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Nodal Count Asymptotics for Separable Geometries

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Abstract. Following earlier works of Blum, Gnutzmann and Smilansky [18], Gnutzmann, Karageorge and Smilansky [30, 31] and Lois and Gnutzmann [32], we generalize nodal count asymptotics for arbitrary dimensionality; in particular, we express the cummulative nodal count as a semiclassical trace formula over the lengths of periodic geodesics of separable compact d-manifolds or domains in $\mathbb{R}^d$ (billiards). We give the explicit mechanical dependence of the leading asymptotic term.

1. Introduction

It is a well known fact that there is a profound relationship between the lengths of the periodic geodesics and the eigenvalues of the Laplace-Beltrami operator (or simply the Laplacian) on compact Riemannian manifolds. In physics terminology, this becomes a duality between classical periodic orbits and quantum energy levels of a particle constrained to move freely on the manifold, reflecting the correspondence between quantum and mechanical motion.

This relationship is expressed in a transparent way by the semiclassical spectral trace formula: a semiclassical expression of the Laplacian spectral density in terms of the lengths of the periodic geodesics, as well as other geometric parameters such as the volume of the manifold or the area of its boundary. In general, one can relate the spectral density of a Schrödinger operator involving some potential, to the periodic orbits of the underlying Hamiltonian dynamics. The eigenvalue-orbit duality has been thoroughly studied for a wide range of settings, and the geometric and dynamical content of the spectrum has been explicitly identified.

In the effort to extract geometric information and dynamical characteristics of the underlying Hamiltonian flow, research has turned to Laplacian eigenfunction morphology as well. Smilansky et al [18, 46] have proposed a programme of studying nodal patterns of real eigenfunctions, revealing the deep geometric and dynamical content of another analytic object related to the Laplacian, its *nodal count*, constituting it rightfully as an important object of study in quantum chaos and inverse spectral theory.

The nodal set of an appropriately smooth function $f : \mathbb{R}^d \to \mathbb{R}$, denoted $Z_f = \{x : f(x) = 0\}$, partitions space into *nodal domains*, which are open, disjoint, maximally connected domains on which it retains a constant sign. The number of

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nodal domains of f, whenever it is finite, is denoted $\nu(f)$, and is called its *nodal count*.

For the Laplacian spectral problem on a compact Riemannian manifold (M, g), or Dirichlet domain $M \subset \mathbb{R}^d$,

$$-\Delta_g \varphi = E\varphi \ , \tag{1}$$

where Dirichlet boundary conditions apply if $\partial M \neq \emptyset$, the *nodal sequence*, ν_k, is the sequence formed by associating to each real eigenfunction its nodal count, having chosen a *specific* spectral ordering $\sigma(-\Delta_g) = \{E_1, E_2, \ldots\}$ and an eigenfunction ordering accordingly $\{\varphi_1, \varphi_2, \ldots\}$,

$$\nu_k := \nu(\varphi_k) \ . \tag{2}$$

We consider the *semiclassical asymptotics* of the nodal sequence, which is the high energy regime $E_k \gg 1$ (equivalently $k \gg 1$), the *classical limit* being the limit $k \to \infty$.

There has been recent progress in this direction, as Aronovitch and Smilansky have shown that one can express the number of intersections of the nodal line with the boundary of a 2-dimensional quantum billiard, both for integrable and ergodic billiards [4].

In analogy to *spectral statistics* [12], *nodal domain statistics* were introduced by Blum, Gnutzmann and Smilansky [18]. The normalized nodal sequence, defined ν_k/k, by Courant's theorem [27] is bounded by 1, $\nu_k \leqslant k$. Blum et al proposed the study of the value distribution of the sequence ν_k/k in the unit interval $[0, 1]$, for a certain set of eigenfunctions corresponding to a growing spectral window, $[E_k, \ldots, E_{k'}]$. It was found that there exists a limiting distribution in the classical limit, $k \to \infty$, and that its features depend crucially on the qualitative type of Hamiltonian dynamics it supports.

If the Hamiltonian flow is separable, thus integrable, the limit distribution displays certain features which are universal, common to all systems of this particular type. If it is ergodic, Berry [14] has shown that the limit distribution is well reproduced by employing a boundary adapted random wave model for the eigenfunctions. Bogomolny and Schmit computed the nodal distribution by using ideas from percolation theory [19], while results from the random wave model and percolation theory were compared by Foltin, Gnutzmann and Smilansky in [28]. Investigations on nodal domain statistics for ergodic toral maps with connections to percolation theory, involving the theory of stochastic Loewner evolution, have been made by Keating, Marklof and Williams [38], while connections between the nodal structure of ergodic eigenfunctions for billiards and stochastic Lowener evolution were made by Bogomolny, Dubertrand and Schmit in [20]. Nodal statistics have been recently generalized by Lois and Gnutzmann for domains and manifolds in $d > 2$ dimensions, both separable and ergodic [32].

Global or local properties of nodal patterns for generic integrable systems, or generally generic Hamiltonian systems (neither integrable nor ergodic), remain essentially terra incognita. There are very few results in this direction, such as an algorithm for counting the nodal domains for the right isosceles triangular billiard by Aronovitch, Band, Fajman and Gnutzmann [5], or the numerical investigations on nodal domain statistics on pseudo-integrable billiards by Sieber, Paulson and Smilansky [45].

There have been continual indications of the geometric importance of the nodal sequence, hence Smilansky and collaborators have put forward the conjecture that the nodal sequence can *resolve isospectrality* [6, 29, 8]. To be more specific, the relevant question here is whether the nodal sequence can distinguish between non-isometric, yet, isospectral systems.

Besides Riemannian manifolds, *quantum graphs* have been employed, as simple and prototype models for quantum chaos [40], to address problems such as the resolution of isospectrality. In [10] Berkolaiko gave a sharp lower bound for the nodal sequence, for a large class of quantum graphs. In [6], Band, Shapira and Smilansky exposed an explicit group theoretic algorithm for the construction of non-isometric, yet, isospectral, quantum graphs and it was proven that their nodal counts differ. A general introduction on nodal domains on quantum (metric) and combinatoric (discrete) graphs is given in Band, Oren and Smilansky [7].

Other studies along this line for manifolds [29], have shown that isospectral tori indeed have different nodal sequences. The above are yet another support of the hypothesis that the geometric information is stored in the nodal and spectral sequences in different ways. One can thus resolve 'drums' which 'sound' the same, by counting their nodal domains.

The first work which addressed the geometric problem of nodal inversion via nodal domain statistics, and revealed the *geometric content* of the nodal sequence, was by Smilansky and Sankaranarayanan [46], where it was shown that the aspect ratio of a planar rectangular domain can be determined by nodal domain statistics for the Dirichlet Laplacian. Following this work, Karageorge and Smilansky [37] showed that the value distribution of the normalized nodal sequence contains enough information to determine the metric of the underlying manifold, for a class of convex surfaces of revolution.

To the author's knowledge, the first work on nodal inversion, was that of Hald and McLaughlin [33]. The authors showed that given a Dirichlet Schrödinger operator on a rectangular planar domain, whose aspect ratio satisfies a certain diophantine property, the potential is determined modulo additive constant by a subset of the nodal line of the eigenfunctions. However, this work focuses on local properties of the nodal set, rather than global ones, such as the nodal count.

Number theoretical arguments pertaining to arithmetic properties of the nodal sequence, were used by Klawonn in [39] to show that the nodal sequence of the Dirichlet Laplacian determines, up to scaling, various classes of two-dimensional manifolds, such as rectangles, flat Klein bottles and flat tori in two and three dimensions.

From these recent results, a new category of problems has risen, *inverse nodal problems*, which are incapsulated in the following question: *given the nodal sequence* $\{\nu_k\}$*, or its statistical properties, what can one say about the geometry (modulo dilations) or the dynamics on the underlying configuration space?* Interest in the novel inversion methods of inverse nodal problems has recently motivated application of these results in other scientific disciplines, such as 3-dimensional shape analysis in neuro-imaging [41].

In this paper, we restrict ourselves to the study of dynamics on manifolds, which admit 'regular' wave motion, underlined by 'regular' classical dynamics; in particular *separable manifolds*, i.e., manifolds or domains on which the Laplacian is separable in the usual sense [35, 36].

Following Gnutzmann et al [30], we derive a semiclassical trace formula for the cummulative nodal sequence, $c_k = \nu_1 + \ldots + \nu_k$, of the form

$$c_k \sim \kappa_* k^2 + k^{\frac{3}{2}-\frac{3}{2d}} \sum_{\substack{\gamma \in \mathbb{N}_0^d \setminus \{0\} \\ \gcd\{\gamma_j\}=1}} \sum_{\mu=1}^{\infty} A_\gamma \frac{\sin\left(\mu(k^{1/d} S_\gamma - \frac{\pi}{2}\alpha_M \cdot \gamma) + \frac{\pi}{4} b_\gamma\right)}{\mu^{\frac{d+1}{2}}} \;, \tag{3}$$

γ signifying a primitive periodic orbit, μ the number of repetitions for a given orbit, taking into account all possible topologies of orbits about a given invariant Liouville-Arnol'd torus; S_γ is the action of the corresponding orbit, and α_M and b_γ are phases determining other geometric and topological data of the flow, and A_γ some bounded coefficients characterized by the orbit family. The leading power of the semiclassical series is universal (mean growth of cummulative nodal count).

