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Moment Operators of Observables  
in Quantum Mechanics,  
with Applications to Quantization and  
Homodyne Detection

by

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# Abstract

The questions studied in this thesis are centered around the moment operators of a quantum observable, the latter being represented by a normalized positive operator measure. The moment operators of an observable are physically relevant, in the sense that these operators give, as averages, the moments of the outcome statistics for the measurement of the observable. The main questions under consideration in this work arise from the fact that, unlike a projection valued observable of the von Neumann formulation, a general positive operator measure cannot be characterized by its first moment operator.

The possibility of characterizing certain observables by also involving higher moment operators is investigated and utilized in three different cases: a characterization of projection valued measures among all the observables is given, a quantization scheme for unbounded classical variables using translation covariant phase space operator measures is presented, and, finally, a mathematically rigorous description is obtained for the measurements of rotated quadratures and phase space observables via the high amplitude limit in the balanced homodyne and eight-port homodyne detectors, respectively.

In addition, the structure of the covariant phase space operator measures, which is essential for the above quantization, is analyzed in detail in the context of a (not necessarily unimodular) locally compact group as the phase space.

# List of articles

This thesis consists of an introductory review and the following seven articles:

- I** J. Kiukas, P. Lahti, K. Ylinen,  
Moment operators of the cartesian marginals of the phase space  
observables,  
*J. Math. Phys.* **46**, 042107 (2005).
- II** J. Kiukas, P. Lahti, K. Ylinen  
Normal covariant quantization maps,  
*J. Math. Anal. Appl.* **319**, 783-801 (2006).
- III** J. Kiukas,  
Covariant observables on a nonunimodular group,  
*J. Math. Anal. Appl.* **324**, 491-503 (2006).
- IV** J. Kiukas, P. Lahti, K. Ylinen,  
Phase space quantization and the operator moment problem,  
*J. Math. Phys.* **47**, 072104 (2006).
- V** J. Kiukas, P. Lahti,  
Quantization and noiseless measurements,  
*J. Phys. A: Math. Theor.* **40**, 2083-2091 (2007).
- VI** J. Kiukas, P. Lahti,  
On the moment limit of quantum observables, with an  
application to the balanced homodyne detection,  
*J. Mod. Opt.*, in press. (ArXiv: 0706.4436.)
- VII** J. Kiukas, P. Lahti,  
A note on the measurement of phase space observables with an  
eight-port homodyne detector,  
ArXiv: 0708.4094. Submitted to *J. Mod. Opt.*

# Chapter 1

## Introduction

It has long been known that the conventional von Neumann formulation of a quantum observable as a selfadjoint operator or, equivalently, a spectral measure, is insufficient in describing many natural properties of measurements, such as measurement inaccuracy. Therefore, the more general concept of a normalized positive operator measure, or semispectral measure, is currently widely used, for instance, in quantum optics, to represent the statistics of measurements.

An essential difference between a spectral measure and a general positive operator measure is the fact that the latter is not, in general, characterized by its first moment operator. In fact, even the entire moment operator sequence is in some cases insufficient to determine a positive operator measure. In particular, a general moment operator cannot be obtained as a power of the first one, as is the case for spectral measures. These complications make it interesting to investigate the relation between a positive operator measure and its moment operator sequence.

As is evident from the above comments, the basic mathematical object in this thesis is a normalized positive operator measure, which is understood as a representation of an observable of some quantum mechanical system. The moment operators of an observable are defined by means of operator integrals, and we pay careful attention to the domains of these (typically unbounded) operators. The basic problem is whether an observable is determined by (some of) its moment operators. However, we do not investigate this difficult mathematical question systematically in a general context, but instead concentrate on some relevant physically motivated special cases.

The introductory review of the thesis is organized around the above mentioned basic problem as follows. In Chapter 2, we give the definitions of the basic concepts, and consider the characterization of spectral measures among all observables in terms of the first and second moment operators



of the observable. Chapter 3 presents a quantization scheme, which maps certain unbounded classical variables to normalized positive operator measures by means of the correspondence of the classical moments of the variable and the moment operators of the resulting quantum observable. Various implementations of this scheme are studied, some of which can be realized by means of positive covariant phase space operator measures. Chapter 4 is devoted to the study of the structure of positive covariant phase space operator measures in the context where the phase space is a (not necessarily unimodular) locally compact group. In Chapter 5, we study the possibility of the convergence of a sequence of quantum observables, given that the moment operators of the observables converge. Then we apply the results to two concrete physically relevant applications, balanced homodyne and eight-port homodyne detectors, thereby giving a mathematically rigorous description of the "high amplitude limit" which leads to a measurement of a quadrature in the former case and a covariant phase space observable in the latter.

# Chapter 2

## Observables, moment operators and sharpness

### 2.1 An observable as a semispectral measure

In standard quantum mechanics, a physical system is described by a complex separable Hilbert space  $\mathcal{H}$ , the *states* of the system being associated with positive operators  $T : \mathcal{H} \rightarrow \mathcal{H}$  of unit trace. We let  $\mathcal{H}$  be fixed throughout the review. Let  $L(\mathcal{H})$  denote the set of bounded operators on  $\mathcal{H}$ , let  $\mathcal{T}(\mathcal{H})$  be the set of trace class operators, and let  $\mathcal{S}(\mathcal{H}) := \{T \in \mathcal{T}(\mathcal{H}) \mid T \geq 0, \text{Tr}[T] = 1\}$  be the set of states. The *pure* states correspond to projections onto one-dimensional subspaces of  $\mathcal{H}$ . If  $\varphi, \psi \in \mathcal{H}$ , we use the symbol  $|\varphi\rangle\langle\psi|$  to denote the operator  $\eta \mapsto \langle\psi|\eta\rangle\varphi$ . In particular, any pure state is of the form  $|\varphi\rangle\langle\varphi|$  for some unit vector  $\varphi \in \mathcal{H}$ .

As mentioned in the Introduction, we adopt the view in which an *observable* of the system is represented by a normalized positive operator measure  $E : \mathcal{A} \rightarrow L(\mathcal{H})$  (see the definition below) with  $\mathcal{A}$  a  $\sigma$ -algebra of subsets of a set  $\Omega$  containing the measurement outcomes for the observable. In this formulation, the outcome probability distribution  $E_T$  for a measurement of an observable  $E : \mathcal{A} \rightarrow L(\mathcal{H})$  in a state  $T \in \mathcal{S}(\mathcal{H})$  is recovered via  $E_T(X) = \text{Tr}[TE(X)]$ ,  $X \in \mathcal{A}$ . In the case where  $T = |\varphi\rangle\langle\varphi|$  for some unit vector  $\varphi \in \mathcal{H}$ , we write simply  $E_\varphi := E_{|\varphi\rangle\langle\varphi|}$ .

We are mainly interested in observables defined on the real line and on the phase space  $\mathbb{R}^2$ , in which cases  $(\Omega, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $(\Omega, \mathcal{A}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ , respectively, where  $\mathcal{B}(\mathbb{R}^n)$  is the Borel  $\sigma$ -algebra of  $\mathbb{R}^n$ ,  $n = 1, 2$ . However, we will also use the concept of normalized positive operator measure as a mathematical tool in a more general setting.

**Definition 1.** Let  $\Omega$  be a set,  $\mathcal{A}$  a  $\sigma$ -algebra of subsets of  $\Omega$ , and let  $E : \mathcal{A} \rightarrow L(\mathcal{H})$  be a map.

- (a) The map  $E$  is a *positive operator measure* if
  - (i)  $E(X) \geq O$  for all  $X \in \mathcal{A}$ ;
  - (ii)  $E(\emptyset) = 0$ , and  $E(\cup_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} E(X_n)$  for any sequence  $(X_n) \subset \mathcal{A}$  of mutually disjoint sets, with the series converging in the weak (or, equivalently, strong) operator topology.
- (b) The map  $E$  is *normalized*, if  $E(\Omega) = I$ . A normalized positive operator measure is also called *semispectral measure*.
- (c) If  $E$  is a semispectral measure, such that  $E(X)$  is a projection for all  $X \in \mathcal{A}$ , then  $E$  is a *spectral measure*.

If  $\Omega$  and  $\mathcal{A}$  are as in the preceding definition,  $\psi, \varphi \in \mathcal{H}$ , and  $E : \mathcal{A} \rightarrow L(\mathcal{H})$  is a positive operator measure, we let  $E_{\psi, \varphi}$  denote the complex measure  $X \mapsto \langle \psi | E(X) \varphi \rangle$ .

Spectral measures correspond to conventional von Neumann type observables; they are in one-to-one correspondence with the selfadjoint operators in  $\mathcal{H}$  according to the spectral theorem. For any selfadjoint operator  $A$  in  $\mathcal{H}$ , we let  $P^A$  denote the spectral measure of  $A$ . Spectral measures, or, equivalently, selfadjoint operators, are also called *sharp* observables. For any linear (not necessarily selfadjoint) operator  $A$  in  $\mathcal{H}$ , we let  $D(A)$  denote the domain of  $A$ , and for a densely defined symmetric operator  $A$ , the symbol  $\bar{A}$  denotes the least closed extension of  $A$ .

## 2.2 The operator integral and moment operators

We will consider the moment operators of observables only in the case  $\Omega = \mathbb{R}$ , with  $\mathcal{A} = \mathcal{B}(\mathbb{R})$ . For an observable  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ , the moment operators are defined as operator integrals of the functions  $x \mapsto x^k$  with respect to the positive operator measure  $E$ . To give a precise definition, we need the following general concept of an operator integral, introduced in [36].

Let  $\Omega$  and  $\mathcal{A}$  be as in Definition 1, let  $E : \mathcal{A} \rightarrow L(\mathcal{H})$  be a positive operator measure, and let  $f : \mathcal{A} \rightarrow \mathbb{C}$  be an  $\mathcal{A}$ -measurable function. It was proved in [36] that the set

$$D(f, E) := \{\varphi \in \mathcal{H} \mid f \text{ is } E_{\psi, \varphi}\text{-integrable for all } \psi \in \mathcal{H}\}$$

is a subspace of  $\mathcal{H}$ , and that there exists a unique linear operator  $L(f, E) : D(f, E) \rightarrow \mathcal{H}$ , such that

$$\langle \psi | L(f, E) \varphi \rangle = \int f dE_{\psi, \varphi}, \quad \psi \in \mathcal{H}, \varphi \in D(f, E).$$

It was also shown in that paper that the *square integrability domain*

$$\tilde{D}(f, E) := \left\{ \varphi \in \mathcal{H} \mid \int |f|^2 dE_{\varphi, \varphi} < \infty \right\}$$

is a subspace of  $D(f, E)$ , and that  $L(f, E)$  is a symmetric operator, provided that  $f$  is real valued. In general, the inclusion  $\tilde{D}(f, E) \subset D(f, E)$  may be proper; see [36] for a trivial example. We let  $\tilde{L}(f, E)$  denote the restriction of  $L(f, E)$  to the square integrability domain  $\tilde{D}(f, E)$ .

It should be noted that the above definition for the operator integral differs slightly from the usual convention. Namely, in the literature the square integrability domain is frequently used as the actual domain (see e.g. [58]). In the case where  $f$  is bounded, however,  $L(f, E) = \tilde{L}(f, E)$  is bounded and defined in all of  $\mathcal{H}$ , and is simply the usual weak operator integral (see e.g. [5, Theorem 9]).

Since the operator integral is an essential mathematical tool in our considerations, we have made some attempts to understand its structure. It is instructive to compare a general operator integral  $L(f, E)$  to the special case where  $E$  is a spectral measure. It was already proved in [36] that for a spectral measure  $E$ , the operator  $L(f, E)$  is exactly the operator given by the usual spectral integral; in particular,  $D(f, E) = \tilde{D}(f, E)$  in that case. In article IV, we investigated, among other things, the possibility of approximating  $f$  with the truncated functions  $\tilde{f}_n$ ,  $n \in \mathbb{N}$ , with  $\tilde{f}_n(x) = f(x)$  whenever  $|f(x)| \leq n$  and  $\tilde{f}_n(x) = 0$  otherwise. It is well known that in the case where  $E$  is a spectral measure, we have

$$D(f, E) = \left\{ \varphi \in \mathcal{H} \mid \lim_{n \rightarrow \infty} L(\tilde{f}_n, E) \varphi \text{ exists} \right\},$$

with  $L(f, E) \varphi = \lim_{n \rightarrow \infty} L(\tilde{f}_n, E) \varphi$  for any  $\varphi \in D(f, E)$ . For a general semispectral measure, this does not hold. (It is easy to give trivial examples, see article IV.) However, the following result holds; for proof<sup>1</sup>, see article IV, Proposition 1. (In the original formulation, we used the truncated versions ( $\tilde{f}_n$ ) of  $f$ , instead of a more general sequence ( $f_n$ ), but the proof shows

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<sup>1</sup>The proof is based on the fact that  $D(f, E)$  is the set of those vectors  $\varphi \in \mathcal{H}$ , for which  $f$  is integrable with respect to the vector valued measure  $X \mapsto E(X) \varphi$ . The vector  $L(f, E) \varphi$  is then the associated integral. (See [62].)

immediately that the result holds also in the following form.) Here  $\chi_Z$  denotes the indicator function of a set  $Z \subset \Omega$ .

**Proposition 1.** *Let  $\Omega$  and  $\mathcal{A}$  be as above, let  $E : \mathcal{A} \rightarrow L(\mathcal{H})$  be a positive operator measure, and let  $f : \Omega \rightarrow \mathbb{C}$  be a measurable function. Let  $(f_n)$  be any sequence of bounded measurable functions  $\Omega \rightarrow \mathbb{C}$  converging pointwise to  $f$ , with  $|f_n(\omega)| \leq |f(\omega)|$  for all  $\omega \in \Omega$ . Then*

$$D(f, E) = \left\{ \varphi \in \mathcal{H} \mid \lim_{n \rightarrow \infty} L(\chi_Z f_n, E)\varphi \text{ exists for each } Z \in \mathcal{A} \right\}, \quad (2.1)$$

and

$$L(f, E)\varphi = \lim_{n \rightarrow \infty} L(f_n, E)\varphi \quad \text{whenever } \varphi \in D(f, E). \quad (2.2)$$

Having defined the operator integral, we can define the moment operators of a real observable.

**Definition 2.** Let  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  be a semispectral measure, and let  $k \in \mathbb{N} (= \{0, 1, 2, \dots\})$ . The  $k$ th *moment operator*  $E[k]$  of the observable  $E$  is defined as  $E[k] := L(x^k, E)$ , where  $x^k$  is a shorthand for the real function  $x \mapsto x^k$ . In addition, we put  $\dot{E}[k] := \dot{L}(x^k, E)$ .

It follows from the above considerations that each moment operator is symmetric, that is,  $E[k] \subset E[k]^*$ . However, it need not be selfadjoint or even densely defined.

Now we have defined the set  $\{E[k] \mid k \in \mathbb{N}\}$  of moment operators for each semispectral measure  $E$ . As mentioned in the Introduction, the basic problem is whether this set uniquely determines  $E$ . It is known that for a compactly supported semispectral measure, the answer is always yes. (In our formulation, for instance, this can be seen immediately from Proposition 2 below.) However, for semispectral measures  $E$  with unbounded support, the moment operator collection  $\{E[k] \mid k \in \mathbb{N}\}$  need not determine  $E$  (see e.g. [27]).

Before proceeding further, we will carefully define some concepts related to the determination of a semispectral measure in terms of its moment operators.

