TURUN YLIOPISTON JULKAISUJA
ANNALES UNIVERSITATIS TURKUENSIS

SARJA - SER. A I OSA - TOM. 441

ASTRONOMICA - CHEMICA - PHYSICA - MATHEMATICA

QUANTUM TOMOGRAPHY WITH PHASE SPACE MEASUREMENTS

by

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Turku 2012

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ISBN 978-951-29-5081-2 (PRINT)
ISBN 978-951-29-5082-9 (PDF)
ISSN 0082-7002
Uniprint - Turku, Finland 2012

Acknowledgments

The research leading to this thesis was done between 2008 and 2012 at the Department of Physics and Astronomy at the University of Turku. I want to thank the people at the Laboratory of Theoretical Physics for a warm and inspiring environment.

I am grateful to my supervisors Dr. Pekka Lahti and Dr. Juha-Pekka Pellonpää for their support, encouragement, and guidance on the way. I also wish to express my gratitude to Prof. Kari Ylinen for his support and help. I want to thank Dr. Jukka Kiukas, Prof. Reinhard F. Werner, and M. Sc. Erkka Haapasalo for collaboration on articles included in this thesis, and Dr. Teiko Heinosaari and Prof. Matteo G. A. Paris for fruitful discussions and collaboration on topics related to this work. Finally, my warmest thanks to Anukaisa whose love, support and patience were essential for the completion of this thesis.

This thesis was financially supported by the Turku University Foundation, the Finnish Cultural Foundation, and the Academy of Finland Grant No. 138135, which I gratefully acknowledge.

Contents

Abstract

This thesis addresses the use of covariant phase space observables in quantum tomography. Necessary and sufficient conditions for the informational completeness of covariant phase space observables are proved, and some state reconstruction formulae are derived. Different measurement schemes for measuring phase space observables are considered. Special emphasis is given to the quantum optical eight-port homodyne detection scheme and, in particular, on the effect of non-unit detector efficiencies on the measured observable. It is shown that the informational completeness of the observable does not depend on the efficiencies.

As a related problem, the possibility of reconstructing the position and momentum distributions from the marginal statistics of a phase space observable is considered. It is shown that informational completeness for the phase space observable is neither necessary nor sufficient for this procedure. Two methods for determining the distributions from the marginal statistics are presented.

Finally, two alternative methods for determining the state are considered. Some of their shortcomings when compared to the phase space method are discussed.

List of articles

This thesis consists of an introductory review and the following six articles:

I J. Kiukas, P. Lahti, J. Schultz,
Position and momentum tomography,
Phys. Rev. A **79** (2009) 052119.

II J. Kiukas, J.-P. Pellonpää, J. Schultz,
Density matrix reconstruction from
displaced photon number distributions,
J. Phys. A: Math. Theor. **43** (2010) 095303.

III P. Lahti, J.-P. Pellonpää, J. Schultz,
Realistic eight-port homodyne detection
and covariant phase space observables,
J. Mod. Opt. **57** (2010) 1171-1179.

IV E. Haapasalo, P. Lahti, J. Schultz,
Weak versus approximate values in
quantum state determination,
Phys. Rev. A **84** (2011) 052107.

V J. Kiukas, P. Lahti, J. Schultz, R. F. Werner,
Characterization of informational completeness
for covariant phase space observables,
submitted, arXiv:1204.3188.

VI J. Schultz,
A note on the Pauli problem in light of
approximate joint measurements,
Phys. Lett. A **376** (2012) 2372-2376.

Chapter 1

Introduction

The purpose of a measurement performed on a physical system is to gain some information about the properties of the system. In quantum tomography, the goal is the complete determination of the state of the system prior to the measurement (see, e.g., [43, 50, 53] for expositions on this topic). The question about the possibility of reconstructing the quantum state from measurement statistics is naturally quite old. Indeed, it was briefly mentioned already in 1933 in Pauli's book [54], though the first systematic approach is usually credited to Fano [25]. A renewed interest to quantum tomography rose after the paper of Vogel and Risken [65], and the quantum optical experiment of Smithey *et al.* [62]. Nowadays, quantum tomography is an immensely wide field of research due to the development of both theoretical and experimental methods.

It is clear that not all measurements allow unique state determination. Therefore one of the fundamental questions in quantum tomography is the informational completeness [57] of sets of observables. Informational completeness of a set of observables means, by definition, that the state can be inferred from the statistics. A remarkable consequence of the modern view of quantum observables as positive operator measures is the existence of single informationally complete observables. An important class of such observables are certain covariant phase space observables, whose significance is undisputed also from a variety of different aspects of quantum mechanics (see, e.g., [59]). Their relevance is further emphasized by the fact that these observables have a quantum optical measurement realization in terms of eight-port homodyne detection. Such a measurement was first performed (in the optical regime) by Walker and Carroll [67].

This thesis addresses the use of covariant phase space observables for the purpose of performing quantum tomography. The introductory review part is organized as follows. In Section 2 the general framework is laid out, and the relevant concepts related to quantum tomography are defined. Section 3 is devoted to characterizing the informational completeness of covariant phase space observables and presenting some reconstruction formulae. In Section 4 three different measurement models for measuring phase space observables are presented, with emphasis on eight-port homodyne detection. The problem of reconstructing the position and momentum distributions from the marginal statistics of phase space observables is considered in Section 5. In Section 6 two alternative methods for quantum tomography are presented for comparison. The conclusions are given in Section 7.

Chapter 2

Tomographic aspects of quantum measurements

2.1 Preliminaries on quantum measurements

In quantum mechanics a complex separable Hilbert space $\mathcal{H}$ is assigned to any physical system. The states of the system are defined as equivalence classes of preparation procedures and they are represented by positive trace class operators ρ with unit trace. We denote by $\mathcal{T}(\mathcal{H})$ the set of trace class operators and by $\mathcal{S}(\mathcal{H})$ the convex set of states. The pure states are the extreme points of $\mathcal{S}(\mathcal{H})$ and they correspond to the one-dimensional projections. For any $\varphi, \psi \in \mathcal{H}$, we define the rank-one operator $|\varphi\rangle\langle\psi| : \mathcal{H} \to \mathcal{H}$ by $|\varphi\rangle\langle\psi|(\eta) = \langle\psi|\eta\rangle\varphi$. In particular, each pure state is of the form $|\varphi\rangle\langle\varphi|$ for some unit vector φ and we occasionally speak of vector states φ.

The observables are defined as equivalence classes of measurements and they are represented by normalized positive operator measures $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ where $\mathcal{A}$ is a σ-algebra of subsets of a measurement outcome set Ω and $\mathcal{L}(\mathcal{H})$ stands for the set of bounded operators on $\mathcal{H}$. In most cases Ω is a topological space such as (a subset of) $\mathbb{R}^n$ and $\mathcal{A}$ is the Borel σ-algebra $\mathcal{B}(\Omega)$.

Definition 1. *An observable is a map* $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ *such that*

(i) $\mathsf{E}(X) \geq 0$ *for all* $X \in \mathcal{A}$*,*

(ii) $\mathsf{E}(\Omega) = I$*,*

(iii) $\mathsf{E}(\cup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} \mathsf{E}(X_i)$ *for any sequence* $(X_i)_{i=1}^{\infty} \subset \mathcal{A}$ *of mutually disjoint sets, where the series converges in the weak operator topology.*

The range of an observable thus consists of operators $\mathsf{E}(X)$ satisfying $0 \leq \mathsf{E}(X) \leq I$. Such operators are called effects. The set of effects is a convex subset of $\mathcal{L}(\mathcal{H})$ and is denoted by $\mathcal{E}(\mathcal{H})$, that is, $\mathcal{E}(\mathcal{H}) = \{B \in \mathcal{L}(\mathcal{H}) \mid 0 \leq B \leq I\}$. Clearly any effect appears in the range of some observable.

An observable is called sharp if it is projection valued, that is, $\mathsf{E}(X)^2 = \mathsf{E}(X)$ for all $X \in \mathcal{A}$. Projection valued measures are also called spectral measures whereas positive operator measures are sometimes referred to as semispectral measures. According to the spectral theorem of selfadjoint operators there is a one-to-one correspondence between sharp observables defined on the real line and selfadjoint operators acting on $\mathcal{H}$. We occasionally write E^A for the spectral measure of a selfadjoint operator A, that is, $A = \int x \, d\mathsf{E}^A(x)$. For future use we note that for all $\psi, \varphi \in \mathcal{H}$ the map $\mathsf{E}_{\psi,\varphi} : \mathcal{A} \to \mathbb{C}$, $\mathsf{E}_{\psi,\varphi}(X) = \langle \psi | \mathsf{E}(X) \varphi \rangle$ is a complex measure.

Each pair (ρ, E) consisting of a state and an observable determines a probability measure $\mathsf{p}_\rho^\mathsf{E} : \mathcal{A} \to [0,1]$ via $\mathsf{p}_\rho^\mathsf{E}(X) = \mathrm{tr}[\rho \mathsf{E}(X)]$, $X \in \mathcal{A}$. We adopt the minimal interpretation according to which $\mathsf{p}_\rho^\mathsf{E}(X)$ is the probability that a measurement of E performed on the system in a state ρ leads to a result in the set X. The empirical content of these probabilities is in terms of relative frequencies: if we have a sufficiently large ensemble of identically prepared systems, then the relative frequency of the measurement outcomes lying in the set X can be approximated by the probability $\mathsf{p}_\rho^\mathsf{E}(X)$.

The statistical description given by states and observables is the crudest level of description of a quantum measurement. By taking into account more details of the measurement it is possible to reach two more levels, each more detailed than the previous one (for an overview of the quantum theory of measurement, see [14]). At the next level, one describes the conditional state changes of the system due to the measurement, conditioned with respect to the pointer values. These are conveniently implemented by the concept of an instrument.

Definition 2. *An instrument is a map $\mathcal{I} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ such that*

(i) $\mathcal{I}(X)$ *is completely positive for all* $X \in \mathcal{A}$,

(ii) $\mathrm{tr}\,[\mathcal{I}(\Omega)(\rho)] = 1$ *for all* $\rho \in \mathcal{S}(\mathcal{H})$,

(iii) $\mathcal{I}\left(\cup_{i=1}^\infty X_i\right)(\rho) = \sum_{i=1}^\infty \mathcal{I}(X_i)(\rho)$ *for any* $\rho \in \mathcal{S}(\mathcal{H})$ *and any sequence* $(X_i)_{i=1}^\infty \subset \mathcal{A}$ *of mutually disjoint sets, where the series converges in trace norm.*

The range of an instrument thus consists of completely positive linear maps $\mathcal{I}(X) : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ satisfying $0 \leq \mathrm{tr}\,[\mathcal{I}(X)(\rho)] \leq 1$ for all $\rho \in \mathcal{S}(\mathcal{H})$.

Such maps are called operations. An operation $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ satisfying $\mathrm{tr}\,[\Phi(\rho)] = 1$ for all $\rho \in \mathcal{S}(\mathcal{H})$ is called a state transformation or a quantum channel.

Each instrument determines uniquely the associated observable via the formula $\mathrm{tr}[\rho \mathsf{E}(X)] = \mathrm{tr}[\mathcal{I}(X)(\rho)]$, or equivalently via the dual instrument as $\mathsf{E}(X) = \mathcal{I}(X)^*(I)$. The instrument contains all the information on the measurement which is relevant to the object system. However, it does not say anything about the measuring apparatus.

The most detailed description of a measurement is obtained when the coupling between the system and the measuring apparatus, the probe, is taken into account. Suppose that we have a probe system with the Hilbert space $\mathcal{K}$ in an initial state $\sigma \in \mathcal{S}(\mathcal{K})$, and we couple the object system and the probe via a state transformation $\Phi : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$. The measurement outcome is then determined by reading the value of a pointer observable $\mathsf{Z} : \mathcal{A} \to \mathcal{L}(\mathcal{K})$. The description of the measurement process is thus given by the 4-tuple $\langle \mathcal{K}, \sigma, \Phi, \mathsf{Z} \rangle$ and the associated observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is determined by the condition

$$\mathrm{tr}\,[\rho \mathsf{E}(X)] = \mathrm{tr}\,[\Phi(\rho \otimes \sigma) I \otimes \mathsf{Z}(X)] \tag{2.1}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $X \in \mathcal{A}$. It is sometimes convenient to allow the pointer observable to have a value space $(\Omega_0, \mathcal{A}_0)$ which is different from that of the measured observable. In that case one needs to introduce a (measurable) pointer function $f : \Omega_0 \to \Omega$ which connects these two spaces. This means that the effect $\mathsf{Z}(X)$ in (2.1) is replaced by $\mathsf{Z}(f^{-1}(X))$. By taking into consideration the pointer function we have arrived at a 5-tuple which we define to be a measurement scheme.