Specifically, from the leading, Weyl term, we extract the analogue of the Pleijel bound, the *cummulative Pleijel bound*, κ_*, which we express in terms of purely dynamical quantities. From the remainder oscillatory part, we show that the cummulative nodal sequence generically determines the lengths of periodic geodesics of the underlying classical flow, up to scaling.

We also make a connection to nodal domain statistics, by giving a representation of the limiting normalized nodal count distribution, relating its features to the Pleijel limit and to the Polterovich conjecture [44].

The limitation on the *geometric information* one can extract from the nodal sequence versus the spectral sequence, is the fact that the former is invariant under uniform scalings of M, while the later would scale with the square of the lengths; thus, we expect to retrieve only scale-invariant quantities from the nodal sequence.

2. The General Setting

2.1. The Geometric and Dynamical Setting

The general framework of this article is the classical and quantum geodesic flow for a certain class of compact, separable Riemannian manifolds M, with $\dim M = d \geqslant 2$. The rare class of separable systems within the class of Liouville-Arnol'd integrable systems, furnish the only problems for which explicit calculations are possible, as there exists an explicit expression for the nodal count in terms of the *quantum numbers*, which give the number of nodes of each product function of the eigenfunctions along the level sets of each coordinate.

The manifold M, or domain $M \subseteq \mathbb{R}^d$ as a special case, is parameterized by a separable coordinate system $x = (x^1, \ldots, x^d)$, a notion explained in what follows; it is equipped with a Riemannian metric tensor, whose covariant components are, in terms of the coordinates (x^j), $(g_{ij}(x))$. If there is a boundary, it is assumed piecewise smooth, with a finite number of edges and vertices, and Dirichlet boundary conditions apply.

The spectral problem of the Laplacian (quantum problem) is closely associated to the geodesic flow on the manifold (mechanical problem), i.e., displacement along the geodesics, a Hamiltonian flow induced by the free Hamiltonian,

$$H(x,p) = \|p\|_x^2 := \sum_{ij} g^{ij}(x) p_i p_j \;, \tag{4}$$

on the cosphere bundle, or the energy shell,

$$\Sigma_E = \Big\{(x,p) \in T'M \, : \, H(x,p) = E\Big\} \ , \tag{5}$$

for some constant $E > 0$.

In the case that $\partial M \neq \emptyset$, we have a billiard system, and assume Fresnel boundary conditions for the rays, i.e. equiangular reflections. In the context of this dynamical setting on M, we have the additional requirements:
1) M is convex; this condition is needed to guarantee that the phase space admits a *global action-angle coordinate system* [2]. This will allow us to use a globally valid Hamiltonian in terms of action coordinates.
2) The energy shell in the action representation is Gauss-positively curved. This technical point will be elaborated on in what follows. We refer to this requirement as the *twist condition* (see, e.g., [15]).

Some examples of separable manifolds complying with the above requirements are the ball and cube endowed with the naturally induced Euclidean metrics, while others without boundary include Liouville tori, surfaces of revolution, Zoll surfaces, etc.

The d independent integrals of the flow, which stem from the isometries of M, are not uniquely chosen. In the quantum level, these symmetries give rise to eigenvalue clustering and degeneracies. By the Liouville-Arnol'd theorem [26], smoothness and independence with respect to the Poisson bracket of the d integrals of the motion guarantees that the invariant manifold $\mathcal{T} \subset \Sigma_E$ is diffeomorphic to a d-torus, $\mathcal{T} \cong \mathbb{T}^d$.

A special choice of canonical phase space coordinates are *the action-angle coordinates*, (ϕ, I) [2]. Each choice of action coordinates $I = (I_1, \ldots, I_d)$ determines an invariant Liouville-Arnol'd torus $\mathcal{T}_I$ in phase space, while the angles $\phi \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ parameterize it; for general integrable flows, the actions read

$$I_j = \frac{1}{2\pi} \int_{\sigma_j} p \cdot dx := \frac{1}{2\pi} \int_{\sigma_j} \sum_i p_i \, dx^i \ , \tag{6}$$

which reduces to $I_j = \frac{1}{2\pi} \int_{\sigma_j} p_j \, dx^j$ for separable flows; σ_j is the j-th irreducible 1-cycle winding around the corresponding invariant Liouville-Arnol'd torus, for which $\frac{1}{2\pi} \int_{\sigma_i} d\phi^j = \delta_{ij}$.

We may isolate the Hamiltonian as a function of the d actions; $h = H \circ \Phi^{-1}$, Φ being a canonical transformation to an action-angle coordinate system in phase space $\Phi : (x,p) \mapsto (\phi, I)$. The angles ϕ are ignorable, as the actions are integrals of the motion, and so we write $I \mapsto h(I)$. The canonical frequencies of the orbits winding around an invariant Liouville-Arnol'd torus $\mathcal{T}_I$ corresponding to I are $\omega(I) = \nabla_I h(I)$. In the case that the components of ω are commensurate, the torus is resonant, and the orbit is periodic.

The Hamiltonian in the action coordinates is homogeneous of degree 2 [25], $h(tI) = t^2 h(I)$ with $t > 0$, and thus, by Euler's theorem, h assumes the polar form

$$h(I) = a(I)\|I\|^2 = a(I)\Big(I_1^2 + \ldots + I_d^2\Big) \ . \tag{7}$$

The function a is positive and homogeneous function of degree zero on a certain sector of the sphere $\mathbb{S}^{d-1}$ in action space $\mathbb{R}^d$. This decomposition of the Hamiltonian is unique, and so the profile function $a(I)$ characterises the dynamics.

As the Hamiltonian is homogeneous in the action coordinates, the dynamics for any value of the energy $E > 0$ are equivalent up to a scaling of the action coordinates $I \mapsto \sqrt{E}I$, so that one can simply consider the flow on the unit energy shell, $\Phi\Sigma_1$. The unit energy shell is isomorphic to a sector of the sphere $\mathbb{S}^{d-1}$, and is actually a smooth, convex deformation of the sphere, by the twist hypothesis.

The importance of integrability on the quantum level lies in the existence of d quantum commuting integrals, generators of the corresponding isometries, to which d *quantum numbers* are associated, forming the quantum lattice, a subset of $\mathbb{Z}^d$.

Separability imposes a strong constraint on the geometric, and subsequently dynamical properties of a Riemannian manifold M. A manifold M, or a domain $M \subset \mathbb{R}^d$ is *separable* if it admits a coordinate system for which the metric constitutes the Laplacian separable in the individual coordinates in the usual sense. This immediately leads to separability of the Hamilton-Jacobi equation as well [35], i.e., that

$$\frac{\partial S}{\partial t} + \|\nabla S\|_x^2 = 0 \tag{8}$$

admits additive-separable solutions of the form $S(x,t) = -Et + \sum W_j(x^j)$, and the separability of the Schrödinger equation, i.e., that (the Planck constant scaled to unity)

$$i\frac{\partial \psi}{\partial t} = -\Delta_g \psi \tag{9}$$

admits multiplicative-separable solutions of the form $\psi(x,t) = c(t)\prod u_j(x^j)$ [35, 36]. Qualitatively, one can say that separable manifolds are those which admit a basis of states of the quantum flow which form an orthogonal grid pattern.

The fact that integrability implies separability can be understood by starting with the Laplacian; separability of the Schrödinger equation implies separability of the Hamilton Jacobi equation, which, in turn, implies that the associated canonical transformation associated to it is a transformation to action-angle coordinates, and thus, the flow is Liouville-Arnol'd integrable.

2.2. The Laplacian Spectrum

We consider the Laplacian spectral problem

$$\begin{aligned} &-\Delta_g \varphi = E\varphi \\ &\varphi_{\restriction \partial M} = 0\ , \quad \text{if} \quad \partial M \neq \emptyset\ , \end{aligned} \tag{10}$$

seeking solutions in the Sobolev space $W^{2,2}(M, \mathbb{R}; dx)$.

In terms of the local coordinates on M, (x^j), the Laplacian is defined as

$$-\Delta_g := -\sum_{ij} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \Big(\sqrt{|g|} g^{ij}(x) \frac{\partial(\cdot)}{\partial x^j} \Big)\ , \tag{11}$$

acting on the twice differentiable functions on M. Here, $|g| = \det(g_{ij}(x))$.

Due to compactness of M, we have $\varphi \in L^2(M, \mathbb{R}; dx) \backslash \{0\}$, wherein the Laplacian is self-adjoint. Thus, the spectrum $\sigma(-\Delta_g)$ is pure point and real, i.e. there are only bound quantum states with allowed energies (eigenvalues) extending from zero to infinity, as it is also non-negative [23].

