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Extensions of subcopulas

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Abstract

In view of Sklar's Theorem the probability distribution function of every (not necessarily continuous) random vector can be uniquely decompose in terms of the marginal distributions of its components and a suitable subcopula. The study of such latter functions is therefore of interest for understanding the dependence information of non–continuous variables. Here, we investigate some analytical properties of the class of subcopulas, including compactness (with respect to a novel metric), approximations and Baire category results. Moreover, under a suitable assumption, we describe all possible extensions from a subcopula to a copula in any dimension.

Key words: Copula, Hausdorff metric, Non–continuous random vectors, Sklar's Theorem, Subcopula.
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1. Introduction

Copulas have been largely employed for describing the association of continuous random variables, as can be witnessed by several contributions devoted to the topic; see, for instance, [11, 18, 21, 25]. Another related aspect of interest is, nowadays, the use of copulas in the determination of statistical models for a random vector $\mathbf{X} = (X_1, \ldots, X_p)$ whose components are possibly not-continuous. In this latter case, however, the copula associated with $\mathbf{X}$ is not anymore unique, a fact that needs special care in several practical problems (see, for instance, [16, 22]). In fact, as can be inferred by the proof of Sklar's Theorem [31] (see also [9, 12, 30]), a copula associated with a non–continuous random vector $\mathbf{X}$ is uniquely determined only on the Cartesian product of the (closure of the) ranges of $X_1, \ldots, X_p$, but various extensions to the whole domain $[0, 1]^p$ are possible. In [31],

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the term *subcopula* was introduced to denote the function (defined on a suitable subset of $[0,1]^p$) that contains the information about the dependence of a not-necessarily continuous random vector.

Various methods to extend a subcopula to a copula have been considered, for instance, in [6, 31], while maximal and minimal extensions can be found in [4, 16]. Such methods are relevant, for instance, in defining various measures of association for non-continuous data [24, 26, 33]. Moreover, they can provide tools for nonparametric estimation of a copula, where it is of interest to smooth the empirical copula while preserving copula properties [13, 17, 27], and/or extend copulas defined on a discrete setting [20, 23, 29]. Moreover, extensions of sub–copulas may be helpful to understand the limit of copula-based inferential procedures when they are applied, without some due changes, to non–continuous data [3, 19, 28].

Here, we continue the study of subcopulas and their extensions by providing a general framework to deal with such problems. Specifically, we introduce a distance ξ in the class of subcopulas that is based on the Hausdorff distance of the respective graphs. As a relevant aspect, we show that the class of subcopulas equipped with the topology induced by ξ is compact (Section 2). Hence, we use continuity arguments to prove, in an alternative way, that any subcopula can be extended to a copula (Section 2.1). Finally, in Section 3 we provide the general analytical expression for all the extensions of a subcopula in a multivariate setting, which generalizes the results presented in [6] for the bivariate case.

2. A metric for subcopulas

For basic definitions and properties of copulas we refer to [11, 25]. Here, we only recall the minimum bare that is necessary to make this manuscript self–contained.

Definition 2.1. Let $A_1,\dots,A_p$ be subsets of $[0,1]$ containing both 0 and 1. Then a *subcopula* is a function $S : A_1 \times \cdots \times A_p \to [0,1]$ such that

(a) $S(u_1,\dots,u_p) = 0$ if $u_j = 0$ for at least one index $j \in \{1,\dots,p\}$;

(b) $S(1,\dots,1,t,1,\dots,1) = t$ for every $t \in A_j$ $(j \in \{1,\dots,p\})$;

(c) For every rectangle $[\mathbf{a},\mathbf{b}]$ having its vertices in $A_1 \times \cdots \times A_p$, the S–volume of $[\mathbf{a},\mathbf{b}]$ is non–negative, namely $V_S([\mathbf{a},\mathbf{b}]) \geq 0$, where

$$V_S([\mathbf{a},\mathbf{b}]) = \sum_{\mathbf{v}\in \mathrm{ver}[\mathbf{a},\mathbf{b}]} \mathrm{sign}(\mathbf{v})\, S(\mathbf{v}),$$

with

$$\mathrm{sign}(\mathbf{v}) = \begin{cases} 1, & \text{if } v_j = a_j \text{ for an even number of indices,} \\ -1, & \text{if } v_j = a_j \text{ for an odd number of indices,} \end{cases}$$

and $\mathrm{ver}([\mathbf{a},\mathbf{b}]) = \{a_1,b_1\} \times \cdots \times \{a_p,b_p\}$ is the set of vertices of $[\mathbf{a},\mathbf{b}]$.

The class of subcopulas will be denoted by $\mathscr{S}$. Every $S \in \mathscr{S}$ is Lipschitz continuous with constant 1 (shortly 1–Lipschitz), i.e.

$$|S(\mathbf{u}) - S(\mathbf{v})| \leq \sum_{i=1}^{p} |u_i - v_i| \tag{1}$$

for every $\mathbf{u}, \mathbf{v} \in \mathrm{Dom}(S)$. Thus, the domain of S can be extended without loss of generality to its closure. In the following, if not otherwise stated, we will therefore assume that $A_1, \ldots, A_p$ are closed.

A *copula* is a subcopula defined on $[0,1]^p$, namely such that $A_j = [0,1]$ for every $j \in \{1, \ldots, p\}$. The class of copulas will be denoted by $\mathscr{C}$. In particular, given $S \in \mathscr{S}$, a copula C is said to be an *extension* of S if $C = S$ on $\mathrm{Dom}(S)$. In the following, we are interested in defining a suitable distance between subcopulas.

Now, $\mathscr{C}$ forms a convex and compact set in the class of all real–valued continuous functions whose domain is $[0,1]^p$ equipped with the distance d_∞, where

$$\forall A, B \in \mathscr{C} \qquad \mathrm{d}_\infty(A,B) = \sup_{\mathbf{u} \in [0,1]^p} |A(\mathbf{u}) - B(\mathbf{u})|.$$

Moreover, in $\mathscr{C}$ pointwise convergence is equivalent to uniform convergence (see, e.g., [7, 9, 11]). However, we cannot define a similar metric in $\mathscr{S}$, since two subcopulas may have different domains.

To overcome this problem, we consider an alternative procedure that is inspired by the fact that, in $\mathscr{C}$ the convergence with respect to d_∞ can be also characterized in terms of level–sets and endograph convergence with respect to Hausdorff metric (see, [14, 34]).

We start with the introduction of a distance in a general metric space.

Let (X, d) be a metric space. For all subsets A and B in X, we adopt the notation

$$\delta^*(A,B) := \sup_{a \in A} \mathrm{d}(a, B)$$

with $\mathrm{d}(\omega, \Omega) := \inf\{\mathrm{d}(\omega, \omega') : \omega' \in \Omega\}$ for given $\omega \in X$ and $\Omega \subseteq X$. Such a δ^* is a pseudo–quasi–metric, i.e. it satisfies all the axioms of a metric with the possible exception of symmetry and it allows $\delta^*(A,B) = 0$ for some $A \neq B$.

Definition 2.2. Let A and B be two compact subsets of the metric space (X, d). The *Hausdorff distance* between A and B is given by

$$\mathrm{d}_H(A,B) := \max\{\delta^*(A,B), \delta^*(B,A)\}.$$

Such a d_H introduces a distance in class $\mathscr{K}(X)$ of all compact sets of X.

For each real–valued function f we denote its graph by $\mathrm{Graph}(f)$, i.e. the set

$$\mathrm{Graph}(f) = \{(\mathbf{x}, y) : \mathbf{x} \in \mathrm{Dom}(f), y = f(\mathbf{x})\}.$$

Since the domain of a subcopula is assumed to be compact, the graph of a subcopula is also compact in $[0,1]^{p+1}$. We are now ready to provide the definition of a distance in $\mathscr{S}$.

