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Multivariate patchwork copulas: A unified approach with applications to partial comonotonicity

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ABSTRACT

We present a general view of patchwork constructions of copulas that encompasses previous approaches based on similar ideas (ordinal sums, gluing methods, piecing-together, etc.). Practical applications of the new methodology are connected with the determination of copulas having specified behaviour in the tails, such as upper comonotonic copulas.

1. Introduction

The search for different families of copulas and/or construction methods is an active research field since copulas represent a convenient way to generate multivariate statistical models (see, e.g., Jaworski et al. (2013, 2010), Joe (1997), Mai and Scherer (2012) and Nelsen (2006) and the references therein). In particular, in the last few years, a number of investigations have focused on the constructions of copulas that describe random vectors with flexible dependence structures (see, e.g., Durante et al. (2010), Nikoloulopoulos et al. (2012) and Okhrin et al. (2013) and the references therein).

A technique aiming at transforming a copula in order to determine more flexible structures is given by the so-called *patchwork construction*. Specifically, given a copula C, a patchwork copula derived from C is any copula whose mass distribution coincides with the mass distribution of C up to a d-dimensional box $B \subseteq \mathbb{I}^d$ (here $\mathbb{I} := [0, 1]$), in which the probability mass is distributed in a different way. In particular, if such a box B has one vertex coinciding with one of the vertices of $\mathbb{I}^d$, then it provides as a by-product a powerful way to change the tail dependence properties of the given C.

One of the first examples of patchwork copulas is the ordinal sum construction, originated in the context of associative functions (see, e.g., Alsina et al. (2006) and Klement et al. (2000)); those copulas are in fact constructed by means of a suitable modification of the comonotone copula M_d (see also Durante and Fernández-Sánchez (2010); Mesiar and Sempi (2010)). Since then, ordinal sums have been extended in different ways under the names orthogonal grid constructions (De Baets and De Meyer, 2007), rectangular patchworks (Durante et al., 2009a; González-Barrios and Hernández-Cedillo, 2013; Zheng et al., 2011), and gluing copulas (Mesiar et al., 2008; Siburg and Stoimenov, 2008). Similar constructions can also be derived from the concepts of upper and partial comonotonicity (Cheung, 2009; Zhang and Duan, 2013) and multivariate piecing-together (Aulbach et al., 2012a,b).

In this paper, we propose a unified approach to patchwork constructions in the multivariate case. Specifically, we show that, by using measure-theoretical tools, these constructions can be presented in a very general setting from which previously considered methods can easily be derived. Several examples and practical considerations about the applicability of the results to risk estimation are included.

The relevance of the presented results in applied stochastic models is (at least) two-fold.

First, patchwork constructions allow us to induce strong positive tail behaviour in a multivariate distribution, a fact that has proved to be useful in a number of cases in order to obtain worst-possible scenarios for various risk measures. For instance, the

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concept of upper comonotonicity (which, as will be seen, is a special patchwork construction) has been used in order to show that several risk measures are additive, not only for the sum of comonotonic risks, but also for the sum of upper-comonotonic risks, provided that the level of probability is greater than a certain threshold. For these notions and properties, see Cheung (2009), Cheung and Lo (2013) and Zhang and Duan (2013). Moreover, notice that patchwork copulas often appear in the literature as the dependence structures that determine upper (lower) bounds for risk measures of a portfolio; see, for instance, Embrechts et al. (2013) and Rüschendorf (2013). In general, patchwork modifications of the tail of a multivariate distribution function may serve to show the well-known fact that the identification of a multivariate model is usually a difficult task when the tail behaviour is to be correctly identified. In fact, the estimation procedures cannot focus only on the middle part of the distribution ignoring the tail, since this will not be a conservative procedure from the point of view of a risk manager (see, for instance, Hua and Joe (2012)).

In the second place, patchwork copulas can be used as well to approximate the dependence structure by means of some basis copulas. Specifically, one may consider a background dependence structure (usually assumed to be the independence or the perfect dependence) and, hence, may modify it in fine partition of the multi-dimensional domain in order to obtain a flexible class of copulas. Such examples include shuffles of Min (Durante and Fernández-Sánchez, 2010, 2012; Durante et al., 2009b), checkerboard copulas (Carley and Taylor, 2002), or Bernstein copulas (Sancetta and Satchell, 2004) (for an application to non-life insurance, see also Diers et al. (2012)), and similar constructions (Zheng et al., 2011).

The paper is organized as follows. Section 2 presents the main ideas about the multivariate patchwork construction focusing on the case when the procedure involves the modification of a dependence structure in only one box. Section 3, instead, presents patchwork constructions in the general case together with some analytical properties of the new method. Finally, an illustration about copulas with orthogonal section is given (Section 4). All these sections contain several examples (and simulating algorithm from the generated models) that show the main features of the methodology. Section 5 concludes.

2. The new approach to patchwork construction

Let C be a copula, i.e., $C : \mathbb{I}^d \to \mathbb{I}$ is the restriction to $\mathbb{I}^d$ of a multivariate distribution function whose univariate margins are uniform on $\mathbb{I}$. As such, it can be extended in a unique way to $\overline{\mathbb{R}}^d$, a fact that will be used extensively in this paper (see, e.g., Jaworski et al. (2013, 2010)). Let $\mu = \mu_C$ be the unique probability measure on the Borel sets of $\mathbb{I}^d$ that is associated with C (see, e.g. Durante and Sempi (2010)) μ_C is a *d-fold stochastic measure* on $\mathbb{I}^d$, namely, the image measure of μ under any projection equals the Lebesgue measure λ on $\mathbb{I}$: for $i \in \{1, \dots, d\}$ and for a Borel subset A_i of $\mathbb{I}$,

$$\mu(\mathbb{I} \times \cdots \times \mathbb{I} \times A_i \times \mathbb{I} \times \cdots \times \mathbb{I}) = \lambda(A_i).$$

Moreover, for every d-dimensional box $B = [\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_d, b_d]$ contained in $\mathbb{I}^d$, one has

$$\mu_C(B) = V_C\,(B)\,, \tag{1}$$

where V_C indicates the C-volume (see Durante and Sempi (2010) for the formal definition). In the following, we denote by $\mathscr{C}_d$ the class of d-copulas.

According to the patchwork construction, by starting with $C \in \mathscr{C}_d$ a different copula can be determined by modifying μ_C in a given d-dimensional box $B = [\mathbf{a}, \mathbf{b}]$ of $\mathbb{I}^d$, as clarified in the following definition.

Definition 1. Let μ be a d-fold stochastic measure. Let B be a d-dimensional box of $\mathbb{I}^d$ with $\mu(B) > 0$ and let μ_B be a measure defined on the Borel subsets of B such that $\mu_B(B) = \mu(B)$. A measure μ^* is called the *patchwork of* μ_B *into* μ if, for every Borel subset A of $\mathbb{I}^d$, one has

$$\mu^*(A) = \mu(A \cap B^c) + \mu_B(A \cap B). \tag{2}$$

The measure μ^* will be denoted by $\langle B, \mu_B\rangle^{\mu}$.

It follows immediately from Definition 1 that $\langle B, \mu_B\rangle^{\mu}$ is a probability measure on the Borel sets of $\mathbb{I}^d$. Moreover, under some additional assumptions, it can be proved that $\langle B, \mu_B\rangle^{\mu}$ is also a d-fold stochastic measure, i.e. it corresponds to a copula.