Since the spectrum is real, we may order it and label each eigenvalue by a counting index, taking into account the degeneracies,

$$\sigma(-\Delta_g) = \{E_k\}_{k\in\mathbb{N}} \ , \quad \text{with} \quad 0 \leqslant E_1 < E_2 \leqslant E_3 \leqslant \ldots \to \infty \ . \tag{12}$$

One must assume a specific ordering convention among the degenerate states. Even if the systematic degeneracies, due to symmetries of the system, grow fast enough, the ordering convention has no effect in semiclassical considerations, while accidental degeneracies are utterly unimportant in the semiclassical regime.

Examples of fast growth of systematic degeneracies is the sphere, where $d_E = O(\sqrt{E})$, while examples of slow degeneracy growth are flat tori (Pythagorean sequences), $d_E = o(E^\varepsilon)$, $\varepsilon > 0$, or irrational flat tori, $d_E = O(1)$ [34]. It has been shown that the maximal growth of spectral degeneracies for such manifolds is $d_E = O(E)$ [24].

Even for degenerate spectra, there is a natural choice, a unique (modulo ordering) preferred real, separable ordered basis [27] in the degeneracy subspace $\mathcal{E}_{E_k} \subset L^2(M)$, corresponding to the eigenvalues

$$E_k = \ldots = E_{k+l-1} \ , \tag{13}$$

where dim $\mathcal{E}_{E_k} = d_{E_k} = l$. One must chose a specific ordering within these elements. The subspace $\mathcal{E}_{E_k}$ has a real, ordered basis $\{\varphi^1_{E_k}, \ldots, \varphi^l_{E_k}\}$, where

$$\varphi^s_{E_k}(x) = \prod_j u_{E_k,j}(x^j) \ , \quad s = 1, \ldots, l \ . \tag{14}$$

We construct the nodal sequence according to this preferable basis, rendering its definition non-ambiguous. For example, consider the problem of the Dirichlet open unit disk in $\mathbb{R}^2$. The spectrum is doubly degenerate (leaving aside accidental degeneracies) due to rotational symmetry. In each 2-dimensional degeneracy subspace, the eigenfunctions are linear combinations (in polar coordinates)

$$\varphi(r,\theta) = \left(c_1 e^{im\theta} + c_2 e^{-im\theta}\right) J_m(j_{mn} r) \ , \tag{15}$$

where $m = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$, and $c_{1,2}$ being normalization constants. The natural preferred basis in each degeneracy subspace spanned by the two different choices is given by

$$\varphi^1(r,\theta) = J_m(j_{mn} r)\cos(m\theta) \ , \quad \varphi^2(r,\theta) = J_m(j_{mn} r)\sin(m\theta) \ , \tag{16}$$

here not L^2-normalized.

The explicit relation between geometry and spectrum lies in the spectral counting function, a starting point of all semiclassical spectral considerations. The spectral counting function is the counting measure of the spectrum [23]

$$N(E) := \#\{k \in \mathbb{N} : \, E_k < E\} \ , \tag{17}$$

supported on $[0,\infty[$, related to the spectral density by $N(E) = \int_0^E \rho(\lambda)\, d\lambda$.

The leading semiclassical behavior of the spectral counting function is given by the Weyl law, a celebrated result in spectral theory, marking the cornerstone of the relation between analytic and geometric properties of Riemannian manifolds [23],

$$N(E) \sim C_* E^{d/2} \ , \quad E \to \infty \ , \tag{18}$$

where *the Weyl coefficient* gives us the volume of M,

$$C_* = \left|\left\{(x,p) :\ H(x,p) < 1\right\}\right| = \frac{|B_1^d|}{(2\pi)^d}|M| = \frac{|M|}{(4\pi)^{d/2}\Gamma(d/2+1)} \ , \tag{19}$$

B_1^d being a unit d-ball, and $|\cdot|$ the Riemannian d-volume.

The Weyl decomposition of the spectral counting functions amounts to isolating the remainder from the leading term,

$$N(E) = C_* E^{d/2} + R(E) \ . \tag{20}$$

The leading term is called the Weyl term, while the remainder is highly oscillatory, with $R(E) = O(E^{\frac{d-1}{2}})$ [23].

The sharpness of the estimate of the remainder varies considerably, depending on the type of the underlying Hamiltonian flow, integrable and ergodic as the two extremes, and its oscillatory behavior is dominated by the geometric and topological features of the periodic orbits of the Hamiltonian flow.

2.3. Quantum Numbers

Integrability allows us to use an alternative labeling of eigenfucntions, i.e., by *quantum numbers*. Associated to each eigenfunction there is a quantum number $q = (q_1, \ldots, q_d) \in \Gamma \subseteq \mathbb{Z}^d$, in essence a multi-index, where Γ is *the quantum lattice*.

The quantum lattice, and thus the quantum numbers, are of course not unique; they are defined modulo canonical transformation of the action angle variables which preserves the actions as constants of the motion. However, one can always make a canonical transformation make the actions range either along the whole real line or the half line.

The range of each quantum number q_j must be in agreement with the sign of the corresponding action I_j; this is what determines the quantum lattice Γ for the particular choice of actions. In particular, $\Gamma = D \cap \mathbb{Z}^d$, where $D \subset \mathbb{R}^d$ is a region of the action space. For each periodic coordinate, it is clear that we have a rotational degeneracy in the spectrum, so that the corresponding quantum number extends to all integer values. Thus, the quantum lattice Γ will be of the form

$$\Gamma = \Gamma_1 \times \ldots \times \Gamma_d \ , \tag{21}$$

where each Γ_j is $\mathbb{Z}_+$ for a nonperiodic coordinate, and $\mathbb{Z}$ for a periodic coordinate, as λ will itself be of the form

$$D = D_1 \times \ldots \times D_d \ , \tag{22}$$

with $D_j = \mathbb{R}_+$ for a nonperiodic coordinate and $D_j = \mathbb{R}$ for a periodic coordinate§.

In this representation, the eigenvalues and eigenfunctions are denoted $E(q)$ and φ_q respectively, and the semiclassical limit is simply the large quantum number limit, $q \to \infty$, identical to the large counting index limit $k \to \infty$, according to the Correspondence Principle.

§ Whether the lattice point 0 is included in Γ_j for a nonperiodic coordinate is of little concern, as the relevant correction vanishes in the semiclassical regime (see Appendix A).

2.4. Semiclassical Spectrum

We employ the Einstein-Brillouin-Keller-Maslov (EBKM) semiclassical approximation, or torus quantization [12], for the eigenvalues in the quantum number representation, according to which those invarinat Liouville-Arnol'd tori $\mathcal{T}_I$ support semiclassical eigenfunctions for which

$$I = q + \frac{1}{4}\alpha_M \ , \tag{23}$$

for $q \in \mathbb{Z}^d$, and $\alpha_M = (\alpha_{M\,1}, \ldots, \alpha_{M\,d})$ the Maslov index, incorporating topological data of the flow. Each $\alpha_{M\,j}$ is the Maslov index for the corresponding degree of freedom, i.e., the number of encounters with caustics of the cycle σ_j along which I_j is defined [11].

This gives the semiclassical eigenvalue asymptotics, as the above is taken on an energy shell for the eigenvalue $\mathcal{T}_I \subset \Sigma_E$. Expressing the Hamiltonian in terms of the action coordinates, we have explicit semiclassical eigenvalue asymptotics,

$$E(q) \sim h(q + \frac{1}{4}\alpha_M) \ , \quad q \to \infty \ . \tag{24}$$

By the quantum number representation, we may define the counting index, $\mathcal{N}$, on the quantum lattice, enumerating the number of states occupying energies below $E(q)$. Explicitly,

$$\mathcal{N}(q) = \#\Big\{ E \in \sigma(-\Delta_g) \in \Gamma : \ E(q) < E \Big\} \ , \tag{25}$$

where clearly, $\mathcal{N}(q) = N(E(q))$. The Weyl law on the quantum lattice, reads

$$\mathcal{N}(q) = C_* h(q + \frac{1}{4}\alpha_M)^{d/2} + O(\|q\|^{d-1}) \ . \tag{26}$$

3. The Nodal Count

Due to separability, there exists a real separable basis of eigenfunctions $\{\varphi_q\}_{q\in\Gamma}$ in $L^2(M)$, such that the nodal set will display a simple, orthogonal, checkerboard pattern on M. This enables us to express the nodal count, $\nu(q) := \nu(\varphi_q) = \nu_{\mathcal{N}(q)}$, on the quantum lattice

$$\nu(q) = \nu_{(d)}(q) + \nu_{(d-1)}(q) + \ldots + \nu_{(0)}(q) \ , \tag{27}$$

each of the $\nu_{(j)}(q)$ being homogenous of degree j in the quantum numbers. Explicitly, we propose the leading behavior to be

$$\nu_{(d)}(q) = 2^\beta |q_1 \ldots q_d| \ , \tag{28}$$

where we propose the β to be the number of rotational symmetries of (M, g); thus, the constant β, which equals the number of separable periodic coordinates parameterizing M, is unique for a given M. Although one can generate a whole find a class of coordinate systems which are equivalent up to certain simple transformations, which constitute the corresponding metric of M separable, β is invariant under such transformations.

