) \mid x - y \equiv 0, 1, \text{ or } 2 \pmod 5\}$

(c) $\{(x, y) \mid x - 3y \equiv 0, 1, 2, 4, 6, \text{ or } 8 \pmod{10}\}$

Figure 4: Optimal $(1, \leq 3)$-LDB codes.

Theorem 4. *The density of optimal* $(1, \leq 3)$*-LDB code is* $\frac{3}{5}$.

Proof. We show that the code in Figure 4(b) is a $(1, \leq 3)$-LDB code. This proves the claim since the density of the code is $\frac{3}{5}$ and we showed in Theorem 3 that the density of every $(1, \leq 3)$-LDB code is at least $\frac{3}{5}$.

Let $F \subseteq V \setminus C$ where $|F| \leq 3$. Now, we observe that the non-codeword $(0, 1)$ belongs to F if and only if $(0, 0)$, $(1, 0)$ and $(1, 1)$ are in $I_1(F)$ and at least one of $(-2, 1)$, $(0, 3)$ and $(3, -2)$ is not. Indeed, if some of the three first mentioned vertices are not in $I_1(F)$, then clearly $(0, 1) \notin F$. If all the three last mentioned vertices are in $I_1(F)$, then $(0, 1) \notin F$, since there is no non-codeword which covers at least two of them. Then the three codewords have to be covered by three non-codewords and none of them is $(0, 1)$. On the other hand, if the three first mentioned codewords are in $I_1(F)$ and $(0, 1) \notin C$, then the non-codewords which cover $(1, 0)$, $(0, 0)$ and $(1, 1)$ also cover $(3, -2)$, $(-2, 1)$ and $(0, 3)$, respectively.

This proves the claim because the neighbourhoods of non-codewords are symmetric. $\square$

Theorem 5. *The density of optimal* $(r, \leq 3)$*-LDB code is* $\frac{r}{r+1}$ *when* $r \geq 2$*.*

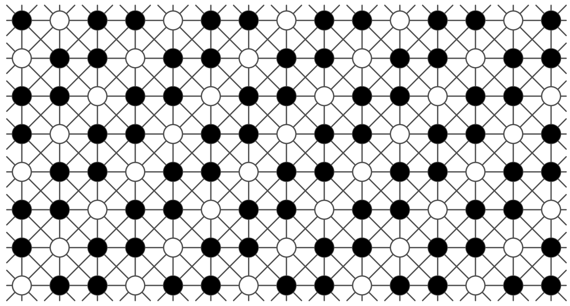


Figure 5: An optimal $(2, \leq 3)$-LDB code.

Proof. It is shown in Theorem 2 that the density of every $(r, \leq 3)$-LDB code is at least $\frac{r}{r+1}$. Now we show that

$$C = \{(x, y) \mid x \not\equiv y \pmod{r+1}\}$$

is such a code when $r \geq 2$. (A part of C is shown in Figure 5 when $r = 2$.) Clearly the density of C is $\frac{r}{r+1}$, so it is enough to show that C is an $(r, \leq 3)$-LDB code.

Assume to the contrary that there are sets $F \subset V \setminus C$ and $F' \subset V \setminus C$ such that $F \neq F'$, $|F| \leq 3$, $|F'| \leq 3$, and $I_r(F) = I_r(F')$. Since C is symmetric for non-codewords, we can assume that $O = (0, 0) \in F \setminus F'$.

First assume that $v = (0, r+1) \in F'$. Then $c_1 = (0, 2r+1) \in I_r(F') = I_r(F)$. Now, v covers neither $(-r, 0) \in C$ nor $(r, 0) \in C$ in $B_r(O)$. Therefore, one non-codeword u_1 of $B_r((-r, 0))$ and one non-codeword u_2 of $B_r((r, 0))$ have to be in F'. Now, $u_1 \neq u_2$ because $O \notin F'$ is only non-codeword which is contained in $B_r((-r, 0)) \cap B_r((r, 0)) = \{(0, y) \mid -r \leq y \leq r\}$. Moreover, u_1 can not cover $(r, -2) \in I_r(F)$ and u_2 can not cover $(-r, -1) \in I_r(F)$. Therefore u_1 must also covers $(-r, -1)$ and so u_1 covers $c_2 = (-r - 1, -1) \in C$ as well since the x-coordinate of u_1 must be negative. In the same way, u_2 has to cover $c_3 = (r + 1, -2) \in C$. Now, $c_1, c_2, c_3 \in I_r(F') = I_r(F)$, but O covers none of them and there is no codeword which covers more than one two of them. Then F has to contain at least four non-codewords which is a contradiction.

Assume then that $(0, r+1) \notin F'$ and also $(0, -r-1) \notin F'$, $(r+1, 0) \notin F'$, and $(-r-1, 0) \notin F'$ by symmetry. Now, no non-codeword except O covers more than two codewords of the set $S = \{(-r, r), (-r, -r+1), (-r+1, -r), (r-1, r), (r, r-1), (r, -r)\}$. Indeed, if F' contains a non-codeword (say v) which covers at least three of codewords in S, then at least one of them must be with Euclidean distance $\sqrt{r^2 + (r-1)^2}$ from O. Now we can assume without loss of generality that v covers $(r, r-1)$. Now, v covers neither $(-r, r) \in S$ nor $(-r, -r+1) \in S$ since F' does not contain non-codewords where the x-coordinate is 0 by the assumption in the beginning of this paragraph. By symmetry, if v also covers $(r-1, r)$, then it can not cover codewords whose y-coordinate is $-r$ and so v would cover only two codewords in S. Then v should be cover $(-r+1, -r)$, but $O \notin F'$ is the only non-codeword of these four vertices which covers both $(r, r-1)$ and $(-r+1, -r)$.

Then F' has to contain exactly three non-codewords and each of them covers exactly two codewords of S. Assume that $u_1 \in F'$ covers $(-r, r)$. Then the other element of S which u_1 covers has to be $(-r, -r+1)$ or $(r-1, r)$. Without loss of generality we can assume that u_1 covers $(-r, -r+1)$. Now u_1 must also cover $c_1 = (-r-1, r)$ since $u_1 \neq O$. Let $u_2 \in F'$ be the non-codeword that covers $(-r+1, -r)$. Then u_2 must also cover $(r, -r) \in S$ (since the other alternative $(-r, -r+1)$ is covered by u_1). Now, u_2 must also cover $c_2 = (r, -r-1)$. Finally, let $u_3 \in F'$ be the non-codeword that covers $(r-1, r)$ and $(r, r-1)$. Then u_3 has to cover $c_3 = (r+1, r)$ as well since $u_3 \notin \{(0,0), (0, r+1), (r+1, 0)\}$. Thus, $I_r(F') = I_r(F)$ contains codewords c_1, c_2, and c_3, but $O \in F$ does not cover any of these three codewords and no vertex covers more than one of them. Then F contains at least four non-codewords which is again a contradiction.

Hence, C is an $(r, \leq 3)$-LDB code. □

References

[1] Auger, D., Charon, I., Hudry, O., Lobstein A.: Watching systems in graphs: An extension of identifying codes, Discrete Applied Mathematics, in press.

[2] Auger, D., Honkala, I.: Watching systems in the king grid, submitted.

[3] Ben-Haim, Y., Gravier, S., Lobstein, A., Moncel, J.: General bounds for identifying codes in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 8(1), R39 (2001).

[4] Charon, I., Honkala, I., Hudry, O., Lobstein, A.: General bounds for identifying codes in some in
nite regular graphs, Electronic Journal of Combinatorics, Vol. 8(1), R39 (2001).

[5] Charon, I., Honkala, I., Hudry, O., Lobstein, A.: The minimum density of an identifying code in the king lattice, Discrete Mathematics, Vol. 276, 95–109 (2004).

[6] Charon, I., Hudry, O. and Lobstein, A.: Identifying codes with small radius in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 9(1), R11 (2002).

[7] Cohen, G., Honkala, I., Lobstein, A., Zémor, G.: On codes identifying vertices in the two-dimensional square lattice with diagonals, IEEE Transactions on Computers, Vol. 50, 174–176 (2001).

[8] Foucaud, F., Laihonen, T., Parreau, A.: An improved lower bound for $(1, \leq 2)$-identifying codes in the king grid, arXiv:1111.2477 (2011).

[9] Honkala, I., Laihonen, T.: Codes for identification in the king lattice, Graphs and Combinatorics, Vol. 19, 505–516 (2003).

[10] Honkala, I., Laihonen, T.: On locating-dominating sets in infinite grids, European Journal of Combinatorics, Vol. 27, 218–227 (2006).

[11] Honkala, I., Laihonen, T., Ranto, S.: On locating-dominating codes in binary Hamming spaces, Discrete Mathematics and Theoretical Computer Science, Vol. 6, 265–282, (2004).

[12] Karpovsky, M. G., Chakrabarty K., Levitin, L. B.: On a new class of codes for identifying vertices in graphs, IEEE Transactions on Information Theory, Vol. IT-44, 599–611 (1998).

[13] Pelto, M.: New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid, Cryptography and Communications, Vol. 2, 41–47 (2010).

[14] Pelto, M.: On locating-dominating codes in the infinite king grid, Ars Combinatoria, in press.

[15] Pelto, M.: On $(r, \leq 2)$-locating-dominating codes in the infinite king grid, Advances in Mathematics of Communications, in press.

[16] Pelto, M.: On locating-dominating codes for locating large numbers of vertices in the infinite king grid, Australasian Journal of Combinatorics, Vol. 50, pp. 127–139 (2011).

[17] Ranto, S.: Identifying and locating-dominating codes in binary Hamming spaces, Ph. D. Thesis, University of Turku, Finland, 95 pages, (2007).

[18] Slater, P. J.: Domination and location in acyclic graphs, Networks, Vol. 17, 55–64 (1987).

[19] Slater, P. J.: Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences, Vol. 22, 445–455 (1988).

[20] http://www.infres.enst.fr/~lobstein/bibLOCDOMetID.html

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On locating-dominating codes for locating large numbers of vertices in the infinite king grid

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Abstract

Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E. The ball $B_r(v)$ denotes the vertices within graphical distance r from v. A subset $C \subseteq V$ is called an $(r, \leq l)$-locating-dominating code of type B if the sets $I_r(F) = \bigcup_{v \in F}(B_r(v) \cap C)$ are distinct for all subsets $F \subseteq V \setminus C$ with at most l vertices. A subset $C \subseteq V$ is an $(r, \leq l)$-locating-dominating code of type A if sets $I_r(F_1)$ and $I_r(F_2)$ are distinct for all subsets $F_1, F_2 \subseteq V$ where $F_1 \neq F_2$, $F_1 \cap C = F_2 \cap C$ and $|F_1|, |F_2| \leq l$. We study $(r, \leq l)$-locating-dominating codes in the infinite king grid when $r \geq 1$ and $l \geq 3$. The infinite king grid is the graph with vertex set $\mathbb{Z}^2$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| \leq 1,\ |y_1 - y_2| \leq 1\}$.

1 Introduction

Let $G = (V, E)$ be an undirected graph where V is a vertex set and E an edge set. Denote by $d(u, v)$ the distance between two vertices u and v, i.e. the number of edges on any shortest path between u and v. The ball with center v and radius r is

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

We call any set C with $C \subseteq V$ a *code.* The vertices of C are called *codewords.* In particular, C is an $(r, \leq l)$*-locating-dominating code of type B* if the sets

$$I_r(F) = \left(\bigcup_{v \in F} B_r(v)\right) \cap C = B_r(F) \cap C$$

are distinct for all subsets $F \subseteq V \setminus C$ with at most l non-codewords. The code is an $(r, \leq l)$*-locating-dominating code of type A* if $I_r(F_1) \neq I_r(F_2)$ for all subsets

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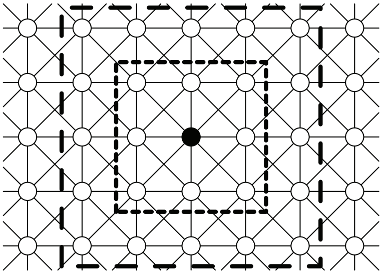

Figure 1: A part of the infinite king grid. The vertices within distance one and two from the black dot are surrounded by the dashed lines.

$F_1 \subseteq V$ and $F_2 \subseteq V$ where $|F_1| \le l$, $|F_2| \le l$, $F_1 \ne F_2$ and $F_1 \cap C = F_2 \cap C$. We call a locating-dominating code of type A (or type B, respectively) an LDA code (or an LDB code, respectively) for short. An LDA code is automatically an LDB code and if $l = 1$, then the definitions of LDA and LDB codes are equivalent. Also, an $(r, \le l)$-LDA code (or -LDB code) is automatically an $(r, \le k)$-LDA code (or -LDB code, respectively) if $k \le l$.

Moreover we say that a codeword is a *special codeword* if exactly one vertex in its r-neighbourhood is a non-codeword. In particular in the case of LDB codes, a special codeword c is in an identifying set $I_r(F)$ if and only if the only non-codeword in the r-neighbourhood of c is in the set $F \subseteq V \setminus C$.

We study codes in the infinite king grid. The infinite king grid is the graph where $V = \mathbb{Z} \times \mathbb{Z}$ and vertices $u = (u_x, u_y)$ and $v = (v_x, v_y)$ are adjacent if $|u_x - v_x| \le 1$ and $|u_y - v_y| \le 1$. Thus vertices u and v are neighbours if the Euclidean distance between u and v is 1 or $\sqrt{2}$.

The density of $C \subseteq \mathbb{Z}^2$ is

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap B_n((0,0))|}{|B_n((0,0))|},$$

where $|C \cap B_n((0,0))|$ is the number of codewords in the ball $\{(x,y) \mid |x| \le n, |y| \le n\}$ and $|B_n((0,0))|$ is the number of all vertices in the ball. We also define $B_n((0,0)) = Q_n$. We search for the minimum density of locating-dominating codes of both types for given r and l in the infinite king grid.

Next, we define that a non-codeword is *isolated* if all the four vertices at Euclidean distance 1 from it are codewords. For example, the circled non-codewords in Figures 2(c) and 2(d) on page 132 are isolated non-codewords.

Moreover, we say that vertices $u_1, u_2, \ldots, u_k$ are *consecutive* in the infinite king grid if all of them have the same x-coordinate (or y-coordinate) and the y-coordinates (or x-coordinates, respectively) are consecutive. Finally, we define that $v_1, v_2, \ldots, v_k$ are k *successive non-codeword neighbours* of (x, y) if they are k successive non-

	$r = 1$		$r = 2$		$r \geq 3$	
$l = 1$	$\frac{1}{5}$	[6]	$\frac{1}{10} \leq D \leq \frac{1}{8}$	[2], [10]	$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ if $2 \mid r$ $\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$ if $2 \nmid r$	[1], [10]
$l = 2$	$\frac{1}{3}$	[11]	$\frac{1}{5} \leq D \leq \frac{1}{4}$	[11]	$\frac{1}{6} \leq D \leq \frac{r+1}{6r+3}$ if $r \equiv 0, 2, 5 \pmod 6$ $\frac{1}{6} \leq D \leq \frac{2r+3}{12r+6}$ if $r \equiv 1, 3, 4 \pmod 6$	[11]
$l \geq 3$	1	a	1	a	1	a

Table 1: The known lower and upper bounds for the density of LDA codes. Reference a refers to Theorem 1.

codewords of cycle

$$(x+1,y),(x+1,y+1),(x,y+1),(x-1,y+1),$$
$$(x-1,y),(x-1,y-1),(x,y-1),(x+1,y-1).$$

Locating-dominating codes (of types A and B) were introduced in the late of 1980s by Slater [13] and [14] when $l = 1$ and in the 2000s by Honkala, Laihonen and Ranto [7] for general l. A motivation for such codes is a safeguard analysis of a facility [13].