Definition 3. *A measurement scheme is a 5-tuple $\mathcal{M} = \langle \mathcal{K}, \sigma, \Phi, \mathsf{Z}, f \rangle$ where $\mathcal{K}$ is the Hilbert space of the probe, $\sigma \in \mathcal{S}(\mathcal{K})$ its initial state, $\Phi : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ a state transformation, $\mathsf{Z} : \mathcal{A}_0 \to \mathcal{L}(\mathcal{H})$ a pointer observable, and $f : \Omega_0 \to \Omega$ a pointer function.*

Each measurement scheme $\mathcal{M}$ defines the associated instrument $\mathcal{I}^{\mathcal{M}} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ via the formula

$$\mathcal{I}^{\mathcal{M}}(X)(\rho) = \mathrm{tr}_{\mathcal{K}}\left[\Phi(\rho \otimes \sigma) I \otimes \mathsf{Z}(f^{-1}(X))\right] \tag{2.2}$$

where $\mathrm{tr}_{\mathcal{K}}[\cdot]$ denotes the partial trace over the Hilbert space $\mathcal{K}$. Thus, the observable $\mathsf{E}^{\mathcal{M}} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ associated with the scheme is determined by the condition

$$\mathrm{tr}\left[\rho \mathsf{E}^{\mathcal{M}}(X)\right] = \mathrm{tr}\left[\mathcal{I}^{\mathcal{M}}(X)(\rho)\right] = \mathrm{tr}\left[\Phi(\rho\otimes\sigma) I \otimes \mathsf{Z}(f^{-1}(X))\right]. \qquad (2.3)$$

The associations $\mathcal{M} \hookrightarrow \mathcal{I} \hookrightarrow \mathsf{E}$ are many-to-one defining natural equivalence classes and reflecting the fact an observable can be measured in a multitude of ways. However, it is a fundamental theorem of the quantum theory of measurement that for any observable E there exists a measurement scheme $\mathcal{M}$ such that $\mathsf{E} = \mathsf{E}^{\mathcal{M}}$ [52]. The measurement can even be chosen in such a way that σ is a pure state $|\phi\rangle\langle\phi|$, Φ is given by a unitary operator U acting on $\mathcal{H}\otimes\mathcal{K}$, the pointer observable Z is sharp, and f is the identity map. In other words, E has a measurement dilation

$$\mathsf{E}(X) = V_\phi^* U^* I \otimes \mathsf{Z}(X) U V_\phi \qquad (2.4)$$

where $V_\phi : \mathcal{H} \to \mathcal{H}\otimes\mathcal{K}$ is the embedding $V_\phi(\varphi) = \varphi\otimes\phi$. In the case of a pure state and a unitary coupling we typically use the notation $\mathcal{M} = \langle \mathcal{K}, \phi, U, \mathsf{Z}, f\rangle$.

Any two measurements $\mathcal{M}_1$ and $\mathcal{M}_2$ may be combined sequentially, e.g., by first performing $\mathcal{M}_1$ and then $\mathcal{M}_2$, to obtain a new measurement $\mathcal{M}_{12}$ (see, e.g., [8]). At the level of instruments this leads to the instrument $\mathcal{I}^{\mathcal{M}_{12}}$ defined on the product space and determined by the condition [23, 8]

$$\mathcal{I}^{\mathcal{M}_{12}}(X\times Y) = \mathcal{I}^{\mathcal{M}_2}(Y)\circ\mathcal{I}^{\mathcal{M}_1}(X), \qquad X\in\mathcal{A}_1,\, Y\in\mathcal{A}_2. \qquad (2.5)$$

The observable determined by this sequential measurement is then

$$\mathsf{E}^{\mathcal{M}_{12}}(X\times Y) = \mathcal{I}^{\mathcal{M}_{12}}(X\times Y)^*(I) = \mathcal{I}^{\mathcal{M}_1}(X)^*\left(\mathsf{E}_2(Y)\right) \qquad (2.6)$$

where E_2 is the observable associated to the subsequent measurement. It should be noted that the observable does not depend on the instrument $\mathcal{I}^{\mathcal{M}_2}$, i.e., on any details of the measurement of E_2. The marginal observables $\mathsf{E}_1^{\mathcal{M}_{12}} : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $\mathsf{E}_2^{\mathcal{M}_{12}} : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ of $\mathsf{E}^{\mathcal{M}_{12}}$ are

$$\begin{aligned} \mathsf{E}_1^{\mathcal{M}_{12}}(X) &= \mathsf{E}^{\mathcal{M}_{12}}(X\times\Omega_2) = \mathsf{E}_1(X), \\ E_2^{\mathcal{M}_{12}}(Y) &= \mathsf{E}^{\mathcal{M}_{12}}(\Omega_1\times Y) = \mathcal{I}^{\mathcal{M}_1}(\Omega_1)^*\left(\mathsf{E}_2(Y)\right), \end{aligned}$$

which shows that the first margin is the observable measured first whereas the second margin is a smeared version of E_2, where the smearing is caused by the first measurement.

2.2 Generalized standard model

We will now present a generalized version of what is known as the standard model of quantum measurement theory. Historically the standard model can be traced back to the book of von Neumann [66]. A modern treatment can be found, for instance in the book [9, Chapter II.3.4] and the survey [13]. This generalization was presented in Article IV.

Let $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be an observable which we wish to measure. Since E has a dilation (2.4) where the map $X \mapsto U^* I \otimes \mathsf{Z}(X) U$ is also a spectral measure, we know that there exists a Hilbert space $\mathcal{H}_0$, a unit vector $\psi \in \mathcal{H}_0$ and a spectral measure $\mathsf{E}^A : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_0)$, with the corresponding selfadjoint operator A, such that

$$\mathsf{E}(X) = V_\psi^* \, \mathsf{E}^A(X) \, V_\psi$$

where $V_\psi : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_0$ is again the embedding $V_\psi(\varphi) = \varphi \otimes \psi$.

Let $\mathcal{K} = L^2(\mathbb{R})$ be the Hilbert space of the probe. For each $\lambda > 0$ define the state transformation $\Phi^\lambda : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ via

$$\Phi^\lambda(\rho \otimes \sigma) = \mathrm{tr}_{\mathcal{H}_0}[e^{-i\lambda A \otimes P} \, (\rho \otimes |\psi\rangle\langle\psi| \otimes \sigma) \, e^{i\lambda A \otimes P}]$$

where P is the momentum operator on $\mathcal{K}$, i.e., $(P\varphi)(x) = -i\varphi'(x)$. Since P generates spatial translations, it is natural to choose as the pointer observable the (sharp) position observable $\mathsf{Q} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{K})$, i.e., the spectral measure of the multiplicative position operator Q. Due to the coupling constant λ it is convenient to choose a pointer function $f^\lambda(x) = \lambda^{-1} x$. The 5-tuple $\mathcal{M} = \langle \mathcal{K}, \sigma, \Phi^\lambda, \mathsf{Q}, f^\lambda \rangle$ then constitutes a measurement scheme with the intention of measuring the system observable E. We call this a generalized standard model for E.

The instrument as well as the observable actually measured can now be computed from (2.2) and (2.3). For simplicity, we assume that the probe system is initially in a (sufficiently regular) vector state ϕ, that is, $\sigma = |\phi\rangle\langle\phi|$. The associated instrument and its dual are

$$\begin{aligned} \mathcal{I}^\lambda(X)(\rho) &= \int_X \mathrm{tr}_{\mathcal{H}_0}[K_x V_\psi \rho V_\psi^* K_x^*] \, dx, \quad \rho \in \mathcal{T}(\mathcal{H}), \\ \mathcal{I}^\lambda(X)^*(B) &= \int_X V_\psi^* K_x^* (B \otimes I) K_x V_\psi \, dx, \quad B \in \mathcal{L}(\mathcal{H}), \end{aligned}$$

where

$$K_x = \sqrt{\lambda}\phi(-\lambda(A - x)) = \int \sqrt{\lambda}\phi(-\lambda(y - x))\, d\mathsf{E}^A(y),$$

for all $x \in \mathbb{R}$. The observable actually measured by this scheme is then an unsharp version $\mu^\lambda * \mathsf{E}$ of the intended one, E, defined as the weak integral

$$\mu^\lambda * \mathsf{E}(X) = \int \mu^\lambda(X - x)\, d\mathsf{E}(x), \quad X \in \mathcal{B}(\mathbb{R}).$$

The convolving probability measure is determined by the initial probe state via $\mu^\lambda(X) = \langle \phi | \mathsf{Q}(\lambda X) \phi \rangle$.

2.3 State distinction power and informational equivalence

When a measurement of an observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is performed on a system prepared in a state ρ, the measurement outcomes are distributed according to the probability measure $\mathsf{p}_\rho^\mathsf{E}$. Thus, if the measurement is performed on a system whose initial state is unknown we can try to deduce the state from the measurement outcome statistics. Typically we are able to single out a subset of possible states, namely those for which the corresponding probability measure agrees with the measurement statistics. In an ideal case this set would consist of a single state, but more often than not this is not the case.

We say that an observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ distinguishes the states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ if $\mathsf{p}_\rho^\mathsf{E} \neq \mathsf{p}_\sigma^\mathsf{E}$. This means that if we know *a priori* that the system under consideration is in either of the states, then the state can be deduced from the statistics of the observable. If an observable E does not distinguish any pairs of states, it is called trivial. This is equivalent to the existence of a probability measure $\mu : \mathcal{A} \to [0, 1]$ such that $\mathsf{E}(X) = \mu(X)I$ for all $X \in \mathcal{A}$. Certain observables can now be compared according to their ability to distinguish different states.

Definition 4. *Let $\mathsf{E} : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $\mathsf{F} : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ be observables. The state distinction power of E is greater than or equal to that of F if for any two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, $\mathsf{p}_\rho^\mathsf{E} = \mathsf{p}_\sigma^\mathsf{E}$ implies $\mathsf{p}_\rho^\mathsf{F} = \mathsf{p}_\sigma^\mathsf{F}$. The observables E and F are informationally equivalent if for any two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, $\mathsf{p}_\rho^\mathsf{E} = \mathsf{p}_\sigma^\mathsf{E}$ if and only if $\mathsf{p}_\rho^\mathsf{F} = \mathsf{p}_\sigma^\mathsf{F}$.*

A typical situation which arises in practice is that one wishes to measure some observable $\mathsf{E} : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ but due to the measurement arrangement one is only able to measure some unsharp version $\mu * \mathsf{E}$ of it. We have already encountered this in Section 2.2 in the case of the standard model. In concrete applications this unsharpness is often due to physical imperfections in the measuring apparatus, such as non-unit efficiencies of photodetectors.

The state distinction power of the observable $\mathsf{E} : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ is always greater than or equal to that of $\mu * \mathsf{E}$. To see this, let ρ, $\sigma \in \mathcal{S}(\mathcal{H})$ be such that $\mathsf{p}^{\mathsf{E}}_\rho = \mathsf{p}^{\mathsf{E}}_\sigma$. It follows that their Fourier transforms are also equal, that is, $\widehat{\mathsf{p}}^{\mathsf{E}}_\rho = \widehat{\mathsf{p}}^{\mathsf{E}}_\sigma$ where

$$\widehat{\mathsf{p}}^{\mathsf{E}}_\rho(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}} \int e^{-i\mathbf{y}\cdot\mathbf{x}}\, d\mathsf{p}^{\mathsf{E}}_\rho(\mathbf{x}).$$

Since $\mathsf{p}^{\mu*\mathsf{E}}_\rho = \mu * \mathsf{p}^{\mathsf{E}}_\rho$ and the Fourier transform maps convolutions into products, that is, $\widehat{\mu * \mathsf{p}^{\mathsf{E}}_\rho} = (2\pi)^{n/2}\widehat{\mu}\,\widehat{\mathsf{p}}^{\mathsf{E}}_\rho$, we conclude by the injectivity of the Fourier transform that $\mathsf{p}^{\mu*\mathsf{E}}_\rho = \mathsf{p}^{\mu*\mathsf{E}}_\sigma$.

Even though $\mu*\mathsf{E}$ cannot distinguish any states that are indistinguishable by E, it may happen that these observables are informationally equivalent. Indeed, if E is a sharp observable, then E and $\mu * \mathsf{E}$ are informationally equivalent if and only if $\operatorname{supp}\widehat{\mu} = \mathbb{R}^n$, that is, the support of the Fourier transform of μ is the whole $\mathbb{R}^n$ [27].

2.4 State determination power and informational completeness

Since quantum tomography deals with the problem of reconstructing the unknown state of the system, the relevant question is obviously whether or not an arbitrary state is uniquely determined by the measurement outcome statistics. We say that a state $\rho \in \mathcal{S}(\mathcal{H})$ is determined by the observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ if for any $\sigma \in \mathcal{S}(\mathcal{H})$, $\mathsf{p}^{\mathsf{E}}_\rho = \mathsf{p}^{\mathsf{E}}_\sigma$ implies $\rho = \sigma$. We denote by $\mathcal{D}_{\mathsf{E}}$ the set of states determined by E.

Definition 5. *Let $\mathsf{E} : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $\mathsf{F} : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ be observables. The state determination power of E is greater than or equal to that of F if $\mathcal{D}_{\mathsf{F}} \subset \mathcal{D}_{\mathsf{E}}$. The observable E is informationally complete if $\mathcal{D}_{\mathsf{E}} = \mathcal{S}(\mathcal{H})$, that is, for any two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, $\mathsf{p}^{\mathsf{E}}_\rho = \mathsf{p}^{\mathsf{E}}_\sigma$ implies $\rho = \sigma$.*

The informational completeness of an observable means that the state is uniquely determined by the measurement statistics. In other words, an observable E is informationally complete if and only if the map $\rho \mapsto \mathsf{p}^{\mathsf{E}}_\rho$ is injective. However, since most of the physically important observables such as position, momentum, energy, etc., are not informationally complete, it is natural to extend the definition to cover sets of observables.

Definition 6. *Let* $\mathsf{E}_i : \mathcal{A}_i \to \mathcal{L}(\mathcal{H})$ *be an observable for all* $i \in \mathcal{I}$*. The set* $\{\mathsf{E}_i \mid i \in \mathcal{I}\}$ *of observables is informationally complete if for any two states* $\rho, \sigma \in \mathcal{S}(\mathcal{H})$*,* $\mathsf{p}_\rho^{\mathsf{E}_i} = \mathsf{p}_\sigma^{\mathsf{E}_i}$ *for all* $i \in \mathcal{I}$ *implies* $\rho = \sigma$*.*

It is now possible to give examples of informationally complete sets of observables. For instance, define for each unit vector $\varphi \in \mathcal{H}$ the two-valued observable E_φ via

$$\mathsf{E}_\varphi(1) = |\varphi\rangle\langle\varphi|, \qquad \mathsf{E}_\varphi(0) = I - |\varphi\rangle\langle\varphi|.$$

The set $\{\mathsf{E}_\varphi \mid \varphi \in \mathcal{H}, \|\varphi\| = 1\}$ is then informationally complete. Indeed, if $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ are such that $\mathsf{p}_\rho^{\mathsf{E}_\varphi} = \mathsf{p}_\sigma^{\mathsf{E}_\varphi}$ for all E_φ then, in particular, $\langle\varphi|\rho\varphi\rangle = \langle\varphi|\sigma\varphi\rangle$ for all unit vectors φ so that $\rho = \sigma$.