A prototype nonseparable planar Dirichlet domain for which the nodal count is not of the above form, is the right isosceles trianglular domain; as was shown by

Aronovitch et al [5], the nodal count is associated with topological characteristics of a graph constructed by joining self-intersection points of the nodal set, exhibiting a highly nonalgebraic behavior.

To illustrate this point, one should take note that for the general form of the eigenfucntion $\varphi(x) = \prod u_j(x^j)$, it is clear that in the domain of the compact coordinate ($\cong \mathbb{S}^1$), say x^i, one can only have an even number of nodal points, and thus an even product for the nodal count.

The Courant theorem [27, 43] states the nodal sequence of the Laplacian for appropriately well behaved compact domains in $\mathbb{R}^d$ or compact d-manifolds alike, has the following upper bound,

$$\nu_k \leqslant k \; . \tag{29}$$

From this we deduce that, $\nu_1 = 1$, i.e., the first eigenfunction must be everywhere non-zero. However, since the set of eigenfunctions $\{\varphi_k\}$ constitutes an orthonormal basis of $L^2(M)$, φ_1 must be orthogonal to all others, something which is possible only if it is the sole eigenfunction with one nodal domain. Thus, for $k \geqslant 2$ the nodal sequence has the trivial lower bound $\nu_k \geqslant 2$.

3.1. The Pleijel Bound

The behavior of the nodal sequence is extremely complex. It is highly oscillatory, and its exacct asymptotic behavior is difficult to probe. Even the question of whether the sequence has a limit at infinity or whether it possesses a non-trivial lower bound are still open.

The roughest nontrivial bound for the nodal sequence is the Courant bound. For the simplest case, that of an open Dirichlet (or Neumann) interval, Sturm's oscillation theorem gives $\nu_k = k$, which is modified for periodic boundary conditions as $\nu_k = 2k$. For the case $d \geqslant 2$, this is replaced by the Courant bound, according to which

$$\nu_k \leqslant k \; , \quad k \in \mathbb{N} \; , \tag{30}$$

which combined with the orthogonality $\varphi_1 \perp \varphi_k$ for $k > 1$, we have the additional nontrivial lower bound $\nu_k \geqslant 2$ for $k \geqslant 2$.

From the Courant bound for the nodal sequence, we incur the Courant bound for the cummulative nodal sequence, $c_k := \nu_1 + \nu_2 + \ldots + \nu_k$,

$$k \leqslant c_k \leqslant \frac{k(k+1)}{2} \; , \tag{31}$$

or, by appropriately normalizing it to the unit interval (this scalling will be justified subsequently)

$$\frac{1}{k} \leqslant \frac{c_k}{k^2} \leqslant \frac{k+1}{2k} \leqslant 1 \; . \tag{32}$$

This bound is attained finitely many times by the sequence c_k/k^2, as we directly see the sharper bound

$$\limsup_{k\to\infty} \frac{c_k}{k^2} \leqslant \frac{1}{2} \; . \tag{33}$$

The only known result of analytic nature, beyond the Courant bound, is the Pleijel theorem. Pleijel [43] showed that for any planar, regular, bounded domain, there exists a sharpening of the Courant bound which is saturated only finitely many times. The limit suppremum of the cummulative nodal sequence is bounded by the *Pleijel constant*, ξ_P,

$$\limsup_{k\to\infty} \frac{\nu_k}{k} \leqslant \xi_P = \frac{4}{j_1^2} = 0.691\ldots < 1 \ , \tag{34}$$

j_1 being the first non-trivial zero of the Bessel function J_0. This bound holds for both Dirichlet and Neumann Laplacians. For example, for a square domain, with the standard separable Dirichlet Laplacian basis, the Courant bound is met only by the first, second and fourth eigenvalues, $\nu_1 = 1,\ \nu_2 = 2$, and $\nu_4 = 4$. This sharpening of the Courant bound finds its probabilistic expression in nodal statistics, the study of the statistical behavior of the nodal count, as will be exposed in section (5).

By following the same arguments used to sharpen the Courant bound to the Pleijel bound, we have for $d = 2$

$$\limsup_{k\to\infty} \frac{c_k}{k^2} \leqslant \frac{2}{j_1^2} = 0.345\ldots \ , \tag{35}$$

i.e., the Pleijel bound for the cummulative nodal sequence is half of the Pleijel bound for the nodal sequence.

The Pleijel theorem holds for general dimensionality, a generalization of the planar case, as an implication of the Faber-Krahn inequality [43, 23]: for compact planar domains in $\mathbb{R}^d$ with piecewise smooth boundary of the same volume, the closed ball has the lowest first Dirichlet eigenvalue‖. So, any such domain M, and for the ball $B_R^d(0)$ of radius R such that $|B_R^d(0)| = |M|$, we have

$$E_1(M) \geqslant E_1(B_R^d(0)) = \frac{j_{d/2-1,1}^2}{R^2} = \frac{\pi j_{d/2-1,1}^2}{\Gamma(d/2+1)^{2/d}|M|^{2/d}} \ , \tag{36}$$

where $j_{d/2-1,1}$ is the first positive root of the fractional order Bessel function $J_{d/2-1}$.

Let M be a domain as described above. Consider a nodal partition of M for the eigenfunction φ_k with eigenvalue E_k, into ν_k nodal domains,

$$\Omega_1^k, \ldots, \Omega_{\nu_k}^k \ . \tag{37}$$

The restriction on a nodal domain $\varphi_{k\restriction\Omega_s^k}$ is the ground state of the Dirichlet Laplacian in that domain and thus $E_1(\Omega_s^k) = E_k$ for $s = 1, 2, \ldots, \nu_k$.

From the Faber Krahn inequality [23],

$$\begin{aligned} &|\Omega_s^k| \geqslant \frac{\pi^{d/2} j_{d/2-1,1}^d}{\Gamma(d/2+1)} \frac{1}{E_k^{d/2}} \\ &\Rightarrow \sum_{s\leqslant\nu_k} |\Omega_s^k| \geqslant \sum_{s\leqslant\nu_k} \frac{\pi^{d/2} j_{d/2-1,1}^d}{\Gamma(d/2+1)} \frac{1}{E_k^{d/2}} \Rightarrow |M| \geqslant \frac{\pi^{d/2} j_{d/2-1,1}^d}{\Gamma(d/2+1)} \frac{\nu_k}{E_k^{d/2}} \ . \end{aligned} \tag{38}$$

From the Weyl law, $E_k^{d/2} \sim C_*^{-1} k$, we finally have [23]

$$\limsup_{k\to\infty} \frac{\nu_k}{k} \leqslant \frac{2^d\ \Gamma(d/2+1)^2}{j_{d/2-1,1}^d} =: \xi_P \ . \tag{39}$$

We call ξ_P *the Pleijel bound*, which is universal, depending only upon the dimensionality. The Pleijel bound is nontrivial as it sharpens the Courant bound;

‖ The theorem applies for general Robin boundary conditions, $\varphi + k\frac{\partial\varphi}{\partial\nu} = 0$ on ∂M.

indeed, for large dimensionalities the first Bessel root grows linearly [1], $j_{d/2-1,1} = d/2 + O(d^{1/3})$, and so

$$\xi_P = \pi d\, e^{-(1-\log 2)d+O(d^{1/3})} \ll 1 \ , \tag{40}$$

while even for small d the Pleijel bound sharpens the Courant bound, $\xi_P < 1$.

A generalized Pleijel bound has been found to hold also for a general class of d-manifolds, which includes the manifolds under consideration in this paper [23].

We generalize the Pleijel bound for the cummulative nodal sequence; starting off from the Faber-Krahn inequality, we have

$$|M|E_k^{d/2} \geqslant \frac{\pi^{d/2} j_{d/2-1,1}^d}{\Gamma(d/2+1)}\nu_k \Rightarrow c_k \leqslant \frac{\Gamma(d/2+1)}{\pi^{d/2} j_{d/2+1,1}^d}|M| \sum_{l\leqslant k} E_l^{d/2} \ . \tag{41}$$

Noting that, by the Weyl law

$$\sum_{l\leqslant k} E_l^{d/2} = C_*^{-1}\frac{k^2}{2}\Big(1+o(1)\Big) \ , \tag{42}$$

we have

$$\limsup_{k\to\infty} \frac{c_k}{k^2} \leqslant \frac{1}{2}\xi_P \ , \tag{43}$$

half of the Pleijel bound for the nodal sequence.

3.2. The Polterovich Conjecture

While the Pleijel bound sharpens the Courant bound, yielding a sharper asymptotic behavior for the nodal sequence, the Pleijel bound in turn seems not to be sharp. As we shall elaborate on in section (5), probabilistic approaches to the asymptotic behavior of the nodal sequence bare strong support that ν_k grows roughly slower than $\xi_P k$. For example, for the case of the planar rectangular domain it has been found that ν_k grows roughly as $\frac{2}{\pi}k$, while for the planar disk as $\frac{1}{2}k$ [18].