We study $(r, \leq l)$-locating-dominating codes for large l. The emphasis of this paper is on LDB codes since a code is an LDA code for $l \geq 3$ only if there are no non-codewords (Theorem 1). For small l, LDA and LDB codes have been studied in the papers [6] and [10]–[12] and Tables 1 and 2 summarize what is known about the density of $(r, \leq l)$-locating-dominating codes of type A and B in the infinite king grid. Here, the upper bound means that there exists such a code with that density and the lower bound means that the density of every such code is at least the value given in the table.

Papers [1]–[4] and [9] study $(r, \leq l)$-identifying codes which is a closely related class of codes in the infinite king grid. More papers on such codes in the infinite king grid and many other graphs can be found in the web bibliography [15].

2 Lower bounds

Theorem 1. *Assume that $r \in \mathbb{N}$ and $l \geq 3$. Then C is an $(r, \leq l)$-LDA code in the infinite king grid if and only if C contains all vertices in the infinite king grid.*

Proof. Clearly, the code containing all vertices is an $(r, \leq l)$-LDA code for all r and l. On the other hand, since $B_r((x,y)) \subseteq B_r(\{(x-1,y),(x+1,y)\})$, we see that $I_r(\{(x-1,y),(x+1,y)\}) = I_r(\{(x-1,y),(x,y),(x+1,y)\})$. Thus, if C is an $(r, \leq l)$-LDA code, then $\{(x-1,y),(x+1,y)\} \cap C \neq \{(x-1,y),(x,y),(x+1,y)\} \cap C$ and so $(x,y) \in C$ for all $(x,y) \in \mathbb{Z}^2$. □

	$r=1$		$r=2$		$r\geq 3$	
$l=1$	$\frac{1}{5}$	[6]	$\frac{1}{10}\leq D\leq\frac{1}{8}$	[2], [10]	$\frac{1}{4r+2}\leq D\leq\frac{1}{4r}$ if $2\mid r$ $\frac{1}{4r+2}\leq D\leq\frac{1}{4r+\frac{2}{r+1}}$ if $2\nmid r$	[1], [10]
$l=2$	$\frac{1}{3}$	[11]	$\frac{1}{6}\leq D\leq\frac{1}{4}$	[11]	$\frac{1}{6}\leq D\leq\frac{r+1}{6r+3}$ if $r\equiv 0,2,5\ (6)$ $\frac{1}{6}\leq D\leq\frac{2r+3}{12r+6}$ if $r\equiv 1,3,4\ (6)$	[11]
$l=3$	$\frac{3}{5}$	[12]	$\frac{2}{3}$	[12]	$\frac{r}{r+1}$	[12]
$4\leq l\leq 4r$	$\frac{2}{3}$	b, d	$\frac{2}{3}\leq D\leq\frac{4}{5}$	[12], d	$\frac{r}{r+1}\leq D\leq\frac{2r}{2r+1}$	[12], d
$l>4r$	$\frac{2}{3}$	b, d	$\frac{4}{5}$	c, d	$\frac{2r}{2r+1}$	c, d

Table 2: The known lower and upper bounds for the density of LDB codes. References b, c and d refer to Theorems 5, 9 and 10, respectively.

Now we have shown when the code is an $(r,\leq l)$-LDA code for any $r\in\mathbb{Z}_+$ and $l\geq 3$. Then, we shall consider only LDB codes in the future.

2.1 LDB codes when $r=1$

Lemma 2. *[12] If C is a $(1,\leq 4)$-LDB code in the infinite king grid, then at most two of four consecutive vertices can be non-codewords, which also means that at least two of four consecutive vertices must be codewords.*

Proof. If there were three non-codewords among four consecutive vertices, then the identifying set of the set of all these three non-codewords would be the same as the identifying set of the set of the outermost non-codewords. □

Corollary 3. *If C is a $(1,\leq 4)$-LDB code in the infinite king grid, then at least one of three consecutive vertices must be a codeword.*

Corollary 4. *If C is a $(1,\leq 4)$-LDB code in the infinite king grid, then there can exist at most three successive non-codeword neighbours.*

In the next proof, we use an averaging method, which is often called voting or discharging. The idea in the method is the following: Initially, each codeword has some fixed number (t_1) of votes. After the initial state, we transfer votes from vertices to others, i.e., we add a certain number of votes to some vertex and subtract the same number votes from another vertex at the same time. Thus, the total number of votes does not change when votes are transferred. Finally, we show that every vertex has at least t_2 votes. This then implies that the density of the code is $\frac{t_2}{t_1}$.

In what follows, we say that a non-codeword v or the neighbourhood of v *covers* the vertex u if $u\in B_r(v)$.

Theorem 5. *The density of a $(1,\leq 4)$-LDB code is at least $\frac{2}{3}$ in the infinite king grid.*

Proof. Let C be a $(1, \leq 4)$-LDB code.

First, every codeword gives one vote to itself and all neighbours. Then, we transfer two more votes from each special codeword to the unique non-codeword in its neighbourhood and half a vote from each codeword with exactly six neighbours in the code to the two non-codewords in its neighbourhood. This is called **Rule 1**.

Now, every codeword with at least five neighbours in the code has at least six votes and we also show in Step 1 that every non-codeword has at least six votes. However, codewords can still have fewer than six votes, but in that case we can transfer extra votes from non-codewords to the codewords by Rule 2 (which will be defined later).

Step 1. Every non-codeword has at least six votes after Rule 1.

Let v be a non-codeword. First, assume that v has a special codeword c in its neighbourhood. Now v has at least four codewords in its neighbourhood. Indeed, the intersection of the neighbourhood of v and the neighbourhood of c contains at least four vertices and apart from v these (at least three) vertices have to be in the code. Furthermore, there are three consecutive vertices that are in the neighbourhood of v, but are not in the neighbourhood of c, and at least one of these consecutive vertices must be in the code by Corollary 3. Thus, there are at least four codewords in the neighbourhood of v and at least one of them is a special codeword. Then, v has at least six votes.

Second, we assume that v has no special codeword in its neighbourhood. Assume further that v is not isolated, i.e., there is another non-codeword u_1 at Euclidean distance one from v. Let v' be the unique vertex with Euclidean distance one from v and two from u_1. Now, v' has to be a codeword or else there would be three consecutive non-codewords, which contradicts Corollary 3. Then v' has another non-codeword neighbour $u_2 \neq v$ since we assume that v does not have a special codeword in its neighbourhood. Now, u_1 and u_2 cover the neighbourhood of v except for one vertex v'' (or zero vertices, but then $B_1(\{v, u_1, u_2\}) = B_1(\{u_1, u_2\})$). Again, since v has no special codeword in its neighbourhood, then v'' has another non-codeword $u_3 \neq v$ in the neighbourhood of v''. If v'' is a non-codeword, then we choose $v'' = u_3$. Thus $B_1(v) \subseteq B_1(\{u_1, u_2, u_3\})$ and so C is not a $(1, \leq 4)$-LDB code. In particular, **non-codewords without special codeword neighbours are isolated.**

Finally, we assume that v is isolated and it has no special codewords in its neighbourhood. If v has at most five codewords in its neighbourhood, then at least three corners in its neighbourhood have to be non-codewords. Let u_1 and u_2 be two of the non-codewords that are in the opposite corners and the third non-codeword be u_3. Let u_4 ($\neq v$) be a non-codeword that covers the fourth corner of the neighbourhood of v. (Possibly, u_4 is the fourth corner.) Then, $I_1(v) \subseteq B_1(\{u_1, u_2, u_4\})$ and again C would not be a $(1, \leq 4)$-LDB code. Hence, v has at least six neighbours in the code and so also at least six votes.

Next, we transfer votes from non-codewords to codewords as in Figure 2 (symmetries such as reflections allowed). The amount of transferred votes is given in Figure

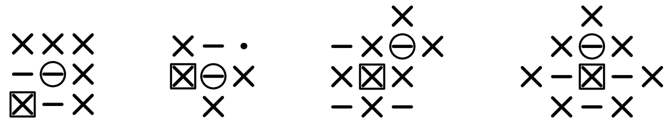

Figure 2: The circled non-codeword gives away 2, 1 or $\frac{1}{4}$ votes to squared codeword. The crosses are codewords and the lines are non-codewords. The black dot may be a codeword or a non-codeword.

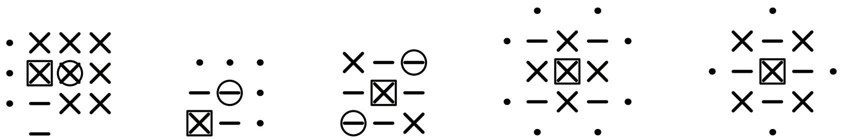

Figure 3: Some cases of the proof of Theorem 5

2. This is called **Rule 2**.

Step 2. Every codeword now has at least six votes after Rule 2.

First, we observe that if a codeword c has two consecutive non-codewords u and v with Euclidean distance one and two, respectively, from c, then c has six votes after Rule 1. Indeed, if c is a special codeword, the claim is clear; so assume that it is not. Because u is not isolated, it must have a special codeword in its neighbourhood (cf. Step 1), and such a special codeword can only be either of two vertices at Euclidean distance one from c and $\sqrt{2}$ from u. Now, we can without loss of generality consider the constellation in Figure 3(a). By Corollary 3, at least one of the black dots has to be a codeword, and c has at least six votes.

Next, we assume that a codeword c has at least three successive non-codeword neighbours and by the Corollary 4, there cannot be more than three successive codewords. See Figure 3(b). If c has at least six votes after Rule 1, then the claim is clear. Therefore, we also assume that c has fewer than six votes after Rule 1. Then, the black dots must be codewords in Figure 3(b). Indeed, the leftmost and lowest black dots are codewords by the observation made in the previous paragraph. Moreover, the black dot in the top right corner has to be a special codeword, because it is the only vertex in the neighbourhood of the circled non-codeword that can be a special codeword (and since the circled codeword is not isolated, there has to be one). Thus, the squared codeword gets two votes by Figure 2(a) in Rule 2. Now, if c has at least four votes after Rule 1, then it has enough votes after Rule 2. Assume then that c has fewer than four votes after Rule 1, then the neighbourhood of c has to be as the constellation of Figure 3(c) or its rotation. Then c gets two votes from both circled codewords and it has seven votes after Rule 2.

Next, we assume that there are two (but not three) successive non-codeword neighbours of the codeword c. Moreover, we assume that c has fewer than six votes after Rule 1. Then, the neighbourhood of the codeword is as in Figure 2(b) or its reflection or rotation where c is the squared codeword. Indeed, the cross on the right has to be a codeword by the observation made at the beginning of Step 2. Thus, the squared codeword gets one vote from the circled non-codeword in Rule 2. Now, if c has at least five votes after Rule 1, then we are done. If c has fewer votes, then there are two pairs of consecutive non-codewords in the neighbourhood of c, and then c gets two votes – one vote from two non-codewords each – by Rule 2. Anyway, c has at least four votes after Rule 1 (or else c has three successive non-codeword neighbours), and six votes after Rule 2.

Assume finally that the codeword c has no two successive non-codeword neighbours. Now there are only two possible constellations given in Figures 3(d) and 3(e) where c has fewer than six votes after Rule 1. All the black dots have to be codewords since at most two of four consecutive vertices can be non-codewords. Now, each of the four non-codewords gives $\frac{1}{4}$ of a vote to c by Figures 2(c) or 2(d) of Rule 2 and so c has enough votes.

Step 3. Every non-codeword still has at least six votes after Rule 2.

First, the circled non-codeword in Figure 2(a) has to have at least eight votes after Rule 1, since it gets one vote from all six codewords and two more votes from a special codeword. The non-codeword must have a special codeword neighbour because it is not isolated. Thus, the circled non-codeword has at least six votes after Rule 1 since it does not give votes to other neighbours than the squared codeword.

The circled non-codeword in Figure 2(b) can only give a vote to the codewords to its left and right. If it gives one vote to the codeword on the right, then the black dot has to also be a codeword. In any case, one of the three lowest vertices in the neighbourhood of the circled non-codeword is a special codeword for the same reason as in the previous case. Then the non-codeword has at least eight votes after Rule 1, if it gives a vote to two codewords, or else at least seven votes after Rule 1. Anyway, it has at least six votes after Rule 2.

Next, we show that the circled non-codeword in Figures 2(c) or 2(d) has at least six votes after the voting. The circled non-codeword can give $\frac{1}{4}$ of a vote to at most four codewords i.e. in total one vote. Then, if the circled non-codeword has at least seven votes after Rule 1, the claim is true. We still assume that the squared codeword is in the origin, so we can use coordinates in the proof.

We show first that non-codeword in $(1,1)$ in Figure 2(c) has at least six votes after Rule 2. Now, $(2,2)$ has to be a codeword since $B_1(\{(1,1)\}) \subseteq B_1(\{(-1,1),(1,-1),(2,2)\})$. Then, $(0,2)$ and $(2,0)$ are non-codewords or else $(1,1)$ has enough votes. Now, $(1,3)$ and $(2,3)$ are also codewords because the union of the neighbourhoods of $(-1,1)$, $(2,0)$ and $(1,3)$ (or $(2,3)$) covers the neighbourhood of $(1,1)$. By symmetry, $(3,1)$ and $(3,2)$ are also codewords. Thus, $(1,1)$ has at least $6\frac{1}{2}$ votes after Rule 1 since it gets in total $1\frac{1}{2}$ votes from $(2,2)$. Moreover, it gives $\frac{1}{4}$ of a vote to only one codeword in Rule 2. Hence, it has at least six votes after Rule 2.

Finally, at least five neighbours of the non-codeword $(0,1)$ in Figure 2(d) have to be codewords. Otherwise three of the non-codewords in the neighbourhood cover all codewords in the neighbourhood. If $(0,1)$ has a special codeword in the neighbourhood, then it has enough votes; so assume that this is not the case. Now, $(-1,2) \in C$ and $(1,2) \in C$ by Step 1. Moreover, $(0,3) \in C$ since $I_1(0,1) \subseteq B_1(\{(0,3),(-1,0),(1,0)\})$. However, $(0,2)$ is not a special codeword, and so $(-1,3)$ or $(1,3)$ is a non-codeword. Without loss of generality, we assume that $(-1,3) \notin C$. Moreover, $(2,1) \in C$ and $(2,2) \in C$: otherwise, the neighbourhoods of one of them, $(-1,0)$ and $(-1,3)$ cover all the codewords in the neighbourhood of $(0,1)$. Thus, $(0,1)$ has at least $6\frac{1}{2}$ votes after Rule 1 since $(1,1)$ in Figure 2(d) gives in total $1\frac{1}{2}$ votes to it. Furthermore, $(0,1)$ gives $\frac{1}{4}$ of a vote only to one codeword in Rule 2.

Hence, all vertices in Q_n have at least six votes and every vertex can get votes only from itself and the vertices within graphical distance two.[1] Then all vertices in Q_{n-2} have at least six votes from vertices in Q_n. On the other hand, the total number of votes given by the vertices in Q_n is $9 \cdot |C \cap Q_n|$. Thus we have

$$9 \cdot |C \cap Q_n| \geq 6 \cdot |Q_{n-2}| = 6 \cdot (|Q_n| - 16n + 8)$$

i.e.

