Kernels for products of L-functions

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Abstract

The Rankin-Cohen bracket of two Eisenstein series provides a kernel yielding products of the periods of Hecke eigenforms at critical values. Extending this idea leads to a new type of Eisenstein series built with a double sum. We develop the properties of these series and their non-holomorphic analogs and show their connection to values of L-functions outside the critical strip.

1 Introduction

In 1952, Rankin [24] introduced the fruitful idea of expressing the product of two critical values of the L-function of a weight k Hecke eigenform f for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ in terms of the Petersson scalar product of f and a product of Eisenstein series:

$$\langle\, E_{k_1} E_{k_2}, f\,\rangle = (-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} L^*(f,1) L^*(f,k_2) \tag{1.1}$$

for $k = k_1 + k_2$, the Bernoulli numbers B_j and the completed, entire L-function of f,

$$L^*(f,s) := \frac{\Gamma(s)}{(2\pi)^s} \sum_{m=1}^{\infty} \frac{a_f(m)}{m^s} = \int_0^\infty f(iy) y^{s-1}\, dy.$$

Zagier [28, p. 149] extended (1.1) to get

$$\langle\, [E_{k_1}, E_{k_2}]_n, f\,\rangle = (-1)^{k_1/2} (2\pi i)^n 2^{3-k} \binom{k-2}{n} \frac{k_1 k_2}{B_{k_1} B_{k_2}} L^*(f,n+1) L^*(f,n+k_2) \tag{1.2}$$

where $k = k_1 + k_2 + 2n$ and $[g_1, g_2]_n$ stands for the *Rankin-Cohen bracket of index* n given by

$$[g_1, g_2]_n := \sum_{r=0}^{n} (-1)^r \binom{k_1+n-1}{n-r} \binom{k_2+n-1}{r} g_1^{(r)} g_2^{(n-r)}. \tag{1.3}$$

The *periods* of f in the critical strip are the numbers

$$L^*(f,1), L^*(f,2), \dots, L^*(f,k-1). \tag{1.4}$$

Zagier in [28, §5] and Kohnen-Zagier in [15] proved important results of the Eichler-Shimura-Manin theory on the algebraicity of these critical values using (1.2). We describe this in more depth in §§2.3, 8.1.

On the face of it, the techniques of [28], employing (1.2), apply only to critical values; an extension to non-critical values, $L^*(f,j)$ for integers $j \leqslant 0$ or $j \geqslant k$, would seem to require Rankin-Cohen brackets of negative index n or holomorphic Eisenstein series of negative weight, neither of which are defined. Analyzing the structure of the Rankin-Cohen bracket of two Eisenstein series in §6 reveals a natural construction which we call a *double Eisenstein series*[1]:

$$\sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma\\ \gamma\delta^{-1}\neq\Gamma_\infty}} \left(c_{\gamma\delta^{-1}}\right)^{l} j(\gamma,z)^{-k_1} j(\delta,z)^{-k_2} \tag{1.5}$$

where, for $\gamma \in \Gamma$, we write

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}, \quad j(\gamma,z) := c_\gamma z + d_\gamma.$$

By comparison, the usual holomorphic Eisenstein series is

$$E_k(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^{-k}. \tag{1.6}$$

The double Eisenstein series (1.5) converges to a weight k_1+k_2 cuspform when $l < k_1-2, k_2-2$. For negative integers l it behaves as a Rankin-Cohen bracket of negative index, see Proposition 2.4. This allows us to further generalize (1.1), (1.2) and in §8 we characterize the field containing an arbitrary value of an L-function in terms of double Eisenstein series and their Fourier coefficients. In the interesting paper [4], Rankin-Cohen brackets are linked to operations on automorphic pseudodifferential operators and may also be reinterpreted in this framework allowing for more general indices.

An extension of Zagier's kernel formula (1.2) in the non-holomorphic direction is given in §9.3. There we show that the holomorphic double Eisenstein series have non-holomorphic counterparts:

$$\sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma\\ \gamma\delta^{-1}\neq\Gamma_\infty}} |c_{\gamma\delta^{-1}}|^{-s-s'} \mathrm{Im}(\gamma z)^s \mathrm{Im}(\delta z)^{s'}. \tag{1.7}$$

These weight 0 functions possess analytic continuations and functional equations resembling those for the classical non-holomorphic Eisenstein series. As kernels, they produce products of L-functions for *Maass cusp forms*, see Theorem 2.9. The main motivation for this construction was its potential use in the rapidly developing study of periods of Maass forms [1, 19, 21, 22]. In developing the properties of (1.7) we require a certain kernel $\mathcal{K}(z;s,s')$ as defined in (9.1). It is interesting to note that Diaconu and Goldfeld [6] needed exactly the same series for their results on second moments of $L^*(f,s)$, see §9.1.

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2 Statement of main results

2.1 Preliminaries

Our notation is as in [7]. In all sections but two, Γ is the modular group $\mathrm{SL}(2,\mathbb{Z})$ acting on the upper half plane $\mathbb{H}$. The definitions we give for double Eisenstein series extend easily to more general groups, so in §4 we prove their basic properties for Γ an arbitrary Fuchsian group of the first kind and in §10 we see how some of our main results are valid in this general context.

Let $S_k(\Gamma)$ be the $\mathbb{C}$-vector space of holomorphic, weight k cusp forms for Γ and $M_k(\Gamma)$ the space of modular forms. These spaces are acted on by the Hecke operators T_m, see (3.6). Let $\mathcal{B}_k$ be the

[1] In the context of multiple zeta functions, the authors in [8] give a different definition of 'double Eisenstein series'. See also [5], for example, for distinct 'double Eisenstein-Kronecker series'.

unique basis of S_k consisting of Hecke eigenforms, normalized to have first Fourier coefficient 1. We assume throughout this paper that $f \in \mathcal{B}_k$. Since $\langle T_m f, f \rangle = \langle f, T_m f \rangle$ it follows that all the Fourier coefficients of f are real and hence $\overline{L^*(f,s)} = L^*(f,\overline{s})$. Also, recall the functional equation

$$L^*(f, k-s) = (-1)^{k/2} L^*(f,s). \tag{2.1}$$

We summarize some standard properties of the non-holomorphic Eisenstein series, see for example [10, Chapters 3, 6]. Throughout this paper we use the variables $z = x+iy \in \mathbb{H}$, $\quad s = \sigma + it \in \mathbb{C}$.

Definition 2.1. *For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the weight zero,* non-holomorphic Eisenstein series *is*

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s = \frac{y^s}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} |cz+d|^{-2s}. \tag{2.2}$$

Let $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$. Then $E(z,s)$ has a Fourier expansion [10, Theorem 3.4] which we may write in the form

$$E(z,s) = y^s + \frac{\theta(1-s)}{\theta(s)} y^{1-s} + \sum_{m \neq 0} \phi(m,s)|m|^{-1/2} W_s(mz) \tag{2.3}$$

where $W_s(mz) = 2(|m|y)^{1/2} K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}$ is the Whittaker function for $z \in \mathbb{H}$ and also $\theta(s)\phi(m,s) = \sigma_{2s-1}(|m|)|m|^{1/2-s}$. As usual, $\sigma_s(m) := \sum_{d|m} d^s$ is the divisor function.

For the weight $k \in 2\mathbb{Z}$, non-holomorphic Eisenstein series, generalizing (2.2), set

$$E_k(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s \left(\frac{j(\gamma,z)}{|j(\gamma,z)|} \right)^{-k}. \tag{2.4}$$

Then (2.4) converges to an analytic function of $s \in \mathbb{C}$, and a smooth function of $z \in \mathbb{H}$, for $\mathrm{Re}(s) > 1$. Also $y^{-k/2}E_k(z,s)$ has weight k in z. Define the *completed non-holomorphic Eisenstein series* as

$$E_k^*(z,s) := \theta_k(s)E_k(z,s) \quad \text{for} \quad \theta_k(s) := \pi^{-s}\Gamma(s+|k|/2)\zeta(2s). \tag{2.5}$$

With (2.3), we see that $E(z,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. The same is true of $E_k(z,s)$, see [7, §2.1] for example. We have the functional equations

$$\theta(s/2) = \theta((1-s)/2), \tag{2.6}$$

$$E_k^*(z,s) = E_k^*(z,1-s). \tag{2.7}$$

2.2 Holomorphic double Eisenstein series

Define the subgroup

$$B := \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | \, n \in \mathbb{Z}\} \subset \mathrm{SL}(2,\mathbb{Z}). \tag{2.8}$$

Then Γ_∞, the subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ fixing ∞, is $B \cup -B$. For $\gamma \in \Gamma_\infty \backslash \Gamma$ the quantities c_γ, d_γ and $j(\gamma,z)$ are only defined up to sign (though even powers are well defined). For $\gamma \in B\backslash\Gamma$ there is no ambiguity in the signs of c_γ, d_γ and $j(\gamma,z)$.

Definition 2.2. *Let $z \in \mathbb{H}$ and $w \in \mathbb{C}$. For integers k_1, $k_2 \geqslant 3$ we define the* double Eisenstein series

$$\boldsymbol{E}_{k_1,k_2}(z,w) := \sum_{\substack{\gamma,\,\delta \in B\backslash\Gamma \\ c_{\gamma\delta^{-1}} > 0}} \left(c_{\gamma\delta^{-1}}\right)^{w-1} j(\gamma,z)^{-k_1} j(\delta,z)^{-k_2}. \tag{2.9}$$

This series is well-defined and, as we see in Proposition 4.2, for $\mathrm{Re}(w) < k_1 - 1, k_2 - 1$ converges to a holomorphic function of z that is a weight $k = k_1 + k_2$ cusp form. It vanishes identically when k_1, k_2 have different parity.

Let k be even. To get the most general kernel, with $s \in \mathbb{C}$, set

$$\boldsymbol{E}_{s,k-s}(z,w) := \sum_{\substack{\gamma,\, \delta \in B\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{w-1} \left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}. \tag{2.10}$$

In the usual convention, for $\rho \in \mathbb{C}$ with $\rho \neq 0$ write $\rho = |\rho| e^{i \arg(\rho)}$ for $-\pi < \arg(\rho) \leqslant \pi$ and

$$\rho^s = |\rho|^s e^{i \arg(\rho) s} \quad \text{for} \quad s \in \mathbb{C}. \tag{2.11}$$

Note that

$$c_{\gamma\delta^{-1}} = \begin{vmatrix} c_\gamma & d_\gamma \\ c_\delta & d_\delta \end{vmatrix} > 0 \implies \frac{j(\gamma,z)}{j(\delta,z)} \in \mathbb{H} \quad \text{for} \quad z \in \mathbb{H}$$

and so $(j(\gamma,z)/j(\delta,z))^{-s}$ in (2.10) is well-defined and a holomorphic function of $s \in \mathbb{C}$ and $z \in \mathbb{H}$. Proposition 4.2 shows that $\boldsymbol{E}_{s,k-s}(z,w)$ converges absolutely and uniformly on compact sets for which $2 < \sigma < k-2$ and $\mathrm{Re}(w) < \sigma - 1, k - 1 - \sigma$.

Define the *completed double Eisenstein series* as

$$\boldsymbol{E}^*_{s,k-s}(z,w) := \left[\frac{e^{si\pi/2}\Gamma(s)\Gamma(k-s)\Gamma(k-w)\zeta(1-w+s)\zeta(1-w+k-s)}{2^{3-w}\pi^{k+1-w}\Gamma(k-1)}\right] \boldsymbol{E}_{s,k-s}(z,w). \tag{2.12}$$

Theorem 2.3. *Let $k \geqslant 6$ be even. The series $\boldsymbol{E}^*_{s,k-s}(z,w)$ has an analytic continuation to all $s, w \in \mathbb{C}$ and as a function of z is always in $S_k(\Gamma)$. For any f in $\mathcal{B}_k$ we have*

$$\langle\, \boldsymbol{E}^*_{s,k-s}(\cdot,w), f\,\rangle = L^*(f,s)L^*(f,w). \tag{2.13}$$

It follows directly from (2.13) and (2.1) that $\boldsymbol{E}^*_{s,k-s}(z,w)$ satisfies eight functional equations generated by:

$$\boldsymbol{E}^*_{s,k-s}(z,w) = \boldsymbol{E}^*_{w,k-w}(z,s), \tag{2.14}$$
$$\boldsymbol{E}^*_{s,k-s}(z,w) = (-1)^{k/2}\boldsymbol{E}^*_{k-s,s}(z,w). \tag{2.15}$$

The next result shows how $\boldsymbol{E}^*_{s,k-s}$ is a generalization of the Rankin-Cohen bracket $[E_{k_1}, E_{k_2}]_n$.