Following this evidence, Polterovich in [44] conjectured that for any regular, bounded domain with piecewise smooth boundary, with either Dirichlet or Neumann boundary conditions, the limit supremum of the normalized nodal sequence is bounded from above by

$$\limsup_{k\to\infty} \frac{\nu_k}{k} \leqslant \frac{2}{\pi} = 0.636\ldots \ , \tag{44}$$

which is well less than the Pleijel constant. This, in addition to Pleijel's proof [43] that for the rectangular domain $\frac{2}{\pi}$ is a lower bound for the limit supremum of the normalized nodal sequence, would imply that $\frac{2}{\pi}$ is sharp for the rectangular domain, a maximum for the limit supremum among all planar domains.

4. Semiclassical Asymptotics of the Nodal Count

In [30, 31], Gnutzmann et al suggested the study a variant of the cumulative nodal count, $c_k = \sum_{l\leqslant k}\nu_l$, in order to recover a semiclassical nodal trace formula, for some prototype separable surfaces, providing an explicit dependence of the nodal count by scaled geometric parameters. This formulated starting from the Berry-Tabor spectral trace formula, and inverting a regularized spectral counting function in order to express the energy in terms of the number of states.

As Gnutzmann et al [30] have shown, the cumulative nodal count has a quadratic leading semiclassical behavior, say const. $\times k^2$, for some system-dependent coefficient.

The object of interest is the staircase function $c(k)$, defined as

$$c(k) := c_{\lfloor k \rfloor} = \sum_{l \leqslant \lfloor k \rfloor} \nu_l \ , \tag{45}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. As the definition of the above is ambiguous in the case of degeneracies, we modify the construction.

Having chosen an ordering of the preferred, separable basis in each degeneracy subspace, we define *the cummulative spectral nodal count*,

$$\tilde{c}(E) = \sum_{q \in \Gamma} \nu(q) \Theta(E - E(q)) \ , \tag{46}$$

where $E(q)$ is the eigenvalue corresponding to the quantum number $q \in \Gamma$. In order to detour the explicit dependence of the spectrum, wanting to refer only to its ordering, we use the (smoothed) spectral counting function

$$N^{\varepsilon}(E) = \sum_{k \in \mathbb{N}} \Theta^{\varepsilon}(E - E_k) \ , \tag{47}$$

where $\Theta^{\varepsilon} = \Theta * \psi^{\varepsilon}$, ψ^{ε} being a positive approximate unity, $\psi^{\varepsilon}(E) = \frac{1}{\varepsilon}\psi\Big(\frac{E}{\varepsilon}\Big)$ and $\psi \in C_0^{\infty}(\mathbb{R}_+, \mathbb{R})$, so that Θ^{ε} is monotonically increasing. Thus, the equation

$$k = N^{\varepsilon}(E) \ , \tag{48}$$

has a unique solution for any $k \in \mathbb{N}$, denoted $E = \lambda^{\varepsilon}(k)$. It is clear that $N^{\varepsilon} \to N$ in $\mathcal{S}'(\mathbb{R}_+, \mathbb{R})$, and $\lambda^{\varepsilon}(k) \to E_k$ as $\varepsilon \to 0^+$.

Thus, we define *the cummulative nodal counting function* as the limiting staircase function in $\mathcal{S}'$,

$$C(k) := \lim_{\varepsilon \to 0^+} \tilde{c}\Big(\lambda^{\varepsilon}(k)\Big) \ , \tag{49}$$

a staircase function, a tempered distribuiton in $\mathbb{R}_+$. We express the above sum as a sum over the quantum lattice Γ, utilizing the Poisson summation formula to estimate its semiclassical asymptotics.

For nondegenerate spectra, we have $C(k) = c(k + \frac{1}{2})$, while for degenerate spectra, at each index k of a degenerate eigenvalue E_k, the cummulative nodal counting function increases by the sum of the nodal counts within the degeneracy class. We will derive a trace formula for this modified cumulative nodal count (omitting the characterization modified in the sequel).

We have, by equation (27)

$$\tilde{c}(E) = \sum_{q \in \Gamma} \Theta(E - E(q)) \nu(q) = \sum_{q \in \Gamma} \Theta(E - E(q)) \Big(\nu_{(d)}(q) + \ldots + \nu_{(0)}(q)\Big) \ . \tag{50}$$

The leading term in the sum is (28) $\tilde{c}(E) \sim \sum_{E(q)<E} \nu_{(d)}(q) = \sum_{E(q)<E} 2^{\beta} |q_1 \ldots q_d|$. In particular (see Appendix A), by disregarding all terms below $\nu_{(d)}(q)$, we have

$$\tilde{c}(E) = \sum_{q \in \Gamma} 2^{\beta} |q_1 \ldots q_d| \Theta(E - E(q)) + O(E^{\frac{d-1}{2}}) \ . \tag{51}$$

By the Poisson summation formula [48], and by taking the EBKM approximation for $E(q)$, the above becomes

$$\tilde{c}(E) \sim \sum_{Q\in\mathbb{Z}^d} \int_D e^{2\pi i Q\cdot I} 2^\beta |I_1 \dots I_d| \Theta\Big(E - h(I + \frac{1}{4}\alpha_M)\Big)\, dI \ . \tag{52}$$

The Weyl term of the above immediately gives us

$$\tilde{c}_{\text{Weyl}}(E) = \int_D 2^\beta |I_1 \dots I_d| \Theta(E - h(I))\, dI \tag{53}$$

4.1. The Weyl Term

Following Berry and Tabor [11], we pass to the coordinate system $I \mapsto (\lambda, \theta)$, on $D \cong \mathbb{R}_+ \times \Phi\Sigma_1$.

The level set $\{\lambda = \text{const.}\}$ is the energy shell $\Phi\Sigma_\lambda$, where explicitly $\lambda = h(I)$. The angles, $\theta = (\theta_1, \dots, \theta_{d-1})$, parameterize the energy shell on a sperical sector determined by the signs of the quantum numbers, $\Phi\Sigma_1 \cong \mathbb{S}^{d-1} \cap D$.

An alternative expression for the Weyl coefficient is achieved by utilizing this coordinate system

$$C_* = \int_{H(x,p)<E} dx dp = \int_D \Theta(1 - h(I))\, dI \ , \tag{54}$$

Passing to the coordinates (λ, θ), we reach a simple explicit representation of the Weyl coefficient as a functional of the amplitude a,

$$C_*(a) := \frac{1}{d} \int_{\mathbb{S}^{d-1}\cap D} \frac{d\Omega(\theta)}{a(n_\theta)^{d/2}} \ , \tag{55}$$

where $n_\theta = I/\|I\|$, stressing that a is defined on the sphere.

Returning to $\tilde{c}$, passing to the new coordinate system, we can write the Weyl term as

$$\begin{aligned} \tilde{c}_{\text{Weyl}}(E) &= 2^{\beta-1} \int_0^E d\lambda \lambda^{d-1} \int_{\mathbb{S}^{d-1}\cap D} \frac{|n_1 \dots n_d|}{C_*^2\, a(n_\theta)^d}\, d\Omega(\theta) \\ &= \left(\frac{2^{\beta-1}}{d} \int_{\mathbb{S}^{d-1}\cap D} \frac{|n_1 \dots n_d| d\Omega(\theta)}{a(n_\theta)^d} \right) E^d \ . \end{aligned} \tag{56}$$

Finally, by the Weyl law and the definition (49), we have

$$C_{\text{Weyl}}(k) = \kappa_* k^2 \ , \tag{57}$$

where the coefficient κ_* is expressed as a functional of a,

$$\kappa_*(a) := \frac{2^{\beta-1}}{d\, C_*(a)^2} \int_{\mathbb{S}^{d-1}\cap D} \frac{|n_1 \dots n_d|}{a^d}\, d\Omega = d\, 2^{\beta-1} \frac{\int_{\mathbb{S}^{d-1}\cap D} \frac{|n_1 \dots n_d|}{a^d} d\Omega}{\left(\int_{\mathbb{S}^{d-1}\cap D} \frac{d\Omega}{a^{d/2}} \right)^2} \ , \tag{58}$$

where we have used the functional representation of the Weyl coefficient (55). We call κ_* the cummulative Pleijel constant, which defines the actual leading asymptotics of the cummulative nodal sequence. In this form, which makes the underlying dynamical contribution explicit, is computable in closed form for simple separable systems.

Since the trace formula is represented as a quasi-asymptotic expansion, the coefficient of the leading term is

$$\kappa_* = \lim_{L\to\infty} \int_{\mathbb{R}_+} \frac{C(Lk)}{L^2} f(k)\, dk = \lim_{L\to\infty} \int_{\mathbb{R}_+} \frac{C_{\mathrm{Weyl}}(Lk)}{L^2} f(k)\, dk \;, \tag{59}$$

for any Schwatz test function f of unit mass.

4.2. The Oscillatory Part

By the Poisson summation formula [48], the above becomes

$$\tilde{c}_{\mathrm{osc}}(E) \sim 2^\beta E^d \sum_{Q\in\mathbb{Z}^d\setminus\{0\}} e^{-\pi i \alpha_M \cdot Q/2} \int_D e^{2\pi i\sqrt{E}Q\cdot I} |I_1 \dots I_d| \Theta\Big(E - h(I)\Big)\, dI \;. \tag{60}$$

Here, $\mathbb{Z}^d$ stands for the dual of the completion of the quantum lattice, and plays the role of the topological lattice, as will be explained in what follows. The residual terms omited are boundary terms of the Poisson formula, and are of smaller order.