Theorem 1. *Let $B = [\mathbf{a}, \mathbf{b}]$. The following statements are equivalent:*

(a) *$\mu^* = \langle B, \mu_B\rangle^{\mu}$ is a d-fold stochastic measure;*
(b) *$\mu_B(B_i) = \mu(B_i)$ for every d-box of the form*

$$B_i = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times \left[a_i', b_i'\right] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_d, b_d]\,.$$

Proof. It is enough to consider the sets of the form

$$D_i = \mathbb{I} \times \cdots \times \mathbb{I} \times [c_i, d_i] \times \mathbb{I} \times \cdots \times \mathbb{I}.$$

Since μ is a d-fold stochastic measure, there is nothing to prove if $D_i \cap B = \emptyset$. Assume now, first,

$$[c_i, d_i] \cap [a_i, b_i] = [a_i, d_i]\,.$$

Then, if $\widetilde{B}_i = \mathbb{I} \times \cdots \times \mathbb{I} \times [a_i, d_i] \times \mathbb{I} \times \cdots \times \mathbb{I}$, one has, since μ is d-fold stochastic:

$$\begin{aligned}\mu^*(D_i) &= \mu^*(\mathbb{I} \times \cdots \times \mathbb{I} \times [c_i, a_i] \times \mathbb{I} \times \cdots \times \mathbb{I}) \\ &\quad + \mu^*(\mathbb{I} \times \cdots \times \mathbb{I} \times [a_i, d_i] \times \mathbb{I} \times \cdots \times \mathbb{I}) \\ &= \mu(\mathbb{I} \times \cdots \times \mathbb{I} \times [c_i, a_i] \times \mathbb{I} \times \cdots \times \mathbb{I}) + \mu^*(\widetilde{B}_i) \\ &= (a_i - c_i) + \mu(\widetilde{B}_i \cap B^c) + \mu_B(\widetilde{B}_i \cap B).\end{aligned}$$

(a)$\Longrightarrow$(b) If μ^* is a d-fold stochastic measure, one has $\mu^*(D_i) = d_i - a_i$, so that

$$d_i - a_i = \mu(\widetilde{B}_i \cap B^c) + \mu_B(\widetilde{B}_i \cap B),$$

whence

$$\begin{aligned}\mu_B(\widetilde{B}_i \cap B) &= (d_i - a_i) - \mu(\widetilde{B}_i \cap B^c) \\ &= \mu(\mathbb{I} \times \cdots \times \mathbb{I} \times [a_i, d_i] \times \mathbb{I} \times \cdots \times \mathbb{I}) \\ &\quad - \mu(\widetilde{B}_i \cap B^c) = \mu(\widetilde{B}_i \cap B).\end{aligned}$$

(b)$\Longrightarrow$(a) If $\mu_B(\widetilde{B}_i \cap B) = \mu(\widetilde{B}_i \cap B)$, then one has

$$\begin{aligned}\mu^*(D_i) &= a_i - c_i + \mu(\widetilde{B}_i \cap B^c) + \mu(\widetilde{B}_i \cap B) \\ &= (a_i - c_i) + \mu(\widetilde{B}_i) = (a_i - c_i) + (d_i - a_i) = d_i - a_i\end{aligned}$$

so that μ^* is indeed d-fold stochastic.

The other cases are dealt with in a similar manner. □

From Theorem 1 it is clear that the possibility of obtaining a d-fold stochastic measure via a patchwork procedure depends on a suitable choice of the measure μ_B satisfying condition (b) of Theorem 1. To this end, it is convenient to reformulate the problem in an equivalent, but easier-to-handle, way.

Let $C \in \mathscr{C}_d$ be a copula; let μ_C be the d-fold stochastic measure induced by C on the family $\mathscr{B}(\mathbb{I}^d)$ of Borel subsets of $\mathbb{I}^d$.

Let μ_B be a measure on the Borel sets $\mathscr{B}(B)$ of $B = [\mathbf{a}, \mathbf{b}]$ that satisfies condition (b) of Theorem 1. Define $\widetilde{\mu}_B : \mathscr{B}(B) \to \mathbb{I}$ by $\widetilde{\mu}_B = \mu_B/\alpha$, where $\alpha := \mu_C(B) > 0$. Obviously, $\widetilde{\mu}_B$ is a probability measure on $\mathscr{B}(B)$ and, hence, the map $\mathbf{x} \mapsto \widetilde{\mu}_B([\mathbf{a}, \mathbf{x}])$ is a d-dimensional distribution function concentrated on B. Thus, in view of Sklar's Theorem (Sklar, 1959) (see also de Amo et al. (2012) and

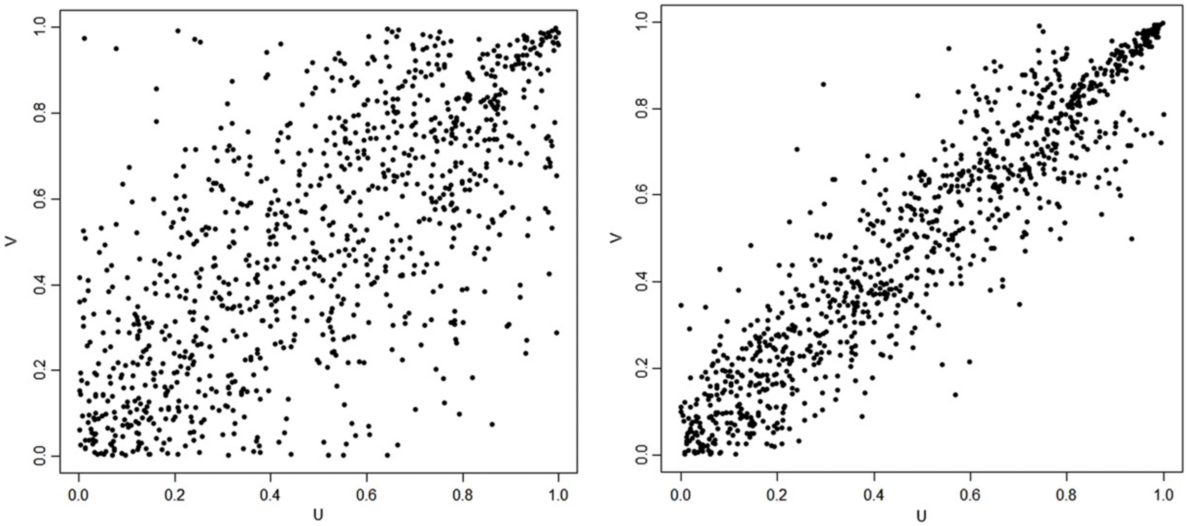

Fig. 1. Pairwise scatterplots from a random sample of 1000 realizations from the copula $\langle B, C_B\rangle^C$ where $B = [0.8, 1]^2$, C is a Frank copula and C_B is a Gumbel copula with Kendall's tau respectively equal to 0.5 and 0.5 (left); 0.75 and 0.75 (right).

Durante et al. (2012, 2013)), there exists a copula C_B such that, for every $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$, one has

$$\widetilde{\mu}_B([\mathbf{a}, \mathbf{x}]) = C_B\left(\widetilde{F}_B^1(x_1), \ldots, \widetilde{F}_B^d(x_d)\right), \tag{3}$$

where the marginal distribution function $\widetilde{F}_B^i$ is given, for every $i \in \{1, \ldots, d\}$, by

$$\begin{aligned}\widetilde{F}_B^i(x_i) &= \widetilde{\mu}_B([a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i, x_i] \\ &\quad \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_d, b_d]) \\ &= \frac{1}{\alpha}\,\mu_C([a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i, x_i] \\ &\quad \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_d, b_d]),\end{aligned}$$

for every $x_i \in [a_i, b_i]$ (here and in the following, we refrain to give the formal expression of $\widetilde{F}_B^i$ outside $[a_i, b_i]$ since this is obvious in view of the properties of a distribution function).