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n\to\infty} \left(\frac{6}{9} - \frac{96n-48}{9 \cdot |Q_n|} \right) = \frac{2}{3}.$$

□

2.2 LDB codes when $r \geq 2$

The next lemma is valid whether or not the code C is an LDB code. Later we shall nevertheless see that the code which satisfies the assumptions of the lemma is always an $(r, \leq l)$-LDB code for all l.

Lemma 6. *Let C be a code in the infinite king grid. If every non-codeword has a special codeword in its r-neighbourhood, then the density of C is at least $\frac{2r}{2r+1}$.*

Proof. Let $c = (a,b)$ be a special codeword and $v = (x,y)$ the unique non-codeword in the neighbourhood of c. Then we mark by v all the vertices

$$J_H(c) = \begin{cases} (a-r,y),(a-r+1,y),\ldots,(a,y) & \text{if } x \leq a, \\ (x-r,y),(x-r+1,y),\ldots,(x,y) & \text{if } x > a \end{cases} \tag{1}$$

and

$$J_V(c) = \begin{cases} (a,y),(a,y+1),\ldots,(a,y+r) & \text{if } y \leq b, \\ (a,b),(a,b+1),\ldots,(a,b+r) & \text{if } y > b. \end{cases} \tag{2}$$

[1]Codewords give votes to the vertices in its neighbourhood by the first rule and votes can be transferred from the vertices which give them by Rule 1 to vertices in their neighbours by the second rules. Therefore, votes stay within distance two from the codewords.

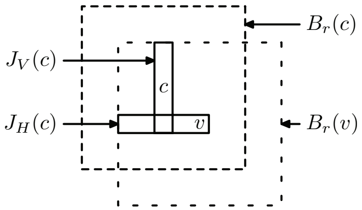

Figure 4: Sets $J_H(c)$ and $J_V(c)$, when $a \leq x$ and $b \geq y$.

Since c is in the r-neighbourhood of v i.e. $|a-x| \leq r$ and $|b-y| \leq r$, then c and v are marked by v. Moreover, both $J_H(c)$ and $J_V(c)$ contain $r+1$ vertices and exactly one of these belongs to both sets. Then $2r+1$ vertices are marked by v. If v has more special codewords in its r-neighbourhood, then there can be more than $2r+1$ vertices marked by v. Furthermore, all vertices in $J_H(c)$ and $J_V(c)$ are in the r-neighbourhood of special codeword c. Therefore v is the only non-codeword which has been marked by v.

Now, we show that a codeword cannot be marked by two non-codewords. Assume that $a \leq x$ and $b \geq y$. (Three other cases are proved in the same way.) See Figure 4. Assume to the contrary that $c' = (a', b')$ is a special codeword in the r-neighbourhood of a non-codeword $v' = (x', y') \neq v$ and w is a vertex which is marked by both v' (with c') and v (with c). Then $d(v, c') > r$ and $d(v', c) > r$ since the r-neighbourhood of c' or c contains only one non-codeword.

If $w \in J_H(c)$, then $b' \in [y-r, y+r]$. Moreover, $a' \in [x-2r, x+r]$. However, a' has to be smaller than $x-r$ since $d(v, c') > r$. Thus $J_V(c')$ cannot contain w (since the x-coordinates of vertices in $J_V(c')$ are a'). Then $J_H(c)$ and $J_H(c')$ have to intersect, but this is possible only if $y' = y$ and $\max\{x', a'\} \geq x-r$. However, this is impossible since $a' < x-r$ (observed above) and $x' < a-r \leq x-r$ because $x' \leq a'+r < x \leq a+r$ and $x' \notin [a-r, a+r]$ where the first inequality follows from the fact $d(v', c') \leq r$ and the latter condition from the fact $d(v', c) > r$.

Thus, w has to be in $J_V(c)$. Now, $x' \in [a-r, a+r]$ and $y' \in [y-r, y+2r]$. However, $y' \notin [b-r, b+r]$ since $d(v', c) > r$. Then $J_H(c')$ cannot contain w. Thus, w is in $J_V(c')$. This is possible only if $a' = a$. Then $b' \notin [y-r, y+r]$ because $d(v, c') > r$. If $b' < y-r$, then $d(c', u) > r$ for all $u \in J_V(c)$. Then $b' > y+r$, but now $J_V(c)$ and $J_V(c')$ can intersect only if $y' \leq y+r$, which is impossible since $d(v', c) > r$ and $d(v', c') \leq r$.

Hence every non-codeword in Q_n has marked at least $2r$ codewords in Q_{n+r} and every codeword has been marked by at most one non-codeword. Therefore we have

$$2r|Q_n \setminus C| \leq |C \cap Q_{n+r}|$$

and so

$$\frac{|C \cap Q_{n+r}|}{|Q_n|} \geq \frac{|C \cap Q_{n+r}|}{|C \cap Q_{n+r}| + |Q_n \setminus C|} = \frac{1}{1 + \frac{|Q_n \setminus C|}{|C \cap Q_{n+r}|}} \geq \frac{1}{1 + \frac{1}{2r}} = \frac{2r}{2r+1}.$$

Then

$$\frac{|C \cap Q_n|}{|Q_n|} \geq \frac{|C \cap Q_{n+r}|}{|Q_n|} - \frac{|Q_{n+r}| - |Q_n|}{|Q_n|} \geq \frac{2r}{2r+1} - \frac{4r(2n+r+1)}{(2n+1)^2}$$

and so

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \frac{2r}{2r+1}.$$

□

Lemma 7. *A code is an $(r, \leq l)$-LDB code in the infinite king grid when $l > 4r$ and $r \geq 2$ if and only if every non-codeword has a special codeword in its r-neighbourhood.*

Proof. First, if every non-codeword has a special codeword in its neighbourhood, then the code is clearly $(r, \leq l)$-LDB code (even for $r = 1$).

Assume then that $O = (0, 0)$ is a non-codeword with no special codewords in its r-neighbourhood. In particular, every vertex — codeword or not — in $B_r(O) \setminus \{O\}$ has at least two non-codewords (one of which is O) within distance r. We show that all codewords in $B_r(O)$ can always be covered by $4r$ non-codewords other than O, or fewer. Then the identifying set of these $4r$ non-codewords is the same as the identifying set of these non-codewords and O.

First, we observe that if two balls of radius r have a non-empty intersection in the infinite king grid, then at least one of the four corners must be in the intersection. Moreover, every row (column, respectively) of $B_r(O) \setminus \{O\}$ can be covered by (at most) two non-codewords none of which is O. Indeed, there exists a non-codeword $(a, b) \neq O$ covering $(-r, y), (-r + 1, y), \ldots, (x, y) \in B_r(O)$, with x somewhere in $[-r, r]$. If $x = r$, we are done; if $x < r$ and if no non-codeword other than O covers $(-r, y)$ and $(x + 1, y)$, then there has to be another non-codeword $(a', b') \neq O$ which covers $(x + 1, y)$ (or $(x + 2, y)$ if $(x + 1, y) = O$). Now, (a', b') has to cover (r, y) as well since (a', b') covers $2r + 1$ consecutive vertices but not $(-r, y)$.

First, we assume that a non-codeword $v \neq O$ covers $(-1, 0)$ and $(1, 0)$. Furthermore, we can assume without loss of generality that, among the four corners, it covers (r, r). Now, vertices in $B_r(O)$ which have not been covered by v can be divided into at most r rows and $r - 1$ columns. These vertices can be covered by at most $2r + 2(r - 1)$ non-codewords none of which is O. Then $4r - 1$ non-codewords (none of which is O) cover all codewords in $B_r(O)$.

Next, we assume that $(-1, 0)$ and $(1, 0)$ ($(0, -1)$ and $(0, 1)$, respectively) are not covered by the same non-codeword other than O. In particular, only corners can be non-codewords in $B_r(O) \setminus \{O\}$. Assume then that a non-codeword $v \neq O$ covers $(-1, 0)$ and a non-codeword $u \neq O$ covers $(1, 0)$. If v and u cover adjacent corners of $B_r(O)$ (for example $(-r, r)$ and (r, r)), then $B_r(O) \setminus (B_r(u) \cup B_r(v))$ can be divided into at most one column and r rows. Thus, codewords in $B_r(O)$ can be covered by at most $2r + 4$ non-codewords other than O, and $2r + 4 \leq 4r$ when $r \geq 2$.

Now, we can assume that every non-codeword except O which covers $(-1, 0)$ ($(1, 0)$, $(0, -1)$ or $(0, 1)$, respectively) always covers the same corner of $B_r(O)$ and the non-codewords $v \neq O$ and $u \neq O$ ($v' \neq O$ and $u' \neq O$, respectively) which cover $(-1, 0)$ and $(1, 0)$ ($(0, -1)$ and $(0, 1)$, respectively) cover the opposite corners of $B_r(O)$. If v and v' cover adjacent corners, then v, v', u, and u' cover all four corners and also all codewords in $B_r(O)$.

Finally, we can assume without loss of generality that v and v' cover $(-r, -r)$ and u and u' cover (r, r). Let $w \neq O$ and $w' \neq O$ be two non-codewords which cover $(-1, 1)$ and $(1, -1)$, respectively. If w (or w', respectively) covers $(-r, -r)$ or (r, r), then we can substitute one of v and u' to w (or one of v' and u to w', respectively). Now, there are at most $r - 1$ columns or rows in $\{(a, b) \in B_r(O) \,|\, a \leq 0, b \geq 0\}$ (or $\{(a, b) \in B_r(O) \,|\, a \geq 0, b \leq 0\}$, respectively) which cannot be covered by four of v, v', u, u', w, and w'. So all in all, if both w and w' cover at least one of $(-r, -r)$ and (r, r), we have at most $2(r - 1) + 2(r - 1) + 4 = 4r$ non-codewords none of which is O covering $B_r(O)$.

If w (or w') covers $(-r, r)$ (or $(r, -r)$, respectively)[2], then w (or w') covers all vertices in $\{(a, b) \in B_r(O) \,|\, a \leq 0, b \geq 0\}$ (or $\{(a, b) \in B_r(O) \,|\, a \geq 0, b \leq 0\}$, respectively). Hence, $B_r(O)$ can be covered by at most $\max\{6, 5 + 2(r - 1)\} \leq 4r$ (when $r \geq 2$) non-codewords, none of which is $O = (0, 0)$. □

Remark 8. The previous lemma is also valid for $(1, \leq l)$-LDB codes in the infinite king grid if $l \geq 7$.

Theorem 9. *The density of an $(r, \leq l)$-LDB code in the infinite king grid is at least $\frac{2r}{2r+1}$ when $l > 4r$ and $r \geq 1$.*

Proof. The claim is proved in Theorem 5 when $r = 1$. Moreover, the claim follows from Lemmas 6 and 7 when $r \geq 2$. □

3 Constructions

Theorem 10. *There exists an $(r, \leq l)$-LDB code with density $\frac{2r}{2r+1}$ for any $r \geq 1$ and $l \geq 1$ in the infinite king grid.*

Proof. Let $C = \{(x, y) \,|\, x - y \equiv 1, 2, \ldots, 4r \pmod{4r + 2}\}$ be a code. A part of the code when $r = 1$ is given in Figure 5. We show that C is an $(r, \leq l)$-LDB code for any $l \geq 1$. Clearly, the density of the code is $\frac{2r}{2r+1}$.

Now, every codeword (x, y) with $x - y \equiv 2r \pmod{4r + 2}$ is a special codeword since $(x - r, y + r)$ is the only non-codeword in its neighbourhood. In the same way, codewords with $x - y = 2r + 1 \pmod{4r + 2}$ are special codewords. On the other hand, every non-codeword (x, y) with $x - y \equiv 0 \pmod{4r + 2}$ has a special codeword $(x + r, y - r)$ in its neighbourhood. In the same way, non-codewords with $x - y \equiv 4r + 1 \pmod{4r + 2}$ have a special codeword in its neighbourhood.

Hence every non-codeword has a special codeword in its neighbourhood and therefore C is an $(r, \leq l)$-LDB code for all $l \geq 1$. □

[2] w does not cover $(r, -r)$ since then w also covers both $(-1, 0)$ and $(1, 0)$.

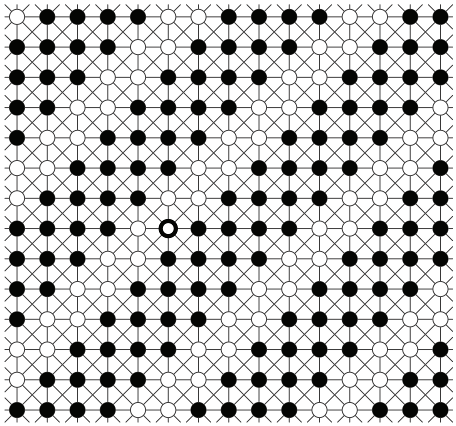

Figure 5: $(1, \leq l)$-LDB code for all $l \in \mathbb{N}_+$. The origin is the non-codeword which is surrounded with a thick circle.

Now, we have seen that the codes which were given in the previous theorem are so-called *optimal* $(r, \leq l)$-LDB codes, when $l > 4r$ or when $l = 4$ and $r = 1$. Indeed, by Theorems 5 and 9, there does not exist any $(r, \leq l)$-LDB code the density of which would be less than the density of the codes given in the previous theorem in the infinite king grid.

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References

[1] I. Charon, I. Honkala, O. Hudry and A. Lobstein, The minimum density of an identifying code in the king lattice, *Discrete Math.* 276 (2004), 95–109.

[2] I. Charon, O. Hudry and A. Lobstein, Identifying codes with small radius in some infinite regular graphs, *Electron. J. Combin.* 9(1), R11 (2002).

[3] G. Cohen, I. Honkala, A. Lobstein and G. Zémor, On codes identifying vertices in the two-dimensional square lattice with diagonals, *IEEE Trans Computers* 50 (2001), 174–176.

[4] I. Honkala and T. Laihonen, Codes for identification in the king lattice, *Graphs Combin.* 19 (2003), 505–516.

[5] I. Honkala and T. Laihonen, On a new class of identifying codes in graphs, *Inf. Proc. Letters* 102 (2007), 92–98.

[6] I. Honkala and T. Laihonen, On locating-dominating sets in infinite grids, *European J. Combin.* 27 (2006), 218–227.

[7] I. Honkala, T. Laihonen and S. Ranto, On locating-dominating codes in binary Hamming spaces, *Discrete Math. Theoret. Comp. Sci.* 6 (2004), 265–282.

[8] M. G. Karpovsky, K. Chakrabarty and L. B. Levitin, On a new class of codes for identifying vertices in graphs, *IEEE Trans. Inf. Theory* IT-44 (1998), 599–611.

[9] M. Pelto, New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid, *Cryptogr. Commun.* 2 (2010), 41–47.

[10] M. Pelto, On locating-dominating codes in the infinite king grid, *Ars Combin.* (to appear).

[11] M. Pelto, On $(r, \leq 2)$-locating-dominating codes in the infinite king grid, (submitted).

[12] M. Pelto, Optimal $(r, \leq 3)$-locating-dominating codes in the infinite king grid, (submitted).

[13] P. J. Slater, Domination and location in acyclic graphs, *Networks* 17 (1987), 55–64.

[14] P. J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sciences* 22 (1988), 445–455.

