Vaasan yliopisto
UNIVERSITY OF VAASA

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Representations and Regularity of Gaussian Processes

ACTA WASAENSIA 329

MATHEMATICS AND STATISTICS 12

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Julkaisija	Julkaisupäivämäärä	
Vaasan yliopisto	Elokuu 2015	
Tekijä(t) Adil Yazigi	**Julkaisun tyyppi** Artikkelikokoelma	
	Julkaisusarjan nimi, osan numero Acta Wasaensia, 329	
Yhteystiedot Vaasan Yliopisto Teknillinen tiedekunta Matemaattisten tieteiden yksikkö PL 700 64101 Vaasa	**ISBN** 978-952-476-628-9 (painettu) 978-952-476-629-6 (verkkojulkaisu)	
	ISSN 0355-2667 (Acta Wasaensia 329, painettu) 2323-9123 (Acta Wasaensia 329, verkkojulkaisu) 1235-7928 (Acta Wasaensia. Mathematics 12, painettu) 2342-9607 (Acta Wasaensia. Mathematics 12, verkkojulkaisu)	
	Sivumäärä 80	**Kieli** Englanti

Julkaisun nimike

Gaussisten prosessien esityslauseet ja säännöllisyys

Tiivistelmä

Tämä työ käsittelee gaussisia prosesseja, niiden esityslauseita ja säännöllisyyttä. Aluksi tarkastelemme gaussisten prosessien polkujen säännöllisyyttä ja annamme riittävät ja välttämättömät ehdot niiden Hölder-jatkuvuudelle. Sitten tarkastelemme itsesimilaaristen gaussisten prosessien kanonista Volterra-esitystä, joka perustuu Lamperti-muunnokselle ja annamme riittävät ja välttämättömät ehdot esityksen olemassaololle käyttäen ns. täyttä epädeterminismiä. Täysi epädeterminismi on luonnollinen ehto stationaarisille prosesseille kuten myös itsesimilaarisille prosesseille. Sitten sovellamme tulosta ja määrittelemme luokan, jolla on sama itsesimilaarisuusindeksi ja jotka ovat ekvivalentteja jakaumamielessä. Lopuksi tarkastelemme gaussisia siltoja yleistetyssä muodossa, jossa gaussiset prosessit on ehdollistettu monella päätearvolla. Yleistetylle sillalle annamme ortogonaalisen ja kanonisen esitysmuodon sekä martingaali- että ei-martingaalitapauksessa. Johdamme kanonisen esityslauseen myös kääntyville gaussisille Volterra-prosesseille.

Asiasanat

Gaussiset prosessit, gaussiset sillat, Hölder-jatkuvuus, kanoninen esityslause, itsesimilaariset prosessit, täysi epädeterminismi

Publisher	**Date of publication**	
Vaasan yliopisto	August 2015	
Author(s) Adil Yazigi	**Type of publication** Selection of articles	
	Name and number of series Acta Wasaensia, 329	
Contact information University of Vaasa Faculty of Technology Department of Mathematics and Statistics P.O. Box 700 FI-65101 Vaasa, Finland	**ISBN** 978-952-476-628-9 (print) 978-952-476-629-6 (online)	
	ISSN 0355-2667 (Acta Wasaensia 329, print) 2323-9123 (Acta Wasaensia 329, online) 1235-7928 (Acta Wasaensia. Mathematics 12, print) 2342-9607 (Acta Wasaensia. Mathematics 12, online)	
	Number of pages 80	**Language** English
Title of publication Representations and Regularity of Gaussian Processes		

Abstract

This thesis is about Gaussian processes, their representations and regularity. Firstly, we study the regularity of the sample paths of Gaussian processes and give a necessary and sufficient condition for Hölder continuity. Secondly, we introduce the canonical Volterra representation for self-similar Gaussian processes based on the Lamperti transformation and on the canonical Volterra representation of stationary Gaussian processes. The necessary and sufficient condition for the existence of this representation is the property of pure non-determinism which is a natural condition for stationary as well as self-similar Gaussian processes. Later, we apply this result to define the class of Gaussian processes that are self-similar with same index and are equivalent in law. Finally, we investigate the Gaussian bridges in a generalized form where the Gaussian processes are conditioned to multiple final values. The generalized Gaussian bridge will be given explicitly in the orthogonal and the canonical representation where both cases of martingale and non-semimartingale will be treated separately. We will also derive the canonical representation of the generalized Gaussian bridge for invertible Gaussian Volterra processes.

Keywords

canonical representation, Gaussian bridges, Gaussian processes, Hölder continuity, purely non-deterministic, self-similar processes

Acknowledgements

First, I would like to express my great appreciation and gratitude to my supervisor, Professor Tommi Sottinen. Thank you for your constant encouragement, guidance, assistance, and above all, kindness during the time of my doctoral studies.

I wish to thank the pre-examiners Professor Ciprian Tudor and Docent Dario Gasbarra for reviewing my thesis and for their feedbacks.

I would like to express my gratitude to my collaborators Lauri Viitasaari and Ehsan Azmoodeh. It is pleasant to remember the beautiful team spirit and the achievements during our joint works. Also, I would like to thank all the members of the Finnish graduate school in Stochastic, and in particular, the Stochastic group in Aalto University for arranging the seminars and for giving me the opportunity to present my work. I acknowledge all the valuable comments and remarks that have been made on my results during the seminars.

I also would like to extend my gratitude to all the members of the department of Mathematics and Statistics at the University of Vaasa for the friendly working environment, and for the moments that we shared during the coffee breaks.

For financial support, I am indebted to the Finnish Doctoral Programme in Stochastic and Statistics (FDPSS), the Finnish Cultural Foundation, and to the University of Vaasa.

Finally, I wish to thank my wife Jaana, my mother-in-law Salme, my parents, my brothers and my little sisters for their love and support.

"*Qui aime n'oublie pas*"

Vaasa, June 2015
Adil Yazigi

CONTENTS

LIST OF PUBLICATIONS

The dissertation is based on the following three refereed articles:

(I) E. Azmoodeh, T. Sottinen, L. Viitasaari, A. Yazigi (2014): Necessary and Sufficient conditions for Hölder continuity of Gaussian Processes, Statistics & Probability Letters, 94, 230-235.

(II) A. Yazigi (2015): Representation of Self-Similar Gaussian Processes, Statistics & Probability Letters, 99, 94-100.

(III) T. Sottinen, A. Yazigi (2014): Generalized Gaussian Bridges, Stochastic Processes and their Applications, 124, Issue 9, 3084-3105.

AUTHOR'S CONTRIBUTION

Publication I: "Necessary and sufficient conditions for Hölder continuity of Gaussian processes"

This article represents a joint discussion and all the results are a joint work with the co-authors.

Publication II: "Representation of self-similar Gaussian processes"

This is an independent work of the author. The topic was proposed by Tommi Sottinen.

Publication III: "Generalized Gaussian bridges"

The article is a joint work with Tommi Sottinen and substantial part of writing and analysis are due to the author. The problem originates from a discussion with Tommi.

1 INTRODUCTION

As models for random phenomena, Gaussian processes represent an important class of stochastic processes in probability theory. Besides the linear structure that Gaussian objects display, they acquire nice properties and geometric features that are simple to analyse, and which leads to interesting results involving the theory of random processes and functional analysis that comes closely connected to different applications in quantum physics, statistics, machine learning, finance and biology.

The modern theory of the Gaussian distribution has known developments since many applications rely on Gaussian distributions, and also since random variables which arise in applications may be approximated by normal distributions which can be controlled by their covariances. Brownian motion or Wiener process which is the most celebrated Gaussian process plays a essential role in the theory of diffusion processes, the stochastic differential equations and sample continuous martingales. It is also the key feature to understand the white noise.

In a historical brief, it was Francis Galton and Karl Pearson during 1889-1893 who first used the term “normal” for the Gaussian distribution. However, the Gaussian distribution first came to the attention of the scientists in the eighteenth century recognized at that time as the “Laplace’s second law” where the bell-shaped curve appeared in his work on an early version of central limit theorem, the “Moivre-Laplace theorem”. Later, C. F. Gauss developed the formula of the distribution through the theory of errors and called it “la loi des erreurs”, which is afterwards adopted by the French school as “Laplace-Gauss’s law”, and as “Gauss’s law” by the English school.

For any stochastic process, series and integral representations provide a powerful tool in the analysis of properties of the process. The well-known Karhunen-Loève series expansion can be applied for Gaussian processes as well as for any second order stochastic process, it is given in terms of eigenvalues and eigenfunctions that are sometimes difficult to express explicitly even for some well-studied processes, moreover, the representation is not unique since there are many ways to expand the process in form of series. On the other hand, to illustrate a Gaussian process with an integral representation, namely the one of Volterra-type, offers a good help to check closely the intrinsic properties of the process in a visualized style which turn out to be useful for good applications especially in the prediction theory. These properties such as sample paths regularity and invariance of the covariance are derived from the deterministic kernel and the Brownian motion that form the Volterra integral representation.

The Volterra-type integral representation of Gaussian processes has been introduced by Paul Lévy in 1955 with a major breakthrough in this area. Lévy’s starting point was to solve a non-linear integral equation that is a factorization of the covariance kernel by using Hilbert space tools. As the solution is naturally a kernel, Lévy’s main interest was to provide a solution that is unique and which preserves the canonicity, an interesting property which lets the linear spaces of the underlying process and that of the Brownian motion to be the same. In 1950, Karhunen studied the sta-

tionary Gaussian processes and introduced their canonical integral representation by using the concept of pure non-determinism under the heavy machinery of complex analysis which was at its golden age at that time. Later, the theory has been developed by Hitsuda in 1960 by involving more probabilistic methods such as the equivalence of Gaussian measures and the Girsanov theorem.
When the question comes to the change of measure, it arises the attention to the construction of Gaussian bridge as a linear transformation that leaves the measure of the underlying Gaussian process invariant. Being as natural model for the insider trading strategy as well as for many different applications, Gaussian bridges exhibit a different treatment to the problem of the enlargement of filtration. Given a Gaussian process X, the bridge of X is a Gaussian process which behaves like X under the condition that the process X reaches a certain value at a fixed time horizon. Gaussian bridges can be also defined through Doob's-h-transform that Doob has introduced in 1957 to investigate conditioned Markov processes. Later, several authors considered the Gaussian bridges in their works, especially the Brownian bridges, pointing out the importance of this process in probability theory.

2 GAUSSIAN PROCESSES

Throughout this thesis, all the processes are real-valued Gaussian processes. First, we recall the basic notions of Gaussian processes and give some of their well-known properties. For more details on Gaussian processes, we refer to Adler (1990), Bogachev (1998), Ibragimov & Rozanov (1970), Kahane (1985), Lifshits (1995) and Neveu (1968).

2.1 General facts

A random variable X defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is Gaussian if its characteristic function $\varphi_X(u) = \mathbb{E}(e^{iuX}), u \in \mathbb{R}$, has the form

$$\varphi_X(u) = e^{imu - \frac{1}{2}\sigma^2 u^2}, \quad m \in \mathbb{R}, \sigma > 0,$$

where $m = \mathbb{E}(X)$ is the *mean* and $\sigma^2 = \mathbb{V}\mathrm{ar}(X)$ is the *variance*. For an n-dimensional random vector $X = (X_1, \ldots, X_n)^\intercal$, the characteristic function is given by

$$\varphi_X(u) = \mathbb{E}(e^{iu^\intercal X}) = e^{iu^\intercal m - \frac{1}{2}u^\intercal R u}, \quad m \in \mathbb{R}^n, R \in \mathbb{R}^{n\times n},$$

for all $u \in \mathbb{R}^n$, where $m = (\mathbb{E}(X_1), \ldots, \mathbb{E}(X_n))^\intercal$ is the mean vector and $R = [R_{ij}]_{i,j=1}^n = [\mathbb{C}\mathrm{ov}(X_i, X_j)]_{i,j=1}^n$ is the *covariance matrix* which is symmetric and non-negative in the sense that

$$a^\intercal R a = \sum_{i=1}^n \sum_{j=1}^n R_{ij} a_i a_j \geq 0$$

holds for any $a = (a_1, \ldots, a_n)^\intercal \in \mathbb{R}^n$. This definition will be extended to the Gaussian processes.

Definition 2.1. Let $\mathbb{T} \subseteq \mathbb{R}$. A stochastic process $X = (X_t)_{t\in\mathbb{T}}$ is a Gaussian process if any finite linear combination $\sum \alpha_i X_{t_i}$, $\alpha_i \in \mathbb{R}$, $t_i \in \mathbb{T}$, $i = 1, \ldots, n$, is a Gaussian random variable. In other words, the law (finite-dimensional distributions) of the random vector $(X_{t_1}, \ldots, X_{t_n})^\intercal$ is Gaussian for any collection of $t_i \in \mathbb{T}$, $i = 1, \ldots, n$.

We denote the equality of finite-dimensional distributions by $X^1 \stackrel{d}{=} X^2$ of two Gaussian processes X^1 and X^2 which defines an equivalence class of equally distributed Gaussian processes. At this point, we note that a Gaussian process $X = (X_t)_{t\in\mathbb{T}}$ is uniquely determined by its mean function $m(t) = \mathbb{E}(X_t)$, $t \in \mathbb{T}$, and by its covariance function $R(t,s) = \mathbb{C}\mathrm{ov}(X_t, X_s)$, $s,t \in \mathbb{T}$. Conversely, for any function $m(t), t \in \mathbb{T}$, and for any symmetric non-negative definite function $R : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, there exists a unique (in law) Gaussian process having mean and covariance that coincide respectively with m and R on $\mathbb{T}$. Upon the non-negative

definiteness property, a Gaussian process is said to be *non-degenerate* if its covariance function is positive definite; otherwise, it is *degenerate*.
We now introduce some of the most important types of stochastic processes.

Definition 2.2. Let $X = (X_t)_{t\in\mathbb{T}}$ be a process, then

1. X is a stationary process if for all $h > 0$ such that $t + h \in \mathbb{T}, t \in \mathbb{T}$,

$$(X_{t+h})_{t\in\mathbb{T}} \overset{d}{=} (X_t)_{t\in\mathbb{T}}.$$

2. X is a self-similar process with index $\beta > 0$ (β-self-similar) if for all $a > 0$ such that $at \in \mathbb{T}, t \in \mathbb{T}$,

$$(X_{at})_{t\in\mathbb{T}} \overset{d}{=} a^\beta (X_t)_{t\in\mathbb{T}}.$$

3. X has stationary increment if for all $h > 0$ such that $t + h \in \mathbb{T}, t \in \mathbb{T}$,

$$(X_{t+h} - X_t)_{t\in\mathbb{T}} \overset{d}{=} (X_h - X_0)_{t\in\mathbb{T}}.$$

Self-similar processes are steadily connected to stationary processes by a deterministic time change. This relationship is expressed by the classical Lamperti transformation which builds a one-to-one correspondence between these two types of processes.

Lemma 2.3 (Lamperti, 1962)**.** *If $(Y_t)_{t\in\mathbb{R}}$ is a stationary process and for some $\beta > 0$*

$$X_t = t^\beta Y_{\log t}, \quad t \geq 0,$$

then $X = (X_t)_{t\geq 0}$ is β-self-similar process. Conversely, if $X = (X_t)_{t\geq 0}$ is a β-self-similar process, $\beta > 0$, and

$$Y_t = e^{-t\beta} X_{e^t}, \quad t \in \mathbb{R},$$

then, the process $Y = (Y_t)_{t\in\mathbb{R}}$ is stationary.

By the mean of Definition 2.1, the Gaussian class is invariant under the Lamperti transformation. For a Gaussian process $X = (X_t)_{t\in\mathbb{T}}$ with mean $m(t)$ and covariance $R(t, s)$, the stationarity asserts that $\mathbb{E}(X_{t+h}) = \mathbb{E}(X_t)$ and $R(t + h, t) = R(h, 0)$ for all t and h, which indicate that the mean function is constant and the covariance function depends only on the difference h. In case of self-similarity with an index $\beta > 0$, we have $\mathbb{E}(X_{at}) = a^\beta \mathbb{E}(X_t)$ and $R(at, as) = a^{2\beta} R(t, s)$, moreover, it follows that $X_0 = 0$ a.s. since $X_0 = X_{a.0} \overset{d}{=} a^\beta X_0$ for any $a > 0$.

Remark 2.4. A process $X \not\equiv 0$ cannot be self-similar and stationary at the same time, if such a process exits, then $\mathbb{E}(X_t X_s) = R(t - s, 0) = \mathbb{E}(X_{t-s} X_0) = 0$ and $\mathbb{E}(X_t) = t^\beta \mathbb{E}(X_1) =$ constant, for all t and s. This implies that $X \equiv 0$.

Theorem 2.5. *Let X be a Gaussian process with a covariance function R. X is Markov if and only if*

$$R(t,s) = \frac{R(t,u)R(s,u)}{R(u,u)}, \quad s \leq u \leq t.$$

Proof. See Kallenberg (1997, p. 204). □

2.2 Abstract Wiener integrals and related Hilbert spaces

Here and in what follows, we take $\mathbb{T} = [0, T]$ for a fixed finite time horizon $T > 0$. We recall some Hilbert spaces related to Gaussian processes.
First, we observe that under the norm $\|f\|_2 = (E(f^2))^{\frac{1}{2}}$, the Gaussian random variables are elements of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of (equivalence classes) square-integrable random variables on Ω, and the *Gaussian Hilbert space* associated with a Gaussian process $X = (X_t)_{t\in[0,T]}$ is to be defined as the first chaos

$$\mathcal{H}_X(T) := \overline{\text{span}}\{X_t; t \in [0,T]\} \subset L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

where the closure is in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.6. Let $t \in [0, T]$. The *linear space* $\mathcal{H}_X(t)$ is the Gaussian closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables X_s, $s \leq t$, i.e. $\mathcal{H}_X(t) = \overline{\text{span}}\{X_s; s \leq t\}$, where the closure is taken in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The linear space is a Gaussian Hilbert space with the inner product $\mathbb{C}\text{ov}[\cdot, \cdot]$.

Definition 2.7 (Mean-continuity)**.** A stochastic process $X = (X_t)_{t\in[0,T]}$ is said to be *mean-continuous* (or mean-square continuous) if $\mathbb{E}\left(|X_t - X_s|^2\right)$ converges to 0 when t tends to s.

The mean-continuity can be well seen as the continuity in t of the curve generated by X_s, $s \leq t$, in the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. On the other hand, the mean-continuity of a Gaussian process $X = (X_t)_{t\in[0,T]}$ with covariance function R is equivalent to the continuity of $R(\cdot, \cdot)$ at the diagonal (t, t) for any $t \in [0, T]$, and by Loève (1978, p. 136), this shall imply that $R(\cdot, \cdot)$ is continuous at every $(t, s) \in [0, T]^2$.
For a mean-continuous stationary Gaussian process $X = (X_t)_{t\in\mathbb{R}}$ with zero-mean and covariance function $R(t - s) = \mathbb{E}(X_t X_s)$, the Bochner theorem asserts that R admits the representation

$$R(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, \mathrm{d}\Delta(\lambda), \tag{2.1}$$

with a unique positive, symmetric finite measure Δ called the *spectral measure* of the stationary Gaussian process X. As pointed out by Doob (1990), if the covariance R is integrable, there exits a continuous *spectral density* $f(\lambda) = \frac{\mathrm{d}\Delta}{\mathrm{d}\lambda}(\lambda)$ which is the inverse Fourier transform of R. We have

$$R(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} f(\lambda)\,\mathrm{d}\lambda = \int_{\mathbb{R}} e^{i\lambda t}(e^{i\lambda s})^* f(\lambda)\,\mathrm{d}\lambda, \tag{2.2}$$

where $(e^{i\lambda s})^*$ is the complex conjugate of $e^{i\lambda s}$. A good account of the spectral representations can be found in Ibragimov & Rozanov (1970) and Yaglom (1962). Another essential approach related to the linear spaces is the construction of the Wiener integral with respect to X.

Definition 2.8. Let $t \in [0,T]$. The *abstract Wiener integrand space* $\Lambda_t(X)$ is the completion of the linear span of the indicator functions $1_s := 1_{[0,s)}$, $s \leq t$, under the inner product $\langle\cdot,\cdot\rangle$ extended bilinearly from the relation

$$\langle 1_s, 1_u\rangle = R(s,u).$$

The elements of the abstract Wiener integrand space are equivalence classes of Cauchy sequences $(f_n)_{n=1}^\infty$ of piecewise constant functions. The equivalence of $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ means that

$$\|f_n - g_n\| \to 0, \text{ as } n \to \infty,$$

where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.
The space $\Lambda_t(X)$ is isometric to $\mathcal{H}_t(X)$. Indeed, the relation

$$\mathcal{I}_t^X[1_s] := X_s, \quad s \leq t, \tag{2.3}$$

can be extended linearly into an isometry from $\Lambda_t(X)$ onto $\mathcal{H}_t(X)$.

Definition 2.9. The isometry $\mathcal{I}_t^X : \Lambda_t(X) \to \mathcal{H}_t(X)$ extended from the relation (2.3) is the *abstract Wiener integral*. We denote

$$\int_0^t f(s)\,\mathrm{d}X_s := \mathcal{I}_t^X[f].$$

2.3 Regularity of Gaussian processes

For a fixed ω, a stochastic process is viewed as a function $X : [0,T] \to \mathbb{R}$ called the *sample path* or the *trajectory* of the process, and we say that a process $X = (X_t)_{t\in[0,T]}$ has continuous sample paths, if the function $X(,\omega)$ is continuous on $[0,T]$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Although the Gaussian processes are uniquely determined in terms of finite-dimensional distributions, this does not suffice to characterize the regularity of their paths, thus, it is natural to put conditions on the sample paths as well as on the finite-dimensional distributions. First, we review some previous works related to the continuity of Gaussian processes.

2.3.1 Earlier works

In prior results, we recite the work of Fernique (1964) where a sufficient condition for the sample paths of a Gaussian process $X = (X_t)_{t\in[0,T]}$ is expressed in terms of incremental variance, i.e. $\mathbb{E}(X_t - X_s)^2$, by assuming that $\mathbb{E}(X_t - X_s)^2 \le \Psi(t-s)$ where Ψ is a nondecreasing function on $[0,\epsilon]$ for some $\epsilon > 0$ and $0 \le s \le t \le \epsilon$, and such that the integral

$$\int_0^{\epsilon} \frac{\Psi(u)}{u(\log u)^{\frac{1}{2}}}\,\mathrm{d}u$$

is finite. In this case, X has continuous sample paths with probability one. Another geometric approach to find a sufficient condition is due to Dudley (1967, 1973) where he employs the *metric entropy* of $[0,T]$, that is, $H(\epsilon) = \log N(\epsilon)$ where $N(\epsilon)$ represents to the smallest number of closed balls of radius ϵ covering $[0,T]$ in the pseudo-metric $d_X(s,t) = (\mathbb{E}(X_t - X_s)^2)^{\frac{1}{2}}$, $s,t \in [0,T]$. The Dudley condition reads

$$\int_0^{\infty} (\log N(\epsilon))^{\frac{1}{2}}\,\mathrm{d}\epsilon < \infty.$$

The Dudley sufficient condition for the continuity of Gaussian processes turn-out to be also necessary in the case of stationary Gaussian processes, c.f. Marcus & Rosen (2006, chap 6.) and Kahane (1985, p. 212). Additionally to this case, we mention the Belyaev dichotomy of stationary Gaussian processes known as Belyaev alternative which shows that a stationary Gaussian process is either continuous a.s. or unbounded a.s. on every compact interval, see Belyaev (1961).
To obtain necessary and sufficient condition, Talagrand (1987) introduces the concept of *majorizing measure*. A probability measure μ defined on the space $([0,T], d_X)$ is called a majorizing measure if

$$\sup_{t\in[0,T]} \int_0^{\infty} \left(\log \frac{1}{\mu(B_{d_X}(t,\epsilon))}\right) \mathrm{d}\epsilon < \infty,$$

where $B_{d_X}(t,\epsilon)$ is the closed ball with radius ϵ and center t in the intrinsic pseudo-metric d_X. Then a Gaussian process $X = (X_t)_{t\in[0,T]}$ is continuous a.s. if and only if there exits a majorizing measure μ on $([0,T], d_X)$ such that

$$\lim_{\delta\to 0} \sup_{t\in[0,T]} \int_0^{\delta} \left(\log \frac{1}{\mu(B_{d_X}(t,\epsilon))}\right) \mathrm{d}\epsilon = 0.$$

2.3.2 Hölder continuity

In the theory of stochastic processes, we often use the Hölder scale to quantify the regularity of the paths of a process. A simpler sufficient condition to guarantee the almost certain continuity of the sample paths of Gaussian processes would be the Hölder continuity of the covariance with any order greater than zero.