Definition 2.3. For every S_1 and S_2 in $\mathscr{S}$, we define

$$\xi(S_1, S_2) := \mathrm{d}_H(\mathrm{Graph}(S_1), \mathrm{Graph}(S_2)),$$

where $\mathrm{Graph}(S_1)$ and $\mathrm{Graph}(S_2)$ are considered subsets of the metric space $[0,1]^{p+1}$ equipped with the Euclidean distance.

Clearly, ξ is a distance in $\mathscr{S}$. Moreover, it holds

Proposition 2.1. *The topology induced by ξ on $\mathscr{C}$ coincides with the topology induced by d_∞ on $\mathscr{C}$.*

Proof. If $C_1, C_2 \in \mathscr{C}$, then $\xi(C_1, C_2) \le \mathrm{d}_\infty(C_1, C_2)$. On the other hand, for every $(x_1, \dots, x_p, C_1(x_1, \dots, x_p))$ there exists $\left(x'_1, \dots, x'_p, C_2\left(x'_1, \dots, x'_p\right)\right)$ such that, for every $i = 1, \dots, p$, $|x_i - x'_i| \le \xi(C_1, C_2)$ and $\left|C_1(x_1, \dots, x_p) - C_2\left(x'_1, \dots, x'_p\right)\right| \le \xi(C_1, C_2)$. The previous inequalities together with the 1–Lipschitz condition for C, imply

$$|C_1(x_1, \dots, x_p) - C_2(x_1, \dots, x_p)| \le (p+1)\,\xi(C_1, C_2).$$

Thus, $\mathrm{d}_\infty(C_1, C_2) \le (p+1)\,\xi(C_1, C_2)$. □

Actually, the class $\mathscr{S}$ equipped with the distance ξ is topologically rich as the following result shows.

Theorem 2.2. *$(\mathscr{S}, \xi)$ is compact.*

The proof of the previous result is based on the following lemma that provides not only a general condition under which the space $(\mathscr{K}(X), \mathrm{d}_H)$ of non-empty compact sets of a metric space X is complete, but also a characterization of the limit of Cauchy sequences in $\mathscr{K}(X)$ (see [2, Theorem 7.1]).

Lemma 2.3 (The Completeness of the Space of Compacts)**.** *Let (X, d) a complete metric space. Then $(\mathscr{K}(X), \mathrm{d}_H)$ is a complete metric space. Moreover, if (K_n) is a Cauchy sequence in $(\mathscr{K}(X), \mathrm{d}_H)$, then the limit*

$$K := \lim_{n\to\infty} K_n \in \mathscr{K}(X),$$

can be characterized as follows:

$$K = \{x \in X : \exists\,(x_n) \text{ a Cauchy sequence with } x_n \in K_n \text{ such that } x_n \to x \text{ as } n \to \infty\}.$$

Proof of Theorem 2.2. We shall prove that, for every sequence (S_n) in $\mathscr{S}$ there exists a subcopula S such that $\xi(S_{\sigma(n)}, S) \longrightarrow 0$ for a suitable subsequence $(S_{\sigma(n)}) \subseteq (S_n)$. Without loss of generality, we can present the proof in the two–dimensional case, since the extension to higher dimensions can be done analogously by suitable modifications.

First, notice that, since $(\mathscr{K}([0,1]^3),\mathrm{d}_H)$ is compact [2, page 38], there exists a subsequence $(S_{\sigma(n)}) \subseteq (S_n)$ and a compact set $K \subseteq [0,1]^3$ such that

$$\mathrm{d}_H(\mathrm{Graph}(S_{\sigma(n)}), K) \longrightarrow 0, \quad \text{as } \sigma(n) \to \infty. \tag{2}$$

By Lemma 2.3, if $(x,y,z) \in K$, then there exists a sequence $(x_{\sigma(n)}, y_{\sigma(n)}, z_{\sigma(n)}) \in \mathrm{Graph}(S_{\sigma(n)})$ that converges to $(x,y,z) \in [0,1]^3$. Moreover, if $(x,y,z),(x,y,z') \in K$, then $z = z'$. In fact, by Lemma 2.3, there exist $(x_{\sigma(n)}, y_{\sigma(n)}, z_{\sigma(n)}),(x'_{\sigma(n)}, y'_{\sigma(n)}, z'_{\sigma(n)}) \in \mathrm{Graph}(S_{\sigma(n)})$ converging to (x,y,z) and (x,y,z'). Thus, for every $\varepsilon > 0$ and for any sufficiently large $\sigma(n)$, it follows that

$$\begin{aligned} &\left|x_{\sigma(n)} - x\right| < \varepsilon, &\quad &\left|x'_{\sigma(n)} - x\right| < \varepsilon, \\ &\left|y_{\sigma(n)} - y\right| < \varepsilon, &\quad &\left|y'_{\sigma(n)} - y\right| < \varepsilon, \\ &\left|z_{\sigma(n)} - z\right| < \varepsilon, &\quad &\left|z'_{\sigma(n)} - z\right| < \varepsilon. \end{aligned}$$

Hence, $\left|x_{\sigma(n)} - x'_{\sigma(n)}\right| < 2\varepsilon$ and $\left|y_{\sigma(n)} - y'_{\sigma(n)}\right| < 2\varepsilon$. Moreover, since S is a Lipschitz function, it follows that $\left|z'_{\sigma(n)} - z_{\sigma(n)}\right| < 4\varepsilon$ and, for the arbitrariness of ε, $z = z'$. The set K is hence the graph of a function S whose domain is contained in $[0,1]^2$. It remains to prove that S is a subcopula.

First, we notice that $\mathrm{Dom}(S) = A \times B$, where A and B include both 0 and 1. To this end, let A be the projection of K with respect to the first coordinate and, analogously, B the projection with respect to the second coordinate. If $(x,y) \in A \times B$, then there exists $(x_{\sigma(n)}, y_{\sigma(n)}, z_{\sigma(n)}) \in \mathrm{Graph}(S_{\sigma(n)})$ such that $(x_{\sigma(n)}, y_{\sigma(n)})$ converges to (x,y). By Lipschitz continuity of S it follows that $(z_{\sigma(n)})$ is a Cauchy sequence and, hence, converges. Thus, $(x,y) \in \mathrm{Dom}(S)$. Moreover, by pointwise convergence, it is clear that S satisfies properties (a) and (b) in Definition 2.1. Therefore, we only need to prove that S is 2-increasing.

Let $\{a,c\} \subset A$, with $a < c$, and $\{b,e\} \subset B$, with $b < e$. We claim that the S–volume of the rectangle $[a,c] \times [b,e]$ is non–negative, i.e.

$$V_S := S(c,e) - S(a,e) - S(c,b) + S(a,b) \geq 0. \tag{3}$$

We proceed by considering each one of the four corner-points $(a,b),(c,b),(a,e)$ and (c,e) in the following way. By (2), there exists $(a_{\sigma(n)}, b_{\sigma(n)}, S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)}))$ converging to $(a,b,S(a,b))$ as n tends to ∞. Thus, for a sufficiently large n, it holds

$$\left|a_{\sigma(n)} - a\right| < \frac{\varepsilon}{16}, \quad \left|b_{\sigma(n)} - b\right| < \frac{\varepsilon}{16}, \quad \left|S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)}) - S(a,b)\right| < \frac{\varepsilon}{16}. \tag{4}$$

In the same way, by (2) there exist the following sequences:

- $(c_{\sigma(n)}, \beta_{\sigma(n)}, S_{\sigma(n)}(c_{\sigma(n)}, \beta_{\sigma(n)}))$ converging to $(c,b,S(c,b))$ as n tends to ∞;
- $(\alpha_{\sigma(n)}, e_{\sigma(n)}, S_{\sigma(n)}(\alpha_{\sigma(n)}, e_{\sigma(n)}))$ converging to $(a,e,S(a,e))$ as n tends to ∞;
- $(\gamma_{\sigma(n)}, \delta_{\sigma(n)}, S_{\sigma(n)}(\gamma_{\sigma(n)}, \delta_{\sigma(n)}))$ converging to $(c,e,S(c,e))$ as n tends to ∞;

from which the analogous of inequalities (4) can be formulated.

