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Representation of Self-Similar Gaussian Processes

Title of publication

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Abstract

We develop the canonical Volterra representation for a self-similar Gaussian process by using the Lamperti transformation of the corresponding stationary Gaussian process, where this latter one admits a canonical integral representation under the assumption of pure non-determinism. We apply the representation obtained for the self-similar Gaussian process to derive an expression for Gaussian processes that are equivalent in law to the self-similar Gaussian process in question.

Contents

REPRESENTATION OF SELF-SIMILAR GAUSSIAN PROCESSES

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Mathematics Subject Classification (2010): 60G15, 60G18, 60G22.

Keywords: Self-similar processes; Gaussian processes; canonical Volterra representation; Lamperti transformation; stationary Gaussian process; equivalence in law; homogeneous kernels.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we will formulate a canonical Volterra representation for self-similar centered Gaussian processes. The role of the canonical Volterra representation which was first introduced by Levy in [13] and [14], and later developed by Hida in [7], is to provide an integral representation for a Gaussian process X in terms of a Brownian motion W and a non-random Volterra kernel k such that the expression

$$X_t = \int_0^t k(t, s)\, \mathrm{d}W_s$$

holds and the Gaussian processes X and W generate the same filtration. It is known, [3], that when the kernel k satisfies the homogeneity property for some degree α, i.e. $k(at, as) = a^\alpha k(t, s)$, $a > 0$, the Gaussian process X is self-similar with index $\alpha + \frac{1}{2}$. Thus, the main goal of this paper is to seek a general construction of the canonical Volterra representation for self-similar Gaussian processes under some suitable conditions, and one way to achieve this, is to use the linear Lamperti transformation that defines the one-one correspondence between stationary processes and self-similar processes. In section 2, we will formulate the explicit form of the canonical Volterra representation for self-similar Gaussian processes in the light of the classical canonical representation of the stationary processes given by Karhunen in [10]. In section 3, we give an application of the representation

obtained to derive an expression for a Gaussian process equivalent in law to the self-similar Gaussian process.

In our mathematical settings, we take $T > 1$ to be a fixed time horizon, and the process $X = (X_t; t \in [0, T])$ to be a centered Gaussian process with covariance $r(t, s) = \mathbb{E}(X_t X_s)$, enjoying the self-similarity property for some $\beta > 0$, i.e.

$$(X_{at})_{0 \leq t \leq T/a} \stackrel{d}{=} (a^\beta X_t)_{0 \leq t \leq T}, \quad \text{for all } a > 0,$$

where $\stackrel{d}{=}$ denotes equality in distributions, or equivalently ,

$$(1.1) \qquad r(t, s) = \mathbb{E}(X_t X_s) = T^{2\beta} \, r\left(\frac{t}{T}, \frac{s}{T}\right), \quad 0 \leq t, s \leq T.$$

In particular, we have $r(t, t) = t^{2\beta} \mathbb{E}(X_1^2)$, which is finite and continuous function at every (t, t) in $[0, T]^2$, and therefore, is continuous at every $(t, s) \in [0, T]^2$, [15].

We denote by $H_\eta(t)$ the closed linear subspace of $L^2([0, T])$ generated by Gaussian random variables η_s for $s \leq t$. We call the *Volterra representation* of X the integral representation of the form

$$(1.2) \qquad\qquad X_t = \int_0^t k(t, s) \, dW_s, \quad t \in [0, T],$$

where $W = (W_t; t \in [0, T])$ is a standard Brownian motion and the kernel $k(t, s)$ is a Volterra kernel, i.e. a measurable function on $[0, T] \times [0, T]$ that satisfies $\int_0^T \int_0^t k(t, s)^2 \, ds \, dt < \infty$, and $k(t, s) = 0$ for $s > t$. The Gaussian process X with such representation is called a *Gaussian Volterra process*, provided with k and W.

Moreover, if the *canonical property*

$$(1.3) \qquad\qquad\qquad H_X(t) = H_W(t)$$

holds for each t, the Volterra representation is said to be *canonical*. An equivalent to the property (1.3) is that if there exits at each t a function ϕ such that $\int_0^t k(t, s)\phi(s) \, ds = 0$, one has $\phi \equiv 0$. This means that the $k(t, \cdot)$'s are linearly independent and the family $\{k(t, \cdot), t \in [0, T]\}$ spans a vector space dense in $L^2([0, T])$.