It follows that $\widetilde{\mu}_B$, and hence μ_B, only depend on the choice of a specific copula C_B and the knowledge of the measure μ_C.

Thus, the copula C^* associated with $\mu^* = \langle B, \mu_B\rangle^\mu$ can be represented, for every $\mathbf{u} \in \mathbb{I}^d$, by

$$C^*(\mathbf{u}) = \mu_C([\mathbf{0}, \mathbf{u}] \cap B^c) + \mu_B([\mathbf{0}, \mathbf{u}] \cap B), \tag{4}$$

which can be rewritten as

$$C^*(\mathbf{u}) = \mu_C\left([\mathbf{0}, \mathbf{u}] \cap B^c\right) + \alpha\, C_B\left(\widetilde{F}_B^1(u_1), \ldots, \widetilde{F}_B^d(u_d)\right). \tag{5}$$

The construction just introduced immediately yields the following result.

Theorem 2. *Let C and C_B be d-dimensional copulas and let $B = [\mathbf{a}, \mathbf{b}]$ be a non-empty box contained in $\mathbb{I}^d$ such that $\mu_C(B) > 0$. The function $C^* : \mathbb{I}^d \to \mathbb{I}$ given by (5) is a copula.*

The copula of (5) is called *patchwork of* (B, C_B) *into* C and it is denoted by the symbol $C^* = \langle B, C_B\rangle^C$. The copula C is called the *background copula* of the patchwork. The measure induced by C^* is $\langle B, \mu_B\rangle^{\mu_C}$, where μ_C is induced by C while μ_B can be constructed by its distribution functions given by

$$\mu_B([\mathbf{a}, \mathbf{x}]) = \alpha\, C_B\left(\widetilde{F}_B^1(x_1), \ldots, \widetilde{F}_B^d(x_d)\right).$$

The method of Theorem 2 has been often used in the literature, as shown in the next examples.

Example 1. Consider the patchwork of copulas of type $\langle B, C_B\rangle^C$, where $B = [\mathbf{a}, \mathbf{1}]$. Then, it follows from (5) that, for all $\mathbf{u} \in \mathbb{I}^d$ one has

$$C^*(\mathbf{u}) = \mu_C([\mathbf{0}, \mathbf{u}] \setminus [\mathbf{a}, \mathbf{1}]) + \alpha\, C_B\left(\widetilde{F}_B^1(u_1), \ldots, \widetilde{F}_B^d(u_d)\right), \tag{6}$$

where $\alpha = V_C(B)$ and, for every $i \in \{1, \ldots, d\}$, one has

$$\widetilde{F}_B^i(x_i) = \frac{1}{\alpha}\,V_C([a_1, 1] \times \cdots [a_i, x_i] \times \cdots \times [a_d, 1]).$$

This construction was introduced in González-Barrios and Hernández-Cedillo (2013, Theorem 3.4) for the multivariate case, although was proved in a more complex way (at least in our opinion). An algorithm for generating a random sample from the copula C^* of (6) goes as follows.

Algorithm 1. 1. Generate $\mathbf{u}$ from the copula C.
2. Generate $\mathbf{v}$ from the copula C_B.
3. For $i = 1, 2, \ldots, d$ set $w_i = (\widetilde{F}_B^i)^{-1}(v_i)$.
4. If $\mathbf{u} \in B$, then return $\mathbf{w}$.
Otherwise, return $\mathbf{u}$.

As is apparent, the efficiency of the algorithm depends both on the ability to generate a random pair from the copulas C and C_B and on the possibility to derive the inverse of the functions $\widetilde{F}_B^i$.

A special feature of this construction should be stressed here. Intuitively, suppose that one wants to induce a specific behaviour of C near the corner $\mathbf{1}$. Then, it is possible to select a constant a close to 1 and to adopt the construction (6) by gluing a copula C_B with a desired tail behaviour (for instance, with a non-trivial tail dependence coefficient) into the probability mass distribution associated C. In Fig. 1 two examples of a bivariate random sample from such copulas are presented showing how the tail behaviour near $(1, 1)$ is modified. For explicit calculations about how patchwork constructions modify the tail dependence coefficients in the bivariate case, we also refer to Durante et al. (2009a). A trivariate example is given in Fig. 2 showing how the behaviour of each pairwise marginal near the upper corner of the unit square is changed. ■

Example 2. A special case related to Example 1 has been used in order to find bounds for functions of dependent risks (Embrechts et al., 2005; Embrechts and Puccetti, 2006, 2010; Embrechts et al., 2013). Specifically, given the vector of losses (L_1, L_2) having fixed marginals, the worst-possible *VaR* (at level α) for the sum $L^+ = L_1$

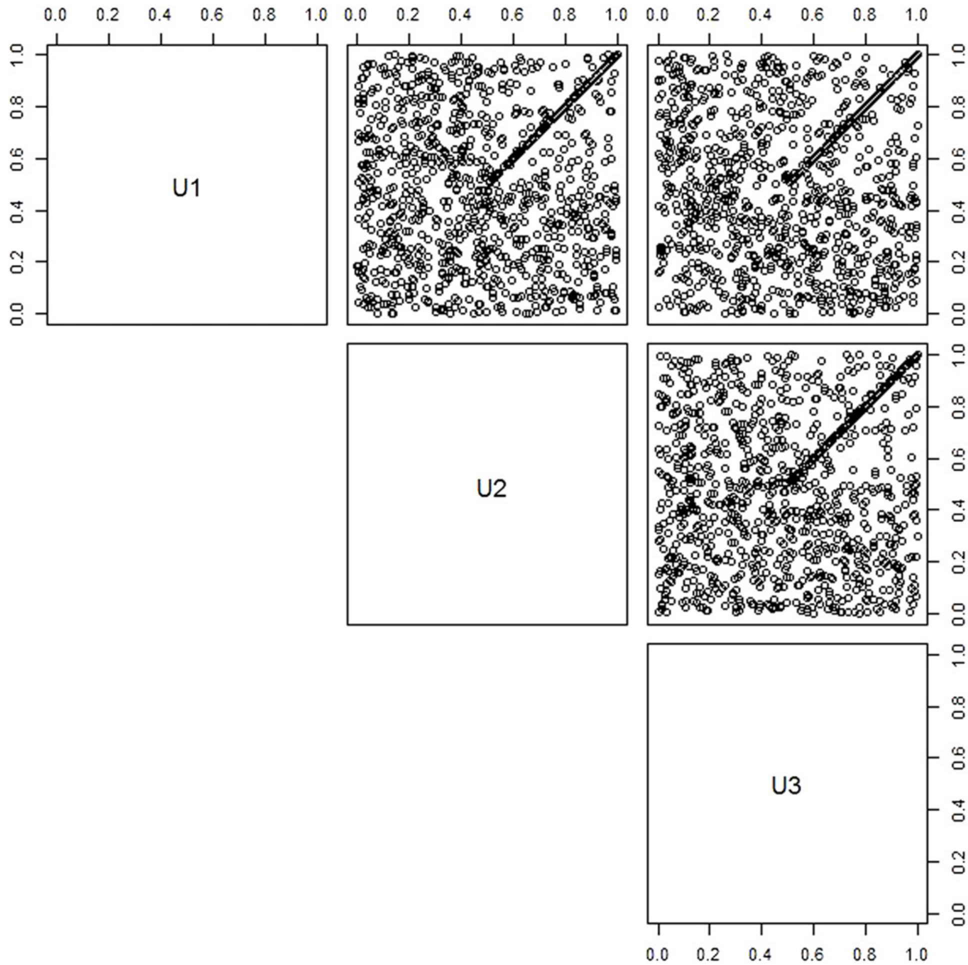

Fig. 2. Random sample of 1000 realizations from the copula $\langle B, C_B\rangle^C$ where $B = [0.5, 1]^3$, C is the independence copula and C_B is the comonotone copula.