By the triangular inequality, it holds

$$\left|a_{\sigma(n)} - \alpha_{\sigma(n)}\right| < \frac{\varepsilon}{8}; \qquad \left|b_{\sigma(n)} - \beta_{\sigma(n)}\right| < \frac{\varepsilon}{8}; \tag{5}$$

$$\left|c_{\sigma(n)} - \gamma_{\sigma(n)}\right| < \frac{\varepsilon}{8}; \qquad \left|e_{\sigma(n)} - \delta_{\sigma(n)}\right| < \frac{\varepsilon}{8}. \tag{6}$$

Now, we will use inequalities (4), (5) and (6), and their analogous versions for the other three corner-points. Therefore, by adding and subtracting the same quantities, we get

$$\begin{aligned} V_S =& S(c,e) - S(a,e) - S(c,b) + S(a,b) \\ & - S_{\sigma(n)}(\gamma_{\sigma(n)}, \delta_{\sigma(n)}) + S_{\sigma(n)}(\gamma_{\sigma(n)}, \delta_{\sigma(n)}) \\ & - S_{\sigma(n)}(\alpha_{\sigma(n)}, e_{\sigma(n)}) + S_{\sigma(n)}(\alpha_{\sigma(n)}, e_{\sigma(n)}) \\ & - S_{\sigma(n)}(c_{\sigma(n)}, \beta_{\sigma(n)}) + S_{\sigma(n)}(c_{\sigma(n)}, \beta_{\sigma(n)}) \\ & - S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)}) + S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)}), \\ & - S_{\sigma(n)}(c_{\sigma(n)}, e_{\sigma(n)}) + S_{\sigma(n)}(c_{\sigma(n)}, e_{\sigma(n)}) \\ & - S_{\sigma(n)}(a_{\sigma(n)}, e_{\sigma(n)}) + S_{\sigma(n)}(a_{\sigma(n)}, e_{\sigma(n)}) \\ & - S_{\sigma(n)}(c_{\sigma(n)}, b_{\sigma(n)}) + S_{\sigma(n)}(c_{\sigma(n)}, b_{\sigma(n)}) \end{aligned}$$

which can be rearranged as

$$\begin{aligned} V_S =& [S(c,e) - S_{\sigma(n)}(\gamma_{\sigma(n)}, \delta_{\sigma(n)})] - [S(a,e) - S_{\sigma(n)}(\alpha_{\sigma(n)}, e_{\sigma(n)})] \\ & - [S(c,b) - S_{\sigma(n)}(c_{\sigma(n)}, \beta_{\sigma(n)})] + [S(a,b) - S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)})] \\ & + [S_{\sigma(n)}(\gamma_{\sigma(n)}, \delta_{\sigma(n)}) - S_{\sigma(n)}(c_{\sigma(n)}, e_{\sigma(n)})] - [S_{\sigma(n)}(\alpha_{\sigma(n)}, e_{\sigma(n)}) - S_{\sigma(n)}(a_{\sigma(n)}, e_{\sigma(n)})] \\ & - [S_{\sigma(n)}(c_{\sigma(n)}, \beta_{\sigma(n)}) - S_{\sigma(n)}(c_{\sigma(n)}, b_{\sigma(n)})] \\ & + S_{\sigma(n)}(c_{\sigma(n)}, e_{\sigma(n)}) - S_{\sigma(n)}(a_{\sigma(n)}, e_{\sigma(n)}) - S_{\sigma(n)}(c_{\sigma(n)}, b_{\sigma(n)}) + S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)}). \end{aligned}$$

By (4), the term

$$\begin{aligned} & [S(c,e) - S_{\sigma(n)}(\gamma_{\sigma(n)}, \delta_{\sigma(n)})] - [S(a,e) - S_{\sigma(n)}(\alpha_{\sigma(n)}, e_{\sigma(n)})] \\ & - [S(c,b) - S_{\sigma(n)}(c_{\sigma(n)}, \beta_{\sigma(n)})] + [S(a,b) - S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)})] \end{aligned}$$

is lower bounded by $-\varepsilon/4$. Since (2) and the Lipschitz condition, the term

$$\begin{aligned} & [S_{\sigma(n)}(\gamma_{\sigma(n)}, \delta_{\sigma(n)}) - S_{\sigma(n)}(c_{\sigma(n)}, e_{\sigma(n)})] - [S_{\sigma(n)}(\alpha_{\sigma(n)}, e_{\sigma(n)}) - S_{\sigma(n)}(a_{\sigma(n)}, e_{\sigma(n)})] \\ & - [S_{\sigma(n)}(c_{\sigma(n)}, \beta_{\sigma(n)}) - S_{\sigma(n)}(c_{\sigma(n)}, b_{\sigma(n)})] \end{aligned}$$

is lower bounded by $-3\varepsilon/8$. Finally, the term

$$S_{\sigma(n)}(c_{\sigma(n)}, e_{\sigma(n)}) - S_{\sigma(n)}(a_{\sigma(n)}, e_{\sigma(n)}) - S_{\sigma(n)}(c_{\sigma(n)}, b_{\sigma(n)}) + S_{\sigma(n)}(a_{\sigma(n)}, b_{\sigma(n)})$$

is non–negative because $S_{\sigma(n)}$ is a subcopula.

Summarizing, for a sufficiently large n, we conclude that $V_S \geq -\varepsilon$, which implies, for the arbitrariness of ε, that $V_S \geq 0$, as we claimed. □

As an immediate consequence, we have

Corollary 2.4. *$(\mathscr{S}, \xi)$ is complete.*

Given a subcopula S, we can provide a standard way to approximate it (in the topology induced by ξ) by another subcopula S^m that is defined on a discrete set of $[0,1]^p$, namely on a mesh (see, e.g., [8]). Specifically, given a closed set $A \subseteq [0,1]$ and the set $\{0, 1/2^m, \dots, (2^m-1)/2^m, 1\}$ for $m \in \mathbb{N}$, we define

$$A^m = \left\{ x \in A : \; d\left(\frac{i}{2^m}, A\right) = d\left(\frac{i}{2^m}, x\right) \text{ for some } i \text{ such that } 0 \le i \le 2^m \right\},$$

where d is the Euclidean distance. Given the subcopula $S \colon A_1 \times \cdots \times A_p \to \mathbb{R}$, we can define the subcopula S^m as the restriction of S to $A_1^m \times \cdots \times A_p^m$.

Lemma 2.5. *Under previous notations, S^m tends to S, as $m \to \infty$, in $(\mathscr{S}, \xi)$.*

Proof. It follows from the fact that

$$\mathrm{d}_H \left(A_1 \times \cdots \times A_p, A_1^m \times \cdots \times A_p^m \right) \le \frac{1}{2^m}$$

and that subcopulas are 1–Lipschitz. □

In the literature, there are various (equivalent) ways to show that a discrete copula, i.e. a subcopula defined on the Cartesian product of p copies of sets of type

$$I_n := \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, 1 \right\},$$

can approximate a copula (see, for instance, Theorems 1 and 2 in [20]). The distance ξ previously introduced allows a unified setting to deal with such problems. In fact, as a direct consequence of Lemma 2.5, it follows that

Corollary 2.6. *Let C be a copula and let C^m be a discrete copula on $I_{2^m} \times \cdots \times I_{2^m}$ such that $C^m = C$ on $\mathrm{Dom}(C^m)$. Then C^m ξ–converges to C, as $m \to \infty$.*

Actually, the previous result does not depend on the specific choice of the domain of the discrete copula, but only on the fact that the diameter of the mesh is getting smaller as m increases.

