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Citation for the original published paper (version of record):
Berbyuk, V. (2004). Numerical Method for Optimization of Semi-Passively Controlled Dynamical Systems. The 1st International Conference ”From Scientific Computing to Computational Engineering”, 2: 866-873

(article starts on next page)

1st International Conference "From Scientific Computing to Computational Engineering"
1st IC-SCCE
Athens, 8-10 September, 2004

NUMERICAL METHOD FOR OPTIMIZATION OF SEMI-PASSIVELY CONTROLLED DYNAMICAL SYSTEMS

Viktor Berbyuk

Department of Machine and Vehicle Systems
Chalmers University of Technology
SE-412 96 Gothenburg, Sweden
e-mail: viktor.berbyuk@me.chalmers.se, web page: http://www.mvs.chalmers.se

Keywords: Semi-Passively Controlled Dynamical System, Underactuation, Overactuation, Optimization, Industrial Robot, Human Locomotor System, Bipedal Locomotion Robot.

Abstract. *Controlled dynamical systems with different type of actuators (e.g. external powered electromotors, magnetostrictive actuators, internal unpowered (passive) spring-damper-like drives, etc.) are considered. These systems are termed semi-passively controlled. Mathematical statement of optimization problem has proposed that is suitable both for modeling of optimal motion and for optimization of structure of semi-passively controlled dynamical systems with different degree of actuation. Numerical method for solving the proposed optimization problem is described. The method was successfully used for solving optimal control problems for several semi-passively controlled dynamical systems (industrial robots, human locomotor system with intelligent lower limb prosthesis, bipedal locomotion robots, others). The results obtained have confirmed the efficiency of the proposed numerical method for solving optimization problems for semi-passively controlled dynamical systems. Analysis of the results gives insight into the study of the role of inherent dynamics in controlled motion and how much a dynamical system should be governed by external drives and how much by a system's inherent dynamics. In particular, it has been shown that complex goal-directed and cost-efficient controlled motion of underactuated dynamical system can be design using optimal interaction between external powered drives and internal unpowered spring-damper-like drives. This constitutes the powerful ability of semi-passively controlled dynamical systems.*

1 INTRODUCTION

To perform cost efficient goal-directed controlled motions many machines and mechanisms can comprise not only different external powered drives, e.g. traditional electromotors, magnetostrictive material based actuators, hydraulic motor and/or pneumatic motors, but also some other force/torque generators like springs, dampers, etc. These generators do not use external power. The forces and/or torques that are exerted by these generators depend only on their inherent dynamics and phase state of the dynamical system modeled the considered mechanism. For the above reasons these generators can be called unpowered (passive) drives. A dynamical system comprises both powered and unpowered drives is termed as semi-passively controlled one. Ground vehicle suspension with passive shock absorber and powered actuator is a classical example of a semi-passively controlled dynamical system.

Efficiency of the control of motion of a dynamical system depends not only on which kinds of drives are used, but also how much the system is actuated. If the number of powered drives is less than the number of degrees of freedom the dynamical system is called underactuated[1]. Opposite, if the number of powered drives exceeds the number of degrees of freedom then the system is called overactuated. The term full actuated dynamical system refers to the system which has the same numbers of degrees of freedom and powered drives.

Usually the analysis of dynamics and design of control algorithms for underactuated systems are much more complicate than for full actuated or overactuated systems[1]. Difficulties can arise due to specific features of underactuated dynamical systems, e.g. luck of their controllability.

In this paper the semi-passively controlled dynamical systems with different degree of actuation are under the study. Mathematical statement of the problem is proposed that is suitable both for modeling of optimal motion and for optimization of structure of semi-passively controlled dynamical systems. Numerical method is described and successfully used to solve energy-optimal control problems for several semi-passively controlled biomechanical systems. Results of numerical analysis and computer simulation of obtained energy-optimal controlled processes of considered systems are presented.