Definition 2.10 (Hölder-continuity). A stochastic process $X = (X_t)_{t\in[0,T]}$ is Hölder continuous of order $\gamma \in [0,1]$ if there exists a finite positive random variable h such that

$$\sup_{s,t\in[0,T];s\neq t} \frac{|X_t - X_s|}{|t-s|^\gamma} \leq h,$$

almost surely.

The most useful tool to study the Hölder regularity is certainly the famous the following Kolmogorov-Čentsov criterion which represents a sufficient condition.

Theorem 2.11 (Kolmogorov-Čentsov). *If a stochastic process $X = (X_t)_{t\in[0,T]}$ satisfies*

$$E\left(|X_t - X_s|^\alpha\right) \leq C\,|t-s|^{1+\delta}, \quad s,t \in [0,T] \tag{2.4}$$

for some $\alpha > 0$, $\delta > 0$ and $C > 0$, then there exists a continuous modification of X which is Hölder continuous of any order $a < \frac{\delta}{\alpha}$.

For the Gaussian case, we have the following corollary:

Corollary 2.12. *Let $X = (X_t)_{t\in[0,T]}$ be a Gaussian process and suppose that there exists a constant C such that*

$$\mathbb{E}\left(|X_t - X_s|^2\right) \leq C\,|t-s|^{2\alpha}, \quad s,t \in [0,T], \tag{2.5}$$

then X has a continuous modification which is Hölder continuous of order $a < \alpha$.

Proof. Since $X_t - X_s$ is Gaussian, it follows from (2.5) that

$$E\left(|X_t - X_s|^p\right) \leq C^p\,|t-s|^{\alpha p}$$

holds for every $p \geq 1$. By Kolmogorov-Čentsov criterion (2.4), X has a continuous modification which is Hölder continuous of order $a < \alpha - \frac{1}{p}$. □

Theorem 2.13. *Let $X = (X_t)_{t\in[0,T]}$ be a β-self-similar and H-Hölder, then $H \leq \beta$.*

Proof. We have

$$\sup_{0\leq s,t\leq T} \frac{|X_t - X_s|}{|t-s|} \geq \sup_{0\leq t\leq T} \frac{|X_t|}{t^H} \stackrel{d}{=} \sup_{0\leq t\leq T} |X_1| \frac{t^\beta}{t^H} = \infty$$

if $H \geq \beta$. □

2.4 Examples

Here we give two interesting and well-studied Gaussian processes : the fractional Brownian motion and the fractional Ornstein-Uhlenbeck processes.

2.4.1 Fractional Brownian motion

The fractional Brownian motion is seen as a generalization of standard Brownian motion with a dependence structure of the increments and the memory of the process. In many applications, empirical data exhibit a so-called *long-range dependence* structure, i.e. the process behaviour after a given time t not only rely on the state of the process at t but also depends on the whole history up to time t. To describe this behaviour, Mandelbrot & van Ness (1968) used a process that they called fractional Brownian motion. However, this process was introduced earlier by Kolmogorov in 1940 as model to study turbulence in fluids. See Molchan (2003) for full historical account, and Biagini et al. (2008), Embrechts & Maejima (2002), Mishura (2008) and Samorodnitsky & Taqqu (1994) for more details on fractional Brownian motion.
A zero mean Gaussian process $B^H = (B^H)_{t\geq 0}$ is a fractional Brownian motion with Hurst index $H \in (0,1)$ if its covariance function $R(t,s)$ has the form

$$R(t,s) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{2.6}$$

Remark 2.14. If $H = \frac{1}{2}$, B^H is the standard Brownian motion, and if $H = 1$, we have $R(t,s) = ts$ or equivalently $B^1 = t\xi$ a.s. for some standard normal random variable ξ.

The fractional Brownian motion is H-self-similar and has stationary increments. Moreover, it has a continuous modification which is Hölder continuous of any order $a < H$. Indeed, the covariance function (2.6) satisfies $R(at,as) = a^{2H}R(t,s)$ for any $a > 0$. Else, we have $\mathbb{E}|B_t^H - B_s^H|^2 = |t-s|^{2H}$, and by Proposition 3 (Lifshits, 1995, Sec.4), B^H has stationary increments. The Hölder continuity follows directly from Corollary 2.12.

Definition 2.15. Let $(\eta_n)_{n\geq 0}$ be a stationary sequence of random variables. $(\eta_n)_{n\geq 0}$ exhibits long-range dependence if its correlation function $\rho(n)$ satisfies

$$\sum_{n=0}^{\infty} \rho(n) = \infty.$$

If $\sum_{n=0}^{\infty} \rho(n) < \infty$, then $(\eta_n)_{n\geq 0}$ exhibits short-range dependence.

From the stationarity of increments of the fractional Brownian motion, it follows that the sequence $(B_n^H - B_{n-1}^H)_{n\in\mathbb{N}}$ which is called *fractional Gaussian noise* is stationary. Denote its autocovarinace function by

$$\rho^H(n) := \mathbb{C}\mathrm{ov}(B_n^H - B_{n-1}^H, B_1^H - B_0^H),$$

we have $\rho^H(n) \sim H(2H-1)n^{2H-2}$. Therefore, if $H > \frac{1}{2}$, it holds that $\rho^H(n) > 0$ and $\sum_{n\in\mathbb{N}} |\rho^H(n)| = \infty$ which means that the increments of the corresponding fractional Brownian motion exhibits a long-range dependence. If $H < \frac{1}{2}$, we have $\rho^H(n) < 0$ and $\sum_{n\in\mathbb{N}} |\rho^H(n)| < \infty$, and in this case the increments exhibits a short-range dependence. When $H = \frac{1}{2}$, it has independent increments since it is a standard Brownian motion.

As a consequence of Theorem 2.5, the fractional Brownian motion is Markovian if and only if $H = \frac{1}{2}$. Note that it is a semimartingale if and only if $H = \frac{1}{2}$, see for instance Biagini et al. (2008).

The representation of fractional Brownian motion as a Wiener integral has been considered by many authors. For a one-sided fractional Brownian motion, we recall the Molchan & Golosov (1969) representation

$$B_t^H = \int_0^t k_H(t,s)\,\mathrm{d}W_s, \quad t \geq 0,$$

where the deterministic kernel $k(t,s)$ is the fractional integral of the form of

$$\begin{aligned} k_H(t,s) &= c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}}\, u^{H-\frac{1}{2}}\,\mathrm{d}u, \quad \text{for } H > \frac{1}{2}, \\ k_H(t,s) &= d_H \Bigg(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \\ &\quad - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}}\,(u-s)^{H-\frac{1}{2}}\,\mathrm{d}u \Bigg), \quad \text{for } H < \frac{1}{2}, \end{aligned}$$

where $c_H = \left(\frac{H(2H-1)}{B(2-2H,H-\frac{1}{2})}\right)^{\frac{1}{2}}$, $d_H = \left(\frac{2H}{(1-2H)B(1-2H,H+\frac{1}{2})}\right)^{\frac{1}{2}}$. B denotes the Beta function.

2.4.2 *Ornstein-Uhlenbeck processes*

One of the most natural example of stationary Gaussian processes is the classical *Ornstein-Uhlenbeck process* which is derived from the Brownian motion by Lamperti transformation, see Cheridito et al. (2003), Embrechts & Maejima (2002), Kaarakka & Salminen (2011) and Lifshits (1995) for more details.

A stationary Gaussian process $(Y_t)_{t\in\mathbb{R}}$ is a Ornstein-Uhlenbeck process if it is continuous, has a zero mean and covariance

$$\mathbb{E}(Y_t^\alpha Y_s^\alpha) = \frac{1}{2\alpha} e^{-\alpha|t-s|}, \tag{2.7}$$

where $\alpha > 0$. It is given by the Lamperti transformation

$$Y_t^\alpha = e^{-\alpha t} W_{a_t},$$

where $W = (W_t)_{t\in\mathbb{R}}$ is two-sided Brownian motion and $a_t = \frac{e^{2\alpha t}}{2\alpha}$. The Ornstein-

Uhlenbeck process can be obtained as a solution of the *Langevin equation*

$$\mathrm{d}Y_t^\theta = -\theta Y_t^\theta \,\mathrm{d}t + \mathrm{d}W_t,$$

which is expressed as

$$Y_t^\theta = \int_{-\infty}^t e^{-\theta(t-s)}\,\mathrm{d}W_s.$$

By checking the covariances, it is easy to see that for $\alpha = \frac{1}{2\theta}$ the processes Y^α and Y^θ are equivalent in law. Nevertheless, it has been proven in Cheridito et al. (2003) and Kaarakka & Salminen (2011) that these two stationary Gaussian processes are not the same if we replace the Brownian motion with a fractional Brownian motion, as they exhibit different dependence structure of the increments.

3 INTEGRAL REPRESENTATIONS OF GAUSSIAN PROCESSES

In this section, we introduce the integral representations of *Fredholm* and *Volterra* types that express the Gaussian processes in terms of the standard Brownian motion. The terminologies of these two types are devoted to the Fredholm and the Volterra integral operators from the theory of integral equations; further properties and applications can be found in Gohberg & Kreĭn (1969, 1970) and Corduneanu (1991).

Definition 3.1 (Fredholm & Volterra representations)**.** Let $X = (X_t)_{t\in[0,T]}$ be a Gaussian process. We call a *Fredholm representation* of X the integral representation of the form

$$X_t = \int_0^T F(t,s)\,\mathrm{d}W_s, \quad 0 \le t \le T, \tag{3.1}$$

where W is a standard Brownian motion and $F \in L^2([0,T]^2)$. If the kernel F is of Volterra type, i.e., $F(t,s) = 0$ when $t < s$, then the representation (3.1) is called a *Volterra* representation of X and we write

$$X_t = \int_0^t F(t,s)\,\mathrm{d}W_s, \quad 0 \le t \le T. \tag{3.2}$$

Denote by $(\mathcal{F}_t^X)_{t\in[0,T]}$ and $(\mathcal{F}_t^W)_{t\in[0,T]}$ the complete filtration of X and W respectively. The difference between the Fredholm and the Volterra representation is that for the construction of X in (3.1) at any point t, one needs the entire path of the underlying Brownian motion W up to time T, i.e., $\mathcal{F}_t^X \subset \mathcal{F}_T^W$, or equivalently $\mathcal{H}_X(t) \subset \mathcal{H}_W(T)$, while in (3.2) the process X at t is generated from the path of W up to t, and which indicates that $\mathcal{F}_t^X \subset \mathcal{F}_t^W$ and $\mathcal{H}_X(t) \subset \mathcal{H}_W(t)$. The interesting case of Volterra representation is when the filtrations coincide (see Definition 3.4 below), in this special case, the representation is dynamically invertible in the sense that the linear spaces $\mathcal{H}_X(t)$ and $\mathcal{H}_W(t)$ are the same at every time t which means that the processes X and W can be constructed from each others without knowing the future-time development of X or W.

The following theorem states that the Fredholm representation of a Gaussian process X exists always under the sufficient and the necessary condition of *trace property* of its covariance.

Theorem 3.2. *Let $X = (X_t)_{t\in[0,T]}$ be a Gaussian process with covariance function R. Then, X admits a Fredholm representation if and only if the covariance R is of trace class, i.e,*

$$\int_0^T R(t,t)\,\mathrm{d}t < \infty.$$

Proof. A complete proof is illustrated in Sottinen & Viitasaari (2014). □

Remark 3.3. The representation is unique in the sense that for any another representation with a kernel F', we have $F'(t,\cdot) = UF(t,\cdot)$ where U is a unitary operator on $L^2([0,T])$.

Definition 3.4 (Canonical representation)**.** The Volterra representation (3.2) is said to be *canonical* if it satisfies

$$\mathcal{F}_t^X = \mathcal{F}_t^W, \quad 0 \le t \le T.$$

An equivalent to the canonical property is that if there exists a random variable $\eta = \int_0^T \phi(s)\,\mathrm{d}W_s$, $\phi \in L^2([0,T])$, such that it is independent of X_t for all $0 \le t \le T$, i.e. $\int_0^t F(t,s)\,\phi(s)\,\mathrm{d}s = 0$, one has $\phi \equiv 0$. This means that the family $\{F(t,\cdot), 0 \le t \le T\}$ is linearly independent and spans a vector space that is dense in $L^2([0,T])$. If we associate with the canonical kernel F a Volterra integral operator $\mathscr{F}$ defined on $L^2([0,T])$ by $\mathscr{F}\phi(t) = \int_0^t F(t,s)\,\phi(s)\,\mathrm{d}s$, it follows from the canonical property that $\mathscr{F}$ is injective and $\mathscr{F}(L^2([0,T]))$ is dense in $L^2([0,T])$. The covariance integral operator, denoted by $\mathscr{R}$, which is associated with the kernel $R(t,s)$ has the decomposition $\mathscr{R} = \mathscr{F}\mathscr{F}^*$, where $\mathscr{F}^*$ is the adjoint operator of $\mathscr{F}$. In this case, the covariance R is *factorizable* and has the factorization

$$R(t,s) = \int_0^{t\wedge s} F(t,u)F(s,u)\,\mathrm{d}u, \quad 0 \le t,s \le T. \tag{3.3}$$

Here we would like to note that in the works of Lévy (1956a,b, 1957) which marked the beginning of the theory of the Volterra representation of Gaussian processes, Lévy introduction of this concept was motivated by solving the non-linear integral equation (3.3) within the Hilbert space settings.

Example 3.5 (Lévy's problem)**.** In a counter-example to the canonicity introduced by Lévy (1957) , we consider the Gaussian process represented by

$$X_t = \int_0^t \left\{3 - 12\frac{u}{t} + 10\left(\frac{u}{t}\right)^2\right\} \mathrm{d}W_u, \quad 0 \le t \le T, \tag{3.4}$$

and the random variable $\eta = \int_0^T s\,\mathrm{d}W_s$. It is easy to see that η is independent of X_t for all t, and therefore, the representation (3.4) is not canonical. Notice that the Gaussian process (3.4) is self-similar with index 0. For this particular problem, there has been a discussion in Long (1968) where the author generalizes the results of Lévy and Hida & Hitsuda on the canonical representations by endowing the linear space with the scale invariant measure $\mathrm{d}m(u) = \frac{\mathrm{d}u}{u}$, instead of the Lebesgue measure $\mathrm{d}u$.

Unlike the Fredholm representation, the canonical Volterra representation requires more assumptions. One of these representations that have been heavily studied in the literature is that of stationary Gaussian processes, see Cramér & Leadbetter (1967), Doob (1990), Dym & McKean (1976), Hida & Hitsuda (1993), Ibragimov & Rozanov (1970) and Karhunen (1950).

Definition 3.6 (PND)**.** Let $\mathbb{T} \subseteq \mathbb{R}$ and consider a finite second moments process $Z = (Z_t)_{t\in\mathbb{T}}$. Let $\mathcal{H}_Z(t)$ be the closed linear L^2-subspace generated by the random variables $Z_s, s \leq t$. Then Z is said to be *purely non-deterministic* when the condition

$$\bigcap_t \mathcal{H}_Z(t) = \{0\} \tag{C}$$

is satisfied, where $\{0\}$ denotes the L^2–subspace spanned by the constants. If

$$\bigcap_t \mathcal{H}_Z(t) = \mathcal{H}_Z(\mathbb{T}),$$

Z is said to be *deterministic*.

The above definition is due to Cramér (1961b) in general L^2-processes framework, where the condition (C) emphasizes that the remote past $\bigcap_t \mathcal{H}_Z(t)$ of process Z is trivial and does not contain any information at all. The most interesting case where this property fails is devoted to the Gaussian process $X_t = t\xi$ where ξ is a standard normal random variable, here we have $\bigcap_t \mathcal{H}_X(t) = \mathrm{span}\{\xi\}$ which is not a trivial.

Remark 3.7. As the remote past assigns the L^2-processes to the determinism or the pure non-determinism, or to both as in the so-called by Lévy the *mixed processes*, this two extreme cases play an important role for decomposition of stationary Gaussian processes. Here we recall the *Wold decomposition* of a given discrete-time stationary process (not necessarily Gaussian) with finite second moments. Following Wold (see e.g. Wold, 1954), an L^2-stationary process $(X_n, n \in \mathbb{Z})$ has a unique decomposition $X_n = X'_n + X''_n$ where X'_n and X''_n are two stationary uncorrelated processes such that $(X'_n, n \in \mathbb{Z})$ is purely non-deterministic and $(X''_n, n \in \mathbb{Z})$ is deterministic. Generalization of Wold decomposition to the continuous-time as well as to the multivariate case has been done by Cramér (1961a) and Hanner (1950).

Proposition 3.8. *Let $Y = (Y_t)_{t\in\mathbb{R}}$ be a stationary Gaussian process and let $X = (X_t)_{t\geq 0}$ be a β-self-similar Gaussian process associated to Y through Lamperti transformation. Then, Y is purely non-deterministic if and only if X is so too.*

Proof. Since $X_t = t^\beta Y_{\log t}$ for all $t \geq 0$, the claim follows from the equality:

$$\bigcap_{t\geq 0} \mathcal{H}_X(t) = \bigcap_{t\geq 0} \mathcal{H}_Y(\log t) = \bigcap_{t\in\mathbb{R}} \mathcal{H}_Y(t).$$

□

Theorem 3.9 (Canonical representation of stationary Gaussian processes)**.** *Let $X = (X_t)_{t\in\mathbb{R}}$ be a mean-continuous stationary Gaussian process. Then, X admits a canonical Volterra representation if and only if it is purely non-deterministic. In this case, X is given by the canonical Volterra representation*

$$X_t = \int_{-\infty}^t G(t-s)\,\mathrm{d}W_s, \quad t \in \mathbb{R},$$

where $G \in L^2(\mathbb{R})$ such that $G(u) = 0$ for all $u < 0$, and $W = (W_t)_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion.

Proof. For the proof, see e.g. Karhunen (1950), Hida & Hitsuda (1993) or Dym & McKean (1976). □

Remark 3.10. The proof of Theorem 3.9 was framed under the mean of complex analysis and Hardy spaces calculus which are beyond the scope of this thesis. However, we would briefly mention that the canonical kernel $G(t-s)$ is constructed via the spectral representation of the covariance $R(t-s) = \mathbb{E}(X_t X_s)$. By Szegö-Kolmogorov theorem (see e.g. Nikol'skiĭ (1986)), the property of pure non-determinism is equivalent to the finiteness of the integral $\int_{\mathbb{R}} \frac{\log f(\lambda)}{1+\lambda^2}\,\mathrm{d}\lambda$ where f is the spectral density of the stationary Gaussian process X; else, a result of Rozanov (Dym & McKean, 1976) shows that the density function f in this case admits the factorization $f(\lambda) = |g(\lambda)|^2$ where g is an *outer* function belonging to the Hardy space H^{2+} of analytic functions in the upper half-plane. The classical Paley-Wiener theorem ensures then the existence of a function $G \in L^2([0,\infty))$ with $G(u) = 0$ for $u < 0$ such that g is the Fourier transform of G. Therefore, $R(t-s) = \int_{\mathbb{R}} G(t-u)G(s-u)\,\mathrm{d}u$. To check the canonicity, we suppose that $G \star h = 0$ for some $h \in L^2(\mathbb{R}_+)$. Then $g\hat{h} = 0$ on the upper-half plane, where $\hat{h}$ is the Fourier transform of h, hence, $\int_0^\infty g(\omega)e^{i\omega t}h(t)\,\mathrm{d}t = 0$ which implies that $h = 0$, since the family $\{g(\omega)e^{i\omega t}, t \geq 0\}$ is dense in H^{2+} by Lax theorem (see e.g. Nikol'skiĭ (1986)).

4 GAUSSIAN BRIDGES

Let $X = (X_t)_{t\in[0,T]}$ be a continuous Gaussian process with covariance function R, mean function m, and $X_0 = 0$, defined on the canonical filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = C([0,T])$, $\mathcal{F}$ is the Borel σ-algebra on $C([0,T])$ with respect to the uniform topology, and $\mathbb{P}$ is the probability measure with respect to which the coordinate process $X_t(\omega) = \omega(t), \omega \in \Omega, t \in [0,T]$, is a centered Gaussian process. Recall that a *bridge measure* $\mathbb{P}^T$ is the regular conditional law

$$\mathbb{P}^T = \mathbb{P}^T\left[X \in \cdot\right] = \mathbb{P}\left[X \in \cdot | X_T = \theta\right], \quad \theta \in \mathbb{R}, \tag{4.1}$$

and a process $X^T = (X_t^T)_{t\in[0,T]}$ is called a *bridge* of X from 0 to θ if it is defined up to distribution in the sense that

$$\mathbb{P}\left[X^T \in \cdot\right] = \mathbb{P}^T\left[X \in \cdot\right] = \mathbb{P}\left[X \in \cdot | X_T = \theta\right], \tag{4.2}$$

with $X_0^T = 0$ and $X_T^T = X_T$ almost surely. Note that we condition on a set of zero measure and that $\mathbb{P}^T(X_T = \theta) = 1$, however, the regular conditional distribution always exists in the Polish space of continuous functions on $[0,T]$, see Shiryaev (1996, p. 227–228).

The bridge X^T can be interpreted as the original process X with an added information drift that bridges the process at the final time T. On the other hand, the bridge can be understood from the initial enlargement of filtration point of view. This dynamic drift interpretation should turn out to be useful in applications such the insider trading in finance. On earlier work related to Gaussian bridges, we mention Baudoin (2002), Baudoin & Coutin (2007) and Gasbarra et al. (2007). One may also refer to Chaleyat-Maurel & Jeulin (1983) and Jeulin & Yor (1990) for more details on the enlargement of filtrations, and to Amendiger (2000), Imkeller (2003) and Gasbarra et al. (2006) for its applications in finance. Furthermore, we would like also to mention other results by Campi et al. (2011) Chaumont & Uribe Bravo (2011) Hoyle et al. (2011) on Markovian and Lévy bridges.

From the definitions (4.1) and (4.2), it is clear that the bridge X^T is Gaussian since the conditional laws of Gaussian processes are Gaussian. Among this class, the Brownian bridge was the most extensively studied bridge, it is given by the equation

$$W_t^T = W_t - \frac{t}{T} W_T, \quad 0 \le t \le T, \tag{4.3}$$

where the conditioning is on the final value $W_T = 0$. The representation (4.3) is called the *orthogonal representation* of the Brownian bridge and it is deduced from the orthogonal decomposition of W with respect to W_T, that is,

$$W_t = \left(W_t - \frac{t}{T}W_T\right) + \frac{t}{T}W_T,$$

where the bridge $W^T = (W_t^T)_{t\in[0,T]}$ has the same law as the conditioned process

$(W_t|W_T = 0)_{t\in[0,T]}$. More generally, the orthogonal representation of the Gaussian bridge X^T is the well-known representation (see Gasbarra et al. (2007)) :

$$X_t^T = \theta\frac{R(T,t)}{R(T,T)} + X_t - \frac{R(T,t)}{R(T,T)}X_T, \quad 0 \le t \le T, \tag{4.4}$$

with mean $\mathbb{E}(X^T) = \theta\frac{R(T,t)}{R(T,T)} + m(t) - \frac{R(T,t)}{R(T,T)}m(T)$ and covariance

$$\mathbb{C}\mathrm{ov}(X_t^T, X_s^T) = R(t,s) - \frac{R(T,t)R(T,s)}{R(T,T)}.$$

Remark 4.1. The covariance is independent of θ and $m(t)$, so, in what follows we may assume that $\theta = 0$ and $m(t) = 0$.

Example 4.2 (Fractional Brownian bridge)**.** If X is a centered fractional Brownian motion with Hurst $H \in (0,1)$ and covariance $R(t,s) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$, the fractional Brownian bridge from 0 to 0 on the interval $[0,T]$ is the process

$$X_t^T = X_t - \frac{t^{2H} + T^{2H} - |t-T|^{2H}}{2T^{2H}}X_T, \quad 0 \le t \le T.$$

The orthogonal representation of the bridge is characterized by

$$\mathcal{H}_{X^T}(t) \perp \mathrm{span}\{X_T\} = \mathcal{H}_X(t), \quad 0 \le t \le T,$$

where $\perp$ indicates the orthogonal direct sum, therefore, the representation is not canonical since the linear spaces $\mathcal{H}_X(t)$ and $\mathcal{H}_{X^T}(t)$ does not coincide at any time $t \in [0,T]$, and moreover, their natural filtrations $\mathcal{F}^{X^T}$ and $\mathcal{F}^X$ are not the same. But the initially $\sigma(X_T)$-enlarged filtrations $\mathcal{F}^{X^T} \vee \sigma(X_T)$ and $\mathcal{F}^X \vee \sigma(X_T)$ are . As a naturally related question, this motivates to write the canonical representation of the bridge in its own filtration, which is surely a different bridge process than (4.4). Indeed, for the case when $X = M$ is a continuous martingale with $M_0 = 0$ and strictly increasing bracket $\langle M\rangle$ for which we have $R(t,s) = \langle M\rangle_{t\wedge s}$, the key feature to the canonical form of the bridge M^T is to use the Girsanov theorem since we have $\mathbb{P}_t^T \sim \mathbb{P}_t$ for all $t \in [0,T)$ and $\mathbb{P}_T^T \perp \mathbb{P}_T$, where $\mathbb{P}_t^T$ and $\mathbb{P}_t$ are the restriction of $\mathbb{P}^T$ and $\mathbb{P}$ on the filtration $\mathscr{F}_t$. As pointed out in Gasbarra et al. (2007), the Girsanov theorem leads to the stochastic differential equation:

$$\mathrm{d}M_t = \mathrm{d}M_t^T - \int_0^t l(t,s)\,\mathrm{d}M_s^T\,\mathrm{d}\langle M\rangle_t, \quad 0 \le t < T, \tag{4.5}$$

where $l(t,s) = -\frac{1}{\langle M\rangle_t - \langle M\rangle_s}$. In addition, $\langle M^T\rangle_t = \langle M\rangle_t$ and $\mathscr{F}_t^{M^T} = \mathscr{F}_t^M$ for all $t \in [0,T)$.