Remark 2.1. Suppose that S_1 and S_2 are two subcopulas that approximate a given target copula C, i.e. $S_1 = C$ on $\mathrm{Dom}(S_1)$ and $S_2 = C$ on $\mathrm{Dom}(S_2)$. If $\mathrm{Dom}(S_1) \subseteq \mathrm{Dom}(S_2)$, then $\xi(S_1, C) \ge \xi(S_2, C)$.

However, if S_1 and S_2 coincide with C on the respective domains, but $\mathrm{Dom}(S_1) \neq \mathrm{Dom}(S_2)$, then it may happen that $\xi(S_1, C) \le \xi(S_2, C)$ even if the (two–dimensional) Lebesgue measure of $\mathrm{Dom}(S_2)$ is strictly greater than the Lebesgue measure of $\mathrm{Dom}(S_1)$. For instance, let C be any bivariate copula and consider a discrete copula C^m as in Corollary

2.6 and a subcopula S_2 defined on $(\{0\} \cup [a,1])^2$ for a fixed $a > 0$. Then, for every a it is possible to find $m > 0$ such that $\xi(C, C^m) < \xi(C, S_2)$, while the domain of C^m has obviously Lebesgue measure equal to 0.

Roughly speaking, the distance ξ provides a way to compare the quality of two different subcopula approximations of a given target copula C taking into account both the domain and the range of the approximating functions.

In order to conclude our overview about analytical aspects of subcopulas, we present a Baire category result for subsets of subcopulas. As discussed in [10], the Baire category of a set allows to understand its relative size compared to the whole space.

Theorem 2.7. *Let $\mathscr{S}$ be the class of subcopulas equipped with he metric ξ. Then*

(a) $\mathscr{C}$ is nowhere dense in $\mathscr{S}$.

(b) The class of discrete copulas is of first category in $\mathscr{S}$.

Proof.
(a): It is well know that $\mathscr{C}$ is a closed set (with respect to d_∞). In view of Proposition 2.1, $\mathscr{C}$ is also a closed subset of $(\mathscr{S}, \xi)$. Moreover, since every $C \in \mathscr{C}$ can be approximated by discrete copulas, as an immediate consequence $\mathscr{C}$ cannot contain any nonempty open subset of $(\mathscr{S}, \xi)$ and, hence, it is nowhere dense in $\mathscr{S}$.
(b): For every integer $k \geq 2$, let $\mathscr{S}_k$ be the class of discrete copulas defined on the subset $I_n \times \cdots \times I_n$, where $n \leq k$.

First, we show that $\mathscr{S}_k$ is closed. To this end, suppose that there exists a sequence $(S_n)_n$ in $\mathscr{S}_k$ that converges to $S \in \mathscr{S}$. Let π^p be the projection of the first p coordinates of a point in $[0,1]^{p+1}$ on $[0,1]^p$. Since $\mathrm{d}_H\left(\pi^p(\mathrm{Graph}(S)), \pi^p(\mathrm{Graph}(S_n))\right) \leq \xi\left(S, S_n\right)$, $\{\pi^p(S_n)\}$ converges to a compact set $\pi^p(S) \subseteq [0,1]^p$. It holds that $\pi^p(S)$ is formed by (at most) k points. In fact, on the contrary, there would exist $K = \{\mathbf{x}_i \in K \colon i = 1, ..., k+1\} \subset \pi^p(S)$ such that the minimal distance among the points in K is $\varepsilon > 0$. Since $\delta^*(\pi^p(S), \pi^p(S_n))$ converges to 0 as $n \to \infty$, it holds that $\delta^*(K, \pi^p(S_n)) < \varepsilon/4$ for a sufficiently large n. However, the condition $\delta^*\left(K, \pi^p\left(S_n\right)\right) < \varepsilon/4$ together with the cardinality of $\pi^p(S)$ implies that there exists a point in $\pi^p\left(S_n\right)$ and a ball centered in it with radius $\varepsilon/4$ that contains two points of K. But, this is a contradiction since the distance among these two points would be less than $\varepsilon/2$. Thus, $S \in \mathscr{S}_k$ and $\mathscr{S}_k$ is a closed set.

Moreover, $\mathscr{S}_k$ has empty interior. In fact, given $S \in \mathscr{S}_k$, we can consider its extension to a copula, denoted by S_{ch}, via checkerboard construction (see, e.g., [5]). For $\varepsilon > 0$ and every $x_{i,k} \in \pi_i(\mathrm{Dom}(S))$, $i = 1, \ldots, p$, π_i the i–th canonical projection, we consider the set $A_i = \bigcup_k B\left(x_{i,k}, \frac{\varepsilon}{4p}\right) \cap [0,1]$. Let S'_{ch} be the subcopula obtained as a restriction of S_{ch} to $A_1 \times \cdots \times A_p$. From the definition of A_i and the 1–Lipschitz condition for a subcopula, we obtain $\delta^*\left(\mathrm{Graph}(S'_{ch}), \mathrm{Graph}(S)\right) < \frac{p\varepsilon}{4p} + \frac{p\varepsilon}{4p} = \frac{\varepsilon}{2}$. Thus, for the arbitrariness of ε, $\mathscr{S}_k$ does not contain any open set because $S'_{ch} \notin \mathscr{S}_k$.

Thus, $\mathscr{S}_k$ is a closed set with empty interior, i.e. it is nowhere dense.

Finally, since the class of discrete copulas is the countable union of all the nowhere dense sets $\mathscr{S}_k$ for $k \in \mathbb{N}$, it follows that it is a set of first category in $(\mathscr{S}, \xi)$. □

2.1. Existence of extensions of subcopulas and Sklar's Theorem

The introduction of the metric ξ in the class of subcopulas also provides an alternative proof of the fact that any subcopula can be extended to a copula. As stressed several times in the literature (see, for instance, [11, Section 2.3.1]), this is the fundamental step to provide an analytical proof of Sklar's Theorem for non–continuous random variables. Before stating this result, we need several preliminary results.

Lemma 2.8. *Let $S \in \mathscr{S}$ and $C \in \mathscr{C}$. If $\delta^*(\mathrm{Graph}(S), \mathrm{Graph}(C)) = 0$, then C is an extension of S. Moreover,*

$$\Delta_S : \mathscr{C} \to \mathbb{R}, \qquad \Delta_S(C) = \delta^*(\mathrm{Graph}(S), \mathrm{Graph}(C))$$

is continuous with respect to the topology induced by ξ in $\mathscr{C}$.

Proof. The first property of δ^* follows directly from the definition. The continuity of Δ_S follows from the fact that, for all $C_1, C_2 \in \mathscr{C}$, $\Delta_S(C_2) \leq \Delta_S(C_1) + \xi(C_1, C_2)$. □

Since $\mathscr{C}$ is a closed set in $(\mathscr{S}, \xi)$, it is also compact. Thus,

$$\inf_{C \in \mathscr{C}} \delta^*(\mathrm{Graph}(S), \mathrm{Graph}(C)) = \min_{C \in \mathscr{C}} \delta^*(\mathrm{Graph}(S), \mathrm{Graph}(C))$$

for every $S \in \mathscr{S}$.