2 METHEMATICAL MODEL AND STATEMENT OF THE PROBLEM

Consider a dynamical system with *n* degrees of freedom. Let the controlled motion of a dynamical system is described by the following equations

$$\mathbf{A(q)\ddot{q} + B(q,\dot{q}) = C(q)u + D(q)p} \tag{1}$$

$$\mathbf{H(q,\dot{q},p,\xi) = 0} \tag{2}$$

Here $\mathbf{q} = [q_1,\ldots,q_n]^T$ be a vector of independent generalized coordinates, $\mathbf{u} = [u_1,\ldots,u_m]^T$ be a vector of controlling stimuli exerted by powered drives, $\mathbf{p} = [p_1,\ldots,p_k]^T$ be a vector of control stimuli exerted by unpowered (passive) drives, **A, B, C, D** are given matrices that describe the inherent dynamics of the considered semi-passively controlled dynamical system, **H** and $\boldsymbol{\xi} = [\xi_1,\ldots,\xi_l]^T$ are matrix and vector of structural parameters that describe the inherent dynamics of unpowered (passive) drives of the system.

The considered system (1)-(2) is underactuated if *m<n;* for *m>n* – overactuated, and in case of *m=n* the considered semi-passively controlled dynamical system is full actuated.

Let assume that the motion of the system is subject to multipoint boundary conditions that can be written as follows

$$\mathbf{q}(t_{j-1}) = \mathbf{q}_{\mathbf{j}-1}, \qquad \dot{\mathbf{q}}(t_{j-1}) = \dot{\mathbf{q}}_{\mathbf{j}-1}, \qquad j = 1,2,\cdots,N \tag{3}$$

Here the vectors $\mathbf{q}_{\mathbf{j}-1}, \dot{\mathbf{q}}_{\mathbf{j}-1}$ determine the phase states of the considered dynamical system at the instants of times t_{j-1}, $j = 1,2,\cdots,N$. All or some of the components of the vectors $\mathbf{q}_{\mathbf{j}-1}, \dot{\mathbf{q}}_{\mathbf{j}-1}$ and the times t_{j-1} can be specified in advanced depend on the requirements imposed on the motion of the system.

Usually, the control stimuli exerted by drives and their structural parameters are not arbitrary and must satisfy some restrictions. In general case these restrictions can be written as

$$\mathbf{u}(t) \in \Omega_u, \qquad t \in [t_0, T] \tag{4}$$

$$\mathbf{p}(t) \in \Omega_p, \qquad t \in [t_0, T] \tag{5}$$

$$\boldsymbol{\xi}(t) \in \Omega_\xi \tag{6}$$

In formulae (4)-(6) Ω_u is given domain in control space $\mathbb{R}_u^m$ of powered drives, Ω_p is given domain in control space $\mathbb{R}_p^k$ of unpowered drives, Ω_ξ is given domain in space $\mathbb{R}_\xi^l$ of the values of the vector of structural parameters of the unpowered drives, t_0, T are initial and final instants of time of the controlled motion of the dynamical system.

In many practical cases the controlled motion of a dynamical system can be restricted not only at the particular instants of time (see boundary conditions (3)) but also during some interval of time $t \in [\tau_1, \tau_2] \subseteq [t_0, T]$. It will lead to the equality and/or inequality constraints that are imposed on the phase trajectories of the system. These restrictions on the phase trajectories can be written as

$$[\mathbf{q}^{\mathbf{T}}(t), \dot{\mathbf{q}}^{\mathbf{T}}(t)] \in \Omega_{q\dot{q}}, \quad t \in [\tau_1, \tau_2] \subseteq [t_0, T] \tag{7}$$

where $\Omega_{q\dot{q}}$ is given domain in phase space $\mathbb{R}_{q\dot{q}}^{2n}$ of a dynamical system.

The equations (1), (2), the boundary conditions (3), the restrictions on control stimuli (4)-(6), and the constraints on the phase trajectories (7) compose the mathematical model of a semi-passively controlled dynamical system. The examples of the described general mathematical model (1)-(7) that were used for modeling and design of optimal processes for different semi-passively controlled robotic and biomechanical

systems can be found in papers[2] – [7].

Assume that there exists a non-empty set of vector-functions $\{\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), t \in [t_0, T]\}$ and a vector of structural parameters $\boldsymbol{\xi}$ which satisfy the equations (1), (2) and the restrictions (3)-(7). The following optimal control problem can be formulated.

Problem A. Given a dynamical system the controlled motion of which is described by equations (1), (2). It is required to determine the vector-function $\mathbf{p}_*(t)$, the motion of the system $\mathbf{q}_*(t)$, the external controlling stimuli $\mathbf{u}_*(t, \mathbf{q}_*, \mathbf{p}_*)$, and the vector $\boldsymbol{\xi}_*$ which alltogether satisfy the equations (1), (2), the restrictions (3)-(7), and which minimize the given objective functional $\Phi[\mathbf{u}]$.