Proposition 2.4. *For $n \in \mathbb{Z}_{\geqslant 1}$ and even $k_1, k_2 \geqslant 4$,*

$$n![E_{k_1}, E_{k_2}]_n = \frac{2(-1)^{k_1/2}\pi^k\Gamma(k-1)}{(2\pi i)^n\zeta(k_1)\zeta(k_2)\Gamma(k_1)\Gamma(k_2)\Gamma(k-n-1)}\boldsymbol{E}^*_{k_1+n,k_2+n}(z,n+1).$$

Another way to understand these double Eisenstein series is through their connections to non-holomorphic Eisenstein series. Any smooth function, transforming with weight k and with polynomial growth as $y \to \infty$ may be projected into S_k with respect to the Petersson scalar product. See [7, §3.2] and the contained references. Denote this holomorphic projection by π_{hol}.

Proposition 2.5. *Let $k = k_1 + k_2 \geqslant 6$ for even $k_1, k_2 \geqslant 0$. Then for all $s, w \in \mathbb{C}$*

$$\boldsymbol{E}^*_{s,k-s}(z,w) = \pi_{hol}\Big[(-1)^{k_2/2} y^{-k/2} E^*_{k_1}(z,u) E^*_{k_2}(z,v)/(2\pi^{k/2})\Big]$$

where

$$u = (s+w-k+1)/2, \quad v = (-s+w+1)/2. \tag{2.16}$$

2.3 Values of L-functions.

For $f \in \mathcal{B}_k$ let K_f be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of f. We will recall Zagier's proof of the next result in §8.1.

Theorem 2.6. (Manin's Periods Theorem) *For each $f \in \mathcal{B}_k$ there exist $\omega_+(f), \omega_-(f) \in \mathbb{R}$ such that*

$$L^*(f,s)/\omega_+(f), \quad L^*(f,w)/\omega_-(f) \in K_f$$

for all s, w with $1 \leqslant s, w \leqslant k-1$ and s even, w odd.

Let $m \in \mathbb{Z}$ satisfy $m \leqslant 0$ or $m \geqslant k$. Then for these values outside the critical strip we have, according to [16, §3.4] and the references therein,

$$L^*(f,m) \in \mathscr{P}[1/\pi]$$

where $\mathscr{P}$ is the ring of periods: complex numbers that may be expressed as an integral of an algebraic function over an algebraic domain. In contrast to the periods (1.4), we do not have much more precise information about the algebraic properties of the values $L^*(f,m)$. A special case of a theorem by Koblitz [14] shows, for example, that

$$L^*(f,m) \quad \notin \quad \mathbb{Z} \cdot L^*(f,1) + \mathbb{Z} \cdot L^*(f,2) + \cdots + \mathbb{Z} \cdot L^*(f,k-1).$$

Let $K\big(\boldsymbol{E}^*_{s,k-s}(\cdot,w)\big)$ be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of $\boldsymbol{E}^*_{s,k-s}(\cdot,w)$ and let $\omega_+(f)$, $\omega_-(f)$ be as given in Theorem 2.6. Then we have

Theorem 2.7. *For all $f \in \mathcal{B}_k$ and $s \in \mathbb{C}$*

$$L^*(f,s)/\omega_+(f) \in K\big(\boldsymbol{E}^*_{s,k-s}(\cdot,k-1)\big)K_f,$$
$$L^*(f,s)/\omega_-(f) \in K\big(\boldsymbol{E}^*_{k-2,2}(\cdot,s)\big)K_f.$$

The above theorem gives the link between Fourier coefficients of double Eisenstein series and arbitrary values of L-functions. We hope that this interesting connection will help shed light on $L^*(f,s)$ for s outside the set $\{1,2,\dots,k-1\}$. See the further discussion in §8.2 for the case when $s \in \mathbb{Z}$.

In §8.3 we also prove results analogous to Theorem 2.7 for the completed L-function of f twisted by $e^{2\pi i m p/q}$ for $p/q \in \mathbb{Q}$:

$$L^*(f,s;p/q) := \frac{\Gamma(s)}{(2\pi)^s} \sum_{m=1}^{\infty} \frac{a_f(m) e^{2\pi i m p/q}}{m^s} = \int_0^{\infty} f(iy+p/q) y^{s-1}\, dy. \tag{2.17}$$

2.4 Non-holomorphic double Eisenstein series

Definition 2.8. *For $z \in \mathbb{H}$, w, s, $s' \in \mathbb{C}$, we define the* non-holomorphic double Eisenstein series *as*

$$\mathcal{E}(z,w;s,s') := \sum_{\substack{\gamma,\, \delta \in \Gamma_\infty \backslash \Gamma \\ \gamma\delta^{-1} \neq \Gamma_\infty}} \frac{\operatorname{Im}(\gamma z)^s \operatorname{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{2.18}$$

A simple comparison with (2.2) shows it is absolutely and uniformly convergent for $\operatorname{Re}(s)$, $\operatorname{Re}(s') > 1$ and $\operatorname{Re}(w) > 0$. (This domain of convergence is improved in Proposition 4.3.) The most symmetric form of (2.18) is when $w = s + s'$. Define

$$\mathcal{E}^*(z;s,s') := 4\pi^{-s-s'}\Gamma(s)\Gamma(s')\zeta(3s+s')\zeta(s+3s')\mathcal{E}(z,s+s';s,s') + 2\theta(s)\theta(s')E(z,s+s'). \tag{2.19}$$

Theorem 2.9. *The completed double Eisenstein series $\mathcal{E}^*(z;s,s')$ has a meromorphic continuation to all $s,s' \in \mathbb{C}$ and satisfies the functional equations*

$$\begin{aligned}\mathcal{E}^*(z;s,s') &= \mathcal{E}^*(z;s',s),\\ \mathcal{E}^*(z;s,s') &= \mathcal{E}^*(z;1-s,1-s').\end{aligned}$$

For any even Maass Hecke eigenform u_j,

$$\langle\, \mathcal{E}^*(z;s,s'), u_j \,\rangle = L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2).$$

3 Further background results and notation

We need to introduce two more families of modular forms.

Definition 3.1. *For $z \in \mathbb{H}$, $k \geqslant 4$ in $2\mathbb{Z}$ and $m \in \mathbb{Z}_{\geqslant 0}$ the holomorphic* Poincaré series *is*

$$P_k(z;m) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma,z)^k} \quad = \quad \frac{1}{2} \sum_{\gamma \in B \backslash \Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma,z)^k}. \tag{3.1}$$

For $m \geqslant 1$ the series $P_k(z;m)$ span $S_k(\Gamma)$. The Eisenstein series $E_k(z) = P_k(z;0)$ is not a cusp form but is in the space $M_k(\Gamma)$. The second family of modular forms is based on a series due to Cohen in [3].

Definition 3.2. *The* generalized Cohen kernel *is given by*

$$\mathcal{C}_k(z,s;p/q) := \frac{1}{2} \sum_{\gamma \in \Gamma} (\gamma z + p/q)^{-s} j(\gamma,z)^{-k} \tag{3.2}$$

for $p/q \in \mathbb{Q}$ and $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$.

In [7, Section 5] we studied $\mathcal{C}_k(z,s;p/q)$ (the factor $1/2$ is included to keep the notation consistent with [7] where $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$). We showed that, for each $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$, $\mathcal{C}_k(z,s;p/q)$ converges to an element of $S_k(\Gamma)$, with a meromorphic continuation to all $s \in \mathbb{C}$. From [7, Prop. 5.4] we have

$$\langle\, \mathcal{C}_k(\cdot,s;p/q), f \,\rangle = 2^{2-k}\pi e^{-si\pi/2} \frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)} L^*(f,k-s;p/q) \tag{3.3}$$

which is a generalization of Cohen's lemma in [15, §1.2]. For simplicity we write $\mathcal{C}_k(z,s)$ for $\mathcal{C}_k(z,s;0)$. The twisted L-functions satisfy

$$\overline{L^*(f,s;p/q)} = L^*(f,\overline{s};-p/q) \tag{3.4}$$

and

$$q^s L^*(f,s;p/q) = (-1)^{k/2} q^{k-s} L^*(f,k-s;-p'/q) \tag{3.5}$$

for $pp' \equiv 1 \mod q$, as in [17, App. A.3].

Define $\mathcal{M}_n := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,\middle|\, a,b,c,d \in \mathbb{Z}, ad-bc=n \right\}$. Thus $\mathcal{M}_1 = \Gamma$. For $k \in \mathbb{Z}$ and $g : \mathbb{H} \to \mathbb{C}$ set

$$(g|_k\gamma)(z) := \det(\gamma)^{k/2} g(\gamma z) j(\gamma,z)^{-k}$$

for all $\gamma \in \mathcal{M}_n$. The weight k Hecke operator T_n acts on $g \in M_k$ by

$$(T_n g)(z) := n^{k/2-1} \sum_{\gamma \in \Gamma \backslash \mathcal{M}_n} (g|_k\gamma)(z) = n^{k-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-k} \sum_{0 \leqslant b < d} g\left(\frac{az+b}{d}\right). \tag{3.6}$$

4 Basic properties of double Eisenstein series

We work more generally in this section with Γ a Fuchsian group of the first kind containing at least one cusp. Set

$$\varepsilon_\Gamma := \#\{\Gamma \cap \{-I\}\}. \tag{4.1}$$

Label the finite number of inequivalent cusps $\mathfrak{a}$, $\mathfrak{b}$ etc and let $\Gamma_\mathfrak{a}$ be the subgroup of Γ fixing $\mathfrak{a}$. There exists a corresponding scaling matrix $\sigma_\mathfrak{a} \in \mathrm{SL}(2,\mathbb{R})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ and

$$\sigma_\mathfrak{a}{}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \begin{cases} B \cup -B & \text{if } -I \in \Gamma \quad (\varepsilon_\Gamma = 1) \\ B & \text{if } -I \notin \Gamma \quad (\varepsilon_\Gamma = 0). \end{cases}$$

Also set $\Gamma^*_\mathfrak{a} := \sigma_\mathfrak{a} B \sigma_\mathfrak{a}{}^{-1}$.

We recall some facts about $E_{k,\mathfrak{a}}(z,s)$, the non-holomorphic Eisenstein series associated to the cusp $\mathfrak{a}$ - see for example [10, Chap. 3], [7, §2.1]. It is defined as

$$E_{k,\mathfrak{a}}(z,s) := \sum_{\gamma \in \Gamma_\mathfrak{a}\backslash\Gamma} \mathrm{Im}(\sigma_\mathfrak{a}{}^{-1}\gamma z)^s \left(\frac{j(\sigma_\mathfrak{a}{}^{-1}\gamma, z)}{|j(\sigma_\mathfrak{a}{}^{-1}\gamma, z)|}\right)^{-k}$$

and absolutely convergent for $\mathrm{Re}(s) > 1$. Put $E^*_{k,\mathfrak{a}}(z,s) := \theta_k(s)E_{k,\mathfrak{a}}(z,s)$ as in (2.5). Then we have the expansion

$$E^*_{0,\mathfrak{a}}(\sigma_\mathfrak{b} z, s) = \delta_{\mathfrak{a}\mathfrak{b}}\theta(s)y^s + \theta(1-s)Y_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + \sum_{l \neq 0} Y_{\mathfrak{a}\mathfrak{b}}(l,s)W_s(lz) \tag{4.2}$$

and

$$E^*_{k,\mathfrak{a}}(\sigma_\mathfrak{b} z, s) = \delta_{\mathfrak{a}\mathfrak{b}}\theta_k(s)y^s + \theta_k(1-s)Y_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + O(e^{-2\pi y}) \tag{4.3}$$

as $y \to \infty$ for all $k \in 2\mathbb{Z}$. Also, its functional equation is

$$E^*_{k,\mathfrak{a}}(z, 1-s) = \sum_{\mathfrak{b}} Y_{\mathfrak{a}\mathfrak{b}}(1-s)E^*_{k,\mathfrak{b}}(z,s). \tag{4.4}$$

We gave the coefficients $Y_{\mathfrak{a}\mathfrak{b}}(s)$ and $Y_{\mathfrak{a}\mathfrak{b}}(l,s)$ explicitly in the case of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ following (2.3), and in general they involve series containing Kloosterman sums, see [10, (3.21),(3.22)].