Lemma 2.9. *The mapping $\Delta : \mathscr{S} \to \mathbb{R}$ with $\Delta(S) = \min_{C \in \mathscr{C}} \Delta_S(C)$ is continuous with respect to $(\mathscr{S}, \xi)$.*

Proof. For every $S_1, S_2 \in \mathscr{S}$ it can be proved that $\Delta_{S_1}(C) \leq \Delta_{S_2}(C) + \xi(S_1, S_2)$ and, analogously, $\Delta(S_2) \leq \Delta(S_1) + \xi(S_1, S_2)$. Thus, $|\Delta(S_1) - \Delta(S_2)| \leq \xi(S_1, S_2)$, from which the continuity of Δ follows. □

Theorem 2.10. *Every subcopula can be extended to a copula.*

Proof. Let $S \in \mathscr{S}$. In view of Lemma 2.5, it follows that there exists a sequence (S^m) of subcopulas defined on a mesh such that $S^m \to S$ as $m \to \infty$. Moreover, by Lemma 2.9, $\Delta(S^m) \to \Delta(S)$ as m goes to ∞. Since the domain of S^m is a mesh, it can be extended to a copula via checkerboard techniques (see, for instance, [8]), so that $\Delta(S^m) = 0 = \Delta(S)$. Thus, there exists $C \in \mathscr{C}$ such that $\Delta_S(C) = 0$, which implies that C is an extension of S. □

The previous result does not describe, obviously, all the ways to construct the copulas that extend a subcopula; this task is considered in the following section.

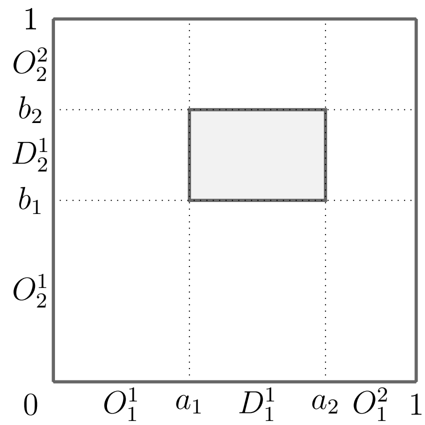

Figure 1: Graphical example of the main notation used with $A_1 = \{0\} \cup [a_1, a_2] \cup \{1\}$ and $A_2 = \{0\} \cup [b_1, b_2] \cup \{1\}$. Here, $T_{1,1} = O_1^1$, $T_{1,1} = D_1^1$ and $T_{1,1} = O_1^2$. Analogously, $T_{2,1} = O_2^1$, $T_{2,2} = D_2^1$ and $T_{2,3} = O_2^2$.

3. Construction of all extensions of a subcopula to a copula

In the following, we provide the analytical description of all possible copulas C that coincide with a subcopula S on $\mathrm{Dom}(S)$.

Consider a subcopula $S\colon A_1 \times \cdots \times A_p \to [0,1]$ where, without loss of generality, $A_1, \ldots, A_p$ are assumed to be closed. For $j = 1, \ldots, p$, A_j is the union of singletons and (countable many) closed intervals D_j^i whose interiors are nonempty. Moreover, $[0,1] \setminus A_j$ is the union of (countable many) disjoint open intervals O_j^i. Define the following subsets:

- $\mathscr{D}_j = \{D_j^1, D_j^2, \ldots\}$;
- $\mathscr{O}_j = \{\overline{O_j^1}, \overline{O_j^2}, \ldots\}$;
- $\mathscr{P}_j = [0,1] \setminus \left(\bigcup_i D_j^i \cup \bigcup_i \overline{O_j^i}\right)$.

Moreover, for $j = 1, \ldots, p$, set

$$\mathscr{T}_j := \{T_{j,t} = [a_{j,t}, b_{j,t}]\,; T_{j,t} \in \mathscr{D}_j \cup \mathscr{O}_j\}_{t \in I_j}\,,$$

where I_j is the index set of the same cardinality as $\mathscr{D}_j \cup \mathscr{O}_j$. In dimension 2, an example is depicted in Figure 1.

Assume that, for every $j = 1, \ldots, p$, the following condition holds:

$$\mathscr{P}_j \text{ has Lebesgue measure equal to 0.} \tag{7}$$

If $C \in \mathscr{C}$ is an extension of the subcopula S, then condition (7) ensures that there exists a countable union of boxes of type $\times_{j=1}^p \left[a_{j,t_j}, b_{j,t_j}\right]$, where $\left[a_{j,t_j}, b_{j,t_j}\right] \in \mathscr{T}_j$ for every $j = 1, \ldots, p$, such that the total C–volume of all such boxes is equal to 1.

Next, let us define auxiliary functions associated to the elements of $\mathscr{T}_j$.

- For every $T_{j,t} \in \mathscr{O}_j$ and $T_{1,t} \in \mathscr{T}_1, \dots, T_{p,t} \in \mathscr{T}_p$, we select a family of distribution functions on $[0,1]$, denoted by $F_{j,t_1,\dots,t_p}$ with $t_j = t$, whose restriction to $]0,1[$ satisfies the condition that, for all $u \in]0,1[$,

$$u = \frac{1}{b_{j,t} - a_{j,t}} \sum_{\substack{t_s \in I_s; s \neq j \\ t_j = t}} \beta_{t_1 \dots t_d} F_{j,t_1 \dots t_p}(u), \tag{8}$$

where

$$\beta_{t_1 \dots t_p} = V_S \left(\times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}] \right). \tag{9}$$

Notice that, by the definition of S, $(b_{j,t} - a_{j,t})$ is the S–volume of the box obtained as Cartesian product of $(p-1)$ copies of $[0,1]$ and the interval $[a_{j,t}, b_{j,t}]$, which occupies the j–th position. As a consequence, there exists at least one family $\{F_{j,t_1,\dots,t_p}\}$ that satisfies (8). In fact, it is enough to assume that each function is the identity function on $[0,1]$, but this family need not be uniquely determined.

- If $T_{j,t_j} \in \mathscr{D}_j$ for every $j = 1, \dots, p$, we set $\beta_{\mathbf{t}} = V_S \left(\times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}] \right)$.
If $\beta_{\mathbf{t}} = 0$, then $F_{j,\mathbf{t}}$ denotes the step function at 1 and $C_{\mathbf{t}}$ denotes any p–dimensioanl copula.
If $\beta_{\mathbf{t}} > 0$, then we consider the distribution function on $[0,1]^p$ given by

$$H_{\mathbf{t}}(x_1, \dots, x_p) = \frac{1}{\beta_{\mathbf{t}}} V_S \left(\times_{1 \le s \le p} [a_{s,t_s}, (b_{s,t_s} - a_{s,t_s}) x_i + a_{s,t_s}] \right)$$

and its univariate marginals

$$F_{j,\mathbf{t}}(x_i) = \frac{1}{\beta_{\mathbf{t}}} V_S \left(\left(\times_{1 \le s < j} \left([a_{s,t_s}, b_{s,t_s}] \right) \times \left[a_{j,t_j}, \left(b_{j,t_j} - a_{j,t_j} \right) x_j + a_{j,t_j} \right] \right) \times \left(\times_{j < s \le p} [a_{s,t_s}, b_{s,t_s}] \right) \right).$$

In view of Sklar's Theorem, since $F_{j,\mathbf{t}}$ are continuous, we denote by $C_{\mathbf{t}}$ the unique copula that satisfies

$$H_{\mathbf{t}}(x_1, \dots, x_p) = C_{\mathbf{t}}(F_{1,\mathbf{t}}(x_1), \dots, F_{p,\mathbf{t}}(x_p)). \tag{10}$$