1.4. **Remark.** If we associate with the canonical kernel k a Volterra integral operator K defined on $L^2([0, T])$ by $K\phi(t) = \int_0^t k(t, s) \, \phi(s) \, ds$, it is injective by (1.3) and $K(L^2([0, T]))$ is dense in $L^2([0, T])$. The covariance operator has the decomposition $R = KK^*$ and the covariance r is factorable, i.e.

$$r(t, s) = \int_0^{t \wedge s} k(t, u)k(s, u) \, du.$$

2. The Canonical Volterra representation and self-similarity

The Gaussian process X is β–self-similar, and according to Lamperti [12], it can be transformed into a stationary Gaussian process Y defined by:

$$(2.1) \qquad\qquad Y(t) := e^{-\beta t} X(e^t), \quad t \in (-\infty, \log T].$$

Conversely, X can be recovered from Y by the inverse Lamperti transformation

$$(2.2) \qquad X(t) = t^{\beta} Y(\log t), \quad t \in [0, T].$$

It is obvious that the mean-continuity of the process Y follows from the fact that

$$\mathbb{E}(Y_t - Y_s)^2 = 2 \left(r(1,1) - e^{-(t-s)\beta} r(e^{t-s}, 1) \right)$$

converges to zero when t approaches s. As was shown by Hida & Hitsuda (§3, [8]), which is a well-known classical result that has been established by Karhunen (§3, Satz 5, [10]), the stationary Gaussian process Y admits the canonical representation

$$(2.3) \qquad Y_t = \int_{-\infty}^{t} G_T(t - s) \, \mathrm{d}W_s^*,$$

where G_T is a measurable function that belongs to $L^2(\mathbb{R}, \mathrm{d}u)$ such that $G_T(u) = 0$ when $u < 0$, and W^* is a standard Brownian motion such that the property $H_Y(t) = H_{W^*}(t)$ holds for each t. A necessary and sufficient condition for the existence of the representation (2.3) is that Y is purely non-deterministic. Following Cramer [4], a process Z is purely non-deterministic if and only if the condition

$$(C) \qquad \bigcap_t H_Z(t) = \{0\},$$

is fulfilled, where $\{0\}$ is the L^2–subspace spanned by the constants. The condition (C) means that $H_Z(t)$ varies with t and the remote past is trivial; see also [10], [6], and [8].

Next, we shall extend the property of pure non-determinism to the self-similar centered Gaussian process X, which will be a main tool to construct Volterra representation for X.

2.4. **Theorem.** *The self-similar centered Gaussian process $X = (X_t; t \in [0, T])$ satisfies the condition* (C) *if and only if there exist a standard Brownian motion W and a Volterra kernel k such that X has the representation*

$$(2.5) \qquad X_t = \int_0^t k(t, s) \, \mathrm{d}W_s,$$

where the Volterra kernel k is defined by

$$(2.6) \qquad k(t, s) = t^{\beta - \frac{1}{2}} F\left(\frac{s}{t}\right)$$

for some function $F \in L^2(\mathbb{R}_+, \mathrm{d}u)$ independent of β, with $F(u) = 0$ for $1 < u$.

Moreover, $H_X(t) = H_W(t)$ holds for each t.

Proof. The fact that X is purely non-deterministic is equivalent to that Y is purely non-deterministic since

$$\bigcap_{t \in (0, T)} H_X(t) = \bigcap_{t \in (0, T)} H_Y(\log t) = \bigcap_{t \in (-\infty, \log T)} H_Y(t).$$