For the natural generalization of (2.10) we define the *double Eisenstein series associated to the cusp* $\mathfrak{a}$ as

$$\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w) := \sum_{\gamma,\,\delta \in \Gamma^*_\mathfrak{a}\backslash\Gamma,\ c_{\sigma_\mathfrak{a}{}^{-1}\gamma\delta^{-1}\sigma_\mathfrak{a}} > 0} \left(c_{\sigma_\mathfrak{a}{}^{-1}\gamma\delta^{-1}\sigma_\mathfrak{a}}\right)^{w-1} \left(\frac{j(\sigma_\mathfrak{a}{}^{-1}\gamma, z)}{j(\sigma_\mathfrak{a}{}^{-1}\delta, z)}\right)^{-s} j(\sigma_\mathfrak{a}{}^{-1}\delta, z)^{-k} \tag{4.5}$$

so that

$$\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a} z, w) = j(\sigma_\mathfrak{a}, z)^k \sum_{\substack{\gamma,\,\delta \in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}} > 0}} \left(c_{\gamma\delta^{-1}}\right)^{w-1} \left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k} \tag{4.6}$$

for $\Gamma' = \sigma_\mathfrak{a}{}^{-1}\Gamma\sigma_\mathfrak{a}$ which is also a Fuchsian group of the first kind. To establish an initial domain of absolute convergence for (4.6) we consider

$$\sum_{\substack{\gamma,\,\delta \in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}} > 0}} \left| \left(c_{\gamma\delta^{-1}}\right)^{w-1} \left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k} \right|. \tag{4.7}$$

Recalling (2.11), we see that

$$|\rho^s| = |\rho|^\sigma e^{-t\arg(\rho)} \ll_t |\rho|^\sigma \quad \text{for} \quad s=\sigma+it\in\mathbb{C}.$$

Therefore, with $r=\mathrm{Re}(w)$ and $\mathrm{Im}(\gamma z)=y|j(\gamma,z)|^{-2}$ we deduce that (4.7) is bounded by a constant depending on s times

$$y^{-k/2}\sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma' \\ \gamma\delta^{-1}\neq\Gamma_\infty}} \left|c_{\gamma\delta^{-1}}\right|^{r-1}\mathrm{Im}(\gamma z)^{\sigma/2}\mathrm{Im}(\delta z)^{(k-\sigma)/2}. \tag{4.8}$$

Lemma 4.1. *There exists a constant $\kappa_\Gamma>0$ so that for all $\gamma,\delta\in\Gamma$ with $c_{\gamma\delta^{-1}}>0$*

$$\kappa_\Gamma \leqslant c_{\gamma\delta^{-1}} \leqslant \mathrm{Im}(\gamma z)^{-1/2}\mathrm{Im}(\delta z)^{-1/2}.$$

Proof. The existence of κ_Γ is described in [10, §§2.5, 2.6] and [26, Lemma 1.25]. Set $\varepsilon(\gamma,z) := j(\gamma,z)/|j(\gamma,z)| = e^{i\arg(j(\gamma,z))}$. It is easy to verify that, for all γ, $\delta\in\Gamma$ and $z\in\mathbb{H}$,

$$\begin{aligned} c_{\gamma\delta^{-1}} &= c_\gamma j(\delta,z)-c_\delta j(\gamma,z) \\ &= \left(\frac{j(\gamma,z)-\overline{j(\gamma,z)}}{2iy}\right)j(\delta,z)-\left(\frac{j(\delta,z)-\overline{j(\delta,z)}}{2iy}\right)j(\gamma,z) \\ &= \left(\varepsilon(\delta,z)^{-2}-\varepsilon(\gamma,z)^{-2}\right)j(\gamma,z)j(\delta,z)/(2iy). \end{aligned}$$

Therefore

$$\begin{aligned} |c_{\gamma\delta^{-1}}| &= \left|\frac{\varepsilon(\gamma,z)}{\varepsilon(\delta,z)}-\frac{\varepsilon(\delta,z)}{\varepsilon(\gamma,z)}\right|\mathrm{Im}(\gamma z)^{-1/2}\mathrm{Im}(\delta z)^{-1/2}/2 \\ &= \left|\mathrm{Im}\left(\frac{\varepsilon(\gamma,z)}{\varepsilon(\delta,z)}\right)\right|\mathrm{Im}(\gamma z)^{-1/2}\mathrm{Im}(\delta z)^{-1/2} \\ &\leqslant \mathrm{Im}(\gamma z)^{-1/2}\mathrm{Im}(\delta z)^{-1/2}. \end{aligned}$$

□

It follows that for $r'=\max(r,1)$ and $\gamma\delta^{-1}\notin\Gamma_\infty$

$$\left|c_{\gamma\delta^{-1}}\right|^{r-1} \ll \mathrm{Im}(\gamma z)^{(1-r')/2}\mathrm{Im}(\delta z)^{(1-r')/2} \tag{4.9}$$

for an implied constant depending on Γ and r. Combining (4.8), (4.9) shows

$$\begin{aligned} \frac{\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a}z,w)}{j(\sigma_\mathfrak{a},z)^k} &\ll y^{-k/2}\sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma' \\ \gamma\delta^{-1}\neq\Gamma_\infty}}\mathrm{Im}(\gamma z)^{(1-r'+\sigma)/2}\mathrm{Im}(\delta z)^{(1-r'+k-\sigma)/2} \\ &= y^{-k/2}\left[E_\mathfrak{a}\left(\sigma_\mathfrak{a}z,\frac{1-r'+\sigma}{2}\right)E_\mathfrak{a}\left(\sigma_\mathfrak{a}z,\frac{1-r'+k-\sigma}{2}\right)-E_\mathfrak{a}\left(\sigma_\mathfrak{a}z,1-r'+k/2\right)\right] \end{aligned} \tag{4.10}$$

on noting that $\mathrm{Im}(\gamma z)=\mathrm{Im}(\delta z)$ for $\gamma\delta^{-1}\in\Gamma_\infty$. Since $E_\mathfrak{a}(z,s)$ is absolutely convergent for $\sigma=\mathrm{Re}(s)>1$, we have proved that the series $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a}z,w)$, defined in (4.6), is absolutely convergent for $2<\sigma<k-2$ and $\mathrm{Re}(w)<\sigma-1, k-1-\sigma$. This convergence is uniform for z in compact sets of $\mathbb{H}$ and s,w in compact sets in $\mathbb{C}$ satisfying the above constraints.

We next verify that $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ has weight k in the z variable. We have

$$f(z)\in M_k(\Gamma) \iff f(\sigma_\mathfrak{a}z)j(\sigma_\mathfrak{a},z)^{-k}\in M_k(\sigma_\mathfrak{a}{}^{-1}\Gamma\sigma_\mathfrak{a})$$

so with (4.6) we must prove that

$$g(z) := \sum_{\substack{\gamma,\,\delta \in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{w-1} \left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}$$

is in $M_k(\Gamma')$. For all $\tau \in \Gamma'$

$$\begin{aligned} \frac{g(\tau z)}{j(\tau,z)^k} &= \sum_{\substack{\gamma,\,\delta \in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{w-1} \left(\frac{j(\gamma,\tau z)}{j(\delta,\tau z)}\right)^{-s} j(\delta,\tau z)^{-k} j(\tau,z)^{-k} \\ &= \sum_{\substack{\gamma,\,\delta \in B\backslash\Gamma' \\ c_{(\gamma\tau)(\delta\tau)^{-1}}>0}} \left(c_{(\gamma\tau)(\delta\tau)^{-1}}\right)^{w-1} \left(\frac{j(\gamma\tau,z)}{j(\delta\tau,z)}\right)^{-s} j(\delta\tau,z)^{-k} = g(z) \end{aligned}$$

as required.

We finally show that $\boldsymbol{E}_{s,k-s}$ is a cusp form. By (4.10), replacing z by $\sigma_{\mathfrak{a}}{}^{-1}\sigma_{\mathfrak{b}} z$ and using (4.3), for any cusp $\mathfrak{b}$ we obtain

$$\begin{aligned} &\frac{\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_{\mathfrak{b}} z, w)}{j(\sigma_{\mathfrak{b}}, z)^k} \\ &\qquad \ll y^{-k/2}\left[E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}} z, \frac{1-r'+\sigma}{2}\right) E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}} z, \frac{1-r'+k-\sigma}{2}\right) - E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}} z, 1-r'+k/2\right)\right] \\ &\qquad\qquad \ll y^{1+\sigma-k} + y^{1-\sigma} + y^{1+r'-k} + y^{r'-k} \end{aligned}$$

and approaches 0 as $y \to \infty$. Thus, by a standard argument (see for example [7, Prop. 5.3]), $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ a cusp form. Assembling these results, we have shown the following:

Proposition 4.2. *Let $z \in \mathbb{H}$, $k \in \mathbb{Z}$ and let $s, w \in \mathbb{C}$ satisfy $2 < \sigma < k-2$ and $\mathrm{Re}(w) < \sigma - 1, k-1-\sigma$. For Γ a Fuchsian group of the first kind with cusp $\mathfrak{a}$, the series $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ is absolutely and uniformly convergent for s, w and z in compact sets satisfying the above constraints. For each such s, w we have $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w) \in S_k(\Gamma)$ as a function of z.*

The same techniques prove the next result, for the non-holomorphic double Eisenstein series. Generalizing (2.18), we set

$$\mathcal{E}_{\mathfrak{a}}(\sigma_{\mathfrak{a}} z, w; s, s') := \sum_{\substack{\gamma,\,\delta \in \Gamma_\infty\backslash\sigma_{\mathfrak{a}}{}^{-1}\Gamma\sigma_{\mathfrak{a}} \\ \gamma\delta^{-1} \neq \Gamma_\infty}} \frac{\mathrm{Im}(\gamma z)^s \mathrm{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{4.11}$$

Proposition 4.3. *Let $z \in \mathbb{H}$, $s, s', w \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s)$ and $\sigma' = \mathrm{Re}(s')$. The series $\mathcal{E}_{\mathfrak{a}}(z,w;s,s')$, defined in* (4.11) *is absolutely and uniformly convergent for z, w, s and s' in compact sets satisfying*

$$\sigma, \sigma' > 1 \quad \textit{and} \quad \mathrm{Re}(w) > 2-2\sigma, 2-2\sigma'.$$

Unlike $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$, the series $\mathcal{E}_{\mathfrak{a}}(z,w;s,s')$ may have polynomial growth at cusps.

5 Further results on double Eisenstein series

5.1 Analytic Continuation

Proof of Theorem 2.3. Our next task is to prove the meromorphic continuation of $\boldsymbol{E}_{s,k-s}(z,w)$ in s and w. For s, w in the initial domain of convergence, we begin with

$$\begin{aligned}
&\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w) \\
&\quad = \sum_{u,v=1}^{\infty} u^{w-1-s}v^{w-1-k+s} \sum_{\substack{a,b,c,d\in\mathbb{Z}\\(a,b)=(c,d)=1\\ad-bc>0}} (ad-bc)^{w-1}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} \\
&\quad = \sum_{u,v=1}^{\infty} \sum_{\substack{a,b,c,d\in\mathbb{Z}\\(a,b)=(c,d)=1\\ad-bc>0}} (au\cdot dv - bu\cdot cv)^{w-1}\left(\frac{au\cdot z+bu}{cv\cdot z+dv}\right)^{-s}(cv\cdot z+dv)^{-k} \\
&\quad = \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ad-bc>0}} (ad-bc)^{w-1}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} \qquad (5.1)\\
&\quad = \sum_{n=1}^{\infty}\frac{1}{n^{1-w}} \sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{M}_n} \left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} \\
&\quad = 2\sum_{n=1}^{\infty}\frac{T_n\mathcal{C}_k(z,s)}{n^{k-w}}, \qquad (5.2)
\end{aligned}$$

recalling (3.2). With Proposition 4.2, we know $\boldsymbol{E}_{s,k-s}(z,w)\in S_k(\Gamma)$ so that

$$\begin{aligned}
\boldsymbol{E}_{s,k-s}(z,w) &= \sum_{f\in\mathcal{B}_k}\frac{\langle\, \boldsymbol{E}_{s,k-s}(\cdot,w), f\,\rangle}{\langle\, f,f\,\rangle} f(z)\\
\implies\quad \zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w) &= 2\sum_{n=1}^{\infty}\frac{1}{n^{k-w}}\sum_{f\in\mathcal{B}_k}\frac{\langle\, T_n\mathcal{C}_k(\cdot,s), f\,\rangle}{\langle\, f,f\,\rangle} f(z).
\end{aligned}$$

Then

$$\langle\, T_n\mathcal{C}_k(z,s), f\,\rangle = \langle\, \mathcal{C}_k(z,s), T_n f\,\rangle = a_f(n)\langle\, \mathcal{C}_k(z,s), f\,\rangle$$

and with (3.3) we obtain

$$\begin{aligned}
\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w) &= 2^{3-w}\pi^{k+1-w}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\\
&\quad\times\sum_{f\in\mathcal{B}_k} L^*(f,k-s)L^*(f,k-w)\frac{f(z)}{\langle\, f,f\,\rangle}. \qquad (5.3)
\end{aligned}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*$ with (2.12). Then (5.3) becomes

$$\boldsymbol{E}^*_{s,k-s}(z,w) = \sum_{f\in\mathcal{B}_k} L^*(f,s)L^*(f,w)\frac{f(z)}{\langle\, f,f\,\rangle}. \qquad (5.4)$$

We also now see from (5.4) that $\boldsymbol{E}^*_{s,k-s}(z,w)$ has a analytic continuation to all s, w in $\mathbb{C}$ and satisfies (2.13) and the two functional equations (2.14), (2.15). The dihedral group D_8 generated by (2.14), (2.15) is described in [7, §4.4]. ☐

5.2 Twisted double Eisenstein series

In this section, we define the *twisted double Eisenstein series* by

$$\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w;p/q) := \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} (ad-bc)^{w-1}\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s}(cz+d)^{-k} \quad (5.5)$$

for $p/q \in \mathbb{Q}$ with $q > 0$ and establish its basic required properties. We remark that the above definition of $\boldsymbol{E}_{s,k-s}(z,w;p/q)$ comes from generalizing (5.1), but it is not clear how it can be extended to general Fuchsian groups.