**Definition 3.** Let  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  be a semispectral measure, and let  $\mathcal{I} \subset \mathbb{N}$ . We say that  $E$  is *determined by the moment operators*  $E[k]$ ,  $k \in \mathcal{I}$ , if  $E' = E$  for any semispectral measure  $E' : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  satisfying  $E[k] = E'[k]$  for all  $k \in \mathcal{I}$ . If  $E$  is determined by the moment operators  $E[k]$ ,  $k \in \mathbb{N}$ , we say that  $E$  is *determinate*.

The above definition is needed, since a phrase such as "determined by the moment operators" can be somewhat ambiguous. For instance, consider a semispectral measure  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ , which is supported in the two-point set  $\{0, 1\}$ . Then  $E$  is completely specified by the first moment operator  $E[1] = E(\{1\})$ , since for any  $X \in \mathcal{B}(\mathbb{R})$ , the operator  $E(X)$  is either  $O$ ,  $E(\{1\})$ ,  $I - E(\{1\})$  or  $I$  according as  $X$  contains neither 0 nor 1, 1 but not 0, 0 but not 1, or both 0 and 1, respectively. Nevertheless, if  $E$  is not projection valued, then  $E$  is *not* determined by the moment operator  $E[1]$  in the sense of Definition 3, since the spectral measure of the selfadjoint operator  $E(\{1\})$  has the same first moment as  $E$ .

In concrete applications, the problem of whether a given semispectral measure is determinate can be approached via probability measures and the classical moment problem. Hence, the following concept is needed.

**Definition 4.** Let  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  be a semispectral measure, and let  $\mathcal{D} \subset \mathcal{H}$  be a subspace. The semispectral measure  $E$  is  $\mathcal{D}$ -*determinate*, if the positive measure  $E_\varphi$  is determinate<sup>2</sup> for each  $\varphi \in \mathcal{D}$ .

It is easy to see that a  $\mathcal{D}$ -determinate semispectral measure is determinate, provided that  $\mathcal{D}$  is dense. (For a proof, see article VI, Remark 2.)

The concept of exponential boundedness has proved to be useful in establishing the determinate character of certain concrete observables, e.g. margins of phase space observables [27, 28]. Recall that a positive measure  $\mu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty)$  is *exponentially bounded* if

$$\int e^{a|x|} d\mu(x) < \infty$$

for some constant  $a > 0$ . An exponentially bounded measure is always determinate (see e.g. [7, Theorem 30.1, p. 406]).

**Definition 5.** For a semispectral measure  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ , we let  $\mathcal{E}(E)$  denote the set of those vectors  $\varphi \in \mathcal{H}$  for which the positive measure  $E_\varphi$  is exponentially bounded.

We proved in article VI (Lemma 1) that  $\mathcal{E}(E)$  is a subspace of  $\mathcal{H}$  for any semispectral measure  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ . Hence, the following holds.

**Proposition 2.** *For any semispectral measure  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ , the set  $\mathcal{E}(E)$  is a subspace of  $\mathcal{H}$ , and  $E$  is  $\mathcal{E}(E)$ -determinate. If  $\mathcal{E}(E)$  is dense, then  $E$  is determinate.*

---

<sup>2</sup>Recall that a positive measure  $\mu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty)$  is *determinate*, if it has finite moments of all orders, and  $\mu = \nu$  for any positive measure  $\nu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty)$  satisfying  $\int x^k d\mu(x) = \int x^k d\nu(x)$  for all  $k \in \mathbb{N}$ .

## 2.3 A characterization of sharp observables in terms of their moment operators

If  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  is a spectral measure, then a well-known consequence of the spectral theorem implies that  $E[1]$  is selfadjoint and  $E[2] = E[1]^2$ . If  $E$  is a general semispectral measure, this need not be the case, i.e. the operator  $N(E) := E[2] - E[1]^2$  may be nonzero.

Consider the case where  $E$  is an observable with  $E[1]$  selfadjoint. Then we may compare  $E$  with the sharp observable  $P^{E[1]}$ , the spectral measure of  $E[1]$ . Now if  $E$  is different from  $P^{E[1]}$ , or, equivalently, if  $E$  is not a spectral measure, then  $E$  is obviously not determined by the moment operator  $E[1]$ .

The difference between  $E$  and  $P^{E[1]}$  is reflected in the measurement statistics: For a pure state  $T = |\varphi\rangle\langle\varphi|$ ,  $\varphi \in D(E[1]) \cap D(E[2])$ ,  $\|\varphi\| = 1$ , the variance  $\text{Var}(E_T)$  of the measurement outcome probability distribution  $E_T$  can be written as

$$\text{Var}(E_T) = \int x^2 dE_\varphi - \left( \int x dE_\varphi \right)^2 = \langle \varphi | N(E) \varphi \rangle + \text{Var}(P_T^{E[1]}),$$

showing that the variance of  $E$  is larger than that of the sharp observable  $P^{E[1]}$  in any (suitable pure) state  $T$ . (This well-known fact was mentioned already in an old preprint by Ingarden [34, p. 87].) Accordingly, the operator  $N(E)$  is sometimes called *intrinsic noise* [12].

It is also thought that the property that  $E[1]$  is selfadjoint with  $E[2] = E[1]^2$ , characterizes the spectral measures among all semispectral measures. In the case where  $E$  is boundedly supported (so that all the operators  $E[k]$  are bounded), this is proved in [51, p. 466], but a proof for the unbounded case seemed difficult to find. Some parts of the proof can be extracted from [1, p. 130], but the result is not very detailed, and, moreover, our definition for the moment operators is slightly different from the usual one as far as the domains are concerned. In article IV (see Proposition 7 and Theorem 5), we generalized the method of [51, p. 466] to prove that the condition  $E[2] = E[1]^2$  is indeed equivalent to  $E$  being a spectral measure, provided that  $\tilde{E}[1]$  is selfadjoint. In view of our definition of the moment operator, this is still somewhat unsatisfactory, because  $\tilde{E}[1]$  is not the entire first moment operator. It appears, however, that this detail is easily corrected, and this is done in the following lemma.

**Lemma 1.** *Let  $(\Omega, \mathcal{A})$  be a measurable space,  $f : \Omega \rightarrow \mathbb{R}$  a measurable function,  $\mathcal{H}$  a Hilbert space and  $E : \mathcal{A} \rightarrow L(\mathcal{H})$  a positive operator measure, such that*

$$L(f^2, E) = L(f, E)^2. \tag{2.3}$$

Then the following are equivalent.

(i)  $L(f, E)$  is selfadjoint (on its entire domain  $D(f, E)$ );

(ii) The restriction  $\tilde{L}(f, E)$  is selfadjoint.

In that case,  $D(f, E) = \tilde{D}(f, E)$ .

*Proof.* Assume first that (ii) holds. Since  $L(f, E)$  is a symmetric extension of  $\tilde{L}(f, E)$ , it follows that  $L(f, E) = \tilde{L}(f, E)$  is selfadjoint, i.e. (i) holds.

Suppose then that (i) holds, and denote  $L(f, E) = A$ . Since  $A$  is selfadjoint, it follows from e.g. [26, p. 1245] that the dense subspace  $D(A^2) \subset D(A)$  is a core for  $A$ , i.e. the closure of the restriction  $A|_{D(A^2)}$  is  $A$  itself. Now, let  $\varphi \in D(f, E) = D(A)$ , and choose a sequence  $(\varphi_n)$  of vectors in  $D(A^2)$  converging to  $\varphi$  such that the sequence  $(A\varphi_n)$  converges to  $A\varphi$ . This is possible, because  $D(A^2)$  is a core for  $A$ . Since  $D(A^2) = D(L(f, E)^2) = D(L(f^2, E))$  by (2.3), we have

$$\int f^2 dE_{\varphi_n, \varphi_n} = \langle \varphi_n | L(f^2, E) \varphi_n \rangle = \langle \varphi_n | L(f, E)^2 \varphi_n \rangle = \|A\varphi_n\|^2, \quad n \in \mathbb{N}$$

where (2.3), and the fact that  $L(f, E)$  is symmetric, have been used. Since  $\lim_n A\varphi_n = A\varphi$ , we get

$$\lim_n \int f^2 dE_{\varphi_n, \varphi_n} = \|A\varphi\|^2. \quad (2.4)$$

Now  $|E_{\varphi_n, \varphi_n}(B) - E_{\varphi, \varphi}(B)| \leq \|\varphi_n - \varphi\|(\|\varphi_n\| + \|\varphi\|)\|E(\mathbb{R})\|$ , so that the sequence of positive measures  $(E_{\varphi_n, \varphi_n})_{n \in \mathbb{N}}$  converges to  $E_{\varphi, \varphi}$  uniformly, and hence in the total variation norm ([25, p. 97]). Since the limit (2.4) exists, it thus follows e.g. by [36, Lemma A.5] that  $f^2$  is  $E_{\varphi, \varphi}$ -integrable, i.e.  $\varphi \in \tilde{D}(f, E)$ . We have proved that  $D(f, E) \subset \tilde{D}(f, E)$ . Hence,  $\tilde{L}(f, E) = L(f, E)$  so  $\tilde{L}(f, E)$  is selfadjoint, i.e. (ii) holds. The proof is complete.  $\square$

**Proposition 3.** *Let  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  be an observable. Then  $E$  is sharp if and only if  $E[1]$  is selfadjoint and  $E[2] = E[1]^2$ .*

*Proof.* The "only if" part follows from the spectral theorem. Assuming that  $E[1]$  is selfadjoint, with  $E[2] = E[1]^2$ , it follows from Lemma 1 that the operator  $\tilde{E}[1]$  is also selfadjoint. It remains to apply Theorem 5 of article IV to complete the proof. (See the discussion above.)  $\square$

Proposition 3 implies, in particular, that sharp observables are determined by their first two moment operators:



**Proposition 4.** *If  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  is a sharp observable, then  $E$  is determined by the moment operators  $E[k]$ ,  $k \in \{1, 2\}$ . In particular,  $E$  is determinate.*

*Proof.* Let  $E' : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  be a semispectral measure, such that  $E[k] = E'[k]$ ,  $k = 1, 2$ . Since  $E$  is a spectral measure, the operator  $E'[1] = E[1]$  is selfadjoint, and  $E'[2] = E[2] = E[1]^2 = E'[1]^2$ . It now follows from the preceding proposition that  $E'$  is a spectral measure, and hence the uniqueness part of the spectral theorem gives  $E = E'$ .  $\square$

Note that a sharp observable is never determined by its first moment operator alone<sup>3</sup>. To demonstrate this with a trivial example, let  $P : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  be a spectral measure, and define an observable  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  by  $E(X) := \frac{1}{2}(P(X-1) + P(X+1))$ . Now  $\varphi \in D(\tilde{E}[1])$  if and only if  $\int \frac{1}{2}((x+1)^2 + (x-1)^2) dP_{\varphi, \varphi} < \infty$ , which happens exactly when  $\int x^2 dP_{\varphi, \varphi} < \infty$ , that is, when  $\varphi \in D(\tilde{P}[1])$ . In that case, clearly  $\tilde{E}[1]\varphi = \tilde{P}[1]\varphi$ , so that  $P[1] = \tilde{P}[1] = \tilde{E}[1] \subset E[1]$ . But  $P[1]$  is selfadjoint and  $E[1]$  symmetric, so that  $P[1] = E[1]$ . Hence, the observable  $E$  has the same first moment operator as the sharp observable  $P$ , and so the latter is not determined by the moment operator  $P[1]$ .

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<sup>3</sup>Recall that the spectral theorem says only that a sharp observable is determined, among all other sharp observables, by its selfadjoint first moment operator.

# Chapter 3

## Quantization as a moment problem

The original idea of *quantization* by Heisenberg and Schrödinger was to provide a method for constructing a quantum mechanical description of a physical system, assuming that its classical description is known. For a particle in one spatial dimension, the dynamical variables are real functions on the phase space  $\mathbb{R}^2$ , and these were quantized by replacing position and momentum coordinates by the multiplication  $\psi \mapsto (x \mapsto x\psi(x))$  and differentiation  $\psi \mapsto -i\frac{d\psi}{dx}$  operators, respectively. These operators provide a solution to the *canonical commutation relation*  $QP - PQ = iI$ , in resemblance to the classical Poisson bracket relation. Weyl transferred the canonical commutation relation into the *Weyl relation*  $e^{itP}e^{isQ} = e^{its}e^{isQ}e^{itP}$  [60], involving the one-parameter unitary groups generated by the selfadjoint operators  $Q$  and  $P$ , and von Neumann proved [44] that the solutions of the Weyl relation essentially determine the generators  $Q$  and  $P$  as the above multiplication and differentiation. Mackey replaced the Weyl relations by the condition  $U(t)QU(t)^* = Q + tI$  [41], which characterizes the position operator  $Q$  in terms of its *covariance* in translations  $(U(t)\psi)(x) = \psi(x+t)$ . He also used the same idea more generally [42], giving rise to a modern understanding where essential quantum observables, such as position, momentum, and angular momentum are defined in terms of their symmetry properties, independently of classical mechanics. In the modern operational approach to quantum mechanics, where quantum observables are represented by semispectral measures, this kind of definition has permitted various so called *unsharp* position and momentum observables, accounting for, e.g. measurement inaccuracies [14].

### 3.1 Quantization as a correspondence

Although no quantization is needed to define position and momentum observables, it is still interesting to obtain *correspondencies* between classical and quantum observables. In that sense, quantization is any procedure that associates a quantum observable to a (suitable) classical variable. Traditionally, the procedure is understood as a map  $f \mapsto \Gamma(f)$ , where  $\Gamma(f)$  is a (preferably) selfadjoint operator. In addition, the map is usually required to quantize correctly the position and momentum variables, and provide a consistent operator ordering for more complex variables. For instance, the classic Weyl quantization is like this.

The usual quantization maps, like Weyl quantization, can be implemented as an integration with respect to suitable operator valued densities. To overcome the mathematical difficulties arising from the integration of unbounded functions, the resulting operators are usually defined on some fixed dense subspace of  $L^2(\mathbb{R})$  consisting of "sufficiently smooth" vectors (see e.g. [23]). Another way to proceed is to ignore unbounded functions, and concentrate on a suitable  $C^*$ -algebra of classical variables, e.g. compactly supported  $C^\infty$ -functions. The reader may wish to consult [37] for an extensive treatment of the  $C^*$ -algebraic approach to quantization. The problem of that approach, however, is that the important canonical variables, being unbounded, seem to be excluded from the treatment.

### 3.2 Quantization in terms of moments

Having discussed the traditional quantizations, we immediately notice that those methods produce only operators as quantized observables. In the von Neumann formulation of quantum observables as spectral measures, this is reasonable, as the quantized observable  $\Gamma(f)$  corresponding to a classical variable  $f$  is obtained as the unique spectral measure having  $\Gamma(f)$  as its first moment operator. However, as we adhere to the view in which quantum observables are represented by semispectral measures, it does not seem to be sufficient to have quantizations that have nothing to say about the majority of observables.

Therefore, instead of having a correspondence  $f \mapsto \Gamma(f)$ , where  $\Gamma(f)$  is an operator, we want a correspondence  $f \mapsto E^f$ , with  $E^f$  a semispectral measure. In particular, such a correspondence could quantize position and momentum variables so that the corresponding observables would be *unsharp* position and momentum, rather than the *operators*  $Q$  and  $P$ , which are sharp observables. Following the basic problem of this thesis, we have formulated

our quantization in terms of moments: instead of quantizing only the first moment, as in traditional quantization, we quantize all the moments, using a fixed map  $\Gamma$ . In the following,  $\Omega$  is the set representing the phase space, with  $\mathcal{A}$  a  $\sigma$ -algebra of its subsets,  $\mathcal{M}(\Omega)$  is the set of classical variables, i.e. real valued  $\mathcal{A}$ -measurable functions on  $\Omega$ , and  $\mathcal{O}(\mathcal{H})$  is the set of all linear (not necessarily bounded) operators in the Hilbert space  $\mathcal{H}$ . We presented this quantization scheme in article IV, and gave some additional considerations and discussions in article V.