By denoting $I = I(\lambda, \theta)$ in the energy coordinates, and by using the homogeneity property, we have

$$\begin{aligned} \tilde{c}_{\mathrm{osc}}(E) &\sim 2^{\beta-1} E^d \sum_{Q\in\mathbb{Z}^d\setminus\{0\}} e^{-\pi i \alpha_M \cdot Q/2} \\ &\quad\times \int_0^1 d\lambda \lambda^{d-1} \int_{\mathbb{S}^{d-1}\cap D} e^{2\pi i\sqrt{E\lambda}Q\cdot I(1,\theta)} \frac{|n_1 \dots n_d|}{a(n_\theta)^d}\, d\Omega(\theta) \end{aligned} \tag{61}$$

where $n_\theta = I/\|I\|$ and $n_j = I_j/\|I\|$.

4.3. The Semiclassical Nodal Trace Frormula

The oscillatory integral of the remainder and can be estimated in terms of the stationary phase method [47]. In particular, for the integral in the series (61),

$$\begin{aligned} &\iint_{\mathbb{S}^{d-1}\cap D\times[0,1]} e^{2\pi i\sqrt{E\lambda}Q\cdot I(1,\theta)} \frac{|n_1 \dots n_d|}{a(n_\theta)^d} \lambda^{d-1}\, d\lambda d\Omega(\theta) \\ &\sim \int_0^1 (E\lambda)^{\frac14 - \frac d4} \left.\frac{|n_1 \dots n_d|}{a(n_\theta)^d \sqrt{|\det_\Omega Q\cdot I''(1,\theta)|}}\right|_{\theta=\theta_0} e^{2\pi i\sqrt{E\lambda}Q\cdot I(1,\theta_0) + ib_Q} \lambda^{d-1} d\lambda \\ &F = \left.\frac{2E^{\frac14-\frac d4} e^{ib_Q} |n_1 \dots n_d|}{a(n_\theta)^d \sqrt{|\det_\Omega Q\cdot I''(1,\theta)|}}\right|_{\theta=\theta_0} \frac{\Gamma(\frac{3d+1}{2}) - \Gamma(\frac{3d+1}{2}, -2\pi i Q\cdot I(1,\theta_0))}{(-2\pi i Q\cdot I(1,\theta_0))^{\frac{3d+1}{2}}} \;. \end{aligned} \tag{62}$$

where $b_Q = \frac{\pi}{4}\mathrm{sgn}\Big(Q\cdot \frac{\partial^2 I}{\partial\theta_i\partial\theta_j}(1,\theta_0)\Big)$, the function $\Gamma(\cdot,\cdot)$ is the upper incomplete Gamma function¶, and $\det_\Omega$ denotes the scalar determinant relative to the uniform measure Ω on the sphere,

$$\det_\Omega Q\cdot I''(1,\theta_0) = \|Q\|^{d-1} \det\left(n_Q \cdot \frac{\partial^2 I}{\partial\theta_i\partial\theta_j}(1,\theta_0)\right)\Big(\frac{d\Omega(\theta_0)}{d\theta}\Big)^2 \;,$$

¶ $\Gamma(s,z) := \int_z^\infty t^{s-1}e^{-t}dt$.

where $\frac{d\Omega(\theta)}{d\theta}$ is the and Radon-Nikodym derivative of the surface measure, and $n_Q = Q/\|Q\|$.

By the asymptotics of the incomplete Gamma function, we finally deduce the asymptotics

$$\iint_{\mathbb{S}^{d-1}\cap D\times[0,1]} e^{2\pi i\sqrt{E\lambda}Q\cdot I(1,\theta)}\frac{|n_1\dots n_d|}{a(n_\theta)^d}\lambda^{d-1}\,d\lambda d\Omega(\theta)$$
$$\sim \frac{E^{\frac{1}{4}-\frac{d}{4}}|n_1\dots n_d|}{a(n_\theta)^d\sqrt{|\det_\Omega Q\cdot I''(1,\theta)|}}\Bigg|_{\theta=\theta_0}\frac{e^{2\pi i\sqrt{E}Q\cdot I(1,\theta_0)+ib_Q}}{\pi i\sqrt{E}Q\cdot I(1,\theta_0)}\ . \tag{63}$$

As in the Berry-Tabor construction, the critical points correspond to the relations

$$Q\cdot\frac{\partial I}{\partial\theta_j}(1,\theta) = 0\ . \tag{64}$$

The frequency vector ω is always perpendicular to the energy shell

$$\omega(I(\theta))\cdot\frac{\partial I}{\partial\theta_j}(1,\theta) = 0\ , \tag{65}$$

as we have $dh_{\restriction\Phi\Sigma_1} = 0$. Thus, on the critical point, we have $\omega(I)\propto Q$, something which, of course, implies that the frequencies $\omega_j(I)$ corresponding to the Liouville-Arnol'd torus $\mathcal{T}_{I_Q}$ are all commensurate, and the corresponding orbit, or rather a ray of orbits, are periodic.

The above immediately verifies the characterization of the Poisson lattice as a topological lattice: the topological lattice of a flow, which is simply the dual of the quantum lattice, determines at each of its points Q the topology of a periodic orbit, i.e., the winding numbers of the orbit around the given invariant Liouville-Arnol'd torus along each independent direction.

To be more precise, the critical points correspond to *rays* of periodic orbits. The lattice points Q and Q', which belong to the same vector ray, i.e. are proportional, belong to the same ray of periodic orbits. Points further from the origin of the lattice, correspond to topologically more complicated families of closed orbits.

A further simplification arises by decomposing the rays of orbits into specific primitive periodic orbits. Each lattice point is written uniquely as $Q = \mu\gamma$, where the components of γ are relatively prime, and μ is the number of repetitions of the primitive orbit γ. Note that the term $2\pi Q\cdot I_Q$, is nothing but μS_γ, where S_γ is the action of the primitive orbit γ. Note that S_γ here is scaled, i.e., invariant under dilations of M.

The function $Q\cdot I(1,\theta)$ has at least one stationary point, due to compactness of the energy shell $\Phi\Sigma_1$; on the other hand, given that $\Phi\Sigma_1$ is convex, it follows that the function has exactly one stationary point. This guarantees that the nodal fluctuations are significantly large in amplitude in the semiclassical limit, and not subalgebraically small.

Proceeding with the calculation, the leading behavior of $\tilde{c}_{\mathrm{osc}}(E)$ is

$$E^{\frac{3(d-1)}{4}}\sum_{Q\in\mathbb{Z}^d\setminus\{0\}}\frac{2^\beta|n_1\dots n_d|}{a(n_\theta)^d\sqrt{|\det_\Omega\, Q\cdot I''(1,\theta)|}}\frac{e^{2\pi i\sqrt{E}Q\cdot I(1,\theta)+ib_Q-\pi i\alpha_M\cdot Q/2}}{2\pi i Q\cdot I(1,\theta)}\Bigg|_{\theta=\theta_0}\ , \tag{66}$$

evaluated at the stationary point $\theta=\theta_0$. At the critical point, the phase $2\pi Q\cdot I(1,\theta)$ becomes the action of the ray corresponding to the lattice point Q.

Finally, by the Weyl law and the definition (49), the semiclassical nodal trace formula is expressed as a Weyl term incorporating the cummulative Pleijel bound, and the oscillatory part as a sum over the primitive topological lattice and repetitions of primitive periodic orbits, by noting additionally that

$$\left|\det_\Omega Q \cdot I''(1,\theta_0)\right| = \|\mu\gamma\|^{d-1} \left(\frac{d\Omega(\theta_0)}{d\theta}\right)^2 K(\theta_0), \tag{67}$$

where $K(\theta_0)$ is the Gaussian curvature at θ_0 on the energy shell $\Phi\Sigma_1$, which is nonzero by the twist condition,

$$C(k) \sim \kappa_* k^2 + k^{\frac{3}{2}-\frac{3}{2d}} \sum_{\substack{\gamma\in\mathbb{N}_0^d\setminus\{0\} \\ \gcd\{\gamma_j\}=1}} \sum_{\mu=1}^{\infty} A_\gamma \frac{\sin\left(\mu(k^{1/d}S_\gamma - \frac{\pi}{2}\alpha_M\cdot\gamma) + \frac{\pi}{4}b_\gamma\right)}{\mu^{\frac{d+1}{2}}} , \tag{68}$$

the amplitudes being

$$A_\gamma = \frac{2^{\beta+1}|n_1\dots n_d|}{a(n_\theta)^d \frac{d\Omega(\theta)}{d\theta}\sqrt{K(\theta)}\|\gamma\|^{\frac{d+1}{2}} S_\gamma}\Big|_{\theta=\theta_0} , \tag{69}$$

where $b_\gamma = b_Q = b_{\mu\gamma}$ is the signature of the Hessian of $Q\cdot I''(1,\theta)$, i.e., the excess of positive over negative eigenvalues of the matrix, and $I_\gamma = I(1,\theta_0)$, the γ-dependence being implicit in θ_0. The power of the leading, Weyl part is universal, do not even depend upon the dimensionality, while the scaling of the fluctuation does.