- Moreover, consider the case where there are intervals $[a_{s,t_s}, b_{s,t_s}] \in \mathscr{D}_s$ and intervals $\left[a_{s',t_{s'}}, b_{s',t_{s'}}\right] \in \mathscr{O}_{s'}$. Without loss of generality, suppose that, for $s \in \{1, \dots, m\}$, $[a_{s,t_s}, b_{s,t_s}] \in \mathscr{D}_s$, and for $s \in \{m+1, \dots, p\}$, $[a_{s,t_s}, b_{s,t_s}] \in \mathscr{O}_s$. Thus, in the box $\times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}]$, the information about the values of S is given by the first m intervals, while the other $p-m$ intervals do not constraint the choice of C and the distribution functions $F_{j,\mathbf{t}}$ can be defined according to (8).
Consider the $m-$dimensional distribution function defined, for $\mathbf{x} \in [0,1]^m$, by

$$H_{\mathbf{t}}(\mathbf{x}) = \frac{1}{\beta_{\mathbf{t}}} V_S \left(\times_{1 \le s \le m} [a_{s,t_s}, (b_{s,t_s} - a_{s,t_s}) x_i + a_{s,t_s}] \times \left(\times_{m < s \le p} [a_{s,t_s}, b_{s,t_s}] \right) \right)$$

and, for $i = 1, \dots, p$, the one-dimensional distribution function

$$F_{i,\mathbf{t}}(x_i) = \frac{1}{\beta_{\mathbf{t}}} V_S \left(\times_{1 \le s < i} [a_{s,t_s}, b_{s,t_s}] \times [a_{i,t_i}, (b_{i,t_i} - a_{i,t_i})\, x_i + a_{i,t_i}] \times (\times_{i < s \le p} [a_{s,t_s}, b_{s,t_s}]) \right).$$

Obviously, $F_{i,\mathbf{t}}$ is the i–th marginal of $H_{\mathbf{t}}$, so that, for $m \geq 2$, there exists an $m-$copula $C'_{\mathbf{t}}$ such that

$$H_{\mathbf{t}}(x_1, \dots, x_m) = C'_{\mathbf{t}}(F_{1,\mathbf{t}}(x_1), \dots, F_{m,\mathbf{t}}(x_m)) \tag{11}$$

Here, for $m = 1$, we interpret a one–dimensional copula as the uniform distribution on $[0, 1]$. Moreover, we set $C_{\mathbf{t}}$ as a p–dimensional extension of $C'_{\mathbf{t}}$.

- Finally, for $j = 1, \dots, p$ and $(t_1 \dots t_p) \in \times_{j=1}^p I_j$, we set

$$U_{j,t_1 \dots t_d}(x_j) = \begin{cases} 0, & \text{if } x_j < a_{j,t_j}, \\ \dfrac{x_j - a_{j,t_j}}{b_{j,t_j} - a_{j,t_j}}, & \text{if } a_{j,t_j} \le x_j \le b_{j,t_j}, \\ 1, & \text{if } x_j > a_{j,t_j}. \end{cases} \tag{12}$$

As a consequence of the functions defined above, a similar procedure as in [6] can be used in the multivariate case to prove the following result.

Theorem 3.1. *Under the previous notations, if condition* (7) *holds, then the following statements are equivalent:*

(a) $C \in \mathscr{C}$ *is an extension of the subcopula* S*;*

(b) for every $\mathbf{u} \in [0, 1]^p$*,* C *can be expressed in the form*

$$C(\mathbf{u}) = \sum_{\mathbf{t} \in \times_{j=1}^p I_j} \beta_{\mathbf{t}} C_{\mathbf{t}} \left(F_{1,\mathbf{t}} \left(U_{1,\mathbf{t}}(u_1) \right), \dots, F_{p,\mathbf{t}} \left(U_{p,\mathbf{t}}(u_p) \right) \right). \tag{13}$$

Proof. If $C \in \mathscr{C}$ is an extension of the subcopula S, then condition (7) ensures that there exists a (finite or countable) union of boxes of type $\times_{j=1}^p \left[a_{j,t_j}, b_{j,t_j}\right]$, where $\left[a_{j,t_j}, b_{j,t_j}\right] \in \mathscr{T}_j$ for every $j = 1, \dots, p$, such that the total C–volume of all such boxes is equal to 1.

Now, if $\mathbf{U}$ is a random vector (on a suitable probability space) distributed according to C, it follows that

$$\begin{aligned} C(\mathbf{u}) &= \sum_{\mathbf{t} \in \times_{j=1}^p I_j} \mathbb{P}\left(\mathbf{U} \in [0, \mathbf{u}] \cap \left(\times_{j=1}^p \left[a_{j,t_j}, b_{j,t_j}\right]\right)\right) \\ &= \sum_{\mathbf{t} \in \times_{j=1}^p I_j, \beta_{\mathbf{t}} > 0} \beta_{\mathbf{t}} \left(\beta_{\mathbf{t}}^{-1} \mathbb{P}\left(\mathbf{U} \in [0, \mathbf{u}] \cap \left(\times_{j=1}^p \left[a_{j,t_j}, b_{j,t_j}\right]\right)\right)\right). \end{aligned}$$

Since $H_{\mathbf{t}}(\mathbf{u}) = \beta_{\mathbf{t}}^{-1}\mathbb{P}\left(\mathbf{U} \in [0,\mathbf{u}] \cap \left(\times_{j=1}^{p} \left[a_{j,t_j}, b_{j,t_j}\right]\right)\right)$ is a distribution function that concentrates the probability mass in $\times_{j=1}^{p} \left[a_{j,t_j}, b_{j,t_j}\right]$, Sklar's Theorem ensures that it can be represented in terms of a copula $C_{\mathbf{t}}$ and its univariate marginals. Now, the expression (13) can be recovered by using the previous notations.

Conversely, first, we show that a function C of type (13) is a copula that extends the subcopula S. To this end, notice that C is d–increasing, since it is the sum of d–increasing functions.

To show that C is an extension of S, we consider that, if $\mathbf{u} \in \mathrm{Dom}(S)$, formulas (8), (9), (10) and (11) allow to reduce C in the form

$$C(\mathbf{u}) = \sum_{\mathbf{t} \in \times_{j=1}^{p} I_j} V_S\left([\mathbf{0},\mathbf{u}] \cap \times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}]\right)$$

If the previous sum consists of a finite number q of addenda, it is possible to cover the set $\bigcup_{\mathbf{t} \in \times_{j=1}^{p} I_j} \left(\times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}]\right)$ with a set of type $\bigcup_{k=1,\dots,q'} R_k$, q' finite, where all R_k's are rectangles such that: their vertices belong to $\mathrm{Dom}(S)$, their interiors are disjoint, their union is equal to $[\mathbf{0},\mathbf{u}]$. Since V_S is finitely additive, it holds

$$\sum_{\mathbf{t} \in \times_{j=1}^{p} I_j} V_S\left([\mathbf{0},\mathbf{u}] \cap \times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}]\right) \le \sum_{k=1,\dots,q'} V_S\left([\mathbf{0},\mathbf{u}] \cap R_k\right) = S(\mathbf{u}).$$

Since this holds true for any finite number q, it holds in general that $C(\mathbf{u}) \le S(\mathbf{u})$.

On the other hand, in view of condition (7), given $\varepsilon > 0$ it is possible to find, for every index s, a finite union of rectangles $[a_{s,t_s}, b_{s,t_s}]$, t_s belonging to a finite index set $I_{s,\varepsilon}$, such that

$$\lambda\left([0,u_s] \bigcap \left(\bigcup_{t_s \in I_{s,\varepsilon}} [a_{s,t_s}, b_{s,t_s}]\right)\right) > u_s - \varepsilon.$$

Thus, again for the finite additivity of V_S and the boundary conditions for a subcopula, it holds

$$\begin{aligned} S(\mathbf{u}) = V_S([\mathbf{0},\mathbf{u}]) &\le \sum_{\mathbf{t} \in \times_{s=1}^{p} I_{s,\varepsilon}} V_S\left([\mathbf{0},\mathbf{u}] \cap \times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}]\right) + p\varepsilon \\ &\le \sum_{\mathbf{t} \in \times_{j=1}^{p} I_j} V_S\left([\mathbf{0},\mathbf{u}] \cap \times_{s=1}^{p} [a_{s,t_s}, b_{s,t_s}]\right) + p\varepsilon = C(\mathbf{u}) + p\varepsilon, \end{aligned}$$

from which $S(\mathbf{u}) \le C(\mathbf{u})$. Hence, $C = S$ on $\mathrm{Dom}(S)$.