**Definition 6.** Let  $\mathcal{F} \subset \mathcal{M}(\Omega)$ , and let  $\Gamma : \mathcal{F} \rightarrow \mathcal{O}(\mathcal{H})$  be a map. A classical variable  $f \in \mathcal{M}(\Omega)$  is *quantizable by*  $\Gamma$ , if  $f^k \in \mathcal{F}$  for all  $k \in \mathbb{N}$ , and there exists a unique semispectral measure  $E^f : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  such that

$$\Gamma(f^k) = E^f[k], \quad k \in \mathbb{N}.$$

The observable  $E^f$ , and the family  $\{\Gamma(f^k) \mid k \in \mathbb{N}\}$  of operators are both referred to as a *quantization* of  $f$ .

By definition, any quantization  $E^f$  of a classical variable  $f$  is determined by the moment operators  $E^f[k]$ ,  $k \in \mathbb{N}$ .

We want to emphasize that Definition 6 should be seen as a natural generalization of the traditional scheme, where we have only the requirement  $\Gamma(f) = E^f[1]$ , which is solved by the unique spectral measure  $E^f : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  of  $\Gamma(f)$ , provided the latter operator is selfadjoint. Instead of one operator, we consider a *sequence* of operators, which, when taken together, determine a unique observable. It is important to note that the operator  $\Gamma(f)$  is, in general, *not* the quantization of  $f$  in the sense of Definition 6, even if it is selfadjoint.

As for the physical relevance of Definition 6, note that  $f^k(\omega)$  is the  $k$ th moment of the (point) probability distribution of the classical observable  $f$  in the pure state  $\omega \in \Omega$ , while  $\langle \varphi | E^f[k] | \varphi \rangle$  is the  $k$ th moment of the probability distribution of the quantum observable  $E^f$  in a pure state  $|\varphi\rangle\langle\varphi|$ . These correspond to each other via  $f^k \mapsto \Gamma(f^k)$ .

The moment approach to quantum observables was also used by Wódkiewicz et al. [29], who have the term "operational observables" for the moments of an observable associated with a quantum measurement. However, it seems that they have not at all addressed the essential question of whether the moment operators contain all information on the observable, i.e. whether the observable is determinate.

## 3.3 Examples of the quantization

It is obvious that a complete analysis of the quantizations permitted by Definition 6 is a difficult mathematical problem, and we have not attempted to do it. Instead, we have concentrated on interesting particular cases.

### 3.3.1 Quantization of questions

For any classical variable  $f \in \mathcal{M}(\Omega)$ , and a set  $X \in \mathcal{B}(\mathbb{R})$ , we get the variable  $\chi_X \circ f$ , which assumes only two values, corresponding to whether  $f(\omega) \in X$  or not. Such variables are traditionally called *questions* [41]. Obviously, the questions are exactly the indicator functions  $\chi_Z$ , where  $Z \in \mathcal{A}$ , so their quantization seems to be somewhat trivial. Nevertheless, it is interesting to compare it with the traditional quantization of questions as projections (going back to Mackey [41]).

Consider a general map  $\Gamma : \mathcal{F} \rightarrow \mathcal{O}(\mathcal{H})$ , where  $\mathcal{F} \subset \mathcal{M}(\Omega)$ , and let  $\chi_Z$ ,  $Z \in \mathcal{A}$  be the question to be quantized. Assuming that  $\chi_Z \in \mathcal{F}$ , put  $A = \Gamma(\chi_Z)$  ( $= \Gamma(\chi_Z^k)$ ), so that the moment problem arising from Definition 6 is simply

$$A = E^{X_Z}[k], \quad k \in \mathbb{N},$$

where the semispectral measure  $E^{X_Z} : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  is the solution to be found. The following simple result was proved in article V (Proposition 1).

**Proposition 5.** *Let  $A \in \mathcal{O}(\mathcal{H})$  be densely defined. Then the moment problem  $E[k] = A$  for all  $k \in \mathbb{N}$ , has a solution as a semispectral measure  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  if and only if  $A$  is bounded and  $0 \leq A \leq I$ . In that case, the only solution is the two-valued semispectral measure defined by  $\{0\} \mapsto I - A$  and  $\{1\} \mapsto A$ .*

Note that the condition that  $D(A)$  is dense is needed; it is easy to give an example of an operator  $A$  with a nondense domain and having multiple solutions to the above moment problem (see the remark following Proposition 1 of article V).

According to Proposition 5, a question  $\chi_Z$  with  $Z \in \mathcal{A}$  is quantizable by  $\Gamma$  provided that  $\Gamma : \mathcal{F} \rightarrow \mathcal{O}(\mathcal{H})$  is such that  $\chi_Z \in \mathcal{F}$ , and the operator  $\Gamma(\chi_Z)$  is bounded, with  $0 \leq \Gamma(\chi_Z) \leq I$ . In that case, the quantized observable is the two-valued observable given by  $\{0\} \mapsto I - \Gamma(\chi_Z)$ ,  $\{1\} \mapsto \Gamma(\chi_Z)$ . It should be emphasized that the bounded selfadjoint operator  $\Gamma(\chi_Z)$  is *not* the quantization of  $\chi_Z$ , except in the special case where its spectral measure satisfies the defining moment problem. According to Proposition 5, this occurs exactly when  $\Gamma(\chi_Z)$  is a projection. Clearly, this special case corresponds to the traditional quantization of questions as projections.

### 3.3.2 Quantization via operator integral

As mentioned before, a map  $\Gamma : \mathcal{F} \rightarrow \mathcal{O}(\mathcal{H})$ , with  $\mathcal{F} \subset \mathcal{M}(\Omega)$ , is traditionally realized as a suitable operator valued integral over the phase space  $\Omega$ . One way to do this is to use the operator integral defined in Chapter 2. Then we have to fix a positive operator measure  $W : \mathcal{A} \rightarrow L(\mathcal{H})$ , and define  $\Gamma_W : \mathcal{M}(\Omega) \rightarrow \mathcal{O}(\mathcal{H})$  by

$$\Gamma_W(f) := L(f, W), \quad f \in \mathcal{M}(\Omega).$$

Now the defining moment problem for the quantization is trivially solved by putting

$$E^f(X) := W(f^{-1}(X)), \quad X \in \mathcal{B}(\mathbb{R}).$$

(This follows from the definition of the operator integral by a change of variables.) Hence, we are left with the nontrivial question of whether this  $E^f$  is determinate. We have not investigated this question in general, and it will be postponed to the more concrete examples.

In view of the idea of our quantization, it is obvious that the generating operator measure<sup>1</sup>  $W$  should not be a spectral measure, since otherwise the quantized observables  $E^f$  would all be mutually commuting spectral measures. In fact, even a single projection in the range of  $W$  would cause the corresponding system of quantized observables to possess a nontrivial classical property (superselection rule), since each operator  $W(Z)$ ,  $Z \in \mathcal{A}$ , would then commute with the projection in question. (See e.g. [4, 35] for discussion on superselection rules of quantum systems.)

Next we will consider the characterizing properties of those maps  $\Gamma : \mathcal{M}(\Omega) \rightarrow \mathcal{O}(\mathcal{H})$  which are of the form  $\Gamma_W$  for some phase space operator measure  $W : \mathcal{A} \rightarrow L(\mathcal{H})$ . The first thing to note is that according to the discussion of the preceding section, any question variable  $\chi_Z$ ,  $Z \in \mathcal{A}$  is quantizable by  $\Gamma_W$ , provided that  $W(\Omega) \leq I$ . This is conveniently realized by requiring that  $W$  is normalized, and we will now do so.

There are different ways to characterize the restrictions of the operator integral maps  $\Gamma_W$  to the set of bounded functions (see e.g. [5, p. 23, p. 39]). Since we are also considering unbounded functions, we need something more.

The following proposition is a combination (and modification) of Proposition 2 of article II, Theorem 2 of article IV, and the above discussed results of article V concerning question variables. The simple proof, relying on Proposition 1, is given for the reader's convenience. The following definition is needed.

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<sup>1</sup>Note that in this context,  $W$  does not represent any observable, although it is a (possibly normalized) positive operator measure. The quantized observables  $E^f$  are semispectral measures on the real line.

**Definition 7.** Let  $\Gamma : \mathcal{M}(\Omega) \rightarrow \mathcal{O}(\mathcal{H})$  be a map.

- (a) The map  $\Gamma$  is *(real) linear*, if  $\Gamma(\alpha f + \beta g) \supset \alpha\Gamma(f) + \beta\Gamma(g)$  for all  $\alpha, \beta \in \mathbb{R}$ .
- (b) The map  $\Gamma$  is *quasicontinuous*, if for each positive function  $f \in \mathcal{M}(\Omega)$  and every increasing sequence  $(f_n) \subset \mathcal{M}(\Omega)$  of positive bounded functions converging pointwise to  $f$ , we have

$$D(\Gamma(f)) = \left\{ \varphi \in \bigcap_{n \in \mathbb{N}} D(\Gamma(f_n)) \mid \lim_{n \rightarrow \infty} \Gamma(\chi_Z f_n) \varphi \text{ exists for each } Z \in \mathcal{A} \right\},$$

and

$$\lim_{n \rightarrow \infty} \Gamma(f_n) \varphi = \Gamma(f) \varphi, \quad \varphi \in D(\Gamma(f)).$$

(We defined quasicontinuity in article II in a slightly different way. That definition was given by Berberian [5, Definition 7] in a similar context, but involving only bounded functions.) It is easy to see that any map  $\Gamma_W$  is linear and quasicontinuous. In fact, the linearity follows immediately from the definition of the operator integral (since  $|f + g| \leq |f| + |g|$ ), and quasicontinuity is a consequence of Proposition 1 and the fact that  $L(f, E) \in L(\mathcal{H})$  for any bounded function  $f \in \mathcal{M}(\Omega)$ .

**Proposition 6.** *A map  $\Gamma : \mathcal{M}(\Omega) \rightarrow L(\mathcal{H})$  coincides with  $\Gamma_W$  for a (clearly unique) normalized positive operator measure  $W$  if and only if the following conditions hold:*

- (i)  $\Gamma(\chi_Z) \in L(\mathcal{H})$ , with  $0 \leq \Gamma(\chi_Z) \leq I$  for all questions  $\chi_Z$ ,  $Z \in \mathcal{A}$ , with  $\Gamma(\chi_\Omega) = I$ ;
- (ii)  $\Gamma$  is linear;
- (iii)  $\Gamma$  is quasicontinuous;
- (iv)  $D(\Gamma(f)) = D(\Gamma(|f|))$  for all  $f \in \mathcal{M}(\Omega)$ .

*Proof.* We have already noted that the conditions (i)-(iii) hold for a map  $\Gamma_W$  with  $W$  a normalized positive operator measure. Condition (iv) follows directly from the definition of the operator integral.

Suppose then that the conditions hold for a map  $\Gamma : \mathcal{M}(\Omega) \rightarrow L(\mathcal{H})$ . Using (i), we can define  $W(Z) := \Gamma(\chi_Z) \in L(\mathcal{H})$ ,  $Z \in \mathcal{A}$ . Then  $W(\Omega) = I$  by (i). Now (ii) implies that  $W : \mathcal{A} \rightarrow L(\mathcal{H})$  so defined is additive, with

$W(\emptyset) = \Gamma(0) = 0$ . If  $(Z_n) \subset \mathcal{A}$  is a decreasing sequence of sets with empty intersection, then

$$\lim_{n \rightarrow \infty} W(Z_n)\varphi = \lim_{n \rightarrow \infty} (\Gamma(\chi_\Omega)\varphi - \Gamma(\chi_{\Omega \setminus Z_n})\varphi) = \varphi - \varphi = 0$$

by (ii) and (iii). Hence,  $W$  is a normalized positive operator measure. It is clear from (i) and (ii) that  $\Gamma(f) = L(f, W) \in L(\mathcal{H})$  for any  $\mathcal{A}$ -simple function  $f$ . Let  $f \in \mathcal{M}(\Omega)$  be positive, and let  $(f_n) \subset \mathcal{M}(\Omega)$  be an increasing sequence of positive  $\mathcal{A}$ -simple functions converging pointwise to  $f$ . Since  $\Gamma(f_n) = L(f_n, E) \in L(\mathcal{H})$  for all  $n$ , it follows from (iii) and Proposition 1 that

$$D(\Gamma(f)) = \left\{ \varphi \in \mathcal{H} \mid \lim_{n \rightarrow \infty} L(\chi_Z f_n, E)\varphi \text{ exists for each } Z \in \mathcal{A} \right\} = D(f, E),$$

and

$$\Gamma(f)\varphi = \lim_{n \rightarrow \infty} \Gamma(f_n)\varphi = \lim_{n \rightarrow \infty} L(f_n, E)\varphi = L(f, E)\varphi$$

for all  $\varphi \in D(\Gamma(f))$ . Hence,  $\Gamma(f) = L(f, E)$ . For a general  $f \in \mathcal{M}(\Omega)$ , we can write  $f = f_+ - f_-$ , where  $f_\pm$  are positive, and so  $\Gamma(f) \supset L(f_+, E) - L(f_-, E) \supset L(f, E)$  by (ii) and the definition of the operator integral (note that  $|f_\pm| \leq |f|$ ). But  $D(\Gamma(f)) = D(\Gamma(|f|)) = D(|f|, E) = D(f, E)$ , so  $\Gamma(f) = L(f, E)$  also in this case.  $\square$

### 3.3.3 Covariant quantization

Now we proceed to a more concrete case, assuming that  $\Omega = G$ , and  $\mathcal{A} = \mathcal{B}(G)$ , where  $G$  is a connected locally compact second countable topological group. We let  $\lambda$  denote a left Haar measure on  $G$ . In addition, we postulate more structure on the phase space  $G$ , assuming that the group structure of  $G$  can be transferred to the group  $\text{Aut}(\mathcal{S}(\mathcal{H}))$  of state automorphisms (i.e. bijective mappings  $\mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$  preserving convex combinations) via a group homomorphism  $\beta : G \rightarrow \text{Aut}(\mathcal{S}(\mathcal{H}))$  which is continuous (with respect to the projective topology induced by the functions  $s \mapsto \text{Tr}[s(T)A]$ ,  $T \in \mathcal{S}(\mathcal{H})$ ,  $0 \leq A \leq I$ ). According to the classic Wigner theorem<sup>2</sup>,  $\beta$  can be written as  $\beta(g)(T) = U(g)TU(g)^*$ , where  $U$  is a projective unitary representation (see the following definition (a)).

**Definition 8.** (a) A map  $U : G \rightarrow L(\mathcal{H})$  is a *projective unitary representation*, if

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<sup>2</sup>The original proof by Wigner can be found in [61, p. 251-254]; for a modern treatment, the reader may wish to consult [17]



- (i)  $U(g)$  is unitary for all  $g \in G$  and  $U(e) = I$ , where  $e$  is the identity element of  $G$ ;
- (ii) the map  $g \mapsto \langle \psi | U(g) \varphi \rangle$  is a Borel function for all  $\psi, \varphi \in \mathcal{H}$ ;
- (iii) there is a Borel map  $m : G \times G \rightarrow \mathbb{C}$ , such that  $U(gh) = m(g, h)U(g)U(h)$  for all  $g, h \in G$ .

The (clearly unique) map  $m$  associated with  $U$  is called the *multiplier* of  $U$ .