In the oscillatory term, we identify the actions of the periodic orbits, which equal their period, or equivalently, the lengths of the corresponding periodic geodesics, which are nothing but the projections of the periodic orbits from the phase space onto M, as incured by the conservation of the energy, $\frac{1}{4}\|v\|_x^2 = 1$; the Riemannian length is $\int_0^{T_\gamma} \|\dot{x}(t)\|_{x(t)}\, dt = 2T_\gamma = 2S_\gamma$.

By appropriate smoothing, the length spectrum can be recovered computationally as the support of the Fourier transform of properly normalized the nodal fluctuations, $k^{\frac{3}{2d}-\frac{3}{2}}C_{\rm osc}(k)$, with respect to $k^{1/d}$,

$$\Sigma(\ell) \sim \sum_{\substack{\gamma\in\mathbb{N}_0^d\setminus\{0\} \\ \gcd\{\gamma_j\}=1}} \sum_{\mu\in\mathbb{Z}_*} 2A_\gamma \frac{e^{\frac{i\pi}{4}(b_\gamma+2\mu\alpha_M)}}{|\mu|^{\frac{d-1}{2}}} \delta\Big(\ell - \mu\frac{S_\gamma}{2\pi}\Big) , \tag{70}$$

where the terms with opposite μ correspond to pairs of orbits which differ in orientation.

5. Connection to Nodal Domain Statistics

Blum, Gnutzmann and Smilansky [18] introduced and studied the value distribution of the normalized nodal sequence ν_k/k for φ_k in a certain set of states, for various Hamiltonian systems, both *separable* and *ergodic*. The probability density reads,

$$P_{kk'}(\xi)\,d\xi = \frac{1}{k'-k+1}\sum_{l=k}^{k'} \delta\Big(\xi - \frac{\nu_l}{l}\Big)\,d\xi\ , \quad \xi\in[0,1]\ , \tag{71}$$

the region of interest being $k \gg 1$.

The distribution of nodal domains has been shown to depend heavily on the qualitative type of the underlying classical motion; for the separable case, there exists a limiting distribution in the classical limit, and a piecewise smooth density function P,

$$\frac{1}{k'-k+1}\sum_{l=k}^{k'} f\Big(\frac{\nu_l}{l}\Big) \to \int_0^1 f(\xi)P(\xi)\, d\xi \ , \tag{72}$$

for appropriate test functions on $[0,1]$.

The limiting distribution's local features have been shown to be universal for separable systems. In particular, there exists some $0 < \xi_* < 1$, such that P is supported on the interval $[0, \xi_*]$, with behavior at the boundary [46, 32],

$$P(\xi) \sim \Big(1 - \xi/\xi_*\Big)^{\frac{d-3}{2}} \ , \ \ \xi \to \xi_*^- \ , \tag{73}$$

and at the other end of the interval

$$\lim_{\xi \to 0^+} P(\xi) = 1 \ . \tag{74}$$

This feature of P, i.e., the fact that it vanishes identically for $\xi > \xi_*$, is a statistical verification of the Pleijel theorem; the normalized nodal sequence asymptotically does not take values in the whole interval $[0, 1]$. This formalism provides a computable 'expectation' of the supremum limit of the normalized nodal sequence, ξ_*, which seems to be always a sharpening of the Pleijel bound ξ_P, as has been conjectured in $d = 2$ by Polterovich.

The density $P_{kk'}(\xi)$ in the semiclassical limit, has been shown to generically determine the geometry (up to scaling) of the configuration, for a class of surfaces of revolution [37].

Following analogous asymptotic calculations as above, we reach an expression for the limiting distribution,

$$P(\xi) = \frac{C_*(a)^{-1}}{d} \int_{\mathbb{S}^{d-1}\cap D} \delta\Big(\xi - 2^\beta \frac{|n_1 \ldots n_d|}{C_*(a) a(n_\theta)^{d/2}}\Big) \frac{d\Omega(\theta)}{a(n_\theta)^{d/2}} \ , \tag{75}$$

a calculation made first by Lois et al [32]. The geometric interpretation of the critical value ξ_* is the maximal volume of the d-rectangles in D whose one corner is in the energy shell $\mathbb{S}^{d-1} \cap D$.

For a given M, and thus a given Hamiltonian and quantum lattice, the critical value ξ_* is explicitly computable for many separable flows,

$$\xi_* = \max_{\theta \in \mathbb{S}^{d-1}\cap D} 2^\beta \frac{|n_1 \ldots n_d|}{C_*(a) a(n_\theta)^{d/2}} \ . \tag{76}$$

6. Simple Illustrations

Finally, we proceed to applications to some simple separable systems.

6.1. The Rectangle

For the planar rectangular domain $M = [0, l] \times [0, l/\sqrt{\alpha}]$, $\sqrt{\alpha}$ being the aspect ratio, with standard Cartesian coordinates, the Hamiltonian is

$$H(x, p) = p_1^2 + p_2^2 \ . \tag{77}$$

A straightforward calculation shows that $I_1 = \frac{l}{\pi} p_1$ and $I_2 = \frac{l}{\pi\sqrt{\alpha}} p_2$, and thus

$$h(I) = \frac{\pi^2}{l^2}(I_1^2 + \alpha I_2^2) \ , \tag{78}$$

which leads to the amplitude $a(I) = \frac{\pi^2}{l^2}\Big(\cos^2\theta + \alpha \sin^2\theta\Big)$.

On the quantum level, the quantum lattice is $\Gamma = \mathbb{N}^2$, and $\beta = 0$. Unsurprisingly, the cummulative Pleijel constant, κ_*, is independent of the aspect ratio α, and thus, the same for all planar rectangles,

$$\kappa_* = \frac{2}{\pi^2} \ , \tag{79}$$

complying with the invariance of the nodal sequence under dilations of M.

A similar calculation gives the exact same value for the rectangular flat 2-torus, while for both systems we have $\xi_* = \frac{2}{\pi}$.

For the d-cube with β periodic conditions (the extremes being $\beta = 0$, the Dirichlet cube, $\beta = d$ the torus, with all the intermediates being Mobiüs strips)

$$a(n_\theta) = \frac{2^{2\frac{d-\beta}{d}}\pi}{\Gamma(d/2+1)^{2/d}} \tag{80}$$

attaining the common maximum value

$$\kappa_* = \frac{2^d\,\Gamma(d/2+1)^2}{d!\pi^d} \ . \tag{81}$$

The critical value for the normalized nodal count distribution is

$$\xi_* = \frac{2^d\,\Gamma(d/2+1)}{d^{d/2}\pi^{d/2}} \ . \tag{82}$$

6.2. The Disk

The calculations for the disk of radius R, in polar coordinates, gives

$$H(r, \theta, p_r, p_\theta) = p_r^2 + \frac{1}{r^2}p_\theta^2 \ . \tag{83}$$

By calculating the action coordinates, we obtain

$$h(I_r, I_\theta) = \frac{I_\theta^2}{R^2}\, f^{-1}\Big(\pi\frac{I_r}{|I_\theta|}\Big)^2 \ , \tag{84}$$

where $f(t) = \sqrt{t^2 - 1} - \mathrm{arcsec}(t)$, $t \geqslant 1$, which leads to,

$$a(I) = \frac{1}{R^2}\sin^2\theta\, f^{-1}\Big(\pi\frac{\cos\theta}{|\sin\theta|}\Big)^2 \ , \tag{85}$$

since $D = \mathbb{R}_+ \times \mathbb{R}$ and $\beta = 1$. Finally, we obtain

$$\kappa_* = \frac{4}{\pi^2} - \frac{1}{4} \ , \quad \text{and} \quad \xi_* = \frac{1}{2} \ . \tag{86}$$

In both (one-dimensional) examples, we note that the assumption $\limsup_k \frac{c_k}{k^2} \leqslant \frac{2}{\pi^2}$ is satisfied, being saturated for the rectangle.

7. Conclusions - Discussion

We have used the Berry-Tabor method [11] to construct a semiclassical nodal trace formula for a class of separable d-manifolds. We generalize the result of Gnutzmann et al [30], in constructing a nodal trace formula in d dimensions. It is to be noted that, while the growth of the cummulative nodal sequence is universal

$$c_k = \nu_1 + \ldots + \nu_k \asymp k^2 \ , \tag{87}$$

the amplitude of the nodal fluctuations growing as the dimensionality grows, as $k^{\frac{3}{2}-\frac{3}{2d}}$.

The Wey term of the nodal trace formula provides us with a proportionality constant, which is the quasi-limit of the distribution $C(k)/k^2$, a lower bound for the cummulative Pleijel limit

$$\limsup_{k\to\infty} \frac{c_k}{k^2} \ . \tag{88}$$

We have given a dynamical argument for the Polterovich conjecture for case of separable flows on d-dimensional manifolds or billiards. Specifically, among all Dirichlet domains, as described in the introduction, the d-rectangle posseses the largest cummulative Pleijel constant.