Writing

$$(ad-bc)^{w-1}\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s} = q^{1-w+s}\big((aq+cp)d-(bq+dp)c\big)^{w-1}\left(\frac{(aq+cp)z+(bq+dp)}{cz+d}\right)^{-s}$$

we see that

$$(5.5) = q^{1-w+s}\sum_{\substack{a',b',c,d\in\mathbb{Z}\\ a'd-b'c>0}} (a'd-b'c)^{w-1}\left(\frac{a'z+b'}{cz+d}\right)^{-s}(cz+d)^{-k}$$

with $a' \equiv cp \mod q$ and $b' \equiv dp \mod q$. Hence $\boldsymbol{E}_{s,k-s}(z,w;p/q)$ is a subseries of $\boldsymbol{E}_{s,k-s}(z,w)$ and, in the same domain of initial convergence, is an element of S_k.

The analog of (5.2) is

$$\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w;p/q) = 2\sum_{n=1}^{\infty}\frac{T_n\mathcal{C}_k(z,s;p/q)}{n^{k-w}}. \quad (5.6)$$

Hence, with (3.3),

$$\begin{aligned}&\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w;p/q)\\ &\quad = 2^{3-w}\pi^{k+1-w}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\sum_{f\in\mathcal{B}_k} L^*(f,k-s;p/q)L^*(f,k-w)\frac{f(z)}{\langle f,f\rangle}. \quad (5.7)\end{aligned}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*_{s,k-s}(z,w;p/q)$ with the same factor as (2.12) and we obtain

$$\langle \boldsymbol{E}^*_{s,k-s}(\cdot,w;p/q), f\rangle = L^*(f,k-s;p/q)L^*(f,k-w) \quad (5.8)$$

for any f in $\mathcal{B}_k$. Then (5.7) implies $\boldsymbol{E}^*_{s,k-s}(z,w;p/q)$ has an analytic continuation to all s,w in $\mathbb{C}$. It satisfies the two functional equations:

$$\begin{aligned}\boldsymbol{E}^*_{s,k-s}(z,k-w;p/q) &= (-1)^{k/2}\boldsymbol{E}^*_{s,k-s}(z,w;p/q),\\ q^s\boldsymbol{E}^*_{k-s,s}(z,w;p/q) &= (-1)^{k/2}q^{k-s}\boldsymbol{E}^*_{s,k-s}(z,w;-p'/q)\end{aligned}$$

for $pp' \equiv 1 \mod q$ using (2.1) and (3.5), respectively.

6 Applying the Rankin-Cohen bracket to Poincaré series

The main objective of this section is to show how double Eisenstein series arise naturally when the Rankin-Cohen bracket is applied to the usual Eisenstein series E_k. Proposition 2.4 will be a consequence of this. In fact, since there is no difficulty in extending these methods, we compute the Rankin-Cohen bracket of two arbitrary Poincaré series

$$[P_{k_1}(z;m_1), P_{k_2}(z;m_2)]_n$$

for $m_1, m_2 \geqslant 0$. The result may be expressed in terms of the *double Poincaré series*, defined below. In this way, the action of the Rankin-Cohen brackets on spaces of modular forms can be completely described. See also Corollary 6.5 at the end of this section.

Definition 6.1. *Let $z \in \mathbb{H}$, $k_1, k_2 \geqslant 3$ in $\mathbb{Z}$ and $m_1, m_2 \in \mathbb{Z}_{\geqslant 0}$. For $w \in \mathbb{C}$ with* $\mathrm{Re}(w) < k_1 - 1, k_2 - 1$, *we define the* double Poincaré series

$$\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2) := \sum_{\substack{\gamma,\,\delta \in B\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{w-1} \frac{e^{2\pi i(m_1\gamma z + m_2 \delta z)}}{j(\gamma,z)^{k_1} j(\delta,z)^{k_2}}. \tag{6.1}$$

The series (6.1) will vanish identically unless k_1 and k_2 have the same parity. Clearly we have $\boldsymbol{E}_{k_1,k_2}(z,w) = \boldsymbol{P}_{k_1,k_2}(z,w;0,0)$. Since $|e^{2\pi i(m_1\gamma z+m_2\delta z)}| \leqslant 1$, it is a simple matter to verify that the work in §4 proves that $\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2)$ converges absolutely and uniformly on compacta to a cusp form in $S_{k_1+k_2}(\Gamma)$.

For $l \in \mathbb{Z}_{\geqslant 0}$ it is convenient to set

$$Q_k(z,l;m) := \begin{cases} P_k(z;m) & \text{if } \ l = 0, \\ \frac{1}{2}\sum_{\gamma \in B\backslash\Gamma} \frac{e^{2\pi i m\gamma z}(c_\gamma)^l}{j(\gamma,z)^{k+l}} & \text{if } \ l \geqslant 1. \end{cases} \tag{6.2}$$

As in the proof of Proposition 4.2, Q_k is an absolutely convergent series for k even and at least 4. The next result may be verified by induction.

Lemma 6.2. *For every $j \in \mathbb{Z}_{\geqslant 0}$, we have the formulas*

$$\begin{aligned} \frac{d^j}{dz^j} E_k(z) &= (-1)^j \frac{(k+j-1)!}{(k-1)!} Q_k(z,j;0), \\ \frac{d^j}{dz^j} P_k(z;m) &= \sum_{l=0}^{j} (-1)^{l+j} (2\pi i m)^l \frac{j!}{l!} \binom{k+j-1}{k+l-1} Q_{k+2l}(z,j-l;m) \qquad (m>0). \end{aligned}$$

Set

$$A_{k_1,k_2}(l,u)_n := \frac{(k_1+n-1)!(k_2+n-1)!}{l!u!(n-l-u)!(k_1+l-1)!(k_2+u-1)!}.$$

Proposition 6.3. *For $m_1, m_2 \in \mathbb{Z}_{\geqslant 1}$*

$$\begin{aligned} &[P_{k_1}(z;m_1), P_{k_2}(z;m_2)]_n \\ &\quad = \sum_{\substack{l,u \geqslant 0 \\ l+u \leqslant n}} A_{k_1,k_2}(l,u)_n (-2\pi i m_1)^l (2\pi i m_2)^u \boldsymbol{P}_{k_1+n+l-u, k_2+n-l+u}(z, n+1-l-u; m_1, m_2)/2 \\ &\qquad\qquad + P_{k_1+k_2+2n}(z; m_1+m_2) \sum_{\substack{l,u \geqslant 0 \\ l+u = n}} A_{k_1,k_2}(l,u)_n (-2\pi i m_1)^l (2\pi i m_2)^u. \end{aligned}$$

Proof. With Lemma 6.2

$$\begin{aligned} &[P_{k_1}(z;m_1), P_{k_2}(z;m_2)]_n \\ &\quad = \sum_{l=0}^{n}\sum_{u=0}^{n} (2\pi i m_1)^l (2\pi i m_2)^u \frac{(k_1+n-1)!(k_2+n-1)!}{l!u!(k_1+l-1)!(k_2+u-1)!} \\ &\qquad \times \sum_{r=l}^{n-u} (-1)^{n+l+u+r} \frac{Q_{k_1+2l}(z,r-l;m_1) Q_{k_2+2u}(z,n-r-u;m_2)}{(r-l)!(n-r-u)!}. \end{aligned} \tag{6.3}$$

The inner sum over r is

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in B\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+2l}j(\delta,z)^{k_2+2u}} \times\sum_{r=l}^{n-u}\binom{n-l-u}{r-l}\left(\frac{c_\gamma}{j(\gamma,z)}\right)^{r-l}\left(\frac{-c_\delta}{j(\delta,z)}\right)^{n-r-u} \tag{6.4}$$

and, employing the binomial theorem, (6.4) reduces to

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in B\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}}\big(c_\gamma j(\delta,z)-c_\delta j(\gamma,z)\big)^{n-l-u} \tag{6.5}$$

for $l+u<n$ and

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in B\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}} \tag{6.6}$$

for $l+u=n$. Noting that $c_\gamma j(\delta,z)-c_\delta j(\gamma,z)=\left|\begin{smallmatrix} c_\gamma & d_\gamma\\ c_\delta & d_\delta\end{smallmatrix}\right|=c_{\gamma\delta^{-1}}$ means that (6.5) becomes

$$\frac{(-1)^l}{2(n-l-u)!}\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,n+1-l-u;m_1,m_2) \tag{6.7}$$

and (6.6) equals

$$\frac{(-1)^l}{(n-l-u)!}\left(\frac{\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,n+1-l-u;m_1,m_2)}{2}+P_{k_1+k_2+2n}(z;m_1+m_2)\right). \tag{6.8}$$

Putting (6.7) and (6.8) into (6.3) finishes the proof. $\square$

In fact, Proposition 6.3 is also valid for m_1 or m_2 equalling 0 provided we agree that $(-2\pi i m_1)^l=1$ in the ambiguous case where $m_1=l=0$ and similarly that $(2\pi i m_2)^u=1$ when $m_2=u=0$. With this notational convention the proof of the last proposition gives

Corollary 6.4. *For $m>0$ we have*

$$\begin{aligned}[E_{k_1}(z),P_{k_2}(z;m)]_n &= \sum_{u=0}^{n}A_{k_1,k_2}(0,u)_n(2\pi i m)^u\boldsymbol{P}_{k_1+n-u,k_2+n+u}(z,n+1-u;0,m)/2\\ &\qquad +P_{k_1+k_2+2n}(z;m)\cdot A_{k_1,k_2}(0,n)_n(2\pi i m)^n,\\ [E_{k_1}(z),E_{k_2}(z)]_n &= A_{k_1,k_2}(0,0)_n\boldsymbol{E}_{k_1+n,k_2+n}(z,n+1)/2+E_{k_1+k_2}(z)\cdot\delta_{n,0}.\end{aligned} \tag{6.9}$$

Proposition 2.4 follows directly from (6.9). Combining Proposition 2.4, with Theorem 2.3 gives a new proof of Zagier's formula (1.2). His original proof in [28, Prop. 6] employed Poincaré series.

Proof of Proposition 2.5. Let $F_{s,w}(z)=(-1)^{k_2/2}y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)/(2\pi^{k/2})$ with $u=(s+w-k+1)/2$, $v=(-s+w+1)/2$ as before in (2.16). Then $F_{s,w}(z)$ has weight k and polynomial growth as $y\to\infty$. It is proved in [7, Prop. 2.1] that

$$\langle\, F_{s,w},f\,\rangle=L^*(f,s)L^*(f,w) \tag{6.10}$$

for all $f\in B_k$. Comparing (6.10) with (2.13) shows that $\boldsymbol{E}^*_{s,k-s}(\cdot,w)=\pi_{hol}(F_{s,w})$ as required. $\square$

A basic property of Rankin-Cohen brackets also naturally emerges from Proposition 6.3 and Corollary 6.4:

Corollary 6.5. *For $g_1 \in M_{k_1}(\Gamma)$ and $g_2 \in M_{k_2}(\Gamma)$ we have $[g_1, g_2]_n \in S_{k_1+k_2+2n}(\Gamma)$ for $n > 0$.*

Proof. The space $M_{k_1}(\Gamma)$ is spanned by E_{k_1} and the Poincaré series $P_{k_1}(z; m)$ for $m \in \mathbb{Z}_{\geqslant 1}$. So we may write g_1, and similarly g_2, as a linear combination of Eisenstein and Poincaré series. Hence $[g_1, g_2]_n$ is a linear combination of the Rankin-Cohen brackets appearing in Proposition 6.3 and Corollary 6.4. By these results $[g_1, g_2]_n$ is a linear combination of double Poincaré and double Eisenstein series which are in $S_{k_1+k_2+2n}(\Gamma)$, as we have already shown. □

It would be interesting to know if $\boldsymbol{P}_{k_1,k_2}(z, w; m_1, m_2)$ has a meromorphic continuation in w. As a corollary of work in the next section we establish the continuation of $\boldsymbol{P}_{k_1,k_2}(z, w; 0, 0)$ to all $w \in \mathbb{C}$.