A remarkable fact is that there exist single observables which are informationally complete. Certain covariant phase space observables, as seen in the next chapter, constitute an important class of such observables. It is even possible to construct discrete observables which are informationally complete [12, 64]. However, it should be emphasized that even though a single observable might be sufficient to determine the state uniquely, a single measurement outcome is never enough. This is due to the fact that for any effect $B \in \mathcal{E}(\mathcal{H})$ there exists a large class of states ρ such that the probability $\mathrm{tr}\,[\rho B]$ is nonzero.

We will next present some mathematical characterizations of the informational completeness of an observable. We start with the following well known lemma which is typically used when determining whether or not a given observable is informationally complete. The proof is a straightforward application of Definition 6 and can be found, for instance, in [7]. Since we use this result explicitly in the next section, we present a proof here.

Proposition 1. *An observable* $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ *is informationally complete if and only if for any* $T \in \mathcal{T}(\mathcal{H})$ *the condition*

$$\mathrm{tr}[T\mathsf{E}(X)] = 0 \text{ for all } X \in \mathcal{A} \tag{2.7}$$

implies $T = 0$*.*

Proof. Assume that E is informationally complete and let $T \in \mathcal{T}(\mathcal{H})$ be such that $\mathrm{tr}\,[T\mathsf{E}(X)] = 0$ for all $X \in \mathcal{A}$. Since T can be decomposed as $T = T_1 + iT_2$ where the $T_j \in \mathcal{T}(\mathcal{H})$ are selfadjoint and thus $\mathrm{tr}\,[T_j\mathsf{E}(X)] \in \mathbb{R}$, it follows that $\mathrm{tr}\,[T_j \mathsf{E}(X)] = 0$ for all $X \in \mathcal{A}$ and $j = 1, 2$. Furthermore, each T_j can be decomposed as $T_j = T_j^+ - T_j^-$ where the $T_j^\pm$ are positive so that $\mathrm{tr}\left[T_j^\pm \mathsf{E}(X)\right] \geq 0$ and thus $\mathrm{tr}\left[T_j^+ \mathsf{E}(X)\right] = \mathrm{tr}\left[T_j^- \mathsf{E}(X)\right]$ for all $X \in \mathcal{A}$ and $j = 1, 2$. Now consider a fixed j. By choosing $X = \Omega$ we have $\mathrm{tr}\left[T_j^+\right] = \mathrm{tr}\left[T_j^-\right]$ so that by the positivity, if either T_j^+ or T_j^- is zero then so is the other one. If they both are nonzero, then the operators $\mathrm{tr}\left[T_j^+\right]^{-1} T_j^\pm$ are positive and of unit trace, and $\mathrm{tr}\left[T_j^+\right]^{-1} \mathrm{tr}\left[T_j^+ \mathsf{E}(X)\right] = \mathrm{tr}\left[T_j^+\right]^{-1} \mathrm{tr}\left[T_j^- \mathsf{E}(X)\right]$ for all $X \in \mathcal{A}$. It follows from the assumption of informational completeness that $T_j^+ = T_j^-$ so that $T_j = 0$. Therefore $T = 0$.

Now assume that $T \in \mathcal{T}(\mathcal{H})$ is zero if and only if $\mathrm{tr}\,[T\mathsf{E}(X)] = 0$ for all $X \in \mathcal{A}$, and let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ be such that $\mathsf{p}_\rho^\mathsf{E} = \mathsf{p}_\sigma^\mathsf{E}$. Then $T = \rho - \sigma \in \mathcal{T}(\mathcal{H})$ and $\mathrm{tr}\,[T\mathsf{E}(X)] = 0$ for all $X \in \mathcal{A}$. This implies that $T = 0$, or, $\rho = \sigma$. □

It is obvious from Proposition 1 that informational completeness is somehow related to the question of the size of the subspace of $\mathcal{L}(\mathcal{H})$, or more precisely, the operator system, generated by the effects $\mathsf{E}(X)$. Indeed, as a consequence of the duality $\mathcal{T}(\mathcal{H})^* \simeq \mathcal{L}(\mathcal{H})$ and the Hahn-Banach theorem, we have that E is informationally complete if and only if the weak*-closure of $\mathrm{lin}\,\{\mathsf{E}(X) | X \in \mathcal{A}\}$ is $\mathcal{L}(\mathcal{H})$ [7, 64].

Since sharp observables are physically an important class of observables, it is natural to address the question of their informational completeness. More generally, we may consider a commutative observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, i.e., one which satisfies $\mathsf{E}(X)\mathsf{E}(Y) = \mathsf{E}(Y)\mathsf{E}(X)$ for all $X, Y \in \mathcal{A}$. Any smearing $\mu * \mathsf{E}^A$ of a sharp observable constitutes an example of a commutative observable which is typically not sharp. It is known that when $\dim(\mathcal{H}) \geq 2$, no commutative observable is informationally complete [12, Theorem 2.1.2]. An even stronger result holds. Indeed, any informationally complete observable E is necessarily totally noncommutative, that is [7, Proposition 3],

$$\mathrm{Com}(\mathsf{E}) = \{\varphi \in \mathcal{H} \mid \mathsf{E}(X)\mathsf{E}(Y)\varphi = \mathsf{E}(Y)\mathsf{E}(X)\varphi \text{ for all } X, Y \in \mathcal{A}\} = \{0\}.$$

Even though no single sharp observable is informationally complete, one can try to find sets of sharp observables which would be such. In that case, it is the noncommutativity which is essential in order to improve the ability to determine the state. In fact, any family of mutually commuting sharp observables is always informationally incomplete.

We will next present a physically significant example of a set of sharp observables, namely the rotated quadrature observables, which satisfies the condition of informational completeness.

2.4.1 Example: Rotated quadrature observables

Let $\mathcal{H} = L^2(\mathbb{R})$ and let Q, $\mathsf{P} : \mathcal{B}(\mathbb{R}) \to \mathcal{H}$ be the position and momentum observables, i.e., the spectral measures of the position and momentum operators Q and P. Since position and momentum are totally noncommutative, it makes sense to ask whether or not (Q, P) is informationally complete. This problem was already mentioned in a footnote in [54], and it has since become known as the Pauli problem. We now know that the answer to this question is negative and several counterexamples are known (see, e.g., [58, 57, 19, 55, 16]). Due to the physical significance of the position and momentum observables it is tempting to ask if we can obtain informational completeness by adding some more physically interesting observables to this set. It seems to be an open question whether or not any triple $(\mathsf{Q}, \mathsf{P}, \mathsf{H})$, where H is a sharp observable, can be informationally complete. However, informational completeness can be reached if we allow the number of observables to be infinite. One possibility for such a completion is given by the rotated quadrature observables.

Let $H = \frac{1}{2}(Q^2 + P^2)$ be the Hamiltonian of the harmonic oscillator and define for each $\theta \in [0, 2\pi)$ the rotated quadrature observable $\mathsf{Q}_\theta : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ via

$$\mathsf{Q}_\theta(X) = e^{i\theta H}\mathsf{Q}(X)e^{-i\theta H}, \qquad X \in \mathcal{B}(\mathbb{R}). \tag{2.8}$$

In particular, we recover $\mathsf{Q} = \mathsf{Q}_0$ and $\mathsf{P} = \mathsf{Q}_{\pi/2}$. For any $\theta, \theta' \in [0, 2\pi)$, $\theta' \neq \theta \pm \pi$, the pair $(\mathsf{Q}_\theta, \mathsf{Q}_{\theta'})$ resembles the position-momentum pair in many ways (see, e.g., [40]). In particular the pair is totally noncommutative and informationally incomplete. However, by taking a larger set $\{\mathsf{Q}_\theta \mid \theta \in \mathcal{I}\}$ where $\mathcal{I} \subset [0, \pi)$ is a dense subset we obtain informational completeness [17, 36].

The significance of the quadrature observables is enhanced by the fact that their measurement has a simple quantum optical realization, namely, the homodyne detection scheme (see, e.g., [43]). In fact, the experimental reconstruction of the state of a single mode electromagnetic field using quadrature measurements was first done by Smithey *et al.* in their pioneering work [62] where the vacuum state and a squeezed state were reconstructed. Afterwards this method has also been used to reconstruct the first number state, i.e., the single photon state, by Lvovsky *et al.* [49].

Chapter 3

Informationally complete covariant phase space observables

In the Hamiltonian formulation of classical mechanics the description of a physical system is based on the phase space Ω consisting of the generalized position and momentum coordinates. The points $(\mathbf{q}, \mathbf{p}) \in \Omega$ represent the pure states and the dynamical variables are given by measurable functions $f : \Omega \to \mathbb{R}$. Furthermore, there are in principle no limitations to measuring the state in a single measurement. In quantum mechanics the situation is obviously quite different. To begin with, position and momentum no longer have definite values but are given merely as probability distributions, and even if one knows these distributions, they do not determine the state uniquely. We have already seen that the latter shortcoming may be overcome by taking into account the set of rotated quadratures, but even in that case one needs infinitely many different measurements to determine the state. We will next consider a different approach which makes use of the fact that position and momentum admit approximate joint measurements. This then leads naturally to study covariant phase space observables.

3.1 Covariant phase space observables

Consider again a quantum system with a Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Physically such a space is associated to a spinless particle moving in a single spacial dimension, or a single mode electromagnetic field. The phase space of the system may be taken to be $\Omega = \mathbb{R}^2$, and it thus consists of pairs (q, p) of “position” and “momentum” coordinates. The phase space translations, i.e., the position shifts and velocity boosts, are represented in $\mathcal{H}$ by the state automorphisms

$\rho \mapsto \alpha_{(q,p)}(\rho) = W(q,p)^*\rho W(q,p)$ where $W(q,p) = e^{i\frac{qp}{2}}e^{-iqP}e^{ipQ}$ are the Weyl operators. The map $(q,p) \mapsto W(q,p)$ is a strongly continuous irreducible projective unitary representation which is square integrable, and thus for each $S, T \in \mathcal{T}(\mathcal{H})$ the function $(q,p) \mapsto \mathrm{tr}\,[SW(q,p)TW(q,p)^*]$ is integrable. In fact, we have (see, e.g., [68, Lemma 3.1])

$$\frac{1}{2\pi}\int \mathrm{tr}\,[SW(q,p)TW(q,p)^*]\; dqdp = \mathrm{tr}\,[S]\,\mathrm{tr}\,[T]\,.$$

In particular, if T is positive and of unit trace, then the map $\mathsf{G}^T : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ defined as

$$\mathsf{G}^T(Z) = \frac{1}{2\pi}\int_Z W(q,p)TW(q,p)^*\, dqdp, \qquad Z \in \mathcal{B}(\mathbb{R}^2),$$

is an observable.

The observables G^T have the important property of being covariant with respect to the phase space translations. This means that given a system in a state ρ, the probability of obtaining a measurement outcome from a translated set $Z + (q,p)$ is the same as the probability of obtaining the result from Z when the system is in the translated state $\alpha_{(q,p)}(\rho)$. This is the defining property of a covariant phase space observable:

Definition 7. *An observable $\mathsf{G} : \mathcal{B}(\mathbb{R}^2) \to \mathcal{H}$ is a covariant phase space observable if*

$$W(q,p)\mathsf{G}(Z)W(q,p)^* = \mathsf{G}(Z + (q,p)) \tag{3.1}$$

for all $Z \in \mathcal{B}(\mathbb{R}^2)$ and $(q,p) \in \mathbb{R}^2$.

We have already noted that any G^T is covariant, but the converse statement is also true. Namely, for any covariant phase space observable G there exists a unique positive trace one operator T such that $\mathsf{G} = \mathsf{G}^T$ [29, 68] (for more recent proofs, see [18, 37]). We say that T is the generating operator of G^T.

The Cartesian margins of G^T are the unsharp position and momentum observables $\mathsf{G}^T_1 = \mu^T * \mathsf{Q}$ and $\mathsf{G}^T_2 = \nu^T * \mathsf{P}$, where the convolving measures are determined by the generating operator via $\mu^T(X) = \mathrm{tr}\,[T\mathsf{Q}(-X)]$ and $\nu^T(Y) = \mathrm{tr}\,[T\mathsf{P}(-Y)]$. Thus, G^T is a joint observable of $\mu^T * \mathsf{Q}$ and $\nu^T * \mathsf{P}$. It is also known that if μ, $\nu : \mathcal{B}(\mathbb{R}) \to [0,1]$ are probability measures, then the corresponding unsharp observables $\mu * \mathsf{Q}$ and $\nu * \mathsf{P}$ have a joint observable if and only if $\mu = \mu^T$ and $\nu = \nu^T$ for some generating operator T [16]. In this sense the covariant phase space observables serve as architypes of approximate joint observables of position and momentum. Indeed, it has been shown in [69] using a distance between observables as the quantifier of the level of approximation, that the optimal approximate joint observable is in fact a covariant phase space observable. The same conclusion was later obtained with a different measure of approximation, namely, the error bar width [15]. The question concerning the optimality of approximate joint observables of course predates these results, and has been addressed for instance in [30]. For a review on these issues we refer to [10].

The fact that certain phase space probability densities arising from covariant phase space observables determine the state uniquely has been known for a long time. Indeed, already in 1940, Husimi introduced the phase space representation which is nowadays known as the Husimi Q-function [31]. In modern terms it is simply the probability density corresponding to the phase space observable generated by the ground state of the harmonic oscillator, the vacuum. Historically this predates the introduction of covariant phase space observables (see, e.g., [22, 28, 30] and references therein). Since these observables are in one-to-one correspondence with the generating operators, it is only natural that the informational completeness of G^T can be characterized in terms of T only. As it turns out, a convenient way to present the condition is via the Weyl transform of T, i.e., the function $(q,p) \mapsto \mathrm{tr}\,[TW(q,p)]$, which uniquely determines T.