The equation (4.5) can be viewed as the *Hitsuda representation* between two equivalent Gaussian processes (c.f. Hitsuda, 1968). By this fact, the Hitsuda representation (4.5) can be inverted to express the solution M^T in terms of M by taking the kernel $l^*(t,s)$ that satisfies the *resolvent equation*

$$l(t,s) + l^*(t,s) = \int_s^t l(t,u)\, l^*(u,s)\, \mathrm{d}\langle M\rangle_u. \tag{4.6}$$

The existence and the uniqueness of the resolvent kernel follows from the theory of integral equations for a given square-integrable kernel, namely $l(t,s)$, (see e.g. Yosida, 1991, p.118), and the resolvent equation is understood as a necessary and sufficient condition to construct the solution M^T in terms of M. In this regard, the Hitsuda representation is unique in the sense that if there exist another canonical representation $\mathrm{d}M_t = \mathrm{d}\widetilde{M}_t - \int_0^t \tilde{l}(t,s)\, \mathrm{d}\widetilde{M}_s\, \mathrm{d}\langle M\rangle_t$, $0 \le t < T$, then $M^T = \widetilde{M}$ and $l(t,s) = \tilde{l}(t,s)$ for almost all $t, s \in [0,T)$.

Theorem 4.3. *The process M^T defined as*

$$\mathrm{d}M_t^T = \mathrm{d}M_t - \int_0^t l^*(t,s)\, \mathrm{d}M_s\, \mathrm{d}\langle M\rangle_t, \quad 0 \le t < T, \tag{4.7}$$

where $l^*(t,s)$ is the kernel defined in (4.6)*, is a bridge of M.*

Proof. Equation (4.7) is the solution to (4.5) if and only if the kernel l^* satisfies the resolvent equation. Indeed, suppose (4.7) is the solution to (4.5). This means that

$$\begin{aligned}\mathrm{d}M_t &= \left(\mathrm{d}M_t - \int_0^t l^*(t,s)\, \mathrm{d}M_s\, \mathrm{d}M\rangle_t\right) \\ &\quad - \int_0^t l(t,s) \left(\mathrm{d}M_s - \int_0^s l^*(s,u)\, \mathrm{d}M_u\, \mathrm{d}\langle M\rangle_s\right) \mathrm{d}\langle M\rangle_t,\end{aligned}$$

or, in the integral form, by using the Fubini's theorem,

$$\begin{aligned}M_t &= M_t - \int_0^t \int_s^t l^*(u,s)\ \mathrm{d}\langle M\rangle_u \mathrm{d}M_s \\ &\quad - \int_0^t \int_s^t l(u,s)\ \mathrm{d}\langle M\rangle_u \mathrm{d}M_s \\ &\quad + \int_0^t \int_s^t \int_u^s l(s,v) l^*(v,u) \mathrm{d}\langle M\rangle_v\ \mathrm{d}\langle M\rangle_u \mathrm{d}M_s.\end{aligned}$$

The resolvent criterion (4.6) follows by identifying the integrands in the $\mathrm{d}\langle M\rangle_u \mathrm{d}M_s$-integrals above. □

The representation (4.7) specifies the Doob-Meyer decomposition of M^T as a semi-martingale with respect to its own filtration. In the particular example of Brownian motion , the canonical representation of the Brownian bridge conditioned by $W_T = 0$ is given by

$$W_t^T = W_t - \int_0^t \int_0^s \frac{1}{T-u}\, \mathrm{d}W_u\, \mathrm{d}s = \int_0^t \frac{T-t}{T-s}\, \mathrm{d}W_s, \tag{4.8}$$

for all $t \in [0, T)$.

Remark 4.4. The singularity at time T between the bridge law and the the underlying process law can be seen from the kernel $\frac{1}{T-u}$ in (4.8) which loses its square-integrability at time T. In general, when the kernel $l(t, s)$ is singular, the corresponding resolvent kernel $l^*(t, s)$ is also singular, see for instance Alili & Wu (2009) and Wu & Yor (2002) for further details on the topic.

5 SUMMARIES OF THE ARTICLES

I. Necessary and sufficient conditions for Hölder continuity of Gaussian processes

In this article we reproduce the Kolmogorov-Čentsov criterion to give a simple necessary and sufficient condition for the Hölder continuity of Gaussian processes. However, this condition is restricted to Gaussian processes. Let $X = (X_t)_{t\in[0,T]}$ be a centered Gaussian process and define

$$\begin{aligned} d_X^2(\tau,\tau') &:= \mathbb{E}[(X_\tau - X_{\tau'})^2], \\ \sigma_X^2(\tau) &:= \mathbb{E}[X_\tau^2]. \end{aligned}$$

Our main result is the following:

Theorem 5.1. *The Gaussian process X is Hölder continuous of any order $a < H$ i.e.*

$$|X_t - X_s| \le C_\epsilon |t-s|^{H-\epsilon}, \quad \textit{for all } \epsilon > 0 \tag{5.1}$$

if and only if there exists constants c_ϵ such that

$$d_X(t,s) \le c_\epsilon |t-s|^{H-\epsilon}, \quad \textit{for all } \epsilon > 0. \tag{5.2}$$

Moreover, the random variables C_ϵ in (5.1) satisfy

$$\mathbb{E}\left[\exp\left(aC_\epsilon^\kappa\right)\right] < \infty \tag{5.3}$$

for any constants $a \in \mathbb{R}$ and $\kappa < 2$; and also for $\kappa = 2$ for small enough positive a. In particular, the moments of all orders of C_ϵ are finite.

The "if" part is obvious since it follows from the Kolmogorov-Čentsov criterion. For the "only if' part by we need to use the following lemma which is a characterization of Gaussian processes.

Lemma 5.2. *Let $\xi = (\xi_\tau)_{\tau\in\mathbb{T}}$ be a centered Gaussian family. If $\sup_{\tau\in\mathbb{T}} |\xi_\tau| < \infty$ then $\sup_{\tau\in\mathbb{T}} \mathbb{E}[\xi_\tau^2] < \infty$.*

For the second part of the theorem we show the finiteness of the exponential moments of the variables C_ϵ by using the Garsia–Rademich–Rumsey inequality:

Lemma 5.3. *Let $p \ge 1$ and $\alpha > \frac{1}{p}$. Then there exists a constant $c = c_{\alpha,p} > 0$ such that for any $f \in C([0,T])$ and for all $0 \le s,t \le T$ we have*

$$|f(t) - f(s)|^p \le cT^{\alpha p-1}|t-s|^{\alpha p-1} \int_0^T\int_0^T \frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p+1}}\,\mathrm{d}x\mathrm{d}y.$$

In the last section, examples are provided applying Theorem 5.1 for these types of Gaussian processes: the stationary processes and the processes with stationary increments as well as the Fredholm and the Volterra processes. We check also the particular case of self-similar Gaussian processes with the canonical Volterra representation.

II. Representation of self-similar Gaussian processes

Let $X = (X_t; t \in [0,T])$ be a β-self-similar Gaussian process. By the inverse Lamperti transformation the process X is associated to a stationary Gaussian process $Y = (Y_t)_{t\in(-\infty,\log T]}$ through the one-to-one correspondence $X_t = t^{\beta} Y_{\log t}$. Following Karhunen (1950), the canonical Volterra representation of Y exists under the condition of pure non-determinism (the condition (C) in Definition 3.6), that is the representation

$$Y_t = \int_{-\infty}^{t} G(t-s)\,\mathrm{d}W_s^*$$

where G is a Volterra kernel and W^* is a standard Brownian motion.
Our main result in this paper is the canonical Volterra representation constructed for the self-similar Gaussian process X by using the pure non-determinism condition. Since the canonical kernel G is constant on the line $\{(t+a, s+a), a \in \mathbb{R}\}$, it is indeed clear that the canonical kernel for X shall satisfy the homogeneity property.

Definition 5.4. We say that a function $f(t,s)$ is homogeneous with degree $\alpha > 0$ if

$$f(at, as) = a^{\alpha} f(t,s)$$

holds.

The pure non-determinism condition will be again necessary and sufficient to construct the canonical Volterra representation for X as it can be extended from Y in the following way:

$$\bigcap_{t\in(0,T)} \mathcal{H}_X(t) = \bigcap_{t\in(0,T)} \mathcal{H}_Y(\log t) = \bigcap_{t\in(-\infty,\log T)} \mathcal{H}_Y(t) = \{0\} \tag{5.4}$$

and by Lamperti transformation and time change we are able to state our main theorem:

Theorem 5.5. *The self-similar centered Gaussian process $X = (X_t; t \in [0,T])$ satisfies the condition* (C) *if and only if there exist a standard Brownian motion W and a Volterra kernel k such that X has the representation*

$$X_t = \int_0^t k(t,s)\,\mathrm{d}W_s, \tag{5.5}$$

where the Volterra kernel k is defined by

$$k(t,s) = t^{\beta-\frac{1}{2}} F\left(\frac{s}{t}\right), \quad s < t, \tag{5.6}$$

for some function $F \in L^2(\mathbb{R}_+, \mathrm{d}u)$ independent of β, with $F(u) = 0$ for $1 < u$. Moreover, $\mathcal{H}_X(t) = \mathcal{H}_W(t)$ holds for each t.

The expression (5.6) shows that the canonical kernel $k(t,s)$ satisfies the homogeneity property of degree $(\beta - \frac{1}{2})$, in addition, the canonical property is preserved under Lamperti transformation since we have that

$$\mathcal{H}_X(t) = \mathcal{H}_Y(\log t) = \mathcal{H}_{\mathrm{d}W^*}(\log t) = \mathcal{H}_{\mathrm{d}W}(t) = \mathcal{H}_W(t).$$

In the last section and as an application, we will use the representation (5.5) to define the class of β-self-similar Gaussian processes that are equivalent in law to X. Let $\widetilde{X} = (\widetilde{X}_t; t \in [0,T])$ be a centered Gaussian process equivalent in law to X. The Hitsuda representation for Volterra processes asserts the existence of a unique Volterra kernel $l(t,s)$ and a unique centered Gaussian process $\widetilde{W} = (\widetilde{W}_t)_{t\in[0,T]}$ equivalent in law to the standard Brownian motion W such that

$$\widetilde{X}_t = \int_0^t k(t,s)\,\mathrm{d}\widetilde{W}_s = X_t - \int_0^t k(t,s) \int_0^s l(s,u)\,\mathrm{d}W_u\,\mathrm{d}s. \tag{5.7}$$

Under the law of $\widetilde{X}$, $\widetilde{W}$ is a standard Brownian motion and $\widetilde{X}$ is β-self-similar since $k(t,s)$ is $(\beta - \frac{1}{2})$-homogeneous. In Picard (2011), it has been proven that a necessary and sufficient condition for $\widetilde{X}$ to be β-self-similar under the law X is that $\widetilde{X}$ has the same law as X. We will prove this fact by using the homogeneity property. From (5.7) we have

$$\widetilde{X}_t = \int_0^t \left(k(t,s) - t^{\beta-\frac{1}{2}} z(t,s)\right)\,\mathrm{d}W_s, \quad 0 \le t \le T,$$

where $z(t,s) = \int_s^t F\left(\frac{u}{t}\right) l(u,s)\,\mathrm{d}u,\ s < t$. This representation is canonical and $\widetilde{X}$ satisfies the pure non-determinism property since the equivalence of laws implies that $\mathcal{H}_{\widetilde{X}}(t) = \mathcal{H}_X(t)$ for all t. It turns out that by using Theorem 5.5, the kernel $\left(k(t,s) - t^{\beta-\frac{1}{2}} z(t,s)\right)$ is $(\beta - \frac{1}{2})$-homogeneous and thus $l(t,s)$ is (-1)-homogeneous. Now we introduce the following lemma:

Lemma 5.6. *If a Volterra kernel on $[0,T]\times[0.T]$ is homogeneous with degree (-1), then it vanishes on $[0,T]\times[0.T]$.*

It follows from the lemma that the that $\widetilde{X}$ has the same law as X.

III. Generalized Gaussian bridges

In this article we combines and extends the results of Alili (2002) and Gasbarra et al. (2007). Let $X = (X_t)_{t\in[0,T]}$ be a continuous Gaussian process with positive definite covariance function R, mean function m of bounded variation, and $X_0 = m(0)$. We define the generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ as (the law of) the Gaussian process X conditioned on the set

$$\left\{\int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y}\right\} = \bigcap_{i=1}^N \left\{\int_0^T g_i(t)\,\mathrm{d}X_t = y_i\right\}, \tag{5.8}$$

where the functions g_i assumed to be linearly independent. For this generalized Gaussian bridge we will give two types of representations: orthogonal and canonical. The orthogonal representation is a straightforward extension of the orthogonal representation of the classical Gaussian bridge which is

$$X_t^{\mathbf{g};\mathbf{y}} = X_t - \langle\!\langle 1_t, \mathbf{g}\rangle\!\rangle^\top \langle\!\langle \mathbf{g}\rangle\!\rangle^{-1}\left(\int_0^T \mathbf{g}(u)\,\mathrm{d}X_u - \mathbf{y}\right), \tag{5.9}$$

where the matrix

$$\langle\!\langle \mathbf{g}\rangle\!\rangle_{ij} := \langle\!\langle g_i, g_j\rangle\!\rangle := \mathbb{C}\mathrm{ov}\left[\int_0^T g_i(t)\,\mathrm{d}X_t\,,\ \int_0^T g_j(t)\,\mathrm{d}X_t\right].$$

Since the covariance of $X^{\mathbf{g};\mathbf{y}}$ is independent of m and $\mathbf{y}$, we may assume that $m = 0$ and $y_i = 0$ for all i. In the simple case when $X = M$ is a martingale we start constructing the canonical representation of the generalized Gaussian bridge $M^{\mathbf{g},0} =: M^{\mathbf{g}}$ by defining the matrix

$$\langle\!\langle \mathbf{g}\rangle\!\rangle_{ij}(t) := \mathbb{E}\left[\int_t^T g_i(s)\,\mathrm{d}X_s \int_t^T g_j(s)\,\mathrm{d}X_s\right] \tag{5.10}$$

and the kernel

$$\ell_{\mathbf{g}}(t,s) := -\mathbf{g}^\top(t)\,\langle\!\langle \mathbf{g}\rangle\!\rangle^{-1}(t)\,\mathbf{g}(s), \tag{5.11}$$

and by the change of probability measure we obtain

$$\mathrm{d}M_t = \mathrm{d}M_t^{\mathbf{g}} - \int_0^t \ell_{\mathbf{g}}(t,s)\,\mathrm{d}M_s^{\mathbf{g}}\,\mathrm{d}\langle M\rangle_t. \tag{5.12}$$

It follows that the canonical representation of the generalized Gaussian bridge $M^{\mathbf{g}}$ is given in terms of the resolvent kernel of $\ell_{\mathbf{g}}(t,s)$, $\ell^*_{\mathbf{g}}(t,s)$, which is expressed explicitly by

$$
\begin{aligned}
\ell^*_{\mathbf{g}}(t,s) &:= -\ell_{\mathbf{g}}(t,s)\frac{|\langle\!\langle\mathbf{g}\rangle\!\rangle|(t)}{|\langle\!\langle\mathbf{g}\rangle\!\rangle|(s)} \\
&= |\langle\!\langle\mathbf{g}\rangle\!\rangle|(t)\mathbf{g}^\top(t)\langle\!\langle\mathbf{g}\rangle\!\rangle^{-1}(t)\ \frac{\mathbf{g}(s)}{|\langle\!\langle\mathbf{g}\rangle\!\rangle|(s)}.
\end{aligned} \tag{5.13}
$$

Clearly, $\ell_{\mathbf{g}}(t,s)$ and $\ell^*_{\mathbf{g}}(t,s)$ satisfy the resolvent equation. Thus $M^{\mathbf{g}}$ is represented canonically as

$$
\mathrm{d}M^{\mathbf{g}}_t = \mathrm{d}M_t - \int_0^t \ell^*_{\mathbf{g}}(t,s)\,\mathrm{d}M_s\,\mathrm{d}\langle M\rangle_t, \tag{5.14}
$$

For the non-semimartingale case, the same approach can be used by employment of the prediction martingale process which is the most natural martingale associated to the Gaussian non-semimartingale. A typical non-semimartingale example of a prediction-invertible Gaussian process would be the fractional Brownian motion.

Definition 5.7. For a Gaussian process $X = (X_t)_{t\in[0,T]}$, the prediction martingale with respect to $(\mathcal{F}^X_t)_{t\in[0,T]}$ is the process given by

$$
\hat{X}_t := \mathbb{E}\left[X_T|\mathcal{F}^X_t\right].
$$

For each t, the element $\hat{X}_t$ belongs to $\mathcal{H}_X(t)$, hence, we can express it by the Wiener integral

$$
\hat{X}_t = \int_0^t p(t,s)\,\mathrm{d}X_s, \quad 0 \le t \le T. \tag{5.15}
$$

Definition 5.8. A Gaussian process X is prediction-invertible if (5.15) is true and there exists for all $t \in [0,T]$ a kernel $p^{-1}(t,s) \in L^2([0,T]^2, \mathrm{d}\langle\hat{X}\rangle)$ such that

$$
X_t = \int_0^t p^{-1}(t,s)\,\mathrm{d}\hat{X}_s, \quad 0 \le t \le T. \tag{5.16}
$$

All martingales are trivially prediction-invertible. The prediction invertibility means that the Gaussian process X and its prediction martingale $\hat{X}$ can be recovered from each others. Now, define the operators P and P^{-1} that extend linearly the kernels p and p^{-1} by the relations

$$
\begin{aligned}
\mathrm{P}[1_t] &= p(t,\cdot), \\
\mathrm{P}^{-1}[1_t] &= p^{-1}(t,\cdot).
\end{aligned}
$$

For a function f such that $\mathrm{P}^{-1}[f] \in L^2([0,T], \mathrm{d}\langle\hat{X}\rangle)$ and a function $\hat{g} \in L^2([0,T], \mathrm{d}\langle\hat{X}\rangle)$, we have

$$\int_0^T f(t)\,\mathrm{d}X_t = \int_0^T \mathrm{P}^{-1}[f](t)\,\mathrm{d}\hat{X}_t, \tag{5.17}$$

$$\int_0^T \hat{g}(t)\,\mathrm{d}\hat{X}_t = \int_0^T \mathrm{P}[\hat{g}](t)\,\mathrm{d}X_t. \tag{5.18}$$

This helps to write the bridge conditioning for the prediction martingale $\hat{X}$.
Our main result in this section is the following theorem:

Theorem 5.9. *Let X be prediction-invertible Gaussian process. Assume that, for all $t \in [0,T]$ and $i = 1,\dots,N$, $g_i 1_t \in \Lambda_t(X)$. Then the generalized bridge $X^{\mathbf{g}}$ admits the canonical representation*

$$X_t^{\mathbf{g}} = X_t - \int_0^t \int_s^t p^{-1}(t,u)\mathrm{P}\left[\hat{\ell}^*_{\hat{\mathbf{g}}}(u,\cdot)\right](s)\,\mathrm{d}\langle \hat{X}\rangle_u\,\mathrm{d}X_s, \tag{5.19}$$

where

$$\begin{aligned}
\hat{g}_i &= \mathrm{P}^{-1}[g_i],\\
\hat{\ell}^*_{\hat{\mathbf{g}}}(u,v) &= |⟪\hat{\mathbf{g}}⟫^{\hat{X}}|(u)\hat{\mathbf{g}}^\top(u)\left(⟪\hat{\mathbf{g}}⟫^{\hat{X}}\right)^{-1}(u)\,\frac{\hat{\mathbf{g}}(v)}{|⟪\hat{\mathbf{g}}⟫^{\hat{X}}|(v)},\\
⟪\hat{\mathbf{g}}⟫^{\hat{X}}_{ij}(t) &= \int_t^T \hat{g}_i(s)\hat{g}_j(s)\,\mathrm{d}\langle M\rangle_s = ⟪\mathbf{g}⟫^X_{ij}(t).
\end{aligned}$$

Next, we apply this result for the invertible Volterra processes.

Definition 5.10. V is an invertible Gaussian Volterra process if it is continuous and there exist Volterra kernels k and k^{-1} such that

$$V_t = \int_0^t k(t,s)\,\mathrm{d}W_s, \tag{5.20}$$

$$W_t = \int_0^t k^{-1}(t,s)\,\mathrm{d}V_s, \tag{5.21}$$

where W is the standard Brownian motion and the Wiener integrals (5.20) and (5.21) are well defined.

Invertible Gaussian Volterra processes are prediction-invertible. Similarly, the Volterra kernels $k(t,s)$ and $k^{-1}(t,s)$ induce the operators

$$\mathrm{K}[1_t] := k(t,\cdot) \quad \text{and} \quad \mathrm{K}^{-1}[1_t] := k^{-1}(t,\cdot)$$

that can be extended linearly, and moreover, we can write

$$\begin{aligned}\int_0^T f(t)\,\mathrm{d}V_t &= \int_0^T \mathrm{K}^{-1}[f](t)\,\mathrm{d}W_t,\\ \int_0^T g(t)\,\mathrm{d}W_t &= \int_0^T \mathrm{K}[g](t)\,\mathrm{d}V_t.\end{aligned}$$

It follows that the operator K and K^{-1} and the operators$\mathrm{P}$ and P^{-1} are connected through the relations

$$\begin{aligned}\mathrm{K}[g] &= k(T,\cdot)\mathrm{P}^{-1}[g],\\ \mathrm{K}^{-1}[g] &= k^{-1}(T,\cdot)\mathrm{P}[g],\end{aligned}$$

and as an application of the theorem (5.9), we have the corollary:

Corollary 5.11. *Let V be an invertible Gaussian Volterra process and let $\mathrm{K}[g_i] \in L^2([0,T])$ for all $i = 1,\dots,N$. Denote*

$$\tilde{\mathbf{g}}(t) := \frac{\mathrm{K}[\mathbf{g}](t)}{k(T,t)}.$$

Then the bridge $V^{\mathbf{g}}$ admits the canonical representation

$$V_t^{\mathbf{g}} = V_t - \int_0^t \int_s^t \frac{k(t,u)k(T,u)}{k^{-1}(T,s)} \mathrm{K}^{-1}\left[\ell^*_{\tilde{\mathbf{g}}}(u,\cdot)\right](s)\,\mathrm{d}u\,\mathrm{d}V_s, \tag{5.22}$$

where

$$\begin{aligned}\tilde{\ell}_{\tilde{\mathbf{g}}}(u,v) &= |\langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle^W|(u)\tilde{\mathbf{g}}^\top(u)(\langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle^W)^{-1}(u)\,\frac{\tilde{\mathbf{g}}(v)}{|\langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle^W|(v)},\\ \langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle^W_{ij}(t) &= \int_t^T \tilde{g}_i(s)\tilde{g}_j(s)\,\mathrm{d}s = \langle\!\langle \mathbf{g}\rangle\!\rangle^X_{ij}(t).\end{aligned}$$

In the final section, we apply the canonical representation of the generalized Gaussian bridges to the insider trading and compute the additional logarithmic utility for the model

$$\frac{\mathrm{d}S_t}{S_t} = a_t\mathrm{d}\langle M\rangle_t + \mathrm{d}M_t, \tag{5.23}$$

where S is a financial asset with $S_0 = 1$, M is a continuous Gaussian martingale with strictly increasing $\langle M\rangle$ with $M_0 = 0$, and the process a is $\mathbb{F}$-adapted satisfying $\int_0^T a_t^2\,\mathrm{d}\langle M\rangle_t < \infty$ $\mathbb{P}$-a.s.