- (b) A projective unitary representation  $U : G \rightarrow L(\mathcal{H})$  is *square integrable*, if there exist nonzero vectors  $\psi, \varphi \in \mathcal{H}$ , such that

$$\int |\langle \psi | U(g) \varphi \rangle|^2 d\lambda(g) < \infty.$$

The irreducibility of a projective unitary representation is defined in the same way as in the case of ordinary unitary representations. For information on projective unitary representations, the Wigner theorem and related concepts, see e.g. [54, 2, 17].

The group  $G$  acts on the phase space  $G$  by means of translations<sup>3</sup>, i.e. for each  $h \in G$ , we have the left translation  $g \mapsto hg$ . Via the homomorphism  $\beta$ , translations act in the state space as transformations  $\beta(h)$ . Considering a map  $\Gamma : G \rightarrow \mathcal{O}(\mathcal{H})$  in this setting, we can require that it behave covariant under translations:

**Definition 9.** Let  $\mathcal{F} \subset \mathcal{M}(G)$ . A map  $\Gamma : \mathcal{F} \rightarrow \mathcal{O}(\mathcal{H})$  is *U-covariant*, if

$$U(g)^* \Gamma(f) U(g) = \Gamma(f(g \cdot)), \quad g \in G, f \in \mathcal{F}.$$

Notice that the above operator equality requires, in particular, that

$$U(g) D(\Gamma(f(g \cdot))) = D(\Gamma(f)).$$

Consider the operator integral map  $\Gamma_W : \mathcal{M}(G) \rightarrow \mathcal{O}(\mathcal{H})$ , where  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  is a normalized positive operator measure. If  $\Gamma_W$  is covariant in the sense of the above definition, then the operator measure  $W$  obviously satisfies the following covariance condition.

**Definition 10.** A positive operator measure  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  is *U-covariant*, if

$$U(g)^* W(Z) U(g) = W(g^{-1}Z), \quad g \in G, Z \in \mathcal{B}(G).$$

---

<sup>3</sup>Thus we consider only the special case where the group acts on itself; the more general setting consists of a (symmetry) group  $G$  acting on a transitive  $G$ -space  $\Omega$  (the phase space) (see e.g. Holevo [33, p. 62]).

Although the following result is rather simple, we did not state it in article IV. We give the proof here for the reader's convenience, since the associated equality of the domains should be carefully verified.

**Proposition 7.** *A semispectral measure  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  is  $U$ -covariant, if and only if the associated map  $\Gamma_W$  is  $U$ -covariant.*

*Proof.* We have already noted that the covariance of  $\Gamma_W$  implies the covariance of  $W$ . Assume now that  $W$  is  $U$ -covariant, let  $g \in G$ , and  $f \in \mathcal{M}(\Omega)$ . First, consider  $\psi, \varphi \in \mathcal{H}$ . By covariance, the complex measure  $W_{U(g)\psi, U(g)\varphi}$  coincides with  $Z \mapsto \langle \psi | W(g^{-1}Z) \varphi \rangle$ . Hence, upon changing the integration variable, one sees that  $f$  is  $W_{U(g)\psi, U(g)\varphi}$ -integrable, if and only if  $f(g \cdot)$  is  $W_{\psi, \varphi}$ -integrable.

Now  $U(g)\varphi \in D(f, W)$  if and only if  $f$  is  $W_{\psi, U(g)\varphi}$ -integrable for all  $\psi \in \mathcal{H}$ . Since  $U(g)$  is unitary, this happens exactly when  $f$  is  $W_{U(g)\psi, U(g)\varphi}$ -integrable for all  $\psi$ , i.e.  $f(g \cdot)$  is  $W_{\psi, \varphi}$ -integrable for all  $\psi$ . The latter is equivalent to  $\varphi \in D(f(g \cdot), W)$ . We have proved that  $U(g)D(f(g \cdot), W) = D(f, W)$ .

The equality

$$\langle \psi | U(g)^* L(f, W) U(g) \varphi \rangle = \langle \psi | L(f(g \cdot), W) \varphi \rangle, \quad \psi \in \mathcal{H}, \varphi \in U(g)^* D(f, W)$$

now follows by a change of variables. □

The above result implies, in particular, that  $D(f, W)$  is an invariant subspace for  $U$  if  $D(f(g \cdot), W) = D(f, W)$  for all  $g$ . Therefore, the following result is immediate<sup>4</sup>. A similar result concerning the square integrability domain also holds; see e.g. Proposition 4 (a) of article IV or [58].

**Proposition 8.** *If  $U$  is irreducible, and  $f \in \mathcal{M}(G)$  is such that  $D(f, W) = D(f(g \cdot), W)$  for all  $g \in G$ , then  $D(f, W)$  is either trivial or dense.*

Each function  $f \in \mathcal{M}(G)$  satisfying  $|f(gh)| \leq M_h |f(g)| + K_h$ , for all  $g, h \in G$ , where  $M_h > 0$  and  $K_h > 0$  depend only on  $h$ , clearly meets the domain condition of the preceding proposition. For example, in the case where  $G = \mathbb{R}^{2n}$ , all the polynomials and exponentials depending only on one coordinate are like this.

For the rest of this subsection, we will assume that the group  $G$  is unimodular, and the projective unitary representation  $U$  is irreducible and square integrable. In this case, the structure of covariant normalized positive operator measures is well known, and there are at least two different ways to obtain it; see [18] for the one and [31, 56] and articles II, III for the other.

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<sup>4</sup>We proved it somewhat awkwardly in article IV (Proposition 4 (b)), using the explicit form for the covariant positive operator measure.

We will discuss the characterization in some detail in Chapter 4, but at this point we will only need the result: Each  $U$ -covariant semispectral measure  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  is of the form  $W = W^T$  for a unique positive operator  $T$  of trace one, where  $W^T$  is defined by

$$W^T(Z) := \frac{1}{d} \int_Z U(g)TU(g)^* d\lambda(g), \quad Z \in \mathcal{B}(G),$$

where the integral is understood in the weak sense, and  $d$  is a constant depending only on  $U$ .

Consider now briefly the quantization of question variables via an operator integral map  $\Gamma_W$  where  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  is a  $U$ -covariant semispectral measure. The covariance brings in the following important property, proved in article IV. (Since the phase space semispectral measures are studied extensively, this result may well be known; however, we have been unable to find it anywhere.)

**Proposition 9.** *Assume that the projective representation  $U$  is continuous with respect to the strong operator topology. Let  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  be a  $U$ -covariant semispectral measure. Then  $\Gamma_W(\chi_Z)$  is never a nontrivial projection.*

Since the quantized observable  $E^f$  corresponding to a classical variable  $f$ , obtained via the map  $\Gamma_W$ , is of the form  $E^f(X) = W(f^{-1}(X))$ , the preceding proposition implies that  $E^f$  is never a spectral measure, if  $W$  is  $U$ -covariant.

To consider some concrete applications of the covariant quantization, we take the simple special case  $G = \mathbb{R}^2$ . Fixing an orthonormal basis  $\{|n\rangle \mid n \in \mathbb{N}\}$  of  $\mathcal{H}$ , we can define the canonical position and momentum operators  $Q = \frac{1}{\sqrt{2}}(a^* + a)$  and  $P = \frac{1}{\sqrt{2}}i(a^* - a)$ , where  $a$  and  $a^*$  are the lowering and rising operators<sup>5</sup> associated with the basis. In addition, we let  $N = a^*a$ . The Weyl operators are defined as  $W(q, p) = e^{\frac{1}{2}iqp}e^{-iqP}e^{ipQ}$ . We use the coordinate representation, i.e.  $\mathcal{H} \simeq L^2(\mathbb{R})$ , via  $|n\rangle \mapsto h_n$ , where  $h_n$  is the  $n$ th Hermite function. In the coordinate representation,  $Q$  and  $P$  act as multiplication  $\psi \mapsto (x \mapsto x\psi(x))$  and differentiation  $\psi \mapsto -i\frac{d\psi}{dx}$ , while the rule for the Weyl operator is  $(W(q, p)\varphi)(x) = e^{-i\frac{1}{2}qp}e^{ipx}\varphi(x - q)$ . The operators  $Q$  and  $P$  are connected by the unitary equivalence  $P = F^*QF$ , where  $F$  is the Fourier-Plancherel operator on  $L^2(\mathbb{R})$ .

The map  $\mathbb{R}^2 \ni (q, p) \mapsto W(q, p) \in L(\mathcal{H})$  is a strongly continuous irreducible square integrable projective unitary representation, so  $W$ -covariant

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<sup>5</sup>See [48, Chapter IV] or [8, Chapter 12] for details concerning these very thoroughly studied operators.

semispectral measures are given by

$$W^T(Z) = \frac{1}{2\pi} \int_Z W(q,p)TW(q,p)^* dqdp, \quad Z \in \mathcal{B}(\mathbb{R}^2), \quad (3.1)$$

where  $T$  is a positive operator of trace one<sup>6</sup>. This form allows some essential operator integrals to be calculated explicitly, as we will see in Subsection 3.3.5.

### 3.3.4 Weyl quantization on $\mathbb{R}^2$

The mathematical formulation of the Weyl quantization has been studied extensively (see e.g. [46, 20, 23] and the references therein). In view of our quantization, we consider the traditional Weyl quantization map as a means to implement a map  $\Gamma : \mathcal{F} \rightarrow \mathcal{O}(\mathcal{H})$ , where  $\mathcal{F} \subset \mathcal{M}(\mathbb{R}^2)$  is a suitable set. As is well known, the Weyl map is formally given by

$$f \mapsto \frac{1}{\pi} \int f(q,p)W(q,p)\mathcal{P}W(q,p)^* dqdp,$$

where  $\mathcal{P}$  is the parity operator. This can be made precise by letting  $\mathcal{F}$  be the set of those  $f \in \mathcal{M}(\mathbb{R}^2)$  for which there is a symmetric operator  $R^f : \mathcal{S} \rightarrow \mathcal{H}$ , defined on the Schwartz space  $\mathcal{S} \subset \mathcal{H}$ , with

$$\langle \varphi | R^f \varphi \rangle = \frac{1}{\pi} \int f(q,p) \langle \varphi | W(q,p)\mathcal{P}W(q,p)^* \varphi \rangle dqdp, \quad \varphi \in \mathcal{S}.$$

Now the Weyl map  $\Gamma^{Weyl}$  can be defined as the association  $\mathcal{F} \ni f \mapsto \Gamma^{Weyl}(f) := \overline{R^f} \in \mathcal{O}(\mathcal{H})$ . In article V, we compared the quantization via  $\Gamma^{Weyl}$  to the covariant quantization via operator integral. In particular, even the quantization of questions is problematic for the Weyl map, as there seems to be no guarantee that  $\chi_Z \in \mathcal{F}$  for all  $Z$ . Only some particular cases are known. For example, if the Lebesgue measure of  $Z$  is finite, then  $\chi_Z \in \mathcal{F}$  and  $\Gamma^{Weyl}(\chi_Z)$  is bounded by the classic result of Pool [46]. However, due to the nonpositivity of the Wigner functions  $(q,p) \mapsto \frac{1}{\pi} \langle \varphi | W(q,p)\mathcal{P}W(q,p)^* \varphi \rangle$ , the condition  $0 \leq \Gamma^{Weyl}(\chi_Z) \leq I$  need not hold, even in this simple case. Another example of sets  $Z \in \mathcal{F}$  are sectors in  $\mathbb{R}^2$ ; see [57].

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<sup>6</sup>In articles I, II, and IV, we defined the Weyl operators so that  $(W(q,p)\varphi)(x) = e^{i\frac{1}{2}qp} e^{ipx} \varphi(x+q)$ , and then used  $W(-q,p)$  in the definition of the phase space observable  $W^T$ , so the result is the same.

### 3.3.5 Quantization of position and momentum in $\mathbb{R}^2$

In article IV, we applied covariant quantization to the position and momentum variables  $(q, p) \mapsto q$  and  $(q, p) \mapsto p$ , which will be denoted by  $x$  and  $y$ , respectively. If  $T$  is a positive operator of trace one, the map  $\Gamma_{W^T}$  defines a covariant quantization, according to the moment problem in Definition 6. The observables corresponding to position and momentum are now  $X \mapsto W^T(x^{-1}(X)) = W^T(X \times \mathbb{R}) =: E^{T,x}(X)$  and  $Y \mapsto W^T(y^{-1}(Y)) = W^T(\mathbb{R} \times Y) =: E^{T,y}(Y)$ . Of course, this does not guarantee that these observables are quantizations of  $x$  and  $y$ ; by definition, the variable  $x$ , for instance, is quantizable by  $\Gamma_{W^T}$  provided that the semispectral measure  $E^{T,x}$  is determined by the moment operators  $E^{T,x}[k] = L(x^k, W^T)$ ,  $k \in \mathbb{N}$ , in the sense of Definition 3. We will discuss this problem shortly, but before that we give the explicit form for the moment operators in question, as obtained in articles I and IV. Related calculations can also be found in [58].

**Proposition 10.** *(a) Let  $k \in \mathbb{N}$ . Then  $\tilde{D}(x^k, W^T) \neq \{0\}$  if and only if  $Q^k \sqrt{T}$  is a Hilbert-Schmidt operator. In that case,  $D(x^k, W^T) = \tilde{D}(x^k, W^T) = D(Q^k)$ , and*

$$L(x^k, W^T) = \sum_{l=0}^k \left( \binom{k}{l} (-1)^{k-l} \text{Tr}[Q^{k-l}T] \right) Q^l.$$

*(b) Part (a) holds true also if "x" and "Q" are replaced with "y" and "P", respectively.*

One should note that the above proposition determines the moment operators only for the case where the square integrability domain is nontrivial. We do not know whether it is possible that e.g.  $\tilde{D}(x^k, W^T) = \{0\}$  with  $D(x^k, W^T) \neq \{0\}$ .

Now we proceed to consider the problem of whether the above moment operator sequences constitute quantizations of position and momentum, i.e. whether the observables  $E^{T,x}$  and  $E^{T,y}$  are determinate. Unfortunately, we have only been able to deal with the stronger condition of exponential boundedness.

In the case where  $T = |n\rangle\langle n|$ , the measures  $E_\varphi^{T,x}$  and  $E_\varphi^{T,y}$ , with  $\varphi \in C_0^\infty(\mathbb{R})$ , were proved in [28] to be exponentially bounded, so that  $E^{T,x}$  and  $E^{T,y}$  are determinate in that case. The following lemma characterizes the exponentially bounded cases completely.