Thus Y admits the representation (2.3) for some square integrable kernel G_T and a standard Brownian motion W^*. By the inverse Lamperti transformation, we obtain

$$X(t) = \int_{-\infty}^{\log t} t^\beta G_T(\log t - s)\, \mathrm{d}W_s^* = \int_0^t t^\beta s^{-\frac{1}{2}} G_T\left(\log \frac{t}{s}\right) \mathrm{d}W_s,$$

where $\mathrm{d}W_s = s^{\frac{1}{2}}\mathrm{d}W_{\log s}^*$. We take the Volterra kernel k to be defined as $k(t,s) = t^{\beta - \frac{1}{2}} F\left(\frac{s}{t}\right)$, where $F(u) = u^{-\frac{1}{2}} G_T(\log u^{-1}) \in L^2(\mathbb{R}_+, \mathrm{d}u)$ vanishing when $u < 1$ since $G_T(u) = 0$ when $u < 0$, i.e. for $t < s$, we have $F(\frac{s}{t}) = 0$, and then, $k(t,s) = 0$. Indeed,

$$\int_0^\infty F(u)^2\, \mathrm{d}u = \int_0^\infty G_T(\log u^{-1})^2 \frac{\mathrm{d}u}{u} = \int_{-\infty}^\infty G_T(v)^2\, \mathrm{d}v < \infty,$$

and

$$\int_0^T \int_0^t F\left(\frac{s}{t}\right)^2 \mathrm{d}s\, \mathrm{d}t \;=\; \int_0^T t\, \mathrm{d}t \int_0^1 F(u)^2\, \mathrm{d}u$$
$$=\; \int_0^T t\, \mathrm{d}t \int_0^\infty G_T(v)^2\, \mathrm{d}v < \infty.$$

Thus,

$$\int_0^T \int_0^t t^{2\beta - 1} F\left(\frac{s}{t}\right)^2 \mathrm{d}s\, \mathrm{d}t \;=\; \left(\int_0^T t^{2\beta}\right)\left(\int_0^1 F(u)^2\, \mathrm{d}u\right) \mathrm{d}t < \infty$$

Considering the closed linear subspace $H_{\mathrm{d}W}(t)$ of $L^2([0,T])$ that is generated by $W_s - W_u$ for all $u \le s \le t$, we have $H_{\mathrm{d}W}(t) = H_W(t)$ since $W_0 = 0$, and therefore, the canonical property follows from the equalities

$$H_X(t) = H_Y(\log t) = H_{\mathrm{d}W^*}(\log t) = H_{\mathrm{d}W}(t) = H_W(t).$$

$\square$

2.7. **Remark.** In the case where the process X is *trivial self-similar*, i.e. $X_t = t^\beta X_1$ a.e., $0 \le t \le T$, the condition (C) is not satisfied since $\bigcap_{t \in (0,T)} H_X(t) = H_X(1)$. Thus, X has no Volterra representation in this case.

A function $f(t,s)$ is said to be homogeneous with degree α if it satisfies the equality $f(at, as) = a^\alpha f(t,s)$, $a > 0$. From the expression (2.6) of canonical kernel, it is easy to see that k is homogeneous with degree $\beta - \frac{1}{2}$, i.e. $k(t,s) = T^{\beta - \frac{1}{2}} k(\frac{t}{T}, \frac{s}{T})$, for all $s < t \in [0,T]$. The next corollary, which follows immediately from theorem (2.4), will characterize the class of the canonical kernels of the self-similar Gaussian Volterra process.

2.8. **Corollary.** *Let* $X = (X_t; t \in [0,T])$ *be a centered Gaussian process that satisfies* (C), *then the following are equivalent:*

(i) X *is* β-*self-similar for some* $\beta > 0$, *i.e.*

$$r(t,s) = T^{2\beta} r(\frac{t}{T}, \frac{s}{T}).$$

(ii) X *is a Gaussian Volterra process with representation* (2.5) *such that the kernel* k *is homogeneous with degree* $\beta - \frac{1}{2}$.

Furthermore, for any bounded unitary endomorphism U on $L^2([0,T])$, with adjoint $U^* = U^{-1}$, the kernel k is homogeneous with degree $\beta - \frac{1}{2}$ if and only if $Uk(t, \cdot)$ is homogeneous with the same degree.