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Statistics and Probability Letters

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Necessary and sufficient conditions for Hölder continuity of Gaussian processes

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ARTICLE INFO

Article history:
Received 3 April 2014
Received in revised form 24 July 2014
Accepted 26 July 2014
Available online 2 August 2014

MSC:
60G15
60G17
60G18

Keywords:
Gaussian processes
Hölder continuity
Kolmogorov–Čentsov condition
Self-similar processes

ABSTRACT

The continuity of Gaussian processes is an extensively studied topic and it culminates in Talagrand's notion of majorizing measures that gives complicated necessary and sufficient conditions. In this note we study the Hölder continuity of Gaussian processes. It turns out that necessary and sufficient conditions can be stated in a simple form that is a variant of the celebrated Kolmogorov–Čentsov condition.

1. Introduction

In what follows X will always be a centered Gaussian process on the interval $[0, T]$. For a centered Gaussian family $\xi = (\xi_\tau)_{\tau\in\mathbb{T}}$ we denote

$$d_\xi^2(\tau, \tau') := \mathbb{E}[(\xi_\tau - \xi_{\tau'})^2],$$

$$\sigma_\xi^2(\tau) := \mathbb{E}[\xi_\tau^2].$$

To put our result in context, we briefly recall the essential results of Gaussian continuity.

One of the earliest results is a sufficient condition due to Fernique (1964): *Assume that for some positive* ε, *and* $0 \le s \le t \le \varepsilon$, *there exists a nondecreasing function* Ψ *on* $[0, \varepsilon]$ *such that* $\sigma_X^2(s, t) \le \Psi^2(t - s)$ *and*

$$\int_0^\varepsilon \frac{\Psi(u)}{u\sqrt{\log u}}\,du < \infty. \tag{1}$$

Then X is continuous. The finiteness of Fernique integral (1) is not necessary for the continuity. Indeed, cf. (Marcus and Shepp, 1970, Sect. 5) for a counter-example.

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http://dx.doi.org/10.1016/j.spl.2014.07.030
0167-7152/

Dudley (1967, 1973) found a sufficient condition for the continuity by using *metric entropy*. Let $N(\varepsilon) := N([0,T], d_X, \varepsilon)$ denote the minimum number of closed balls of radius ε in the (pseudo) metric d_X needed to cover $[0,T]$. *If*

$$\int_0^\infty \sqrt{\log N(\varepsilon)}\,\mathrm{d}\varepsilon < \infty, \tag{2}$$

then X is continuous. Like in the case of Fernique's condition, the finiteness of the Dudley integral (2) is not necessary for continuity, cf. (Marcus and Rosen, 2006, Ch 6.). However, for stationary processes (2) is necessary and sufficient.

Finally, necessary and sufficient conditions were obtained by Talagrand (1987). Denote $B_{d_X}(t,\varepsilon)$ a ball with radius ε at center t in the metric d_X. A probability measure μ on $([0,T], d_X)$ is called a *majorizing measure* if

$$\sup_{t\in[0,T]} \int_0^\infty \sqrt{\log \frac{1}{\mu\left(B_{d_X}(t,\varepsilon)\right)}}\,\mathrm{d}\varepsilon < \infty. \tag{3}$$

The Gaussian process X is continuous if and only if there exists a majorizing measure μ on $([0,T], d_X)$ such that

$$\lim_{\delta\to 0} \sup_{t\in[0,T]} \int_0^\delta \sqrt{\log \frac{1}{\mu(B_{d_X}(t,\varepsilon))}}\,\mathrm{d}\varepsilon = 0.$$

2. Main theorem

Talagrand's necessary and sufficient condition (3) for the continuity of a Gaussian process is rather complicated. In contrast, the general Kolmogorov–Čentsov condition for continuity is very simple. It turns out that for Gaussian processes the Kolmogorov–Čentsov condition is very close to being necessary for Hölder continuity:

Theorem 1. *The Gaussian process X is Hölder continuous of any order $a < H$ i.e.*

$$|X_t - X_s| \le C_\varepsilon |t-s|^{H-\varepsilon}, \quad \textit{for all } \varepsilon > 0 \tag{4}$$

if and only if there exist constants c_ε such that

$$d_X(t,s) \le c_\varepsilon |t-s|^{H-\varepsilon}, \quad \textit{for all } \varepsilon > 0. \tag{5}$$

Moreover, the random variables C_ε in (4) satisfy

$$\mathbb{E}\left[\exp\left(aC_\varepsilon^\kappa\right)\right] < \infty \tag{6}$$

for any constants $a \in \mathbb{R}$ and $\kappa < 2$; and also for $\kappa = 2$ for small enough positive a. In particular, the moments of all orders of C_ε are finite.

The differences between the classical Kolmogorov–Čentsov continuity criterion and Theorem 1 are: (i) Theorem 1 deals only with Gaussian processes, (ii) there is an ε-gap to the classical Kolmogorov–Čentsov condition and (iii) as a bonus we obtain that the Hölder constants C_ε must have light tails by the estimate (6). Note that the ε-gap cannot be closed. Indeed, let

$$X_t = f(t)B_t,$$

where B is the fractional Brownian motion with Hurst index H and $f(t) = (\log\log 1/t)^{-1/2}$. Then, by the law of the iterated logarithm due to Arcones (1995), X is Hölder continuous of any order $a < H$, but (5) does not hold without an $\varepsilon > 0$.

The proof of the first part Theorem 1 is based on the classical Kolmogorov–Čentsov continuity criterion and the following elementary lemma:

Lemma 1. *Let $\xi = (\xi_\tau)_{\tau\in\mathbb{T}}$ be a centered Gaussian family. If $\sup_{\tau\in\mathbb{T}} |\xi_\tau| < \infty$ then $\sup_{\tau\in\mathbb{T}} \mathbb{E}[\xi_\tau^2] < \infty$.*

Proof. Since $\sup_{\tau\in\mathbb{T}} |\xi_\tau| < \infty$, $\mathbb{P}[\sup_{\tau\in\mathbb{T}} |\xi_\tau| < x] > 0$ for a large enough $x \in \mathbb{R}$. Now, for all $\tau \in \mathbb{T}$, we have that

$$\begin{aligned}
\mathbb{P}\left[\sup_{\tau\in\mathbb{T}} |\xi_\tau| < x\right] &\le \mathbb{P}\left[|\xi_\tau| < x\right] \\
&= \mathbb{P}\left[\left|\frac{\xi_\tau}{\sigma_\xi(\tau)}\right| < \frac{x}{\sigma_\xi(\tau)}\right] \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{x/\sigma_\xi(\tau)} e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&\le \frac{2}{\sqrt{2\pi}} \frac{x}{\sigma_\xi(\tau)}.
\end{aligned}$$

Consequently,

$$\sigma_\xi^2(\tau) \le \frac{2x^2}{\pi \mathbb{P}\left[\sup_{\tau\in\mathbb{T}} |\xi_\tau| < x\right]^2},$$

and the claim follows from this. □

The second part on the exponential moments of the Hölder constants of Theorem 1 follows from the following Garsia–Rademich–Rumsey inequality (Garsia et al., 1970). Let us also note, that this part is intimately connected to the Fernique's theorem (Fernique, 1978) on the continuity of Gaussian processes.

Lemma 2. *Let $p \ge 1$ and $\alpha > \frac{1}{p}$. Then there exists a constant $c = c_{\alpha,p} > 0$ such that for any $f \in C([0,T])$ and for all $0 \le s, t \le T$ we have*

$$|f(t) - f(s)|^p \le cT^{\alpha p-1}|t-s|^{\alpha p-1} \int_0^T\int_0^T \frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p+1}}\,\mathrm{d}x\mathrm{d}y.$$

Proof of Theorem 1. The if part follows from the Kolmogorov–Čentsov continuity criterion. For the only-if part assume that X is Hölder continuous of order $a = H - \varepsilon$, i.e.

$$\sup_{t,s\in[0,T]} \frac{|X_t - X_s|}{|t-s|^{H-\varepsilon}} < \infty.$$

Define a family $\xi = (\xi_{t,s})_{(t,s)\in[0,T]^2}$ by setting

$$\xi_{t,s} = \frac{X_t - X_s}{|t-s|^{H-\varepsilon}}.$$

Since ξ is a centered Gaussian family that is bounded by the Hölder continuity of X, we obtain, by Lemma 1, that $\sup_{(t,s)\in[0,T]^2} \sigma_\xi^2(t,s) < \infty$. This means that

$$\sup_{t,s\in[0,T]} \frac{d_X^2(t,s)}{|t-s|^{2H-2\varepsilon}} < \infty,$$

or

$$d_X(t,s) \le C_\varepsilon |t-s|^{H-\varepsilon}.$$

The property (6) follows from the Garsia–Rademich–Rumsey inequality of Lemma 2. Indeed, by choosing $\alpha = H - \frac{\varepsilon}{2}$ and $p = \frac{2}{\varepsilon}$ we obtain

$$|X_t - X_s| \le c_{H,\varepsilon} T^{H-\varepsilon}|t-s|^{H-\varepsilon}\xi,$$

where

$$\xi = \left(\int_0^T\int_0^T \frac{|X_u - X_v|^{\frac{2}{\varepsilon}}}{|u-v|^{\frac{2H}{\varepsilon}}}\,\mathrm{d}u\mathrm{d}v\right)^{\frac{\varepsilon}{2}}. \quad (7)$$

Let us first estimate moments of ξ. First we recall the fact that for a Gaussian random variable $Z \sim \mathcal{N}(0,\sigma^2)$ and any number $q > 0$ we have

$$\mathbb{E}\left[|Z|^q\right] = \sigma^q \frac{2^{\frac{q}{2}}\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}},$$

where Γ denotes the Gamma function. Let now $\delta < \frac{\varepsilon}{2}$ and $p \ge \frac{2}{\varepsilon}$. By Minkowski inequality and estimate (5) we obtain

$$\begin{aligned}\mathbb{E}\left[|\xi|^p\right] &\le \left(\int_0^T\int_0^T \frac{\left(\mathbb{E}|X_u - X_v|^p\right)^{\frac{2}{p\varepsilon}}}{|u-v|^{\frac{2H}{\varepsilon}}}\mathrm{d}v\mathrm{d}u\right)^{\frac{p\varepsilon}{2}} \\ &\le \left(\int_0^T\int_0^T \frac{\left(c_p c_\delta |u-v|^{p(H-\delta)}\right)^{\frac{2}{p\varepsilon}}}{|u-v|^{\frac{2H}{\varepsilon}}}\,\mathrm{d}v\mathrm{d}u\right)^{\frac{p\varepsilon}{2}}\end{aligned}$$

$$= c_p c_\delta 2^{\frac{p\varepsilon}{2}} \left(\int_0^T \int_0^u (u-v)^{-\frac{2\delta}{\varepsilon}} \mathrm{d}v \mathrm{d}u \right)^{\frac{p\varepsilon}{2}}$$
$$= c_p c_\delta 2^{\frac{p\varepsilon}{2}} \left(\frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} \left(1 - \frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} T^{q(\varepsilon-\delta)},$$

where c_δ is the constant from (5) and

$$c_q = \frac{2^{\frac{q}{2}} \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}}. \tag{8}$$

Hence, we may take

$$C_\varepsilon = c_{H,\varepsilon} T^{H-\varepsilon} \xi,$$

where $c_{H,\varepsilon}$ is the constant from Garsia–Rademich–Rumsey inequality and ξ is given by (7). Moreover, for any $p \geq \frac{2}{\varepsilon}$ and any $\delta < \frac{\varepsilon}{2}$ we have estimate

$$\mathbb{E}\left[|\xi|^p\right] \leq c_p c_\delta 2^{\frac{p\varepsilon}{2}} \left(\frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} \left(1 - \frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} T^{q(\varepsilon-\delta)}.$$

Consequently,

$$\mathbb{E}\left[|C_\varepsilon|^p\right] \leq c^p \Gamma\left(\frac{p+1}{2} \right)$$

for some constant $c = c_{\varepsilon,\delta,T}$. Thus, by plugging in (8) to the series expansion of the exponential we obtain

$$\mathbb{E}\left[\exp\left(aC_\varepsilon^\kappa\right)\right] \leq \sum_{j=0}^{\infty} a^j c^{\kappa j} \frac{\Gamma\left(\frac{\kappa j+1}{2}\right)}{\Gamma(j+1)}.$$

So, to finish the proof we need to show that the series above converges. Now, by Stirling's approximation

$$\Gamma(z) = \frac{\sqrt{2\pi}}{\sqrt{z}} \left(\frac{z}{e} \right)^z (1 + O(1/z)),$$

we obtain (the constant c may vary from line to line)

$$\frac{\Gamma\left(\frac{\kappa j+1}{2}\right)}{\Gamma(j+1)} \sim \frac{\left(\frac{\kappa j+1}{2}\right)^{-\frac{1}{2}} e^{-\frac{\kappa j+1}{2}} \left(\frac{\kappa j+1}{2}\right)^{\frac{\kappa j+1}{2}}}{(j+1)^{-\frac{1}{2}} e^{-j-1} (j+1)^{j+1}}$$
$$\leq c^j \frac{1}{\sqrt{j+1}} \frac{(\kappa j+1)^{\frac{\kappa j}{2}}}{(j+1)^j}$$
$$\leq c^j \frac{1}{\sqrt{j+1}} \frac{(2j+2)^{\frac{\kappa j}{2}}}{(j+1)^j}$$
$$= (2c)^j \frac{1}{\sqrt{j+1}} (j+1)^{\left(\frac{\kappa}{2}-1\right)j}$$

which is clearly summable since $\kappa < 2$. If $\kappa = 2$, then in the approximation above we obtain that $\Gamma\left(\frac{2j+1}{2}\right) / \Gamma(j+1) \sim c^j$ for some constant c. Hence, depending on constant $c_{\varepsilon,\delta,T}$, we obtain that $\mathbb{E}\left[\exp\left(aC_\varepsilon^2\right)\right] < \infty$ for small enough $a > 0$. □

3. Applications and examples

Stationary-increment processes. This case is simple:

Corollary 1. *If X has stationary increments then it is Hölder continuous of any order $a < H$ if and only if*

$$\sigma_X^2(t) \leq c_\varepsilon t^{2H-\varepsilon}, \quad \textit{for all } \varepsilon > 0.$$

Stationary processes. For a stationary process $\mathbb{E}[X_t X_s] = r(t-s)$, where, by the Bochner's theorem,

$$r(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \, \Delta(\mathrm{d}\lambda),$$

where Δ, the *spectral measure* of X, is finite and symmetric. Since now

$$d_X^2(t,s) = 2\big(r(0) - r(t-s)\big)$$

we have the following corollary.

Corollary 2. *If X is stationary with spectral measure Δ then it is Hölder continuous of any order $a < H$ if and only if*

$$\int_0^\infty \big(1-\cos(\lambda t)\big)\,\Delta(\mathrm{d}\lambda) \le c_\varepsilon t^{2H-\varepsilon} \quad \textit{for all } \varepsilon > 0.$$

Fredholm processes. A bounded process can be viewed as an $L^2([0,T])$-valued random variable. Hence, the covariance operator admits a square root with kernel K, and we may represent X as a *Gaussian Fredholm process*:

$$X_t = \int_0^T K(t,s)\,\mathrm{d}W_s, \tag{9}$$

where W is a Brownian motion and $K \in L^2([0,T]^2)$.

Corollary 3. *A Gaussian process X is Hölder continuous of any order $a < H$ if and only if it admits the representation (9) with K satisfying*

$$\int_0^T |K(t,u) - K(s,u)|^2\,\mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon} \quad \textit{for all } \varepsilon > 0.$$

Proposition 1. *Let X be Gaussian Fredholm process with kernel K.*

(i) *If for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in L^2([0,T])$ such that*

$$|K(t,u) - K(s,u)| \le f_\varepsilon(u)|t-s|^{H-\varepsilon}$$

then X is Hölder continuous of any order $a < H$.

(ii) *If X is Hölder continuous of any order $a < H$ then*

$$f_\varepsilon := \liminf_{s\to t} \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t-s|^{2H-\varepsilon}} \in L^1([0,T]).$$

Proof. The first part follows from Corollary 3. Consider then the second part and assume that X is Hölder continuous of any order $a < H$ and

$$\liminf_{s\to t} \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t-s|^{2H-\varepsilon}} \notin L^1([0,T]).$$

By Corollary 3 we know that

$$\int_0^T \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t-s|^{2H-\varepsilon}}\mathrm{d}u \le c_\varepsilon.$$

On the other hand, by Fatou Lemma we have

$$\liminf_{s\to t} \int_0^T \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t-s|^{2H-\varepsilon}}\mathrm{d}u \ge \int_0^T \liminf_{s\to t} \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t-s|^{2H-\varepsilon}}\mathrm{d}u = \infty$$

which is a contradiction. □

Volterra processes. A Fredholm process is a *Volterra process* if its kernel K satisfies $K(t,s) = 0$ if $s > t$. In this case Corollary 3 becomes:

Corollary 4. *A Gaussian Volterra process X with kernel K is Hölder continuous of any order $a < H$ if and only if, for all $s < t$ and $\varepsilon > 0$*

(i) $\int_s^t K(t,u)^2\mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon}$,
(ii) $\int_0^s |K(t,u) - K(s,u)|^2\,\mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon}$.

By Alòs et al. (2001, p. 779) the following is a sufficient condition:

Proposition 2. *Let X be a Gaussian Volterra process with kernel K that satisfies*

(i) $\int_s^t K(t,u)^2\,\mathrm{d}u \le c(t-s)^{2H}$,

(ii) $K(t,s)$ *is differentiable in t and* $\left|\frac{\partial K}{\partial t}(t,s)\right| \le c(t-s)^{H-\frac{3}{2}}$.

Then X is Hölder continuous of any order $a < H$.

Self-similar processes. A process X is *self-similar* with index $\beta > 0$ if

$$(X_{at})_{0\le t\le T/a} \overset{d}{=} (a^\beta X_t)_{0\le t\le T}, \quad \text{for all } a > 0.$$

In the Gaussian case this means that

$$d_X(t,s) = a^{-\beta} d_X(at, as) \quad \text{for all } a > 0.$$

So, it is clear that X cannot be Hölder continuous of order $H > \beta$.

Let $\mathcal{H}_t^X$ be the closed linear subspace of $L^2(\Omega)$ generated by the Gaussian random variables $\{X_s; s \le t\}$. Denote $\mathcal{H}_{0+}^X := \cap_{t\in(0,T]} \mathcal{H}_t^X$. Then X is *purely non-deterministic* if $\mathcal{H}_{0+}^X$ is trivial. By Yazigi (2014) a purely non-deterministic Gaussian self-similar process admits the representation

$$X_t = \int_0^t t^{\beta-\frac{1}{2}} F\left(\frac{u}{t}\right) \mathrm{d}W_u, \tag{10}$$

where $F \in L^2([0,1])$ is positive. Consequently:

Corollary 5. *Let X be a purely non-deterministic Gaussian self-similar process with index β and representation (10). Then X is Hölder continuous of any order* $a < H$ *if and only if*

(i) $\int_s^t t^{2\beta-1} F(\frac{u}{t})^2\,\mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon}$,

(ii) $\int_0^s \left| t^{\beta-\frac{1}{2}} F\left(\frac{u}{t}\right) - s^{\beta-\frac{1}{2}} F\left(\frac{u}{s}\right)\right|^2 \mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon}$

for all $s < t$ *and* $\varepsilon > 0$.

Proposition 3. *Let X be a purely non-deterministic Gaussian self-similar process with index β and representation (10). Then X is Hölder continuous of any order* $a < H$ *if*

(i) $F(x) \le c\, x^{\beta-H}(1-x)^{H-\frac{1}{2}}$, $0 < x < 1$,

(ii) $\left|1 - \frac{F(x)}{F(y)}\right| \le \left|\left(\frac{y}{x}\right)^{H-\beta}\left(\frac{1-x}{1-y}\right)^{H-\frac{1}{2}} - 1\right|$, $0 < y < x < 1$.

Proof. Condition (i) of Corollary 4 follows from assumption (i) and condition (ii) of Corollary 4 follows from assumptions (i) and (ii) applied to the estimate

$$\left|t^{\beta-\frac{1}{2}} F\left(\frac{u}{t}\right) - s^{\beta-\frac{1}{2}} F\left(\frac{u}{s}\right)\right| \le F\left(\frac{u}{t}\right) t^{\beta-\frac{1}{2}} \left|1 - \frac{F\left(\frac{u}{s}\right)}{F\left(\frac{u}{t}\right)}\right| + F\left(\frac{u}{s}\right)\left|t^{\beta-\frac{1}{2}} - s^{\beta-\frac{1}{2}}\right|.$$

The details are left to the reader. □

Acknowledgment

A. Yazigi was funded by the Finnish Doctoral Programme in Stochastics and Statistics.

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Yazigi, A., Representation of self-similar Gaussian processes. Preprint, arXiv:1401.3236, 2014.

Statistics and Probability Letters 99 (2015) 94–100

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Representation of self-similar Gaussian processes

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ARTICLE INFO

Article history:
Received 23 July 2014
Received in revised form 12 January 2015
Accepted 12 January 2015
Available online 20 January 2015

MSC:
60G15
60G18
60G22

Keywords:
Canonical Volterra representation
Equivalence in law
Gaussian processes
Lamperti transformation
Self-similar processes

ABSTRACT

We develop the canonical Volterra representation for a self-similar Gaussian process by using the Lamperti transformation of the corresponding stationary Gaussian process, where this latter one admits a canonical integral representation under the assumption of pure non-determinism. We apply the representation obtained to the equivalence in law for self-similar Gaussian processes.

1. Introduction and preliminaries

In this paper, we will construct the canonical Volterra representation for given self-similar centered Gaussian processes. The role of the canonical Volterra representation which was first introduced by Lévy in Lévy (1956a,b), and later developed by Hida in Hida (1960), is to provide an integral representation for a Gaussian process X in terms of a Brownian motion W and a non-random Volterra kernel k such that the expression

$$X_t = \int_0^t k(t,s)\,\mathrm{d}W_s$$

holds for all t and the Gaussian processes X and W generate the same filtration. It is known, see Jost (2007) and Lévy (1956a), that if the kernel k satisfies the homogeneity property for some degree α, i.e. $k(at,as) = a^{\alpha}k(t,s),\ a > 0$, the Gaussian process X is self-similar with index $\alpha + \frac{1}{2}$. Thus, the main goal of this paper is to give, under some suitable conditions, a general construction of the canonical Volterra representation for self-similar Gaussian processes, and which also guaranties the homogeneity property of the kernel. In Section 2, the linear Lamperti transform that defines the one–one correspondence between stationary processes and self-similar processes, will be used to express the explicit form of the canonical Volterra representation for self-similar Gaussian processes in the light of the classical canonical representation of the stationary processes given by Karhunen in Karhunen (1950). In Section 3, we give an application of the representation obtained to a Gaussian process equivalent in law to the self-similar Gaussian process.

In our mathematical settings, we take $T > 1$ to be a fixed time horizon, and on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ we consider a centered Gaussian process $X = (X_t;\, t \in [0,T])$ that enjoys the self-similarity property for some $\beta > 0$, i.e.

$$(X_{at})_{0\le t\le T/a} \overset{d}{=} (a^{\beta}X_t)_{0\le t\le T}, \quad \text{for all } a > 0,$$

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http://dx.doi.org/10.1016/j.spl.2015.01.012

where $\stackrel{d}{=}$ denotes equality in distributions, or equivalently,

$$r(t,s) = \mathbb{E}(X_t X_s) = T^{2\beta}\, r\left(\frac{t}{T}, \frac{s}{T}\right), \quad 0 \le t, s \le T. \tag{1.1}$$

In particular, we have $r(t,t) = t^{2\beta}\mathbb{E}(X_1^2)$, which is finite and continuous function at every (t,t) in $[0,T]^2$, and therefore, is continuous at every $(t,s) \in [0,T]^2$, see Loève (1978). A consequence of the continuity of the covariance function r is that X is mean-continuous.

We denote by $\mathrm{H}_X(t)$ the closed linear subspace of $L^2([0,T])$ generated by the Gaussian random variables X_s for $s \le t$, and by $(\mathscr{F}_t^X)_{t\in[0T]}$, where $\mathscr{F}_t^X := \sigma(X_s, s \le t)$, the completed natural filtration of X. We call the *Volterra representation* of X the integral representation of the form

$$X_t = \int_0^t k(t,s)\,\mathrm{d}W_s, \quad t \in [0,T], \tag{1.2}$$

where $W = (W_t; t \in [0,T])$ is a standard Brownian motion and the kernel $k(t,s)$ is a Volterra kernel, i.e. a measurable function on $[0,T] \times [0,T]$ that satisfies $\int_0^T \int_0^t k(t,s)^2\,\mathrm{d}s\,\mathrm{d}t < \infty$, and $k(t,s) = 0$ for $s > t$. The Gaussian process X with such representation is called a *Gaussian Volterra process*, provided with k and W.