[15] `http://www.infres.enst.fr/~lobstein/bibLOCDOMetID.html`

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Publication VII

Mikko Pelto: Optimal identifying codes in the infinite 3-dimensional king grid, submitted 2011.

Optimal identifying codes in the infinite 3-dimensional king grid

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Abstract

A subset $C \subseteq V$ is an r-identifying code in a graph $G = (V, E)$ if the sets $I_r(v) = \{c \in C \mid d(c, v) \leq r\}$ are distinct and non-empty for all vertices $v \subseteq V$. Here, $d(c, v)$ denotes the number of edges on any shortest path from c to v. We consider the infinite n-dimensional king grid, i.e., the graph with vertex set $V = \mathbb{Z}^n$ and the edge set

$$E = \{\{x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)\} \mid |x_i - y_i| \leq 1 \text{ for } i = 1, \ldots, n,\ x \neq y\},$$

and give some lower bounds on the density of an r-identifying code. In particular, we prove that for $n = 3$ and for all $r \geq 15$, the optimal density of an r-identifying code is $\frac{1}{8r^2}$.

Key words

identifying code, king grid, graph, density, combinatorial geometry

1 Introduction

Let $G = (V, E)$ be an undirected graph and the distance $d(u, v)$ between two vertices u and v be the number of edges on any shortest path from u to v. The distance between a vertex u and set S of vertices is

$$d(u, S) = \min\{d(u, v) \mid v \in S\}.$$

Furthermore, the ball with center v and radius r is

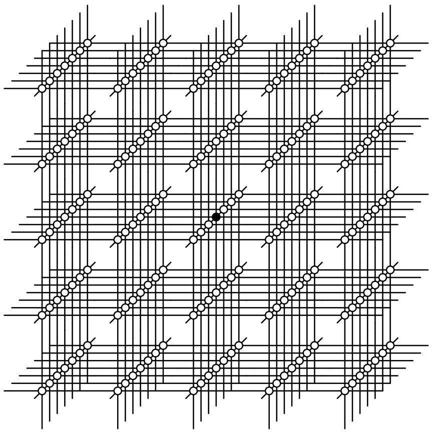
$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

We call any $C \subseteq V$ a *code*. The vertices of C are called *codewords*. In particular, C is an *r-identifying code* if the sets

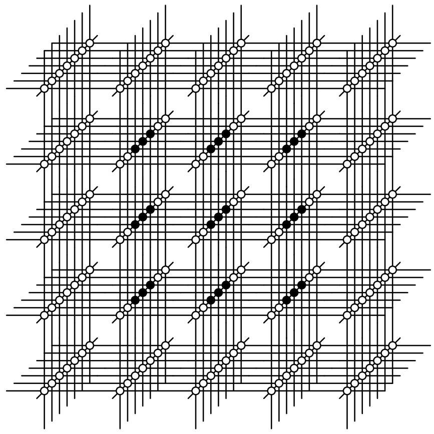
$$I_r(v) = B_r(v) \cap C$$

are distinct and non-empty for all vertices $v \in V$.

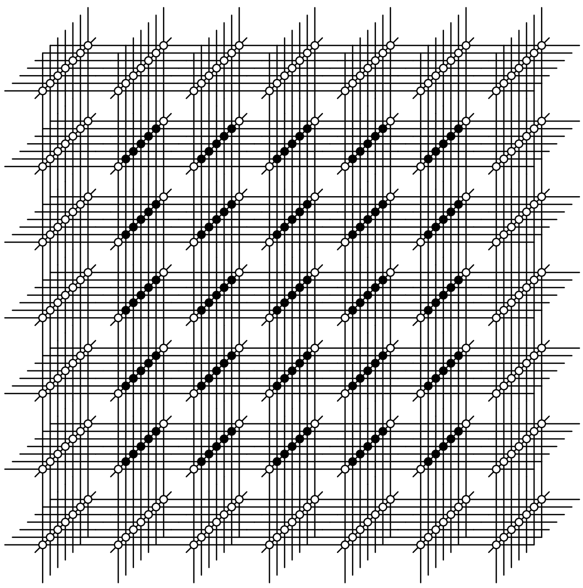
In this paper, we study r-identifying codes in the infinite n-dimensional king grid. The infinite n-dimensional king grid is the graph where $V = \mathbb{Z}^n$ and vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are adjacent if $|u_i - v_i| \leq 1$ for all $i \in \{1, \ldots, n\}$ and $u \neq v$. Figure 1 illustrates the 3-dimensional king grid.



(a) A vertex v in the infinite 3-dimensional lattice



(b) The black dots belong to $B_1(v)$ in the infinite 3-dimensional king grid.



(c) The black dots belong to $B_2(v)$ in the infinite 3-dimensional king grid.

Figure 1: Balls in the infinite 3-dimensional king grid are cubes each side of which contains $2r + 1$ vertices. Figures 1(b) and 1(c) illustrate the vertices in the balls with radius 1 and 2, respectively.

A special interest of the theory of identifying codes is to search for an identifying code with the smallest possible cardinality. However, the number of codewords is infinite for all identifying codes in the infinite n-dimensional king grid, and we instead of consider the density of C defined by

$$D(C) = \limsup_{k\to\infty} \frac{|C \cap Q_k^n|}{|Q_k^n|},$$

where $|C\cap Q_k^n|$ is the number of codewords in the ball $Q_k^n = B_k(\mathbf{0}) = \{(x_1,\dots,x_n) \mid |x_i| \leq k, \forall i \in \{1,\dots,n\}\}$ and $|Q_k^n|$ is the number of vertices in the ball. Often, we simply write Q_k instead of Q_k^n if the dimension n is clear from the context. In particular, an identifying code the density of which is the smallest is called *optimal*.

The concept of identifying codes was introduced in the late 1990s by Karpovsky, Chakrabarty and Levitin [7]. Two motivations of such codes are a fault diagnosis of a multiprocessor system [7] and a safeguard analysis of a facility using sensor networks [9]. The identifying codes in the 3-dimensional graphs can for example be used observing location of one object in the three-dimensional space.

In this paper, we construct two families of r-identifying codes and prove that the codes in one of the families are optimal r-identifying codes in the 3-dimensional king grid for all $r \geq 15$. In particular, we shall also show that the density of optimal identifying codes in the infinite 3-dimensional king grid is $\frac{1}{8r^2}$ when $r \geq 3$. Moreover, we show a general lower bound for such codes for all values of r and n. In particular, if $r \gg n$, then we shall see that these lower bounds are very near to the upper bounds obtained from these constructions. Finally, we give an idea how one can prove better bounds on r-identifying codes in the 3-dimensional king grid when $r = 1$ or $4 \leq r \leq 14$. Since there probably exist no r-identifying codes with such density, we do not prove the lower bounds in detail.

The r-identifying codes in the 2-dimensional king grid (which is often called simply the king grid) has been studied in [2–6, 8], for example. The infinite one-dimensional king grid is simply the doubly infinite path, where identifying codes are studied in [1], for instance. Identifying codes in the n-dimensional graphs have been studied for example in [11]. In [11] codes in n-dimensional square lattices where $V = \mathbb{Z}^n$ and

$$E = \left\{ \{(x_1,\dots,x_n),(y_1,\dots,y_n)\} \;\middle|\; \sum_{i=1}^{n} |x_i - y_i| = 1 \right\}$$

are studied. In 3-dimensional square lattices, the balls are octahedra, whereas the balls are cubes in the 3-dimensional king grid. More papers on identifying codes and closely related classes of codes in many graphs can be found in [12].

Table 1 summarizes what is now known about the density of r-identifying codes in the infinite n-dimensional king grid.

2 Constructions

Theorem 1. *There is an r-identifying code with density*

$$D(C) = \min\left\{ \frac{r}{(2r)^n}, \frac{2^n - 1}{2\cdot(4r+2)^{n-1}} \right\}$$

in the infinite n-dimensional king grid.

r	$n=2$ lower and upper bounds		$n=3$ lower bound	$n=3$ upper bound	$n \geq 4$ lower bound	$n \geq 4$ upper bound
1	$\frac{2}{9}$	[3], [4]	$\frac{1}{12}$	$\frac{7}{72}$	$\frac{3}{4} \cdot \frac{1}{3^{n-1}}$	$\frac{2^n-1}{2^n} \cdot \frac{1}{3^{n-1}}$
2	$\frac{1}{8}$	[2], [3]	$\frac{1}{40}$	$\frac{1}{32}$	$\frac{5}{8} \cdot \frac{1}{5^{n-1}}$	$\frac{2^n-1}{2^n} \cdot \frac{1}{5^{n-1}}$
3	$\frac{1}{12}$	[2], [3]	$\frac{1}{84}$	$\frac{1}{72}$	$\frac{7}{12} \cdot \frac{1}{7^{n-1}}$	$\begin{cases} \frac{3}{6^n} & \text{if } n \leq 5 \\ \frac{2^n-1}{2^n} \cdot \frac{1}{7^{n-1}} & \text{if } n \geq 6 \end{cases}$
$4, \ldots, 14$	$\frac{1}{4r}$	[2]	$\frac{1}{8r^2+10\frac{2}{3}}$	$\frac{1}{8r^2}$	$\frac{1}{(2r+1)^{n-3}(8r^2+10\frac{2}{3})}$	$\min\{\frac{r}{(2r)^n}, \frac{2^n-1}{2\cdot(4r+2)^{n-1}}\}$
≥ 15	$\frac{1}{4r}$	[2]	$\frac{1}{8r^2}$	$\frac{1}{8r^2}$	$\frac{1}{8r^2\cdot(2r+1)^{n-3}}$	$\min\{\frac{r}{(2r)^n}, \frac{2^n-1}{2\cdot(4r+2)^{n-1}}\}$

Table 1: The known bounds for identifying codes in the infinite n-dimensional king grid. Here, the upper bound means that there exists an identifying code with that density and the lower bound means that density of every identifying code is at least the value given in the table.

–	–	–	–	–	–	–	–	–	–	–	–
–	–	4	–	–	–	4	–	–	–	4	–
–	–	–	–	–	–	–	–	–	–	–	–
2	–	–	–	2	–	–	–	2	–	–	–
–	–	–	–	–	–	–	–	–	–	–	–
–	–	**4**	–	–	–	4	–	–	–	4	–
–	–	–	–	–	–	–	–	–	–	–	–
2	–	–	–	2	–	–	–	2	–	–	–

Figure 2: A part of the 2-identifying code $C_1(2,3)$ in the infinite 3-dimensional king grid. The vertex (x,y,z) is a codeword if the cell (x,y) contains the number which is congruent modulo 4 with z. The origin is the place where the number is bold.

Proof. We show that codes

$$C_1(r,n) = \{(x_1,\ldots,x_n) \,|\, x_1 \equiv \ldots \equiv x_n \pmod{2r},\ x_1 \equiv 0 \pmod 2\}$$

and

$$C_2(r,n) = \{(x_1,\ldots,x_n) \,|\, x_1 \equiv \ldots \equiv x_n \pmod{2r+1},\ x_1 \cdots x_n \equiv 0 \pmod 2\}$$

are r-identifying codes in the infinite n-dimensional king grid. A part of the codes $C_1(2,3)$ and $C_2(1,3)$ are shown as Figures 2 and 3, respectively.

Let $v = (x_1,\ldots,x_n)$ and $u = (y_1,\ldots,y_n)$. First, we observe that $I_r(v) \neq \emptyset \neq I_r(u)$. Assume then that $x_1 < y_1$. (Case $x_1 > y_1$ is proved in the same way.)

If $y_1 - x_1 > 2r$, then $B_r(v) \cap B_r(u) = \emptyset$ and so also $I_r(v) \neq I_r(u)$. Then we assume $0 < y_1 - x_1 \leq 2r$. Now,

$$A_1 = B_r(v) \cap \{(x_1 - r, a_2, \ldots, a_n) \,|\, a_2, \ldots, a_n \in \mathbb{Z}\} \subseteq B_r(v) \setminus B_r(u)$$

and

$$A_2 = B_r(u) \cap \{(x_1 + r + 1, b_2, \ldots, b_n) \,|\, b_2, \ldots, b_n \in \mathbb{Z}\} \subseteq B_r(u) \setminus B_r(v).$$

–	–	2	–	–	2, 5	–	–	2	–	–	2, 5
–	1, 4	–	–	1, 4	–	–	1, 4	–	–	1, 4	–
6	–	–	3, 6	–	–	6	–	–	3, 6	–	–
–	–	2, 5	–	–	2, 5	–	–	2, 5	–	–	2, 5
–	1, 4	–	–	4	–	–	1, 4	–	–	4	–
3, 6	–	–	3, 6	–	–	3, 6	–	–	3, 6	–	–
–	–	2	–	–	2, 5	–	–	2	–	–	2, 5
–	1, 4	–	–	1, 4	–	–	1, 4	–	–	1, 4	–
6	–	–	3, 6	–	–	6	–	–	3, 6	–	–
–	–	2, 5	–	–	2, 5	–	–	2, 5	–	–	2, 5
–	1, 4	–	–	4	–	–	1, 4	–	–	4	–
3, 6	–	–	**3,6**	–	–	3, 6	–	–	3, 6	–	–

Figure 3: A part of the 1-identifying code $C_2(1,3)$ in the infinite 3-dimensional king grid. The vertex (x,y,z) is a codeword if the cell (x,y) contains the number which is congruent modulo 6 with z. The origin is the place where the numbers are bold.

We show that A_1 contains a codeword if $x_1 - r$ is even. (Otherwise $x_1 + r + 1$ is even and A_2 contains a codeword.) First, the second condition in both $C_1(r,n)$ and $C_2(r,n)$ is valid for all vertices in A_1. Now, A_1 contains exactly one vertex for every element of $(x-r) \oplus \mathbb{Z}_{2r+1}^{n-1}$ and then also at least one vertex for every element of $(x-r) \oplus \mathbb{Z}_{2r}^{n-1}$. Then, $x_1 \equiv \ldots \equiv x_n \pmod{2r+1}$ and $x_1 \equiv \ldots \equiv x_n \pmod{2r}$ are valid for at least one vertex in A_1.

Thus, $I_r(v) \neq I_r(u)$, if the first coordinates are distinct. In the same way, we can prove that $I_r(v) \neq I_r(u)$ if v and u are distinct in any other coordinate. Notice, that the second condition of the definition of $C_1(r,n)$ can be replaced by the condition $x_i \equiv 0 \pmod 2$ for any $i \in \{1,\ldots,n\}$ since if the first condition is valid, then all coordinates are either even or odd.

Moreover, the densities of the codes are

$$D(C_1(r,n)) = \frac{r}{(2r)^n}$$

and

$$D(C_2(r,n)) = \frac{2^n - 1}{2 \cdot (4r+2)^{n-1}}.$$

Indeed, $C_1(r,n)$ consists of tiles with $(2r)^n$ vertices, and for every even number in $\{2,\ldots,2r\}$, there is exactly one combination of coordinates where all coordinates are congruent modulo $2r$. Then a tile contains exactly r codewords. In the same way, $C_2(r,n)$ consists of tiles with $2\cdot(2(2r+1))^{n-1}$ vertices where the first coordinate has two choices and the other coordinates have $4r+2$ choices. For each coordinate except the first, there are two choices – even and odd – which are congruent to the first coordinate modulo $2r+1$ irrespective of the first coordinate. Then the tile contains 2^n vertices that satisfy the first condition of the definition of $C_1(r,n)$. However, exactly one choice of these has all coordinates odd and therefore this is not a codeword. Thus, the tile contains $2^n - 1$ codewords. □

3 Lower bounds in the n-dimensional king grid

Lemma 2. *Assume that the density of each r-identifying code in the n-dimensional king grid is at least D, then $\alpha(Q_k^n)$ contains at least*

$$|Q_{k+r+1}^n| \cdot D - |Q_{k+r+1}^n \setminus Q_k^n|$$

codewords for all isometric α.