$+ L_2$ (write: $\overline{VaR}_\alpha(L^+)$) is given when (L_1, L_2) is coupled by $\langle[\alpha, 1]^2, W_2\rangle^{M_2}$. Interestingly, it is well known that

$$VaR_\alpha(L_1) + VaR_\alpha(L_2) \le \overline{VaR}_\alpha(L^+),$$

where the left hand side of the previous inequality corresponds to the case when (L_1, L_2) is coupled by M_2. Now, for any copula C, consider the patchwork $C^* = \langle[\alpha, 1]^2, C\rangle^{M_2}$. This copula can be used in order to interpolate between the comonotonic scenario and the worst-case scenario for $VaR_\alpha(L^+)$. Numerically, such a property is illustrated in Table 1.

Remark 1. A general advice is needed here about Example 1. Suppose that a vector $\mathbf{X}$ of random losses is associated with a copula of type $C^* = \langle[\mathbf{a}, \mathbf{1}], C_B\rangle^C$, where each a_i corresponds to the α-quantile of X_i (think at $\alpha = 0.95$, for instance). Then, intuitively, B represents a risky region for the loss portfolio. In view of the patchwork construction, $\mathbf{P}_{C^*}(\mathbf{X} \in B) = \mathbf{P}_C(\mathbf{X} \in B)$. In other words, the probability that the losses are jointly in the risky region is the same for the two dependence structures. Therefore, the effect of this specific patchwork modification concerns the way how the probability mass is spread around the corner and not the probability of having joint extreme losses (i.e. losses falling in the region B).

Example 3. Let C_B be an arbitrary d-copula and let $M_d(\mathbf{u}) = \min\{u_1, \dots, u_d\}$ be the comonotone copula. Consider the patchwork of copulas of type $\langle B, C_B\rangle^{M_d}$, where $B = [\mathbf{0}, \mathbf{a}]$. Then, taking into account that M_d concentrates the probability mass entirely along the main diagonal of $\mathbb{I}^d$, it follows that, for every $i \in \{1, \dots, d\}$, one has

$$\widetilde{F}_B^i(x) = \frac{1}{\alpha} V_{M_d}([0, a_1] \times \cdots \times [0, x] \times \cdots \times [0, a_d]) = \frac{1}{\alpha} \min\{a_1, \dots, x, \dots, a_d\}.$$

Here and in the following the M_d-volume of a box is calculated using the formula of Nelsen (2006, Exercise 2.3.5), namely

$$V_{M_d}([\mathbf{a}, \mathbf{b}]) = \max\{\min\{b_1, \dots, b_d\} - \max\{a_1, \dots, a_d\}, 0\}.$$

Thus Eq. (5) may be further simplified as

$$C^*(\mathbf{u}) = \mu_C\left([\mathbf{0}, \mathbf{u}] \cap B^c\right) + \alpha\, C_B\left(\frac{\min\{a_1, \dots, u_1, \dots, a_d\}}{\alpha}, \dots, \frac{\min\{a_1, \dots, u_d, \dots, a_d\}}{\alpha}\right).$$

Notice that in this case, $\alpha = V_{M_d}(B) = \min\{a_1, \dots, a_d\}$.

When all the components of $\mathbf{a}$ are equal to a, constructions of copulas of this type describe upper comonotonic random vectors. For the study of these dependence structures and their applications, we refer to Cheung (2009); Zhang and Duan (2013). Examples are illustrated in Fig. 3. ■

Example 4. Consider the d-box $B =]a, b[^d$, a copula $C \in \mathscr{C}_d$ and the background copula M_d. Since

$$\widetilde{F}_B^i(u_i) = \frac{1}{b - a} \max\{\min\{b, x_i\} - a, 0\},$$

Table 1
Numerical approximation of $VaR_{0.90}(L_1^{C^*}, L_2^{C^*})$ where $L_1, L_2, \sim N(0, 1)$, $C^* = \langle [0.90, 1]^2, C\rangle^{M_2}$ for a Clayton copula C with Kendall's τ equal to the indicated value. Results based on 10^6 simulation from the given copula.

	$\tau = 1$	$\tau = 0.50$	$\tau = 0.00$	$\tau = -0.50$	$\tau = -1$
$VaR_\alpha(L_1^{C^*}, L_2^{C^*})$	2.5631	2.5663	2.5749	3.0340	3.2897

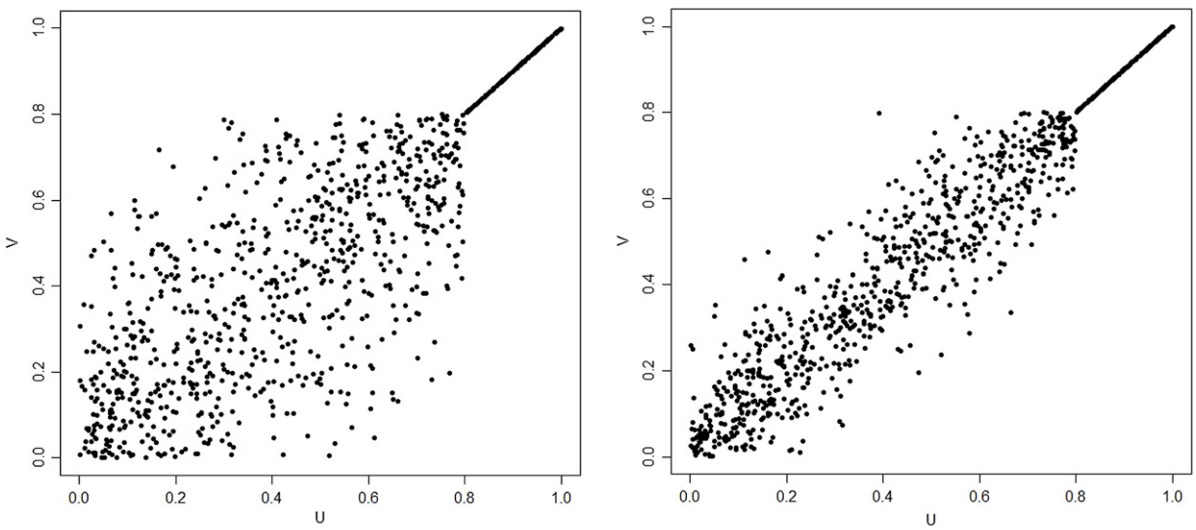

Fig. 3. Random sample of 1000 realizations from the copula $\langle B, C_B\rangle^{M_2}$ where $B = [0, 0.8]^2$, C_B is a Frank with Kendall's tau equal to 0.5 (left) and 0.75 (right).

for every $u_i \in \mathbb{I}$, from Eq. (5) one has, for the patchwork of C into M_d

$$\langle]a, b[^d, C\rangle^{M_d}(\mathbf{u}) = (b-a)\, C\left(\widetilde{F}_B^i(u_1), \dots, \widetilde{F}_B^i(u_d)\right) + \mu_{M_d}\left([\mathbf{0}, \mathbf{u}] \cap B^c\right).$$

This copula is an ordinal sum (see Durante and Fernández-Sánchez (2010); Jaworski and Rychlik (2008); Mesiar and Sempi (2010)). In particular, such a method includes the lower and tail comonotonicity presented in Zhang and Duan (2013). ■

3. Patchwork construction: the general case

The procedure described in the previous section may also be applied when the measure induced by a copula is modified in several boxes of $\mathbb{I}^d$. Let S be equal to either $\{1, \dots, n\}$ or to $\mathbb{N}$. Let λ_d be the d-dimensional Lebesgue measure on $\mathbb{I}^d$.