Moreover, the Volterra representation is said to be *canonical* if the *canonical property*

$$\mathscr{F}_t^X = \mathscr{F}_t^W$$

holds for all t, or equivalently

$$\mathrm{H}_X(t) = \mathrm{H}_W(t), \quad \text{for all } t. \tag{1.3}$$

Remark 1.1. (i) An equivalent to the canonical property is that if there exists a random variable $\eta = \int_0^T \phi(s)\,\mathrm{d}W_s$, $\phi \in L^2([0,T])$, such that it is independent of X_t for all $0 \le t \le T$, i.e. $\int_0^t k(t,s)\,\phi(s)\,\mathrm{d}s = 0$, one has $\phi \equiv 0$. This means that the family $\{k(t,\cdot), 0 \le t \le T\}$ is free and spans a vector space that is dense in $L^2([0,T])$. If we associate with the canonical kernel k a Volterra integral operator $\mathscr{K}$ defined on $L^2([0,T])$ by $\mathscr{K}\phi(t) = \int_0^t k(t,s)\,\phi(s)\,\mathrm{d}s$, it follows from the canonical property (1.3) that $\mathscr{K}$ is injective and $\mathscr{K}(L^2([0,T]))$ is dense in $L^2([0,T])$. The covariance integral operator $\mathscr{R}$ associated with the kernel $r(t,s)$ has the decomposition $\mathscr{R} = \mathscr{K}\mathscr{K}^*$, where $\mathscr{K}^*$ is the adjoint operator of $\mathscr{K}$. In this case, the covariance r is factorable, i.e.

$$r(t,s) = \int_0^{t\wedge s} k(t,u)k(s,u)\,\mathrm{d}u.$$

(ii) A special property for a Volterra integral operator is that it has no eigenvalues, see Gohberg and Krein (1969).

2. The canonical Volterra representation and self-similarity

The Gaussian process X is β-self-similar, and according to Lamperti (1962), it can be transformed into a stationary Gaussian process Y defined by:

$$Y(t) := e^{-\beta t}X(e^t), \quad t \in (-\infty, \log T]. \tag{2.1}$$

Conversely, X can be recovered from Y by the inverse Lamperti transformation

$$X(t) = t^{\beta}Y(\log t), \quad t \in [0,T]. \tag{2.2}$$

As a consequence of (2.1) and the mean-continuity of X, it is easy to see that Y is mean-continuous since

$$\mathbb{E}(Y_t - Y_s)^2 = 2\left(r(1,1) - e^{-(t-s)\beta}r(e^{t-s},1)\right)$$

converges to zero when t approaches s.

We denote by $\mathrm{H}_Y(t)$ the closed linear subspace of $L^2((-\infty, \log T])$ generated by Y_s, $s \le t$, to this end, we need to recall the concept of pure non-determinism which is required to construct the canonical representation of the stationary Gaussian process Y.

Definition 2.1. Let Z be a process with finite second moments and let $H_Z(t)$ be the closed linear L^2-subspace generated by the random variables Z_s, $s \le t$. Then Z is said to be purely non-deterministic when the condition

$$\bigcap_t \mathrm{H}_Z(t) = \{0\} \tag{C}$$

is satisfied, where $\{0\}$ denotes the L^2-subspace spanned by the constants.

The above definition is due to Cramer in general L^2-processes framework, see Cramer (1961), where the condition (C) emphasizes that the remote past $\bigcap_t \mathrm{H}_Z(t)$ of process Z is trivial and does not contain any information. In the Gaussian case which falls naturally into this class, application of the condition (C) that has been investigated was mainly for Gaussian processes of stationary type; one may refer to Dym and McKean (1979) and Hida and Hitsuda (1993).

As was shown by Hida and Hitsuda (Section 3, Hida and Hitsuda, 1993), which is a well-known classical result that was first established by Karhunen (Section 3, Satz 5, Karhunen, 1950), a mean-continuous stationary Gaussian process admits a canonical representation if and only if it is purely non-deterministic. Under this necessary and sufficient condition, and following the construction used in Hida and Hitsuda (1993), the stationary Gaussian process Y can be represented canonically by the form

$$Y_t = \int_{-\infty}^{t} G_T(t-s)\,\mathrm{d}W_s^*, \tag{2.3}$$

where G_T is a measurable function in $L^2(\mathbb{R}, \mathrm{d}u)$ such that $G_T(u) = 0$ for $u < 0$, and W^* is a standard Brownian motion satisfying the canonical property, i.e.,

$$\mathrm{H}_Y(t) = \mathrm{H}_{W^*}(t), \quad t \in (-\infty, \log T].$$

Next, we shall extend the property of pure non-determinism to the self-similar centered Gaussian process X.

Theorem 2.2. *The self-similar centered Gaussian process $X = (X_t; t \in [0,T])$ satisfies the condition (C) if and only if there exist a standard Brownian motion W and a Volterra kernel k such that X has the representation*

$$X_t = \int_0^t k(t,s)\,\mathrm{d}W_s, \tag{2.4}$$

where the Volterra kernel k is defined by

$$k(t,s) = t^{\beta-\frac{1}{2}} F\left(\frac{s}{t}\right), \quad s < t, \tag{2.5}$$

for some function $F \in L^2(\mathbb{R}_+, \mathrm{d}u)$ independent of β, with $F(u) = 0$ for $1 < u$.

Moreover, $\mathrm{H}_X(t) = \mathrm{H}_W(t)$ holds for each t.

Remark 2.3. In the case where the process X is *trivial self-similar*, i.e. $X_t = t^\beta W_1,\ 0 \le t \le T$, the condition (C) is not satisfied since $\bigcap_{t\in(0,T)} \mathrm{H}_X(t) = \mathrm{H}_W(1)$. Thus, X has no Volterra representation in this case.

Proof. The fact that X is purely non-deterministic is equivalent to that Y is purely non-deterministic since

$$\bigcap_{t\in(0,T)} \mathrm{H}_X(t) = \bigcap_{t\in(0,T)} \mathrm{H}_Y(\log t) = \bigcap_{t\in(-\infty,\log T)} \mathrm{H}_Y(t).$$

Thus Y admits the representation (2.3) for some square integrable kernel G_T and a standard Brownian motion W^*. By the inverse Lamperti transformation, we obtain

$$X(t) = \int_{-\infty}^{\log t} t^\beta G_T(\log t - s)\,\mathrm{d}W_s^* = \int_0^t t^\beta s^{-\frac{1}{2}} G_T\left(\log\frac{t}{s}\right)\mathrm{d}W_s,$$

where $\mathrm{d}W_s = s^{\frac{1}{2}}\mathrm{d}W^*_{\log s}$. We take the Volterra kernel k to be defined as $k(t,s) = t^{\beta-\frac{1}{2}} F\left(\frac{s}{t}\right)$, where $F(u) = u^{-\frac{1}{2}} G_T(\log u^{-1}) \in L^2(\mathbb{R}_+, \mathrm{d}u)$ vanishing when $u < 1$ since $G_T(u) = 0$ when $u < 0$, i.e. for $t < s$, we have $F(\frac{s}{t}) = 0$, and then, $k(t,s) = 0$. Indeed,

$$\int_0^\infty F(u)^2\,\mathrm{d}u = \int_0^\infty G_T(\log u^{-1})^2 \frac{\mathrm{d}u}{u} = \int_{-\infty}^\infty G_T(v)^2\,\mathrm{d}v < \infty,$$

and

$$\begin{aligned}\int_0^T \int_0^t F\left(\frac{s}{t}\right)^2 \mathrm{d}s\,\mathrm{d}t &= \int_0^T t\,\mathrm{d}t \int_0^1 F(u)^2\,\mathrm{d}u \\ &= \int_0^T t\,\mathrm{d}t \int_0^\infty G_T(v)^2\,\mathrm{d}v < \infty.\end{aligned}$$

Thus,

$$\int_0^T \int_0^t t^{2\beta-1} F\left(\frac{s}{t}\right)^2 \mathrm{d}s\,\mathrm{d}t = \left(\int_0^T t^{2\beta}\right)\left(\int_0^1 F(u)^2\,\mathrm{d}u\right)\mathrm{d}t < \infty.$$

Considering the closed linear subspace $\mathrm{H}_{\mathrm{d}W}(t)$ of $L^2([0,T])$ that is generated by $W_s - W_u$ for all $u \le s \le t$, we have $\mathrm{H}_{\mathrm{d}W}(t) = \mathrm{H}_W(t)$ since $W_0 = 0$, and therefore, the canonical property follows from the equalities

$$\mathrm{H}_X(t) = \mathrm{H}_Y(\log t) = \mathrm{H}_{\mathrm{d}W^*}(\log t) = \mathrm{H}_{\mathrm{d}W}(t) = \mathrm{H}_W(t). \quad \square$$

Example 2.4 (*Fractional Brownian Motion*)**.** The fractional Brownian motion (fBm) on $[0, T]$ with index $H \in (0, 1)$ is a centered Gaussian process $B^H = (B_t; 0 \le t \le T)$ with the covariance function $R_H(t, s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})$. The fBm is H-self-similar, and following Alòs et al. (2001) and Decreusefond and Üstünel (1999), it admits the canonical Volterra representation with the canonical kernel

$$k_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, du, \quad \text{for } H > \frac{1}{2},$$

$$k_H(t, s) = d_H \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t - s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u - s)^{H-\frac{1}{2}} \, du \right), \quad \text{for } H < \frac{1}{2},$$

where $c_H = \left(\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}$, $d_H = \left(\frac{2H}{(1-2H)B(1-2H, H+\frac{1}{2})} \right)^{\frac{1}{2}}$, here B denotes the Beta function. So, the function F that corresponds to the canonical Volterra representation of fBm has the expressions:

$$F(u) = c_H \left(u^{\frac{1}{2}-H} \int_u^1 (z - u)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} \, dz \right), \quad \text{for } H > \frac{1}{2},$$

and

$$F(u) = d_H \left(\left(\frac{1}{u} - 1\right)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) (u)^{\frac{1}{2}-H} \int_u^1 z^{H-\frac{3}{2}} (z - u)^{H-\frac{1}{2}} \, dz \right),$$

for $H < \frac{1}{2}$.

A function $f(t, s)$ is said to be homogeneous with degree α if the equality

$$f(at, as) = a^\alpha f(t, s), \quad a > 0,$$

holds for all t, s in $[0, T]$. From the expression (2.5) of the canonical kernel, it is easy to see that k is homogeneous with degree $\beta - \frac{1}{2}$, i.e. $k(t, s) = T^{\beta-\frac{1}{2}} k(\frac{t}{T}, \frac{s}{T})$, for all $s < t \in [0, T]$.

Given X with the canonical Volterra representation (2.4), let $\mathscr{U}$ to be a bounded unitary endomorphism on $L^2([0, T])$ with adjoint $\mathscr{U}^* = \mathscr{U}^{-1}$, and define the process $B = (B)_t := (\mathscr{U}^*(W))_t$ for each $t \in [0, T]$. Indeed, B is a standard Brownian motion since the Gaussian measure is preserved under the unitary transformations. With the notation $k_t(\cdot) := k(t, \cdot)$, the Gaussian process associated with the kernel $(\mathscr{U} k_t)(s)$ and the standard Brownian motion B has same law as X. For the covariance operator, we write

$$\mathscr{R} = \mathscr{K}\mathscr{K}^* = \mathscr{K}\mathscr{U}^*\mathscr{U}\mathscr{K}^* = (\mathscr{K}\mathscr{U}^*)(\mathscr{K}\mathscr{U}^*)^*,$$

where the operator $\mathscr{K}\mathscr{U}^*$ is defined by

$$(\mathscr{K}\mathscr{U}^*)\phi(t) = \int_0^t k(t, s) \, (\mathscr{U}^*\phi)(s) \, ds = \int_0^T (\mathscr{U} k_t)(s) \, \phi(s) \, ds, \quad \phi \in L^2([0, T]).$$

The associated Gaussian process has then the integral representation $\int_0^T (\mathscr{U} k_t)(s) \, dB_s$ for all $t \in [0, T]$.

Corollary 2.5. *For any bounded unitary endomorphism $\mathscr{U}$ on $L^2([0, T])$, the homogeneity of k is preserved under $\mathscr{U}$.*

Proof. Let $\mathscr{U}$ be a bounded unitary endomorphism on $L^2([0, T])$, and let the scaling operator $\mathscr{S} f(t) = T^{\frac{1}{2}} f(Tt)$ with adjoint $\mathscr{S}^* f(t) = T^{-\frac{1}{2}} f(\frac{t}{T})$ to be defined for all $f \in L^2([0, T])$. The homogeneity of k means that

$$k_t(s) = T^\beta \left(\mathscr{S}^* k_{\frac{t}{T}} \right)(s),$$

then we have

$$\mathscr{U} k_t(s) = T^\beta \left(\mathscr{U}\mathscr{S}^* k_{\frac{t}{T}} \right)(s) = T^{\beta-\frac{1}{2}} \left(\mathscr{S}\mathscr{U}\mathscr{S}^* k_{\frac{t}{T}} \right)\left(\frac{s}{T}\right).$$

To show the equality $\mathscr{S}\mathscr{U}\mathscr{S}^* k_{\frac{t}{T}} = \mathscr{U} k_{\frac{t}{T}}$, we will use the Mellin transform

$$\begin{aligned} \int_0^\infty \left(\mathscr{S}\mathscr{U}\mathscr{S}^* k_{\frac{t}{T}} \right)(s) \, s^{p-1} \, ds &= \int_0^\infty \left(\mathscr{U}\mathscr{S}^* k_{\frac{t}{T}} \right)(s) \, (\mathscr{S}^* s^{p-1}) \, ds \\ &= T^{\frac{1}{2}-p} \int_0^\infty \left(\mathscr{U}\mathscr{S}^* k_{\frac{t}{T}} \right)(s) \, s^{p-1} \, ds \end{aligned}$$

$$= T^{\frac{1}{2}-p}\int_0^\infty \left(\mathscr{S}^* k_{\frac{t}{T}}\right)(s)\,(\mathscr{U}^* s^{p-1})\,\mathrm{d}s$$
$$= T^{-p}\int_0^\infty k_{\frac{t}{T}}\left(\frac{s}{T}\right)(\mathscr{U}^* s^{p-1})\,\mathrm{d}s$$
$$= \int_0^\infty k_{\frac{t}{T}}(u)\,(\mathscr{U}^* u^{p-1})\,\mathrm{d}u = \int_0^\infty \mathscr{U} k_{\frac{t}{T}}(u)\,u^{p-1}\,\mathrm{d}u,$$

and the uniqueness property of the Mellin transform implies that

$$\mathscr{S}\mathscr{U}\mathscr{S}^* k_{\frac{t}{T}} = \mathscr{U} k_{\frac{t}{T}}. \quad \square$$

Remark 2.6. The fact that the β-self-similar Gaussian process X satisfies the condition (C), guaranties the existence of the canonical kernel k which is homogeneous with degree $\beta - \frac{1}{2}$, and its homogeneity is preserved under unitary transformation. If we consider again the example in Remark 2.3, one has the representation

$$X_t = \int_0^T t^\beta 1_{[0,1]}(s)\,\mathrm{d}W_s, \quad 0 \le t \le T,$$

where $1_{[0,1]}(s)$ is the indicator function. In this case, we see that the kernel $t^\beta 1_{[0,1]}(s)$ does not satisfy the homogeneity property of any degree.

3. Application to the equivalence in law

In this section, we shall emphasize the self-similarity property under the equivalence of laws of Gaussian processes. First, we recall the results shown by Hida–Hitsuda in the case of Brownian motion, see Hida and Hitsuda (1993) and Hitsuda (1968). Following Hitsuda's representation theorem, a centered Gaussian process $\widetilde{W} = (\widetilde{W}_t; t \in [0, T])$ is equivalent in law to a standard Brownian motion $W = (W_t; t \in [0, T])$ if and only if $\widetilde{W}$ can be represented in a unique way by

$$\widetilde{W}_t = W_t - \int_0^t \int_0^s l(s,u)\,\mathrm{d}W_u\,\mathrm{d}s, \tag{3.1}$$

where $l(s, u)$ is a Volterra kernel, i.e.

$$\int_0^T \int_0^t l(t,s)^2\,\mathrm{d}s\,\mathrm{d}t < \infty, \quad l(t,s) = 0 \text{ for } t < s, \tag{3.2}$$

and such that the equality $\mathrm{H}_{\widetilde{W}}(t) = \mathrm{H}_W(t)$ holds for each t. We note here that the uniqueness of the canonical decomposition (3.1) is in the sense that if l' is a Volterra kernel and $W' = (W'_t; t \in [0, T])$ is a standard Brownian motion such that for $0 \le t \le T$

$$W'_t - \int_0^t \int_0^s l'(s,u)\,\mathrm{d}W'_u\,\mathrm{d}s = W_t - \int_0^t \int_0^s l(s,u)\,\mathrm{d}W_u\,\mathrm{d}s,$$

then $l = l'$ and $W = W'$.

If we denote by $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ the laws of W and $\widetilde{W}$ respectively, these two processes are equivalent in law if $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, and the Radon–Nikodym density is given by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left\{\int_0^T \int_0^s l(s,u)\mathrm{d}W_u\,\mathrm{d}W_s - \frac{1}{2}\int_0^T \left(\int_0^s l(s,u)\mathrm{d}W_s\right)^2 \mathrm{d}s\right\}.$$

The centered Gaussian process $\widetilde{W}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$ with $\widetilde{\mathbb{E}}(\widetilde{W}_t\widetilde{W}_s) = \mathbb{E}(W_t W_s)$, hence, it is self-similar with index $\frac{1}{2}$ under $\widetilde{\mathbb{P}}$. It follows from (3.1) that the covariance of $\widetilde{W}$ under $\mathbb{P}$ has the form of

$$\mathbb{E}(\widetilde{W}_t\widetilde{W}_s) = t \wedge s - \int_0^{t\wedge s}\int_u^s l(v,u)\,\mathrm{d}v\,\mathrm{d}u - \int_0^{t\wedge s}\int_u^t l(v,u)\,\mathrm{d}v\,\mathrm{d}u$$
$$+ \int_0^t \int_0^s \int_0^{v_1\wedge v_2} l(v_1,u)\,l(v_2,u)\,\mathrm{d}u\,\mathrm{d}v_1\,\mathrm{d}v_2.$$

The Hitsuda representation can be extended to the class of the canonical Gaussian Volterra processes, see Baudoin and Nualart (2003) and Sottinen (2004). A centered Gaussian process $\widetilde{X} = (\widetilde{X}_t; t \in [0, T])$ is equivalent in law to a Gaussian Volterra process X if and only if there exists a unique centered Gaussian process, namely $\widetilde{W}$, satisfying (3.1) and (3.2), and such that

$$\widetilde{X}_t = \int_0^t k(t,s)\,\mathrm{d}\widetilde{W}_s = X_t - \int_0^t k(t,s)\int_0^s l(s,u)\,\mathrm{d}W_u\,\mathrm{d}s, \tag{3.3}$$

where the kernel $k(t, s)$ and the standard Brownian motion stand for (1.2), the canonical Volterra representation of X. Moreover, we have $\mathrm{H}_{\widetilde{X}}(t) = \mathrm{H}_X(t)$ for all t.

Under the condition (C), the kernel k is $(\beta - \frac{1}{2})$-homogeneous, and the centered Gaussian process $\widetilde{X}$ is β-self-similar under $\widetilde{\mathbb{P}}$ since $\widetilde{W}$ is a standard Brownian motion. It is obvious that if $\widetilde{X}$ has same law as X, it is β-self-similar under $\mathbb{P}$, and this condition is also necessary, see Picard (2011). However, in the next proposition, we will use the homogeneity property of the Volterra kernel l as a necessary and sufficient condition for the self-similarity for the process $\widetilde{X}$, and equivalently for $\widetilde{W}$, under the law $\mathbb{P}$.

Proposition 3.1. *Let $X = (X_t; t \in [0, T])$ be a centered β-self-similar Gaussian process satisfying the condition (C), then*

(i) *a centered Gaussian process $\widetilde{X} = (\widetilde{X}_t; t \in [0, T])$ is equivalent in law to X if and only if $\widetilde{X}$ admits a representation of the form of*

$$\widetilde{X}_t = X_t - t^{\beta-\frac{1}{2}} \int_0^t z(t, s)\, dW_s, \quad 0 \le t \le T, \tag{3.4}$$

where W is a standard Brownian motion on $[0, T]$, and the kernel $z(t, s)$ is independent of β and expressed by

$$z(t, s) = \int_s^t F\left(\frac{u}{t}\right) l(u, s)\, du, \quad s < t,$$

for a Volterra kernel l and some function $F \in L^2(\mathbb{R}_+, du)$ vanishing on $(1, \infty]$.

(ii) *In addition, $\widetilde{X}$ is β-self-similar if and only if $l \equiv 0$.*

For the proof, we need the following lemma.

Lemma 3.2. *If a Volterra kernel on $[0, T] \times [0.T]$ is homogeneous with degree (-1), then it vanishes on $[0, T] \times [0.T]$.*

Proof. Let a Volterra kernel h be (-1)-homogeneous. Combining the square integrability and the homogeneity property $h(t, s) = \frac{1}{a} h(\frac{t}{a}, \frac{s}{a}),\ a > 0,\ 0 \le s < t \le T$, yields

$$\int_0^T \int_0^t h(t, s)^2\, ds\, dt = \int_0^{\frac{T}{a}} \int_0^{\frac{t}{a}} h\left(\frac{t}{a}, \frac{s}{a}\right)^2 \frac{1}{a^2}\, ds\, dt = \int_0^{\frac{T}{a}} \int_0^{t'} h(t', s')^2\, ds'\, dt'$$

which is finite for all $a > 0$. This implies that h vanishes on $[0, T] \times [0.T]$. □

Proof. (i) X satisfies the condition (C), and by Theorem 2.2, it admits a canonical Volterra representation with a standard Brownian motion W and a kernel of the form of $k(t, s) = t^{\beta-\frac{1}{2}} F\left(\frac{s}{t}\right)$, $F \in L^2(\mathbb{R}_+, du)$ vanishing on $(1, \infty]$. By using Fubini theorem, (3.3) gives

$$\widetilde{X}_t = X_t - \int_0^t \int_s^t k(t, u) l(u, s)\, du\, dW_s, \quad 0 \le t \le T,$$

which proves the claim.

(ii) Suppose that $\widetilde{X}$ is β-self-similar. From (i), $\widetilde{X}$ has the representation

$$\widetilde{X}_t = \int_0^t \left(k(t, s) - t^{\beta-\frac{1}{2}} z(t, s)\right) dW_s, \quad 0 \le t \le T,$$

which is a canonical Volterra representation. Indeed, if $\mathscr{L}$ denotes the Volterra integral operator associated with the Volterra kernel $l(t, s)$, the integral operator $\mathscr{K} - \mathscr{K}\mathscr{L} = \mathscr{K}(\mathscr{I} - \mathscr{L})$ that corresponds to the Volterra kernel $k(t, s) - t^{\beta-\frac{1}{2}} z(t, s)$ is also a Volterra integral operator, Gohberg and Krein (1969). Here, $\mathscr{I}$ denotes the Identity operator. In particular, if we let $f \in L^2([0, T])$ be such that $\mathscr{K}(\mathscr{I} - \mathscr{L})f = 0$. By (i) in Remark 1.1, the operator $\mathscr{K}$ is injective, hence, $(\mathscr{I} - \mathscr{L})f = 0$, i.e., $\mathscr{L}f = f$. Therefore, the Volterra integral operator $\mathscr{L}$ admits an eigenvalue, which is a contradiction by (ii) in Remark 1.1. So, $f \equiv 0$.

Now, using the fact that $H_{\widetilde{X}}(t) = H_X(t)$ for all t, $\widetilde{X}$ satisfies also the condition (C), and by Theorem 2.2, the canonical kernel $k(t, s) - t^{\beta-\frac{1}{2}} z(t, s)$ is $(\beta - \frac{1}{2})$-homogeneous. For $a > 0$, we write

$$k(t, s) - t^{\beta-\frac{1}{2}} z(t, s) = a^{\beta-\frac{1}{2}} \left(k\left(\frac{t}{a}, \frac{s}{a}\right) - t^{\beta-\frac{1}{2}} z\left(\frac{t}{a}, \frac{s}{a}\right)\right),$$

which implies that $z(t, s) = z(\frac{t}{a}, \frac{s}{a})$, and by the change of variable, we have

$$\int_s^t F\left(\frac{u}{t}\right) l(u, s)\, du = \int_{\frac{s}{a}}^{\frac{t}{a}} F\left(\frac{u}{\frac{t}{a}}\right) l\left(u, \frac{s}{a}\right) du = \int_s^t F\left(\frac{v}{t}\right) \frac{1}{a} l\left(\frac{v}{a}, \frac{s}{a}\right) dv, \quad s < t,$$

which is equivalent to

$$\int_0^t F\left(\frac{u}{t}\right) l(u, s)\, du = \int_0^t F\left(\frac{u}{t}\right) \frac{1}{a} l\left(\frac{u}{a}, \frac{s}{a}\right) dv, \quad s < u < t.$$

Taking derivatives with respect to t on both sides, and since $F\left(\frac{u}{t}\right) \neq 0$, we obtain

$$l(u,s) = \frac{1}{a}\, l\left(\frac{u}{a}, \frac{s}{a}\right), \quad s < u,$$

which means that l is homogeneous with degree (-1). By applying Lemma 3.2, we get $l \equiv 0$.