Proof. Assume to the contrary that C is an r-identifying code and $|C \cap Q_k^n| = d < |Q_{k+r+1}^n| \cdot D - |Q_{k+r+1}^n \setminus Q_k^n|$. Let $A = (C \cap Q_k^n) \cup (Q_{k+r+1}^n \setminus Q_k^n) \subseteq \mathbb{Z}^n$ be a set. Now,

$$B = \left\{ \mathbf{u} + \mathbf{v} \,\middle|\, \mathbf{u} \in A \text{ and } \frac{\mathbf{v}}{2k + 2r + 3} \in \mathbb{Z}^n \right\}$$

is an r-identifying code with density $\frac{d + |Q_{k+r+1}^n \setminus Q_k^n|}{|Q_{k+r+1}^n|} < D$, which is a contradiction. Indeed, sets $I_{r,B}(v)$ are non-empty and distinct for all $v \in Q_{k+r+1}^n$ since $(C \cap Q_{k+2r+2}^n) \subseteq (B \cap Q_{k+2r+2}^n)$. Moreover, if $B_r(v) \cap B_r(u) \neq \emptyset$ for some $v \in Q_{k+r+1}^n$ and $u \notin Q_{k+r+1}^n$, then $(B_r(v) \setminus B_r(u)) \cap (Q_{k+1}^n \setminus Q_k^n) \neq \emptyset$ and so $I_{r,B}(v) \neq I_{r,B}(u)$ since $(Q_{k+1}^n \setminus Q_k^n) \subseteq B$. Then, B is an r-identifying code by symmetry. □

Theorem 3. *If the density of each r-identifying code in the $(n-1)$-dimensional king grid is at least D, then the density of each r-identifying code in the n-dimensional king grid is at least $\frac{D}{2r+1}$.*

Proof. Let C be an r-identifying code in the n-dimensional king grid. Moreover, we assume that the density of every r-identifying code is at least D in the $(n-1)$-dimensional king grid. Next, we define code C_a in $(n-1)$-dimensional king grid by

$$\mathbf{c} \in C_a \iff (\mathbf{c} \oplus b) \in C \text{ for some } b \in \{a - r, \ldots, a + r\}.$$

Now, if $(\mathbf{c} \oplus b) \in I_{r,C}(\mathbf{v} \oplus a) \setminus I_{r,C}(\mathbf{u} \oplus a)$ then $d(\mathbf{c} \oplus b, \mathbf{v} \oplus a) \leq r$ and so $b \in \{a - r, \ldots, a + r\}$ and $d(\mathbf{c}, \mathbf{v}) \leq r$ i.e. $\mathbf{c} \in I_{r,C_a}(\mathbf{v})$. On the other hand, $d(\mathbf{c}, \mathbf{u}) > r$ since $b \in \{a - r, \ldots, a + r\}$. Therefore, $\mathbf{c} \in I_{r,C_a}(\mathbf{v}) \setminus I_{r,C_a}(\mathbf{u})$. In the same way, we also see that if $I_{r,C}(\mathbf{v} \oplus a) \neq \emptyset$, then $I_{r,C_a}(\mathbf{v}) \neq \emptyset$. Thus, C_a is an r-identifying code in the $(n-1)$-dimensional king grid for all $a \in \mathbb{Z}$.

By Lemma 2, the number of codewords in $C_a \cap Q_k^{n-1}$ is at least $|Q_{k+2r}^{n-1}| \cdot D - |Q_{k+r+1}^{n-1} \setminus Q_k^{n-1}|$. Moreover, a codeword in $C \cap Q_k^n$ can contribute one codeword into at most $2r + 1$ codes C_a, where $a \in \{-k + r, \ldots, k - r\}$. On the other hand, every codeword in C_a ($a \in \{-k + r, \ldots, k - r\}$) has been contributed by some codeword in $C \cap Q_k^n$. Hence,

$$|C \cap Q_k^n| \geq \frac{\sum_{a=-k+r}^{k-r} |C_a \cap Q_k^{n-1}|}{2r + 1} \geq \frac{(2k - 2r + 1)(|Q_{k+2r}^{n-1}| \cdot D - |Q_{k+r+1}^{n-1} \setminus Q_k^{n-1}|)}{2r + 1}.$$

and

$$\begin{aligned} D(C) &= \limsup_{k \to \infty} \frac{|C \cap Q_k^n|}{|Q_k^n|} \\ &\geq \limsup_{k \to \infty} \left(\frac{D}{2r + 1} \cdot \frac{(2k - 2r + 1)|Q_{k+2r}^{n-1}|}{|Q_k^n|} - \frac{(2k - 2r + 1)|Q_{k+r+1}^{n-1} \setminus Q_k^{n-1}|}{(2r + 1)|Q_k^n|} \right) \\ &= \frac{D}{2r + 1}. \end{aligned}$$

□

Corollary 4. *The densities of* 2*-identifying and* 3*-identifying codes in the infinite* 3*-dimensional king grid are at least* $\frac{1}{40}$ *and at least* $\frac{1}{84}$*, respectively.*

Theorem 3 gives the lower bound $\frac{1}{4r(2r+1)^{n-2}}$ for $r \geq 2$ when we use the known lower bound $\frac{1}{4r}$ for r-identifying code in 2-dimensional king grid. Moreover, we can improve the lower bound a little with the help of the results in the next section when $r \geq 4$. On the other hand, we saw in the previous section that there exists an r-identifying code whose density is $\frac{r}{(2r)^n}$. Now, the ratio of the lower and upper bounds is $(\frac{2r}{2r+1})^{n-2}$, which is very near to 1 when $r \gg n$. Therefore, it is strong possibility that the next conjecture is valid.

Conjecture. For every n, there exists a number r_n s.t. the density of optimal r-identifying code in the infinite n-dimensional king grid is $\frac{r}{(2r)^n}$ for all $r \geq r_n$.

For large n, the density of the code $C_1(r,n)$ is less than the density of the code $C_2(r,n)$ when $r > \frac{n}{2\ln 2}$. Therefore, it is an interesting additional question whether r_n can be chosen in such a way that $r_n = \alpha n$ where α is constant.

4 Lower bound for the r-identifying codes in the 3-dimensional king grid when $r \geq 15$

In this section, we consider the case $n = 3$ and show that the density of r-identifying code is at least $\frac{1}{8r^2}$ when $r \geq 15$. The proof is based on studying the symmetric differences of eight vertices whose pairwise graphical distances of these vertices are all 1. The vertices which belong to the symmetric differences form the outer layer of a cube. We will notice that the faces of all these cubes contain at least three codewords or else another cube contains more than three codewords in its faces. However, we will see in Theorem 9 that the faces of the cubes contain at least three codewords on average.

In the future, the term *cube* always refers to the symmetric differences of these eight vertices as defined in Definition 6.

Definition 5. The set

$$M((x,y,z)) = \{(a,b,c) \mid a \in \{x, x+1\},\ b \in \{y, y+1\} \text{ and } c \in \{z, z+1\}\}$$

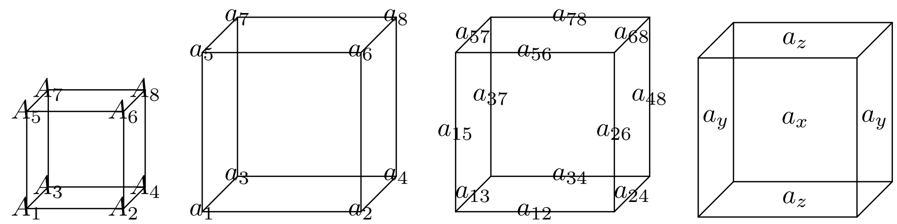
is called a *minicube*.

Definition 6. The set

$$C_r(v) = \{u \in \mathbb{Z}^3 \mid d(u, M(v)) = r\}$$

is called a *cube*. We partition a cube into three sets: *faces* (F), *corners* (O) and *sides* (S) as follows:

$$\begin{aligned} F(v) &= \{u \in C_r(v) \mid \text{there are four vertices } w \text{ in } M(v) \text{ s.t. } d(u,w) = r\}, \\ S(v) &= \{u \in C_r(v) \mid \text{there are two vertices } w \text{ in } M(v) \text{ s.t. } d(u,w) = r\}, \\ O(v) &= \{u \in C_r(v) \mid \text{there is only one vertex } w \text{ in } M(v) \text{ s.t. } d(u,w) = r\}. \end{aligned}$$



(a) Vertices of a minicube (b) Corners of a cube (c) Sides of a cube (d) Faces of a cube

Figure 4: The names of the vertices of a minicube and of the parts of a cube

We say that $u \in F$ and $v \in F$ are *in the parallel faces* if either $d(u, w) = d(v, w)$ for all $w \in M(x, y, z)$ or $d(u, w) \neq d(v, w)$ for all $w \in M(x, y, z)$. Moreover, the last vertices in the sides are called *end-of-sides*. Notice that the end-of-sides and corners are not the same vertices and that the distance from an end-of-side to the nearest corner is one.

In addition, we give names $A_1, \ldots, A_8$ to the vertices of a minicube and names a_i or a_{ij} to corners and sides, respectively, if $d(u, A_i) = r$ (and $d(u, A_j) = r$) when $u \in a_i$ or $u \in a_{ij}$. Similarly, we can give the name of type a_{ijkl} to faces. However, we often call the faces by names a_x, a_y and a_z. Then, we call two parallel faces by the same name, but this is not a problem since two codewords in parallel faces separate the vertices $A_1, \ldots, A_8$ in the same way. Figure 4 illustrates the names of the parts of the cubes.

Lemma 7. *If C is an r-identifying code in the 3-dimensional king grid, then every cube contains a codeword in a corner, in two different parallel sides, or in three non-parallel faces.*

Proof. If there are three non-parallel faces which contain at least one codeword, then the claim is clear. Thus, we can assume by symmetry, that the right- and left-hand side faces a_y do not contain any codeword. Then, the codewords in the faces do not separate any of the following four pairs: A_1–A_2, A_3–A_4, A_5–A_6 and A_7–A_8. Moreover, we can assume that none of the corners contains any codeword since otherwise the claim is clear. Therefore all of these four pairs have to be separated by the codewords in the sides.

First, A_1 and A_2 can be separated from each other by only the codewords in a_{13}, a_{15}, a_{24} and a_{26}. Without loss of generality, we can assume that a_{13} contains a codeword. Furthermore, A_7 and A_8 must be separated. Then a_{37}, a_{57}, a_{48} or a_{68} contains a codeword. However, a_{57} and a_{68} are parallel with a_{13}, then if one of them contains codeword, we are done. Therefore we can assume that a_{37} or a_{48} contains a codeword.

However, none of the codewords in a_{13}, a_{37} or a_{48} separates A_5 and A_6. Therefore, a_{15}, a_{57}, a_{26} or a_{68} has to also contain a codeword. Thus, there must be two parallel sides containing a codeword. Indeed, a_{57} and a_{68} are parallel with a_{13} and a_{15} and a_{26} are parallel with both a_{37} and a_{48}. This proves the claim. $\square$

Remark 8. In the previous proof, we show in fact that if none of the corners nor the face a_y contain any codeword, then there are at least two parallel sides which both contain a codeword. To be exact, these sides must also be parallel with a_y since the other sides do not separate the pairs A_1–A_2, A_3–A_4, A_5–A_6 and A_7–A_8.

In the next theorem, we use an averaging method which is often called voting or discharging. In the method, every codeword has some fixed number (t_1) of votes and non-codewords have no votes initially. Next, we transfer votes from vertices to another vertices such that the total number of votes does not change. Finally, we show that every vertex has at least t_2 votes and furthermore we can conclude that the density of the code is at least $\frac{t_2}{t_1}$.

Theorem 9. *The density of each r-identifying code in the infinite 3-dimensional king grid is at least $\frac{1}{8r^2}$ for all $r \geq 15$.*

Proof. Assume that C is an r-identifying code in the infinite 3-dimensional king grid and $r \geq 15$.

Voting rules: First, let every codeword have $24r^2$ votes and non-codewords have no votes. Then there are $24r^2|C \cap Q_k|$ votes in Q_k. Next, the codewords give votes for cubes s.t. codeword c gives one vote to cube $C_r(v)$ if $c \in F(v)$. In particular, since $|F(v)| = 24r^2$, then every codeword gives exactly all its votes to the cubes. This is called the initial state.

We will give three voting rules and show that every cube has at least three votes when the rules have been applied.

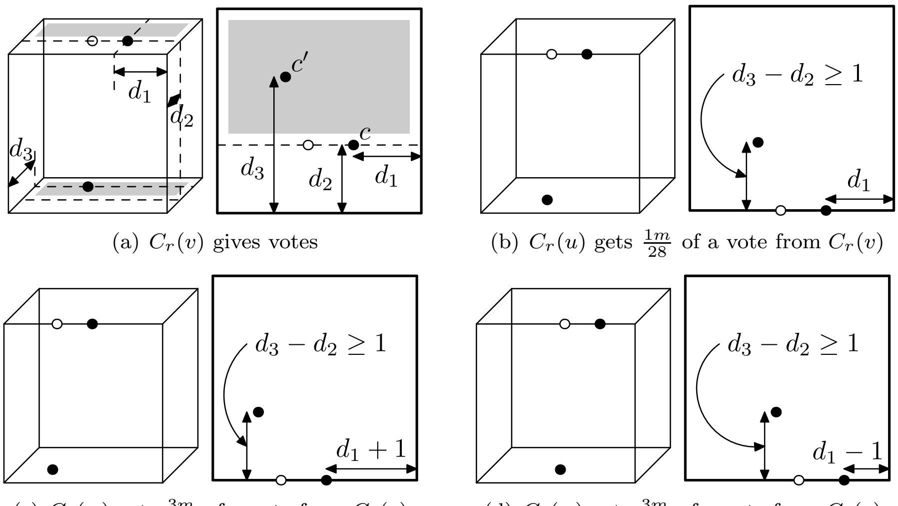
During the first voting rule, cube $C_r(v)$ gives $\frac{1}{28}$ of a vote to cube $C_r(u)$ if two coordinates of v and u are the same and there are at least two codewords c and c' in two parallel faces $f_v \in \{a_x, a_y, a_z\}$ of $C_r(v)$ s.t. c also belongs to side s of $C_r(u)$ and c' belongs to face f_u of $C_r(u)$ where f_u and f_v are in the same plane. See Figures 5(a) and 5(b). In particular, if codeword c can be chosen in m different ways, then $C_r(v)$ gives $\frac{m}{28}$ of a vote to $C_r(u)$. Also, $C_r(v)$ gives $\frac{m-1}{28}$ of a vote to cube $C_r(u)$, if codeword c can be chosen in m different ways, but there is no codeword c'. See Figures 6(a) and 6(b).