Before discussing this further, we note that the existence of informationally complete phase space observables opens up the possibility of constructing classical phase space representations of quantum mechanics. The idea of representing quantum mechanics in phase space has a long history dating all the way back to Wigner's famous paper [71], and its subsequent developments such as [51, 56]. However, in these approaches one is lead to representing states as quasiprobability distributions, i.e., functions which take also negative values. This is due to the requirement of obtaining the position and momentum distributions as the margins. Of course this is an artificial requirement in quantum mechanics and thus it is natural to drop it and to consider instead the operational probability distributions arising from informationally complete covariant phase space observables. Such an approach was initiated by Ali and Prugovečki in [2] (see also [64]). More recently, the idea of representing quantum mechanics in terms of operationally sensible probability distributions has regained some attention due to the development of the so called tomographic representation of Man'ko *et al.* (see [32] and references therein).

3.2 Characterization of informational completeness

Recall that the informational completeness of the observable G^T means that any state $\rho \in \mathcal{S}(\mathcal{H})$ is uniquely determined by the probability measure $\mathsf{p}_\rho^{\mathsf{G}^T}$. Since this probability measure is absolutely continuous with with respect to the Lebesgue measure on the phase space, an equivalent condition is that ρ is determined by the probability density $(q,p) \mapsto \frac{1}{2\pi}\mathrm{tr}\,[\rho W(q,p)TW(q,p)^*]$. The key observation needed for obtaining a characterization for informational completeness is that the symplectic Fourier transform of this density is

$$\frac{1}{2\pi}\int e^{-i(qp'-q'p)}\mathrm{tr}\,[\rho W(q',p')TW(q',p')^*]\;dq'dp' = \mathrm{tr}\,[\rho W(q,p)]\,\overline{\mathrm{tr}\,[TW(q,p)]}. \tag{3.2}$$

From this it is clear that the relevant property is the size of the zero set

$$Z(T) = \{(q,p) \in \mathbb{R}^2 \mid \mathrm{tr}\,[TW(q,p)] = 0\}$$

of the Weyl transform of the generating operator T. Indeed, we make the following simple observation.

Proposition 2. *Let G^T and G^S be covariant phase space observables such that $Z(T) \subset Z(S)$. Then the state distinction power of G^T is greater than or equal to that of G^S.*

Proof. Let $\rho,\sigma \in \mathcal{S}(\mathcal{H})$ be such that $\mathsf{p}_\rho^{\mathsf{G}^T} = \mathsf{p}_\sigma^{\mathsf{G}^T}$. By taking the symplectic Fourier transforms, we see that $\mathrm{tr}\,[\rho W(q,p)] = \mathrm{tr}\,[\sigma W(q,p)]$ for all $(q,p) \in Z(T)^c = \mathbb{R}^2 \setminus Z(T)$. Since $Z(S)^c \subset Z(T)^c$, we have $\mathrm{tr}\,[\rho W(q,p)] = \mathrm{tr}\,[\sigma W(q,p)]$ for all $(q,p) \in Z(S)^c$, that is, $\mathsf{p}_\rho^{\mathsf{G}^S} = \mathsf{p}_\sigma^{\mathsf{G}^S}$ by (3.2). □

Since we know that there exist informationally complete covariant phase space observables, Proposition 2 shows that any G^T for which $Z(T) = \emptyset$ must be informationally complete. This was recognized in [68], where the corresponding generating operators were called regular. Despite some claims concerning the necessity of regularity [21], Ali and Prugovečki actually proved in [3] that the weaker condition of $Z(T)$ being of Lebesgue measure zero is already sufficient. It was shown in Article V that the condition can be further relaxed so that the necessary and sufficient condition is that $Z(T)$ does not contain an open set, or equivalently, that $Z(T)$ has a dense complement.The proof is included here for convenience.

Proposition 3. *The covariant phase space observable* G^T *is informationally complete if and only if* $Z(T)$ *does not contain an open set.*

Proof. We use Proposition 1 to prove the result. Assume first that $Z(T)$ does not contain an open set, and let $S \in \mathcal{T}(\mathcal{H})$ be such that $\mathrm{tr}\left[S\mathsf{G}^T(Z)\right] = 0$ for all $Z \in \mathcal{B}(\mathbb{R}^2)$, so that $\mathrm{tr}\,[SW(q,p)]\,\overline{\mathrm{tr}\,[TW(q,p)]} = 0$ for all $(q,p) \in \mathbb{R}^2$ by (3.2). It follows that $\mathrm{tr}\,[SW(q,p)] = 0$ for all $(q,p) \in Z(T)^c$ which is a dense set. Since the Weyl transform is a continuous function, it must be identically zero. Thus, $S = 0$.

Suppose now that $Z(T)$ contains an open set U. We must construct a nonzero trace class operator such that its Weyl transform is nonzero only inside U. To that end, let $S_0 \in \mathcal{T}(\mathcal{H})$ be regular. For any $f \in L^1(\mathbb{R}^2)$ define

$$f * S_0 = \int f(q,p)W(q,p)S_0W(q,p)^*\,dqdp$$

so that $f * S_0 \in \mathcal{T}(\mathcal{H})$ [68]. Now $\mathrm{tr}\,[f * S_0W(q,p)] = 2\pi\,\widehat{f}(-p,q)\mathrm{tr}\,[S_0W(q,p)]$ so that $Z(T) = \{(q,p) \in \mathbb{R}^2 \mid \widehat{f}(-p,q) = 0\}$. We can now choose $\widehat{f}$ to be, for instance, a nonzero compactly supported C^∞-function such that $\widehat{f}(-p,q) = 0$ for all $(q,p) \in U^c$. We then obtain f via the inverse Fourier transform, and our choice of $\widehat{f}$ guarantees that $f \in L^1(\mathbb{R}^2)$. Hence, we have a nonzero operator $f * S_0 \in \mathcal{T}(\mathcal{H})$ such that $\mathrm{tr}\left[f * S_0\mathsf{G}^T(Z)\right] = 0$ for all $Z \in \mathcal{B}(\mathbb{R}^2)$. That is, G^T is not informationally complete. □

Note that Proposition 3 does not, in itself, show that the condition of Ali and Prugovečki is not necessary, since it could happen that these conditions were actually equivalent. This question was resolved in Article V where a thorough analysis was performed on the three possible characterizations of the smallness of the zero set, namely, the algebraic, the measure theoretic, and the topological one:

(i) $Z(T)$ is empty,

(ii) $Z(T)$ is of measure zero,

(iii) $Z(T)$ does not contain an open set.

One of the important results in Article V was the following proposition, where the failure of the converse implications was proved by constructing explicit counterexamples. The first claim concerning the implications is clearly true.

Proposition 4. *Let $T \in \mathcal{T}(\mathcal{H})$ be positive and of trace one. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$, but neither of the implications can be reversed.*

This treatment can naturally be carried out also in the case of a more general phase space. Indeed, the generalization for a phase space $\mathcal{G} \times \widehat{\mathcal{G}}$ where $\mathcal{G}$ is a locally compact Abelian group (which is Hausdorff and second countable) and $\widehat{\mathcal{G}}$ is the dual group, is straightforward and Proposition 3 still holds. However, the statement about the failure of the converse implications in Proposition 4 is not always true. In Article V this question was considered in the case that $\mathcal{G}$ is a finite product of copies of $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, and a finite group $\mathbb{F}$, that is, $\mathcal{G} = \mathbb{R}^k \times \mathbb{T}^l \times \mathbb{Z}^m \times \mathbb{F}^n$. It was shown that if either of the converse implications $(ii) \Rightarrow (i)$ or $(iii) \Rightarrow (ii)$ is true, then $\mathcal{G}$ is necessarily finite.

Mathematically, the problem of informational completeness of phase space observables can be viewed as part of a more general framework. Indeed, the informational completeness of G^T is equivalent to the weak*-density of the linear combinations of the translates $W(q,p)TW(q,p)^*$ in $\mathcal{L}(\mathcal{H})$. Therefore it is natural to ask about the density of the linear combinations in the trace class $\mathcal{T}(\mathcal{H})$, the Hilbert-Schmidt class $\mathcal{HS}(\mathcal{H})$, or more generally in the Schatten p-classes $\mathcal{T}_p(\mathcal{H})$, with respect to the appropriate topologies. As shown in Proposition 4 of Article V, the conditions $(i)-(iii)$ give the characterization of the density in $\mathcal{T}(\mathcal{H})$, $\mathcal{HS}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$. This is the operator analogue of the classic problem in harmonic analysis concerning the density of linear combinations of translates of a function $f \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ in $L^p(\mathbb{R}^2)$ and, in particular, the possibility of characterizing this density property in terms of the zero set $Z(f) = \{(q,p) \in \mathbb{R}^2 \mid \widehat{f}(q,p) = 0\}$. This problem dates back to Wiener's classic paper [70] where conditions (i) and (ii) for $Z(f)$ were shown to be necessary and sufficient for the density in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$, respectively. Condition (iii) is again equivalent to the weak*-density in $L^\infty(\mathbb{R}^2)$ as shown for instance in [24]. As a matter of fact, this whole problem concerning the operators can be mapped to the corresponding problem for functions (see Proposition 5 of Article V). In particular, based on a recent result for functions [46], it was shown in Proposition 7 of Article V that for $1 < p < 2$ the density property in $\mathcal{T}_p(\mathcal{H})$ cannot be characterized in terms of the zero set $Z(T)$.

3.3 Reconstruction formulae

Even though the measurement of an informationally complete observable forms the foundation of quantum tomography, from the practical point of view it is merely the first step. In many cases, particularly those where the probability distribution is not the typical Q-function, it may be difficult to say anything about the state based on the statistics. For this purpose, it is desirable to cast the data into a more suitable form. In other words, one seeks to find a reconstruction formula which allows the computation of the state in some suitable representation. We present two methods for deriving such formulae: the Fourier theory, as already suggested by the above treatment, and the method of infinite matrix inversion. Our main concern is finding the matrix elements of the state with respect to the basis $\{h_n \mid n = 0, 1, 2, \ldots\}$ consisting of the Hermite functions.

3.3.1 Fourier theory

Let G^T be informationally complete so that $Z(T)$ does not contain an open set, and let $\rho \in \mathcal{S}(\mathcal{H})$ be an arbitrary state. From (3.2) we obtain the Weyl transform of ρ as

$$\mathrm{tr}\,[\rho W(q,p)] = \frac{1}{\mathrm{tr}\,[TW(q,p)^*]}\,\frac{1}{2\pi}\int e^{-i(qp'-q'p)}\,d\mathsf{p}_\rho^{\mathsf{G}^T}(q',p') \tag{3.3}$$

for all $(q,p) \in Z(T)^c$. Note that we have here used the fact that $\overline{\mathrm{tr}\,[TW(q,p)]} = \mathrm{tr}\,[TW(q,p)^*]$ since T is self-adjoint. Now the set $Z(T)^c$ is dense in $\mathbb{R}^2$, and the Weyl transform $(q,p) \mapsto \mathrm{tr}\,[\rho W(q,p)]$ is a continuous function, so that by taking appropriate limits on the right-hand side of (3.3) we are able to determine $\mathrm{tr}\,[\rho W(q,p)]$ for all $(q,p) \in \mathbb{R}^2$.

The Weyl transform for trace class operators is very much the analog of the Fourier transform for integrable functions (see, e.g., [68]). In particular, it maps trace class operators into continuous functions vanishing at infinity. However, due to the square-integrability of the Weyl representation, the Weyl transform is also square-integrable. The Plancherel theorem then states that it can be extended to a unitary operator $\mathcal{HS}(\mathcal{H}) \to L^2(\mathbb{R})$. In particular, for any $A, B \in \mathcal{T}(\mathcal{H}) \subset \mathcal{HS}(\mathcal{H})$, the Hilbert-Schmidt inner product is given by

$$\langle A|B\rangle_{\mathcal{HS}(\mathcal{H})} = \mathrm{tr}\,[A^*B] = \frac{1}{2\pi}\int \overline{\mathrm{tr}\,[AW(q,p)]}\,\mathrm{tr}\,[BW(q,p)]\;dqdp. \tag{3.4}$$

Now the matrix elements $\rho_{m,m+k} = \langle h_m|\rho h_{m+k}\rangle$ can be calculated from the Weyl transform of ρ using (3.4). In the case that $Z(T)$ is of measure zero, we can combine this with (3.3) to get the expression

$$\rho_{m,m+k} = \frac{1}{4\pi^2} \int \frac{\langle h_m | W(q,p)^* h_{m+k} \rangle}{\operatorname{tr}\left[T W(q,p)^* \right]} \left[\int e^{-i(qp'-q'p)} \, d\mathsf{p}_\rho^{\mathsf{G}^T}(q',p') \right] dqdp. \quad (3.5)$$

In the case that $Z(T)$ merely has dense complement but positive measure we may still consider (3.5) but the integrand needs to be determined using the limit procedure described at the beginning of this section.

3.3.2 Method of infinite matrix inversion

Another possible way to derive reconstruction formulae is given by the so-called method of infinite matrix inversion used in Article II (see also [38, 60]). We demonstrate this in the case where the generating operator is the number state $T_s = |h_s\rangle\langle h_s|$, $s = 0, 1, 2, \ldots$. In this case it is convenient to consider the situation in the complex plane. Indeed, we identify $\mathbb{C} \simeq \mathbb{R}^2$ via $z = \frac{1}{\sqrt{2}}(q + ip)$ and use the notation $D(z) = W(q,p)$. In particular, we may use the annihilation and creation operators $a = \frac{1}{\sqrt{2}}(Q + iP)$ and $a^* = \frac{1}{\sqrt{2}}(Q - iP)$ to write $D(z) = e^{za^* - \bar{z}a}$. Let $N = a^*a$ be the number operator related to the basis in question and define the rotation operators $R(\theta) = e^{i\theta N}$ for all $\theta \in [0, 2\pi)$. Then, by switching to the polar coordinate representation $z = re^{i\theta}$ we have $D(re^{i\theta}) = R(\theta)D(r)R(\theta)^*$.