Definition 2. Let μ be a d-fold stochastic measure on $\mathscr{B}(\mathbb{I}^d)$. Given S, let $(B_s)_{s\in S}$ be a family of d-boxes contained in $\mathbb{I}^d$ such that $\mu(B_s) > 0$ $(s \in S)$ and $\lambda_d(B_s \cap B_{s'}) = 0$ if $s \neq s'$. Set $B = \cup_{s\in S} B_s$ and, for each $s \in S$, let μ_s be a measure defined on the Borel sets $\mathscr{B}(B_s)$ of B_s such that $\mu_s(B_s) = \mu(B_s) > 0$. A measure μ^* is called the *patchwork of* $(\mu_s)_{s\in S}$ into μ if, for every Borel subset A of $\mathbb{I}^d$, one has

$$\mu^*(A) := \mu\left(A \cap B^c\right) + \sum_{s\in S} \mu_s\left(A \cap B_s\right). \tag{7}$$

The measure μ^* will be denoted by $\langle B_s, \mu_s\rangle_{s\in S}^{\mu}$.

It follows immediately from the definition (7) that $\langle B_s, \mu_s\rangle_{s\in S}^{\mu}$ is a probability measure on the Borel sets of $\mathbb{I}^d$. By an argument analogous to that of Theorem 1 one proves the following theorem.

Theorem 3. *Assume the notations of Definition 2. Moreover, for every $s \in S$, let $B_s = [\mathbf{a}^s, \mathbf{b}^s]$. If $\mu_s(B_i) = \mu(B_i)$ for every d-box of the form*

$$B_i = \left[a_1^s, b_1^s\right] \times \cdots \times \left[a_{i-1}^s, b_{i-1}^s\right] \times \left[a_i', b_i'\right] \times \left[a_{i+1}^s, b_{i+1}^s\right] \times \cdots \times \left[a_d^s, b_d^s\right]$$

contained in B_s, then $\mu^* = \langle B_s, \mu_s\rangle_{s\in S}^{\mu}$ is a d-fold stochastic measure.

As is apparent from the definition, given a system of Borel sets $(B_s)_{s\in\{1,\dots,n\}}$ (with $\lambda_d(B_s \cap B_{s'}) = 0$ if $s \neq s'$) the patchwork of $(\mu_s)_{s\in\{1,\dots,n\}}$ into μ can be obtained by considering the patchwork of μ_n into $\langle B_s, \mu_s\rangle_{s\in\{1,\dots,n-1\}}^{\mu}$. This is also true when one deals with countable many measures, in the sense specified below. Consider, in fact, the family of finite real (also called signed) measures on $\mathscr{B}(\mathbb{I}^d)$. This is a real linear space that can be endowed with the norm of total variation given by

$$\|\mu\|_{\text{tv}} := \sup\left\{\sum_{n\in\mathbb{N}} |\mu(E_n)|\right\},$$

where the supremum is taken over all the countable and measurable partitions $(E_n)_{n\in\mathbb{N}}$ of $\mathbb{I}^2$. The following result holds.

Theorem 4. *Let $\langle B_s, \mu_s\rangle_{s\in\overline{\mathbb{Z}}_+}^{\mu}$ be the patchwork of $(\mu_{B_s})_{s\in S}$ into μ. Then the sequence*

$$\left(\langle B_s, \mu_s\rangle_{s\in\{1\dots,n\}}^{\mu}\right)_{n\in\mathbb{N}}$$

converges in total variation to $\langle B_s, \mu_s\rangle_{s\in\mathbb{N}}^{\mu}$.

Proof. On account of the definition of patchwork one has

$$\left\| \langle B_s, \mu_s\rangle_{s\in\{1\dots,n\}}^{\mu} - \langle B_s, \mu_s\rangle_{s\in\mathbb{N}}^{\mu} \right\|_{\text{tv}} \le 2 \sum_{j\ge n+1} \mu(B_j),$$

which is the n-th remainder of a convergent series, and, as a consequence, tends to zero as n goes to $+\infty$. □

Now, let $C \in \mathscr{C}_d$ be a copula, and let μ_C be the d-fold stochastic measure induced by C on the family $\mathscr{B}(\mathbb{I}^d)$ of Borel subsets of $\mathbb{I}^d$.

For $s \in S$ let $B_s = [\mathbf{a}^s, \mathbf{b}^s]$ be a d-box contained in $\mathbb{I}^d$, and let μ_s be a measure on the Borel sets $\mathscr{B}(B_s)$ of B_s such that $\mu_s(B_i) = \mu(B_i)$ for every d-box of the form

$$B_i = \left[a_1^s, b_1^s\right] \times \cdots \times \left[a_{i-1}^s, b_{i-1}^s\right] \times \left[a_i', b_i'\right] \times \left[a_{i+1}^s, b_{i+1}^s\right] \times \cdots \times \left[a_d^s, b_d^s\right]$$

contained in B_s. Define $\widetilde{\mu}_s : \mathscr{B}(B_s) \to \mathbb{I}$ by $\widetilde{\mu}_s = \mu_s/\alpha_s$, where $\alpha_s = \mu_C(B_s) > 0$. Obviously, $\widetilde{\mu}_{B_s}$ is a probability measure on $\mathscr{B}(B_s)$. As above, the maps $\mathbf{x} \mapsto \widetilde{\mu}_s([\mathbf{a}^s, \mathbf{x}])$ are distribution functions, and, hence, for every $\mathbf{x} \in [\mathbf{a}^s, \mathbf{b}^s]$, one has

$$\widetilde{\mu}_s\left(\left[\mathbf{a}^s, \mathbf{x}\right]\right) = C_s\left(\widetilde{F}_{B_s}^1(x_1), \ldots, \widetilde{F}_{B_s}^d(x_d)\right), \tag{8}$$

where the marginal distribution function $\widetilde{F}_{B_s}^i$ is given, for every $i \in \{1, \ldots, d\}$, by

$$\begin{aligned}\widetilde{F}_{B_s}^i(x_i) &= \widetilde{\mu}_s\left(\left[a_1^s, b_1^s\right] \times \cdots \times \left[a_{i-1}^s, b_{i-1}^s\right] \times \left[a_i^s, x_i\right] \times \left[a_{i+1}^s, b_{i+1}^s\right] \times \cdots \times \left[a_d^s, b_d^s\right]\right) \\ &= \frac{1}{\alpha_s}\, \mu_C\left(\left[a_1^s, b_1^s\right] \times \cdots \times \left[a_{i-1}^s, b_{i-1}^s\right] \times \left[a_i^s, x_i\right] \times \left[a_{i+1}^s, b_{i+1}^s\right] \times \cdots \times \left[a_d^s, b_d^s\right]\right).\end{aligned}$$

It follows that $\widetilde{\mu}_s$, and hence μ_s, only depends on the choice of a specific copula C_s and the knowledge of the measure μ.