Furthermore during the first voting rule, $C_r(v)$ gives $\frac{3}{28}$ of a vote to $C_r(w)$ if the following three condition are valid: $C_r(u)$ gets $\frac{1}{28}$ of a vote from $C_r(v)$ by the way which is given in the previous paragraph; and Euclidean distance between u and w is 1; and c belongs to $O(w) \cup S(w)$. Again, if $C_r(v)$ gives $\frac{m}{28}$ of a vote to $C_r(u)$, then it gives $\frac{3m}{28}$ of a vote to $C_r(w)$. See Figures 5(c), 5(d), 6(c) and 6(d).

In particular, the cube gives $\frac{1}{28}$ of a vote to another cube per each other codeword in a_z except the codeword which is the furthest (or one of the furthest if there are more than one) from side a_{12} or a_{56}. Moreover, it gives $\frac{3}{28}$ of a vote to two other cubes per each other codeword in a_z except the furthest codeword. And so it gives $\frac{7}{28} = \frac{1}{4}$ of a vote to one direction per each codeword in a_z except one. Therefore it gives $\frac{m-1}{4}$ votes to one direction by the codewords in a_z if a_z contains $m \geq 1$ codewords. However, there are four symmetric directions, and then the cube gives $m-1$ votes by the codewords in a_z during the first voting rule. Thus, the cube has $n_1 + n_2$ votes after the first voting rule, where $n_1 \in \{0, 1, 2, 3\}$ is the number of non-parallel faces which contain at least one codeword, and n_2 is the number of votes, which the cube gets from other cubes during the first voting rule.

Most of the cubes have at least three votes now. There may nevertheless be some cubes which have fewer than three votes. Therefore we still need the second and the third voting rules which are illustrated in Figures 7 and 8, respectively.

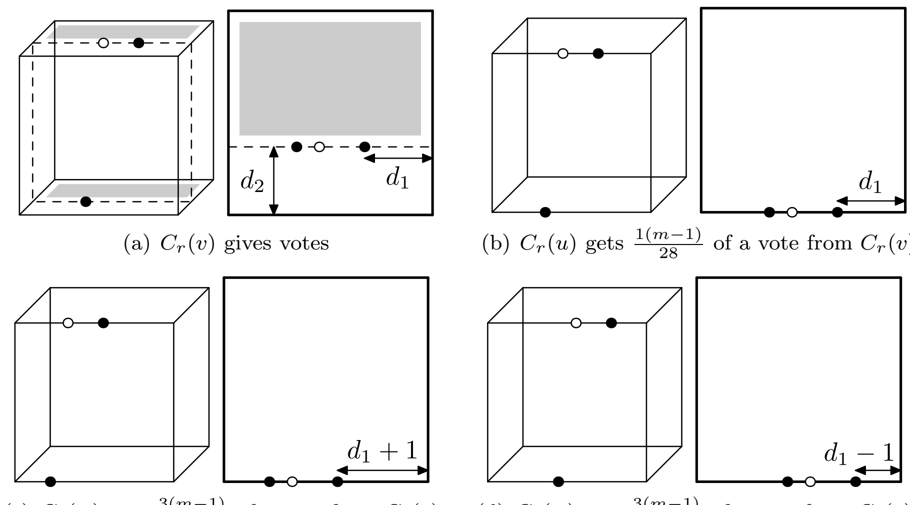
When we have first shown that every cube has at least three votes after three voting rules have been applied. Then, we can still transfer all votes from the cube $C_r(v)$ to vertex v if necessary. Then, we can observe that there have to be at least $3|Q_k|$ votes in Q_k finally. However, some of the vertices in Q_k may have got some



(a) $C_r(v)$ gives votes (b) $C_r(u)$ gets $\frac{1m}{28}$ of a vote from $C_r(v)$

(c) $C_r(w)$ gets $\frac{3m}{28}$ of a vote from $C_r(v)$ (d) $C_r(w)$ gets $\frac{3m}{28}$ of a vote from $C_r(v)$

Figure 5: Voting rule 1 if the grey area contains at least one codeword c': There are two pictures of the cubes – one shows the whole sphere of the cube and the other show only the lowest and the highest faces (i.e. a_z) and the sides and corners which are parallel with these faces. The amount of votes which $C_r(v)$ gives to other cubes is given in the caption of figures 5(b)–5(d). Number m is the number of codewords whose distance is d_2 from the front sides a_{12} or a_{56} in Figure 5(a). Distances $d_1, d_2, d_3 \in \{1, \ldots, 2r\}$. The black dots are codewords, and circles illustrates possible other codewords, if $m \geq 2$. Possible codewords within distance $d_2 - 1$ from a_{12} or a_{56} in Figure 5(a) do not influence how many votes $C_r(u)$ and $C_r(w)$ get.



(a) $C_r(v)$ gives votes (b) $C_r(u)$ gets $\frac{1(m-1)}{28}$ of a vote from $C_r(v)$

(c) $C_r(w)$ gets $\frac{3(m-1)}{28}$ of a vote from $C_r(v)$ (d) $C_r(w)$ gets $\frac{3(m-1)}{28}$ of a vote from $C_r(v)$

Figure 6: Voting rule 1 if the grey area does not contain any codeword c'.

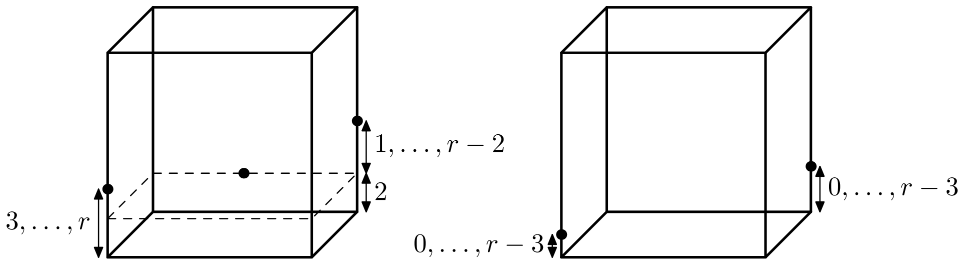


Figure 7: Voting rule 2: The left-hand side cube gives three votes the right-hand side cube. The distance between the left-hand side and the right-hand side cube is three. The left-hand side cube contains two codewords in the opposite vertical sides at distance $3, \ldots, r$ from the lowest face a_{1234} and one codeword in faces or sides at distance two from a_{1234}. All isometries of the cubes give votes in the same way. Moreover there may extra codewords in the cube, then the left-hand side cube can give three votes more than one direction, but it gets at most three votes from one direction.

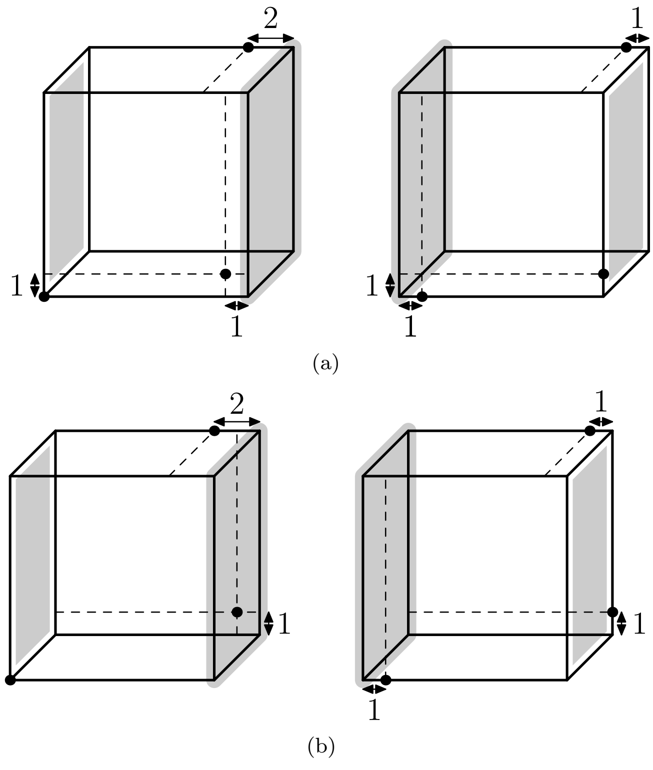
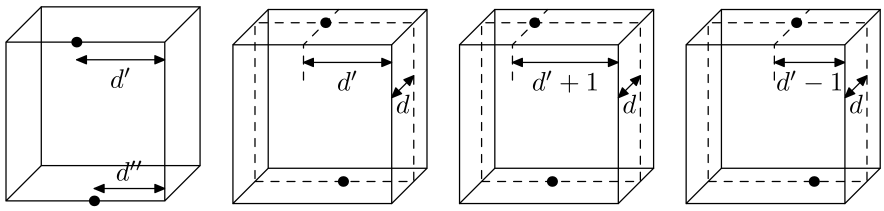


Figure 8: Voting rule 3: The left-hand side cubes gives one vote to the right-hand side cubes. The black dots are the only codewords in the white areas. The grey areas can contain any codewords. The distance between the left-hand side and the right-hand side cube is one. All isometries of the cubes give votes in the same way.



(a) $C_r(u)$ (b) $C_r(v_d)$ (c) $C_r(w_d)$ (d) $C_r(w'_d)$

Figure 9: Cube $C_r(u)$ which has two codewords in two adjacent parallel sides gets at least seven votes all in all if $d' \in [2, r-1]$. Indeed, $C_r(v_d)$ gives at least $\frac{1}{28}$ of a vote and $C_r(w_d)$ and $C_r(w'_d)$ give $\frac{3}{28}$ of a vote each by voting rule 1 for all values of $d \in [1, 2r]$.

votes from outside of Q_k. Indeed, cube $C_r(v)$ (or the vertex v) gets votes from vertices within distance $r+1$ in the initial state; within distance $2r+1$ in the first voting rule; within distance three in the second voting rule; and within distance one in the third voting rule. Therefore, the votes of v might come from the codewords within distance $3r+6$ from v. In any case, there were $24r^2|C \cap Q_{k+3r+6}|$ votes in Q_{k+3r+6} initially, and finally, at least $3|Q_k|$ votes in Q_k and all these votes come from the codewords in Q_{k+3r+6}. Thus we have

$$3|Q_k| \leq 24r^2|C \cap Q_{k+3r+6}| \leq 24r^2(|C \cap Q_k| + |Q_{k+3r+6} \setminus Q_k|)$$

and so

$$D(C) = \limsup_{k\to\infty} \frac{|C \cap Q_k|}{|Q_k|} \geq \limsup_{k\to\infty} \left(\frac{3}{24r^2} - \frac{|Q_{k+3r+6} \setminus Q_k|}{|Q_k|} \right) = \frac{1}{8r^2}.$$

The main ideas in transferring votes: Next, we prove that every cube has at least three votes after all three voting rules have been applied. The main idea in the proof is the following: If there are two codewords in parallel adjacent sides, for example a_{12} and a_{56}, in cube $C_r(u)$ (cf. Figure 9), then $C_r(u)$ gets at least $\frac{1}{28}$ votes from $2r$ cubes and at least $\frac{3}{28}$ votes from $4r$ cubes. Indeed, $C_r(u)$ gets votes for each of $C_r(v_d)$, $C_r(w_d)$ and $C_r(w'_d)$ and for all values of $d \in \{1, 2, \ldots, 2r\}$. Thus, $C_r(u)$ gets at least $\frac{14r}{28} = \frac{r}{2} \geq \frac{15}{2} > 7$ votes all in all by the codewords in two adjacent parallel sides. However, if one or both of the codewords is located in the end-of-sides of $C_r(u)$ (i.e., $d' \in \{1, 2r\}$ or $d'' \in \{1, 2r\}$ in 9(a)), then one of the codewords in Figure 9(c) or 9(d) is located in a side, not in the face. Therefore, $C_r(w)$ or $C_r(w')$ does not give any votes to $C_r(u)$ by the first voting rule. These cases are studied later in the part *Special cases*.

The second important idea is the case where $C_r(u)$ contains two parallel opposite sides, say for example a_{12} and a_{78}, which both have a codeword (c and c', resp.). See Figure 10. Now, if in addition at least one of the parallel faces a_z (or a_x) contains a codeword c'' and $d(c'', a_{1256}) = t \in \{1, \ldots 2r\}$ (or $d(c'', a_{1234}) = t \in \{1, \ldots 2r\}$, resp.), then there are $2r - t$ cubes, which contain both c and c'' in the face a_z and they give $\frac{1}{28}$ votes to $C_r(u)$. On the other hand, there are $t-1$ cubes, which contain both c' and c'' in the face a_z and they also give $\frac{1}{28}$ votes to $C_r(u)$. And so $C_r(u)$ gets at least $\frac{1}{28}$ votes from $2r - 1 \geq 29$ different cubes, when $r \geq 15$. Moreover, if none of c, c' and c'' is within graphical distance one from a_y, then $C_r(u)$ also gets

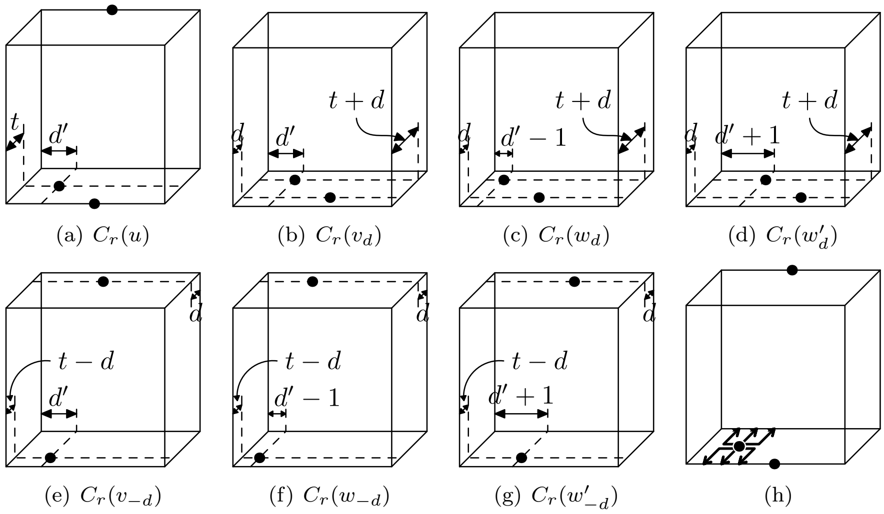


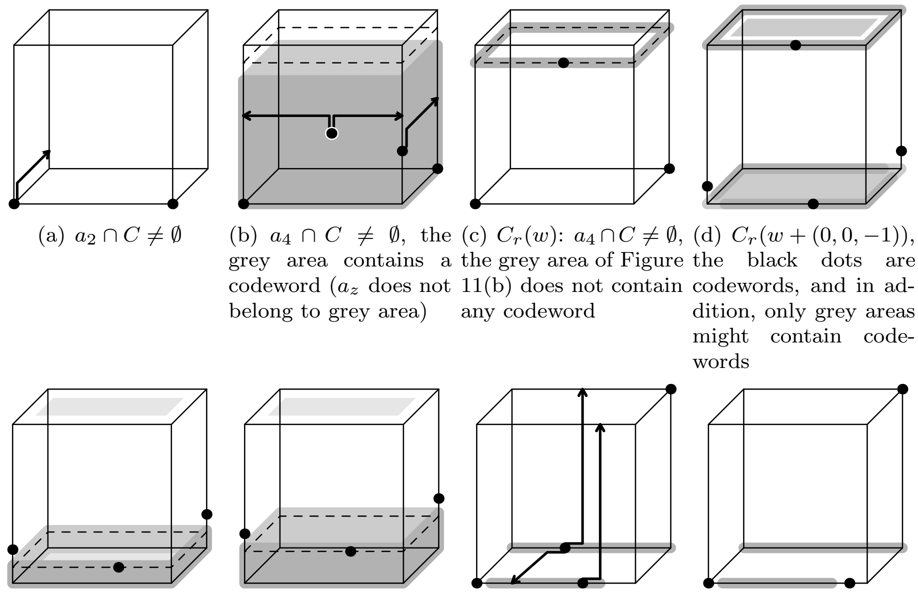
Figure 10: Cube $C_r(u)$ which has two codewords in two opposite parallel sides gets at least seven votes all in all if $d' \in [2, r-1]$ and none of the black dots in 10(a) is in the end-of-sides. Indeed, $C_r(v_d)$ gives at least $\frac{1}{28}$ of a vote and $C_r(w_d)$ and $C_r(w'_d)$ give $\frac{3}{28}$ of a vote each by voting rule 1 for all values of d for which $t+d \in [1, 2r] \setminus \{t\}$.

at least $\frac{3}{28}$ votes from $2 \cdot (2r-1)$ cubes by Figures 10(c), 10(d), 10(f) and 10(g). The arrows in Figure 10(h) illustrate all the cubes which give votes to $C_r(u)$. Thus, $C_r(u)$ gets at least $\frac{7(2r-1)}{28} \geq 7$ votes in total when $r \geq 15$.