7 The Hecke action

The expression (5.2), giving $\boldsymbol{E}_{s,k-s}$ in terms of $\mathcal{C}_k$ acted upon by the Hecke operators, can be studied further and yields an interesting relation between $\boldsymbol{E}_{s,k-s}(z, w)$ and the generalized Cohen kernel $\mathcal{C}_k(z, s; p/q)$.

We have

$$\begin{aligned} T_n \mathcal{C}_k(z, s; p/q) &= n^{k-1} \sum_{\rho \in \Gamma \backslash \mathcal{M}_n} \mathcal{C}_k(\rho z, s; p/q) \cdot j(\rho, z)^{-k} \\ &= \frac{1}{2} n^{k-1} \sum_{\gamma \in \mathcal{M}_n} \left(\gamma z + \frac{p}{q}\right)^{-s} j(\gamma, z)^{-k}. \end{aligned}$$

To decompose $\mathcal{M}_n$ into left Γ-cosets, set $\mathcal{H} := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \,\middle|\, a, b, d \in \mathbb{Z}_{\geqslant 0},\ ad = n,\ 0 \leqslant b < a \right\}$ so that $\mathcal{M}_n = \bigcup_{\rho \in \mathcal{H}} \rho\Gamma$, a disjoint union. Hence

$$\begin{aligned} T_n \mathcal{C}_k(z, s; p/q) &= \frac{1}{2} n^{k-1} \sum_{\rho \in \mathcal{H}} \sum_{\gamma \in \Gamma} \left(\rho\gamma z + \frac{p}{q}\right)^{-s} j(\rho, \gamma z)^{-k} j(\gamma, z)^{-k} \\ &= \frac{1}{2} n^{k-1} \sum_{a|n} \left(\frac{n}{a}\right)^{-k} \left(\frac{a^2}{n}\right)^{-s} \sum_{0 \leqslant b < a} \sum_{\gamma \in \Gamma} \left(\gamma z + \frac{b}{a} + \frac{n}{a^2}\frac{p}{q}\right)^{-s} j(\gamma, z)^{-k} \\ &= n^{s-1} \sum_{a|n} a^{k-2s} \sum_{0 \leqslant b < a} \mathcal{C}_k\left(z, s; \frac{b}{a} + \frac{n}{a^2}\frac{p}{q}\right). \end{aligned} \tag{7.1}$$

Combining (7.1) in the case $p/q = 0$, with (5.2) we find

$$\begin{aligned} \frac{\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z, w)}{2} &= \sum_{n=1}^{\infty} \frac{T_n \mathcal{C}_k(z, s)}{n^{k-w}} \\ &= \sum_{n=1}^{\infty} n^{s+w-k-1} \sum_{a|n} a^{k-2s} \sum_{0 \leqslant b < a} \mathcal{C}_k\left(z, s; \frac{b}{a}\right) \\ &= \sum_{a=1}^{\infty} a^{k-2s} \sum_{v=1}^{\infty} (av)^{s+w-k-1} \sum_{0 \leqslant b < a} \mathcal{C}_k\left(z, s; \frac{b}{a}\right) \\ &= \zeta(k+1-s-w) \sum_{a=1}^{\infty} a^{w-s-1} \sum_{0 \leqslant b < a} \mathcal{C}_k\left(z, s; \frac{b}{a}\right). \end{aligned}$$

Consequently, for $2 < \sigma < k-2$ and $\mathrm{Re}(w) < \sigma - 1, k-1-\sigma$

$$\zeta(1-w+s)\boldsymbol{E}_{s,k-s}(z,w) = 2\sum_{a=1}^{\infty} a^{w-s-1} \sum_{b=0}^{a-1} \mathcal{C}_k\left(z, s; \frac{b}{a}\right). \tag{7.2}$$

Upon taking the inner product of both sides with $f \in \mathcal{B}_k$, by using (2.13), (3.3), and then simplifying we obtain

$$\frac{(2\pi)^{k-w}}{\Gamma(k-w)} L^*(f,s)L^*(f,w) = \zeta(k+1-s-w)\sum_{a=1}^{\infty} a^{w-s-1}\sum_{b=0}^{a-1} L^*\left(f, k-s; \frac{b}{a}\right). \tag{7.3}$$

Since the eigenforms f in $\mathcal{B}_k$ span S_k, we may verify (7.2) by giving another proof of (7.3). Note that the right side of (7.3) equals

$$\begin{aligned}
\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}} \sum_{a=1}^{\infty} a^{w-s-1} \sum_{b=0}^{a-1}\sum_{m=1}^{\infty} \frac{a_f(m) e^{2\pi i m b/a}}{m^{k-s}} \\
= \zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}} \sum_{m=1}^{\infty}\sum_{a|m} a^{w-s}\frac{a_f(m)}{m^{k-s}} \\
= \zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}} \sum_{m=1}^{\infty} \frac{a_f(m)\sigma_{w-s}(m)}{m^{k-s}}.
\end{aligned}$$

The series

$$L(f\otimes E(\cdot,v), k-s) := \sum_{m=1}^{\infty} \frac{a_f(m)\sigma_{w-s}(m)}{m^{k-s}}$$

is a convolution L-series involving the Fourier coefficients of $f(z)$ and $E(z,v)$ for $2v = -s+w+1$ (as in (2.16)) and, recalling [28, (72)] or [7, (2.11)],

$$\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}} L(f\otimes E(\cdot,v), k-s) = \frac{(2\pi)^{k-w}}{\Gamma(k-w)} L^*(f,k-s)L^*(f,k-w). \tag{7.4}$$

Applying the functional equation, (2.1), confirms that the right side of (7.4) equals the left side of (7.3).

Looking to simplify (7.2) leads to the natural question: what are the relations between the $\mathcal{C}_k(z,s;p/q)$ for rational p/q in the interval $[0,1)$? For example, it is a simple exercise with (3.3) and (3.5) to show that

$$q^{-s}\mathcal{C}_k(z,s;p/q) = e^{-si\pi} q^{-k+s}\mathcal{C}_k(z,k-s;-p'/q)$$

for $pp' \equiv 1 \mod q$. With $s = k/2$ at the center of the critical strip we get an even simpler relation:

$$\mathcal{C}_k(z,k/2;p/q) = (-1)^{k/2}\mathcal{C}_k(z,k/2;-p'/q). \tag{7.5}$$

A more interesting, but speculative, possibility would be to argue in the reverse direction in order to derive information about L-functions twisted by exponentials with *non-rational* exponents. Specifically, if we established, by other means, relations between the $\mathcal{C}_k(z,s;x)$ for $x \notin \mathbb{Q}$, then (7.2) and other results proven here might lead to relations for L-functions twisted by exponentials with non-rational exponents. That would be important because such L-functions play a prominent role in Kaszorowski and Perelli's programme of classifying the Selberg class (see e.g. [13]). Relations between these L-functions seem to be necessary for the extension of Kaszorowski and Perelli's classification to degree 2, to which L-functions of $\mathrm{GL}(2)$ cusp forms belong.

8 Periods of cusp forms

8.1 Values of L-functions inside the critical strip

We first review Zagier's proof in [28, §5] of Manin's Periods Theorem. This exhibits a general principle of proving algebraicity we will be using in the next sections.

For all $s, w \in \mathbb{C}$ it is convenient to define $H_{s,w} \in S_k$ by the conditions

$$\langle H_{s,w}, f \rangle = L^*(f,s)L^*(f,w) \quad \text{for all} \quad f \in \mathcal{B}_k.$$

We need the following result.

Lemma 8.1. *For $g \in S_k$ with Fourier coefficients in the field K_g and $f \in \mathcal{B}_k$ with coefficients in K_f,*

$$\langle g, f \rangle / \langle f, f \rangle \in K_g K_f.$$

Proof. See Shimura's general result [25, Lemma 4]. It is also a simple extension of [7, Lemma 4.3]. □

Let $K_{critical}$ be the field obtained by adjoining to $\mathbb{Q}$ all the Fourier coefficients of

$$\Big\{ H_{s,k-1}, H_{k-2,w} \;\Big|\; 1 \leqslant s, w \leqslant k-1, s \text{ even }, w \text{ odd} \Big\}.$$

Thus, with $f \in \mathcal{B}_k$ and employing Lemma 8.1,

$$L^*(f,k-1)L^*(f,k-2) = \langle H_{k-1,k-2}, f \rangle = c_f \langle f, f \rangle \tag{8.1}$$

for $c_f \in K_{critical} K_f$ and the left side of (8.1) is nonzero because the Euler product for $L^*(f,s)$ converges for $\mathrm{Re}(s) > k/2 + 1/2$. Set

$$\omega_+(f) := \frac{c_f \langle f, f \rangle}{L^*(f,k-1)}, \quad \omega_-(f) := \frac{\langle f, f \rangle}{L^*(f,k-2)}. \tag{8.2}$$

Then $\omega_+(f)\omega_-(f) = \langle f, f \rangle$ and we have:

Lemma 8.2. *For each $f \in \mathcal{B}_k$*

$$L^*(f,s)/\omega_+(f), \quad L^*(f,w)/\omega_-(f) \in K_{critical} K_f$$

for all s, w with $1 \leqslant s, w \leqslant k-1$ and s even, w odd.

Proof. For such s and w,

$$\begin{aligned}
\frac{L^*(f,s)}{\omega_+(f)} &= \frac{L^*(f,s)L^*(f,k-1)}{c_f \langle f, f \rangle} = \frac{\langle H_{s,k-1}, f \rangle}{c_f \langle f, f \rangle} = \frac{c'_f \langle f, f \rangle}{c_f \langle f, f \rangle} \in K_{critical} K_f \\
\frac{L^*(f,w)}{\omega_-(f)} &= \frac{L^*(f,w)L^*(f,k-2)}{c_f \langle f, f \rangle} = \frac{\langle H_{k-2,w}, f \rangle}{c_f \langle f, f \rangle} = \frac{c''_f \langle f, f \rangle}{c_f \langle f, f \rangle} \in K_{critical} K_f. \quad \square
\end{aligned}$$

To deduce Manin's Theorem from Lemma 8.2, we use Zagier's explicit expression for $H_{s,w}$. For $n \geqslant 0$, even $k_1, k_2 \geqslant 4$ and $k = k_1 + k_2 + 2n$, (1.2) implies

$$(-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} \binom{k-2}{n} H_{n+1,n+k_2} = \frac{[E_{k_1}, E_{k_2}]_n}{(2\pi i)^n}. \tag{8.3}$$

The Fourier coefficients of E_{k_1}, E_{k_2} are rational and hence the right side of (8.3) has rational coefficients. Then $H_{n+1,n+k_2}$ has Fourier coefficients in $\mathbb{Q}$ (and also for k_1, $k_2 = 2$ as described in [15, p. 214]). It follows that $K_{critical} = \mathbb{Q}$ and Lemma 8.2 becomes Theorem 2.6, Manin's Periods Theorem.

8.2 Arbitrary L-values

With the results of the last section we may now give the proof of Theorem 2.7, restated here:

Theorem 8.3. *For all $f \in \mathcal{B}_k$ and $s \in \mathbb{C}$, with $\omega_+(f)$, $\omega_-(f)$ as in Manin's Theorem,*

$$L^*(f,s)/\omega_+(f) \in K\big(\boldsymbol{E}^*_{s,k-s}(\cdot,k-1)\big)K_f,$$
$$L^*(f,s)/\omega_-(f) \in K\big(\boldsymbol{E}^*_{k-2,2}(\cdot,s)\big)K_f.$$

Proof. By Theorem 2.3, we have $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z,w)$ for all $s, w \in \mathbb{C}$. Thus, arguing as in Lemma 8.2 with $\boldsymbol{E}^*_{s,k-s}(\cdot,k-1) = H_{s,k-1}$ and $\boldsymbol{E}^*_{k-2,2}(\cdot,s) = H_{k-2,s}$ yields the theorem. ☐

We indicate briefly how the double Eisenstein series Fourier coefficients required to define $K\big(\boldsymbol{E}^*_{s,k-s}(\cdot,k-1)\big)$ and $K\big(\boldsymbol{E}^*_{k-2,2}(\cdot,s)\big)$ in Theorem 2.7 may be calculated when $s \in \mathbb{Z}$, using a slight extension of the methods in [7, §3]. We wish to find the l-th Fourier coefficient, $a_{s,w}(l)$, of $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z,w)$ for s even and w odd (and we assume $s, w \geqslant k/2 > 1$). With Proposition 2.5, this is $(-1)^{k_2/2}/(2\pi^{k/2})$ times the l-th Fourier coefficient of

$$\pi_{hol}\Big[y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)\Big]$$

for $u = (s+w-k+1)/2$ and $v = (-s+w+1)/2$ both in $\mathbb{Z}$. Let

$$\begin{aligned} F(z) := {} & y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v) \\ & - \frac{\theta_{k_1}(u)\theta_{k_2}(1-v)}{\theta_k(s+1-k/2)} y^{-k/2}E^*_k(z,s+1-k/2) - \frac{\theta_{k_1}(u)\theta_{k_2}(v)}{\theta_k(w+1-k/2)} y^{-k/2}E^*_k(z,w+1-k/2). \end{aligned}$$

Then $\pi_{hol}\left(y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)\right) = \pi_{hol}\left(F(z)\right)$ because $\pi_{hol}\left(y^{-k/2}E^*_k(z,s)\right) = 0$ for every s. We have constructed F so that $F(z) \ll y^{-\epsilon}$ as $y \to \infty$ and we may use [7, Lemma 3.3] to obtain

$$a_{s,w}(l) = \frac{(-1)^{k_2/2}(4\pi l)^{k-1}}{(2\pi^{k/2})(k-2)!} \int_0^\infty F_l(y)e^{-2\pi l y}y^{k-2}\,dy,$$

on writing $F(z) = \sum_{l\in\mathbb{Z}} e^{2\pi i l x}y^{-k/2}F_l(y)$. The functions $F_l(y)$ are sums involving the Fourier coefficients of $E^*_{k_1}(z,u)$ and $E^*_{k_2}(z,v)$ with $u, v \in \mathbb{Z}$. As shown in [7, Theorem 3.1] these coefficients are simply expressed in terms of divisor functions, Bernoulli numbers and a combinatorial part. For s, w in the critical strip, this calculation yields an explicit finite formula for $a_{s,w}(l)$ in [7, Theorem 1.3] (and another proof that $H_{s,w}$ in (8.3) has rational Fourier coefficients and that $K_{critical} = \mathbb{Q}$). For s, w outside the critical strip, we obtain infinite series representations for $a_{s,w}(l)$, but again involving nothing more complicated than divisor functions and Bernoulli numbers. Further details of this computation will appear in [23].