As a result of the solution of Problem A the optimal control process of a dynamical system having both powered and unpowered drives will be designed. Both internal $\mathbf{p}_*(t)$ and external controlling stimuli $\mathbf{u}_*(t, \mathbf{q}_*, \mathbf{p}_*)$ for the system are found which minimize the given objective functional.

One of the primary goals for the incorporation of unpowered drives into the structure of the dynamical systems is an improvement of their control processes. It means that the validity of the following inequality is expected $\Phi[\mathbf{u}_*(t, \mathbf{q}_*, \mathbf{p}_*)] < \Phi[\mathbf{u}_0(t, \mathbf{q}_0)]$, where $\mathbf{q_0}(t), \mathbf{u_0}(t)$ are the optimal motion and the respective controlling stimuli obtained under the restrictions (3), (4), (7) for the dynamical system without the unpowered drives. In this sense the solution of Problem A could help to estimate the limiting possibility of improvement of the external control strategies for the dynamical systems due to incorporation into their structure different unpowered drives determined by the equation (2) and restrictions (5), (6).

3 METHODOLOGY

We have formulated the optimal control problem for a semi-passively controlled dynamical system. The key feature of the proposed mathematical statement of the problem is direct utilization of the equations describing the inherent dynamics of internal unpowered drives together with all other constraints that are imposed on the state vector and the controlling stimuli of a system. It leads to the non-uniqueness of the solution of the direct and the inverse dynamics problems as well as makes it possible to design optimal unpowered drives for a dynamical system.

In general case for the dynamical systems with many degrees-of-freedom powerful numerical algorithms are needed to solve Problem A. Futhermore, during the calculation of optimal control for a dynamical system it is necessary to design at the same time the optimal structure of the unpowered drives taking into account the equation (2) and the restrictions (5), (6). This can significantly increase the complexity of the computation.

The numerical method has been developed for the solution of Problem A for the dynamical systems, which model semi-passively controlled manipulator robots and bipedal locomotion systems with unpowered drives at their joints [2], [6], [7]. The method is based on a special procedure to convert the initial optimal control problem (Problem A) into a standard nonlinear programming problem:

$$F(\mathbf{C}) \Rightarrow \mathbf{min}_{\mathbf{C}}, \qquad \mathbf{g}(\mathbf{C}) \le \mathbf{0} \tag{8}$$

where **C** is a vector of varying parameters. This is made by an approximation of the independently varying functions by a combination of the fifth order polynomial and Fourier series, i.e. by using the following expression:

$$\begin{aligned} x(t) &= \sum_{i=0}^{5} C_{xi}^{j}(t - t_{j-1})^i + \sum_{k=1}^{N_{xj}} [a_{xk}^{j} \cos(k\omega_j(t - t_{j-1})) + b_{xk}^{j} \sin(k\omega_j(t - t_{j-1}))] \\ t &\in [t_{j-1}, t_j], \qquad j = 1, 2, ..., N, \qquad t_N = T \end{aligned} \tag{9}$$

Here $x(t)$ is an independent varying function, $\omega_j = 2\pi/(t_j - t_{j-1})$, and N_{xj} is given positive integer. The list of independent varying functions can include generalized coordinates $q_i(t)$ and/or some of the components of the vector of controlling stimuli of powered drives $\mathbf{u}(t)$.

Considering the boundary conditions (3) and the restriction (4), the vector of independently varying parameters **C** can be determined. For instance, in case of two-point boundary conditions imposed on the state

vector $[\mathbf{q}(t), \dot{\mathbf{q}}(t)]^T$ of the system, from the formulae (3) and (9) with $N=2$ follows that the parameters $C_{x4}^j, C_{x5}^j, a_{xk}^j, b_{xk}^j, \quad j=1,2, \quad k=1,...,N_{xj}$ can serve as independently varying parameters.

To solve the nonlinear programming problem (8) different algorithms have been used that are based on the Rozenbrock's method[8], and the sequential quadratic programming method[9] implemented in the software package TOMLAB[10].

The key features of the proposed method for solving Problem A is its high numerical efficiency and the possibility to satisfy a lot of restrictions imposed on the phase coordinates of the system automatically and accurately. The efficiency of the developed method has been illustrated by solution of the energy-optimal control problems for several semi-passively controlled robotic and biomechanical systems[2], [6], [7].