Notice, that if two opposite sides contain a codeword s.t. the codewords are not in the end-of-sides, then $C_r(u)$ nevertheless gets at least three votes by the first voting rule although faces a_z and a_x contain codewords only within distance one from a_y or do not contain any codewords. Indeed, A_4 and A_6 must nevertheless be separated from each other. Then a_{24}, a_{26}, a_{48}, a_{68}, a_4, a_6, a_x or a_z contains a codeword (or else a_{34} or a_{56} contains a codeword, but then there are two parallel and adjacent sides containing a codeword). Then all $2r-1$ cubes $C_r(w)$ (or all $2r-1$ cubes $C_r(w')$) give $\frac{3}{28}$ of a vote each to $C_r(u)$ and so $C_r(u)$ gets at least three votes.

Special cases: Now we have seen that every cube which does not have codewords in its corners or end-of-sides has at least three votes after the first voting rule. Indeed, by Lemma 7 we have already observed that this is valid for the cubes which have a codeword in its three non-parallel faces or which have a codeword in two different parallel sides if none of them is in the end-of-side. Furthermore, we shall prove the theorem for cubes which give votes away by the second and the third voting rules.

First, the cube $C_r(v)$ which gives three votes away by the second voting rule gets at least seven votes during the first voting rule. Therefore, the cube has at least four votes after the second voting rule. Moreover, if the cube gives three votes to more than one direction by the second voting rule, then the cube gets at least seven votes per each direction for which $C_r(v)$ gives votes away during the second voting rule. Indeed, if $C_r(v)$ gives votes to the cube $C_r(u)$ which is below $C_r(v)$, then there has to be two sides containing a codeword s.t. the sides are perpendicular with the



(a) $a_2 \cap C \neq \emptyset$

(b) $a_4 \cap C \neq \emptyset$, the grey area contains a codeword (a_z does not belong to grey area)

(c) $C_r(w)$: $a_4 \cap C \neq \emptyset$, the grey area of Figure 11(b) does not contain any codeword

(d) $C_r(w+(0,0,-1))$, the black dots are codewords, and in addition, only grey areas might contain codewords

(e) $C_r(w+(0,0,-2))$

(f) $C_r(w+(0,0,-3))$ gives three votes to $C_r(u)$ by voting rule 2.

(g) $a_8 \cap C \neq \emptyset$, the grey area contains a codeword

(h) $a_8 \cap C \neq \emptyset$, the grey area does not contain any codeword

Figure 11: At least two corners containing a codeword.

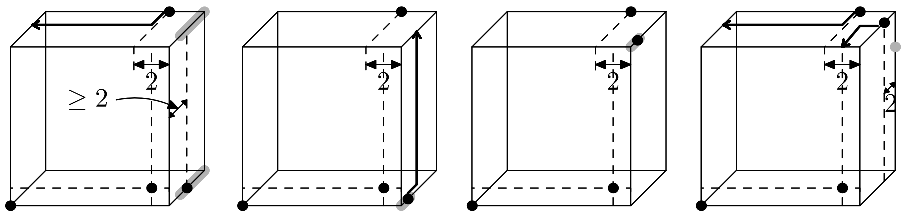
lowest face and both codewords are nearer the lowest face than the highest face. Therefore, only one direction can get votes from one codeword that is in a side.

Before we show that cubes which give votes away by the third voting rule, we prove that the cubes which have at least two codewords in corners have at least three votes after voting rule 2.

First, we assume that two adjacent corners a_1 and a_2 of $C_r(w)$ contain two codewords c and c', resp. See Figure 11(a). Now, $C_r(v)$ gets at least $\frac{3}{28}$ of a vote from $2r$ cubes, i.e., at least three votes all in all, when $r \geq 15$.

Second, we assume that $C_r(w)$ has two corners (say, a_1 and a_4) which are at Euclidean distance $2r\sqrt{2}$ from each other and which contain codewords c and c', respectively. However, c and c' do not separate A_2 and A_3. Therefore, there must be another codeword c'' in the cube and, in particular, the codeword can not be in the highest corners a_5–a_8, sides a_{56}–a_{78} nor face a_z. If c'' is not within distance one from a_{5678}, then the cube gets at least three votes by the arrows in Figure 11(b). (Notice, that there may be only one of the two upper codewords which are drawn in Figure 11(b).) Otherwise, we can assume that all codewords which are able to separate A_2 and A_3 are within distance one from a_{5678}. See Figure 11(c) where the two black dots are the only codewords below the dashed line. Now, none of the codewords in $C_r(w)$ can separate the vertices A_2 and A_3 of $M(w+(0,0,1))$. See Figure 11(d). Therefore, a_{12}, a_{13}, a_{24}, a_{34}, a_2 or a_3 of $C_r(w+(0,0,1))$ must contain another codeword c^*. Then, c, c' and c^* are situated in $C_r(w+(0,0,3))$ as the black dots in Figure 11(f) and so $C_r(w+(0,0,3))$ gives three votes to $C_r(w)$ by the second voting rule.

We will also need the similar argument as in the previous paragraph in the future, and we shall call the *argument of type* a_1a_4. In particular, the argument



(a) (b) (c) (d)

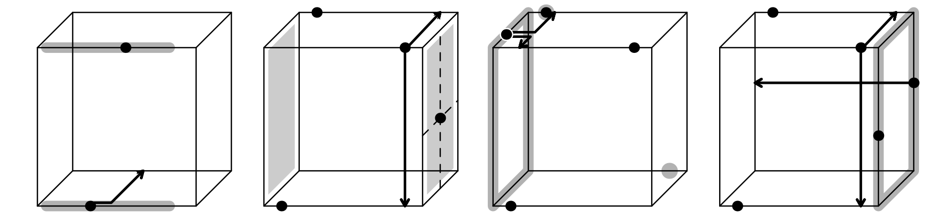
Figure 12: Cubes which gives votes by the third voting rule.

would also be valid although one or both of c and c' are situated in the side a_{15} or a_{48} of $C_r(w)$, respectively, if they are nevertheless within distance $r-3$ from the lowest face a_{1234} of $C_r(w)$.

Third, we assume that only the opposite corners a_1 and a_8 of $C_r(w)$ contain codewords c and c', respectively. Now, these codewords separate A_1 and A_8 from A_2–A_7. However, there must be at least three sides and non-parallel faces which contain a codeword so that A_2–A_7 can be separated from each other. If there are three codewords in three non-parallel faces, then we are done. So we can assume that there is at least one side which contains a codeword c^*. By symmetry, we can assume that $c^* \in a_{12}$ or $c^* \in a_{34}$. If $d(c^*, a_2) > 1$, then $C_r(w)$ gets at least three votes by one of the arrows in Figure 11(g). Otherwise, c^* is situated in the end-of-side of a_{12}. See Figure 11(h). Then the cube gets at least three votes by the argument of type a_1a_4 and by the location of codewords c' and c^*.

Now, we show that cube $C_r(u)$ which gives votes away by the third voting rule has at least three votes finally. See the left-hand side constellations in Figure 8. First, none of the black dots separates A_2 and A_6. Then some codeword in the grey area has to be separates these codewords. To be precise, this codeword must be situated in a_2, a_6, a_{24} or a_{68}. If there is a codeword, which separates A_2 and A_6, and the distance to both a_2 and a_6 is at least two (cf. Figure 12(a)), then the cube gets at least $\frac{3}{28}$ of a vote from $2r-2$ cubes and so at least three votes totally, when $r \geq 15$. If the codeword is within distance one from a_2 (in a_2 or a_{24}), then the cube gets at least three votes by the arrow in Figure 12(b). Otherwise, a_6 or the end-of-side in a_{68} at distance one from a_6 contains a codeword. Then the cube gets at least three votes by the argument of type a_1a_4. In every case, the cube gets one more vote in the initial state from the codeword in a_x and so the cube has at least four votes after the second voting rule. (Notice that no cube can give votes by both the second and the third voting rule.) Therefore, it can give one vote away in the third voting rule. However, it is possible that the cube gives more votes away during the third voting rule, but this is possible only, if at least two distinct sides contain a codeword at distance 2 from a_8. See Figure 12(d). Then the cube gets at least six votes by the first voting rule and it nevertheless gives at most one vote to three directions by the third voting rule.

Next, we study cubes which have no *important* codewords in a corner. We say that codewords in corners are *unimportant* if the vertices A_1–A_8 can also be separated without the codewords in corners. In particular, we can in addition assume that there is at most one codeword in corners since the other cases have already been considered. Then we know by Lemma 7 that there are two parallel sides which contain at least one codeword each (or there are three non-parallel faces

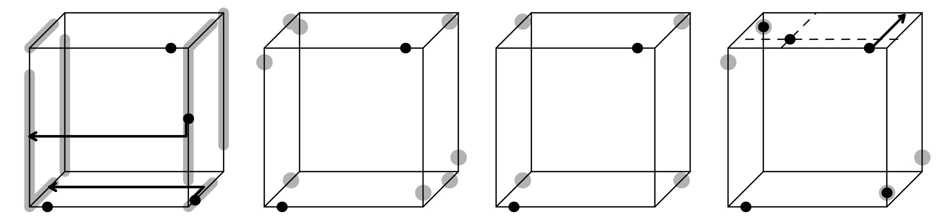


(a) $d(c, a_{2468}) \geq 2$, $d(c', a_{2468}) \geq 2$

(b) $a_{78} \cap C \neq \emptyset$, $(a_y) \cap C \neq \emptyset$

(c) $a_{78} \cap C \neq \emptyset$, $(a_{13} \cup a_{15} \cup a_{37} \cup a_{57}) \cap C \neq \emptyset$

(d) $a_{78} \cap C \neq \emptyset$, $(a_y \cup a_{13} \cup a_{15} \cup a_{37} \cup a_{57}) \cap C \neq \emptyset$



(e) $(a_{34} \cup a_{78}) \cap C = \emptyset$, grey areas contain at least one codeword in total.

(f) Only the grey dots can be codewords in addition to black dots in sides.

(g) $(a_z \cup a_{34} \cup a_{78}) \cap C = \emptyset$, at least two of the grey dots are codewords.

(h) $(a_{34} \cup a_{78}) \cap C = \emptyset$, $a_z \cap C \neq \emptyset$

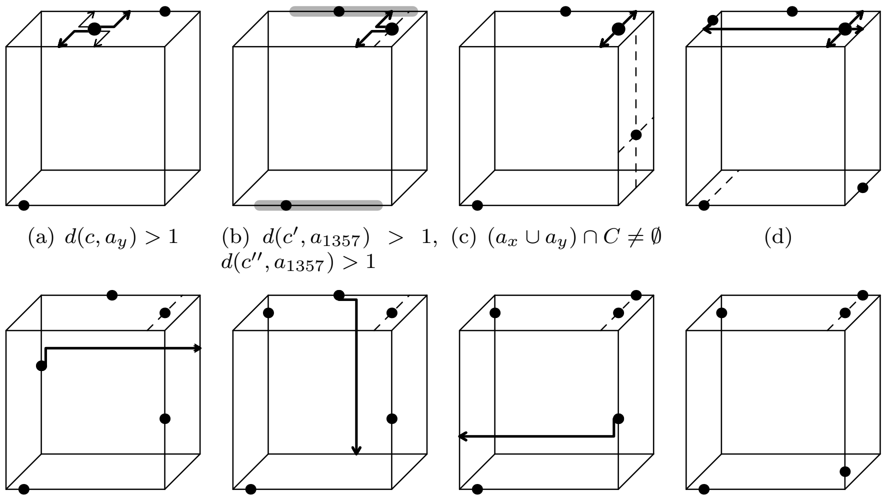
Figure 13: Two parallel and adjacent sides containing a codeword. None of the corners contains an important codeword.

with a codeword and then we are done).

First, we assume that two adjacent and parallel sides contain codewords c and c'. Let $c \in a_{12}$ and $c' \in a_{56}$, for example. We prove that the cube has at least three votes in Figure 13. First, if none of c and c' is within distance 1 from a_{2468} (or a_{1357}), then the cube gets at least three votes as in Figure 13(a).

Then, we can assume that $d(c, a_{1357}) = 1$ and $d(c', a_{2468}) = 1$. Moreover, we first assume that there is also a third parallel side, say a_{78}, which contains a codeword c^*. Now, c^* has to be in the end-of-side or else the case is of the type of Figure 13(a). See Figures 13(b)–13(d). Now, the cube gets two votes by the two arrows in Figure 13(b). Moreover, it gets one more vote initially if a_y contains a codeword (cf. Figure 13(b)) or three more votes if at least one of a_{13}, a_{15}, a_{37} and a_{57} contains a codeword (cf. Figure13(c)). Otherwise, both of a_{24} and a_{68} or both of a_{26} and a_{48} contain a codeword by Remark 8. Then the cube gets the third vote by the horizontal arrow in Figure 13(d).

Now, we can assume that a_{34} and a_{78} do not contain any codewords. First, if some of the vertices which are marked by the grey lines in Figure 13(e) is a codeword, then the cube gets at least three votes as the arrows in the figure illustrate. Now, we can assume that only the vertices marked with grey (or black) dots in Figure 13(f) can be codewords in the sides. Assume first that a_z does not contain any codeword. Then vertices A_3 and A_7 (and A_4 and A_8) can be separated only if a_{13} or a_{57} (and a_{24} or a_{68}, resp.) contains a codeword. Then, at least two of the grey dots in Figure 13(g) have to be codewords. Now, if these two codewords are in adjacent sides, then the cube is of the type in Figure 13(a). On the other hand, if two opposite sides contain a codeword, then the cube gets at least three votes by the argument of $a_1 a_4$. Otherwise, we can assume that a_z contains a codeword.



(a) $d(c, a_y) > 1$ (b) $d(c', a_{1357}) > 1$, $d(c'', a_{1357}) > 1$ (c) $(a_x \cup a_y) \cap C \neq \emptyset$ (d)

(e) (f) (g) (h)

Figure 14: Two parallel and opposite sides and face which is the parallel with them containing a codeword. None of the corners contains an important codeword and there are no two parallel and adjacent sides which both contain a codeword.