It seems reasonable to expect, as a generalization of the conjecture made by Polterovich [44], that the Pleijel limit supremum is maximized for the case of the rectangle in any dimensionality, not simply for separable flows. This is reinforced by the evidence that among chaotic (ergodic) flows provide a slower growth, $\nu_k \asymp k^{1/2}$ [18], and certain integrable, nonseparable integrable flows, provide nodal counts of maximal growth [5]; one could expect that a maximal growth of the nodal count gives rise to a maximum Pleijel constant.

Appendix A. The Poisson Summation Formula [48]

Our convention for the Fourier transform is $\hat{f}(p) = \int_{\mathbb{R}^d} e^{2\pi i p\cdot x} f(x)\, dx$. For real, smooth, rapidly decaying functions, the Poisson formula for finite sums is

$$\sum_{a\leqslant n\leqslant b} f(n) = \sum_{n\in\mathbb{Z}} \int_a^b e^{2\pi i n x} f(x)\, dx + \frac{1}{2}\Big(f(a) + f(b)\Big) \ , \tag{A.1}$$

with $a < b$ integers, which derives from the formula $\int_{\mathbb{R}_+} \delta(x) f(x)\, dx = \frac{1}{2} f(0)$.

We thus generalize the Poisson summation formula for a simple boundary case in

$\mathbb{R}^d$; if $\Gamma = D \cap \mathbb{Z}^d$, where D is the rectangle $\{x : x_j \geqslant l_j\}$ with l_j integers, then

$$\sum_{n\in\Gamma} f(n) = \sum_{n\in\mathbb{Z}^d} \int_D e^{2\pi i n\cdot x} f(x)\, dx$$
$$+\sum_{s=1}^{d-1} \frac{1}{2^{d-s}} \sum_{n_1,\ldots,n_s\in\mathbb{Z}} \int_{D_s} e^{2\pi i(n_1x_1+\ldots+n_sx_s)} f(x_1,\ldots,x_s,l_{s+1},\ldots,l_d)\, dx_1\ldots dx_s$$
$$+\frac{1}{2^d} f(l_1,\ldots,l_d)\ , \tag{A.2}$$

where $D_s = \{x_1 \geqslant l_1, \ldots, x_s \geqslant l_s\}$ is a rectangle in $\mathbb{R}^s$.

For generic lattices $\Gamma' = \mathbb{Z}^d \cap A$, where the full measure convex domain of $\mathbb{R}^d$ lies entirely in D, we substitue in the above the function $f(x) = g(x)\chi_A(x)$.

Now, consider the sum

$$\sum_{n\in LD\cap\mathbb{Z}^d} f(n)\ , \tag{A.3}$$

where D is convex, $L \gg 1$, and the function f is smooth and homogeneous of degree $\ell > 0$. By applying the boundary form Poisson summation formula, we have

$$\sum_{n\in LD\cap\mathbb{Z}^d} f(n) = L^{(1+\ell)d} \sum_{n\in\mathbb{Z}^d} \int_D e^{2\pi i L n\cdot x} f(x)\, dx + o(L^{(\ell+1)d})\ . \tag{A.4}$$

Further, the 'bulk' term is

$$L^{(1+\ell)d} \sum_{n\in\mathbb{Z}^d} \int_D e^{2\pi i L n\cdot x} f(x)\, dx = L^{(1+\ell)d} \int_D f(x)\, dx + L^{(1+\ell)d} \sum_{n\neq 0} \hat{\chi}_D(Ln)\ , \tag{A.5}$$

$\hat{\chi}_D$ being the Fourier transform of the characteristic function of D, for which we have the estimate [25]

$$\hat{\chi}_D(p) = O\Big((1+\|p\|)^{-\frac{d+1}{2}}\Big)\ , \tag{A.6}$$

so that

$$\sum_{n\neq 0} \hat{\chi}_D(Ln) = O\Big(L^{-\frac{d+1}{2}}\Big)\ , \tag{A.7}$$

which leads to

$$\sum_{n\in LD\cap\mathbb{Z}^d} f(n) = L^{(1+\ell)d} \int_D f(x)\, dx + O\Big(L^{(1+\ell)d-\frac{d+1}{2}}\Big)\ . \tag{A.8}$$

Appendix B. Quasi-Asymptotic Expansions [42]

A positive continuous function on the half line, S, is called slowly varying at infinity if for $L > 0$ we have

$$\lim_{k\to\infty} S(Lk)/k = 1\ . \tag{B.1}$$

A tempered real distribution, F, on the positive axis, has the *quasi-asymptotic* behavior at infinity with respect to $k^m S(k)$, S being a slowly varying function, with limit the tempered distribution G if for any test function f

$$\lim_{L\to\infty} \int_{\mathbb{R}_+} \frac{F(Lk)}{L^m S(L)} f(k)\, dk = \int_{\mathbb{R}_+} G(k) f(k)\, dk\ . \tag{B.2}$$

We write $F \sim G$ at ∞, with respect to $k^m S(k)$, for an appropriate $a \in \mathbb{R}$ and some slowly varying function S.

Appendix C. Spherical Coordinates [23]

The spherical coordinate chart on $\mathbb{S}^{d-1}$ is $\theta = (\theta_1, \ldots, \theta_{d-1}) \in [0,\pi]^{d-2} \times [0, 2\pi[$:

$$\begin{aligned}
x_1 &= \cos\theta_1 \\
x_2 &= \sin\theta_1 \cos\theta_2 \\
&\ldots \\
x_{d-1} &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2} \cos\theta_{d-1} \\
x_d &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2} \sin\theta_{d-1} \ .
\end{aligned} \tag{C.1}$$

The uniform measure on the sphere is

$$d\Omega(\theta) = \sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \ldots \sin\theta_{d-2}\, d\theta_1 \ldots d\theta_{d-1} \ , \tag{C.2}$$

and so the $\mathbb{R}^d$ Lebesgue measure in spherical coordinates becomes $dx = r^{d-1}\, dr d\Omega(\theta)$, and the Radon-Nikodym derivative is

$$\frac{d\Omega(\theta)}{d\theta} = \sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \ldots \sin\theta_{d-2} \ . \tag{C.3}$$

The Lebesgue measure in the energy coordinates for a given Hamiltonian h, with the unique decomposition $h(I) = a(I)\|I\|^2$, becomes

$$dx = \frac{\lambda^{d/2-1}}{2a(n_\theta)^{d/2}}\, d\lambda d\Omega(\theta) \ , \tag{C.4}$$

where $n_\theta = x/\|x\|$.

Appendix D. Useful References

The d-volume of the unit ball is

$$|B_1^d(0)| = \int_{\|x\|<1} dx = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \ , \tag{D.1}$$

while the $(d-1)$-area of the unit sphere is $|\mathbb{S}^{d-1}|_{d-1} = d|B^d| = \frac{d\pi^{d/2}}{\Gamma(d/2+1)}$.

The spectrum of the Dirichlet d-rectangle $[0, l_1] \times \ldots \times [0, l_d]$, and d-torus $(\mathbb{R}/l_1\mathbb{Z}) \times \ldots \times (\mathbb{R}/l_d\mathbb{Z})$ are, respectively,

$$\left\{\pi^2 q \cdot Kq :\ q \in \mathbb{N}^d\right\} \ , \tag{D.2}$$

and

$$\left\{4\pi^2 q \cdot Kq :\ q \in \mathbb{Z}^d\right\} \ , \tag{D.3}$$

where the quadratic form is given by the matrix $K = \mathrm{diag}\left(\frac{1}{l_1^2}, \ldots, \frac{1}{l_d^2}\right)$.

The $\mathbb{R}^d$ Laplacian in spherical coordinates becomes

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{\mathbb{S}^{d-1}} \ . \tag{D.4}$$

The spectrum of the spherical Laplacian is

$$\sigma(-\Delta_{\mathbb{S}^{d-1}}) = \left\{l(l+d-2)\right\}_{l\in\mathbb{N}_0} \ , \tag{D.5}$$

while the Dirichlet spectrum of the open ball $B^d_R(0)$ is

$$\sigma(-\Delta_{B^d_R(0)}) = \left\{ \frac{j^2_{b,m}}{R^2} \right\}_{(m,l)\in\mathbb{N}\times\mathbb{N}_0} , \tag{D.6}$$

where $j_{b,m}$ is the m-th positive root of the Bessel function J_b, the order here being $b = l + d/2 - 1$.

Finally, a useful integral

$$\int_{\|x\|<1} |x_1 \ldots x_d|\, dx = \int_0^1 drr^{2d-1} \int_0^{2\pi} |\cos\theta_{d-1} \sin\theta_{d-1}|\, d\theta_{d-1}$$
$$\times \prod_{j=1}^{d-2} \int_0^{\pi} |\cos\theta_j| \sin^{2(d-j)-1}\theta_j\, d\theta_j = \frac{1}{d!} \; . \tag{D.7}$$

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