If $l \equiv 0$, we have $\mathbb{E}(\widetilde{X}_t\widetilde{X}_s) = \mathbb{E}(X_tX_s)$ which means that $\widetilde{X} \stackrel{d}{=} X$. Therefore, $\widetilde{X}$ is β-self-similar. □

Remark 3.3. The importance of the condition (C) in Proposition 3.1 can be seen in the case of the fBm with index $H = 1$, i.e. $B_t^H = tB_1^H$, $0 \le t \le T$. Here the condition (C) fails. Since fBm is Gaussian, each process is determined by its covariance $\mathbb{E}(B_t^HB_s^H) = ts\,\mathbb{E}((B_1^H)^2)$. However, the laws of processes that correspond to different values of $\mathbb{E}((B_1^H)^2)$ are equivalent, on the other hand, these laws are different.

Acknowledgments

The author wishes to thank Tommi Sottinen for discussions and helpful remarks. Thanks are also due to the referee for useful comments. The author would like to thank the Finnish Doctoral Programme in Stochastics and Statistics (contract number 2730350) and the Finnish Cultural Foundation (grant number 00121045) for financial support.

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Stochastic Processes and their Applications 124 (2014) 3084–3105

stochastic processes and their applications

www.elsevier.com/locate/spa

Generalized Gaussian bridges

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Received 16 April 2013; received in revised form 3 April 2014; accepted 4 April 2014
Available online 13 April 2014

Abstract

A generalized bridge is a stochastic process that is conditioned on N linear functionals of its path. We consider two types of representations: orthogonal and canonical. The orthogonal representation is constructed from the entire path of the process. Thus, the future knowledge of the path is needed. In the canonical representation the filtrations of the bridge and the underlying process coincide. The canonical representation is provided for prediction-invertible Gaussian processes. All martingales are trivially prediction-invertible. A typical non-semimartingale example of a prediction-invertible Gaussian process is the fractional Brownian motion. We apply the canonical bridges to insider trading.

MSC: 60G15; 60G22; 91G80

Keywords: Canonical representation; Enlargement of filtration; Fractional Brownian motion; Gaussian process; Gaussian bridge; Hitsuda representation; Insider trading; Orthogonal representation; Prediction-invertible process; Volterra process

1. Introduction

Let $X = (X_t)_{t\in[0,T]}$ be a continuous Gaussian process with positive definite covariance function R, mean function m of bounded variation, and $X_0 = m(0)$. We consider the conditioning, or bridging, of X on N linear functionals $\mathbf{G}_T = [G^i_T]_{i=1}^N$ of its paths:

$$\mathbf{G}_T(X) = \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \left[\int_0^T g_i(t)\,\mathrm{d}X_t\right]_{i=1}^N. \tag{1.1}$$

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http://dx.doi.org/10.1016/j.spa.2014.04.002

We assume, without any loss of generality, that the functions g_i are linearly independent. Indeed, if this is not the case then the linearly dependent, or redundant, components of $\mathbf{g}$ can simply be removed from the conditioning (1.2) without changing it.

The integrals in the conditioning (1.1) are the so-called abstract Wiener integrals (see Definition 2.5 later). The abstract Wiener integral $\int_0^T g(t)\,\mathrm{d}X_t$ will be well-defined for functions or generalized functions g that can be approximated by step functions in the inner product $⟪\cdot,\cdot⟫$ defined by the covariance R of X by bilinearly extending the relation $⟪1_{[0,t)}, 1_{[0,s)}⟫ = R(t,s)$. This means that the integrands g are equivalence classes of Cauchy sequences of step functions in the norm $|||\cdot|||$ induced by the inner product $⟪\cdot,\cdot⟫$. Recall that for the case of Brownian motion we have $R(t,s) = t \wedge s$. Therefore, for the Brownian motion, the equivalence classes of step functions are simply the space $L^2([0,T])$.

Informally, the generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ is (the law of) the Gaussian process X conditioned on the set

$$\left\{\int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y}\right\} = \bigcap_{i=1}^N \left\{\int_0^T g_i(t)\,\mathrm{d}X_t = y_i\right\}. \tag{1.2}$$

The rigorous definition is given in Definition 1.3 later.

For the sake of convenience, we will work on the canonical filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\Omega = C([0,T])$, $\mathscr{F}$ is the Borel σ-algebra on $C([0,T])$ with respect to the supremum norm, and $\mathbb{P}$ is the Gaussian measure corresponding to the Gaussian coordinate process $X_t(\omega) = \omega(t)$: $\mathbb{P} = \mathbb{P}[X \in \cdot\,]$. The filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ is the intrinsic filtration of the coordinate process X that is augmented with the null-sets and made right-continuous.

Definition 1.3. The *generalized bridge measure* $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ is the regular conditional law

$$\mathbb{P}^{\mathbf{g};\mathbf{y}} = \mathbb{P}^{\mathbf{g};\mathbf{y}}[X \in \cdot\,] = \mathbb{P}\left[X \in \cdot \;\middle|\; \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y}\right].$$

A *representation of the generalized Gaussian bridge* is any process $X^{\mathbf{g};\mathbf{y}}$ satisfying

$$\mathbb{P}\left[X^{\mathbf{g};\mathbf{y}} \in \cdot\,\right] = \mathbb{P}^{\mathbf{g};\mathbf{y}}[X \in \cdot\,] = \mathbb{P}\left[X \in \cdot \;\middle|\; \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y}\right].$$

Note that the conditioning on the $\mathbb{P}$-null-set (1.2) in Definition 1.3 is not a problem, since the canonical space of continuous processes is a Polish space and all Polish spaces are Borel spaces and thus admit regular conditional laws, cf. [20, Theorems A1.2 and 6.3]. Also, note that as a *measure* $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ the generalized Gaussian bridge is unique, but it has several different *representations* $X^{\mathbf{g};\mathbf{y}}$. Indeed, for any representation of the bridge one can combine it with any $\mathbb{P}$-measure-preserving transformation to get a new representation.

In this paper we provide two different representations for $X^{\mathbf{g};\mathbf{y}}$. The first representation, given by Theorem 3.1, is called the *orthogonal representation*. This representation is a simple consequence of orthogonal decompositions of Hilbert spaces associated with Gaussian processes and it can be constructed for any continuous Gaussian process for any conditioning functionals. The second representation, given by Theorem 4.25, is called the *canonical representation*. This representation is more interesting but also requires more assumptions. The canonical representation is dynamically invertible in the sense that the linear spaces $\mathscr{L}_t(X)$ and $\mathscr{L}_t(X^{\mathbf{g};\mathbf{y}})$ (see Definition 2.1 later) generated by the process X and its bridge representation $X^{\mathbf{g};\mathbf{y}}$ coincide for all times $t \in [0,T)$. This means that at every time point $t \in [0,T)$ the bridge and the underlying process can be constructed from each others without knowing the future-time development of the underlying process or the bridge. A typical example of a non-semimartingale Gaussian process for which we can provide the canonically represented generalized bridge is the fractional Brownian motion.

The canonically represented bridge $X^{\mathbf{g};\mathbf{y}}$ can be interpreted as the original process X with an added "information drift" that bridges the process at the final time T. This dynamic drift interpretation should turn out to be useful in applications. We give one such application in connection to insider trading in Section 5. This application is, we must admit, a bit classical.

On earlier work related to bridges, we would like to mention first Alili [1], Baudoin [5], Baudoin and Coutin [6] and Gasbarra et al. [13]. In [1] generalized Brownian bridges were considered. It is our opinion that our article extends [1] considerably, although we do not consider the "non-canonical representations" of [1]. Indeed, Alili [1] only considered Brownian motion. Our investigation extends to a large class of non-semimartingale Gaussian processes. Also, Alili [1] did not give the canonical representation for bridges, i.e. the solution to Eq. (4.9) was not given. We solve Eq. (4.9) in (4.14). The article [5] is, in a sense, more general than this article, since we condition on fixed values $\mathbf{y}$, but in [5] the conditioning is on a probability law. However, in [5] only the Brownian bridge was considered. In that sense our approach is more general. In [6, 13] (simple) bridges were studied in a similar Gaussian setting as in this article. In this article we generalize the results of [6] and [13] to generalized bridges. Second, we would like to mention the articles [9,11,14,17] that deal with Markovian and Lévy bridges and [12] that studies generalized Gaussian bridges in the semimartingale context and their functional quantization.

This paper is organized as follows. In Section 2 we recall some Hilbert spaces related to Gaussian processes. In Section 3 we give the orthogonal representation for the generalized bridge in the general Gaussian setting. Section 4 deals with the canonical bridge representation. First we give the representation for Gaussian martingales. Then we introduce the so-called prediction-invertible processes and develop the canonical bridge representation for them. Then we consider invertible Gaussian Volterra processes, such as the fractional Brownian motion, as examples of prediction-invertible processes. Finally, in Section 5 we apply the bridges to insider trading. Indeed, the bridge process can be understood from the initial enlargement of filtration point of view. For more information on the enlargement of filtrations we refer to [10,19].

2. Abstract Wiener integrals and related Hilbert spaces

In this section $X = (X_t)_{t\in[0,T]}$ is a continuous (and hence separable) Gaussian process with positive definite covariance R, mean zero and $X_0 = 0$.

Definitions 2.1 and 2.2 give us two central separable Hilbert spaces connected to separable Gaussian processes.

Definition 2.1. Let $t \in [0, T]$. The *linear space* $\mathscr{L}_t(X)$ is the Gaussian closed linear subspace of $L^2(\Omega, \mathscr{F}, \mathbb{P})$ generated by the random variables $X_s,\ s \le t$, i.e. $\mathscr{L}_t(X) = \overline{\mathrm{span}}\{X_s; s \le t\}$, where the closure is taken in $L^2(\Omega, \mathscr{F}, \mathbb{P})$.

The linear space is a Gaussian Hilbert space with the inner product $\mathbb{C}\mathrm{ov}[\cdot,\cdot]$. Note that since X is continuous, R is also continuous, and hence $\mathscr{L}_t(X)$ is separable, and any orthogonal basis $(\xi_n)_{n=1}^{\infty}$ of $\mathscr{L}_t(X)$ is a collection of independent standard normal random variables. (Of course, since we chose to work on the canonical space, $L^2(\Omega, \mathscr{F}, \mathbb{P})$ is itself a separable Hilbert space.)

Definition 2.2. Let $t \in [0, T]$. The *abstract Wiener integrand space* $\Lambda_t(X)$ is the completion of the linear span of the indicator functions $1_s := 1_{[0,s)},\ s \le t$, under the inner product $\langle\!\langle\!\langle \cdot,\cdot \rangle\!\rangle\!\rangle$

extended bilinearly from the relation

$$\langle\!\langle\!\langle 1_s, 1_u \rangle\!\rangle\!\rangle = R(s,u).$$

The elements of the abstract Wiener integrand space are equivalence classes of Cauchy sequences $(f_n)_{n=1}^{\infty}$ of piecewise constant functions. The equivalence of $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ means that

$$|\!|\!| f_n - g_n |\!|\!| \to 0, \quad \text{as } n \to \infty,$$

where $|\!|\!| \cdot |\!|\!| = \sqrt{\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle}$.

Remark 2.3. (i) The elements of $\Lambda_t(X)$ cannot in general be identified with functions as pointed out e.g. by Pipiras and Taqqu [22] for the case of fractional Brownian motion with Hurst index $H > 1/2$. However, if R is of bounded variation one can identity the function space $|\Lambda_t|(X) \subset \Lambda_t(X)$:

$$|\Lambda_t|(X) = \left\{ f \in \mathbb{R}^{[0,t]};\ \int_0^t \int_0^t |f(s)f(u)|\,|R|(\mathrm{d}s, \mathrm{d}u) < \infty \right\}.$$

(ii) While one may want to interpret that $\Lambda_s(X) \subset \Lambda_t(X)$ for $s \le t$ it may happen that $f \in \Lambda_t(X)$, but $f 1_s \notin \Lambda_s(X)$. Indeed, it may be that $|\!|\!| f 1_s |\!|\!| > |\!|\!| f |\!|\!|$. See Bender and Elliott [7] for an example in the case of fractional Brownian motion.

The space $\Lambda_t(X)$ is isometric to $\mathscr{L}_t(X)$. Indeed, the relation

$$\mathscr{I}_t^X[1_s] := X_s, \quad s \le t, \tag{2.4}$$

can be extended linearly into an isometry from $\Lambda_t(X)$ onto $\mathscr{L}_t(X)$.

Definition 2.5. The isometry $\mathscr{I}_t^X : \Lambda_t(X) \to \mathscr{L}_t(X)$ extended from the relation (2.4) is the *abstract Wiener integral*. We denote

$$\int_0^t f(s)\,\mathrm{d}X_s := \mathscr{I}_t^X[f].$$

Let us end this section by noting that the abstract Wiener integral and the linear spaces are now connected as

$$\mathscr{L}_t(X) = \{\mathscr{I}_t[f];\ f \in \Lambda_t(X)\}.$$

In the special case of the Brownian motion this relation reduces to the well-known Itô isometry with

$$\mathscr{L}_t(W) = \left\{ \int_0^t f(s)\,\mathrm{d}W_s;\ f \in L^2([0,t]) \right\}.$$

3. Orthogonal generalized bridge representation

Denote by $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ the matrix

$$\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle_{ij} := \langle\!\langle\!\langle g_i, g_j \rangle\!\rangle\!\rangle := \mathbb{C}\mathrm{ov}\left[\int_0^T g_i(t)\,\mathrm{d}X_t, \int_0^T g_j(t)\,\mathrm{d}X_t \right].$$

Note that $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ does not depend on the mean of X nor on the conditioned values $\mathbf{y}$: $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ depends only on the conditioning functions $\mathbf{g} = [g_i]_{i=1}^N$ and the covariance R. Also, since $g_1, \ldots, g_N$ are linearly independent and R is positive definite, the matrix $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ is invertible.

Theorem 3.1. *The generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ can be represented as*

$$X_t^{\mathbf{g};\mathbf{y}} = X_t - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \left(\int_0^T \mathbf{g}(u)\, \mathrm{d}X_u - \mathbf{y} \right). \tag{3.2}$$

Moreover, $X^{\mathbf{g};\mathbf{y}}$ is a Gaussian process with

$$\mathbb{E}\left[X_t^{\mathbf{g};\mathbf{y}}\right] = m(t) - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \left(\int_0^T \mathbf{g}(u)\, \mathrm{d}m(u) - \mathbf{y} \right),$$
$$\mathbb{C}\mathrm{ov}\left[X_t^{\mathbf{g};\mathbf{y}}, X_s^{\mathbf{g};\mathbf{y}}\right] = \langle\!\langle\!\langle 1_t, 1_s \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \langle\!\langle\!\langle 1_s, \mathbf{g} \rangle\!\rangle\!\rangle.$$

Proof. It is well-known (see, e.g., [24, p. 304]) from the theory of multivariate Gaussian distributions that conditional distributions are Gaussian with

$$\mathbb{E}\left[X_t \middle| \int_0^T \mathbf{g}(u)\mathrm{d}X_u = \mathbf{y}\right] = m(t) + \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \left(\mathbf{y} - \int_0^T \mathbf{g}(u)\, \mathrm{d}m(u) \right),$$
$$\mathbb{C}\mathrm{ov}\left[X_t, X_s \middle| \int_0^T \mathbf{g}(u)\, \mathrm{d}X_u = \mathbf{y}\right] = \langle\!\langle\!\langle 1_t, 1_s \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \langle\!\langle\!\langle 1_s, \mathbf{g} \rangle\!\rangle\!\rangle.$$

The claim follows from this. □

Corollary 3.3. *Let X be a centered Gaussian process with $X_0 = 0$ and let m be a function of bounded variation. Denote $X^{\mathbf{g}} := X^{\mathbf{g};\mathbf{0}}$, i.e., $X^{\mathbf{g}}$ is conditioned on $\{\int_0^T \mathbf{g}(t)\mathrm{d}X_t = \mathbf{0}\}$. Then*

$$(X+m)_t^{\mathbf{g};\mathbf{y}} = X_t^{\mathbf{g}} + \left(m(t) - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \int_0^T \mathbf{g}(u)\, \mathrm{d}m(u) \right) + \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \mathbf{y}.$$

Remark 3.4. Corollary 3.3 tells us how to construct, by adding a deterministic drift, a general bridge from a bridge that is constructed from a centered process with conditioning $\mathbf{y} = \mathbf{0}$. So, in what follows, we shall almost always assume that the process X is centered, i.e. $m(t) = 0$, and all conditionings are with $\mathbf{y} = \mathbf{0}$.

Example 3.5. Let X be a zero mean Gaussian process with covariance function R. Consider the conditioning on the final value and the average value:

$$X_T = 0,$$
$$\frac{1}{T}\int_0^T X_t\, \mathrm{d}t = 0.$$

This is a generalized Gaussian bridge. Indeed,

$$X_T = \int_0^T 1\, \mathrm{d}X_t =: \int_0^T g_1(t)\, \mathrm{d}X_t,$$
$$\frac{1}{T}\int_0^T X_t\, \mathrm{d}t = \int_0^T \frac{T-t}{T}\, \mathrm{d}X_t =: \int_0^T g_2(t)\, \mathrm{d}X_t.$$

Now,

$$\langle\!\langle\!\langle 1_t, g_1 \rangle\!\rangle\!\rangle = \mathbb{E}[X_t X_T] = R(t,T),$$
$$\langle\!\langle\!\langle 1_t, g_2 \rangle\!\rangle\!\rangle = \mathbb{E}\left[X_t \frac{1}{T}\int_0^T X_s\,\mathrm{d}s\right] = \frac{1}{T}\int_0^T R(t,s)\,\mathrm{d}s,$$
$$\langle\!\langle\!\langle g_1, g_1 \rangle\!\rangle\!\rangle = \mathbb{E}[X_T X_T] = R(T,T),$$
$$\langle\!\langle\!\langle g_1, g_2 \rangle\!\rangle\!\rangle = \mathbb{E}\left[X_T \frac{1}{T}\int_0^T X_s\,\mathrm{d}s\right] = \frac{1}{T}\int_0^T R(T,s)\,\mathrm{d}s,$$
$$\langle\!\langle\!\langle g_2, g_2 \rangle\!\rangle\!\rangle = \mathbb{E}\left[\frac{1}{T}\int_0^T X_s\,\mathrm{d}s \frac{1}{T}\int_0^T X_u\,\mathrm{d}u\right] = \frac{1}{T^2}\int_0^T\int_0^T R(s,u)\,\mathrm{d}u\mathrm{d}s,$$
$$|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle| = \frac{1}{T^2}\int_0^T\int_0^T R(T,T)R(s,u) - R(T,s)R(T,u)\,\mathrm{d}u\,\mathrm{d}s$$

and

$$\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} = \frac{1}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|}\begin{bmatrix} \langle\!\langle\!\langle g_2, g_2 \rangle\!\rangle\!\rangle & -\langle\!\langle\!\langle g_1, g_2 \rangle\!\rangle\!\rangle \\ -\langle\!\langle\!\langle g_1, g_2 \rangle\!\rangle\!\rangle & \langle\!\langle\!\langle g_1, g_1 \rangle\!\rangle\!\rangle \end{bmatrix}.$$

Thus, by Theorem 3.1,

$$\begin{aligned} X_t^{\mathbf{g}} &= X_t - \frac{\langle\!\langle\!\langle 1_t, g_1 \rangle\!\rangle\!\rangle \langle\!\langle\!\langle g_2, g_2 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle 1_t, g_2 \rangle\!\rangle\!\rangle \langle\!\langle\!\langle g_1, g_2 \rangle\!\rangle\!\rangle}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|}\int_0^T g_1(t)\,\mathrm{d}X_t \\ &\quad - \frac{\langle\!\langle\!\langle 1_t, g_2 \rangle\!\rangle\!\rangle \langle\!\langle\!\langle g_1, g_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle 1_t, g_1 \rangle\!\rangle\!\rangle \langle\!\langle\!\langle g_1, g_2 \rangle\!\rangle\!\rangle}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|}\int_0^T g_2(t)\,\mathrm{d}X_t \\ &= X_t - \frac{\int_0^T\int_0^T R(t,T)R(s,u) - R(t,s)R(T,s)\mathrm{d}s\,\mathrm{d}u}{\int_0^T\int_0^T R(T,T)R(s,u) - R(T,s)R(T,u)\,\mathrm{d}s\,\mathrm{d}u} X_T \\ &\quad - \frac{T\int_0^T R(T,T)R(t,s) - R(t,T)R(T,s)\mathrm{d}s}{\int_0^T\int_0^T R(T,T)R(s,u) - R(T,s)R(T,u)\,\mathrm{d}s\,\mathrm{d}u}\int_0^T \frac{T-t}{T}\,\mathrm{d}X_t. \end{aligned}$$

Remark 3.6. (i) Since Gaussian conditionings are projections in Hilbert space to a subspace, it is well-known that they can be done iteratively. Indeed, let $X^n := X^{g_1,\ldots,g_n;y_1,\ldots,y_n}$ and let $X^0 := X$ be the original process. Then the orthogonal generalized bridge representation X^N can be constructed from the rule

$$X_t^n = X_t^{n-1} - \frac{\langle\!\langle\!\langle 1_t, g_n \rangle\!\rangle\!\rangle_{n-1}}{\langle\!\langle\!\langle g_n, g_n \rangle\!\rangle\!\rangle_{n-1}}\left[\int_0^T g_n(u)\,\mathrm{d}X_u^{n-1} - y_n\right],$$

where $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle_{n-1}$ is the inner product in $\mathscr{L}_T(X^{n-1})$.

(ii) If $g_j = 1_{t_j}$, $j = 1,\ldots,N$, then the corresponding generalized bridge is a *multibridge*. That is, it is pinned down to values y_j at points t_j. For the multibridge $X^N = X^{1_{t_1},\ldots,1_{t_N};y_1,\ldots,y_N}$ the orthogonal bridge decomposition can be constructed from the iteration

$$X_t^0 = X_t,$$
$$X_t^n = X_t^{n-1} - \frac{R_{n-1}(t,t_n)}{R_{n-1}(t_n,t_n)}\left[X_{t_n}^{n-1} - y_n\right],$$

where

$$R_0(t,s) = R(t,s),$$
$$R_n(t,s) = R_{n-1}(t,s) - \frac{R_{n-1}(t,t_n)R_{n-1}(t_n,s)}{R_{n-1}(t_n,t_n)}.$$

4. Canonical generalized Bridge representation

The problem with the orthogonal bridge representation (3.2) of $X^{\mathbf{g};\mathbf{y}}$ is that in order to construct it at any point $t \in [0,T)$ one needs the whole path of the underlying process X up to time T. In this section we construct a bridge representation that is canonical in the following sense:

Definition 4.1. The bridge $X^{\mathbf{g};\mathbf{y}}$ is of *canonical representation* if, for all $t \in [0,T)$, $X_t^{\mathbf{g};\mathbf{y}} \in \mathscr{L}_t(X)$ and $X_t \in \mathscr{L}_t(X^{\mathbf{g};\mathbf{y}})$.

Example 4.2. Consider the classical Brownian bridge. That is, condition the Brownian motion W with $\mathbf{g} = g = 1$. Now, the orthogonal representation is

$$W_t^1 = W_t - \frac{t}{T} W_T.$$

This is not a canonical representation, since the future knowledge W_T is needed to construct W_t^1 for any $t \in (0,T)$. A canonical representation for the Brownian bridge is, by calculating the $\ell_{\mathbf{g}}^*$ in Theorem 4.12,

$$\begin{aligned} W_t^1 &= W_t - \int_0^t \int_0^s \frac{1}{T-u}\,\mathrm{d}W_u\,\mathrm{d}s \\ &= (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}W_s. \end{aligned}$$

Remark 4.3. Since the conditional laws of Gaussian processes are Gaussian and Gaussian spaces are linear, the assumptions $X_t^{\mathbf{g};\mathbf{y}} \in \mathscr{L}_t(X)$ and $X_t \in \mathscr{L}_t(X^{\mathbf{g};\mathbf{y}})$ of Definition 4.1 are the same as assuming that $X_t^{\mathbf{g};\mathbf{y}}$ is $\mathscr{F}_t^X$-measurable and X_t is $\mathscr{F}_t^{X^{\mathbf{g};\mathbf{y}}}$-measurable (and, consequently, $\mathscr{F}_t^X = \mathscr{F}_t^{X^{\mathbf{g};\mathbf{y}}}$). This fact is very special to Gaussian processes. Indeed, in general conditioned processes such as generalized bridges are not linear transformations of the underlying process.