The copula C^* associated with μ^* can be represented, for every $\mathbf{u} \in \mathbb{I}^d$, by

$$C^*(\mathbf{u}) = \mu_C([0, \mathbf{u}] \cap B^c) + \sum_{s \in S} \mu_{B_s}([\mathbf{0}, \mathbf{u}] \cap B_s), \tag{9}$$

where $B = \cup_{s\in S} B_s$; this can be rewritten as

$$C^*(\mathbf{u}) = \mu_C\left([0, \mathbf{u}] \cap B^c\right) + \sum_{s \in S} \alpha_s\, C_s\left(\widetilde{F}_{B_s}^1(u_1), \ldots, \widetilde{F}_{B_s}^d(u_d)\right), \tag{10}$$

where $B = \cup_{s\in S} B_s$. The construction just introduced immediately yields the following result.

Theorem 5. *Let C and C_{B_s} $(s \in S)$ be d-dimensional copulas and let B_s $(s \in S)$ be a system (finite or countable) of non-empty boxes contained in $\mathbb{I}^d$ such that $\lambda_d(B_s \cap B_{s'}) = 0$ if $s \neq s'$. Then the function $C^* : \mathbb{I}^d \to \mathbb{I}$ given by (10) is a copula.*

The copula of (10) is called the *patchwork of* $(B_s, C_{B_s})_{s\in S}$ *into* C and it is denoted by the symbol $C^* = \langle B_s, C_{B_s}\rangle_{s\in S}^C$. Notice that the measure induced by C^* is $\langle B_s, \mu_{B_s}\rangle_{s\in S}^{\mu}$, where μ is induced by C while μ_{B_s} can be constructed by its distribution functions given by

$$\mu_{B_s}\left([\mathbf{a}_s, \mathbf{x}]\right) = \alpha_s\, C_s\left(\widetilde{F}_{B_s}^1(x_1), \ldots, \widetilde{F}_{B_s}^d(x_d)\right).$$

Example 5. Let $\theta \in]0, 1[$ and let C_1 and C_2 be bivariate copulas. Consider the boxes B_1 and B_2 defined by

$$B_1 := [0, \theta] \times \mathbb{I}, \qquad B_2 = [\theta, 1] \times \mathbb{I},$$

respectively and consider the patchwork

$$C := \langle B_i, C_i\rangle_{i=1,2}^{\Pi_2}.$$

Eq. (10) now yields, for every point $(u_1, u_2) \in \mathbb{I}^2$,

$$C(u_1, u_2) = \theta\, C_1\left(\frac{u_1}{\theta}, u_2\right) + (1-\theta)\, C_2\left(\frac{u_1 - \theta}{1-\theta}, u_2\right).$$

The previous expression coincides with the method for constructing gluing copulas (Siburg and Stoimenov, 2008); by using similar arguments it can be extended as well to d-dimensional copulas and to the case when more boxes are involved, as in the case of gluing ordinal sums (Durante and Jaworski, 2012; Mesiar et al., 2008). ■

Remark 2. It is now easy to extend the construction of ordinal sums to the case in which more d-boxes $]a_s, b_s[^d$ $(s \in S)$ are given having their main diagonal lying on that of the unit box $\mathbb{I}^d$; let $C_s \in \mathscr{C}_d$ be a set of copulas indexed by S. Set $B = \cup_{s\in S}]a_s, b_s[^d$. Then the same argument of Example 4 yields that $\langle B_s, C_s\rangle_{s\in S}^{M_d}$ is an ordinal sum of copulas in the sense of Durante and Fernández-Sánchez (2010), Jaworski and Rychlik (2008) and Mesiar and Sempi (2010). In particular, the interval comonotonicity of Zhang and Duan (2013) can be expressed in this form.

Remark 3. The general patchwork construction presented in Theorem 5 may be used as well to modify the tail behaviour of a copula in two or more corners of $\mathbb{I}^d$. The importance for such constructions has been stressed in the bivariate case in Zhang (2008) (see also Durante et al. (2009a)).

Now, let us consider the patchwork as a general operator among function spaces. Define $\mathscr{C}^S := \{(C_s)_{s\in S}\}$, where, for every $s \in S$, C_s is a d-copula. Let $C \in \mathscr{C}_d$ and let B_s $(s \in S)$ be a system of d-boxes as in Theorem 5. Formally, the patchwork is defined as the mapping $T_C : \mathscr{C}^S \to \mathscr{C}_d$ given by

$$T_C\left((C_s)_{s\in S}\right) := \langle B_s, C_s\rangle_{s\in S}^C. \tag{11}$$

Such a mapping is continuous, in the sense that small changes components of the generating copula family $(C_s)_{s\in S}$ does not amplify in the patchwork process, as shown below.

Theorem 6. *Let $(B_s)_{s\in S}$ be a family of d-boxes contained in $\mathbb{I}^d$, indexed by S, and such that $\lambda_d(B_s \cap B_{s'}) = 0$ for $s \neq s'$; let C be a d-copula such that $\mu_C(B_s) > 0$ for every $s \in S$. Then the mapping T_C given by (11) is uniformly continuous when the space $\mathscr{C}_d$ is endowed by the uniform distance d_∞, and $\mathscr{C}^S$ by the distance*

$$\begin{aligned}d_S\left((C_s)_{s\in S}, (\widetilde{C}_s)_{s\in S}\right) &:= \sup_{s\in S} \max_{\mathbf{u}\in\mathbb{I}^d} |C_s(\mathbf{u}) - \widetilde{C}_s(\mathbf{u})| \\ &= \sup_{s\in S} d_\infty(C_s, \widetilde{C}_s).\end{aligned}$$

Proof. Let $B = \cup_{s\in S} B_s$. Given $\varepsilon > 0$ define $\delta := \varepsilon/\mu_C(B)$ and consider $(C_s)_{s\in S}$ and $(\widetilde{C}_s)_{s\in S}$ in $\mathscr{C}^S$ with

$$d_S\left((C_s)_{s\in S}, (\widetilde{C}_s)_{s\in S}\right) < \delta.$$

With reference to (10), one now has

$$\begin{aligned}&d_\infty\left(T_C((C_s)_{s\in S}), T_C((\widetilde{C}_s)_{s\in S})\right) \\ &= \max_{\mathbf{u}\in\mathbb{I}^d} \left| \langle B_s, C_s\rangle_{s\in S}^C(\mathbf{u}) - \langle B_s, \widetilde{C}_s\rangle_{s\in S}^C(\mathbf{u})\right| \\ &\le \max_{\mathbf{u}\in\mathbb{I}^d} \sum_{s\in S} \alpha_s \left| C_s\left(\widetilde{F}_{B_s}^1(u_1), \ldots, \widetilde{F}_{B_s}^d(u_d)\right) - \widetilde{C}_s\left(\widetilde{F}_{B_s}^1(u_1), \ldots, \widetilde{F}_{B_s}^d(u_d)\right)\right| \\ &\le \sum_{s\in S} \alpha_s\, d_S\left((C_s)_{s\in S}, (\widetilde{C}_s)_{s\in S}\right) < \delta \sum_{s\in S} \alpha_s = \delta\, \mu_C(B) = \varepsilon,\end{aligned}$$

which proves the assertion. □

An interesting result is obtained when $(C_s)_{s\in S} \in \mathscr{C}^S$ is such that $C_s = C' \in \mathscr{C}_d$ for every $s \in S$. In this case, in fact, one can show that the patchwork generates a variety of different dependence structures up to a single case.