Finally, C has uniform marginals. In fact, consider without loss of generality $u_1 \in [0,1]$. If $u_1 \in \mathrm{Dom}(S)$, then $C(u_1, 1, \dots, 1) = u_1$. Otherwise, consider the maximal point such that $b_{1,t_1} \le u_1$. Conditions (8) and (9) together with $C(b_{1,t_1}, 1, \dots, 1) = \sum_{\mathbf{t} \in \times_{j=1}^{p} I_j} \beta_{\mathbf{t}} C_{\mathbf{t}}\left(F_{1,\mathbf{t}}\left(U_{1,\mathbf{t}}(u_1)\right), 1, \dots, 1\right)$ imply that $C(u_1, 1, \dots, 1) = u_1$, which concludes the proof. □

Remark 3.1. Suppose that the following conditions are satisfied:

- all the distribution functions $F_{j,\mathbf{t}}$ in (8) are uniform on $[0,1]$;
- Every $C_{\mathbf{t}}$ that extends an m–dimensional copula C_m to a d–dimensional copula with $d > m \geq 2$ is obtained by multiplying C_m with the remaining variables (i.e. by assuming independence from C_m).
- All the copulas that can be freely chosen in eq. (13) are assumed to be equal to the independence copula.

Then the copula of (13) coincides with the multilinear copula extension by [31].

Remark 3.2. Here we provide an example where condition (7) is not satisfied. To this end, we consider a generalized version of Cantor set, also known as Smith–Volterra–Cantor set. We start with the closed interval $[0,1]$. In the first iteration, we remove an open interval of length $1/2^2$ centered at $1/2$ from $[0,1]$. In the second iteration, we remove an open interval of length $1/2^4$ from the center of any of the closed subintervals obtained in the previous step. In general, at the k–th iteration we remove an open interval of length $1/2^{2k}$ from any of the closed intervals obtained at previous step. The Smith–Volterra–Cantor set is then formed by all the points that are never removed by the previous iterations.

Now, if A_1 is a set of previous type, then it contains no intervals and therefore has empty interior (thus it cannot contain any open interval D_1^i). A_1 is also the intersection of a sequence of closed sets, which means that it is closed. It has a positive Lebesgue measure. Moreover, $\lambda(\mathscr{P}_1) = 1/2$. For more details about this construction, see for instance [15].

3.1. The bivariate case

In the two dimensional case, Theorem 3.1 has been proved in [6] under the assumption (not explicitly stated in the paper) given in (7) (see also [1], where this fact was noticed for the first time). However, as it was seen before, there are cases when this assumption is not satisfied. In this latter case, the representation of all subcopula extensions of Theorem 3.1 can be modified as follows (for an alternative expression, see also [1, Theorem 3.2.1]).

Let C be a copula that extends S.

- Suppose that $\lambda(\mathscr{P}_2) > 0$. In view of the Disintegration theorem (see, e.g., [14]), for every Borel $B \subseteq [0,1]^2$, the probability measure μ_C associated with C can be expressed as

  $$\mu_C(B) = \int_{[0,1]} K_C(B_v, v) dv,$$

  where $B_v := \{u \in [0,1] : (u,v) \in B\}$ and K_C is the so-called Markov kernel of C.

  Let c be the boundary point of one of the intervals of $\mathscr{T}_1$. Let $\varphi_c(z) = C(c,z)$ be the measure-generating function of $K_C(\cdot, z)$ on the Borel sets of $[0,c]$. Since C is a Lipschitz function, it follows that φ_c is absolutely continuous.

We can define the function $f_c : [0,1] \to [0,c]$, given by

$$f_c(z) = \sup \{S(c,v) : v \le z \text{ and } v \in \mathscr{P}_2\}.$$

Such a f_c is a monotone function, it is derivable almost everywhere in its domain and, furthermore, the set of points of $\mathscr{P}_2$ where it admits a derivative has Lebesgue measure $\lambda(\mathscr{P}_2)$. Now, f_c coincides with φ_c on $[0,1] \setminus \cup_t T_{2,t}$. Moreover, it is the measure-generating function of $K_C(\cdot, z)$ for $z \in \mathscr{P}_2$. In other words, f_c generates a measure on $[0,1]$ that has an absolutely continuous component and a discrete component that concentrates the probability of all intervals of type $T_{2,t}$ in its extreme (upper) points.

Thus, if $[a_t, b_t] \in \mathscr{O}_1$, then, following the previous results, we replace condition (8) with

$$u = \frac{1}{b_t - a_t} \left(\sum_j \beta_{tj} F_{t,j}(u) + \int_0^1 \left(f'_{b_t}(s) - f'_{a_t}(s)\right) F_{t,s}(u) ds \right) \text{ for all } u \in [0,1]. \quad (14)$$

Now, $F_{t,j}$ and $F_{t,s}$ are distribution functions. Actually, one can choose $F_{t,s}$ only for the elements of $\mathscr{P}_2$, since in the complementary set the derivative vanishes almost everywhere.

- Analogously, let $\lambda(\mathscr{P}_1) > 0$. Thus, for every $[c_t, \mathrm{d}_t] \in \mathscr{O}_2$ we replace condition (8) with

$$u = \frac{1}{d_j - c_j} \left(\sum_t \beta_{tj} G_{t,j}(u) + \int_0^1 \left(g'_{d_j}(s) - g'_{c_j}(s)\right) G_{s,j}(u) ds \right) \text{ for all } u \in [0,1], \quad (15)$$

where $g_c : [0,1] \to [0,c]$, given by

$$g_c(z) = \sup \{S(u,c) : u \le z \text{ and } u \in \mathscr{P}_1\}.$$

Under the previous notations, a bivariate copula C extends a subcopula S if, and only if, it can be written in the following form:

- if $(u,v) \in \mathrm{Dom}(S)$, then $C(u,v) = S(u,v)$;
- if $(u,v) \notin \mathrm{Dom}(S)$ and $(u,v) \in [a_i, b_i] \times [c_j, d_j] = T_{1,i} \times T_{2,j}$, with $T_{1,i} \in \mathscr{T}_1$ and $T_{2,j} \in \mathscr{T}_2$, then

$$\begin{aligned} C(u,v) =& S(a_i, c_j) + \beta_{ij} C_{ij} \left(F_{i,j}\left(\frac{u - a_i}{b_i - a_i}\right), G_{i,j}\left(\frac{v - c_j}{d_j - c_j}\right)\right) \\ &+ \sum_{i' \in S_i} \beta_{i'j} G_{i',j}\left(\frac{v - c_j}{d_j - c_j}\right) + \sum_{j' \in Z_j} \beta_{ij'} F_{i,j'}\left(\frac{u - a_i}{b_i - a_i}\right) \\ &+ \int_0^v \left(f'_{b_i}(s) - f'_{a_i}(s)\right) F_{i,s}(u) ds + \int_0^u \left(g'_{d_j}(s) - g'_{c_j}(s)\right) G_{s,j}(v) ds. \end{aligned}$$

where $C_{tj} \in \mathscr{C}$, $F_{i,j}$ and $G_{i,j}$ are distribution functions satisfying (14) and (15), with $S_i = \{t' : a_{t'} < a_i\}$ and $Z_j = \{j' : c_{j'} < c_j\}$.