We shall require that the restricted measures $\mathbb{P}_t^{\mathbf{g},\mathbf{y}} := \mathbb{P}^{\mathbf{g};\mathbf{y}}|\mathscr{F}_t$ and $\mathbb{P}_t := \mathbb{P}|\mathscr{F}_t$ are equivalent for all $t < T$ (they are obviously singular for $t = T$). To this end we assume that the matrix

$$\begin{aligned} \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle_{ij}(t) &:= \mathbb{E}\left[\left(G_T^i(X) - G_t^i(X)\right)\left(G_T^j(X) - G_t^j(X)\right)\right] \\ &= \mathbb{E}\left[\int_t^T g_i(s)\,\mathrm{d}X_s \int_t^T g_j(s)\,\mathrm{d}X_s\right] \end{aligned} \tag{4.4}$$

is invertible for all $t < T$.

Remark 4.5. On notation: in the previous section we considered the matrix $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$, but from now on we consider the function $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(\cdot)$. Their connection is of course $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle = \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)$. We hope that this overloading of notation does not cause confusion to the reader.

Gaussian martingales

We first construct the canonical representation when the underlying process is a continuous Gaussian martingale M with strictly increasing bracket $\langle M\rangle$ and $M_0 = 0$. Note that the bracket is strictly increasing if and only if the covariance R is positive definite. Indeed, for Gaussian martingales we have $R(t,s) = \mathbb{V}\mathrm{ar}(M_{t\wedge s}) = \langle M\rangle_{t\wedge s}$.

Define a Volterra kernel

$$\ell_{\mathbf{g}}(t,s) := -\mathbf{g}^\top(t)\,\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\,\mathbf{g}(s). \tag{4.6}$$

Note that the kernel $\ell_{\mathbf{g}}$ depends on the process M through its covariance $\langle\!\langle\!\langle\cdot,\cdot\rangle\!\rangle\!\rangle$, and in the Gaussian martingale case we have

$$\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle_{ij}(t) = \int_t^T g_i(s)g_j(s)\,\mathrm{d}\langle M\rangle_s.$$

Lemma 4.7 is the key observation in finding the canonical generalized bridge representation. Actually, it is a multivariate version of Proposition 6 of [13].

Lemma 4.7. *Let $\ell_{\mathbf{g}}$ be given by (4.6) and let M be a continuous Gaussian martingale with strictly increasing bracket $\langle M\rangle$ and $M_0 = 0$. Then the Radon–Nikodym derivative $\mathrm{d}\mathbb{P}_t^{\mathbf{g}}/\mathrm{d}\mathbb{P}_t$ can be expressed in the form*

$$\frac{\mathrm{d}\mathbb{P}_t^{\mathbf{g}}}{\mathrm{d}\mathbb{P}_t} = \exp\left\{\int_0^t\int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\mathrm{d}M_s - \frac{1}{2}\int_0^t\left(\int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\right)^2\mathrm{d}\langle M\rangle_s\right\}$$

for all $t \in [0,T)$.

Proof. Let

$$p(\mathbf{y};\boldsymbol{\mu},\boldsymbol{\Sigma}) := \frac{1}{(2\pi)^{N/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\}$$

be the Gaussian density on $\mathbb{R}^N$ and let

$$\alpha_t^{\mathbf{g}}(\mathrm{d}\mathbf{y}) := \mathbb{P}\left[\mathbf{G}_T(M)\in\mathrm{d}\mathbf{y}\,\middle|\,\mathscr{F}_t^M\right]$$

be the conditional law of the conditioning functionals $\mathbf{G}_T(M) = \int_0^T \mathbf{g}(s)\,\mathrm{d}M_s$ given the information $\mathscr{F}_t^M$.

First, by Bayes' formula, we have

$$\frac{\mathrm{d}\mathbb{P}_t^{\mathbf{g}}}{\mathrm{d}\mathbb{P}_t} = \frac{\mathrm{d}\alpha_t^{\mathbf{g}}}{\mathrm{d}\alpha_0^{\mathbf{g}}}(\mathbf{0}).$$

Second, by the martingale property, we have

$$\frac{\mathrm{d}\alpha_t^{\mathbf{g}}}{\mathrm{d}\alpha_0^{\mathbf{g}}}(\mathbf{0}) = \frac{p\Big(\mathbf{0};\mathbf{G}_t(M),\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(t)\Big)}{p\Big(\mathbf{0};\mathbf{G}_0(M),\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(0)\Big)},$$

where we have denoted $\mathbf{G}_t(M) = \int_0^t \mathbf{g}(s)\,\mathrm{d}M_s$.

Third, denote

$$\frac{p\Big(\mathbf{0}; \mathbf{G}_t(M), \langle\langle\langle \mathbf{g}\rangle\rangle\rangle(t)\Big)}{p\Big(\mathbf{0}; \mathbf{G}_0(M), \langle\langle\langle \mathbf{g}\rangle\rangle\rangle(0)\Big)} =: \left(\frac{|\langle\langle\langle \mathbf{g}\rangle\rangle\rangle|(0)}{|\langle\langle\langle \mathbf{g}\rangle\rangle\rangle|(t)}\right)^{\frac{1}{2}} \exp\left\{F(t, \mathbf{G}_t(M)) - F(0, \mathbf{G}_0(M))\right\},$$

with

$$F(t, \mathbf{G}_t(M)) = -\frac{1}{2}\left(\int_0^t \mathbf{g}(s)\, \mathrm{d}M_s\right)^{\top} \langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}(0) \left(\int_0^t \mathbf{g}(s)\, \mathrm{d}M_s\right).$$

Then, straightforward differentiation yields

$$\int_0^t \frac{\partial F}{\partial s}(s, \mathbf{G}_s(M))\, \mathrm{d}s = -\frac{1}{2}\int_0^t \left(\int_0^s \ell_{\mathbf{g}}(s,u)\, \mathrm{d}M_u\right)^2 \mathrm{d}\langle M\rangle_s,$$

$$\int_0^t \frac{\partial F}{\partial x}(s, \mathbf{G}_s(M))\, \mathrm{d}M_s = \int_0^t \int_0^s \ell_{\mathbf{g}}(s,u)\, \mathrm{d}M_u\, \mathrm{d}M_s,$$

$$-\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, \mathbf{G}_s(M))\, \mathrm{d}\langle M\rangle_s = \log\left(\frac{|\langle\langle\langle \mathbf{g}\rangle\rangle\rangle|(t)}{|\langle\langle\langle \mathbf{g}\rangle\rangle\rangle|(0)}\right)^{\frac{1}{2}}$$

and the form of the Radon–Nikodym derivative follows by applying the Itô formula. □

Corollary 4.8. *The canonical bridge representation $M^{\mathbf{g}}$ satisfies the stochastic differential equation*

$$\mathrm{d}M_t = \mathrm{d}M_t^{\mathbf{g}} - \int_0^t \ell_{\mathbf{g}}(t,s)\, \mathrm{d}M_s^{\mathbf{g}}\, \mathrm{d}\langle M\rangle_t, \tag{4.9}$$

where $\ell_{\mathbf{g}}$ is given by (4.6). Moreover $\langle M\rangle = \langle M^{\mathbf{g}}\rangle$.

Proof. The claim follows by using Girsanov's theorem. □

Remark 4.10. (i) Note that for all $\varepsilon > 0$,

$$\int_0^{T-\varepsilon} \int_0^t \ell_{\mathbf{g}}(t,s)^2\, \mathrm{d}\langle M\rangle_s\, \mathrm{d}\langle M\rangle_t < \infty.$$

In view of (4.9) this means that the processes M and $M^{\mathbf{g}}$ are equivalent in law on $[0, T-\varepsilon]$ for all $\varepsilon > 0$. Indeed, Eq. (4.9) can be viewed as the *Hitsuda representation* between two equivalent Gaussian processes, cf. Hida and Hitsuda [16]. Also note that

$$\int_0^T \int_0^t \ell_{\mathbf{g}}(t,s)^2\, \mathrm{d}\langle M\rangle_s\, \mathrm{d}\langle M\rangle_t = \infty$$

meaning that the measures $\mathbb{P}$ and $\mathbb{P}^{\mathbf{g}}$ are singular on $[0, T]$.

(ii) In the case of the Brownian bridge, cf. Example 4.2, the item (i) above can be clearly seen. Indeed,

$$\ell_{\mathbf{g}}(t,s) = \frac{1}{T-t}$$

and $\mathrm{d}\langle W\rangle_s = \mathrm{d}s$.

(iii) In the case of $\mathbf{y} \neq \mathbf{0}$, the formula (4.9) takes the form

$$\mathrm{d}M_t = \mathrm{d}M_t^{\mathbf{g};\mathbf{y}} + \left(\mathbf{g}^{\top}(t)\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}(t)\mathbf{y} - \int_0^t \ell_{\mathbf{g}}(t,s)\, \mathrm{d}M_s^{\mathbf{g};\mathbf{y}}\right) \mathrm{d}\langle M\rangle_t. \tag{4.11}$$

Next we solve the stochastic differential equation (4.9) of Corollary 4.8. In general, solving a Volterra–Stieltjes equation like (4.9) in a closed form is difficult. Of course, the general theory of Volterra equations suggests that the solution will be of the form (4.14) of Theorem 4.12, where $\ell^*_{\mathbf{g}}$ is the resolvent kernel of $\ell_{\mathbf{g}}$ determined by the resolvent equation (4.15). Also, the general theory suggests that the resolvent kernel can be calculated implicitly by using the Neumann series. In our case the kernel $\ell_{\mathbf{g}}$ factorizes in its argument. This allows us to calculate the resolvent $\ell^*_{\mathbf{g}}$ explicitly as (4.13). (We would like to point out that a similar SDE was treated in [2,15].)

Theorem 4.12. *Let $s \le t \in [0, T]$. Define the Volterra kernel*

$$\begin{aligned} \ell^*_{\mathbf{g}}(t,s) &:= -\ell_{\mathbf{g}}(t,s)\frac{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)} \\ &= |\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t)\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\,\frac{\mathbf{g}(s)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)}. \end{aligned} \tag{4.13}$$

Then the bridge $M^{\mathbf{g}}$ has the canonical representation

$$\mathrm{d}M^{\mathbf{g}}_t = \mathrm{d}M_t - \int_0^t \ell^*_{\mathbf{g}}(t,s)\,\mathrm{d}M_s\,\mathrm{d}\langle M\rangle_t, \tag{4.14}$$

i.e., (4.14) *is the solution to* (4.9).

Proof. Eq. (4.14) is the solution to (4.9) if the kernel $\ell^*_{\mathbf{g}}$ satisfies the *resolvent equation*

$$\ell_{\mathbf{g}}(t,s) + \ell^*_{\mathbf{g}}(t,s) = \int_s^t \ell_{\mathbf{g}}(t,u)\ell^*_{\mathbf{g}}(u,s)\,\mathrm{d}\langle M\rangle_u. \tag{4.15}$$

This is well-known if $\mathrm{d}\langle M\rangle_u = \mathrm{d}u$, cf. e.g. Riesz and Sz.-Nagy [23]. In the $\mathrm{d}\langle M\rangle$ case the resolvent equation can be derived as in the classical $\mathrm{d}u$ case. We show the derivation here, for the convenience of the reader:

Suppose (4.14) is the solution to (4.9). This means that

$$\begin{aligned} \mathrm{d}M_t &= \left(\mathrm{d}M_t - \int_0^t \ell^*_{\mathbf{g}}(t,s)\,\mathrm{d}M_s\,\mathrm{d}\langle M\rangle_t\right) \\ &\quad - \int_0^t \ell_{\mathbf{g}}(t,s)\left(\mathrm{d}M_s - \int_0^s \ell^*_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\,\mathrm{d}\langle M\rangle_s\right)\mathrm{d}\langle M\rangle_t, \end{aligned}$$

or, in the integral form, by using Fubini's theorem,

$$\begin{aligned} M_t &= M_t - \int_0^t\int_s^t \ell^*_{\mathbf{g}}(u,s)\;\mathrm{d}\langle M\rangle_u \mathrm{d}M_s - \int_0^t\int_s^t \ell_{\mathbf{g}}(u,s)\;\mathrm{d}\langle M\rangle_u \mathrm{d}M_s \\ &\quad + \int_0^t\int_s^t\int_u^s \ell_{\mathbf{g}}(s,v)\ell^*_{\mathbf{g}}(v,u)\mathrm{d}\langle M\rangle_v\,\mathrm{d}\langle M\rangle_u \mathrm{d}M_s. \end{aligned}$$

The resolvent criterion (4.15) follows by identifying the integrands in the $\mathrm{d}\langle M\rangle_u\mathrm{d}M_s$-integrals above.

Finally, let us check that the resolvent equation (4.15) is satisfied with $\ell_{\mathbf{g}}$ and $\ell^*_{\mathbf{g}}$ defined by (4.6) and (4.13), respectively:

$$\begin{aligned} &\int_s^t \ell_{\mathbf{g}}(t,u)\ell^*_{\mathbf{g}}(u,s)\,\mathrm{d}\langle M\rangle_u \\ &\quad = -\int_s^t \mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(u)\,|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(u)\mathbf{g}^\top(u)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(u)\frac{\mathbf{g}(s)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)}\,\mathrm{d}\langle M\rangle_u \end{aligned}$$

$$
\begin{aligned}
&= -\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\frac{\mathbf{g}(s)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)}\int_s^t \mathbf{g}(u)|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(u)\mathbf{g}^\top(u)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(u)\,\mathrm{d}\langle M\rangle_u \\
&= \mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\frac{\mathbf{g}(s)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)}\int_s^t \langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(u)|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(u)\mathrm{d}\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(u) \\
&= \mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\frac{\mathbf{g}(s)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)}\Big(|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t) - |\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)\Big) \\
&= \mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(s)\frac{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)} - \mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(s) \\
&= \ell^*_{\mathbf{g}}(t,s) + \ell_{\mathbf{g}}(t,s),
\end{aligned}
$$

since

$$\mathrm{d}\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(t) = -\mathbf{g}^\top(t)\mathbf{g}(t)\mathrm{d}\langle M\rangle_t.$$

So, the resolvent equation (4.15) holds. □

Gaussian prediction-invertible processes

To construct a canonical representation for bridges of Gaussian non-semimartingales is problematic, since we cannot apply stochastic calculus to non-semimartingales. In order to invoke the stochastic calculus we need to associate the Gaussian non-semimartingale with some martingale. A natural martingale associated with a stochastic process is its prediction martingale:

For a (Gaussian) process X its *prediction martingale* is the process $\hat{X}$ defined as

$$\hat{X}_t = \mathbb{E}\left[X_T|\mathscr{F}^X_t\right].$$

Since for Gaussian processes $\hat{X}_t \in \mathscr{L}_t(X)$, we may write, at least informally, that

$$\hat{X}_t = \int_0^t p(t,s)\,\mathrm{d}X_s,$$

where the abstract kernel p depends also on T (since $\hat{X}$ depends on T). In Definition 4.16 we assume that the kernel p exists as a real, and not only formal, function. We also assume that the kernel p is invertible.

Definition 4.16. A Gaussian process X is *prediction-invertible* if there exists a kernel p such that its prediction martingale $\hat{X}$ is continuous, can be represented as

$$\hat{X}_t = \int_0^t p(t,s)\,\mathrm{d}X_s,$$

and there exists an inverse kernel p^{-1} such that, for all $t \in [0,T]$, $p^{-1}(t,\cdot) \in L^2([0,T],\mathrm{d}\langle\hat{X}\rangle)$ and X can be recovered from $\hat{X}$ by

$$X_t = \int_0^t p^{-1}(t,s)\,\mathrm{d}\hat{X}_s.$$

Remark 4.17. In general it seems to be a difficult problem to determine whether a Gaussian process is prediction-invertible or not. In the discrete time non-degenerate case all Gaussian processes are prediction-invertible. In continuous time the situation is more difficult, as Example 4.18 illustrates. Nevertheless, we can immediately see that if the centered Gaussian process X with covariance R is prediction-invertible, then the covariance must satisfy the relation

$$R(t,s) = \int_0^{t\wedge s} p^{-1}(t,u)\, p^{-1}(s,u)\, \mathrm{d}\langle \hat{X}\rangle_u,$$

where the bracket $\langle \hat{X}\rangle$ can be calculated as the variance of the conditional expectation:

$$\langle \hat{X}\rangle_u = \mathbb{V}\mathrm{ar}\left(\mathbb{E}\left[X_T|\mathscr{F}_u\right]\right).$$

However, this criterion does not seem to be very helpful in practice.

Example 4.18. Consider the Gaussian slope $X_t = t\xi,\ t \in [0,T]$, where ξ is a standard normal random variable. Now, if we consider the "raw filtration" $\mathscr{G}_t^X = \sigma(X_s; s \le t)$, then X is not prediction invertible. Indeed, then $\hat{X}_0 = 0$ but $\hat{X}_t = X_T$, if $t \in (0,T]$. So, $\hat{X}$ is not continuous. On the other hand, the augmented filtration is simply $\mathscr{F}_t^X = \sigma(\xi)$ for all $t \in [0,T]$. So, $\hat{X} = X_T$. Note, however, that in both cases the slope X can be recovered from the prediction martingale: $X_t = \frac{t}{T}\hat{X}_t$.

In order to represent abstract Wiener integrals of X in terms of Wiener–Itô integrals of $\hat{X}$ we need to extend the kernels p and p^{-1} to linear operators:

Definition 4.19. Let X be prediction-invertible. Define operators P and P^{-1} by extending linearly the relations

$$\mathrm{P}[1_t] = p(t,\cdot),$$
$$\mathrm{P}^{-1}[1_t] = p^{-1}(t,\cdot).$$

Now the following lemma is obvious.

Lemma 4.20. *Let f be such a function that $\mathrm{P}^{-1}[f] \in L^2([0,T], \mathrm{d}\langle\hat{X}\rangle)$ and let $\hat{g} \in L^2([0,T], \mathrm{d}\langle\hat{X}\rangle)$. Then*

$$\int_0^T f(t)\,\mathrm{d}X_t = \int_0^T \mathrm{P}^{-1}[f](t)\,\mathrm{d}\hat{X}_t, \tag{4.21}$$

$$\int_0^T \hat{g}(t)\,\mathrm{d}\hat{X}_t = \int_0^T \mathrm{P}[\hat{g}](t)\,\mathrm{d}X_t. \tag{4.22}$$

Remark 4.23. (i) Eqs. (4.21) or (4.22) can actually be taken as the definition of the Wiener integral with respect to X.

(ii) The operators P and P^{-1} depend on T.

(iii) If $p^{-1}(\cdot,s)$ has bounded variation, we can represent P^{-1} as

$$\mathrm{P}^{-1}[f](s) = f(s)p^{-1}(T,s) + \int_s^T \left(f(t) - f(s)\right)\, p^{-1}(\mathrm{d}t, s).$$

A similar formula holds for P also, if $p(\cdot,s)$ has bounded variation.

(iv) Let $⟪\mathbf{g}⟫^X(t)$ denote the remaining covariance matrix with respect to X, i.e.,

$$⟪\mathbf{g}⟫_{ij}^X(t) = \mathbb{E}\left[\int_t^T g_i(s)\,\mathrm{d}X_s \int_t^T g_j(s)\,\mathrm{d}X_s\right].$$

Let $⟪\hat{\mathbf{g}}⟫^{\hat{X}}(t)$ denote the remaining covariance matrix with respect to $\hat{X}$, i.e.,

$$⟪\hat{\mathbf{g}}⟫_{ij}^{\hat{X}}(t) = \int_t^T \hat{g}_i(s)\hat{g}_j(s)\,\mathrm{d}\langle\hat{X}\rangle_s.$$

Then

$$⟪\mathbf{g}⟫_{ij}^{X}(t) = ⟪\mathrm{P}^{-1}[\mathbf{g}]⟫_{ij}^{\hat{X}}(t) = \int_t^T \mathrm{P}^{-1}[g_i](s)\mathrm{P}^{-1}[g_j](s)\,\mathrm{d}\langle \hat{X}\rangle_s.$$

Now, let $X^{\mathbf{g}}$ be the bridge conditioned on $\int_0^T \mathbf{g}(s)\,\mathrm{d}X_s = \mathbf{0}$. By Lemma 4.20 we can rewrite the conditioning as

$$\int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \int_0^T \mathrm{P}^{-1}[\mathbf{g}](t)\,\mathrm{d}\hat{X}(t) = \mathbf{0}. \tag{4.24}$$

With this observation the following theorem, that is the main result of this article, follows.

Theorem 4.25. *Let X be prediction-invertible Gaussian process. Assume that, for all $t \in [0, T]$ and $i = 1, \ldots, N$, $g_i 1_t \in \Lambda_t(X)$. Then the generalized bridge $X^{\mathbf{g}}$ admits the canonical representation*

$$X_t^{\mathbf{g}} = X_t - \int_0^t \int_s^t p^{-1}(t,u)\mathrm{P}\left[\hat{\ell}_{\hat{\mathbf{g}}}^*(u,\cdot)\right](s)\,\mathrm{d}\langle \hat{X}\rangle_u\,\mathrm{d}X_s, \tag{4.26}$$

where

$$\hat{g}_i = \mathrm{P}^{-1}[g_i],$$

$$\hat{\ell}_{\hat{\mathbf{g}}}^*(u,v) = |⟪\hat{\mathbf{g}}⟫^{\hat{X}}|(u)\hat{\mathbf{g}}^{\top}(u)(⟪\hat{\mathbf{g}}⟫^{\hat{X}})^{-1}(u)\,\frac{\hat{\mathbf{g}}(v)}{|⟪\hat{\mathbf{g}}⟫^{\hat{X}}|(v)},$$

$$⟪\hat{\mathbf{g}}⟫_{ij}^{\hat{X}}(t) = \int_t^T \hat{g}_i(s)\hat{g}_j(s)\,\mathrm{d}\langle \hat{X}\rangle_s = ⟪\mathbf{g}⟫_{ij}^{X}(t).$$

Proof. Since $\hat{X}$ is a Gaussian martingale and because of the equality (4.24) we can use Theorem 4.12. We obtain

$$\mathrm{d}\hat{X}_s^{\hat{\mathbf{g}}} = \mathrm{d}\hat{X}_s - \int_0^s \hat{\ell}_{\hat{\mathbf{g}}}^*(s,u)\,\mathrm{d}\hat{X}_u\,\mathrm{d}\langle \hat{X}\rangle_s.$$

Now, by using the fact that X is prediction invertible, we can recover X from $\hat{X}$, and consequently also $X^{\mathbf{g}}$ from $\hat{X}^{\hat{\mathbf{g}}}$ by operating with the kernel p^{-1} in the following way:

$$\begin{aligned} X_t^{\mathbf{g}} &= \int_0^t p^{-1}(t,s)\,\mathrm{d}\hat{X}_s^{\hat{\mathbf{g}}} \\ &= X_t - \int_0^t p^{-1}(t,s)\left(\int_0^s \hat{\ell}_{\hat{\mathbf{g}}}^*(s,u)\,\mathrm{d}\hat{X}_u\right)\mathrm{d}\langle \hat{X}\rangle_s. \end{aligned} \tag{4.27}$$

The representation (4.27) is a canonical representation already but it is written in terms of the prediction martingale $\hat{X}$ of X. In order to represent (4.27) in terms of X we change the Wiener integral in (4.27) by using Fubini's theorem and the operator P:

$$\begin{aligned} X_t^{\mathbf{g}} &= X_t - \int_0^t p^{-1}(t,s)\int_0^s \mathrm{P}\left[\hat{\ell}_{\hat{\mathbf{g}}}^*(s,\cdot)\right](u)\,\mathrm{d}X_u\,\mathrm{d}\langle \hat{X}\rangle_s \\ &= X_t - \int_0^t \int_s^t p^{-1}(t,u)\mathrm{P}\left[\hat{\ell}_{\hat{\mathbf{g}}}^*(u,\cdot)\right](s)\,\mathrm{d}\langle \hat{X}\rangle_u\,\mathrm{d}X_s. \quad \square \end{aligned}$$

Remark 4.28. Recall that, by assumption, the processes $X^{\mathbf{g}}$ and X are equivalent on $\mathscr{F}_t,\ t < T$. So, the representation (4.26) is an analogue of the Hitsuda representation for prediction-invertible processes. Indeed, one can show, just like in [25,26], that a zero mean Gaussian process $\tilde{X}$ is equivalent in law to the zero mean prediction-invertible Gaussian process X if it admits the representation

$$\tilde{X}_t = X_t - \int_0^t f(t,s)\,\mathrm{d}X_s$$

where

$$f(t,s) = \int_s^t p^{-1}(t,u)\mathrm{P}\left[v(u,\cdot)\right](s)\,\mathrm{d}\langle\hat{X}\rangle_u$$

for some Volterra kernel $v \in L^2([0,T]^2, \mathrm{d}\langle\hat{X}\rangle \otimes \mathrm{d}\langle\hat{X}\rangle)$.