4 OPTIMIZATION OF CONTROLLED MOTION OF A BIPEDAL LOCOMOTION SYSTEM

Here the application of methodology of optimization of semi-passively controlled dynamical systems described in paragraph 3 is demonstrated by solving the design problem of lower limb prostheses and by optimization of controlled motion of bipedal locomotion system models amputee with above-knee prosthesis.

There is an important difference between the dynamics of an intact limb and a prosthetic limb of an amputee. Here the mathematical modelling of a human gait of an amputee with above-knee prosthesis is considered based on an assumption that the force moments at the knee and at the ankle joints of the prosthetic leg are passive ones. The values of these moments depend not only on the gait pattern, but also on the prosthesis construction. The considered model of the amputee system belongs to the class of underactuated semi-passively controlled dynamical systems.

The sketch of the bipedal locomotion system that models the amputee with above-knee prosthesis is depicted in Figure 1. The system is modeled as the mechanical system of seven rigid bodies connected by ideal cylindrical hinges. The bodies HG, HK_i, and K_iA_i (i=1,2), which model the torso, thighs, and shins respectively, are assumed to have weight and inertia, and the bodies $A_iT_iH_i$ (the feet) are weightless and inertialess.

In addition to the weights of the trunk, thighs and the shanks, the external forces acting on the system include the interaction forces between the feet and the ground, which are replaced by the resultant forces. It is also assumed that the control torques $q_i(t)$, $u_i(t)$, $p_i(t)$ acting at the hip (point H), knee (point K_i) and the ankle (point A_i) joints, respectively.

As generalized coordinates that jointly determined the position of the given mechanical system we chose the following: x and y, the Cartesian coordinates of the point of attachment of the legs (the point H); $\psi, \alpha_i, \beta_i, \gamma_i$ the angles of deviation of the link HG, HK_i, K_iA_i, and $A_iT_iH_i$, (i=1,2) from vertical (Figure 1).

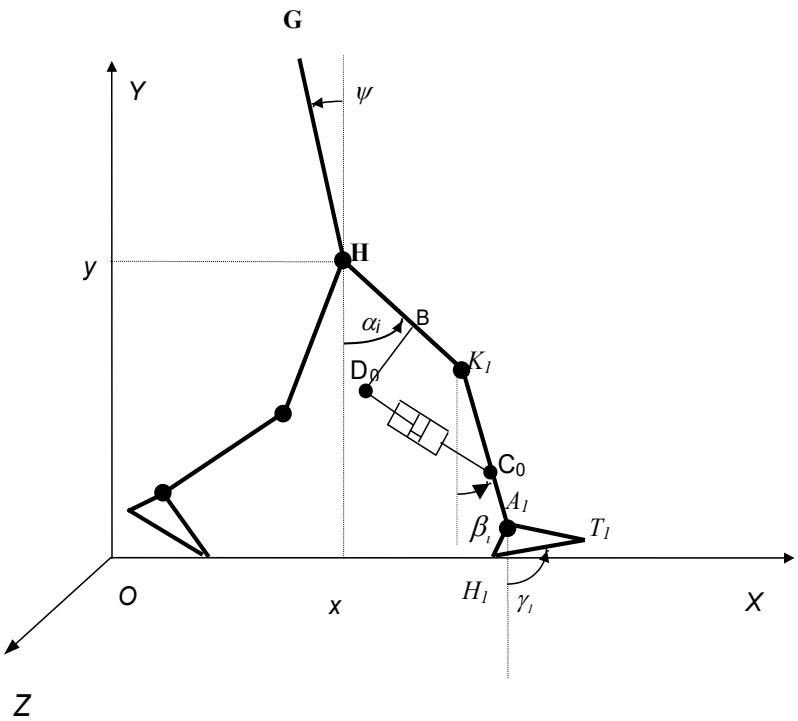

Figure 1. Sketch of an amputee with above-knee prosthesis

The above-knee prosthesis comprises the linear-viscoelastic ankle mechanism and the hydraulic or the pneumatic knee mechanism that are assumed to be weightless and inertialess.
During locomotion of the amputee with above-knee prosthesis the unpowered (passive) control torques $p_1(t)$, $u_1(t)$ are generated at the ankle and at the knee joints of the prosthetic leg, respectively. These torques are determined as follows[6]

$$\begin{aligned} p_1(t) &= C(\beta_1 - \gamma_1 + \pi/2) + K(\dot{\beta}_1 - \dot{\gamma}_1) + D \\ u_1(t) &= (P_2 - P_1) S_p d_2 (d_1^2 + l_0^2)^{1/2} \sin(\alpha_1 - \beta_1 + \eta)/l_1 \end{aligned} \tag{10}$$