Now consider the observable G^{T_s}. For any state $\rho \in \mathcal{S}(\mathcal{H})$ the corresponding probability density is then (up to a scaling)

$$g_\rho^s(re^\theta) = \langle h_s | D(re^{i\theta})^* \rho D(re^{i\theta}) h_s \rangle.$$

If we now expand the state as $\rho = \sum_{m,n=0}^\infty \rho_{m,n} |h_m\rangle\langle h_n|$ we obtain the form

$$\begin{aligned} g_\rho^s(re^\theta) &= \sum_{m,n=0}^\infty \rho_{m,n} e^{i\theta(n-m)} \langle h_s | D(r)^* h_m \rangle \langle h_n | D(r) h_s \rangle \\ &= \sum_{m,n=0}^\infty \rho_{m,n} e^{i\theta(n-m)} f_{n,m}^s(r) \end{aligned}$$

where $f_{n,m}^s(r) = \langle h_s | D(r)^* h_m \rangle \langle h_n | D(r) h_s \rangle$. The idea behind the method of infinite matrix inversion is to manipulate the above expression into a suitable (infinite) matrix relation which then yields the matrix elements of the state through the inversion of the matrix in question.

The first step is to calculate the cyclic moments of the distribution:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta} g_\rho^s(re^{i\theta})\,d\theta = \sum_{n=0}^{\infty} \rho_{n+k,n} f_{n,n+k}^s(r)$$

In general, the next step would be to multiply the above by a suitable function, and then to either integrate or differentiate with respect to the free variable r. The goal is to manipulate the relation into a form

$$y_p = \sum_{n=0}^{\infty} a_{pn} x_n,$$

so that if the matrix $(a_{pn})_{p,n=0}^{\infty}$ has an inverse $(b_{np})_{n,p=0}^{\infty}$, we can recover the x_n:s as

$$x_n = \sum_{p=0}^{\infty} b_{np} y_p.$$

For the present purpose, we need to consider separately the cases where k is even or odd. In both cases we first make the change of variables $x = r^2$. For the even case ($k = 2l$), we multiply the kth cyclic moment by e^x and differentiate with respect to x. After performing the necessary matrix inversion, we end up with the reconstruction formula

$$\rho_{n+2l,n} = \sum_{p=0}^{\infty} b_{np}^{s,2l} \frac{\partial^{p+s+l}}{\partial x^{p+s+l}} \left[e^x \frac{1}{2\pi} \int_0^{2\pi} e^{i2l\theta} g_\rho^s(\sqrt{x}e^{i\theta})\,d\theta \right] \Bigg|_{x=0}.$$

In the odd case ($k = 2l+1$) the suitable function for multiplication is $\sqrt{x}e^x$ which then eventually gives the formula

$$\rho_{n+2l+1,n} = \sum_{p=0}^{\infty} b_{np}^{s,2l+1} \frac{\partial^{p+s+l+1}}{\partial x^{p+s+l+1}} \left[\sqrt{x}e^x \int_0^{2\pi} e^{i(2l+1)\theta} g_\rho^s(\sqrt{x}e^{i\theta})\,d\theta \right] \Bigg|_{x=0}.$$

The details concerning the derivation and in particular the matrices $(b_{np}^{s,k})_{n,p=0}^{\infty}$ are given in Article II.

Chapter 4

Measuring covariant phase space observables

In this chapter we present three measurement models realizing the measurement of a covariant phase space observable. The first one is the Arthurs-Kelly model of an approximate joint measurement of position and momentum, the second one is a sequential measurement scheme, and the third one is a quantum optical realization using an eight-port homodyne detector.

4.1 Arthurs-Kelly model

The Arthurs-Kelly model [4] is probably the first measurement model for an approximate joint measurement of position and momentum. This model was first presented by Arthurs and Kelly and has been further developed by Busch [5, 6] (see also [9, pp. 150-152]). In this model, a system with the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ is coupled to two independent probe systems with the Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R})$ via a unitary coupling of the form

$$U = e^{-i\lambda Q\otimes P_1\otimes I_2 + i\mu P\otimes I_1\otimes Q_2}.$$

As the pointer observables we choose the position Q_1 of probe 1 and the momentum P_2 of probe 2, together with the pointer function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (\lambda^{-1}x, \mu^{-1}y)$. If we then assume that all three systems are initially in the vector states $\varphi \in \mathcal{H}$, $\phi_1 \in \mathcal{H}_1$ and $\phi_2 \in \mathcal{H}_2$, and denote by $\Psi = \varphi \otimes \phi_1 \otimes \phi_2$ the initial vector state of the combined system, the measured phase space observable $\mathsf{G} : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ is determined by the condition

$$\langle \varphi | \mathsf{G}(X \times Y)\varphi \rangle = \langle U\Psi | I \otimes \mathsf{Q}_1(\lambda X) \otimes \mathsf{P}_2(\mu Y) U\Psi \rangle,$$

for all $X, Y \in \mathcal{B}(\mathbb{R})$.

Using the fact that the state after the measurement interaction is given in the position representation as

$$(U\psi)(x, y, z) = \varphi(x + \mu z)\phi_1(y - \lambda x - \tfrac{1}{2}\lambda\mu z)\phi_2(z),$$

one can verify that G transforms covariantly under the action of the Weyl operators. Thus, $\mathsf{G} = \mathsf{G}^T$ for some generating operator T. To calculate the explicit form of T, we note that

$$\langle \varphi | \mathsf{G}^T(X \times Y)\varphi \rangle = \int_{\mathbb{R} \times \lambda X \times \mu Y} \left| (I \otimes I \otimes F_2 U\Psi)(x, y, z) \right|^2 dxdydz$$

where the Fourier transform is taken with respect to the third coordinate,

$$(I \otimes I_1 \otimes F_2 U\psi)(x, y, z) = \frac{1}{\sqrt{2\pi}} \int e^{-izw} \varphi(x + \mu w)\phi_1(y - \lambda x - \tfrac{1}{2}\lambda\mu w)\phi_2(w)\, dw.$$

By assuming that the initial probe states ϕ_1 and ϕ_2 are sufficiently well behaving, we get by a direct calculation

$$\begin{aligned} &\langle \varphi | \mathsf{G}^T(X \times Y)\varphi \rangle \\ &= \frac{1}{2\pi} \int_{X \times Y} \left(\int \overline{(W(y, z)^*\varphi)(w)} T(w, w')(W(y, z)^*\varphi)(w')\, dwdw' \right) dydz \end{aligned}$$

where

$$T(w, w') = \int \overline{\phi_1^\lambda(\tfrac{1}{2}(x - w))\phi_2^\mu(x + w)} \phi_1^\lambda(\tfrac{1}{2}(x - w'))\phi_2^\mu(x + w')\, dx \qquad (4.1)$$

and we have denoted $\phi_1^\lambda(x) = \sqrt{\lambda}\phi_1(\lambda x)$ and $\phi_2^\mu(x) = \frac{1}{\sqrt{\mu}}\phi_2(\frac{x}{\mu})$. In conclusion, the phase space observable measured in the Arthurs-Kelly model is G^T where T is the integral operator defined by the kernel (4.1).

4.2 Sequential measurement scheme

The sequential measurement scheme consists of a standard model type measurement of unsharp position followed by any sharp momentum measurement. Recall from Section 2.2 that the measurement scheme for the standard measurement of unsharp position is $\mathcal{M} = \langle L^2(\mathbb{R}), \phi, e^{-i\lambda Q\otimes P_p}, \mathsf{Q}, f^\lambda \rangle$ where P_p is the momentum operator on the Hilbert space of the probe and the pointer function is $f^\lambda(x) = \lambda^{-1}x$. The instrument associated to this scheme is then in the dual form

$$\mathcal{I}^\lambda(X)^*(B) = \int_X K_x^* B K_x \, dx, \quad B \in \mathcal{L}(\mathcal{H}),$$

where $K_x = \sqrt{\lambda}\phi(-\lambda(Q - x))$, and the measured observable is the unsharp position $\mu^\lambda * \mathsf{Q}$ with $\mu^\lambda(X) = \langle \phi | \mathsf{Q}(\lambda X)\phi \rangle$.

Suppose now that after the first measurement we perform any measurement of the sharp momentum P. This results in the sequential observable

$$\mathsf{G}(X \times Y) = \mathcal{I}^\lambda(X)^*(\mathsf{P}(Y)), \qquad X, Y \in \mathcal{B}(\mathbb{R}),$$

which is again a covariant phase space observable so that $\mathsf{G} = \mathsf{G}^T$ for some T. Now for any $\varphi \in \mathcal{H}$ we have

$$\langle \varphi | \mathcal{I}^\lambda(X)^*(\mathsf{P}(Y))\varphi \rangle = \int_{X\times Y} \left| (FK_x\varphi)(y) \right|^2 dx\, dy,$$

and a direct calculation gives

$$(FK_x\varphi)(y) = \frac{1}{\sqrt{2\pi}} e^{-i\frac{xy}{2}} \langle \phi_\lambda | W(x,y)^* \varphi \rangle$$

where $\phi_\lambda(x) = \sqrt{\lambda}\, \overline{\phi(-\lambda x)}$ so that

$$\langle \varphi | \mathsf{G}(X \times Y)\varphi \rangle = \frac{1}{2\pi} \int_{X\times Y} \left| \langle \phi_\lambda | W(x,y)^* \varphi \rangle \right|^2 dx\, dy.$$

In other words, $\mathsf{G} = \mathsf{G}^T$ with $T = |\phi_\lambda\rangle\langle\phi_\lambda|$.

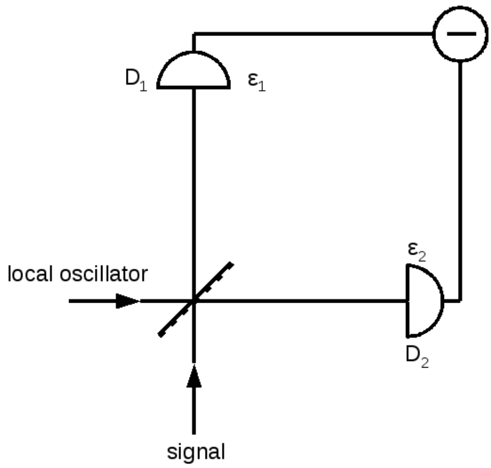

Figure 4.1: Balanced homodyne detector

4.3 Eight-port homodyne detection

4.3.1 Balanced homodyne detection

The balanced homodyne detector is a standard tool in quantum optics (see, e.g., [43, pp. 85-97]). The success of this scheme is based on the fact that it can be used to measure the field quadratures, that is, the quadrature observables Q_θ. As depicted in Figure 4.1, in balanced homodyne detection a signal field is mixed with an auxiliary field of a local oscillator by means of a $50:50$ beam splitter, and the (scaled) photon number difference between the output fields is measured. By assuming that the auxiliary field is a strong coherent field, that is, a laser with high intensity, the measurement statistics is approximately that of a quadrature observable.

To make things more precise, let $\mathcal{H}$ and $\mathcal{H}_{\rm aux}$ be the Hilbert spaces of the signal field and the auxiliary field, respectively. We use the photon number basis $\{|n\rangle \,|\, n = 0, 1, 2, \ldots\}$ for each mode and denote by N and $N_{\rm aux}$ the number operators for the modes. Assume that the auxiliary field is in the coherent state $|z\rangle$, $z \in \mathbb{C}$, defined by the expression

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle.$$

The beam splitter is modelled by the unitary operator U determined by its action on the tensor product of coherent states:

$$U|w\rangle \otimes |z\rangle = \left|\tfrac{1}{\sqrt{2}}(w-z)\right\rangle \otimes \left|\tfrac{1}{\sqrt{2}}(w+z)\right\rangle \tag{4.2}$$

for all $w, z \in \mathbb{C}$.

The detector involves two photodetectors D_1 and D_2 with quantum efficiencies $\epsilon_1, \epsilon_2 \in (0,1]$. Each photodetector thus measures the approximate photon number, given by the detection observable (see [9, pp. 177-180] or [43, pp. 79-83])

$$n \mapsto \mathsf{E}_n^{\epsilon_j} = \sum_{m=n}^{\infty} \binom{m}{n} \epsilon_j^n (1-\epsilon_j)^{m-n} |m\rangle\langle m|,$$

and we are interested in the scaled number differences. In particular, the set of possible measurement outcomes is taken to be

$$\Omega = \left\{ \frac{1}{\sqrt{2}|z|} \left(\frac{n}{\epsilon_2} - \frac{m}{\epsilon_1} \right) \,\middle|\, m, n = 0, 1, 2, \ldots \right\}$$

and the detection statistics is represented by the observable $\mathsf{E}_{\epsilon_1,\epsilon_2} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_{\mathrm{aux}})$,

$$\mathsf{E}_{\epsilon_1,\epsilon_2}(X) = \sum_X \mathsf{E}_m^{\epsilon_1} \otimes \mathsf{E}_n^{\epsilon_2}$$

where the summation is over those m, n for which $\frac{1}{\sqrt{2}|z|}\left(\frac{n}{\epsilon_2} - \frac{m}{\epsilon_1}\right) \in X$.