**Lemma 2.** *Let  $T$  be a positive operator of trace one.*

(a)  $\mathcal{E}(E^{T,x}) \neq \{0\}$  if and only if  $e^{\frac{1}{2}a_0|Q|}\sqrt{T}$  is a Hilbert-Schmidt operator for some constant  $a_0 > 0$ . In that case,

$$\mathcal{E}(E^{T,x}) = \left\{ \varphi \in \mathcal{H} \mid \int e^{a|t|} |\varphi(t)|^2 dt < \infty \text{ for some } a \in (0, a_0] \right\}.$$

(b)  $\mathcal{E}(E^{T,y}) \neq \{0\}$  if and only if  $e^{\frac{1}{2}a_0|P|}\sqrt{T}$  is a Hilbert-Schmidt operator for some constant  $a_0 > 0$ . In that case,

$$\mathcal{E}(E^{T,y}) = \left\{ \varphi \in \mathcal{H} \mid \int e^{a|t|} |F\varphi(t)|^2 dt < \infty \text{ for some } a \in (0, a_0] \right\}.$$

*Proof.* To prove (a), we show first that for any  $a > 0$ ,

$$\int e^{a|t|} dE_{\varphi}^{T,x}(t) < \infty \text{ if and only if} \quad (3.2)$$

$$e^{\frac{1}{2}a|Q|}\sqrt{T} \text{ is Hilbert-Schmidt and } \int e^{a|t|} |\varphi(t)|^2 dt < \infty.$$

Write  $T$  in the form  $T = \sum_{n=1}^{\infty} w_n |\eta_n\rangle\langle\eta_n|$ , where  $\sum_n w_n = 1$ ,  $w_n \geq 0$  and  $(\eta_n)$  is an orthonormal sequence in  $\mathcal{H}$ . A calculation similar to one in the proof of Proposition 6 of article IV shows that

$$\int e^{a|t|} dE_{\varphi}^{T,x}(t) = \int \int \sum_n e^{a|t-q|} w_n |\eta_n(q)|^2 |\varphi(t)|^2 dt dq (\leq \infty)$$

for any  $\varphi \in \mathcal{H}$ . If this expression is finite, then Fubini's theorem implies that  $\int \sum_n e^{a|t-q|} w_n |\eta_n(q)|^2 dq < \infty$  for almost all  $t \in \mathbb{R}$ , and  $\int e^{a|t-q|} |\varphi(t)|^2 dt < \infty$  for almost all  $q \in \mathbb{R}$ . Since e.g.  $|t| \leq |q| + |t - q|$  for any  $q, t \in \mathbb{R}$ , this implies

$$\sum_{n=0}^{\infty} w_n \|e^{\frac{1}{2}a|Q|}\eta_n\|^2 = \int \sum_n e^{a|q|} w_n |\eta_n(q)|^2 dq < \infty, \quad (3.3)$$

and

$$\int e^{a|t|} |\varphi(t)|^2 dt < \infty. \quad (3.4)$$

(The monotone convergence theorem is used to obtain the equality in (3.3), which is always true, under the convention that  $\|e^{\frac{1}{2}a|Q|}\eta_n\| := \infty$  whenever  $\eta_n$  is not in the domain of  $e^{\frac{1}{2}a|Q|}$ .) On the other hand, if (3.3) and (3.4) hold,

then  $\int e^{a|t|} dE_\varphi^{T,x}(t) < \infty$ . But by Lemma 1 of article IV, the condition (3.3) holds exactly when  $e^{\frac{1}{2}a|Q|}\sqrt{T}$  is Hilbert-Schmidt. Hence, we have proved (3.2).

Since the subspace  $\{\varphi \in \mathcal{H} \mid \int e^{a|t|}|\varphi(t)|^2 dt < \infty\}$  is clearly non-empty for any  $a > 0$  (it contains, e.g. all compactly supported continuous functions), the result (3.2) immediately gives the first claim of part (a). Now assume that there is  $a_0 > 0$  such that  $e^{\frac{1}{2}a_0|Q|}\sqrt{T}$  is Hilbert-Schmidt. First, let  $\varphi \in \mathcal{H}$  be such that  $E_\varphi^{T,x}$  is exponentially bounded. This means that  $\int e^{a|t|} dE_\varphi^{T,x}(t) < \infty$  for some  $a > 0$ , and so (3.2) gives that  $\int e^{a|t|}|\varphi(t)|^2 dt < \infty$ . If  $a \geq a_0$ , then  $e^{a_0|t|} \leq e^{a|t|}$  for all  $t$ , so  $\int e^{a_0|t|}|\varphi(t)|^2 dt < \infty$ . Hence, in any case,  $\int e^{a|t|}|\varphi(t)|^2 dt < \infty$  for some  $a \leq a_0$ . Conversely, let  $\varphi \in \mathcal{H}$  be such that  $\int e^{a|t|}|\varphi(t)|^2 dt < \infty$  for some  $a \leq a_0$ . Since  $e^{\frac{1}{2}a_0|Q|}\sqrt{T}$  was assumed to be Hilbert-Schmidt, and now  $a \leq a_0$ , we have  $\int \sum_n e^{a|q|} w_n |\eta_n(q)|^2 dq < \infty$ , so that also  $e^{\frac{1}{2}a|Q|}\sqrt{T}$  is Hilbert-Schmidt. But now (3.2) implies that  $\int e^{a|t|} dE_\varphi^{T,x}(t) < \infty$ , so that  $E_\varphi^{T,x}$  is exponentially bounded. Now (a) is proved.

Part (b) follows in a similar way, since  $P = F^*QF$ , and

$$\int e^{a|t|} dE_\varphi^{T,y}(t) = \int \int \sum_n e^{a|t-q|} w_n |F\eta_n(q)|^2 |F\varphi(t)|^2 dt dq (\leq \infty)$$

for any  $\varphi \in \mathcal{H}$ . (See the proof of Proposition 6 of article IV.)  $\square$

One should note that in the case where  $T = |\eta\rangle\langle\eta|$  for some unit vector  $\eta \in \mathcal{H}$ , the above Hilbert-Schmidt condition reduces to the requirement that  $\int e^{a_0|t|}|\eta(t)|^2 dt < \infty$  for some constant  $a_0 > 0$ .

The following proposition presents a class of covariant quantizations which permit the quantization of position and momentum variables.

**Proposition 11.** *Let  $T$  be a positive operator of trace one.*

(a) *If  $e^{a_0|Q|}\sqrt{T}$  is a Hilbert-Schmidt operator for some constant  $a_0 > 0$ , then the position variable  $x$  is quantizable by  $\Gamma_{WT}$ .*

(b) *If  $e^{a_0|P|}\sqrt{T}$  is a Hilbert-Schmidt operator for some constant  $a_0 > 0$ , then the momentum variable  $y$  is quantizable by  $\Gamma_{WT}$ .*

*Proof.* By the preceding lemma,

$$\mathcal{E}(E^{T,x}) = \left\{ \varphi \in \mathcal{H} \mid \int e^{a|t|}|\varphi(t)|^2 dt < \infty \text{ for some } a \in (0, 2a_0] \right\}$$

with the assumptions of (a) and

$$\mathcal{E}(E^{T,y}) = \left\{ \varphi \in \mathcal{H} \mid \int e^{a|t|}|F\varphi(t)|^2 dt < \infty \text{ for some } a \in (0, 2a_0] \right\}$$

with the assumptions of (b). The subspaces in the right hand sides are both dense, as the first contains all the compactly supported functions, and the second is the image of the first in the inverse Fourier-Plancherel transform  $F^{-1}$ . Hence, an application of Proposition 2 completes the proof.  $\square$

Note that the requirement in part (a) that  $e^{a_0|Q|}\sqrt{T}$  be Hilbert-Schmidt implies, in particular, that each operator  $Q^k\sqrt{T}$  is a Hilbert-Schmidt operator for all  $k \in \mathbb{N}$  (use the integral formula in the above proof, and the inequality  $x^{2k} \leq \frac{(2k)!}{(2a_0)^k} e^{2a_0|x|}$ ). Similarly, the requirement in part (b) implies that  $P^k\sqrt{T}$  is Hilbert-Schmidt for all  $k \in \mathbb{N}$ .

Hence, under the assumptions of Proposition 11, which thus ensure the quantizability of  $x$  and  $y$ , the moment operators that constitute the quantizations are all selfadjoint, and can be written as in Proposition 10. The corresponding observables  $E^{T,x}$  and  $E^{T,y}$  are known to possess the symmetry properties which make them *unsharp* position and momentum observables.

As a final remark concerning the quantization of position and momentum, consider the Weyl map  $\Gamma^{Weyl}$ . It is known that  $\Gamma^{Weyl}(x^k) = Q^k$  and  $\Gamma^{Weyl}(y^k) = P^k$  for all  $k \in \mathbb{N}$ . [23, Proposition 8.31]. The moment problems corresponding to the quantizations of position and momentum are now solved by the spectral measures  $E^Q$  and  $E^P$ , respectively. Since any spectral measure is determinate by Proposition 4, position and momentum variables are quantizable by the map  $\Gamma^{Weyl}$ , and the resulting observables are just the standard position and momentum observables. Hence, our quantization scheme contains the traditional Weyl quantizations for position and momentum variables as a special case.



# Chapter 4

## On the structure of covariant phase space observables

We have already utilized the general form of covariant phase space semispectral measures as a mathematical tool in our quantization scheme. However, these semispectral measures have an essential role as phase space *observables* in quantum mechanics; in particular, they constitute important examples of approximate joint measurements of position and momentum observables [21, 32, 11, 53]. The importance of these examples has been emphasized by the recent result of Werner [59], which says that for *any* approximate joint observable of position and momentum of a quantum object there is a covariant phase space observable which approximates them to a degree at least as good as that of the original observable. (For a precise formulation, see the original paper [59], as well as subsequent developments [15, 12, 13].)

In this chapter, we concentrate on the structure of covariant phase space observables in the more general context where the phase space is a locally compact second countable topological group  $G$ , and the covariance is with respect to a square integrable projective unitary representation. There are at least two different ways to obtain the result (Theorem 1 below). A direct method was outlined by Holevo [31] for a unimodular group  $G$ , and further elaborated by Werner [56] in the case where  $G = \mathbb{R}^{2n}$ . Werner's proof relies on the fact that the Banach space of trace class operators has the Radon-Nikodým property. The other approach [18] (Cassinelli et al.), is group theoretical in flavour, being based on Mackey's imprimitivity theorem. Cassinelli's proof also covers nonunimodular cases.

In article II, we generalized Werner's proof to unimodular groups  $G$ , thereby filling in the details missing from Holevo's original sketch. In article III, we further generalized the result to not necessarily unimodular groups, achieving Cassinelli's result by using an entirely different method.

We use same notations as in Chapter 3, where the characterization was already used in the unimodular case. Let  $G$  be a locally compact second countable topological group, and fix  $\lambda$  to be a left Haar measure on  $G$ . Let  $\tilde{\lambda}$  denote the corresponding right Haar measure  $Z \mapsto \lambda(Z^{-1})$ , and let  $\Delta : G \rightarrow \mathbb{R}$  be the modular function. As in Chapter 3, we assume that there is an irreducible square integrable projective unitary representation  $U : G \rightarrow L(\mathcal{H})$ , which will remain fixed throughout the chapter. The projective representation  $U$  determines a densely defined, injective, positive selfadjoint operator  $K$ , called the *formal degree* of  $U$ , such that  $U(g)K = \Delta(g)^{-1}KU(g)$  for all  $g \in G$ , and

$$\int |\langle \psi | U(g)\varphi \rangle|^2 d\lambda(g) = \|\psi\|^2 \|K^{-\frac{1}{2}}\varphi\|^2, \quad \varphi, \psi \in \mathcal{H},$$

with the understanding that  $\|K^{-\frac{1}{2}}\varphi\| = \infty$  whenever  $\varphi \notin D(K^{-\frac{1}{2}})$ . We let  $C_U$  denote the square root of the formal degree, i.e.  $C_U := K^{\frac{1}{2}}$ . Now also  $C_U$  is densely defined, selfadjoint and injective. The operator  $C_U$  is bounded if and only if  $G$  is unimodular, and in that case, it is a multiple of the identity. (See [24, Theorem 3] and e.g. the discussion preceding Lemma 3 of article III.) The characterization in question is the following:

**Theorem 1.** (a) *For each positive operator  $T$  of trace one, there is a (clearly unique)  $U$ -covariant semispectral measure  $W^T : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ , such that*

$$\langle \varphi | W^T(Z)\psi \rangle = \int_Z \langle C_U\varphi | U(g)TU(g)^*C_U\psi \rangle d\tilde{\lambda}(g), \quad Z \in \mathcal{B}(G).$$

for all  $\varphi, \psi \in D(C_U)$ .

(b) *Each  $U$ -covariant semispectral measure  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  is of the form  $W = W^T$  for a unique positive operator  $T$  of trace one.*

In the case where  $G$  is unimodular, the left Haar measure is the same as the right one, and the operator  $C_U$  is a multiple of the identity. The above theorem then gives the canonical form

$$W^T(Z) = d^{-1} \int_Z U(g)TU(g)^* d\lambda(g), \quad Z \in \mathcal{B}(G),$$

of covariant semispectral measures we already presented in Chapter 3. The integral actually exists in the ultraweak sense in this case.

In article II, we generalized Werner's approach [56], and treated the unimodular case by characterizing the weak-\*continuous covariant maps

$\Gamma : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$ , with  $\Gamma(1) = I$ .<sup>1</sup> Werner's idea was to use the fact that the Banach space  $\mathcal{T}(\mathcal{H})$  of trace class operators on a separable Hilbert space has the *Radon-Nikodým property* [22, p. 61, 79]. This property is the following: for any finite measure space  $(\Omega, \mathcal{A}, \nu)$  and a  $\nu$ -continuous vector measure  $\mu : \mathcal{A} \rightarrow \mathcal{T}(\mathcal{H})$  of bounded variation, there is a  $\nu$ -(Bochner-)integrable function  $f_\mu : \Omega \rightarrow \mathcal{T}(\mathcal{H})$ , such that  $\mu(X) = \int_X f_\mu d\nu$  for all  $X \in \mathcal{A}$ . The function  $f_\mu$  is  $\nu$ -essentially unique. The above concepts and results concerning the theory of vector measures can be found in the book of Diestel and Uhl [22, pp. 1-2, 11, 45-47].

In the proof of Theorem 2 of article III, we used semispectral measures directly to obtain the characterization also in the case of nonunimodular groups. The proof is still based on the Radon-Nikodým property of  $\mathcal{T}(\mathcal{H})$ . We will outline this proof in the following section. In section 4.2, we present a slightly different proof, which shows that the problem can be understood as a special case of a more general setting, already considered in article II but applied there only for the unimodular case.

As the proof of part (a) of the theorem is straightforward (see Theorem 1 of article III), we will concentrate on part (b).

Since the treatment involves integration of Banach space valued functions, one must pay careful attention to the correct notion of measurability in this context: If  $B$  is a Banach space, a function  $f : G \rightarrow B$  is  $\lambda$ -measurable, if for each  $Z \in \mathcal{B}(G)$  of finite  $\lambda$ -measure there is a sequence of  $\lambda$ -simple functions converging to  $\chi_Z f$  in  $\lambda$ -measure (or, equivalently,  $\lambda$ -almost everywhere) [25, p. 106,150]. The measurability with respect to the right Haar measure  $\tilde{\lambda}$  is, of course, defined similarly. Since  $G$  is  $\sigma$ -compact, and both  $\lambda$  and  $\tilde{\lambda}$  are finite on compact sets, it follows that  $\lambda$ -measurability and  $\tilde{\lambda}$ -measurability are equivalent.

If  $B$  is separable (in particular, if  $B$  is a scalar field), then  $\lambda$ -measurability is equivalent to the measurability with respect to the Lebesgue extension of the  $\sigma$ -algebra  $\mathcal{B}(G)$  associated with  $\lambda$  [25, p. 148]. In the following section, we will use the Banach space  $\mathcal{T}(\mathcal{H})$ , which is indeed separable<sup>2</sup>. (This is probably well known, but we gave a simple proof in article II, Lemma 5.)

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<sup>1</sup>Since each such map defines a covariant semispectral measure via  $\mathcal{B}(G) \ni Z \mapsto \Gamma(\chi_Z) \in L(\mathcal{H})$ , and each covariant semispectral measure  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  defines a weak- $*$ -continuous covariant map  $L^\infty(G, \lambda) \ni f \mapsto L(f, W) \in L(\mathcal{H})$ , this approach also gives the characterization of covariant observables.

<sup>2</sup>This requires that  $\mathcal{H}$  be separable, which we assumed in the first chapter.

## 4.1 A proof of the characterization

First we need the following lemma; for its proof, see Lemma 4 of article III. In the unimodular case, this result was given in the proof of Lemma 6(b) of II, which is based on [31].