Here C, K are the torsion spring and the damping coefficients of the ankle mechanism; D is determined by the free angle of the spring and torsion spring coefficients; P_1, P_2 are the chamber pressures of the hydraulic or the pneumatic actuator that can be calculated by using the equations of dynamics of the knee mechanism of the prosthesis [6], S_p is the cylinder piston cross-area, and

$$\begin{aligned} l_1 &= (d_1^2 + d_2^2 + l_0^2 + 2d_2(d_1^2 + l_0^2)^{1/2} \cos(\alpha_1 - \beta_1 + \eta))^{1/2} \\ \eta &= a \tan(l_0/d_1), \quad d_1 = |BK_1|, \quad d_2 = |K_1C_0|, \quad l_0 = |BD_0| \end{aligned} \tag{11}$$

The detailed description of the considered model of amputee with above-knee prosthesis can be found in [2], [6].

The design problem of the above-knee prosthesis can be formulated in the same way as Problem A. It should be taken into account that the considered semi-passively controlled dynamical system has the state vector $[x, \dot{x}, y, \dot{y}, \psi, \alpha_i, \dot{\alpha}_i, \beta_i, \dot{\beta}_i, \gamma_i, \dot{\gamma}_i, i = 1,2]^T$, the vector of controlling stimuli of the powered drives $\mathbf{u} = [q_1, q_2, u_2, p_2]^T$, and the vector of the constructive parameters of the unpowered drives $\mathbf{C}_p = [C, K, D, d_1, d_2, l_0, S_p, S_0]^T$.

The controlled motion of amputee with above-knee prosthesis is described by Lagrange equations [2], [6], and by the expressions (10), (11). The boundary conditions and other constraints on the phase coordinates of the system have been given based on known experimental data on the human gait [11].

The following functional

$$E = \frac{1}{2L} \int_0^T \{ \sum_{i=1}^{2} |q_i(\dot{\psi} - \dot{\alpha}_i)| + |u_2(\dot{\alpha}_2 - \dot{\beta}_2)| + |p_2(\dot{\beta}_2 - \dot{\gamma}_2)| \} dt \tag{12}$$

is used for solving Problem A. The objective functional (12) estimates the energy expenditure per unit of distance traveling of bipedal locomotion systems [12], [13]. The same approach as described in paragraph 3 has been used for solving the problem of design energy-optimal above-knee prostheses. Due to the dynamic constraints (10) the procedure of converting the Problem A into the standard nonlinear programming problem (8) includes the solution of the semi-inverse dynamics problem for the controlled mechanical system that models amputee with above-knee prosthesis. It sufficiently increases the time consumption of the numerical algorithm for designing the energy-optimal above-knee prosthesis.

We now present the individual results of mathematical modeling of motion of amputee with energy-optimal above-knee prosthesis obtained in the context of the proposed formulation of the optimal control problem (Problem A) and the methodology for solving it numerically.

The computation was carried out for a model of amputee of height 1.76 m and mass 73.2 kg. The respective values of linear and mass-inertia characteristics of links of amputee were calculated on the basis of known experimental data[14].

Problem A has been solved numerically for two types of the prostheses: the above-knee prosthesis with the hydraulic actuator at the knee and the prosthesis with the pneumatic knee mechanism. For both of these prostheses three types of human gait have been studied, characterized by different values for the duration of a double step T, velocity V, and length of step L[11] : slow walking with T_S=1.383 s, V_S=0.998 m/s, L_S=0.69 m; walking at a normal pace T_N=1.1396 s, V_N=1.325 m/s, L_N=0.755 m; and fast walking at T_F=0.9733 s, V_F=1.685 m/s, L_F=0.82 m.