8.3 Twisted Periods

There is an analog of Manin's Periods Theorem for twisted L-functions. Let $p/q \in \mathbb{Q}$ and let u be an integer with $1 \leqslant u \leqslant k-1$. Manin shows in [20, (13)] (see also [18, Chapter 5]) that $i^u \int_0^{p/q} f(iy)y^{u-1}\,dy$ is an integral linear combination of periods $i^v \int_0^\infty f(iy)y^{v-1}\,dy$ for $v = 1, \dots, k-1$. With (2.17) this proves

$$i^u q^{k-2} L^*(f,u;p/q) \quad \in \quad \mathbb{Z}\cdot iL^*(f,1) + \mathbb{Z}\cdot i^2 L^*(f,2) + \cdots + \mathbb{Z}\cdot i^{k-1}L^*(f,k-1).$$

Therefore, Theorem 2.6 implies the next result.

Proposition 8.4. *For all $f \in \mathcal{B}_k$, $p/q \in \mathbb{Q}$ and integers u with $1 \leqslant u \leqslant k-1$,*

$$L^*(f, u; p/q) \in K_f(i)\omega_+(f) + K_f(i)\omega_-(f).$$

Employing (5.8), a similar proof to that of Theorem 2.7 in the last section shows the following.

Proposition 8.5. *For all $f \in \mathcal{B}_k$, $p/q \in \mathbb{Q}$ and $s \in \mathbb{C}$, with $\omega_+(f)$, $\omega_-(f)$ as in Manin's Theorem,*

$$L^*(f, s; p/q)/\omega_+(f) \in K\big(\boldsymbol{E}^*_{k-s,s}(\cdot, 1; p/q)\big) K_f,$$
$$L^*(f, s; p/q)/\omega_-(f) \in K\big(\boldsymbol{E}^*_{k-s,s}(\cdot, 2; p/q)\big) K_f.$$

9 The non-holomorphic case

9.1 Background results and notation

We will need a non-holomorphic analog of the Cohen kernel $\mathcal{C}_k(z, s)$:

Definition 9.1. *With $z \in \mathbb{H}$, s, $s' \in \mathbb{C}$ define the* non-holomorphic kernel *$\mathcal{K}$ as*

$$\mathcal{K}(z; s, s') := \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{\mathrm{Im}(\gamma z)^{s+s'}}{|\gamma z|^{2s}}. \tag{9.1}$$

Following directly from the results in [7, §5.2], it is absolutely convergent, uniformly on compacta, for $z \in \mathbb{H}$ and $\mathrm{Re}(s), \mathrm{Re}(s') > 1/2$. The kernel $\mathcal{K}(z; s, s')$ was introduced by Diaconu and Goldfeld in [6, (2.1)], (though they describe it there as a Poincaré series and their kernel is a product of Γ factors). Starting with the identity ([6, Prop. 3.5])

$$\langle\, f \cdot \mathcal{K}(\cdot; s, s'), g \,\rangle = \frac{\Gamma(s+s'+k-1)}{2^{s+s'+k-1}} \int_{-\infty}^{\infty} \frac{L^*(f, \alpha+i\beta) L^*(g, -s+s'+k-\alpha-i\beta)}{\Gamma(s+\alpha+i\beta)\Gamma(-s+s'+k-\alpha-i\beta)}\, d\beta$$

for f, g in $\mathcal{B}_k$, they provide a new method to establish estimates for the second moment of $L^*(f, s)$ along the critical line $\mathrm{Re}(s) = k/2$. They give similar results for $L^*(u_j, s)$, the L-function associated to a Maass form u_j as defined below.

The spectral decomposition of $\mathcal{K}(z; s, s')$ and its meromorphic continuation in the s, s' variables is shown in [6, §5]. We do the same; our treatment is slightly different and we include it in §9.2 for completeness.

For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the discrete spectrum of the Laplace operator $\Delta = -4y^2 \partial_z \partial_{\overline{z}}$ is given by u_0, the constant eigenfunction, and u_j for $j \in \mathbb{Z}_{\geqslant 1}$ an orthogonal system of Maass cuspforms (see e.g. [10, Chapters 4,7]) with Fourier expansions

$$u_j(z) = \sum_{n \neq 0} |n|^{-1/2} \nu_j(n) W_{s_j}(nz)$$

where u_j has eigenvalue $s_j(1-s_j)$ and by Weyl's law [10, (11.5)]

$$\#\{j : |\mathrm{Im}(s_j)| \leqslant T\} = T^2/12 + O(T \log T). \tag{9.2}$$

We may assume the u_j are Hecke eigenforms normalized to have $\nu_j(1) = 1$. Necessarily we have $\nu_j(n) \in \mathbb{R}$. Let ι be the antiholomorphic involution $(\iota u_j)(z) := u_j(-\overline{z})$. We may also assume each u_j is an eigenfunction of this operator, necessarily with eigenvalues ± 1. If $\iota u_j = u_j$ then $\nu_j(n) = \nu_j(-n)$ and u_j is called *even*. If $\iota u_j = -u_j$ then $\nu_j(n) = -\nu_j(-n)$ and u_j is *odd*.

The L-function associated to the Maass cusp form u_j is $L(u_j, s) = \sum_{n=1}^{\infty} \nu_j(n)/n^s$, convergent for $\mathrm{Re}(s) > 3/2$ since $\nu_j(n) \ll n^{1/2}$ by [10, (8.8)]. The completed L-function for an even form u_j is

$$L^*(u_j, s) := \pi^{-s}\Gamma\left(\frac{s+s_j-1/2}{2}\right)\Gamma\left(\frac{s-s_j+1/2}{2}\right)L(u_j, s) \tag{9.3}$$

and it satisfies

$$L^*(u_j, 1-s) = L^*(u_j, s) = \overline{L^*(u_j, \overline{s})}. \tag{9.4}$$

See [2, p. 107] for (9.3), (9.4) and the analogous odd case.

To $E(z, s)$ (recall (2.3)) we associate the L-function

$$L(E(\cdot, s), w) := \sum_{m=1}^{\infty} \frac{\phi(m, s)}{m^w}$$

The well-known identity $\sum_{m=1}^{\infty} \sigma_x(m)/m^w = \zeta(w)\zeta(w-x)$ implies

$$L(E(\cdot, s), w) = \frac{2\pi^s}{\Gamma(s)} \frac{\zeta(w+s-1/2)\zeta(w-s+1/2)}{\zeta(2s)}. \tag{9.5}$$

9.2 The non-holomorphic kernel $\mathcal{K}$

Throughout this section we use $s = \sigma + it$, $s' = \sigma' + it'$. Recall $\mathcal{K}(z; s, s')$ defined in (9.1) for $\mathrm{Re}(s)$, $\mathrm{Re}(s') > 1/2$. Our goal is to find the spectral decomposition of $\mathcal{K}(z; s, s')$ and prove its meromorphic continuation in s and s'. See [6, §5] and also [10, §7.4] for a similar decomposition and continuation of the automorphic Green function.

A routine verification (using [11, Lemma 9.2] for example) yields

$$\Delta\mathcal{K}(z; s, s') = (s+s')(1-s-s')\mathcal{K}(z; s, s') + 4ss'\mathcal{K}(z; s+1, s'+1). \tag{9.6}$$

Put

$$\xi_{\mathbb{Z}}(z, s) := \sum_{m \in \mathbb{Z}} \frac{1}{|z+m|^{2s}}.$$

Then

$$\mathcal{K}(z; s, s') = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^{s+s'} \xi_{\mathbb{Z}}(\gamma z, s). \tag{9.7}$$

Use the Poisson summation formula as in [10, §3.4] or [9, Th. 3.1.8] to see that

$$\xi_{\mathbb{Z}}(z, s) = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{1/2-s} \sum_{m \neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x} \tag{9.8}$$

for $\mathrm{Re}(s) > 1/2$. Set

$$\xi^{\sharp}_{\mathbb{Z}}(z, s) := \sum_{m \neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}. \tag{9.9}$$

Let $B_\rho := \{z \in \mathbb{C} : |z| \leqslant \rho\}$. Then with [12, Lemma 6.4]

$$\sqrt{y} K_{s-1/2}(2\pi y) \ll e^{-2\pi y}\left(y^{\rho+3} + y^{-\rho-3}\right)$$

for all $s \in B_\rho$ and $\rho, y > 0$ with the implied constant depending only on ρ. Hence

$$\xi^{\sharp}_{\mathbb{Z}}(z, s) \ll \sum_{m=1}^{\infty} e^{-2\pi m y}\left(m^{\rho+\sigma+2} y^{\rho+5/2} + m^{-\rho+\sigma-4} y^{-\rho-7/2}\right).$$

We also have [12, Lemma 6.2]

$$\sum_{m=1}^{\infty} m^\rho e^{-2m\pi y} \ll e^{-2\pi y}\left(1+y^{-\rho-1}\right)$$

for all $y>0$ with the implied constant depending only on $\rho \geqslant 0$. Therefore

$$\xi_{\mathbb{Z}}^{\sharp}(z,s) \ll e^{-2\pi y}\left(y^{\rho+5/2}+y^{-\rho-9/2}\right). \tag{9.10}$$

Consider the weight 0 series

$$\mathcal{K}^{\sharp}(z;s,s') := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^{s'+1/2}\xi_{\mathbb{Z}}^{\sharp}(\gamma z,s). \tag{9.11}$$

With (9.10), we have

$$\mathcal{K}^{\sharp}(z;s,s') \ll \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left(\operatorname{Im}(\gamma z)^{\sigma'+\rho+3}+\operatorname{Im}(\gamma z)^{\sigma'-\rho-4}\right)e^{-2\pi\operatorname{Im}(\gamma z)} \tag{9.12}$$

so that $\mathcal{K}^{\sharp}(z;s,s')$ is absolutely convergent for $\operatorname{Re}(s')>\rho+5$.