Then the cube gets one vote initially and it also gets at least one vote by the arrow in Figure 13(h). Now, if a_y contains a codeword, then the cube gets the third vote already in the initial state. Therefore, we can assume that a_y has no codewords. Then, there must again be two more parallel sides which contain a codeword by Remark 8. If they are adjacent sides, the cube gets one more vote as in the case of Figure 13(d); if they are parallel with a_{13}, the cube gets one more vote as in Figure 13(g); otherwise the case is as in Figure 13(h) and so the cube gets one more vote by the third voting rule.

Now, we have shown that if the cube has two adjacent sides which contain at least one codeword each, then the cube has at least three votes finally. Next we assume that there are two parallel and opposite sides which contain a codeword but no two parallel and adjacent sides containing a codeword. We study this case in Figures 14 and 15.

First, we assume that at least one of the faces which is parallel with the sides contains a codeword. Now, without loss of generality we can assume that a_z, a_{12} and a_{78} contain codewords c, c' and c'', respectively. See Figure 14. By Figures 14(a) and 14(b) and symmetry, we can still assume that $d(c, a_{2468}) = 1$ and $d(c', a_{1357}) = 1$ (or $d(c'', a_{1357}) = 1$). In any case, the cube has finally at least one vote from the codewords in a_z and the second vote from the nearby cubes by the first voting rule (cf. the arrows in Figure 14(c).) Thus, if a_x or a_y contains a codeword, then the cube gets enough votes.

Therefore, we can assume that a_y does not contain any codeword. Again by Remark 8, there are at least two parallel sides more which each contain a codeword. Figure 14(d) shows how the cube gets the third vote if these sides are parallel with a_{15}. Otherwise, either a_{24} and a_{57} or a_{13} and a_{68} contain at least one codeword each. Since both cases are proved similarly, assume that a_{24} and a_{57} each have a codeword. Now, the codewords are as in one of Figures 14(e)–14(h) depending on

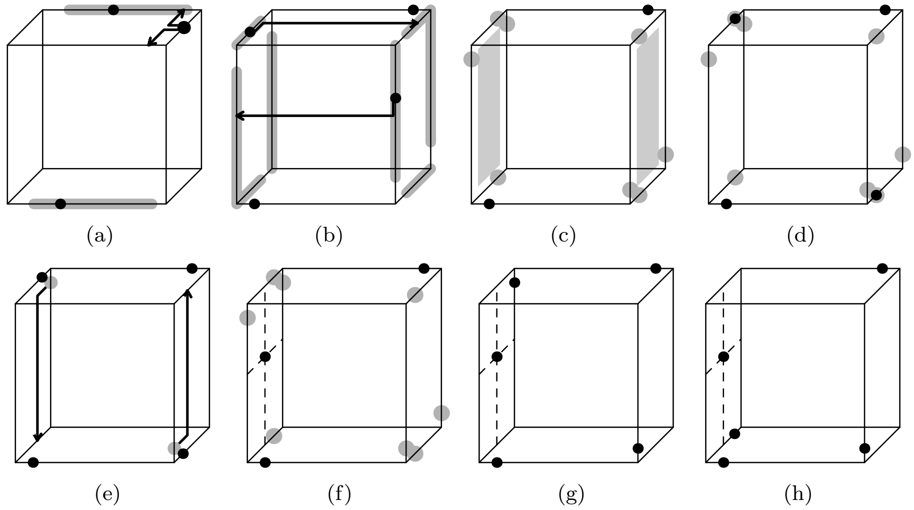


Figure 15: Two parallel and opposite sides containing a codeword, but none of the faces which are parallel to these sides contains any codeword. None of the corners contains an important codeword and there are no two parallel and adjacent sides which both contain a codeword.

which codewords are in the end-of-sides. If the cube is one of the type of Figures 14(e)–14(g), then it gets at least three votes by the arrows in the figures. If the cube is as in Figure 14(h), then it gets the third vote (and also the fourth vote) from the upper and lower cubes by the third voting rule.

Second, we assume that both a_{12} and a_{78} still have a codeword but the faces a_x and a_z and the sides a_{34} and a_{56} which are parallel with a_{12} do not contain any codeword. Now, at least one of a_{13}, a_{15}, a_{37} and a_{57} and at least one of a_{24}, a_{26}, a_{48} and a_{68} have to contain at least one codeword, since pairs A_3–A_5 and A_4–A_6 have to be separated. Let $c' \in a_{12}$ and $c'' \in a_{78}$ again be two codewords. Then, if $d(c', a_{1357}) > 1$ and $d(c'', a_{1357}) > 1$, then the cube gets enough votes as in Figure 15(a). Therefore, we can assume that c' and c'' are as in Figure 15(b). Now, if some codeword is in the grey area in Figure 15(b), then the cube again gets enough votes.

Now we can assume that only a_y and the end-of-sides which are marked with grey dots in Figure 15(c) can be codewords in addition to c' and c''. Assume first, that a_y does not contain any codeword. Again, there has to be two more parallel sides containing a codeword by Remark 8. Without loss of generality, we can assume that these sides are a_{24} and a_{57}. However, the codewords in these four sides, which have a black dot in Figure 15(d), do not separate A_3 from A_6. Therefore, at least one of the grey dots in Figure 15(e) has to be contain a codeword. Now, the cube gets at least three votes by one of the arrows.

Next, we assume that at least one codeword belongs to faces a_y. See Figures 15(f). Still, pairs A_3–A_5 and A_4–A_6 have to be separated and there have to be two more sides that contain codewords. If the sides are parallel (cf. Figure 15(g)), then the situation is of the type in Figure 13 or 14. Otherwise, the cube is apart symmetry as in Figure 15(h). Then it gets one vote initially and one more vote both from left-hand side and from right-hand side cubes by the third voting rule.

Now, we have shown that if the cube has no important codeword in its corners or it has at least two corners containing a codeword, then the cube has at least three votes when all voting rules have been applied.

Finally, we study cubes which have exactly one codeword in a corner. Without loss of generality we assume that a_1 contains a codeword c. Moreover we assume that the codeword is important for the cube, i.e., $I_r(A_i) = I_r(A_1) \setminus \{c\}$ for some $i \in \{2, \ldots, 8\}$. First, if there is a codeword in the side covering the grey lines in Figure 16(a), then the cube gets at least three votes in total by the first voting rule.

Now, we assume that some side which is adjacent with a_1 contains a codeword c'. By the previous paragraph, this codeword has to be located in the end-of-side at distance $2r$ from a_1. Without loss of generality, we can assume that the side is a_{12}. See Figure 16(b). Now, none of a_y, a_{13}, a_{15}, a_{24} and a_{26} contains any codeword since otherwise c' would be not important. See Figure 16(b). Now, pairs A_3–A_4 and A_5–A_6 can be separated from each other by only the vertices in the grey area in Figure 16(c). If some vertex which is not within distance one from a_{78} is a codeword, then the cube gets three votes as the arrow shows in Figure 16(c). Otherwise at least two of the four grey dots must be a codeword and so the cube gets enough votes by the arrows in Figure 16(d).

Next we can assume that a_{12}, a_{13} and a_{15} do not contain any codeword. Assume in addition that at least one of a_{24}, a_{26}, a_{56}, a_{57}, a_{37} and a_{34} contains a codeword. Without loss of generality we can assume that a_{24} does. Now, Figure 16(e) shows vertices which may contain codewords. First, if there is a codeword in a_{56}, a_{48}, a_{68} or a_{78} except the end-of-sides marked with grey dots in 16(g), then the cube gets enough votes. See Figure 16(f).

Now, we know that only seven vertices which are marked with grey dots in 16(g) can be codewords in addition to the vertices marked with black dots or else the case has already been considered. First, if one of the dots which are marked with an arrow is a codeword, then we can again use the argument of type a_1a_4. If a_{68} contained a codeword, then none of the other grey dots in 16(h) could be a codeword or else the cube gets enough votes. However, none of the codewords in the sides and the corners separates any of the pairs A_5–A_7, A_7–A_3 and A_3–A_4. Then these pairs have to be separated by codewords in the faces. The pair A_5–A_7 (A_7–A_3 and A_3–A_4) can be separated only by the face a_x (a_z and a_y, respectively). And so the cube has enough votes. Similarly, if a_{24} contains a codeword, then a_{34} and a_{57} does not contain any codeword and so the pairs A_3–A_7, A_5–A_7 and A_7–A_8 have to be separated by the codewords in three non-parallel faces. See Figure 16(i). Finally, vertices in a_{34} and a_{57} do not separate any of the pairs A_2–A_6, A_3–A_4 and A_5–A_7 and so we again need three non-parallel faces which contain a codeword. See Figure 16(j).

In the last case, we can assume that only the faces, a_1, a_{48}, a_{68} and a_{78} can contain codewords. See Figure 16(k). First, if there is at least one codeword in a_{48}, a_{68} and a_{78} s.t. the distance between a_8 and this codeword is at least $r + 3$, then the cube gets enough votes by the argument of type a_1a_4. See Figure 16(l). Otherwise there are three non-parallel faces containing a codeword or at least two of the sides a_{48}, a_{68} and a_{78} contain a codeword s.t. the codewords are within distance $r + 3$ from a_8. See Figure 16(m). Now, the arrows show that the cube gets at least $2r - (r + 3) = r - 3$ times $\frac{3}{28}$ of a vote per each arrow. Therefore the cube gets at least $2(r - 3) \cdot \frac{3}{28} \geq 2$ votes when $r \geq 15$. Moreover the cube gets at least one vote initially since A_2 and A_3 can be separated only by the codewords in the faces.

Hence, we have shown that every cube has at least three votes finally. □

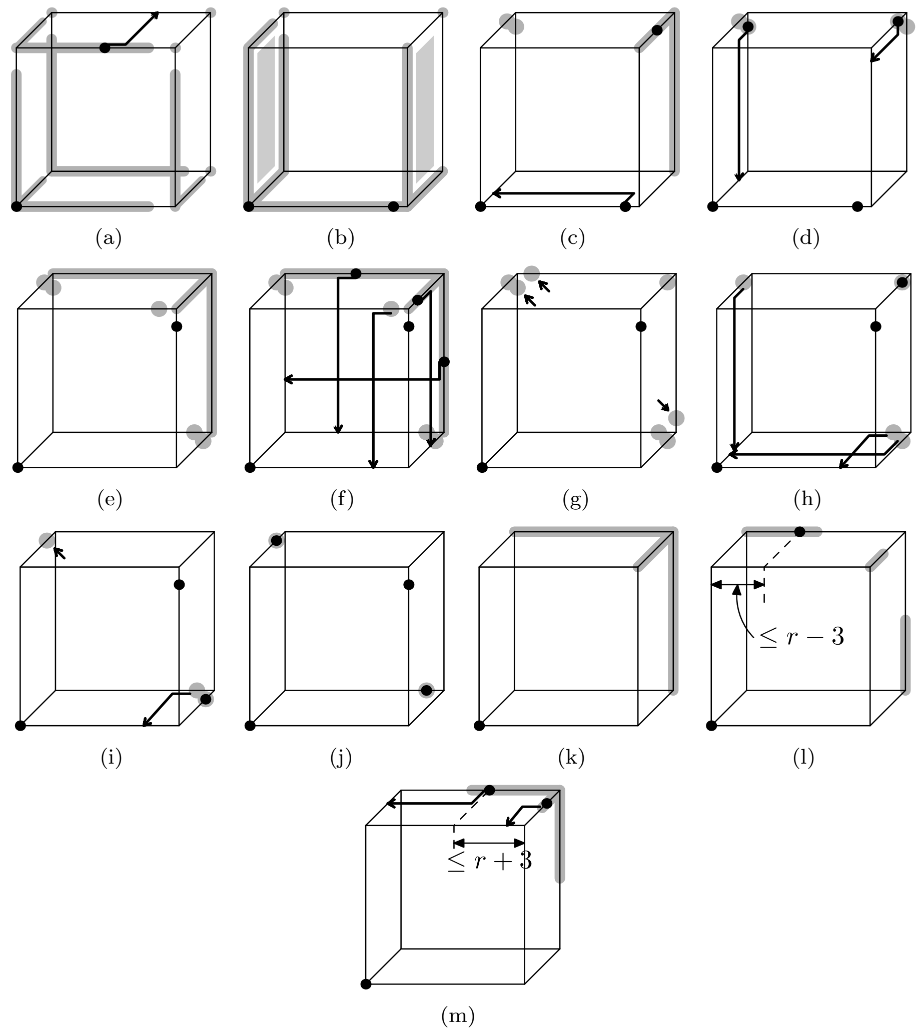


Figure 16: The one corner contains an important codeword but the other corners do not contain any codewords.

Remark 10. It is possible to show that the faces, corners and end-of-sides contain at least three codewords on average when $r \geq 4$. Then, with the help of this fact, it is possible to prove that the density of r-identifying code in the 3-dimensional king grid is at least $\frac{1}{8r^2+10\frac{2}{3}}$, when $r \geq 4$. The claim can be proved in a way similar to the proof of Theorem 9, but the voting rules are different.

Remark 11. The lower bound $\frac{1}{12}$ for 1-identifying code in the 3-dimensional king grid, which has already been mentioned in Table 1, can be proved with the help of the share technique. The technique has been introduced in [10] by Slater. In this technique, we count the share, i.e., the sum of $\sum_{v \in B_1(c)} \frac{1}{|B_1(v) \cap C|}$ for all codewords $c \in C$. After quite simple but boring case analysis, we see that the share of any codeword is at most 12. By this fact, we are able to show that the lower bound is valid.

References

[1] Bertrand, N., Charon, I., Hudry, O., Lobstein A.: Identifying and locating-dominating codes on chains and cycles, European Journal of Combinatorics, Vol. 25, 969–987 (2004).

[2] Charon, I., Honkala, I., Hudry, O., Lobstein A.: The minimum density of an identifying code in the king lattice, Discrete Mathematics, Vol. 276, 95–109 (2004).

[3] Charon, I., Hudry, O. and Lobstein, A.: Identifying codes with small radius in some infinite regular graphs, Electronic Journal of Combinatorics, Vol. 9(1), R11 (2002).

[4] Cohen, G., Honkala, I., Lobstein, A., Zémor, G.: On codes identifying vertices in the two-dimensional square lattice with diagonals, IEEE Transactions on Computers, Vol. 50, 174–176 (2001).

[5] Honkala, I., Laihonen, T.: Codes for identification in the king lattice, Graphs and Combinatorics, Vol. 19, 505–516 (2003).

[6] Honkala, I., Laihonen, T.: On a new class of identifying codes in graphs, Information Processing Letters, Vol. 102, 92–98 (2007).

[7] Karpovsky, M. G., Chakrabarty K., Levitin, L. B.: On a new class of codes for identifying vertices in graphs, IEEE Transactions on Information Theory, Vol. IT-44, 599–611 (1998).

[8] Pelto, M.: New bounds for $(r, \leq 2)$-identifying codes in the infinite king grid, Cryptography and Communications, Vol. 2, 41–47 (2010).

[9] Slater, P. J.: Domination and location in acyclic graphs, Networks, Vol. 17, 55–64 (1987).

[10] Slater, P. J.: Fault-tolerant locating-dominating sets, Discrete Mathematics, Vol. 249, pp. 179–189 (2002).

[11] Stanton, B.: Vertex identifying codes for the n-dimensional lattice, arXiv:1008.4892v2 (2011).

[12] http://www.infres.enst.fr/~lobstein/bibLOCDOMetID.html

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