**Lemma 3.** *Let  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  be a  $U$ -covariant semispectral measure. Then*

$$\tilde{\lambda}(Z) = \|C_U^{-1}W(Z)^{\frac{1}{2}}\|_{HS}^2, \quad Z \in \mathcal{B}(G),$$

where  $\|\cdot\|_{HS}$  denotes the Hilbert-Schmidt norm, with the notation  $\|S\|_{HS} := \infty$  whenever the (not necessarily bounded) operator  $S$  is not Hilbert-Schmidt. In particular, if  $G$  is unimodular, then

$$\lambda(Z) = \text{Tr}[W(Z)] d, \quad Z \in \mathcal{B}(G),$$

where the constant  $d$  is given by  $C_U = d^{-\frac{1}{2}}I$ .

To begin a proof of Theorem 1 (b), assume that  $W : \mathcal{B}(G) \rightarrow L(\mathcal{H})$  is a  $U$ -covariant semispectral measure. Take  $Z \in \mathcal{B}(G)$  with  $\tilde{\lambda}(Z) < \infty$ . Define  $A(Z) = (C_U^{-1}W(Z)^{\frac{1}{2}})(C_U^{-1}W(Z)^{\frac{1}{2}})^*$ . By the above lemma, this operator is everywhere defined, and in the trace class, with  $\text{Tr}[A(Z)] = \tilde{\lambda}(Z)$ . In addition, one easily sees that  $A(Z)$  coincides with the densely defined operator  $C_U^{-1}W(Z)C_U^{-1}$  on the subspace  $D(C_U^{-1})$ . Using this fact, a straightforward calculation confirms that the covariance of  $W$  is reflected by

$$A(hZ) = \Delta(h)^{-1}U(h)A(Z)U(h)^*, \quad h \in G, Z \in \mathcal{B}(G), \tilde{\lambda}(Z) < \infty. \quad (4.1)$$

Since  $\tilde{\lambda}$  is  $\sigma$ -finite, we can write  $G = \bigcup_{n=0}^{\infty} K_n$ , with  $K_n \in \mathcal{B}(G)$  and  $\tilde{\lambda}(K_n) < \infty$ .

Then we use an adaptation of an idea of Werner [56] to get a representation

$$A(Z) = \int_Z v d\tilde{\lambda}, \quad Z \in \mathcal{B}(G), \tilde{\lambda}(Z) < \infty, \quad (4.2)$$

where  $v : G \rightarrow \mathcal{T}(\mathcal{H})$  is a  $\tilde{\lambda}$ -measurable function and the integral is a  $\mathcal{T}(\mathcal{H})$ -valued Bochner integral. Namely, the relation  $\text{Tr}[A(Z)] = \tilde{\lambda}(Z)$  implies that for any  $n \in \mathbb{N}$ , the map  $\mathcal{B}(K_n) \ni Z \mapsto A(Z) \in \mathcal{T}(\mathcal{H})$  is a  $\tilde{\lambda}$ -continuous vector measure of bounded variation, so by the the Radon-Nikodým property of  $\mathcal{T}(\mathcal{H})$ , there exist ( $\tilde{\lambda}$ -essentially unique)  $\tilde{\lambda}$ -measurable functions  $v_n : G \rightarrow \mathcal{T}(\mathcal{H})$ , with  $A(Z) = \int_Z v_n d\tilde{\lambda}$  for any  $Z \in \mathcal{B}(K_n)$ . Then  $v := \sum_n v_n$  (pointwise) is  $\tilde{\lambda}$ -measurable and satisfies (4.2).

It follows from the construction that  $v(g) \geq 0$  and  $\|v(g)\|_{\text{tr}} = 1$  for  $\lambda$ -almost all  $g \in G$ . In particular,  $v$  is  $\lambda$ -essentially bounded. Now define

$v_0 : G \rightarrow \mathcal{T}(\mathcal{H})$  by  $v_0(g) = U(g^{-1})v(g)U(g^{-1})^*$ . Lemma 2 of article III, the results (4.1) and (4.2), together with the fact that  $g \mapsto U(g)$  is a projective representation, imply that  $v_0$  is  $\lambda$ -measurable,  $\lambda$ -essentially bounded, and satisfies

$$\text{for each } h \in G, v_0(g) = v_0(hg) \text{ for } \lambda\text{-almost all } g \in G.$$

The next step is to show that  $v_0$  is constant, i.e. there exists a  $T \in \mathcal{T}(\mathcal{H})$ , such that  $v_0(g) = T$  for  $\lambda$ -almost all  $g \in G$ . This is not immediately obvious from the above result; although  $v_0(g) = v_0(hg)$  for  $g$  in a set  $S_h \in \mathcal{B}(G)$  whose complement is a null set, the set  $S_h$  could be different for each  $h$ . We proved the following result in article II as Lemma 4.

**Lemma 4.** *Let  $B$  be a Banach space, and  $f : G \rightarrow B$  a  $\lambda$ -measurable  $\lambda$ -essentially bounded function such that for each  $h \in G$ , the function  $f(h \cdot)$  coincides with  $f$   $\lambda$ -almost everywhere. Then there is an  $s \in B$ , such that  $f(g) = s$  for  $\lambda$ -almost all  $g \in G$ .*

This lemma establishes that  $v_0(g) = T$  for  $\lambda$ -almost all  $g \in G$ , for a fixed  $T \in \mathcal{T}(\mathcal{H})$ . Now it follows that  $v(g) = U(g)TU(g)^*$  for  $\lambda$ -almost all  $g \in G$ , so that

$$A(Z) = \int_Z U(g)TU(g)^* d\tilde{\lambda}(g), \quad Z \in \mathcal{B}(G), \tilde{\lambda}(Z) < \infty. \quad (4.3)$$

Notice that the integral is a Bochner integral with respect to the trace norm. It is also worth mentioning that the cumbersome unbounded operator  $C_U$  is not explicitly present in this formula. It enters only when we return to the original observable  $W$  by using the above mentioned fact that  $A(Z)\varphi = C_U^{-1}W(Z)C_U^{-1}\varphi$  for any  $\varphi \in D(C_U^{-1})$ . To derive the desired formula for  $W(Z)$ , we also have to use the fact that

$$\int \langle \varphi | U(g)TU(g)^* \varphi \rangle d\tilde{\lambda}(g) = \text{Tr}[T] \|C_U^{-1}\varphi\|^2 < \infty$$

for  $\varphi \in D(C_U^{-1})$  (see Lemma 3 (a) of article III), which implies that the positive function  $g \mapsto \langle \varphi | U(g)SU(g)^* \varphi \rangle$  is integrable over the group  $G$ , even though the trace class valued Bochner integral in (4.3) exists only when  $Z$  is of finite  $\tilde{\lambda}$ -measure.

## 4.2 More general covariant maps

In article II, we investigated the problem of covariant maps also in a slightly more general setting which need not involve a Hilbert space, and proved the

following result (Proposition 1 in that paper). The essential part of the proof is again Lemma 4 above. For a Banach space  $B$ , let  $\text{Aut}(B)$  denote the group of linear homeomorphisms from  $B$  onto itself.

**Proposition 12.** *Let  $B$  be a Banach space having the Radon-Nikodým property, and assume that there is a homomorphism  $\alpha : G \rightarrow \text{Aut}(B)$ , such that*

$$(i) \sup_{g \in G} \|\alpha(g)\| < \infty;$$

(ii) *for all  $w \in B$ , the map  $g \mapsto \alpha(g^{-1})(w)$  is  $\lambda$ -measurable.*

*If  $\Gamma : L^1(G, \lambda) \rightarrow B$  is a continuous linear map satisfying*

$$\alpha(g)(\Gamma(f)) = \Gamma(f(g^{-1}\cdot)), \quad f \in L^1(G, \lambda), g \in G,$$

*then there is a unique vector  $s \in B$ , such that*

$$\Gamma(f) = \int f(g)\alpha(g)(s) d\lambda(g), \quad f \in L^1(G, \lambda).$$

*If each  $\alpha(g)$  is an isometry, then  $\|s\| = \|\Gamma\|$ .*

We used this result in article II to derive Theorem 1 (b) in the unimodular case. It is interesting to notice, however, that Proposition 12 can also be employed in the case where  $G$  need not be unimodular. Since this was not done in the articles, and because the procedure further illustrates the structure of covariant semispectral measures, we will do it here.

To begin this alternative proof of Theorem 1 (b), we first proceed as before, by constructing the positive trace class operator  $A(Z)$  for any  $Z \in \mathcal{B}(G)$  of finite  $\tilde{\lambda}$ -measure according to Lemma 3. Hence, (4.1) holds, and for any  $D \in \mathcal{B}(G)$  with finite  $\tilde{\lambda}$ -measure, the map  $\mathcal{B}(D) \ni X \mapsto A(X) \in \mathcal{T}(\mathcal{H})$  is additive and satisfies  $\text{Tr}[A(X)] = \tilde{\lambda}(X)$ . Let  $\mathcal{F}$  denote the linear space of complex valued  $\mathcal{B}(G)$ -simple  $\tilde{\lambda}$ -integrable functions on  $G$ .

Now additivity and the above trace relation imply that the rule

$$\mathcal{F} \ni \sum_{n=0}^k c_n \chi_{Z_n} \mapsto \sum_{n=0}^k c_n A(Z_n) \in \mathcal{T}(\mathcal{H})$$

unambiguously defines a linear map from  $\mathcal{F}$  to  $\mathcal{T}(\mathcal{H})$ . If  $f = \sum_{n=0}^k c_n \chi_{Z_n} \in \mathcal{F}$ , then

$$\left\| \sum_{n=0}^k c_n A(Z_n) \right\|_{\text{tr}} \leq \sum_{n=0}^k |c_n| \text{Tr}[A(Z_n)] = \int |f| d\tilde{\lambda},$$

implying that the linear map in question is well-defined and continuous, when  $\mathcal{F}$  is considered as a (dense) subspace of the Banach space  $L^1(G, \tilde{\lambda})$ . Hence, it can be uniquely extended to a continuous linear map  $\tilde{\Gamma} : L^1(G, \tilde{\lambda}) \rightarrow \mathcal{T}(\mathcal{H})$ . Define  $\Gamma : L^1(G, \lambda) \rightarrow \mathcal{T}(\mathcal{H})$  by  $\Gamma(f) = \tilde{\Gamma}(f\Delta)$ , where  $\Delta$  is the modular function. This is possible, since  $f \in L^1(G, \lambda)$  implies  $\int |f|\Delta d\tilde{\lambda} = \int |f| d\lambda < \infty$ , i.e.  $f\Delta \in L^1(G, \tilde{\lambda})$ . Moreover, the (clearly linear) map  $\Gamma$  is continuous, having the same norm as  $\tilde{\Gamma}$ .

Now define  $\alpha(g)(T) = U(g)TU(g)^*$  for  $g \in G$  and  $T \in \mathcal{T}(\mathcal{H})$ . Each  $\alpha(g)$  is a linear homeomorphism from  $\mathcal{T}(\mathcal{H})$  onto  $\mathcal{T}(\mathcal{H})$ . Since  $U$  is a projective unitary representation, the map  $\alpha : G \rightarrow \text{Aut}(\mathcal{T}(\mathcal{H}))$  thus defined is a homomorphism. Condition (i) of Proposition 12 is clearly satisfied, since  $\|\alpha(g)\| = 1$  for any  $g$ . Condition (ii) follows from Lemma 2 of article III.

It follows from (4.1) that for all  $g \in G$  and  $Z \in \mathcal{B}(G)$  with  $\tilde{\lambda}(Z) < \infty$ , we have  $\alpha(g)(A(Z)) = \Delta(g)A(gZ)$ , i.e.  $\alpha(g)(\tilde{\Gamma}(\chi_Z)) = \Delta(g)\tilde{\Gamma}(\chi_Z(g^{-1}\cdot))$ . By the linearity and continuity of  $\tilde{\Gamma}$  and each  $\alpha(g)$ , we get

$$\alpha(g)(\tilde{\Gamma}(f)) = \Delta(g)\tilde{\Gamma}(f(g^{-1}\cdot)), \quad f \in L^1(G, \tilde{\lambda}).$$

As a consequence,  $\Gamma$  is covariant:

$$\alpha(g)(\Gamma(f)) = \Gamma(f(g^{-1}\cdot)), \quad f \in L^1(G, \lambda).$$

Thus, Proposition 12 can be invoked to get a unique trace class operator  $T$  of trace one, such that

$$\Gamma(f) = \int f(g)\alpha(g)(T) d\lambda(g), \quad f \in L^1(G, \lambda).$$

Now for any  $X \in \mathcal{B}(G)$  with  $\tilde{\lambda}(X) < \infty$ , we get

$$A(X) = \tilde{\Gamma}(\chi_X) = \Gamma(\Delta^{-1}\chi_X) = \int_X \alpha(g)(T)\Delta(g)^{-1} d\lambda(g) = \int_X \alpha(g)(T) d\tilde{\lambda}(g),$$

thereby arriving at (4.3).

# Chapter 5

## Homodyne detection as an application of the moment limit of observables

In this chapter, we demonstrate that the rotated quadrature observable  $\frac{1}{\sqrt{2}}(\overline{e^{-i\theta}a} + e^{i\theta}a^*)$  of a single mode electromagnetic field can be determined in the so called "high amplitude limit" of the balanced homodyne detection, by using the moment operators of observables actually measured by the detector. This is done by considering a sequence  $(E^n)_{n \in \mathbb{N}}$  of such observables, the limit  $n \rightarrow \infty$  corresponding to the high amplitude limit. It turns out that the moment operators  $E^n[k]$  converge (in a natural sense) to the corresponding moment operators of the quadrature. Moreover, it also turns out that the operators  $E^n(X)$ ,  $X \in \mathcal{B}(\mathbb{R})$ , themselves converge for certain sets  $X$ . Both limits have an operational meaning, the first corresponding to the convergence of the scalar moments of the measurement statistics, and the latter implying the convergence of the actual statistics.

We will first formulate and investigate the above mentioned two limit concepts in general, and then apply the results to the balanced homodyne detection scheme. As another related application, we use the results to rigorously prove that the high amplitude limit in the eight-port homodyne detector permits the measurement of all the covariant phase space observables on  $\mathbb{R}^2$ .

The results presented in this chapter are from articles VI and VII.



## 5.1 Moment limit of semispectral measures

Recall that  $\mathcal{H}$  is a fixed complex separable Hilbert space. The following definition specifies what we mean by the "moment limit of observables". It appeared as Definition 1 in article VI.

**Definition 11.** Let  $E^n$ ,  $n \in \mathbb{N}$ , and  $E$  be semispectral measures  $\mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ . If  $\mathcal{D} \subset \bigcap_{n,k \in \mathbb{N}} D(E^n[k])$  is a dense subspace, such that

$$\lim_{n \rightarrow \infty} \langle \psi | E^n[k] \varphi \rangle = \langle \psi | E[k] \varphi \rangle, \quad k \in \mathbb{N}, \psi \in \mathcal{H}, \varphi \in \mathcal{D}$$

(in particular, each limit in the left hand side exists), then we say that  $E$  is a *moment limit for  $(E^n)_{n \in \mathbb{N}}$  on  $\mathcal{D}$* .

A sequence  $(E^n)$  can have various different moment limits on a subspace  $\mathcal{D}$  (consider e.g. a constant sequence  $E^n = \mu I$ , with  $\mu$  a probability measure which is not determinate). However, the following result holds (Proposition 2 of article VI).