It seems that, except in [13], the prediction-invertible Gaussian processes have not been studied at all. Therefore, we give a class of prediction-invertible processes that is related to a class that has been studied in the literature: the Gaussian Volterra processes. See, e.g., Alòs et al. [3], for a study of stochastic calculus with respect to Gaussian Volterra processes.

Definition 4.29. V is an *invertible Gaussian Volterra process* if it is continuous and there exist Volterra kernels k and k^{-1} such that

$$V_t = \int_0^t k(t,s)\,\mathrm{d}W_s, \tag{4.30}$$

$$W_t = \int_0^t k^{-1}(t,s)\,\mathrm{d}V_s. \tag{4.31}$$

Here W is the standard Brownian motion, $k(t,\cdot) \in L^2([0,t]) = \Lambda_t(W)$ and $k^{-1}(t,\cdot) \in \Lambda_t(V)$ for all $t \in [0,T]$.

Remark 4.32. (i) The representation (4.30), defining a Gaussian Volterra process, states that the covariance R of V can be written as

$$R(t,s) = \int_0^{t\wedge s} k(t,u)k(s,u)\,\mathrm{d}u.$$

So, in some sense, the kernel k is the square root, or the Cholesky decomposition, of the covariance R.

(ii) The inverse relation (4.31) means that the indicators $1_t,\ t \in [0,T]$, can be approximated in $L^2([0,t])$ with linear combinations of the functions $k(t_j,\cdot),\ t_j \in [0,t]$. I.e., the indicators 1_t belong to the image of the operator K extending the kernel k linearly as discussed below.

Precisely as with the kernels p and p^{-1}, we can define the operators K and K^{-1} by linearly extending the relations

$$\mathrm{K}[1_t] := k(t,\cdot) \quad \text{and} \quad \mathrm{K}^{-1}[1_t] := k^{-1}(t,\cdot).$$

Then, just like with the operators P and P^{-1}, we have

$$\int_0^T f(t)\,\mathrm{d}V_t = \int_0^T \mathrm{K}[f](t)\,\mathrm{d}W_t,$$
$$\int_0^T g(t)\,\mathrm{d}W_t = \int_0^T \mathrm{K}^{-1}[g](t)\,\mathrm{d}V_t.$$

The connection between the operators K and K^{-1} and the operators P and P^{-1} are

$$\mathrm{K}[g] = k(T,\cdot)\mathrm{P}^{-1}[g],$$
$$\mathrm{K}^{-1}[g] = k^{-1}(T,\cdot)\mathrm{P}[g].$$

So, invertible Gaussian Volterra processes are prediction-invertible and the following corollary to Theorem 4.25 is obvious:

Corollary 4.33. *Let V be an invertible Gaussian Volterra process and let $\mathrm{K}[g_i] \in L^2([0,T])$ for all $i = 1,\ldots,N$. Denote*

$$\tilde{\mathbf{g}}(t) := \mathrm{K}[\mathbf{g}](t).$$

Then the bridge $V^{\mathbf{g}}$ admits the canonical representation

$$V_t^{\mathbf{g}} = V_t - \int_0^t \int_s^t k(t,u)\mathrm{K}^{-1}\left[\tilde{\ell}^*_{\tilde{\mathbf{g}}}(u,\cdot)\right](s)\,\mathrm{d}u\,\mathrm{d}V_s, \tag{4.34}$$

where

$$\tilde{\ell}_{\tilde{\mathbf{g}}}(u,v) = |\langle\!\langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W|(u)\tilde{\mathbf{g}}^\top(u)(\langle\!\langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W)^{-1}(u)\,\frac{\tilde{\mathbf{g}}(v)}{|\langle\!\langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W|(v)},$$
$$\langle\!\langle\!\langle \tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W_{ij}(t) = \int_t^T \tilde{g}_i(s)\tilde{g}_j(s)\,\mathrm{d}s = \langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^V_{ij}(t).$$

Example 4.35. The *fractional Brownian motion* $B = (B_t)_{t\in[0,T]}$ with Hurst index $H \in (0,1)$ is a centered Gaussian process with $B_0 = 0$ and covariance function

$$R(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Another way of defining the fractional Brownian motion is that it is the unique centered Gaussian H-self-similar process with stationary increments normalized so that $\mathbb{E}[B_1^2] = 1$.

It is well-known that the fractional Brownian motion is an invertible Gaussian Volterra process with

$$\mathrm{K}[f](s) = c_H s^{\frac{1}{2}-H}\mathrm{I}_{T-}^{H-\frac{1}{2}}\left[(\,\cdot\,)^{H-\frac{1}{2}} f\right](s), \tag{4.36}$$

$$\mathrm{K}^{-1}[f](s) = \frac{1}{c_H} s^{\frac{1}{2}-H}\mathrm{I}_{T-}^{\frac{1}{2}-H}\left[(\,\cdot\,)^{H-\frac{1}{2}} f\right](s). \tag{4.37}$$

Here $\mathrm{I}_{T-}^{H-\frac{1}{2}}$ and $\mathrm{I}_{T-}^{\frac{1}{2}-H}$ are the *Riemann–Liouville fractional integrals* over $[0, T]$ of order $H - \frac{1}{2}$ and $\frac{1}{2} - H$, respectively:

$$\mathrm{I}_{T-}^{H-\frac{1}{2}}[f](t) = \begin{cases} \dfrac{1}{\Gamma\left(H-\frac{1}{2}\right)} \displaystyle\int_t^T \frac{f(s)}{(s-t)^{\frac{3}{2}-H}}\,\mathrm{d}s, & \text{for } H > \dfrac{1}{2}, \\ \dfrac{-1}{\Gamma\left(\frac{3}{2}-H\right)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_t^T \frac{f(s)}{(s-t)^{H-\frac{1}{2}}}\,\mathrm{d}s, & \text{for } H < \dfrac{1}{2}, \end{cases}$$

and c_H is the normalizing constant

$$c_H = \left(\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(2-2H)} \right)^{\frac{1}{2}}.$$

Here

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}t$$

is the Gamma function. For the proofs of these facts and for more information on the fractional Brownian motion we refer to the monographs by Biagini et al. [8] and Mishura [21].

One can calculate the canonical representation for generalized fractional Brownian bridges by using the representation (4.34) by plugging in the operators K and K^{-1} defined by (4.36) and (4.37), respectively. Unfortunately, even for a simple bridge the formula becomes very complicated. Indeed, consider the standard fractional Brownian bridge B^1, i.e., the conditioning is $g(t) = 1_T(t)$. Then

$$\tilde{g}(t) = \mathrm{K}[1_T](t) = k(T, t)$$

is given by (4.36). Consequently,

$$\langle\!\langle\!\langle \tilde{g} \rangle\!\rangle\!\rangle^W(t) = \int_t^T k(T, s)^2\,\mathrm{d}s,$$

$$\tilde{\ell}_{\tilde{g}}^*(u, v) = k(T, u) \frac{k(T, v)}{\int_v^T k(T, w)^2\,\mathrm{d}w}.$$

We obtain the canonical representation for the fractional Brownian bridge:

$$B_t^1 = B_t - \int_0^t \int_s^t k(t, u) k(T, u) \mathrm{K}^{-1}\left[\frac{k(T, \cdot)}{\int_\cdot^T k(T, w)^2\,\mathrm{d}w} \right](s)\,\mathrm{d}u\,\mathrm{d}B_s.$$

This representation can be made "explicit" by plugging in the definitions (4.36) and (4.37). It seems, however, very difficult to simplify the resulting formula.

5. Application to insider trading

We consider insider trading in the context of initial enlargement of filtrations. Our approach here is motivated by Amendiger [4] and Imkeller [18], where only one condition was used. We extend that investigation to multiple conditions although otherwise our investigation is less general than in [4].

Consider an insider who has at time $t = 0$ some insider information of the evolution of the price process of a financial asset S over a period $[0, T]$. We want to calculate the additional expected utility for the insider trader. To make the maximization of the utility of terminal wealth reasonable we have to assume that our model is arbitrage-free. In our Gaussian realm this boils down to assuming that the (discounted) asset prices are governed by the equation

$$\frac{\mathrm{d}S_t}{S_t} = a_t \,\mathrm{d}\langle M\rangle_t + \mathrm{d}M_t, \tag{5.1}$$

where $S_0 = 1$, M is a continuous Gaussian martingale with strictly increasing $\langle M\rangle$ with $M_0 = 0$, and the process a is $\mathbb{F}$-adapted satisfying $\int_0^T a_t^2 \,\mathrm{d}\langle M\rangle_t < \infty$ $\mathbb{P}$-a.s.

Assuming that the trading ends at time $T - \varepsilon$, the insider knows some functionals of the return over the interval $[0, T]$. If $\varepsilon = 0$ there is obviously arbitrage for the insider. The insider information will define a collection of functionals $G_T^i(M) = \int_0^T g_i(t)\,\mathrm{d}M_t$, where $g_i \in L^2([0, T], \mathrm{d}\langle M\rangle)$, $i = 1, \ldots, N$, such that

$$\int_0^T \mathbf{g}(t)\frac{\mathrm{d}S_t}{S_t} = \mathbf{y} = [y_i]_{i=1}^N, \tag{5.2}$$

for some $\mathbf{y} \in \mathbb{R}^N$. This is equivalent to the conditioning of the Gaussian martingale M on the linear functionals $\mathbf{G}_T = [G_T^i]_{i=1}^N$ of the log-returns:

$$\mathbf{G}_T(M) = \int_0^T \mathbf{g}(t)\,\mathrm{d}M_t = \left[\int_0^T g_i(t)\,\mathrm{d}M_t\right]_{i=1}^N.$$

Indeed, the connection is

$$\int_0^T \mathbf{g}(t)\,\mathrm{d}M_t = \mathbf{y} - \langle\!\langle\!\langle a, \mathbf{g}\rangle\!\rangle\!\rangle =: \mathbf{y}',$$

where

$$\langle\!\langle\!\langle a, \mathbf{g}\rangle\!\rangle\!\rangle = [\langle\!\langle\!\langle a, g_i\rangle\!\rangle\!\rangle]_{i=1}^N = \left[\int_0^T a_t g_i(t)\,\mathrm{d}\langle M\rangle_t\right]_{i=1}^N.$$

As the natural filtration $\mathbb{F}$ represents the information available to the ordinary trader, the insider trader's information flow is described by a larger filtration $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ given by

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(G_T^1, \ldots, G_T^N).$$

Under the augmented filtration $\mathbb{G}$, M is no longer a martingale. It is a Gaussian semimartingale with the *semimartingale decomposition*

$$\mathrm{d}M_t = \mathrm{d}\tilde{M}_t + \left(\int_0^t \ell_{\mathbf{g}}(t, s)\,\mathrm{d}M_s - \mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\mathbf{y}'\right)\mathrm{d}\langle M\rangle_t, \tag{5.3}$$

where $\tilde{M}$ is a continuous $\mathbb{G}$-martingale with bracket $\langle M\rangle$, and which can be constructed through the formula (4.11).

In this market, we consider the portfolio process π defined on $[0, T - \varepsilon] \times \Omega$ as the fraction of the total wealth invested in the asset S. So the dynamics of the discounted value process associated to a self-financing strategy π is defined by $V_0 = v_0$ and

$$\frac{\mathrm{d}V_t}{V_t} = \pi_t \frac{\mathrm{d}S_t}{S_t}, \quad \text{for } t \in [0, T - \varepsilon],$$

or equivalently by

$$V_t = v_0 \exp\left(\int_0^t \pi_s \,\mathrm{d}M_s + \int_0^t \left(\pi_s a_s - \frac{1}{2}\pi_s^2 \right) \mathrm{d}\langle M \rangle_s \right). \tag{5.4}$$

Let us denote by $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle_\varepsilon$ and $|||\cdot|||_\varepsilon$ the inner product and the norm on $L^2([0, T - \varepsilon], \mathrm{d}\langle M \rangle)$.

For the ordinary trader, the process π is assumed to be a non-negative $\mathbb{F}$-progressively measurable process such that

(i) $\mathbb{P}[|||\pi|||_\varepsilon^2 < \infty] = 1$.
(ii) $\mathbb{P}[\langle\!\langle\!\langle \pi, f \rangle\!\rangle\!\rangle_\varepsilon < \infty] = 1$, for all $f \in L^2([0, T - \varepsilon], \mathrm{d}\langle M \rangle)$.

We denote this class of portfolios by $\Pi(\mathbb{F})$. By analogy, the class of the portfolios disposable to the insider trader shall be denoted by $\Pi(\mathbb{G})$, the class of non-negative $\mathbb{G}$-progressively measurable processes that satisfy the conditions (i) and (ii) above.

The aim of both investors is to maximize the expected utility of the terminal wealth $V_{T-\varepsilon}$, by finding an optimal portfolio π on $[0, T - \varepsilon]$ that solves the optimization problem

$$\max_\pi \mathbb{E}\left[U(V_{T-\varepsilon}) \right].$$

Here, the utility function U will be the logarithmic utility function, and the utility of the process (5.4) valued at time $T - \varepsilon$ is

$$\begin{aligned} \log V_{T-\varepsilon} &= \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s + \int_0^{T-\varepsilon} \left(\pi_s a_s - \frac{1}{2}\pi_s^2 \right) \mathrm{d}\langle M \rangle_s \\ &= \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s + \frac{1}{2} \int_0^{T-\varepsilon} \pi_s \,(2a_s - \pi_s)\, \mathrm{d}\langle M \rangle_s \\ &= \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s + \frac{1}{2} \langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon. \end{aligned} \tag{5.5}$$

From the ordinary trader's point of view M is a martingale. So, $\mathbb{E}\left(\int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s \right) = 0$ for every $\pi \in \Pi(\mathbb{F})$ and, consequently,

$$\mathbb{E}\left[U(V_{T-\varepsilon}) \right] = \log v_0 + \frac{1}{2} \mathbb{E}\left[\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon \right].$$

Therefore, the ordinary trader, given $\Pi(\mathbb{F})$, will solve the optimization problem

$$\max_{\pi \in \Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon}) \right] = \log v_0 + \frac{1}{2} \max_{\pi \in \Pi(\mathscr{F})} \mathbb{E}\left[\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon \right]$$

over the term $\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon = 2\langle\!\langle\!\langle \pi, a \rangle\!\rangle\!\rangle_\varepsilon - |||\pi|||_\varepsilon^2$. By using the polarization identity we obtain

$$\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon = |||a|||_\varepsilon^2 - |||\pi - a|||_\varepsilon^2 \le |||a|||_\varepsilon^2.$$

Thus, the maximum is obtained with the choice $\pi_t = a_t$ for $t \in [0, T-\varepsilon]$, and maximal expected utility value is

$$\max_{\pi \in \Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] = \log v_0 + \frac{1}{2}\mathbb{E}\left[\lvert\lvert\lvert a \rvert\rvert\rvert_\varepsilon^2\right].$$

From the insider trader's point of view the process M is not a martingale under his information flow $\mathbb{G}$. The insider can update his utility of terminal wealth (5.5) by considering (5.3), where $\tilde{M}$ is a continuous $\mathbb{G}$-martingale. This gives

$$\log V_{T-\varepsilon} = \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}\tilde{M}_s + \frac{1}{2}\langle\langle\langle \pi, 2a - \pi \rangle\rangle\rangle_\varepsilon + \left\langle\!\!\left\langle\!\!\left\langle \pi, \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, t)\,\mathrm{d}M_t - \mathbf{g}^\top \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}\mathbf{y}' \right\rangle\!\!\right\rangle\!\!\right\rangle_\varepsilon .$$

Now, the insider maximizes the expected utility over all $\pi \in \Pi(\mathbb{G})$:

$$\max_{\pi \in \Pi(\mathbb{G})} \mathbb{E}\left[\log V_{T-\varepsilon}\right] = \log v_0 + \frac{1}{2}\max_{\pi \in \Pi(\mathbb{G})} \mathbb{E} \times \left[\left\langle\!\!\left\langle\!\!\left\langle \pi, 2\left(a + \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, t)\,\mathrm{d}M_t - \mathbf{g}^\top \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}\mathbf{y}'\right) - \pi \right\rangle\!\!\right\rangle\!\!\right\rangle_\varepsilon\right].$$

The optimal portfolio π for the insider trader can be computed in the same way as for the ordinary trader. We obtain the optimal portfolio

$$\pi_t = a_t + \int_0^t \ell_{\mathbf{g}}(t, s)\,\mathrm{d}M_s - \mathbf{g}^\top(t)\langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(t)\mathbf{y}', \quad t \in [0, T - \varepsilon].$$

Let us then calculate the additional expected logarithmic utility for the insider trader. Since

$$\mathbb{E}\left[\left\langle\!\!\left\langle\!\!\left\langle a, \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, s)\,\mathrm{d}M_s - \mathbf{g}^\top \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}\mathbf{y}' \right\rangle\!\!\right\rangle\!\!\right\rangle_\varepsilon\right] = 0,$$

we obtain that

$$\begin{aligned}\Delta_{T-\varepsilon} &= \max_{\pi \in \Pi(\mathbb{G})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] - \max_{\pi \in \Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] \\ &= \frac{1}{2}\mathbb{E}\left[\left|\left|\left| \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, s)\,\mathrm{d}M_s - \mathbf{g}^\top \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}\mathbf{y}' \right|\right|\right|_\varepsilon^2\right].\end{aligned}$$

Now, let us use the short-hand notation

$$\begin{aligned}&\mathbf{G}_t := \int_0^t \mathbf{g}(s)\,\mathrm{d}M_s, \\ &\langle\langle\langle \mathbf{g} \rangle\rangle\rangle(t, s) := \langle\langle\langle \mathbf{g} \rangle\rangle\rangle(t) - \langle\langle\langle \mathbf{g} \rangle\rangle\rangle(s), \\ &\langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(t, s) := \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(t) - \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(s).\end{aligned}$$

Then, by expanding the square $\lvert\lvert\lvert \cdot \rvert\rvert\rvert_\varepsilon^2$, we obtain

$$\begin{aligned}2\Delta_{T-\varepsilon} &= \mathbb{E}\left[\left|\left|\left| \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, s)\,\mathrm{d}M_s - \mathbf{g}^\top \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}\mathbf{y}' \right|\right|\right|_\varepsilon^2\right] \\ &= \mathbb{E}\left[\left|\left|\left| \mathbf{g}^\top \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}\left(\mathbf{y}' + \mathbf{G}\right) \right|\right|\right|_\varepsilon^2\right]\end{aligned}$$

$$= \mathbb{E}\left[\int_0^{T-\varepsilon} \mathbf{y}'^{\top} \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^{\top}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{y}'\,\mathrm{d}\langle M\rangle_t\right]$$
$$+\mathbb{E}\left[\int_0^{T-\varepsilon} \mathbf{G}_t^{\top} \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^{\top}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{G}_t\,\mathrm{d}\langle M\rangle_t\right].$$

Now the formula $\mathbb{E}[\mathbf{x}^{\top}\mathbf{A}\mathbf{x}] = \mathrm{Tr}[\mathbf{A}\mathbb{C}\mathrm{ov}\mathbf{x}] + \mathbb{E}[\mathbf{x}]^{\top}\mathbf{A}\mathbb{E}[\mathbf{x}]$ yields

$$\begin{aligned}
2\Delta_{T-\varepsilon} &= \int_0^{T-\varepsilon} \mathbf{y}'^{\top} \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^{\top}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{y}'\,\mathrm{d}\langle M\rangle_t \\
&\quad + \int_0^{T-\varepsilon} \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^{\top}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0,t)\right]\mathrm{d}\langle M\rangle_t \\
&= \mathbf{y}'^{\top} \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon,0)\mathbf{y}' \\
&\quad + \int_0^{T-\varepsilon} \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^{\top}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)\right]\mathrm{d}\langle M\rangle_t \\
&\quad - \int_0^{T-\varepsilon} \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^{\top}(t)\right]\mathrm{d}\langle M\rangle_t \\
&= (\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle)^{\top}\,\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon,0)\,(\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle) \\
&\quad + \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon,0)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)\right] + \log\frac{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(T-\varepsilon)}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(0)}.
\end{aligned}$$

We have proved the following proposition:

Proposition 5.6. *The additional logarithmic utility in the model* (5.1) *for the insider with information* (5.2) *is*

$$\begin{aligned}
\Delta_{T-\varepsilon} &= \max_{\pi\in\Pi(\mathbb{G})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] - \max_{\pi\in\Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] \\
&= \frac{1}{2}(\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle)^{\top}\left(\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon) - \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(0)\right)(\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left[\left(\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon) - \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(0)\right)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)\right] + \frac{1}{2}\log\frac{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(T-\varepsilon)}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(0)}.
\end{aligned}$$

Example 5.7. Consider the classical Black and Scholes pricing model:

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t, \qquad S_0 = 1,$$

where $W = (W_t)_{t\in[0,T]}$ is the standard Brownian motion. Assume that the insider trader knows at time $t = 0$ that the total and the average return of the stock price over the period $[0, T]$ are both zeros and that the trading ends at time $T - \varepsilon$. So, the insider knows that

$$G_T^1 = \int_0^T g_1(t)\,\mathrm{d}W_t = \frac{y_1}{\sigma} - \frac{\mu}{\sigma}\langle\!\langle\!\langle g_1, 1_T \rangle\!\rangle\!\rangle = -\frac{\mu}{\sigma}\langle\!\langle\!\langle g_1, 1_T \rangle\!\rangle\!\rangle$$
$$G_T^2 = \int_0^T g_2(t)\,\mathrm{d}W_t = \frac{y_2}{\sigma} - \frac{\mu}{\sigma}\langle\!\langle\!\langle g_2, 1_T \rangle\!\rangle\!\rangle = -\frac{\mu}{\sigma}\langle\!\langle\!\langle g_2, 1_T \rangle\!\rangle\!\rangle,$$

where

$$g_1(t) = 1_T(t),$$
$$g_2(t) = \frac{T-t}{T}.$$

Then, by Proposition 5.6,

$$\Delta_{T-\varepsilon} = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \langle\langle\langle \mathbf{g}, 1_T \rangle\rangle\rangle^\top \left(\langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(T-\varepsilon) - \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(0)\right) \langle\langle\langle \mathbf{g}, 1_T \rangle\rangle\rangle + \frac{1}{2}\mathrm{Tr}\left[\left(\langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(T-\varepsilon) - \langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(0)\right) \langle\langle\langle \mathbf{g} \rangle\rangle\rangle(0)\right] + \frac{1}{2}\log\frac{|\langle\langle\langle \mathbf{g} \rangle\rangle\rangle|(T-\varepsilon)}{|\langle\langle\langle \mathbf{g} \rangle\rangle\rangle|(0)},$$

with

$$\langle\langle\langle \mathbf{g} \rangle\rangle\rangle^{-1}(t) = \begin{bmatrix} \frac{4}{T}\left(\frac{T}{T-t}\right) & -\frac{6}{T}\left(\frac{T}{T-t}\right)^2 \\ -\frac{6}{T}\left(\frac{T}{T-t}\right)^2 & \frac{12}{T}\left(\frac{T}{T-t}\right)^3 \end{bmatrix}$$

for all $t \in [0, T-\varepsilon]$. We obtain

$$\Delta_{T-\varepsilon} = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \left\{3T\left(\frac{T}{\varepsilon}\right)^3 - 6T\left(\frac{T}{\varepsilon}\right)^2 + 4T\left(\frac{T}{\varepsilon}\right) - T\right\} + 2\left(\frac{T}{\varepsilon}\right)^3 - 3\left(\frac{T}{\varepsilon}\right)^2 + 2\left(\frac{T}{\varepsilon}\right) - 2\log\left(\frac{T}{\varepsilon}\right) - 1.$$

Here it can be nicely seen that $\Delta_0 = 0$ (no trading at all) and $\Delta_T = \infty$ (the knowledge of the final values implies arbitrage).

Acknowledgments

We thank the referees for their helpful comments.

A. Yazigi was funded by the Finnish Doctoral Programme in Stochastics and Statistics and by the Finnish Cultural Foundation.

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