Theorem 7. *Let B_s $(s \in S)$ be a system (finite or countable) of non-empty boxes contained in $\mathbb{I}^d$ such that $\lambda_d(B_s \cap B_{s'}) = 0$ and $B = \cup_{s\in S} B_s$. Let C be a copula. Suppose that $\mu_C(B) < 1$. Then there exists a unique copula $\widetilde{C}$ that is invariant under the map $F : \mathscr{C}_d \to \mathscr{C}_d$ defined by*

$$F\left((C')_{s\in S}\right) := \langle B_s, C'\rangle_{s\in S}^C.$$

Proof. Let C' and C'' be two d-copulas. Then

$$\begin{aligned} d_\infty\big((F(C'), F(C''))\big) &= \max_{\mathbf{u}\in\mathbb{I}^d} \left|\langle B_s, C'\rangle^C_{s\in S}(\mathbf{u}) - \langle B_s, C''\rangle^C_{s\in S}(\mathbf{u})\right| \\ &\le \max_{\mathbf{u}\in\mathbb{I}^d} \sum_{s\in S}\alpha_s \left| C'\left(\widetilde{F}^1_{B_s}(u_1),\dots,\widetilde{F}^d_{B_s}(u_d)\right)\right. \\ &\quad \left. - C''\left(\widetilde{F}^1_{B_s}(u_1),\dots,\widetilde{F}^d_{B_s}(u_d)\right)\right| \\ &\le \sum_{s\in S}\alpha_s\, d_\infty(C', C'') = \mu_C(B)\, d_\infty(C', C''). \end{aligned}$$

Thus F is a contraction mapping if $\mu_C(B) < 1$. Moreover, the space $\mathscr{C}_d$ is a compact (see, e.g., Durante et al. (2012)), and, hence, complete, metric space. Therefore, by the Banach fixed point theorem, there exists a unique copula $\widetilde{C} \in \mathscr{C}_d$ that is invariant under F, $F(\widetilde{C}) = \widetilde{C}$. □

4. Illustration: copulas with a given orthogonal section

The construction of copulas that take given values on specific sections has been extensively considered in the literature (see, e.g., Nelsen (2006) and the references therein). Since most of these constructions are limited to the 2-dimensional case, we wish to extend the results of Klement et al. (2007) (see also Durante et al. (2007) and Jaworski (2013)) and to consider multivariate copulas that have a prescribed orthogonal section. We start by the very definition.

Definition 3. Given a copula $C \in \mathscr{C}_d$ and $b \in\ ]0, 1]$ the function $h^C_{j,b} : \mathbb{I}^{d-1} \to \mathbb{I}$ defined, for $j = 1, \dots, d$, by

$$h^C_{j,b}(u_1,\dots,u_{j-1},u_{j+1},\dots,u_n) := C(u_1,\dots,u_{j-1},b,u_{j+1},\dots,u_n)$$

will be said to be the *j-th orthogonal section of C at b*.

Obviously $h^C_{j,b}$ satisfies the following conditions:

(a) $h^{W_d}_{j,b} \le h^C_{j,b} \le h^{M_d}_{j,b}$, where W_d and M_d are the Fréchet–Hoeffding bounds;
(b) $h^C_{j,b}$ is $(d-1)$-increasing;
(c) $h^C_{j,b}$ is 1-Lipschitz.

On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ let $U_1, \dots, U_d$ be random variables uniformly distributed on $(0, 1)$ and having C as their distribution function. Then

$$\begin{aligned} h^C_{j,b}(u_1,\dots,u_{j-1},u_{j+1},\dots,u_n) &= \mathbb{P}\left(\bigcap_{\substack{i=1\\ i\ne j}}^{d}\{U_i\le u_i\}\cap\{U_j\le b\}\right) \\ &= b\,\mathbb{P}\left(\bigcap_{\substack{i=1\\ i\ne j}}^{d}\{U_i\le u_i\}\mid U_j\le b\right) \\ &= b\,F_{U_1,\dots,U_{j-1},U_{j+1},\dots,U_d\mid U_j\le b}(u_1,\dots,u_{j-1},u_{j+1},\dots,u_n). \end{aligned}$$

Thus the section $h^C_{j,b}$ is b times the conditional distribution function of the vector

$$(U_1,\dots,U_{j-1},U_{j+1},\dots,U_d)$$

under the condition $U_j \le b$.

In general, when no restrictions are imposed on the continuous random variables $X_1, \dots, X_d$ linked by the copula C, one has, for each $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \mathbb{R}^{d-1}$,

$$\begin{aligned} &h^C_{j,b}\left(F_{X_1}(x_1),\dots,F_{X_{j-1}}(x_{j-1}),F_{X_{j+1}}(x_{j+1}),\dots,F_{X_d}(x_d)\right) \\ &\quad = b\,F_{X_1,\dots,X_{j-1},X_{j+1},\dots,X_d\mid X_j\le Q_{X_j}(b)}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_d), \end{aligned}$$

where Q_{X_j} is the quantile function of X_j. Thus orthogonal sections express one's knowledge about the vector $(X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_d)$ under the condition that X_j does not exceed a prescribed fixed value. This corresponds to truncation (on the right) of the random variable X_j, a well-known statistical practice. As such, copulas with a given orthogonal section have been, for instance, considered in some extensions of the Koziol–Green model (see Gaddah and Braekers (2010, 2011) and the references therein).

Moreover, notice that the knowledge of the conditional distribution of $\mathbf{X}$ given $X_j \le Q_{X_j}(b)$ is also of interest in the derivation of novel risk measures in the financial sectors like CoVaR (see, e.g., Bernard et al. (2013)).

Now, the construction of copulas with a given orthogonal section can be obtained by using the following result, which generalizes Proposition 2.1 in Klement et al. (2007). The result below deals with the orthogonal section related to the d-th coordinate; one might as well have chosen a different coordinate.

Theorem 8. *If a function $h_b : \mathbb{I}^{d-1} \to \mathbb{I}$ satisfies conditions* (a)–(c) *of Definition 3, then there exists a copula $C \in \mathscr{C}_d$ of which h_b is the d-th orthogonal section at b.*

Proof. Given $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{I}^d$ set $\mathbf{u}' := (u_1, \dots, u_{d-1})$. The function defined by

$$\widetilde{C}_b(\mathbf{u}) := \begin{cases} \dfrac{u_d\, h_b(\mathbf{u}')}{b}, & u_d \le b, \\ h_b(\mathbf{u}') + \dfrac{u_d - b}{(1-b)^{d-1}} \\ \quad \times \displaystyle\prod_{i=1}^{d-1}\left(u_i - h_b(1,\dots,1,u_i,1,\dots,1)\right), & u_d > b, \end{cases}$$

is a d-copula. It is immediately seen that the d-th orthogonal section of $\widetilde{C}_b$ at b equals h_b. It follows from (a) of Definition 3 that

$$h_b(u_1,\dots,u_{i-1},0,u_{i+1},\dots,u_{d-1}) = 0;$$

therefore

$$\widetilde{C}_b(u_1,\dots,u_{i-1},0,u_{i+1},\dots,u_d) = 0,$$

for $i \ne d$, while the definition of $\widetilde{C}_b$ yields $\widetilde{C}_b(\mathbf{u}', 0) = 0$. On the other hand, since $h_b(1, \dots, 1) = b$ one has, for $i \ne d$,

$$\begin{aligned} \widetilde{C}_b(1,\dots,1,u_i,1,\dots,1) &= h_i + \frac{1-b}{(1-b)^{d-1}}(1-b)^{d-2}(u_i - h_i) \\ &= h_i + u_i - h_i = u_i, \end{aligned}$$

where $h_i := h_b(1, \dots, 1, u_i, 1, \dots, 1)$. For $u_d = t$ one has $\widetilde{C}_b(1, \dots, 1, t) = t$, if $t \le b$, while, if $t > b$,

$$\widetilde{C}_b(1,\dots,1,t) = b + \frac{t-b}{(1-b)^{d-1}}(1-b)^{d-1} = b + t - b = t.$$

