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Abstract

The continuity of Gaussian processes is extensively studied topic and it culminates in the Talagrand's notion of majorizing measures that gives complicated necessary and sufficient conditions. In this note we study the Hölder continuity of Gaussian processes. It turns out that necessary and sufficient conditions can be stated in a simple form that is a variant of the celebrated Kolmogorov–Centsov condition.

Contents

NECESSARY AND SUFFICIENT CONDITIONS FOR HÖLDER CONTINUITY OF GAUSSIAN PROCESSES

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ABSTRACT. The continuity of Gaussian processes is extensively studied topic and it culminates in the Talagrand's notion of majorizing measures that gives complicated necessary and sufficient conditions. In this note we study the Hölder continuity of Gaussian processes. It turns out that necessary and sufficient conditions can be stated in a simple form that is a variant of the celebrated Kolmogorov–Čentsov condition.

1. INTRODUCTION

In what follows X will always be a centered Gaussian process on the interval $[0, T]$. For a centered Gaussian family $\xi = (\xi_\tau)_{\tau \in \mathbb{T}}$ we denote

$$
\begin{aligned}
d_\xi^2(\tau, \tau') &:= \mathbb{E}[(\xi_\tau - \xi_{\tau'})^2], \\
\sigma_\xi^2(\tau) &:= \mathbb{E}[\xi_\tau^2].
\end{aligned}
$$

To put our result in context, we briefly recall the essential results of Gaussian continuity.

One of the earliest results is a sufficient condition due to Fernique [5]: *Assume that for some positive ε, and $0 \le s \le t \le \varepsilon$, there exists a nondecreasing function Ψ on $[0, \varepsilon]$ such that $\sigma_X^2(s, t) \le \Psi^2(t - s)$ and*

$$
(1) \qquad \int_0^\varepsilon \frac{\Psi(u)}{u\sqrt{\log u}} \, \mathrm{d}u < \infty.
$$

Then X is continuous. The finiteness of Fernique integral (1) is not necessary for the continuity. Indeed, cf. [9, Sect. 5] for a counter-example.

Dudley [3, 4] found a sufficient condition for the continuity by using *metric entropy*. Let $N(\varepsilon) := N([0, T], d_X, \varepsilon)$ denote the minimum number of closed balls of radius ε in the (pseudo) metric d_X needed to cover $[0, T]$. *If*

$$
(2) \qquad \int_0^\infty \sqrt{\log N(\varepsilon)} \, \mathrm{d}\varepsilon < \infty,
$$

then X is continuous. Like in the case of the Fernique's condition, the finiteness of the Dudley integral (2) is not necessary for continuity, cf. [8, Ch 6.]. However, for stationary processes (2) is necessary and sufficient.

Finally, necessary and sufficient conditions were obtained by Talagrand [10]. Denote $B_{d_X}(t, \varepsilon)$ a ball with radius ε at center t in the metric d_X. A probability measure μ on $([0, T], d_X)$ is called a *majorizing measure* if

$$(3) \qquad \sup_{t \in [0,T]} \int_0^\infty \sqrt{\log \frac{1}{\mu\left(B_{d_X}(t, \varepsilon)\right)}}\, \mathrm{d}\varepsilon < \infty.$$

The Gaussian process X is continuous if and only if there exists a majorizing measure μ on $([0, T], d_X)$ such that

$$\lim_{\delta \to 0} \sup_{t \in [0,T]} \int_0^\delta \sqrt{\log \frac{1}{\mu(B_{d_X}(t, \varepsilon))}}\, \mathrm{d}\varepsilon = 0.$$

2. MAIN THEOREM

The Talagrand's necessary and sufficient condition (3) for the continuity of a Gaussian process is rather complicated. In contrast, the general Kolmogorov–Čentsov condition for continuity is very simple. It turns out that for Gaussian processes the Kolmogorov–Čentsov condition is very close to being necessary for Hölder continuity:

Theorem 1. *The Gaussian process X is Hölder continuous of any order $a < H$ i.e.*

$$(4) \qquad |X_t - X_s| \le C_\varepsilon |t - s|^{H - \varepsilon}, \quad \text{for all } \varepsilon > 0$$

if and only if there exists constants c_ε such that

$$(5) \qquad d_X(t, s) \le c_\varepsilon |t - s|^{H - \varepsilon}, \quad \text{for all } \varepsilon > 0.$$

Moreover, the random variables C_ε in (4) satisfy

$$(6) \qquad \mathbb{E}\left[\exp\left(a C_\varepsilon^\kappa\right)\right] < \infty$$

for any constants $a \in \mathbb{R}$ and $\kappa < 2$; and also for $\kappa = 2$ for small enough positive a. In particular, the moments of all orders of C_ε are finite.