The analysis of the solutions obtained has shown that the kinematic, dynamic, and energetic characteristics of controlled motion of amputee are strongly sensitive to the essential prosthesis' parameters. For a given

individual and pace of a gait there exist optimal values of constructive parameters of the prosthesis' knee and ankle mechanisms $\mathbf{C}_p^* = [C^*, K^*, D^*, d_1^*, d_2^*, l_0^*, S_p^*, S_0^*]^T$. These parameters give minimum energy expended per unit of distance traveled. For above mentioned types of human walking the following minimal values for the energy consumption were obtained for slow, normal and fast paces of motion respectively: E_S=117 J/m, E_N=114 J/m, E_F=147 J/m (for pneumatic knee mechanism), and E_S=103 J/m, E_N=96 J/m, E_F=125 J/m (for hydraulic knee mechanism). Comparison of these data shows that the normal pace of the amputee's gait gives a minimum to the energy expended per unit of distance traveled comparing to the amount of energy needed for the slow or fast gaits. This is valued for both energy-optimal pneumatic and hydraulic knee mechanisms.

Some kinematic and dynamic characteristics of the energy-optimal motion of amputee with optimal structure of the above-knee prosthesis that were obtained by the numerical solving of Problem A for the gait with normal pace are shown in Figures 2 - 5 (solid thin curves correspond to the prosthesis with the hydraulic actuator at the knee, dashed curves - to the prosthesis with the pneumatic knee mechanism). Knee angle ($\alpha_1 - \beta_1$) and hip angle ($\alpha_1 - \psi$) of the prosthetic leg are depicted in Figures 2-3, respectively. Hip torque of the prosthetic leg, ($q_1(t)/M$), and knee torque of the healthy leg, ($u_2(t)/M$) are presented in Figures 4-5, respectively (here M is a total mass of the amputee). For the comparison purposes in Figures 2-5 the domains of the values of the respective kinematic and dynamic characteristics obtained by the biomechanical experiments for a human normal gait are depicted by heavy solid curves[11].