The signal observable $\mathsf{E}^z_{\epsilon_1,\epsilon_2} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is now given by the measurement dilation formula

$$\mathsf{E}^z_{\epsilon_1,\epsilon_2}(X) = V_z^* U^* \mathsf{E}_{\epsilon_1,\epsilon_2}(X) U V_z \tag{4.3}$$

where $V_z : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_{\mathrm{aux}}$ is the embedding $\varphi \mapsto \varphi \otimes |z\rangle$. To consider rigorously the high-amplitude limit $|z| \to \infty$ we adopt the approach presented in [35], that is, we consider the limit weakly in the sense of probabilities. In the following definition the symbol ∂X denotes the boundary of the set X, that is, $\partial X = \overline{X} \cap \overline{X^c}$ where X^c is the complement of X.

Definition 8. *A sequence $(\mathsf{E}_j)_{j\in\mathbb{N}}$ of observables $\mathsf{E}_j : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ converges to an observable $\mathsf{E} : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ weakly in the sense of probabilities if*

$$\lim_{j\to\infty} \mathsf{E}_j(X) = \mathsf{E}(X)$$

in the weak operator topology for all $X \in \mathcal{B}(\mathbb{R}^n)$ such that $\mathsf{E}(\partial X) = 0$.

This convergence means that given any signal state ρ, the corresponding sequence of probability measure $(\mathsf{p}_\rho^{\mathsf{E}_j})_{j\in\mathbb{N}}$ converges weakly to the measure $\mathsf{p}_\rho^{\mathsf{E}}$. In other words, for any bounded continuous function $f:\mathbb{R}\to\mathbb{R}$ we have

$$\lim_{j\to\infty}\int f(x)\,d\mathsf{p}_\rho^{\mathsf{E}_j}(x)=\int f(x)\,d\mathsf{p}_\rho^{\mathsf{E}}(x).$$

Now fix the phase $\theta\in[0,2\pi)$ of the local oscillator and let $z_j=r_je^{i\theta}$ where $(r_j)_{j\in\mathbb{N}}$ is an arbitrary increasing sequence of positive numbers such that $\lim_{j\to\infty}r_j=\infty$. If $\epsilon_1<1$ or $\epsilon_2<1$ define the probability measure $\mu_{\epsilon_1,\epsilon_2}:\mathcal{B}(\mathbb{R})\to[0,1]$ via

$$\mu_{\epsilon_1,\epsilon_2}(X)=\sqrt{\tfrac{2\epsilon_1\epsilon_2}{\pi(\epsilon_1-2\epsilon_1\epsilon_2+\epsilon_2)}}\int_X e^{-\frac{2\epsilon_1\epsilon_2}{\epsilon_1-2\epsilon_1\epsilon_2+\epsilon_2}x^2}\,dx,$$

and let $\mu_{1,1}$ be the Dirac measure concentrated at the origin. We can now state the result concerning the observable measured in the high-amplitude limit. The result was proved in [35] for ideal detectors ($\epsilon_1=\epsilon_2=1$), and this was generalized to the case of non-unit quantum efficiencies in Article III.

Proposition 5. *For all $\epsilon_1,\epsilon_2\in(0,1]$ the sequence $(\mathsf{E}^{z_j}_{\epsilon_1,\epsilon_2})_{j\in\mathbb{N}}$ converges to $\mu_{\epsilon_1,\epsilon_2}*\mathsf{Q}_\theta$ weakly in the sense of probabilities.*

4.3.2 Eight-port homodyne detection

The eight-port homodyne detector is, vaguely speaking, a combination of two balanced homodyne detectors (see, e.g., [43, pp. 147-159]). A signal field is mixed with a reference field using a beam splitter, and the quadratures Q and P are measured on the output fields using balanced homodyne detection. The measured observable then turns out to be a covariant phase space observable whose generating operator depends on the state of the reference field, and by varying this state any phase space observable can be measured [34] (This result can also be found already in [30, Chapter III.6] although no direct reference to the quantum optical realization has been made.). For this reason this scheme is also known as double homodyne detection. Now let us go into the details of this setup.

The eight-port homodyne detector consists of four input modes, four $50:50$ beam splitters, a phase shifter, and four photodetectors D_j with quantum efficiencies $\epsilon_j\in(0,1]$, see Figure 4.2. Let ρ and σ be the states of the signal and reference fields and let the local oscillator be in the coherent state $|\sqrt{2}z\rangle$. In this more complicated setup we use the notation U_{ij} for the beam splitter,

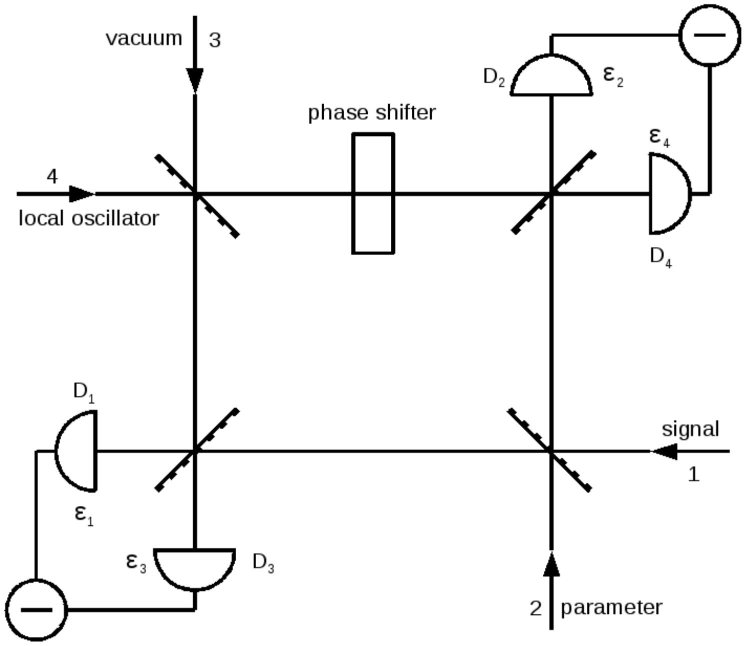

Figure 4.2: Eight-port homodyne detector

where the subscripts refer to the primary and secondary input modes, that is, the first and second terms in the tensor product in (4.2). In Figure 4.2 the dashed line represents the primary input mode. The phase shifter providing a phase shift $\theta \in [0, 2\pi)$ is modelled by the rotation operator $R(\theta) = e^{i\theta N}$.

We assign to each detector D_j a quantum efficiency $\epsilon_j \in (0, 1]$ so that the detection is represented by the biobservable

$$(X, Y) \mapsto \mathsf{E}_{\epsilon_1,\epsilon_3}(X) \otimes \mathsf{E}_{\epsilon_2,\epsilon_4}(Y).$$

The unique signal observable $\mathsf{E}^{\sigma,z,\theta} : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ determined by this setup can be expressed in terms of the homodyne observables (4.3). Indeed, the observable is determined by the condition

$$\mathrm{tr}[\rho \mathsf{E}^{\sigma,z,\theta}(X \times Y)] = \mathrm{tr}[U_{12}(\rho \otimes \sigma)U_{12}^{*} \mathsf{E}^{z}_{\epsilon_1,\epsilon_3}(X) \otimes \mathsf{E}^{ze^{i\theta}}_{\epsilon_2,\epsilon_4}(Y)]$$

for all $X \times Y \in \mathcal{B}(\mathbb{R}^2)$.

Proposition 5 now allows one to calculated the high-amplitude limit observable. Assume that the phase shift provided by the phase shifter is $\theta = \frac{\pi}{2}$ (an arbitrary phase shift leading to covariance with respect to a different representation has been treated in [42]). Let ϵ be a collective symbol for the set $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ of efficiencies and define the probability measure $\mu_\epsilon : \mathcal{B}(\mathbb{R}^2) \to [0,1]$ by

$$\mu_\epsilon(X \times Y) = \mu_{\epsilon_1,\epsilon_3}(\tfrac{1}{\sqrt{2}}X)\mu_{\epsilon_2,\epsilon_4}(\tfrac{1}{\sqrt{2}}Y), \qquad X \times Y \in \mathcal{B}(\mathbb{R}^2).$$

Denote by C the conjugation map, that is, $(C\varphi)(x) = \overline{\varphi(x)}$. Let $(r_j)_{j\in\mathbb{N}}$ be again an arbitrary increasing sequence of positive numbers such that $\lim_{j\to\infty} r_j = \infty$. The following proposition then shows that the high-amplitude limit observable is a smeared version of a covariant phase space observable. The proposition was first proved in [34] for ideal detectors and the general case was proved in Article III.

Proposition 6. *The sequence* $(\mathsf{E}^{\sigma,r_j,\pi/2})_{j\in\mathbb{N}}$ *coverges to* $\mu_\epsilon * \mathsf{G}^{C\sigma C^{-1}}$ *weakly in the sense of probabilities.*

It should be noted that the covariance property of an observable is not affected by the smearing. Indeed, the observable $\mu_\epsilon * \mathsf{G}^{C\sigma C^{-1}}$ still satisfies the covariance condition (3.1), so it is generated by a unique positive trace one operator. It was shown in Proposition 4.1 of Article III that $\mu_\epsilon * \mathsf{G}^{C\sigma C^{-1}} = \mathsf{G}^{\mu_\epsilon * C\sigma C^{-1}}$ where the convoluted state [68] is defined as

$$\mu_\epsilon * C\sigma C^{-1} = \int W(q,p)C\sigma C^{-1}W(q,p)^*\, d\mu_\epsilon(q,p).$$

From the tomographic point of view an important point is that the state distinction power of the high-amplitude limit observable does not depend on the quantum efficiencies of the detectors. This is due to the Gaussian form of the convolving measure μ_ϵ which in turn results from the binomial form of the unsharp photon number observable. All in all, we have the following result from Article III.

Proposition 7. *The observables* $\mu_\epsilon * \mathsf{G}^{C\sigma C^{-1}}$ *and* $\mathsf{G}^{C\sigma C^{-1}}$ *are informationally equivalent. In particular,* $\mu_\epsilon * \mathsf{G}^{C\sigma C^{-1}}$ *is informationally complete if and only if* $\mathsf{G}^{C\sigma C^{-1}}$ *is informationally complete.*

Using Proposition 3 and the simple observation that $C^{-1}W(q,p)C = W(q,-p)$ we get the following corollary.

Corollary 1. *The observable* $\mu_\epsilon * \mathsf{G}^{C\sigma C^{-1}}$ *is informationally complete if and only if* $Z(\sigma)$ *does not contain an open set.*

Chapter 5

Position and momentum tomography

Even though the noncommutativity of position and momentum observables denies the possibility of measuring them jointly, the existence of informationally complete observables guarantees that the position and momentum distributions can be reconstructed from the statistics of a single measurement. However, since the pair (Q,P) is far from being informationally complete, one might expect that a great deal of redundancy is included in the statistics of an informationally complete observable. Indeed, one should be able to determine the position and momentum distributions more directly and, in particular, with less information. In this chapter we consider the possibility of reconstructing the distributions from the marginal statistics of covariant phase space observables.

5.1 On the connection to informational completeness

Consider a covariant phase space observable G^T with the margins $\mu^T * \mathsf{Q}$ and $\nu^T * \mathsf{P}$. A minimal requirement for reconstructing the position and momentum distributions from the marginal statistics is that the marginal observables are informationally equivalent with Q and P. Written in terms of the convolving measures, we need to require that $\operatorname{supp}\widehat{\mu}^T = \operatorname{supp}\widehat{\nu}^T = \mathbb{R}$. The Fourier transforms of the measures can be given in terms of the Weyl transform of the generating operator:

$$\widehat{\mu}^T(p) = \frac{1}{\sqrt{2\pi}}\mathrm{tr}[TW(0,-p)], \qquad \widehat{\nu}^T(q) = \frac{1}{\sqrt{2\pi}}\mathrm{tr}[TW(q,0)].$$

Now the informational completeness of G^T is obtained whenever the support of $(q,p) \mapsto \mathrm{tr}[TW(q,p)]$ is $\mathbb{R}^2$ whereas the above condition for informational equivalence concerns only the restrictions of this function to the coordinate axes. Therefore there is no *a priori* reason to assume that only informational complete measurements allow the reconstruction of the position and momentum distributions. The following example from Article VI shows that informational completeness is in fact unnecessary.

Let $\varphi = \chi_{[-1/2,1/2]}$, the characteristic function of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and define

$$T = \frac{1}{2}|\varphi\rangle\langle\varphi| + \frac{1}{2}|\widehat{\varphi}\rangle\langle\widehat{\varphi}|.$$

Consider the margins of G^T. Now the Fourier transforms of the convolving measures are given by

$$\widehat{\mu}^T(p) = \widehat{\nu}^T(p) = \begin{cases} \frac{1}{2\sqrt{2\pi}}\left(1 - |p| + \frac{\sin(p/2)}{p/2}\right), & \text{when } |p| \leq 1 \\ \frac{1}{2\sqrt{2\pi}}\frac{\sin(p/2)}{p/2}, & \text{otherwise.} \end{cases}$$

This shows that the set of zeros for the above function is countable and thus the support is $\mathbb{R}$ (see Figure 5.1). In other words, the margins are informationally equivalent with sharp position and momentum. However, G^T is not informationally complete, since $\mathrm{tr}[TW(q,p)] = 0$ whenever $|q| \geq 1$ and $|p| \geq 1$.