The differences between the classical Kolmogorov–Čentsov continuity criterion and Theorem 1 are: (i) Theorem 1 deals only with Gaussian processes, (ii) there is an ε-gap to the classical Kolmogorov–Čentsov condition and (iii) as a bonus we obtain that the Hölder constants C_ε must have light tails by the estimate (6). Note that the ε-gap cannot be closed. Indeed, let

$$X_t = f(t) B_t,$$

where B is the fractional Brownian motion with Hurst index H and $f(t) = (\log \log 1/t)^{-1/2}$. Then, by the law of the iterated logarithm due to Arcones [2], X is Hölder continuous of any order $a < H$, but (5) does not hold without an $\varepsilon > 0$.

The proof of the first part Theorem 1 is based on the classical Kolmogorov–Čentsov continuity criterion and the following elementary lemma:

Lemma 1. *Let $\xi = (\xi_\tau)_{\tau \in \mathbb{T}}$ be a centered Gaussian family. If $\sup_{\tau \in \mathbb{T}} |\xi_\tau| < \infty$ then $\sup_{\tau \in \mathbb{T}} \mathbb{E}[\xi_\tau^2] < \infty$.*

Proof. Since $\sup_{\tau \in \mathbb{T}} |\xi_\tau| < \infty$, $\mathbb{P}[\sup_{\tau \in \mathbb{T}} |\xi_\tau| < x] > 0$ for a large enough $x \in \mathbb{R}$. Now, for all $\tau \in \mathbb{T}$, we have that

$$
\begin{aligned}
\mathbb{P}\left[\sup_{\tau \in \mathbb{T}} |\xi_\tau| < x\right] &\leq \mathbb{P}\left[|\xi_\tau| < x\right] \\
&= \mathbb{P}\left[\left|\frac{\xi_\tau}{\sigma_\xi(\tau)}\right| < \frac{x}{\sigma_\xi(\tau)}\right] \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{x/\sigma_\xi(\tau)} e^{-\frac{1}{2}z^2} \, \mathrm{d}z \\
&\leq \frac{2}{\sqrt{2\pi}} \frac{x}{\sigma_\xi(\tau)}.
\end{aligned}
$$

Consequently,

$$
\sigma_\xi^2(\tau) \leq \frac{2x^2}{\pi \mathbb{P}\left[\sup_{\tau \in \mathbb{T}} |\xi_\tau| < x\right]^2},
$$

and the claim follows from this. $\qquad\square$

The second part on the exponential moments of the Hölder constants of Theorem 1 follows from the following Garsia–Rademich–Rumsey inequality [7]. Let us also note, that this part is intimately connected to the Fernique's theorem [6] on the continuity of Gaussian processes.

Lemma 2. *Let $p \geq 1$ and $\alpha > \frac{1}{p}$. Then there exists a constant $c = c_{\alpha,p} > 0$ such that for any $f \in C([0,T])$ and for all $0 \leq s, t \leq T$ we have*

$$
|f(t) - f(s)|^p \leq cT^{\alpha p - 1}|t - s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 1}} \, \mathrm{d}x\mathrm{d}y.
$$

Proof of Theorem 1. The if part follows from the Kolmogorov–Čentsov continuity criterion. For the only-if part assume that X is Hölder continuous of order $a = H - \varepsilon$, i.e.

$$
\sup_{t,s \in [0,T]} \frac{|X_t - X_s|}{|t - s|^{H - \varepsilon}} < \infty.
$$