Proof. (i)$\Rightarrow$ (ii) follows from theorem (2.4). (ii)$\Rightarrow$ (i): If the kernel k is homogeneous with degree $\beta - \frac{1}{2}$, it implies that

$$r(t,s) = \int_0^{t \wedge s} k(t,u)k(s,u)\,\mathrm{d}u = T^{2\beta} r(\frac{t}{T}, \frac{s}{T}).$$

Let the scaling operator $\mathcal{S}f(t) = T^{\frac{1}{2}} f(Tt)$ with adjoint $\mathcal{S}^* f(t) = T^{-\frac{1}{2}} f(\frac{t}{T})$ to be defined for all $f \in L^2([0,T])$, and let the notation $k_t(\cdot) := k(t, \cdot)$. The homogeneity of k means that $k_t(s) = T^\beta (\mathcal{S}^* k_{\frac{t}{T}})(s)$, then we have

$$Uk_t(s) = T^\beta (U\mathcal{S}^* k_{\frac{t}{T}})(s) = T^{\beta - \frac{1}{2}} (\mathcal{S}U\mathcal{S}^* k_{\frac{t}{T}})(\frac{s}{T}).$$

To show the equality $\mathcal{S}U\mathcal{S}^* k_{\frac{t}{T}} = Uk_{\frac{t}{T}}$, we will use the Mellin transform

$$\begin{aligned}
\int_0^\infty (\mathcal{S}U\mathcal{S}^* k_{\frac{t}{T}})(s)\, s^{p-1}\,\mathrm{d}s &= \int_0^\infty (U\mathcal{S}^* k_{\frac{t}{T}})(s)\, (\mathcal{S}^* s^{p-1})\,\mathrm{d}s \\
&= T^{\frac{1}{2}-p} \int_0^\infty (U\mathcal{S}^* k_{\frac{t}{T}})(s)\, s^{p-1}\,\mathrm{d}s \\
&= T^{\frac{1}{2}-p} \int_0^\infty (\mathcal{S}^* k_{\frac{t}{T}})(s)\, (U^* s^{p-1})\,\mathrm{d}s \\
&= T^{-p} \int_0^\infty k_{\frac{t}{T}}(\frac{s}{T})\, (U^* s^{p-1})\,\mathrm{d}s \\
&= \int_0^\infty k_{\frac{t}{T}}(u)\, (U^* u^{p-1})\,\mathrm{d}u = \int_0^\infty Uk_{\frac{t}{T}}(u)\, u^{p-1}\,\mathrm{d}u,
\end{aligned}$$

and the uniqueness property of the Mellin transform implies that

$$\mathcal{S}U\mathcal{S}^* k_{\frac{t}{T}} = Uk_{\frac{t}{T}}.$$

For the last part of the proof, since we have that $Uk(t, \cdot)$ is homogeneous, it is enough to take $U = I$, the Identity operator, then k is homogeneous. $\square$

3. Application to the equivalence in law

In this section, we shall emphasize the self-similarity property under the equivalence of laws of Gaussian processes. It is known that the laws of two Gaussian processes are either equivalent or singular. Therefore, as we are interested in the case of equivalence, we shall recall the case of the Brownian motion, see [8] and [9]. By the Hitsuda representation theorem, a centered Gaussian process $\widetilde{W} = (\widetilde{W}_t; t \in [0,T])$ is equivalent in law to the standard Brownian motion $W = (W_t; t \in [0,T])$ if and only if $\widetilde{W}$ can be represented in a unique way by

$$(3.1) \qquad \widetilde{W}_t = W_t - \int_0^t \int_0^s l(s,u)\,\mathrm{d}W_u\,\mathrm{d}s,$$

where $l(s,u)$ is a Volterra kernel, i.e.

$$(3.2) \qquad \int_0^T \int_0^t l(t,s)^2 \, \mathrm{d}s \, \mathrm{d}t < \infty, \qquad l(t,s) = 0 \quad \text{for} \quad t < s,$$

and such that the equality $H_{\widetilde{W}}(t) = H_W(t)$ holds for each t. If we denote by $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ the laws of W and $\widetilde{W}$ respectively, these two processes are equivalent in law if $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, and the Radon-Nikodym density is given by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left\{ \int_0^T \int_0^s l(s,u) \mathrm{d}W_u \, \mathrm{d}W_s - \frac{1}{2} \int_0^T \left(\int_0^s l(s,u)\mathrm{d}W_s \right)^2 \mathrm{d}s \right\}.$$