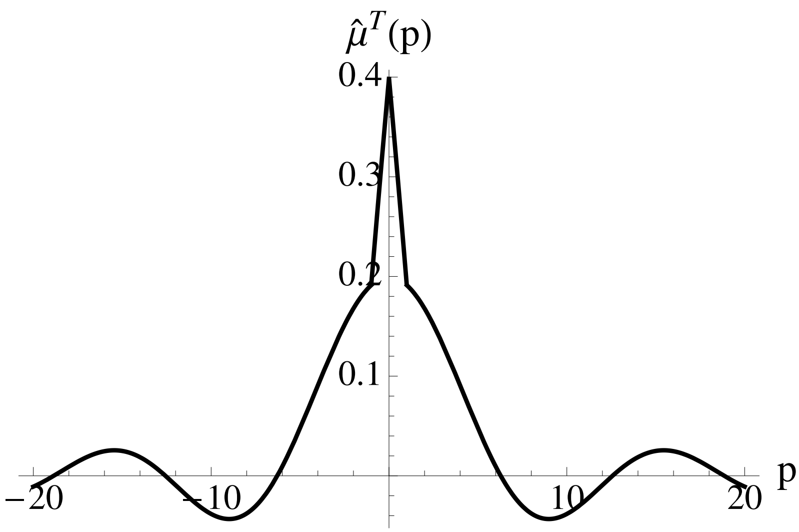

Figure 5.1: The plot of $\widehat{\mu}^T$ as a function of p.

Article VI contains also another result, which shows that the informational completeness of a G^T does not guarantee that the margins are informationally equivalent with their sharp counterparts. This means that if one wishes to reconstruct the position and momentum distributions from the statistics, then it is not necessarily the informational completeness which ought to be sought. That is, even though these distributions can be obtained by first reconstructing the state, the methods presented in the next section do not require this. Indeed, the distributions can be reconstructed directly from the marginal statistics.

5.2 Reconstructing position and momentum

We will next consider reconstructing the position and momentum distributions from the marginal statistics of a covariant phase space observable. We focus on two methods: using the Fourier theory and using the statistical method of moments.

5.2.1 Fourier theory

Let G^T be a covariant phase space observable such that $\mu^T * \mathsf{Q}$ and $\nu^T * \mathsf{P}$ are informationally equivalent with Q and P, that is, $\operatorname{supp}\widehat{\mu}^T = \operatorname{supp}\widehat{\nu}^T = \mathbb{R}$. Let ρ be an arbitrary state and consider the first marginal observable.

By taking the Fourier transform of the corresponding probability measure we obtain $(\widehat{\mu^T * \mathsf{p}_\rho^\mathsf{Q}})(p) = \sqrt{2\pi}\widehat{\mu}^T(p)\widehat{\mathsf{p}}_\rho^\mathsf{Q}(p)$ for all $p \in \mathbb{R}$. It follows that for all $p \in \mathbb{R}$ such that $\widehat{\mu}^T(p) \neq 0$ we have

$$\widehat{\mathsf{p}}_\rho^\mathsf{Q}(p) = \frac{1}{\sqrt{2\pi}} \frac{(\widehat{\mu^T * \mathsf{p}_\rho^\mathsf{Q}})(p)}{\widehat{\mu}^T(p)}. \tag{5.1}$$

Since $\widehat{\mathsf{p}}_\rho^\mathsf{Q}$ is a bounded continuous function and $\operatorname{supp}\widehat{\mu}^T = \mathbb{R}$, we may take appropriate limits to determine $\widehat{\mathsf{p}}_\rho^\mathsf{Q}(p)$ for all $p \in \mathbb{R}$. Since the Fourier transform is injective, this then determines $\mathsf{p}_\rho^\mathsf{Q}$ uniquely.

Of course, if we want to obtain explicitly the position distribution, then we need the integrability of the right-hand side of (5.1) which is not always guaranteed. However, if for instance $\rho = \sum_{n=1}^k c_n|\psi_n\rangle\langle\psi_n|$ where each ψ_n is a finite linear combination of Hermite functions, then the position distribution $x \mapsto \sum_{n=1}^k c_n|\psi_n(x)|^2$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$, and hence also the Fourier transform is in $\mathcal{S}(\mathbb{R})$. In particular, since $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ we may use the Fourier inversion theorem to obtain the position distribution

$$\sum_{n=1}^{k} c_n |\psi_n(x)|^2 = \frac{1}{2\pi} \int e^{ixp} \frac{(\widehat{\mu^T * \mathsf{p}_\rho^{\mathsf{Q}}})(p)}{\widehat{\mu}^T(p)} \,\mathrm{d}p$$

for almost all $x \in \mathbb{R}$. Note that by [26, Theorem 1] the states of the above form are dense in the set of all states. A similar treatment shows that we can reconstruct the momentum distribution from $\nu^T * \mathsf{p}_\rho^{\mathsf{P}}$.

5.2.2 Method of moments

The statistical method of moments was presented in this context in [11], and was further studied in Article I. The method is based on the fact that under suitable assumptions it is possible to reconstruct, for instance, the moments $\mathsf{p}_\rho^{\mathsf{Q}}[k] = \int q^k \, d\mathsf{p}_\rho^{\mathsf{Q}}(q)$ of the position distribution from the moments of the measured marginal distribution. The obvious downside of this method is that one needs to make the strong assumption that all the moments are finite. Even under such assumption it is not guaranteed that the probability distribution is uniquely determined by its moment sequence. Indeed, it may happen that two different probability measures have the same moment sequences. However, these problems vanish if the measures in question are exponentially bounded.

Recall that a probability measure $\mu : \mathcal{B}(\mathbb{R}) \to [0,1]$ is said to be exponentially bounded if there exists an $a > 0$ such that

$$\int e^{a|x|} \, d\mu(x) < \infty.$$

According to [41, Proposition 2] this is the case if and only if there exist constants $C, R > 0$ such that the moment inequality

$$|\mu[k]| \leq C R^k k! \tag{5.2}$$

holds for all $k \in \mathbb{N}$. As shown, for instance in [61, Proposition 5] an exponentially bounded measure is always uniquely determined by its moment sequence. That is, if $\nu : \mathcal{B}(\mathbb{R}) \to [0,1]$ is another probability measure such that $\nu[k] = \mu[k]$ for all $k \in \mathbb{N}$, then $\nu = \mu$.

Consider again G^T and suppose that ρ is a state such that marginal probability measures $\mu^T * \mathsf{p}_\rho^{\mathsf{Q}}$ and $\nu^T * \mathsf{p}_\rho^{\mathsf{P}}$ are exponentially bounded. The kth moment of, say, the measure $\mu^T * \mathsf{p}_\rho^{\mathsf{Q}}$ can now be calculated as

$$\left(\mu^T * \mathsf{p}_\rho^{\mathsf{Q}}\right)[k] = \sum_{n=0}^{k} \binom{k}{n} \mu^T[k-n]\mathsf{p}_\rho^{\mathsf{Q}}[n].$$

From this expression the moments of the position distribution can be solved recursively which gives

$$\mathsf{p}_\rho^{\mathsf{Q}}[k] = \left(\mu^T * \mathsf{p}_\rho^{\mathsf{Q}}\right)[k] - \sum_{n=0}^{k-1} \binom{k}{n} \mu^T[k-n]\mathsf{p}_\rho^{\mathsf{Q}}[n].$$

In other words, we are able to calculate the moments from the moments of the measured marginal statistics. If we know *a priori* that $\mathsf{p}_\rho^{\mathsf{Q}}$ is determined by the moment sequence, then we have actually determined the position distribution. The same treatment can of course be carried out for the second margin. Note that if the convolving measures μ^T and ν^T are exponentially bounded, then the marginal distributions are exponentially bounded whenever $\mathsf{p}_\rho^{\mathsf{Q}}$ and $\mathsf{p}_\rho^{\mathsf{P}}$ are such. Since this is the case for all finite mixtures of vector states which are finite linear combinations of Hermite functions, we find that in the case of exponentially bounded convolving measures, this method works for a dense set of states.

Chapter 6

Remarks on other reconstruction methods

Apart from the use of informationally complete phase space observables, there are also other methods for determining the unknown state of a system. In this chapter we consider two methods, the first one uses the rotated quadrature observables whereas the second one uses the so-called weak values of observables.

6.1 Rotated quadrature observables

In quantum optics, perhaps the most used method for quantum tomography uses the set of rotated quadrature observables $\{\mathsf{Q}_\theta \,|\, \theta \in [0, 2\pi)\}$. From the experimentalist's point of view this is quite appealing since the experimental realization of a quadrature measurement is simple; one may use a single balanced homodyne detector where by varying the phase of the coherent reference field one can measure each Q_θ. There are several possible ways to reconstruct the state from the quadrature statistics. The original idea of Vogel and Risken [65] is based on the observation that the quadrature distributions are obtained from the Wigner function of the state via the Radon transform. Therefore, if one is able to invert this transform, then one can calculate the Wigner function from the measurement statistics.

In [20], D'Ariano *et al.* proposed a different method which allowed the calculation of the matrix elements with respect to the number basis directly from the quadrature data. This was improved and generalized in [45] allowing the calculation of the matrix elements with respect to any basis. The main idea is that suitable "pattern functions", depending on the chosen basis, are averaged with respect to the quadrature data. In [38] a mathematically rigorous derivation lead to the same reconstruction formula for the number basis. We will now present this formula.

Let $\rho \in \mathcal{S}(\mathcal{H})$ and for each $\theta \in [0, 2\pi)$ let $x \mapsto q_\rho(x,\theta)$ be the density of the probability measure $X \mapsto \mathrm{tr}\,[\rho \mathsf{Q}_\theta(X)]$. The function $q_\rho : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}$ is then integrable and, in particular, the integral

$$q_\rho^k(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} q_\rho(x,\theta)\, d\theta$$

exists for all $k = 0, 1, 2, \ldots$ and almost all $x \in \mathbb{R}$. Now let $\mathrm{daw} : \mathbb{R} \to \mathbb{R}$ denote the so-called Dawson's integral [63, Chapter 42], that is,

$$\mathrm{daw}(x) = e^{-x^2} \int_0^x e^{t^2}\, dt$$

and define the quantity $\Xi_\rho^{k,m}$, for $k, m = 0, 1, 2, \ldots$, via

$$\Xi_\rho^{k,m} = 2 \int \mathrm{daw}^{(k+2m+1)}(x) q_\rho^k(x)\, dx$$

where $\mathrm{daw}^{(k+2m+1)}$ denotes the $(k + 2m + 1)$th derivative of Dawson's integral. Then, the matrix elements of ρ can be expressed as

$$\rho_{n+k,n} = (-1)^k \sqrt{\frac{(n+k)!}{2^k n!}} \sum_{m=0}^{n} \binom{n}{m} \frac{\Xi_\rho^{k,m}}{2^m (k+m)!}.$$

This is the reconstruction formula appearing in both [45] and [38].

Note that even though a finite number of quadratures is never enough to determine a completely unknown state, some prior knowledge may allow unique state determination from finitely many measurements. Indeed, if we know that the matrix representation with respect to the number basis is finite, then only finitely many quadratures are needed [44, 38]. Since any state can be approximated by a finite matrix, it is often in practise justified to make the *a priori* assumption of finiteness. In any case, no real detector can detect arbitrarily high energies and only finitely many measurements can be performed.

6.2 Reconstruction via weak values

The concept of the weak value of an observable originates from the work of Aharonov *et al.* [1]. The intuitive idea behind the weak value of a sharp observable $\mathsf{E}^A : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is that in addition to preparing the system to a state $\varphi \in \mathcal{H}$, which is viewed as a preselection, and performing a measurement, we postselect "only those systems which are in a given state $\eta \in \mathcal{H}$". If the measurement performed between the pre- and postselection is weak (in some appropriate sense), then the average value obtained from the statistics should be the real part of the weak value $\langle A \rangle_w$, defined as

$$\langle A \rangle_w = \frac{\langle \eta | A\varphi \rangle}{\langle \eta | \varphi \rangle}. \tag{6.1}$$

Similarly, the imaginary part of $\langle A \rangle_w$ can be obtained from a weak measurement using a different pointer observable [33]. In Article IV this idea was analyzed rigorously using the theory of sequential measurements, where the postselection refers to conditioning with respect to some fixed subset of outcomes of the second measurement.

From the tomographic point of view an important work is the experimental paper [48] of Lundeen *et al.* where the weak values were used for reconstructing the pointwise values of the wavefunction. This method was further developed in [47] to cover also the case where the state is allowed to be mixed. Though this method works well in finite dimensions (see below), in the infinite dimensional case it has several disadvantages when compared to the use of phase space observables. The most crucial shortcoming is the fact that the method does not work for all initial states[1]. Before going into the analysis of this reconstruction method, we consider the (generalized) weak values and how they can be obtained from measurements.

6.2.1 Weak values as limits of conditional averages

We begin by generalizing the formal definition (6.1) into the case where the observables in question are positive operator measures. The selfadjoint operator A must therefore be replaced by the first moment operator $\mathsf{E}[1]$ of the observable E, whereas the postselection corresponds to conditioning with respect to some subset $Y \in \mathcal{B}(\mathbb{R})$ of values of the subsequent measurement or with respect to some effect $B \in \mathcal{E}(\mathcal{H})$.

For an observable $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ denote by $\mathcal{D}(x, \mathsf{E}) \subset \mathcal{H}$ the subspace of those φ for which the identity map $x \mapsto x$ is $\mathsf{E}_{\psi,\varphi}$-integrable for all $\psi \in \mathcal{H}$.

[1] Here the use of the term "informationally complete" is avoided since the method uses the (conditional) average values of a large number of measurements rather than the full statistics. In other words, the set of measured observables may be informationally complete but the average values do not determine the state.

The first moment operator of E is then the linear operator $\mathsf{E}[1] : \mathcal{D}(x, \mathsf{E}) \to \mathcal{H}$ defined as

$$\langle\psi|\mathsf{E}[1]\varphi\rangle = \int x \, d\mathsf{E}_{\psi,\varphi}(x), \qquad \varphi \in \mathcal{D}(x, \mathsf{E}),\ \psi \in \mathcal{H}.$$

The domain $\mathcal{D}(x, \mathsf{E})$ contains as a subspace the so-called square-integrability domain $\widetilde{\mathcal{D}}(x, \mathsf{E})$ consisting of the those $\varphi \in \mathcal{H}$ for which the function $x \mapsto x^2$ is integrable with respect to the positive measure $\mathsf{E}_{\varphi,\varphi}$ (see, e.g., [39]).