Define a family $\xi = (\xi_{t,s})_{(t,s) \in [0,T]^2}$ by setting

$$
\xi_{t,s} = \frac{X_t - X_s}{|t - s|^{H - \varepsilon}}.
$$

Since ξ is a centered Gaussian family that is bounded by the Hölder continuity of X, we obtain, by Lemma 1, that $\sup_{(t,s) \in [0,T]^2} \sigma_\xi^2(t,s) < \infty$. This means that

$$
\sup_{t,s \in [0,T]} \frac{d_X^2(t,s)}{|t - s|^{2H - 2\varepsilon}} < \infty,
$$

or

$$
d_X(t,s) \leq C_\varepsilon |t - s|^{H - \varepsilon}.
$$

The property (6) follow from the Garsia–Rademich–Rumsey inequality of Lemma 2. Indeed, by choosing $\alpha = H - \frac{\varepsilon}{2}$ and $p = \frac{2}{\varepsilon}$ we obtain

$$
|X_t - X_s| \leq c_{H,\varepsilon} T^{H - \varepsilon} |t - s|^{H - \varepsilon} \xi,
$$

where

$$(7) \qquad \xi = \left(\int_0^T \int_0^T \frac{|X_u - X_v|^{\frac{2}{\varepsilon}}}{|u - v|^{\frac{2H}{\varepsilon}}} \, \mathrm{d}u \mathrm{d}v \right)^{\frac{\varepsilon}{2}}.$$

Let us first estimate moments of ξ. First we recall the fact that for a Gaussian random variable $Z \sim \mathcal{N}(0, \sigma^2)$ and any number $q > 0$ we have

$$\mathbb{E}\left[|Z|^q\right] = \sigma^q \frac{2^{\frac{q}{2}} \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}},$$

where Γ denotes the Gamma function. Let now $\delta < \frac{\varepsilon}{2}$ and $p \geq \frac{2}{\varepsilon}$. By Minkowski inequality and estimate (5) we obtain

$$
\begin{aligned}
\mathbb{E}\left[|\xi|^p\right] & \leq \left(\int_0^T \int_0^T \frac{\left(\mathbb{E}|X_u - X_v|^p\right)^{\frac{2}{p\varepsilon}}}{|u - v|^{\frac{2H}{\varepsilon}}} \mathrm{d}v \mathrm{d}u \right)^{\frac{p\varepsilon}{2}} \\
& \leq \left(\int_0^T \int_0^T \frac{\left(c_p c_\delta |u - v|^{p(H-\delta)}\right)^{\frac{2}{p\varepsilon}}}{|u - v|^{\frac{2H}{\varepsilon}}} \mathrm{d}v \mathrm{d}u \right)^{\frac{p\varepsilon}{2}} \\
& = c_p c_\delta 2^{\frac{p\varepsilon}{2}} \left(\int_0^T \int_0^u (u - v)^{-\frac{2\delta}{\varepsilon}} \mathrm{d}v \mathrm{d}u \right)^{\frac{p\varepsilon}{2}} \\
& = c_p c_\delta 2^{\frac{p\varepsilon}{2}} \left(\frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} \left(1 - \frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} T^{q(\varepsilon - \delta)},
\end{aligned}
$$

where c_δ is the constant from (5) and

$$(8) \qquad c_q = \frac{2^{\frac{q}{2}} \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}}.$$

Hence, we may take

$$C_\varepsilon = c_{H,\varepsilon} T^{H-\varepsilon} \xi,$$

where $c_{H,\varepsilon}$ is the constant from Garsia–Rademich–Rumsey inequality and ξ is given by (7). Moreover, for any $p \geq \frac{2}{\varepsilon}$ and any $\delta < \frac{\varepsilon}{2}$ we have estimate

$$\mathbb{E}\left[|\xi|^p\right] \leq c_p c_\delta 2^{\frac{p\varepsilon}{2}} \left(\frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} \left(1 - \frac{\varepsilon}{2\delta} \right)^{\frac{p\varepsilon}{2}} T^{q(\varepsilon - \delta)}.$$

Consequently,

$$\mathbb{E}\left[|C_\varepsilon|^p\right] \leq c^p \Gamma\left(\frac{p+1}{2}\right)$$

for some constant $c = c_{\varepsilon, \delta, T}$. Thus, by plugging in (8) to the series expansion of the exponential we obtain

$$\mathbb{E}\left[\exp\left(a C_\varepsilon^\kappa\right)\right] \leq \sum_{j=0}^\infty a^j c^{\kappa j} \frac{\Gamma\left(\frac{\kappa j + 1}{2}\right)}{\Gamma\left(j+1\right)}.$$