The centered Gaussian process $\widetilde{W}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$ with $\widetilde{\mathbb{E}}(\widetilde{W}_t\widetilde{W}_s) = \mathbb{E}(W_tW_s)$, hence, it is self-similar with index $\frac{1}{2}$ under $\widetilde{\mathbb{P}}$. It follows from (3.1) that the covariance of $\widetilde{W}$ under $\mathbb{P}$ has the form of

$$
\begin{aligned}
\mathbb{E}(\widetilde{W}_t\widetilde{W}_s) &= t \wedge s - \int_0^{t\wedge s} \int_u^s l(v,u) \, \mathrm{d}v \, \mathrm{d}u - \int_0^{t\wedge s} \int_u^t l(v,u) \, \mathrm{d}v \, \mathrm{d}u \\
&\quad + \int_0^t \int_0^s \int_0^{v_1 \wedge v_2} l(v_1,u) \, l(v_2,u) \, \mathrm{d}u \, \mathrm{d}v_1 \, \mathrm{d}v_2.
\end{aligned}
$$

This last formula was first appeared in [7].

3.3. **Remark.** The standard Brownian motion W is a purely non-deterministic process, and from the equality $H_{\widetilde{W}}(t) = H_W(t)$, it follows that $\widetilde{W}$ is also purely non-deterministic.

The class of Hitsuda representation can be extended to the class of the Gaussian Volterra processes, see [2] and [18]. A centered Gaussian process $\widetilde{X} = (\widetilde{X}_t; t \in [0,T])$ is equivalent in law to a Gaussian Volterra process X if and only if there exits a unique centered Gaussian process, namely $\widetilde{W}$, satisfying (3.1) and (3.2), and such that

$$(3.4) \qquad \widetilde{X}_t = \int_0^t k(t,s) \, \mathrm{d}\widetilde{W}_s = X_t - \int_0^t k(t,s) \int_0^s l(s,u) \, \mathrm{d}W_u \, \mathrm{d}s,$$

where the kernel $k(t,s)$ and the standard Brownian motion stand for the Volterra representation of X, i.e. $X_t = \int_0^t k(t,s) \, \mathrm{d}W_s$.

3.5. **Proposition.** *Let $X = (X_t; t \in [0,T])$ be a centered Gaussian β-self-similar satisfying the condition (C), then*

(i) *A centered Gaussian process $\widetilde{X} = (\widetilde{X}_t; t \in [0,T])$ is equivalent in law to X if and only if $\widetilde{X}$ admits a representation of the form of*

$$(3.6) \qquad\qquad \widetilde{X}_t = X_t - t^{\beta - \frac{1}{2}} \int_0^t z(t,s) \, \mathrm{d}W_s,$$

where W is a standard Brownian motion, and the kernel $z(t,s)$ is independent of β provided with the expression

$$z(t,s) = \int_s^t F\left(\frac{v}{t}\right) l(v,s) \, \mathrm{d}v, \quad s < t,$$

for some function $F \in L^2(\mathbb{R}_+, \mathrm{d}u)$ and Volterra kernel $l(v,s)$.

(ii) *In addition, $\widetilde{X}$ is β-self–similar if and only if $\widetilde{X} = X$.*

Proof. i) By theorem (2.4), X has a Volterra representation with a kernel $k(t,s) = t^{\beta - \frac{1}{2}} F\left(\frac{s}{t}\right)$, $F \in L^2(\mathbb{R}_+, \mathrm{d}u)$, and a standard Brownian motion W. By rewriting (3.4) as

$$\widetilde{X}_t = X_t - \int_0^t \int_s^t k(t,u)l(u,s)\,\mathrm{d}u\,\mathrm{d}W_s,$$

proves the claim.

ii) Since the kernel k is $(\beta - \frac{1}{2})$-homogeneous, $\widetilde{X}$ is β-self–similar if and only if $\widetilde{W}$ is $\frac{1}{2}$-self-similar. Firstly, we will show that the necessary and the sufficient condition for the claim to be true for $\widetilde{W}$ is that l is homogeneous with degree -1.