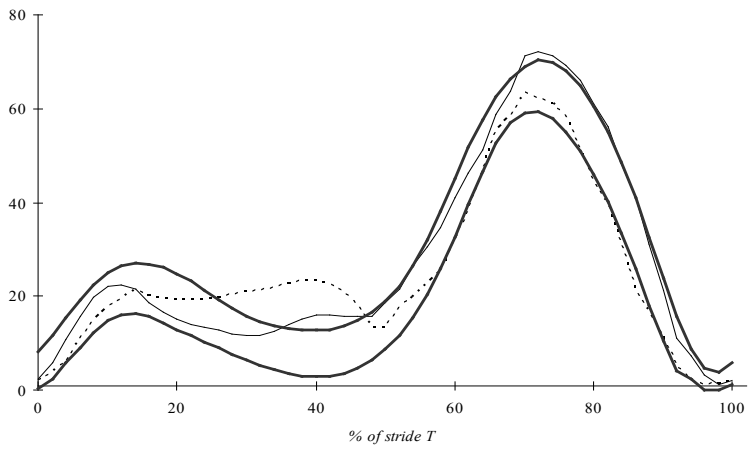

Figure 2. Knee angle of the prosthetic leg, ($\alpha_1 - \beta_1$), in degrees

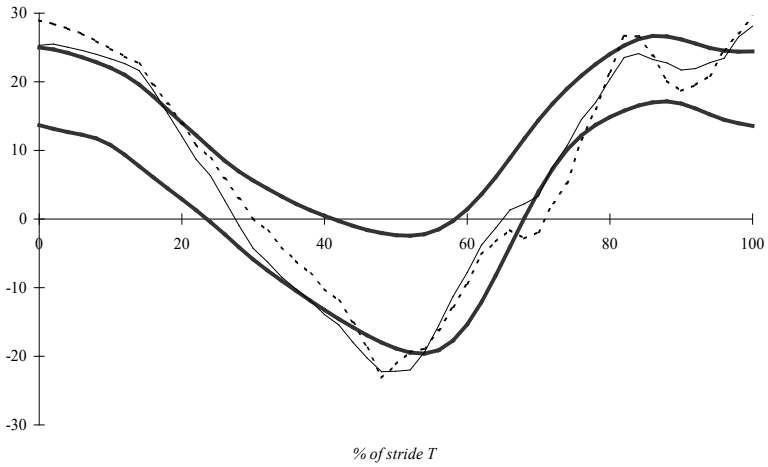

Figure 3. Hip angle of the prosthetic leg, ($\alpha_1 - \psi$), in degrees

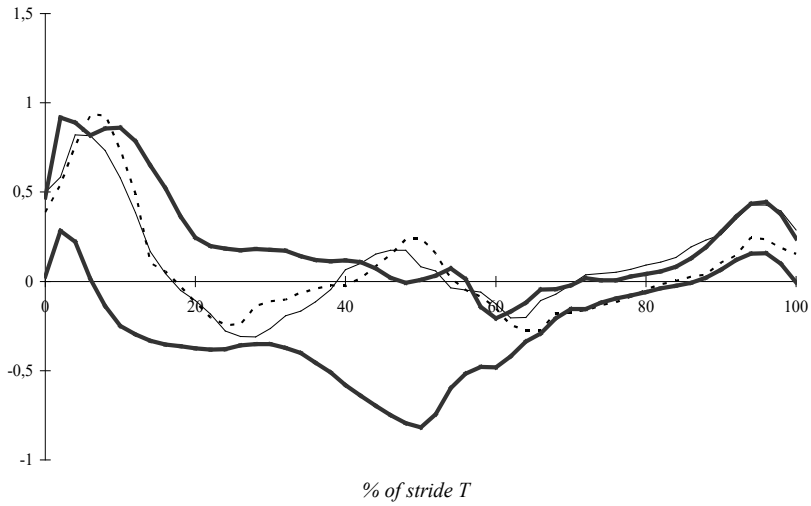

Figure 4. Hip torque of the prosthetic leg, ($q_1(t)/M$), in Nm/kg

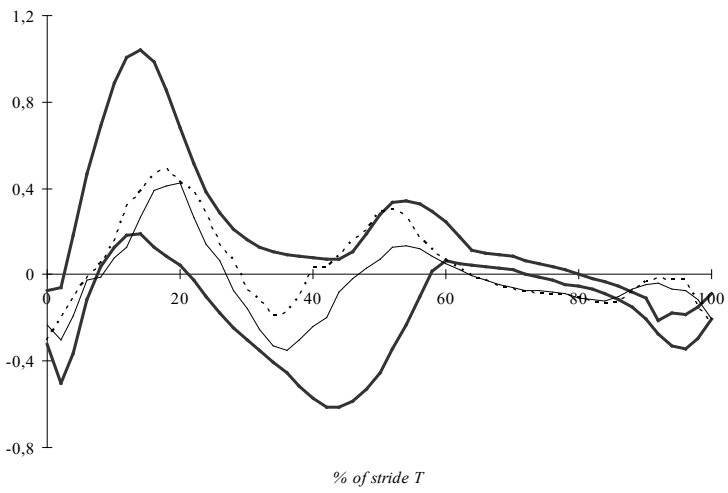

Figure 5. Knee torque of the healthy leg, ($u_2(t)/M$), in Nm/kg

Analysis of the plots depicted in Figures 2-5 shows that the kinematic and dynamic characteristics of the motion of amputee with obtained energy-optimal structure of above-knee prostheses are within reasonable proximity to the respective characteristics of a human normal gait[11].

4 CONCLUSIONS

The optimal control problem (Problem A) has formulated for semi-passively controlled dynamical systems. The key feature of the proposed mathematical statement of the problem is the direct utilization of the equations describing the inherent dynamics of unpowered (passive) drives together with all other constraints imposed on the state vector and the controlling stimuli of the system. It leads to the non-uniqueness of the solution of the direct and inverse dynamics problems and makes it possible to design optimally both structure (passive drives) and external control of a dynamical system.

For solving optimization problems of general type of semi-passively controlled dynamical system the numerical method has been presented. Efficiency of the proposed method is illustrated by the solution of design problem of the energy-optimal above-knee prostheses with two types of passively controlled knee mechanisms. Analysis of the numerical results obtained has shown that during the optimal motion of the considered underactuated semi-passively controlled system there is a strong interaction between the gravity force, the

external control torque exerted by the powered drives and the internal torque exerted by the passive linear spring-damper actuators. The kinematic, dynamic, and energetic characteristics of controlled motion of considered system that models amputee with above-knee prosthesis are strongly sensitive to the essential parameters of the passive drives of the prosthesis. For a given individual and pace of a gait there exist optimal values of the spring and damper parameters of the prosthesis's ankle and knee mechanisms. These parameters give minimum energy expended per unit of distance travelled.

Results obtained give some insight into the study of questions about the role of inherent dynamics in controlled motion, and how much the controlled dynamical systems should be governed by the external powered drives and how much by the system's inherent dynamics. They can also be used to design energy efficient passively controlled mechanisms of the lower limb prostheses.

5 ACKNOWLEDGEMENTS

I am grateful to M. Lidberg and N. Nishchenko who have joined with me in the work described in this paper.

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