So, to finish the proof we need to show that the series above converges. Now, by the Stirling's approximation

$$\Gamma(z) = \frac{\sqrt{2\pi}}{\sqrt{z}} \left(\frac{z}{e} \right)^z \left(1 + O(1/z) \right),$$

we obtain (the constant c may vary from line to line)

$$
\begin{aligned}
\frac{\Gamma\left(\frac{\kappa j+1}{2}\right)}{\Gamma(j+1)} &\sim \frac{\left(\frac{\kappa j+1}{2}\right)^{-\frac{1}{2}} e^{-\frac{\kappa j+1}{2}} \left(\frac{\kappa j+1}{2}\right)^{\frac{\kappa j+1}{2}}}{(j+1)^{-\frac{1}{2}} e^{-j-1} (j+1)^{j+1}} \\
&\leq c^j \frac{1}{\sqrt{j+1}} \frac{(\kappa j+1)^{\frac{\kappa j}{2}}}{(j+1)^j} \\
&\leq c^j \frac{1}{\sqrt{j+1}} \frac{(2j+2)^{\frac{\kappa j}{2}}}{(j+1)^j} \\
&= (2c)^j \frac{1}{\sqrt{j+1}} (j+1)^{\left(\frac{\kappa}{2}-1\right)j}
\end{aligned}
$$

which is clearly summable since $\kappa < 2$. If $\kappa = 2$, then in the approximation above we obtain that $\Gamma\left(\frac{2j+1}{2}\right)/\Gamma(j+1) \sim c^j$ for some constant c. Hence, depending on constant $c_{\varepsilon,\delta,T}$, we obtain that $\mathbb{E}\left[\exp\left(aC_\varepsilon^2\right)\right] < \infty$ for small enough $a > 0$. $\qquad\square$

3. Applications and Examples

Stationary-Increment Processes. This case is simple:

Corollary 1. *If X has stationary increments then it is Hölder continuous of any order $a < H$ if and only if*

$$\sigma_X^2(t) \leq c_\varepsilon t^{2H-\varepsilon}, \quad \text{for all } \varepsilon > 0.$$

Stationary Processes. For a stationary process $\mathbb{E}[X_t X_s] = r(t-s)$, where, by the Bochner's theorem,

$$r(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, \Delta(\mathrm{d}\lambda),$$

where Δ, the *spectral measure* of X, is finite and symmetric. Since now

$$d_X^2(t,s) = 2\big(r(0) - r(t-s)\big)$$

we have the following corollary.

Corollary 2. *If X is stationary with spectral measure Δ then it is Hölder continuous of any order $a < H$ if and only if*

$$\int_0^{\infty} \big(1 - \cos(\lambda t)\big)\, \Delta(\mathrm{d}\lambda) \leq c_\varepsilon t^{2H-\varepsilon} \quad \text{for all } \varepsilon > 0.$$

Fredholm Processes. A bounded process can be viewed as an $L^2([0,T])$-valued random variable. Hence, the covariance operator admits a square root with kernel K, and we may represent X as a *Gaussian Fredholm process*:

$$(9) \qquad\qquad X_t = \int_0^T K(t,s)\, \mathrm{d}W_s,$$

where W is a Brownian motion and $K \in L^2([0,T]^2)$.

Corollary 3. *A Gaussian process X is Hölder continuous of any order $a < H$ if and only if it admits the representation (9) with K satisfying*

$$\int_0^T |K(t,u) - K(s,u)|^2 \, du \le c_\varepsilon |t - s|^{2H-\varepsilon} \quad \text{for all } \varepsilon > 0.$$

Proposition 1. *Let X be Gaussian Fredholm process with kernel K.*

(i) *If for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in L^2([0,T])$ such that*

$$|K(t,u) - K(s,u)| \le f_\varepsilon(u)|t - s|^{H-\varepsilon}$$

then X is Hölder continuous of any order $a < H$.

(ii) *If X is Hölder continuous of any order $a < H$ then*

$$f_\varepsilon := \liminf_{s \to t} \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t - s|^{2H-\varepsilon}} \in L^1([0,T])$$

Proof. The first part follows from Corollary 3. Consider then the second part and assume that X is Hölder continuous of any order $a < H$ and

$$\liminf_{s \to t} \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t - s|^{2H-\varepsilon}} \notin L^1([0,T]).$$

By Corollary 3 we know that

$$\int_0^T \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t - s|^{2H-\varepsilon}} du \le c_\varepsilon.$$

On the other hand, by Fatou Lemma we have

$$\liminf_{s \to t} \int_0^T \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t - s|^{2H-\varepsilon}} du \ge \int_0^T \liminf_{s \to t} \frac{|K(t,\cdot) - K(s,\cdot)|^2}{|t - s|^{2H-\varepsilon}} du = \infty$$

which is a contradiction. □

Volterra Processes. A Fredholm process is a *Volterra process* if its kernel K satisfies $K(t,s) = 0$ if $s > t$. In this case Corollary 3 becomes:

Corollary 4. *A Gaussian Volterra process X with kernel K is Hölder continuous of any order $a < H$ is and only if, for all $s < t$ and $\varepsilon > 0$*

(i) $\int_s^t K(t,u)^2 \, du \le c_\varepsilon |t - s|^{2H-\varepsilon}$,

(ii) $\int_0^s |K(t,u) - K(s,u)|^2 \, du \le c_\varepsilon |t - s|^{2H-\varepsilon}$.

By [1, p. 779] the following is a sufficient condition:

Proposition 2. *Let X be a Gaussian Volterra process with kernel K that satisfies*

(i) $\int_s^t K(t,u)^2 \, du \le c(t - s)^{2H}$,

(ii) $K(t,s)$ *is differentiable in t and $\left| \frac{\partial K}{\partial t}(t,s) \right| \le c(t - s)^{H - \frac{3}{2}}$.*

Then X is Hölder continuous of any order $a < H$.

Self-Similar Processes. A process X is *self-similar* with index $\beta > 0$ if

$$(X_{at})_{0 \leq t \leq T/a} \stackrel{d}{=} (a^\beta X_t)_{0 \leq t \leq T}, \quad \text{for all } a > 0.$$

In the Gaussian case this means that

$$d_X(t,s) = a^{-\beta} d_X(at, as) \quad \text{for all } a > 0.$$

So, it is clear that X cannot be Hölder continuous of order $H > \beta$.

Let $\mathscr{H}_t^X$ be the closed linear subspace of $L^2(\Omega)$ generated by the Gaussian random variables $\{X_s; s \leq t\}$. Denote $\mathscr{H}_{0+}^X := \cap_{t \in (0,T]} \mathscr{H}_t^X$. Then X is *purely non-deterministic* if $\mathscr{H}_{0+}^X$ is trivial. By [11] a purely non-deterministic Gaussian self-similar process admits the representation

$$(10) \qquad X_t = \int_0^t t^{\beta - \frac{1}{2}} F\left(\frac{u}{t}\right) dW_u,$$

where $F \in L^2([0,1])$ is positive. Consequently:

Corollary 5. *Let X be a purely non-deterministic Gaussian self-similar process with index β and representation (10). Then X is Hölder continuous of any order $a < H$ if and only if*

  (i) $\int_s^t t^{2\beta-1} F(\frac{u}{t})^2 \, du \leq c_\varepsilon |t-s|^{2H-\varepsilon}$,

  (ii) $\int_0^s \left| t^{\beta-\frac{1}{2}} F\left(\frac{u}{t}\right) - s^{\beta-\frac{1}{2}} F\left(\frac{u}{s}\right) \right|^2 du \leq c_\varepsilon |t-s|^{2H-\varepsilon}$

for all $s < t$ and $\varepsilon > 0$.

Proposition 3. *Let X be a purely non-deterministic Gaussian self-similar process with index β and representation (10). Then X is Hölder continuous of any order $a < H$ if*

  (i) $F(x) \leq c\, x^{\beta-H}(1-x)^{H-\frac{1}{2}}, \quad 0 < x < 1$,

  (ii) $\left| 1 - \frac{F(x)}{F(y)} \right| \leq \left| \left(\frac{y}{x}\right)^{H-\beta} \left(\frac{1-x}{1-y}\right)^{H-\frac{1}{2}} - 1 \right|, \quad 0 < y < x < 1$,

Proof. Condition (i) of Corollary 4 follows from assumption (i) and condition (ii) of Corollary 4 follows from assumption (i) and (ii) applied to the estimate

$$\left| t^{\beta-\frac{1}{2}} F\left(\frac{u}{t}\right) - s^{\beta-\frac{1}{2}} F\left(\frac{u}{s}\right) \right|$$

$$\leq \quad F\left(\frac{u}{t}\right) t^{\beta-\frac{1}{2}} \left| 1 - \frac{F\left(\frac{u}{s}\right)}{F\left(\frac{u}{t}\right)} \right| + F\left(\frac{u}{s}\right) \left| t^{\beta-\frac{1}{2}} - s^{\beta-\frac{1}{2}} \right|.$$

The details are left to the reader. $\qquad\square$

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