**Proposition 13.** Let  $E^n$ ,  $n \in \mathbb{N}$ , and  $E$  be semispectral measures  $\mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ , and  $\mathcal{D} \subset \mathcal{H}$  be a dense subspace, such that  $E$  is a moment limit for  $(E^n)_{n \in \mathbb{N}}$  on  $\mathcal{D}$ . If  $E$  is  $\mathcal{D}$ -determinate, then  $E$  is the only moment limit for  $(E^n)_{n \in \mathbb{N}}$  on  $\mathcal{D}$ .

## 5.2 Convergence of semispectral measures in the sense of probabilities

Throughout this section, we let  $\Omega$  be a metric space, with  $\mathcal{B}(\Omega)$  the associated Borel  $\sigma$ -algebra. In this context, we formulate a convergence concept of semispectral measures, which corresponds to the weak convergence of the associated probability measures. Recall that a sequence  $(\mu_n)$  of probability measures on  $\mathcal{B}(\Omega)$  converges weakly to a probability measure  $\mu : \mathcal{B}(\Omega) \rightarrow [0, 1]$ , if  $\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu$  for all bounded continuous functions  $f : \Omega \rightarrow \mathbb{R}$  [6, p. 11]. This can be characterized in terms of probabilities: A sequence  $(\mu_n)$  of probability measures converges weakly to a probability measure  $\mu$ , if and only if  $\lim_{n \rightarrow \infty} \mu_n(X) = \mu(X)$  whenever  $X \in \mathcal{B}(\Omega)$  is such that the boundary<sup>1</sup>  $\partial X$  of  $X$  has zero  $\mu$ -measure [6, Theorem 2.1].

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<sup>1</sup>The boundary  $\partial X$  of a set  $X \subset \Omega$  is the intersection of the closures of  $X$  and its complement.

**Definition 12.** Let  $E^n : \mathcal{B}(\Omega) \rightarrow L(\mathcal{H})$  be a semispectral measure for each  $n \in \mathbb{N}$ . We say that the sequence  $(E^n)_{n \in \mathbb{N}}$  converges to a semispectral measure  $E : \mathcal{B}(\Omega) \rightarrow L(\mathcal{H})$  weakly in the sense of probabilities, if

$$\lim_{n \rightarrow \infty} E^n(X) = E(X)$$

in the weak operator topology, for all  $X \in \mathcal{B}(\Omega)$  with  $E(\partial X) = 0$ .

The following proposition (Proposition 10 of article VI) characterizes this convergence in terms of weak convergence of the associated probability measures. Since the weak limit of a sequence of probability measures is unique, this shows, in particular, that a sequence of semispectral measures can converge to at most one semispectral measure weakly in the sense of probabilities.

**Proposition 14.** Let  $E^n : \mathcal{B}(\Omega) \rightarrow L(\mathcal{H})$  be a semispectral measure for each  $n \in \mathbb{N}$ , and let also  $E : \mathcal{B}(\Omega) \rightarrow L(\mathcal{H})$  be a semispectral measure. Then the following conditions are equivalent.

- (i)  $(E^n)$  converges to  $E$  weakly in the sense of probabilities;
- (ii) for each positive operator  $T$  of trace one, the sequence  $E_T^n$  of probability measures converges weakly to  $E_T$ ;
- (iii) there exists a dense subspace  $\mathcal{D} \subset \mathcal{H}$ , such that the sequence  $(E_\varphi^n)$  of probability measures converges weakly to  $E_\varphi$  for any unit vector  $\varphi \in \mathcal{D}$ ;
- (iv)  $\lim_{n \rightarrow \infty} L(f, E^n) = L(f, E)$  in the weak operator topology for each bounded continuous function  $f : \Omega \rightarrow \mathbb{R}$ .

Our goal is to connect the concept introduced above of the moment limit of a sequence  $(E^n)_{n \in \mathbb{N}}$  of semispectral measures  $\mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  to the convergence of  $(E^n)_{n \in \mathbb{N}}$  in the sense of probabilities (Proposition 16 below). The corresponding result in the case of probability measures is a part of classical probability theory (see the original paper [30] by Fréchet and Shohat, and also [49], or [7, p. 405-408] for a more modern formulation). The relevant concept in this context is relative compactness. In probability theory, a family  $\mathcal{P}$  of probability measures on  $\mathcal{B}(\Omega)$  is said to be *relatively compact*, if every sequence of elements of  $\mathcal{P}$  contains a weakly convergent subsequence [6, p. 35].

The following proposition (Proposition 11 of article VI) brings the concept of relative compactness of probability measures to the level of semispectral measures. It is needed in the proof of Proposition 16.

**Proposition 15.** *Let  $\mathcal{D} \subset \mathcal{H}$  be a dense subspace, and let  $\mathcal{M}$  be a collection of semispectral measures  $E : \mathcal{B}(\Omega) \rightarrow L(\mathcal{H})$ . Suppose that the set  $\{E_\varphi \mid E \in \mathcal{M}\}$  of probability measures is relatively compact for each unit vector  $\varphi \in \mathcal{D}$ . Then every sequence of elements of  $\mathcal{M}$  contains a subsequence which converges weakly in the sense of probabilities.*

## 5.3 An "asymptotic measurement" scheme

We proceed to formulate a measurement scheme which consists of a sequence of measurements, having the property that the moments of the measurement outcome statistics converge. In the next section we show that balanced homodyne detection can be understood as an example of such a measurement.

The formulation is based on the following proposition, which was the goal of the developments of the preceding sections.

**Proposition 16.** *Let  $E^n : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  be a semispectral measure for each  $n \in \mathbb{N}$ . Assume that there is a dense subspace  $\mathcal{D} \subset \bigcap_{n,m \in \mathbb{N}} D(E^n[m])$ , such that the limit*

$$\lim_{n \rightarrow \infty} \int x^m dE_\varphi^n(x)$$

*exists in  $\mathbb{R}$  for each  $m \in \mathbb{N}$  and  $\varphi \in \mathcal{D}$ .*

- (a) *There exists a semispectral measure  $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ , which is a moment limit for  $(E^n)_{n \in \mathbb{N}}$  on  $\mathcal{D}$ .*
- (b) *Suppose that  $E$  is  $\mathcal{D}$ -determinate. Then  $E$  is the only moment limit for  $(E^n)_{n \in \mathbb{N}}$  on  $\mathcal{D}$ , and the sequence  $(E^n)_{n \in \mathbb{N}}$  converges to  $E$  weakly in the sense of probabilities.*

The proof of this proposition is based on Proposition 15 above, which is used to establish the existence of  $E$  in part (a). (The proposition can be applied, since each set  $\{E_\varphi^n \mid n \in \mathbb{N}\}$ ,  $\varphi \in \mathcal{D}$ , is relatively compact by the limit assumption in the proposition and certain basic results in probability theory.) See Proposition 5 of article VI for the complete proof.

Suppose now that we have a measurement setup which can be configured to measure various observables  $E^n : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ ,  $n \in \mathbb{N}$ , the idea being that each observable represents, in some sense, an approximate version of a single observable we actually want to measure. As  $n$  increases, the approximation is supposed to get better. Such is the situation in homodyne detection (see the next section), where the desired measurement of a rotated quadrature is supposed to be realized only in the "high amplitude limit".

We make the following, physically reasonable assumptions on the sequence  $(E^n)$ , and the measurement setup.

- (1) Suppose that we can prepare a set of calibration states, which can be identified with the unit sphere  $\mathcal{D}_1$  of some dense subspace  $\mathcal{D} \subset \mathcal{H}$ .
- (2) Let  $M_{n,\varphi}^k$  be the  $k$ th moment of the measurement statistics of the observable  $E^n$  in the state  $\varphi$ , for each  $n, k \in \mathbb{N}$ ,  $\varphi \in \mathcal{D}_1$ . Here we have assumed that  $\mathcal{D}_1$  can be chosen so that each  $M_{n,\varphi}^k$  exists. Notice that the moments can be calculated directly from the measurement statistics.
- (3) Assume that  $\lim_{n \rightarrow \infty} M_{n,\varphi}^k$  exists for each  $k \in \mathbb{N}$  and  $\varphi \in \mathcal{D}_1$ . Then Proposition 16 implies the existence of a moment limit  $E$ .
- (4) In order to assure that the limits determine a unique observable, we have to assume that  $E$  is  $\mathcal{D}$ -determinate.

Under these assumptions, Proposition 16 tells us that the observable  $E$  is uniquely determined as a moment limit of  $(E^n)_{n \in \mathbb{N}}$ , and, moreover, the sequence  $(E^n)_{n \in \mathbb{N}}$  converges to  $E$  in the sense of probabilities. By proposition 14, the latter fact implies that the measurement statistics  $(E_T^n)$  converge to  $E_T$  for *any* state  $T$ , not just for the calibration states in  $\mathcal{D}_1$ .

## 5.4 Balanced homodyne detection

The balanced homodyne detection, introduced in [64, 63] is an important technique in quantum optics, because it is assumed to provide a means to measure the rotated quadrature amplitudes  $\frac{1}{\sqrt{2}}(e^{-i\theta}a + e^{i\theta}a^*)$  of a single mode electromagnetic field. Perhaps the most notable application of this method is quantum state estimation, i.e. quantum tomography. (See [45] for a collection of articles concerning this topic.) The significance of the homodyne measurement in that context is due to the fact that the combined measurement statistics of all the rotated quadratures determine the state of the system.

The measurement scheme is the following: A signal field is mixed with an auxiliary field by means of a 50-50 beam splitter (possibly followed by a phase shifter), and the difference of photon numbers on the output ports of the splitter is detected. The auxiliary field is taken to be an oscillator in a coherent state, operating with the same frequency as the signal field. Assuming that the beam splitter is lossless and causes no phase shift between the modes, the process is described by a unitary operator  $U$  (see [47, 39]), transforming the field annihilation operators  $a$  (signal) and  $b$  (auxiliary) into  $\tilde{a} = U^*aU = \frac{1}{\sqrt{2}}(a - b)$ , and  $\tilde{b} = U^*bU = \frac{1}{\sqrt{2}}(a + b)$ . With respect to the

initial state of the two-mode field (Heisenberg picture), the photon number difference observable is then  $\tilde{b}^*\tilde{b} - \tilde{a}^*\tilde{a} = ab^* + a^*b$ , provided that the photon detectors are ideal. (Of course, these formal operator relations are made precise by restricting the operators to a dense subspace, see (5.4) below.)

When the amplitude of the auxiliary field is high, the above mentioned photon difference statistics are considered to resemble those of a rotated field quadrature, the rotation angle being identified as the fixed phase difference between the input signal and auxiliary fields. This is usually justified by the following heuristic explanation: the strong auxiliary field can be treated "classically", by replacing  $b$  with the complex field amplitude  $\beta = re^{i\theta}$  in the operator  $ab^* + a^*b$ , with  $\theta$  identified as the phase relative to the input signal field. Consequently, by suitably scaling the resulting operator with a factor proportional to the intensity of the auxiliary field, one recovers the rotated quadrature  $\frac{1}{\sqrt{2}}(e^{-i\theta}a + e^{i\theta}a^*)$ .

A more theoretical explanation takes into account the quantum nature of the auxiliary field. In addition to the original paper [63], this has been done in [10, 19, 3, 55]. These papers calculate the characteristic functions of the probability measures associated with the photon difference statistics, and then take the high amplitude limit at this level. Vogel [55] applies the Levy-Cramer theorem to prove that the associated probability measures converge weakly. However, in the calculations leading to the above mentioned characteristic functions, some of the problematic mathematical questions (such as formal operator series expansions and term-by-term integration of a series) were not addressed, making it difficult to verify whether the treatment could be made rigorous.

In the following, we demonstrate that the balanced homodyne detection can be understood rigorously in terms of the moment operators of the signal field observables arising from the photon difference statistics. The moment operators were also used in the context of homodyne detection in [3, 43], where the authors consider the effects of imperfectness of the photon detectors by using the concepts of [29]. Unfortunately, they dismiss the limit problem by applying the heuristic classical approximation of the auxiliary field to the characteristic functions, so that their results do not contribute to our problem.

We let  $\mathcal{H}$  and  $\mathcal{H}_{aux}$  be separable complex Hilbert spaces corresponding to the signal and auxiliary field modes, respectively. We use the following usual notations. Fix orthonormal bases of the form  $\{|n\rangle \mid n \in \mathbb{N}\}$  for both  $\mathcal{H}$  and  $\mathcal{H}_{aux}$ , representing the photon number states. Let  $a, a^*$  and  $b, b^*$  be the creation and annihilation operators for the aforementioned bases of  $\mathcal{H}$  and  $\mathcal{H}_{aux}$ , respectively, and let  $N = a^*a$  and  $N_{aux} = b^*b$  be the photon number operators for the two modes. The operators  $a, a^*, b, b^*, N, N_{aux}$  are considered

as being defined on their natural domains, e.g.

$$D(a) = D(a^*) = \{\varphi \in \mathcal{H} \mid \sum_{n \in \mathbb{N}} n |\langle n | \varphi \rangle|^2 < \infty\};$$

$$D(N) = D(a^*a) = \{\varphi \in \mathcal{H} \mid \sum_{n \in \mathbb{N}} n^2 |\langle n | \varphi \rangle|^2 < \infty\}.$$

For any  $z \in \mathbb{C}$  the *coherent state*  $|z\rangle \in \mathcal{H}$  is defined by

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle,$$

and we use the same symbols for the coherent states in  $\mathcal{H}_{aux}$ . The subspace  $D_{coh} := \text{lin}\{|z\rangle \mid z \in \mathbb{C}\}$  is dense in  $\mathcal{H}$ , and so is the corresponding subspace  $D_{coh}^{aux}$  in  $\mathcal{H}_{aux}$ . (Here the symbol "lin" denotes the (algebraic) linear span of the set in question.) The (algebraic) tensor product  $D_{coh}^2 := D_{coh} \otimes D_{coh}^{aux}$  can be identified with  $\text{lin}\{|\beta\rangle \otimes |z\rangle \mid \beta, z \in \mathbb{C}\}$ , which is dense in  $\mathcal{H} \otimes \mathcal{H}_{aux}$ .

Denote by  $Q$  and  $P$  the signal field quadrature operators  $\frac{1}{\sqrt{2}}(a^* + a)$  and  $\frac{i}{\sqrt{2}}(a^* - a)$ , respectively.

Define the *rotated quadrature operators*  $Q_\theta$ , with  $\theta \in [0, 2\pi)$ , via

$$Q_\theta = e^{i\theta N} Q e^{-i\theta N} = \frac{1}{\sqrt{2}}(e^{-i\theta} a + e^{i\theta} a^*). \quad (5.1)$$

In particular, each  $Q_\theta$  is selfadjoint on its domain  $D(Q_\theta) = e^{i\theta N} D(Q) \supset D(a) = D(a^*)$ , and the restriction  $Q_\theta|_{D(a)} = \frac{1}{\sqrt{2}}(e^{-i\theta} a + e^{i\theta} a^*)$  is essentially selfadjoint. The ordinary quadratures are given by  $Q_0 = Q$  and  $Q_{\frac{\pi}{2}} = P$ . We will need the following lemma (Lemma 2 in article VI.)