- if $(u, v) \in (T_{1,i} \times \mathscr{P}_2) \setminus \mathrm{Dom}(S)$, then

$$\begin{aligned} C(u,v) =& S(a_i, v) + \beta_{ij} C_{ij}\left(F_{i,j}\left(\frac{u-a_i}{b_i-a_i}\right), G_{i,j}\left(\frac{v-c_j}{d_j-c_j}\right)\right) \\ &+ \sum_{j' \in Z_j} \beta_{ij'} F_{i,j'}\left(\frac{u-a_i}{b_i-a_i}\right) + \int_0^v \left(f'_{b_i}(s) - f'_{a_i}(s)\right) F_{i,s}(u) ds. \end{aligned}$$

- if $(u, v) \in (\mathscr{P}_1 \times T_{2,t}) \setminus \mathrm{Dom}(S)$, then

$$\begin{aligned} C(u,v) =& S(u, b_j) + \beta_{ij} C_{ij}\left(F_{i,j}\left(\frac{u-a_i}{b_i-a_i}\right), G_{i,j}\left(\frac{v-c_j}{d_j-c_j}\right)\right) \\ &+ \sum_{i' \in S_i} \beta_{i'j} G_{i',j}\left(\frac{v-c_j}{d_j-c_j}\right) + \int_0^u \left(g'_{d_j}(s) - g'_{c_j}(s)\right) G_{s,j}(v) ds. \end{aligned}$$

The proof can be done analogous to [6].

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References

[1] A. Ajavon. *Sur les prolongements de sous-copules.* PhD thesis, Université de Montréal, 2015.

[2] M. F. Barnsley. *Fractals everywhere.* Academic Press, Boston, 1988.

[3] A. Bücher and I. Kojadinovic. An overview of nonparametric tests of extreme-value dependence and of some related statistical procedures. In D. Dey and J. Yan, editors, *Extreme Value Modeling and Risk Analysis: Methods and Applications*, pages 377–398. CRC Press, Boca Raton, FL, 2016.

[4] H. Carley. Maximum and minimum extensions of finite subcopulas. *Comm. Statist. Theory Methods*, 31(12):2151–2166, 2002.

[5] H. Carley and M. D. Taylor. A new proof of Sklar's theorem. In C. M. Cuadras, J. Fortiana, and J. A. Rodríguez-Lallena, editors, *Distributions with given marginals and statistical modelling*, pages 29–34. Kluwer Acad. Publ., Dordrecht, 2002.

[6] E. de Amo, M. Díaz-Carrillo, and J. Fernández-Sánchez. Characterization of all copulas associated with non-continuous random variables. *Fuzzy Sets and Systems*, 191:103–112, 2012.

[7] P. Deheuvels. Propriétés d'existence et propriétés topologiques des fonctions de dépendance avec applications à la convergence des types pour des lois multivariées. *C. R. Acad. Sci. Paris Sér. A-B*, 288(2):A145–A148, 1979.

[8] F. Durante, J. Fernández-Sánchez, J. J. Quesada-Molina, and M. Úbeda-Flores. Convergence results for patchwork copulas. *European J. Oper. Res.*, 247(2):525–531, 2015.

[9] F. Durante, J. Fernández-Sánchez, and C. Sempi. Sklar's theorem obtained via regularization techniques. *Nonlinear Anal.*, 75(2):769–774, 2012.

[10] F. Durante, J. Fernández-Sánchez, and W. Trutschnig. A typical copula is singular. *J. Math. Anal. Appl.*, 430:517–527, 2015.

[11] F. Durante and C. Sempi. *Principles of copula theory.* CRC Press, Boca Raton, FL, 2016.

[12] O. P. Faugeras. Sklar's theorem derived using probabilistic continuation and two consistency results. *J. Multivariate Anal.*, 122:271–277, 2013.

[13] J.-D. Fermanian, D. Radulović, and M. Wegkamp. Weak convergence of empirical copula processes. *Bernoulli*, 10(5):847–860, 2004.

[14] J. Fernández-Sánchez and W. Trutschnig. Conditioning-based metrics on the space of multivariate copulas and their interrelation with uniform and levelwise convergence and iterated function systems. *J. Theoret. Probab.*, 28(4):1311–1336, 2015.

[15] B. R. Gelbaum and J. M. H. Olmsted. *Counterexamples in analysis.* Dover Publications, Inc., Mineola, NY, 2003. Corrected reprint of the second (1965) edition.

[16] C. Genest and J. Nešlehová. A primer on copulas for count data. *Astin Bull.*, 37(2):475–515, 2007.

[17] C. Genest, J.G. Nešlehová, and B. Rémillard. On the empirical multilinear copula process for count data. *Bernoulli*, 20(3):1344–1371, 2014.

[18] H. Joe. *Dependence modeling with copulas*, volume 134 of *Monographs on Statistics and Applied Probability.* CRC Press, Boca Raton, FL, 2015.

[19] I. Kojadinovic. Some copula inference procedures adapted to the presence of ties. *Comput. Statist. Data Anal.*, in press, 2017.

[20] A. Kolesárová, R. Mesiar, J. Mordelová, and C. Sempi. Discrete copulas. *IEEE T. Fuzzy Systems*, 14(5):698–705, 2006.

[21] J.-F. Mai and M. Scherer. *Simulating copulas.* Imperial College Press, London, 2012.

[22] A. W. Marshall. Copulas, marginals, and joint distributions. In L. Rüschendorf, B. Schweizer, and M.D. Taylor, editors, *Distributions with fixed marginals and related topics (Seattle, WA, 1993)*, volume 28 of *IMS Lecture Notes Monogr. Ser.*, pages 213–222. Inst. Math. Statist., Hayward, CA, 1996.

[23] G. Mayor, J. Suñer, and J. Torrens. Copula–like operations on finite settings. *IEEE Trans. Fuzzy Syst.*, 13:468–477, 2005.

[24] M. Mesfioui and J.-F. Quessy. Concordance measures for multivariate non-continuous random vectors. *J. Multivariate Anal.*, 101(10):2398–2410, 2010.

[25] R. B. Nelsen. *An Introduction to Copulas.* Springer Series in Statistics. Springer, New York, second edition, 2006.

[26] J. Nešlehová. On rank correlation measures for non-continuous random variables. *J. Multivariate Anal.*, 98(3):544–567, 2007.

[27] M. Omelka, I. Gijbels, and N. Veraverbeke. Improved kernel estimation of copulas: weak convergence and goodness-of-fit testing. *Ann. Stat.*, 37(5B):3023–3058, 2009.

[28] R. Pappadà, F. Durante, and G. Salvadori. Quantification of the environmental structural risk with spoiling ties: is randomization worthwhile? *Stoch. Environ. Res Risk Asess.*, 2016.

[29] J. J. Quesada Molina and C. Sempi. Discrete quasi–copulas. *Insurance Math. Econom.*, 37(1):27–41, 2005.

[30] L. Rüschendorf. On the distributional transform, Sklar's Theorem, and the empirical copula process. *J. Statist. Plan. Infer.*, 139(11):3921–3927, 2009.

[31] B. Schweizer and A. Sklar. Operations on distribution functions not derivable from operations on random variables. *Studia Math.*, 52:43–52, 1974.

[32] J. Segers, M. Sibuya and H. Tsukahara. The empirical beta copula. *J. Multivariate Anal.*, 155:35–51, 2017.

[33] Q. Shan, T. Wongyang, T. Wang, and S. Tasena. A measure of mutual complete dependence in discrete variables through subcopula. *Internat. J. Approx. Reason.*, 65:11–23, 2015.

[34] W. Trutschnig. Some results on the convergence of (quasi–)copulas. *Fuzzy Sets and Systems*, 191:113–121, 2012.