Definition 9. *Let $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be an observable, $\varphi \in \mathcal{D}(x, \mathsf{E})$, $\|\varphi\| = 1$ and let $B \in \mathcal{E}(\mathcal{H})$ be such that $\langle\varphi|B\varphi\rangle \neq 0$. The weak value of E in a vector state φ conditioned by B is*

$$\mathsf{E}_w(\varphi, B) = \frac{\langle\varphi|B\mathsf{E}[1]\varphi\rangle}{\langle\varphi|B\varphi\rangle} \tag{6.2}$$

Notice that by choosing $\mathsf{E} = \mathsf{E}^A$ and $B = |\eta\rangle\langle\eta|$ so that $\mathsf{E}[1] = A$, we have the original form (6.1).

In Article IV we showed that weak value in the form of Definition 9 can be obtained as a limit of conditional averages in sequential measurements if the initial state belongs to the square-integrability domain of the moment operator. The essential measurement scheme for this purpose is the generalized standard model $\mathcal{M}^\lambda$ (where we have now explicitly indicated the λ-dependence) which realizes the measurement of the unsharp observable $\mu^\lambda * \mathsf{E}$ for an arbitrary observable $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$. Recall that $\mathcal{M}^\lambda = \langle L^2(\mathbb{R}), \phi, \Phi^\lambda, \mathsf{Q}, f^\lambda\rangle$ where Φ^λ is defined through the specific dilation of E. We also need a modification of this scheme where instead of choosing the probe's position to be the pointer observable we choose its momentum P. The resulting scheme is denoted by $\mathcal{N}^\lambda = \langle L^2(\mathbb{R}), \phi, \Phi^\lambda, \mathsf{P}, f^\lambda\rangle$. Note that the observable measured with $\mathcal{N}^\lambda$ is the trivial one $X \mapsto \langle\phi|\mathsf{P}(\lambda X)\phi\rangle I$, but the state change caused by the measurement is nontrivial. Indeed, the instrument $\mathcal{J}^\lambda$ associated to this scheme is in the dual form

$$\mathcal{J}^\lambda(X)^*(B) = \int_X |\sqrt{\lambda}\,\widehat{\phi}(\lambda x)|^2 \, V_\psi^* L_x^*(B \otimes I) L_x V_\psi \, dx, \qquad B \in \mathcal{L}(\mathcal{H}),$$

where $L_x = e^{-i\lambda^2 x A}$ and the selfadjoint operator A is obtained from the dilation of E.

In order to obtain the weak value (6.2), we choose any observable $\mathsf{F} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ such that $B \in \mathcal{E}(\mathcal{H})$ is contained in the range of F, that is,

$B = \mathsf{F}(Y_0)$ for some $Y_0 \in \mathcal{B}(\mathbb{R})$. If we now perform a sequential measurement where the first measurement is $\mathcal{M}^\lambda$ and the second one is any measurement of F, the sequential observable M is, as usual, determined by the instrument $\mathcal{I}^\lambda$ associated to $\mathcal{M}^\lambda$ as $\mathsf{M}(X \times Y) = \mathcal{I}^\lambda(X)^*(\mathsf{F}(Y))$. If we then fix the set Y_0 of values of the subsequent measurement and normalize the probabilities, we end up with the conditional probability measure $\Lambda_1^\lambda : \mathcal{B}(\mathbb{R}) \to [0,1]$,

$$\Lambda_1^\lambda(X) = \frac{\langle \varphi | \mathsf{M}^\lambda(X \times Y_0)\varphi \rangle}{\langle \varphi | \mathsf{M}^\lambda(\mathbb{R} \times Y_0)\varphi \rangle} = \frac{\langle \varphi | \mathcal{I}^\lambda(X)^*(B)\varphi \rangle}{\langle \varphi | \mathcal{I}^\lambda(\mathbb{R})^*(B)\varphi \rangle}$$

from which we may calculate the first moment $\Lambda_1^\lambda[1] = \int x \, d\Lambda_1^\lambda(x)$. If the initial state of the system belongs to the square-integrability domain $\widetilde{\mathcal{D}}(x, \mathsf{E})$, and the initial probe state is chosen suitably, for instance, by taking $\phi = h_0$, then we get the limit

$$\lim_{\lambda \to 0} \Lambda_1^\lambda[1] = \mathrm{Re}\, \mathsf{E}_w(\varphi, B).$$

That is, the real part of the weak value is obtained as a limit of conditional averages.

The above procedure can be repeated also in the case that the first measurement is $\mathcal{N}^\lambda$. In this case we end up with the conditional probability measure $\Lambda_2^\lambda : \mathcal{B}(\mathbb{R}) \to [0,1]$ whose first moment we can again calculate. As before, under suitable regularity assumptions we get the limit

$$\lim_{\lambda \to 0} \Lambda_2^\lambda[1] = \mathrm{Im}\, \mathsf{E}_w(\varphi, B)$$

so that we obtain the imaginary part of the weak value. We combine these considerations into the following proposition. For more details and less strict assumptions, see Article IV.

Proposition 8. *Let $\mathcal{M}^\lambda$ and $\mathcal{N}^\lambda$ be as above with the initial probe state $\phi = h_0$. Then*

$$\mathsf{E}_w(\varphi, B) = \lim_{\lambda \to 0} \left(\Lambda_1^\lambda[1] + i\Lambda_2^\lambda[1] \right)$$

for all $\varphi \in \widetilde{\mathcal{D}}(x, \mathsf{E})$ such that $\langle \varphi | B\varphi \rangle \neq 0$.

Note that the entire weak value can also be obtained using a single measurement scheme with a suitably chosen phase space observable as the pointer. However, for the purpose of demonstrating the reconstruction method of [48], it is not needed. Details can be found in Article IV.

6.2.2 Direct measurement of the wavefunction

In [48] Lundeen *et al.* presented a method for determining the pointwise values of the wavefunction using weak values, and reported an experiment where they applied the method for determining the wavefunction of a photon. In order to avoid the problems concerning photon localization we consider instead a spin-$\frac{1}{2}$ particle. We restrict only to one spacial degree of freedom, say, the z-direction, so that the Hilbert space may be taken to be $L^2(\mathbb{R}) \otimes \mathbb{C}^2$. Here the spin degree of freedom will play the role of the probe.

In order to determine the values of the wavefunction, the position space is divided into disjoint intervals $(I_i)_{i\in\mathbb{N}}$ with the assumption that the intervals are of equal length and the center is labelled by x_i. For each $i \in \mathbb{N}$ we will then perform a measurement of the two-valued observable $1 \mapsto \mathsf{Q}(I_i), 0 \mapsto I - \mathsf{Q}(I_i) = \mathsf{Q}(\mathbb{R} \setminus I_i)$ thus scanning the whole position space. The spin state of the particle is prepared to the σ_z eigenstate $|+\rangle$ and the measurement interaction is given by the unitary operator $e^{-i\lambda \mathsf{Q}(I_i)\otimes\sigma_y}$. As the pointer observable we choose the spin in either x- or y-direction. This corresponds to the choice of the position or momentum pointer observables of the previous section. The pointer function is chosen to be $f(x) = \frac{x}{2\sin\lambda}$. The subsequently measured observable is chosen to be the momentum P of the particle where only the values which lie in the small interval $J_\epsilon = (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ are postselected. After calculating the conditional averages and the limit $\lambda \to 0$, we will arrive at the two values

$$\begin{aligned} \xi_i &= \mathrm{Re}\langle\varphi|\mathsf{P}(J_\epsilon)\mathsf{Q}(I_i)\varphi\rangle \\ \eta_i &= \mathrm{Im}\langle\varphi|\mathsf{P}(J_\epsilon)\mathsf{Q}(I_i)\varphi\rangle \end{aligned}$$

where ξ_i and η_i refer to the measurements of σ_x and σ_y, respectively, with the fixed position interval I_i. If the wavefunction is assumed to be sufficiently regular and the lengths of the intervals I_i and J_ϵ are sufficiently small, then we have

$$\langle\varphi|\mathsf{P}(J_\epsilon)\mathsf{Q}(I_i)\varphi\rangle \simeq \text{constant} \cdot \varphi(x_i).$$

More precisely, if we set $I_i = (x_i - \frac{\epsilon'}{2}, x_i + \frac{\epsilon'}{2})$ with $\epsilon' > 0$, then

$$\begin{aligned} \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0}\frac{1}{\epsilon'\epsilon}\langle\varphi|\mathsf{P}(J_\epsilon)\mathsf{Q}(I_i)\varphi\rangle &= \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0}\frac{1}{\epsilon'\epsilon}\int_{J_\epsilon}\overline{(F\varphi)(p)}(F\mathsf{Q}(I_i)\varphi)(p)\,dp \\ &= \frac{\overline{\widehat{\varphi}(0)}}{\sqrt{2\pi}}\lim_{\epsilon'\to 0}\frac{1}{\epsilon}\int_{I_i}\varphi(x)\,dx = \frac{\overline{\widehat{\varphi}(0)}}{\sqrt{2\pi}}\varphi(x_i) \end{aligned}$$

provided that the limits exist. This is the case, for instance, when $\varphi \in \mathcal{S}(\mathbb{R})$, the Schwartz space of C^∞-functions of rapid decrease.

The proposed method has obvious limitations as a method of state determination. First of all, due to the postselection of momentum values, only in the case that the wavefunction satisfies $\mathsf{P}(J_\epsilon)\varphi \neq 0$ can this method succeed. Indeed, if the momentum of the system is localized outside this small interval, then no information can be obtained from the measurement. Moreover, if it happens that $\mathsf{P}(J_\epsilon)\varphi = \varphi$, then it is known from basic Fourier theory that all the component vectors $\mathsf{Q}(I_i)\varphi$, $i \in \mathbb{N}$, are nonzero and one has to scan through all the intervals I_i in order to determine the state. However, the advantage of this method is that it provides the value of the wavefunction directly as the limit of these average, and thus we do avoid heavy processing of the measurement data.

We close this section by considering this reconstruction method in the finite dimensional case (see [48]). Let $\mathcal{H}_d$ be the d-dimensional Hilbert space ($d < \infty$) of a quantum system and let $\varphi \in \mathcal{H}_d$ be a unit vector. Consider a fixed orthonormal basis $\mathcal{B} = \{\xi_k \mid k = 0, \ldots, d-1\}$. We are interested in determining the Fourier components $\langle \xi_k | \varphi \rangle$ using weak values. To that end, let E^k denote the two-valued observable with $\mathsf{E}^k(1) = |\xi_k\rangle\langle\xi_k|$ and $\mathsf{E}^k(0) = I - |\xi_k\rangle\langle\xi_k|$ for all k. The first moment is then simply $\mathsf{E}^k[1] = |\xi_k\rangle\langle\xi_k|$. If we now assert $\mathsf{F}(Y) = |\psi\rangle\langle\psi|$ for some unit vector $\psi \in \mathcal{H}_d$ with $\langle\psi|\varphi\rangle \neq 0$, the weak value is given by

$$\mathsf{E}^k_w(\varphi, \psi) = \frac{\langle\psi|\xi_k\rangle}{\langle\psi|\varphi\rangle}\langle\xi_k|\varphi\rangle.$$

The crucial step now is to choose ψ in a suitable way. This can be done by using another basis $\mathcal{B}' = \{\eta_j \mid j = 0, \ldots, d-1\}$ such that $\mathcal{B}$ and $\mathcal{B}'$ are mutually unbiased, i.e., they satisfy $|\langle\eta_j|\xi_k\rangle|^2 = \frac{1}{d}$. An example of such a basis is given by

$$\eta_j = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} e^{-2\pi i jk/d}\xi_k$$

so that $\langle\eta_j|\xi_k\rangle = \frac{1}{\sqrt{d}}e^{2\pi ijk/d}$. Furthermore, the fact that $\mathcal{B}'$ is a basis guarantees that $\langle\eta_j|\varphi\rangle \neq 0$ for some j. Let η_j now be a fixed basis vector such that $\langle\eta_j|\varphi\rangle \neq 0$, and for all $k = 0, \ldots, d-1$ let $\theta(j,k) \in [0, 2\pi)$ be such that $\langle\eta_j|\xi_k\rangle = \frac{1}{\sqrt{d}}e^{i\theta(j,k)}$. Then the Fourier coefficients can be expressed in terms of the weak values as

$$\langle\xi_k|\varphi\rangle = \sqrt{d}\, e^{-i\theta(j,k)}\langle\eta_j|\varphi\rangle\mathsf{E}^k_w(\varphi, \eta_j).$$

In this way the pure state φ can be determined from the weak values.

Chapter 7

Conclusions

This thesis has dealt with the problem of quantum tomography, that is, reconstructing the unknown state of a quantum system, using measurements of phase space observables. The results are theoretical in nature though some are directly related to practical questions concerning actual measurements.

The condition of informational completeness for covariant phase space observables has been thoroughly analyzed and a necessary and sufficient condition has been proved, thus sharpening an earlier result on this matter. Reconstruction formulae for calculating the state from the measurement statistics have also been derived. Different measurement schemes for measuring phase space observables have been considered, with emphasis on the quantum optical realization via eight-port homodyne detection. A detailed analysis of the effect of detector efficiencies in homodyne detection has been performed, and as a result, it has been shown that the informational completeness of the measured observable does not depend on the efficiencies of the detectors.

In addition to full state reconstruction, the problem of reconstructing the position and momentum distributions from the marginal statistics of phase space observables has been considered. It has been shown that informational completeness is neither necessary nor sufficient for this procedure. Indeed, the relevant feature is that the marginal observables are informationally equivalent with position and momentum, and a counterexample has been presented showing that this may occur without informational completeness. Two methods for determining the position and momentum distributions have also been presented.

Finally, two alternative methods for quantum tomography have been reviewed, namely, the standard quadrature tomography, and the method of weak values. Some of their shortcomings when compared to phase space measurements have been discussed.

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