Proposition 9.2. *Let $\rho>0$ and $s,s'\in\mathbb{C}$ satisfy $s\in B_\rho$, $\operatorname{Re}(s)>1/2$ and $\operatorname{Re}(s')>\rho+5$. Then*

$$\mathcal{K}(z;s,s') = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)}E(z,s'-s+1)+\frac{2\pi^s}{\Gamma(s)}\mathcal{K}^{\sharp}(z;s,s') \tag{9.13}$$

and, for an implied constant depending only on s,s',

$$\mathcal{K}^{\sharp}(z;s,s') \ll y^{5+\rho-\sigma'} \quad \textit{as} \quad y\to\infty. \tag{9.14}$$

Proof. It is clear that (9.13) follows from (9.7), (9.8), (9.9) and (9.11) when s and s' are in the stated range. With (9.12) and employing (4.3) we deduce that as $y\to\infty$

$$\begin{aligned}\mathcal{K}^{\sharp}(z;s,s') &\ll \left(y^{\sigma'+\rho+3}+y^{\sigma'-\rho-4}\right)e^{-2\pi y}+\sum_{\gamma\in\Gamma_\infty\backslash\Gamma,\gamma\neq\Gamma_\infty}\left(\operatorname{Im}(\gamma z)^{\sigma'+\rho+3}+\operatorname{Im}(\gamma z)^{\sigma'-\rho-4}\right)\\ &\ll e^{-\pi y}+y^{1-(\sigma'+\rho+3)}+y^{1-(\sigma'-\rho-4)}\\ &\ll y^{5+\rho-\sigma'}.\end{aligned}$$

□

Clearly, for $\operatorname{Re}(s')>\rho+5$, (9.13) gives the meromorphic continuation of $\mathcal{K}(z;s,s')$ to all $s\in B_\rho$. For these s,s' it follows from (9.14) that $\mathcal{K}^{\sharp}$, as a function of z, is bounded. Also use (9.6) and (9.13) to show that

$$\Delta\mathcal{K}^{\sharp}(z;s,s') = (s+s')(1-s-s')\mathcal{K}^{\sharp}(z;s,s')+4\pi s'\mathcal{K}^{\sharp}(z;s+1,s'+1)$$

and hence $\Delta\mathcal{K}^{\sharp}$ is also bounded. Therefore, with [10, Theorems 4.7, 7.3], $\mathcal{K}^{\sharp}$ has the spectral decomposition

$$\mathcal{K}^{\sharp}(z;s,s') = \sum_{j=0}^{\infty}\frac{\langle \mathcal{K}^{\sharp}(\cdot;s,s'),u_j\rangle}{\langle u_j,u_j\rangle}u_j(z)+\frac{1}{4\pi i}\int_{(1/2)}\langle \mathcal{K}^{\sharp}(\cdot;s,s'),E(\cdot,r)\rangle E(z,r)\,dr \tag{9.15}$$

where the integral is from $1/2-i\infty$ to $1/2+i\infty$ and the convergence of (9.15) is pointwise absolute in z and uniform on compacta.

Lemma 9.3. *For $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$ we have*

$$\langle \mathcal{K}^\sharp(\cdot; s, s'), u_j \rangle = \frac{\pi^{1/2-s}}{4\Gamma(s')} L^*(u_j, s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)$$

when u_j is an even Maass cuspform. If u_j is odd or constant then the inner product is zero.

Proof. Unfolding,

$$\begin{aligned}
\langle \mathcal{K}^\sharp(\cdot; s, s'), u_j \rangle &= \int_{\Gamma\backslash\mathbb{H}} \mathcal{K}^\sharp(z; s, s')\overline{u_j(z)}\, d\mu(z) \\
&= \int_0^\infty \int_0^1 \left(\sum_{m\neq 0} y^{s'+1/2}|m|^{s-1/2} K_{s-1/2}(2\pi|m|y)e^{2\pi i m x} \right) \overline{u_j(z)}\, \frac{dxdy}{y^2} \\
&= 2\sum_{m\neq 0} \nu_j(m)|m|^{s-1/2} \int_0^\infty y^{s'} K_{s-1/2}(2\pi|m|y) K_{\overline{s_j}-1/2}(2\pi|m|y)\frac{dy}{y}.
\end{aligned}$$

Evaluating the integral [10, p. 205] yields

$$\langle \mathcal{K}^\sharp(\cdot; s, s'), u_j \rangle = \frac{L(u_j, s'-s+1/2)}{4\pi^{s'}\Gamma(s')} \prod \Gamma\left(\frac{s' \pm (s-1/2) \pm (\overline{s_j}-1/2)}{2}\right).$$

Using (9.3) and that $\overline{s_j} = 1 - s_j$ finishes the proof. □

In the same way, when $\mathrm{Re}(r) = 1/2$,

$$\langle \mathcal{K}^\sharp(\cdot; s, s'), E(\cdot, r) \rangle = \frac{L(\overline{E(\cdot, r)}, s'-s+1/2)}{4\pi^{s'}\Gamma(s')} \prod \Gamma\left(\frac{s' \pm (s-1/2) \pm (\overline{r}-1/2)}{2}\right).$$

Further, $\overline{E(z,r)} = E(z, \overline{r}) = E(z, 1-r)$ and with (9.5) we have shown the following.

Lemma 9.4. *For $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$*

$$\begin{aligned}
\langle \mathcal{K}^\sharp(\cdot; s, s'), E(\cdot, r) \rangle &= \frac{\pi^{1/2-s}}{2\Gamma(s')\theta(1-r)} \\
&\quad \times \Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right).
\end{aligned}$$

Recall that $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$ as in §2.1. Let

$$\begin{aligned}
\mathcal{K}_1(z; s, s') &:= \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} E(z, s'-s+1) \\
\mathcal{K}_2(z; s, s') &:= \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)\frac{u_j(z)}{\langle u_j, u_j \rangle} \\
\mathcal{K}_3(z; s, s') &:= \frac{\pi^{1/2}}{\Gamma(s)\Gamma(s')}\frac{1}{4\pi i}\int_{(1/2)} \Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right) \\
&\qquad \times \theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right)\frac{E(z,r)}{\theta(1-r)}\, dr.
\end{aligned}$$

Assembling Proposition 9.2, (9.15) and Lemmas 9.3, 9.4 we have proven the decomposition

$$\mathcal{K}(z; s, s') = \mathcal{K}_1(z; s, s') + \mathcal{K}_2(z; s, s') + \mathcal{K}_3(z; s, s') \tag{9.16}$$

for $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$. This agrees exactly with [6, (5.8)].

Clearly $\mathcal{K}_1(z; s, s')$ is a meromorphic function of s and s' in all of $\mathbb{C}$. The same is true for $\mathcal{K}_2(z; s, s')$ since the factors $L(u_j, s'-s+1/2)\frac{u_j(z)}{\langle u_j, u_j\rangle}$ have at most polynomial growth as $\mathrm{Im}(s_j) \to \infty$ while the Γ factors have exponential decay by Stirling's formula. See (9.2) and [10, §§7,8] for the necessary bounds. The next result was first established in [6, §5].

Theorem 9.5. *The non-holomorphic kernel $\mathcal{K}(z; s, s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$.*

Proof. As we have discussed, $\mathcal{K}_1(z; s, s')$ and $\mathcal{K}_2(z; s, s')$ are meromorphic functions of $s, s' \in \mathbb{C}$. The poles of $\Gamma(w)$ are at $w = 0, -1, -2, \dots$ and $\theta(w)$ has poles exactly at $w = 0, 1/2$ (with residues $-1/2$, $1/2$ respectively). Therefore, the integral in $\mathcal{K}_3(z; s, s')$ is certainly an analytic function of s, s' for $\sigma' > \sigma + 1/2$ and $\sigma > 1/2$ since the Γ and θ factors have exponential decay as $|r| \to \infty$. Next consider s fixed (with $\sigma > 1/2$) and s' varying. Consider a point r_0 with $\mathrm{Re}(r_0) = 1/2$. Let $B(r_0)$ be a small disc centered at r_0 and $B(1 - r_0)$ an identical disc at $1 - r_0$. By deforming the path of integration to a new path C to the left of $B(r_0)$ and to the right of $B(1 - r_0)$, we may, by Cauchy's theorem, analytically continue $\mathcal{K}_3(z; s, s')$ to s' with $s' - s \in B(r_0)$. Let C_1 be a clockwise contour around the left side of $B(r_0)$ and C_2 be a counter-clockwise contour around the right side of $B(1 - r_0)$ so that $C = (1/2) + C_1 + C_2$. For $s' - s$ inside C_1 (and $1 - (s' - s)$ inside C_2) we have

$$\pi^{-1/2}\Gamma(s)\Gamma(s') \cdot \mathcal{K}_3(z; s, s') = \frac{1}{4\pi i}\int_C * = \frac{1}{4\pi i}\int_{(1/2)} * + \frac{1}{4\pi i}\int_{C_1} * + \frac{1}{4\pi i}\int_{C_2} *$$

with $*$ denoting the integrand in the definition of $\mathcal{K}_3$. Then

$$\begin{aligned}
\frac{1}{4\pi i}\int_{C_1} &= \frac{-2\pi i}{4\pi i}\left(\operatorname*{Res}_{r=s'-s} \theta\left(\frac{s'-s+1-r}{2}\right)\right)\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z, s'-s) \\
&= \frac{1}{2}\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z, s'-s) \\
&= \frac{1}{2}\Gamma(s)\Gamma(s'-1/2)E(z, s-s'+1).
\end{aligned}$$

We get the same result for $\frac{1}{4\pi i}\int_{C_2}$ and it follows that for all s' with $\sigma - 1/2 < \mathrm{Re}(s') < \sigma + 1/2$, the continuation of $\mathcal{K}_3(z; s, s')$ is given by

$$\pi^{-1/2}\Gamma(s)\Gamma(s') \cdot \mathcal{K}_3(z; s, s') = \Gamma(s)\Gamma(s'-1/2)E(z, s-s'+1) + \frac{1}{4\pi i}\int_{(1/2)} *. \tag{9.17}$$

Similarly, as s' crosses the line with real part $\sigma - 1/2$, the term $-\Gamma(s-1/2)\Gamma(s')E(z, s'-s+1)$ must be added to the right side of (9.17). Thus, for all s' with $1/2 < \mathrm{Re}(s') < \sigma - 1/2$, the continuation of $\mathcal{K}(z; s, s')$ is

$$\mathcal{K}(z; s, s') = \frac{\pi^{1/2}\Gamma(s'-1/2)}{\Gamma(s')}E(z, s-s'+1) + \mathcal{K}_2(z; s, s') + \mathcal{K}_3(z; s, s'). \tag{9.18}$$

Clearly, with (9.17), (9.18) we have demonstrated the meromorphic continuation of $\mathcal{K}(z; s, s')$ to all $s, s' \in \mathbb{C}$ with $\mathrm{Re}(s), \mathrm{Re}(s') > 1/2$. The continuation to all $s, s' \in \mathbb{C}$ follows in the same way with further terms in the expression for $\mathcal{K}(z; s, s')$ appearing from the residues of the poles of $\Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)$ as $\mathrm{Re}(s'+s) \to -\infty$. □

Proposition 9.6. *We have the functional equation*

$$\mathcal{K}(z; s, s') = \mathcal{K}(z; s', s). \tag{9.19}$$

Proof. We may verify (9.19) by comparing (9.16) with (9.18) and using that $\mathcal{K}_2(z;s,s') = \mathcal{K}_2(z;s',s)$ by (9.4), and $\mathcal{K}_3(z;s,s') = \mathcal{K}_3(z;s',s)$ by (2.6). There is a second, easier proof: with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, replace γ in (9.1) by $S\gamma$. $\square$

Proposition 9.7. *For all $s, s' \in \mathbb{C}$ and any even Maass Hecke eigenform u_j,*

$$\langle\, \mathcal{K}(\cdot;s,s'), u_j \,\rangle = \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')}\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)\cdot L^*(u_j, s'-s+1/2).$$

Proof. Since each u_j is orthogonal to Eisenstein series we have by (9.16) (for $s \in B_\rho$ and $\mathrm{Re}(s') > \rho+5$) that

$$\langle\, \mathcal{K}(\cdot;s,s'), u_j \,\rangle = \langle\, \mathcal{K}_2(\cdot;s,s'), u_j \,\rangle.$$

The result follows, extending to all $s, s' \in \mathbb{C}$ by analytic continuation. $\square$

9.3 Non-holomorphic double Eisenstein series

A similar argument to the proof of (5.2) shows that, for $\mathrm{Re}(s), \mathrm{Re}(s') > 1$ and $\mathrm{Re}(w) \geqslant 0$,

$$\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z,w;s,s') = \frac{1}{2}\sum_{n=1}^{\infty}\frac{T_n\mathcal{K}(z;s,s')}{n^{w-1/2}} \tag{9.20}$$

where, in this context [9, (3.12.3)], the appropriately normalized Hecke operator acts as

$$T_n\mathcal{K}(z) = \frac{1}{n^{1/2}}\sum_{\gamma\in\Gamma\backslash\mathcal{M}_n}\mathcal{K}(\gamma z).$$

For each Maass form we have $T_n u_j = \nu_j(n)u_j$ and for the Eisenstein series [9, Prop. 3.14.2] implies $T_nE(z,s) = n^{s-1/2}\sigma_{1-2s}(n)E(z,s)$. Therefore, as in (9.5),

$$\sum_{n=1}^{\infty}\frac{T_nE(z,s)}{n^{w-1/2}} = E(z,s)\sum_{n=1}^{\infty}\frac{\sigma_{1-2s}(n)}{n^{w-s}} = E(z,s)\zeta(w-s)\zeta(w+s-1).$$

Now choose any $\rho > 0$. For $s \in B_\rho$, $\mathrm{Re}(s) > 1$, $\mathrm{Re}(s') > \rho+5$ and $\mathrm{Re}(w) \geqslant 0$ we may apply T_n to both sides of (9.16) and obtain

$$\begin{aligned}\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z,w;s,s') &= \frac{\pi^{1/2}\Gamma(s-1/2)}{2\Gamma(s)}\zeta(s'-s+w)\zeta(s-s'+w-1)E(z,s'-s+1)\\ &+\frac{\pi^{1/2}}{4\Gamma(s)\Gamma(s')}\sum_{\substack{j=1\\ u_j \text{ even}}}^{\infty} L^*(u_j,s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)L(u_j,w-1/2)\frac{u_j(z)}{\langle\, u_j,u_j\,\rangle}\\ &+\frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')}\frac{1}{4\pi i}\int_{(1/2)}\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right)\Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)\\ &\qquad\times\zeta(w-r)\zeta(w-1+r)\frac{E(z,r)}{\theta(1-r)}\,dr.\end{aligned} \tag{9.21}$$

Put

$$\Omega(s,s';r) := \theta\left(\frac{s'+s-r}{2}\right)\theta\left(\frac{s'+s-1+r}{2}\right)\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right)\Big/\theta(1-r).$$

Define the completed double Eisenstein series as in (2.19) and write

$$U(z;s,s') := \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s+s'-1/2) L^*(u_j, s'-s+1/2) \frac{u_j(z)}{\langle u_j, u_j \rangle}.$$

As in the last section, Ω and U have exponential decay as $|r|$ and $|\mathrm{Im}(s_j)| \to \infty$. Specializing (9.21) to $w = s + s'$, we have proved the next result.