The boundary conditions of a copula are thus satisfied. In order to show that $\widetilde{C}_b$ is d-increasing consider first a box $R = R' \times [u', u'']$ included in $\mathbb{I}^{d-1} \times [0, b]$; here $R' = \prod_{i=1}^{d-1}[a_i, b_i]$. The vertices of R are $(\mathbf{v}', u')$ and $(\mathbf{v}', u'')$, where $\mathbf{v}' = (v'_1, \dots, v'_{d-1})$ with $v'_i \in \{a_i, b_i\}$ $(i = 1, \dots, d-1)$. Then, if $s := \mathrm{card}(\{i : v'_i = a_i\})$

$$\begin{aligned} V_{\widetilde{C}_b}(R) &= \sum_{\mathbf{v}'}(-1)^s\left(\widetilde{C}_b(\mathbf{v}', u'') - \widetilde{C}_b(\mathbf{v}', u')\right) \\ &= \frac{u''}{b}\sum_{\mathbf{v}'}(-1)^s h_b(\mathbf{v}') - \frac{u'}{b}\sum_{\mathbf{v}'}(-1)^s h_b(\mathbf{v}') \\ &= \frac{u''-u'}{b}\sum_{\mathbf{v}'}(-1)^s h_b(\mathbf{v}') = \frac{u''-u'}{b} V_{h_b}(R') \ge 0, \end{aligned}$$

since h_b is $(d-1)$-increasing.

Consider now a box $R = R' \times [u'', u']$ contained in $\mathbb{I}^{d-1} \times\ ]b, 1]$. For the ease of notation, set $h'_i := h_b(1, \dots, 1, a_i, 1, \dots, 1)$ and

$h_i'' := h_b(1, \dots, 1, b_i, 1, \dots, 1)$. Then an easy, but fairly long, calculation yields

$$\begin{aligned} V_{\widetilde{C}_b}(R' \times [u', u'']) &= \frac{u''-u'}{(1-b)^{d-1}} \sum_{\mathbf{v}'} (-1)^s \prod_{i=1}^{d-1} (b_i - h_i'')(a_i - h_i') \\ &= \frac{u''-u'}{(1-b)^{d-1}} \prod_{i=1}^{d-1} \left\{(b_i - a_i) - (h_i'' - h_i')\right\} \\ &\geq \frac{u''-u'}{(1-b)^{d-1}} \prod_{i=1}^{d-1} \{(b_i - a_i) - (b_i - a_i)\} = 0, \end{aligned}$$

where use has been made of the 1-Lipschitz property of h_b.

The general case follows from the additivity of $\widetilde{C}_b$-volumes. □

Notice that the function $\mathbb{I}^{d-1} \ni \mathbf{u}' \mapsto \frac{h_b(\mathbf{u}')}{b}$ is a $(d-1)$-dimensional continuous distribution function on $\mathbb{I}^d$, as is immediately seen; therefore, there exists a unique copula $C_b \in \mathscr{C}_{d-1}$ such that, for every $\mathbf{u}' = (u_1, \dots, u_{d-1}) \in \mathbb{I}^{d-1}$,

$$\frac{h_b(\mathbf{u}')}{b} = C_b\left(F_1^b(u_1), \dots, F_{d-1}^b(u_{d-1})\right), \tag{12}$$

where, for $i = 1, \dots, d-1$, F_i^b is its i-th marginal

$$F_i^b(u_i) = \frac{h_b(1, \dots, 1, u_i, 1, \dots, 1)}{b}.$$

Recourse to the construction of patchwork copulas allows us to characterize all copulas in $\mathscr{C}_d$ that have h_b as their d-th orthogonal section.

Theorem 9. *Given the orthogonal section h_b the following statements are equivalent for a copula $C \in \mathscr{C}_d$:*

(a) *C has h_b as its d-th orthogonal section;*
(b) *C has the representation*

$$C(\mathbf{u}) = b\, C_1\left(F_1^b(u_1), \dots, F_{d-1}^b(u_{d-1}), \frac{u_d}{b}\right),$$

for $u_n \leq b$, and

$$h_b(\mathbf{u}') + (1-b)\, C_2\left(\frac{u_1 - h_b(u_1, 1, \dots, 1)}{1-b}, \dots, \frac{u_{d-1} - h_b(1, \dots, 1, u_{d-1})}{1-b}, \frac{u_d - b}{1-b}\right),$$

for $u_n > b$, where C_1 is any copula that has C_b of Eq. (12) as marginal, viz., $C_1(\mathbf{v}', 1) = C_b(\mathbf{v}')$ for every $\mathbf{v}' \in \mathbb{I}^{d-1}$, while C_2 is any d-copula.

Proof. (a) $\Longrightarrow$ (b) It will be seen that C can be written as $\langle C_j, R_j \rangle_{j=1,2}^{\widetilde{C}_b}$. To this end, take $R_1 = \mathbb{I}^{d-1} \times [0, b]$ and $R_2 = \mathbb{I}^{d-1} \times [b, 1]$. The function

$$\mathbb{I}^d \ni \mathbf{u} \mapsto H(\mathbf{u}) := \frac{1}{b}\, C(\mathbf{u}', b u_d)$$

is a distribution function concentrated on R_1 whose d-th marginal is given by $H(\mathbf{u}', 1) = h_b(\mathbf{u}')$. Therefore, H must necessarily be expressed in the form

$$H(\mathbf{u}) = C_1\left(F_1^b(u_1), \dots, F_{d-1}^b(u_{d-1}), \frac{u_d}{b}\right),$$

where $C_1(\mathbf{v}', 1) = C_b(\mathbf{v}')$ for every $\mathbf{v}' \in \mathbb{I}^{d-1}$. In a similar manner, the function

$$\mathbb{I}^d \ni \mathbf{u} \mapsto H_2(\mathbf{u}) := \frac{1}{1-b}\left(C\left[(1-b)(u_1 + h_1^b(u_1)), \dots, (1-b)(u_{d-1} + h_{d-1}^b), u_d - b\right] - h_b(\mathbf{u}')\right)$$

is a distribution function concentrated on R_2; therefore there exists a copula C_2 such that

$$H_2(\mathbf{u}) = C_2\left(\frac{u_1 - h_b(u_1, 1, \dots, 1)}{1-b}, \dots, \frac{u_{d-1} - h_b(1, \dots, 1, u_{d-1})}{1-b}, \frac{u_d - b}{1-b}\right).$$

The proof of implication (b) $\Longrightarrow$ (a) is obvious. □

5. Conclusions

We have presented a method to construct copulas by modifying the probability mass distribution of a given copula in some suitable subsets of the domain. The methodology presented here includes as special cases a number of constructions presented in the literature under different names, including upper and partial comonotonicity. In particular, the method shows that it is possible to spread the probability mass distribution of a copula in the tail of a distribution in a multitude of ways. Therefore, modelling and estimating risks when the tails are involved should be an exercise that need special care, since the tail behaviour may be much more complex than standard copula families are able to describe.

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