**Lemma 5.** *Let  $\theta \in [0, 2\pi)$ . Then  $D_{coh} \subset \mathcal{E}(P^{Q_\theta})$ . In particular, the spectral measure  $P^{Q_\theta}$  is  $D_{coh}$ -determinate (and thereby determinate).*

The 50-50-beam splitter is described by the unitary operator  $U \in L(\mathcal{H} \otimes \mathcal{H}_{aux})$ , defined by its acting in the coordinate representation (see e.g. [39]):

$$L^2(\mathbb{R}^2) \ni \Psi \mapsto \left( (x_1, x_2) \mapsto \Psi\left(\frac{1}{\sqrt{2}}(x_1 + x_2), \frac{1}{\sqrt{2}}(-x_1 + x_2)\right) \right) \in L^2(\mathbb{R}^2). \quad (5.2)$$

We have assumed that the beam splitter does not generate any phase shift<sup>2</sup>. Under this transformation, the coherent states change according to

$$|\alpha\rangle \otimes |\beta\rangle \mapsto \left| \frac{1}{\sqrt{2}}(\alpha - \beta) \right\rangle \otimes \left| \frac{1}{\sqrt{2}}(\alpha + \beta) \right\rangle. \quad (5.3)$$

---

<sup>2</sup>This is not a restriction, since any shift can be easily accomplished by changing the phase of the auxiliary coherent state.

This implies the equally well-known transformation rules for annihilation operators:

$$\begin{aligned} U^*(a \otimes I)|_{D_{coh}^2} U &= \frac{1}{\sqrt{2}}(a - b)|_{D_{coh}^2}; \\ U^*(I \otimes b)|_{D_{coh}^2} U &= \frac{1}{\sqrt{2}}(a + b)|_{D_{coh}^2}. \end{aligned} \quad (5.4)$$

Let  $T$  be the input state of the signal field, and let  $|z\rangle$  be the input coherent state of the auxiliary field. The photon difference operator is  $N_- := \overline{I \otimes N_{aux}} - N \otimes I$  (where  $I$  is the identity operator), so our detection observable is the spectral measure

$$\mathcal{B}(\mathbb{R}) \ni P^{(\sqrt{2}|z|)^{-1}N_-}(X) \in L(\mathcal{H} \otimes \mathcal{H}_{aux})$$

of the scaled operator  $(\sqrt{2}|z|)^{-1}N_-$ , the scale  $|z|$  being the amplitude of the auxiliary oscillator. (Factor  $\sqrt{2}$  appears for convenience.)

Since the state emerging from the beam splitter is  $U(T \otimes |z\rangle\langle z|)U^*$ , the detection statistics are given by the probability measures

$$X \mapsto \text{Tr}[U(T \otimes |z\rangle\langle z|)U^* P^{(\sqrt{2}|z|)^{-1}N_-}(X)].$$

For a fixed coherent state  $|z\rangle$ , these probability measures define an observable  $E^z : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$  on the signal field according to

$$\text{Tr}[TE^z(X)] := \text{Tr}[T \otimes |z\rangle\langle z| P^{|z|^{-1}A}(X)], \quad X \in \mathcal{B}(\mathbb{R}), \quad (5.5)$$

where

$$A := \frac{1}{\sqrt{2}} \overline{(a \otimes b^* + a^* \otimes b)} = \frac{1}{\sqrt{2}} U^* N_- U.$$

Let  $V_z : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_{aux}$  be the linear isometry  $\varphi \mapsto \varphi \otimes |z\rangle$ . Then we have simply

$$E^z(X) = V_z^* P^{|z|^{-1}A}(X) V_z, \quad X \in \mathcal{B}(\mathbb{R}). \quad (5.6)$$

Now the idea in balanced homodyne detection is that one takes the "high amplitude limit"  $|z| \rightarrow \infty$  on the observable  $E^z$ , with a fixed phase  $\theta$ , i.e.  $z = |z|e^{i\theta}$ . We do this at the level of the moment operators  $E^z[k]$  of  $E^z$ , and then apply the general "asymptotic measurement" scheme presented in the preceding section.

The following result (Proposition 9 of article VI) justifies the consideration of the moment operators  $E^z[k]$  in place of the observable  $E^z$ . It should be compared to Lemma 5.

**Proposition 17.** *Let  $z \in \mathbb{C}$ . Then  $D_{coh} \subset \mathcal{E}(E^z)$ . In particular, the semispectral measure  $E^z$  is  $\mathcal{D}_{coh}$ -determinate (and thereby determinate).*

We were only able to determine the restrictions  $\tilde{E}^z[k]$  of the moment operators  $E^z[k]$ . In view of the general scheme, this is, however, sufficient. The result is presented in the following proposition, which is a combination of Propositions 6 and 8 of article VI.

**Proposition 18.** *Let  $z = re^{i\theta} \in \mathbb{C}$ ,  $r \geq 1$ , and  $k \in \mathbb{N}$ . Then*

$$\tilde{E}^z[k] = |z|^{-k} V_z^* A^k V_z \supset Q_\theta^k|_{D(a^k)} + \frac{1}{r^2} C_k(r, \theta),$$

with  $D(a^k) \subset D(\tilde{E}^z[k])$ , where

$$C_k(r, \theta) = \sum_{\substack{n, m \in \mathbb{N}, \\ n+m \leq k}} c_{nm}^k(r, \theta) (a^*)^n a^m,$$

and each function  $c_{nm}^k : [1, \infty) \times [0, 2\pi) \rightarrow \mathbb{C}$  is bounded.

The first two moment operators assume the explicit forms  $\tilde{E}^z[1] \supset Q_\theta|_{D(a)}$ ,  $\tilde{E}^z[2] \supset Q_\theta^2|_{D(a^2)} + \frac{1}{2}|z|^{-2}N$ . (See proposition 7 of article VI.) This shows, in particular, that the associated intrinsic noise operator (see section 2.3 for the definition) coincides with the selfadjoint operator  $\frac{1}{2}|z|^{-2}N$  (Corollary 1 of article VI).

Proposition 18 immediately implies that  $\lim_{r \rightarrow \infty} E^{re^{i\theta}}[k]\varphi = Q_\theta^k\varphi$ , if  $\varphi \in D(a^k)$ . Combined with Lemma 5, and the general results of section 5.3, we get the following conclusion on the high amplitude limit in the balanced homodyne detection. It was given in section 8 of article VI.

**Proposition 19.** *Let  $(r_n)$  be any sequence of positive numbers tending to infinity, let  $\theta \in [0, 2\pi)$ , and set  $z_n = r_n e^{i\theta}$ ,  $n \in \mathbb{N}$ .*

- (a) *The spectral measure  $P^{Q_\theta}$  is the unique moment limit for  $(E^{z_n})_{n \in \mathbb{N}}$  on  $D_{coh}$ .*
- (b) *The sequence  $(E^{z_n})_{n \in \mathbb{N}}$  converges to  $P^{Q_\theta}$  weakly in the sense of probabilities.*
- (c) *The sequence  $(E^{z_n})_{n \in \mathbb{N}}$  constitutes an example of the "asymptotic measurement" scheme presented in the preceding section.*

Since the spectral measure  $P^{Q_\theta}$  has the same null sets as the Lebesgue measure, result (b) implies that

$$\lim_{n \rightarrow \infty} E^{z_n}(X) = P^{Q_\theta}(X)$$

in the weak operator topology, whenever  $X \in \mathcal{B}(\mathbb{R})$  is such that  $\partial X$  has zero Lebesgue measure. It is worth noting that this limit relation does indeed *not* hold for all Borel sets  $X$ . (See Remark 8 of article VI.)



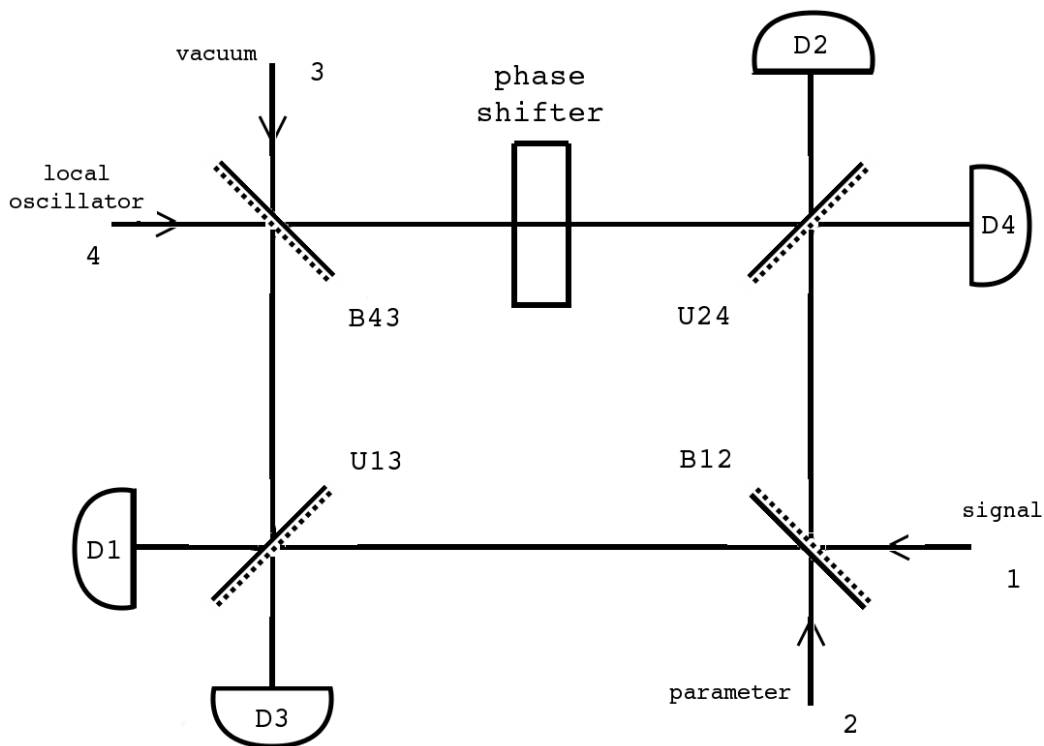


Figure 5.1: The eight port homodyne detector

## 5.5 Measurement of covariant phase space observables with the eight-port homodyne detector

In this section, we give another example of a concrete experimental setup, the eight-port homodyne detector, which can be understood rigorously by using some of the above concepts. The results of this section are from article VII.

The eight-port homodyne detector consists of the setup shown in Figure 5.1 (see [39], and [38, p. 147-155]). The detector involves four modes as indicated in the picture, and we will denote the associated (complex separable) Hilbert spaces accordingly by  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ ,  $\mathcal{H}_3$ ,  $\mathcal{H}_4$ . Mode 1 corresponds to the signal field (i.e. the object system with respect to which the measured observable will be interpreted), the input state for mode 2 serves as a parameter which determines (as will be seen below) the observable to be measured,

and mode 4 is the reference beam in a coherent state. (The input for mode 3 is left empty, corresponding to the vacuum state.)

It is well known that using the heuristic classical "high amplitude" approximation for mode 4, and feeding the vacuum state to mode 2, the Husimi Q-function, i.e. the covariant phase space observable  $W^{|0\rangle\langle 0|}$ , is obtained as the measured observable with respect to the signal field. Recall from (3.1) that the covariant phase space observables are of the form

$$W^S(Z) := \frac{1}{2\pi} \int_Z W(q,p) S W(q,p)^* dqdp, \quad Z \in \mathcal{B}(\mathbb{R}^2),$$

where  $S$  is a positive operator of trace one. In the following, we describe the "high amplitude" limit rigorously, by using the results already obtained for a single homodyne detector. In fact, we show that any phase space observable can be obtained as a limit of eight-port detector observables.

We fix the photon number bases  $\{|n\rangle \mid n \in \mathbb{N}\}$  for each  $\mathcal{H}_i$ , so that the annihilation operators  $a_j$ , as well as the quadratures  $Q_j = \frac{1}{\sqrt{2}}(a_j^* + a_j)$ ,  $P_j = \frac{1}{\sqrt{2}}i(a_j^* - a_j)$  and photon number operators  $N_j = a_j^* a_j$  are defined for each mode  $j = 1, 2, 3, 4$ .

The photon detectors  $D_j$  shown in the picture are considered to be ideal, so that each detector  $D_j$  measures the sharp photon number  $N_j$ . The phase shifter in mode 4 is represented by the unitary operator  $e^{i\phi N_4}$ , where  $\phi$  is the shift.

There are four 50-50-beam splitters  $B_{12}$ ,  $B_{43}$ ,  $U_{13}$ ,  $U_{24}$ , each of which is defined by its acting in the coordinate representation according to (5.2). In the picture, the dashed line in each beam splitter indicates the input port of the "primary mode", i.e. the mode associated with the first component of the tensor product  $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^2)$  in the description of equation (5.2). The beam splitters  $B_{12}$ ,  $B_{43}$ ,  $U_{13}$  and  $U_{24}$  are indexed so that the first index indicates the primary mode.

Let  $|\sqrt{2}z\rangle$  be the coherent input state for mode 4. We detect the scaled number differences  $\frac{1}{|z|}N_{13}^-$  and  $\frac{1}{|z|}N_{24}^-$ , where  $N_{13}^- := \overline{I_1 \otimes N_3 - N_1 \otimes I_3}$  and  $N_{24}^- := \overline{I_2 \otimes N_4 - N_2 \otimes I_4}$ , so that the joint detection statistics are described by unique spectral measure extending the set function

$$X \times Y \mapsto P^{|z|^{-1}N_{13}^-}(X) \otimes P^{|z|^{-1}N_{24}^-}(Y),$$

where the operator acts on the entire four-mode field.

Let  $T$  and  $S$  be the input states for modes 1 and 2, respectively. Then the state of the four-mode field after the combination of the beam splitters and the phase shifter is  $U_{13} \otimes U_{24} W_{T,S,z,\phi} U_{13}^* \otimes U_{24}^*$ , where

$$W_{T,S,z,\phi} := B_{12}(T \otimes S) B_{12}^* \otimes |z\rangle\langle z| \otimes |ze^{i\phi}\rangle\langle ze^{i\phi}|.$$

We regard  $S$ ,  $|\sqrt{2}z\rangle$  and  $\phi$  as fixed parameters, while  $T$  is the state of the object system, i.e. signal field. The detection statistics then define an observable  $G^{z,S,\phi} : \mathcal{B}(\mathbb{R}^2) \rightarrow L(\mathcal{H}_1)$  on the signal field via

$$\mathrm{Tr}[TG^{z,S,\phi}(Z)] = \mathrm{Tr}[W_{T,S,z,\phi}P^z(\frac{1}{\sqrt{2}}Z)], \quad (5.7)$$

where  $P^z : \mathcal{B}(\mathbb{R}^2) \rightarrow L(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4)$  is defined as the unique spectral measure extending the set function  $X \times Y \mapsto P^{|z|^{-1}A_{13}}(X) \otimes P^{|z|^{-1}A_{24}}(Y)$ , where  $A_{ij} := \frac{1}{\sqrt{2}}(a_i \otimes a_j^* + a_j^* \otimes a_i)$ . (See article VII for details.) The observable  $G^{z,S,\phi}$  is the one actually measured by the detector.

Let  $C : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  denote the conjugation map, i.e.  $(C\varphi)(x) = \overline{\varphi(x)}$  in the coordinate representation. Now we are ready to present the conclusion concerning the measurement of phase space observables with the eight-port detector. The proof of the following proposition is in article VII.

**Proposition 20.** *Let  $S \in L(\mathcal{H}_2)$  be any positive operator of trace one, and let  $(r_n)$  be any sequence of positive numbers tending to infinity. Then the sequence  $(G^{r_n,S,\frac{\pi}{2}})_{n \in \mathbb{N}}$  of observables converges to the covariant phase space observable  $W^{CSC^{-1}}$  weakly in the sense of probabilities.*

Since any phase space observable  $W^S$  is absolutely continuous with respect to the Lebesgue measure, it follows from the preceding proposition that

$$\lim_{n \rightarrow \infty} G^{r_n,S,\frac{\pi}{2}}(Z) = E^{CSC^{-1}}(Z)$$

in the weak operator topology, for any  $Z \in \mathcal{B}(\mathbb{R}^2)$  such that  $\partial Z$  has zero Lebesgue measure.

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