Lemma 9.8. *For $s \in B_\rho$, $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s') > \rho + 5$*

$$\mathcal{E}^*(z;s,s') = 2\theta(s)\theta(s')E(z;s+s') + 2\theta(1-s)\theta(s')E(z,s'-s+1) \\ + U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)} \Omega(s,s';r)E(z,r)\,dr. \quad (9.22)$$

From this we show the following.

Theorem 9.9. *The completed double Eisenstein series $\mathcal{E}^*(z;s,s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$ and we have the functional equations*

$$\mathcal{E}^*(z;s,s') = \mathcal{E}^*(z;s',s), \quad (9.23)$$
$$\mathcal{E}^*(z;s,s') = \mathcal{E}^*(z;1-s,1-s'). \quad (9.24)$$

Proof. First note that (9.22) gives the meromorphic continuation of $\mathcal{E}^*(z;s,s')$ to all s, s' with $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$. As in the proof of Theorem 9.5, we see that the further continuation in s' is given by (9.22) along with residues that are picked up as the line of integration is crossed: for $s \in B_\rho$ fixed and $\mathrm{Re}(s') \to -\infty$ the continuation of $\mathcal{E}^*(z;s,s')$ is given by (9.22) plus each of the following

$$\begin{aligned} 2\theta(s)\theta(1-s')E(z,s-s'+1) &\quad \text{when} \quad \mathrm{Re}(s') < \sigma + 1/2, \\ -2\theta(1-s)\theta(s')E(z,s'-s+1) &\quad \text{when} \quad \mathrm{Re}(s') < \sigma - 1/2, \\ 2\theta(1-s)\theta(1-s')E(z,2-s-s') &\quad \text{when} \quad \mathrm{Re}(s') < -\sigma + 1/2, \\ -2\theta(s)\theta(s')E(z,s+s') &\quad \text{when} \quad \mathrm{Re}(s') < -\sigma - 1/2. \end{aligned}$$

We have therefore shown the meromorphic continuation of $\mathcal{E}^*(z;s,s')$ to all $s \in B_\rho$ and $s' \in \mathbb{C}$. Hence, for all s' with $\mathrm{Re}(s') < -\rho - 4$, say, we have

$$\mathcal{E}^*(z;s,s') = 2\theta(1-s)\theta(1-s')E(z,2-s-s') + 2\theta(s)\theta(1-s')E(z,s-s'+1) \\ + U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)} \Omega(s,s';r)E(z,r)\,dr. \quad (9.25)$$

The functional equation (9.24) is a consequence of the easily checked symmetries $U(z;1-s,1-s') = U(z;s,s')$, $\Omega(1-s,1-s';r) = \Omega(s,s';r)$ and a comparison of (9.22) and (9.25). The equation (9.23) has a similar proof, or more simply follows from the definition (2.19). □

Proposition 9.10. *For any even Maass Hecke eigenform u_j (as in §9.1) and all $s, s' \in \mathbb{C}$*

$$\langle \mathcal{E}^*(\cdot;s,s'), u_j \rangle = L^*(u_j, s+s'-1/2) L^*(u_j, s'-s+1/2).$$

Proof. As in Proposition 9.7, only $U(z;s,s')$ in (9.22) will contribute to the inner product. □

With Theorem 9.9 and Proposition 9.10, we have completed the proof of Theorem 2.9.

10 Double Eisenstein series for general groups

We proved in §5.1 that for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ the holomorphic double Eisenstein series $\boldsymbol{E}_{s,k-s}(z,w)$ may be continued to all s, w in $\mathbb{C}$ and satisfies a family of functional equations. That proof does not extend to groups where Hecke operators are not available. To show the continuation of $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ for Γ an arbitrary Fuchsian group of the first kind we first demonstrate a generalization of Proposition 2.5. Recall the definitions of u, v in (2.16) and ε_Γ in (4.1).

Theorem 10.1. *For s, w in the initial domain of convergence and even $k_1, k_2 \geqslant 0$ with $k = k_1 + k_2$ we have*

$$\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w) = 2^{\varepsilon_\Gamma - 1}\pi_{hol}\left[(-1)^{k_2/2} y^{-k/2} E^*_{k_1,\mathfrak{a}}(\cdot, 1-u) E^*_{k_2,\mathfrak{a}}(\cdot, 1-v)/(2\pi^{k/2})\right]. \tag{10.1}$$

Proof. Let $g \in S_k(\Gamma)$ and set $\Gamma' = \sigma_\mathfrak{a}{}^{-1}\Gamma\sigma_\mathfrak{a}$. Then

$$\begin{aligned}\langle \boldsymbol{E}_{s,k-s,\mathfrak{a}}(\cdot,w), g\rangle &= \int_{\Gamma'\backslash\mathbb{H}} \mathrm{Im}(\sigma_\mathfrak{a} z)^k \overline{g}(\sigma_\mathfrak{a} z)\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a} z, w)\, d\mu z \\ &= \int_{\Gamma'\backslash\mathbb{H}} y^k \frac{\overline{g}(\sigma_\mathfrak{a} z)}{\overline{j}(\sigma_\mathfrak{a}, z)^k} \sum_{\delta \in B\backslash\Gamma'} j(\delta, z)^{-k}\left[\sum_{\substack{\gamma\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{w-1}\left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s}\right] d\mu z. \end{aligned} \tag{10.2}$$

Since $g(\sigma_\mathfrak{a} z) j(\sigma_\mathfrak{a}, z)^{-k} \in S_k(\Gamma')$ we have

$$y^k \frac{\overline{g}(\sigma_\mathfrak{a} z)}{\overline{j}(\sigma_\mathfrak{a}, z)^k j(\delta, z)^k} = \mathrm{Im}(\delta z)^k \frac{\overline{g}(\sigma_\mathfrak{a}\delta z)}{\overline{j}(\sigma_\mathfrak{a}, \delta z)^k}.$$

Note also that $j(\gamma, z)/j(\delta, z) = j(\gamma\delta^{-1}, \delta z)$. Hence (10.2) equals

$$2^{\varepsilon_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}} y^k \frac{\overline{g}(\sigma_\mathfrak{a} z)}{\overline{j}(\sigma_\mathfrak{a}, z)^k}\left[\sum_{\substack{\gamma\in B\backslash\Gamma' \\ c_\gamma>0}} (c_\gamma)^{w-1} j(\gamma,z)^{-s}\right] d\mu z. \tag{10.3}$$

Writing

$$\sum_{\substack{\gamma\in B\backslash\Gamma' \\ c_\gamma>0}} (c_\gamma)^{w-1} j(\gamma,z)^{-s} = \sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} (c_\gamma)^{w-1} \sum_{m\in\mathbb{Z}} j(\gamma, z+m)^{-s}$$

and using the Fourier expansion of g at $\mathfrak{a}$: $j(\sigma_\mathfrak{a}, z)^{-k} g(\sigma_\mathfrak{a} z) = \sum_{n=1}^\infty a_{g,\mathfrak{a}}(n) e^{2\pi i n z}$, we get that

$$\begin{aligned}(10.3) \quad &= \quad 2^{\varepsilon_\Gamma}\sum_{n=1}^\infty \overline{a_{g,\mathfrak{a}}}(n)\sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{1}{(c_\gamma)^{s+1-w}}\int_0^\infty\int_{-\infty}^\infty y^{k-2}\frac{e^{-2\pi i n x - 2\pi n y}}{(x + d_\gamma/c_\gamma + iy)^s}\, dx dy \\ &= \quad 2^{\varepsilon_\Gamma} I_k(s)\sum_{n=1}^\infty \frac{\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s}}\sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{e^{2\pi i n d_\gamma/c_\gamma}}{(c_\gamma)^{s+1-w}}\end{aligned}$$

for

$$I_k(s) := \int_0^\infty\int_{-\infty}^\infty y^{k-2}\frac{e^{-2\pi i x - 2\pi y}}{(x+iy)^s}\, dx dy.$$

The inner integral over x may be evaluated with a formula of Laplace [27, p. 246]:

$$\int_{-\infty}^\infty \frac{e^{-2\pi i x}}{(x+iy)^s}\, dx = e^{-2\pi y}\frac{(2\pi)^s}{\Gamma(s) e^{s i\pi/2}}$$

so that

$$I_k(s) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{(2\pi)^s}{\Gamma(s)e^{si\pi/2}}.$$

With (4.2) and, for example [10, Chap. 3], we recognize

$$\sum_{\substack{\gamma \in B\backslash\Gamma'/B \\ c_\gamma > 0}} \frac{e^{2\pi i n d_\gamma / c_\gamma}}{(c_\gamma)^{2s}} = \sum_{\substack{\gamma \in \Gamma_\infty\backslash\Gamma'/\Gamma_\infty \\ c_\gamma > 0}} \frac{e^{2\pi i n d_\gamma / c_\gamma}}{(c_\gamma)^{2s}} = \frac{Y_{\mathfrak{a}\mathfrak{a}}(n,s)}{\zeta(2s) n^{s-1}}.$$

It follows that we have shown

$$\langle\, \boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(\cdot, w), g \,\rangle = 2^{\varepsilon_\Gamma - 1} \frac{\zeta(2-2u)\Gamma(k-s)\Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^\infty \frac{Y_{\mathfrak{a}\mathfrak{a}}(n, 1-v)\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}.$$

Reasoning as in the proof of [7, (2.10)] we also find, for all even k_1, $k_2 \geqslant 0$ with $k_1 + k_2 = k$,

$$\begin{aligned} &\langle\, (-1)^{k_2/2} y^{-k/2} E^*_{k_1,\mathfrak{a}}(\cdot, 1-u) E^*_{k_2,\mathfrak{b}}(\cdot, 1-v)/(2\pi^{k/2}), g \,\rangle \\ &\qquad = \frac{\zeta(2-2u)\Gamma(k-s)\Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^\infty \frac{Y_{\mathfrak{b}\mathfrak{a}}(n, 1-v)\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}. \end{aligned}$$

Since $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w) \in S_k(\Gamma)$ and $g \in S_k(\Gamma)$ is arbitrary, (10.1) follows. □

Corollary 10.2. *The double Eisenstein series $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w)$ has a meromorphic continuation to all $s, w \in \mathbb{C}$ and as a function of z is always in $S_k(\Gamma)$. It satisfies the functional equation*

$$\boldsymbol{E}^*_{k-s,s,\mathfrak{a}}(z,w) = (-1)^{k/2} \boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w). \tag{10.4}$$

Proof. Since $E^*_{k,\mathfrak{a}}(z,s)$ has a well-known continuation to all $s \in \mathbb{C}$, due to Selberg, the continuation of $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w)$ follows from (10.1). The change of variables $(s,w) \to (k-s, w)$ corresponds to $(u,v) \to (v,u)$ and so (10.4) is also a consequence of (10.1). □

If Γ has more than one cusp then $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w)$ does not appear to possess a functional equation of the type (2.14) as $(s,w) \to (w,s)$. This corresponds on the right of (10.1) to $(u,v) \to (u, 1-v)$ and the functional equation for $E^*_{k_2,\mathfrak{a}}(\cdot, 1-v)$ involves a sum over cusps as in (4.4).

We remark that the functional equation (10.4) also follows directly from (4.6) if $-I \in \Gamma$: replace γ and δ in the sum by $-\delta$ and γ respectively.

Finally, it would be interesting to find the continuation in s, s' of the non-holomorphic double Eisenstein series $\mathcal{E}^*_{\mathfrak{a}}(z; s, s')$ for general groups. We expect that a similar decomposition to (9.22) should be true.

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