If we rewrite the representation (3.1) as

$$(3.7) \qquad \widetilde{W}_t = W_t - \int_0^t L(t,s)\,\mathrm{d}W_s,$$

where $L(t,s) := \int_s^t l(u,s)\,\mathrm{d}u$, we see the fact that l is homogeneous with degree -1 is equivalent to that L is homogeneous with degree 0. Suppose now that L is 0-homogeneous. The covariance of $\widetilde{W}$ is expressed as

$$
\begin{aligned}
\mathbb{E}(\widetilde{W}_t \widetilde{W}_s) \;=\; & t \wedge s - \int_0^{t \wedge s} L(t,u)\,\mathrm{d}u - \int_0^{t \wedge s} L(s,u)\,\mathrm{d}u \\
& + \int_0^{t \wedge s} L(t,u)L(s,u)\,\mathrm{d}u,
\end{aligned}
$$

and by the change of variables: $u = vT$ and the 0-homogeneity of L, we have $L(t,u) = L(\frac{t}{T}, v)$, $L(s,u) = L(\frac{s}{T}, v)$ and

$$\int_0^{t \wedge s} L(t,u)L(u,s)\,\mathrm{d}u = T \int_0^{\frac{T}{t} \wedge \frac{s}{T}} L(\frac{t}{T}, v)L(\frac{s}{T}, v)\,\mathrm{d}v.$$

Similarly, $\int_0^{t \wedge s} L(t,u)\,\mathrm{d}u = T \int_0^{\frac{T}{t} \wedge \frac{s}{T}} L(\frac{t}{T}, v)\,\mathrm{d}v$ and $\int_0^{t \wedge s} L(s,u)\,\mathrm{d}u = T \int_0^{\frac{T}{t} \wedge \frac{s}{T}} L(\frac{s}{T}, v)\,\mathrm{d}v$. Thus,

$$\mathbb{E}(\widetilde{W}_t \widetilde{W}_s) = T\,\mathbb{E}(\widetilde{W}_{\frac{t}{T}} \widetilde{W}_{\frac{s}{T}}),$$

which means that $\widetilde{W}$ is $\frac{1}{2}$–self-similar. Now, suppose that $\widetilde{W}$ is $\frac{1}{2}$–self-similar and consider the centered Gaussian process $(\widetilde{W}_t - W_t)_t$, it is a $\frac{1}{2}$–self-similar process since its Lamperti transformation $(e^{-\frac{t}{2}}(\widetilde{W}_{e^t} - W_{e^t}))_t$ is stationary. On the other hand, $H_{\widetilde{W} - W}(t) = H_W(t)$ for all t, and hence, it is satisfies the condition (C). By theorem (2.4), there exist a Volterra kernel $\overline{k}$, which is homogeneous with degree 0, and a standard Brownian motion $\overline{W}$ such that

$$\widetilde{W}_t - W_t = \int_0^t \overline{k}(t,s)\,\mathrm{d}\overline{W}_s.$$

Due to the uniqueness of the representation (3.7) that follows from (3.1), we have $L = \overline{k}$ and $W = \overline{W}$, and thus L is 0-homogeneous, i.e. l is (-1)-homogeneous.

Secondly, combining the square integrability condition (3.2) with the homogeneity property $l(t,s) = \frac{1}{a}\, l(\frac{t}{a}, \frac{s}{a})$, $a > 0$, gives

$$\int_0^T \int_0^t l(t,s)^2 \,\mathrm{d}s\,\mathrm{d}t = \int_0^{\frac{T}{a}} \int_0^{\frac{t}{a}} l\left(\frac{t}{a}, \frac{s}{a}\right)^2 \frac{1}{a^2}\,\mathrm{d}s\,\mathrm{d}t = \int_0^{\frac{T}{a}} \int_0^{t'} l(t',s')^2\,\mathrm{d}s'\,\mathrm{d}t'$$

which is finite for all $a > 0$. This implies that $l = 0$.

Finally, we conclude that $\widetilde{W} = W$, and consequently $\widetilde{X} = X$. $\square$

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