A Note on the Ichoua et al (2003) Travel Time Model

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Abstract

In this paper we exploit some properties of the travel time model proposed by Ichoua et al (2003), on which most of the current time-dependent vehicle routing literature relies. Firstly, we prove that any continuous piecewise linear travel time model can be generated by an appropriate Ichoua et al (2003) model. We also show that the model parameters can be obtained by solving a system of linear equations for each arc. Then such parameters are proved to be nonnegative if the continuous piecewise linear travel time model satisfies the FIFO property, which allows to interpret them as (dummy) speeds. Finally, we illustrate the procedure through a numerical example. As a by-product, we are able to link the travel time models of a *road graph* and the associated *complete graph* over which vehicle routing problems are usually formulated.

Keywords: Time-Varying Travel Times, Vehicle Routing

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1. Introduction

Most of the literature on time-dependent vehicle routing relies on the stepwise speed model proposed by Ichoua, Gendreau and Potvin in 2003 (IGP model, in the following). The main point in their model is that they do not assume a constant speed over the entire length of a link. Rather, the speed changes when the boundary between two consecutive time periods is crossed. This feature guarantees that if a vehicle leaves a node i for a node j

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at a given time, any identical vehicle leaving node i for node j at a later time will arrive later at node j (*no-passing* or *first-in-first-out* (FIFO) property). In this paper we prove that any continuous piecewise linear travel time model can be generated by an appropriate IGP model, and show how to compute the model parameters. We also prove that such parameters can be interpreted as speeds if the time model satisfies the FIFO property. These results allow us to link the travel time models of a *road graph* and the associated *complete graph* over which vehicle routing problems are usually formulated. This is quite interesting because, while the hypothesis of instantaneous speed variation over an arc is quite realistic for the arcs of the *road graph* (at least if the corresponding streets are not too long), it is not so intuitive that this assumption may be reasonable for the associated complete graph as well.

The literature on Time-Dependent Vehicle Routing is fairly limited and can be divided, for the sake of convenience, into four broad areas: travel time modeling and estimation; the *Time-Dependent Shortest Path Problem* (TDSPP); the *Time-Dependent Traveling Salesman Problem* (TDTSP) and its variants; and the *Time-Dependent Vehicle Routing Problem* (TDVRP). Here we focus on the first research stream. [1] proposed a model for time-dependent travel speeds and several approaches for estimating the parameters of this model. The modeling approach has been implemented in a commercial courier vehicle scheduling system and was judged to be "very useful" by users in a number of different metropolitan areas in the United States. [2] proposed a travel time modeling approach based on a continuous piecewise linear travel time function (the IGP model). Later, [3] investigated the assumptions that this function must satisfy to ensure that travel times satisfy the FIFO property. They also described the derivation of travel time data from modern traffic information systems. In particular, they presented a general framework for the implementation of time-varying travel times in various vehicle-routing algorithms. Finally, they reported on computational tests with travel time data obtained from a traffic information system in the city of Berlin. [4] investigated exact and approximate methods for estimating time-minimizing vehicular movements in road network models where link speeds vary over time. The assumptions made about network conditions recognize the intrinsic relationship between speed and travel duration and are substantiated by elementary methods to obtain link travel duration. The assumptions also imply a condition of FIFO consistency, which justifies the use of Dijkstra's algorithm for path-finding purposes.

This paper is organized as follows. In Section 2, we gain some insight into a constant stepwise travel speed model *with constant distances* and illustrate a procedure for deriving any continuous piecewise linear travel time model from a suitable *IGP* model. We also show that the model parameters can be obtained by solving a system of linear equations for each arc. In Section 3, we illustrate a numerical example, while in Section 4 we exploit the relationship between the travel time models of a *road graph* and the associated *complete graph* over which vehicle routing problems are usually formulated. Conclusions and future research issues are reported in Section 5.

2. Continuous piecewise linear travel times and the *IGP* model

In this section, we prove that any continuous piecewise linear travel time model can be generated by an appropriate *IGP* model. Let $G = (V, A)$ be a graph, where $V = \{1, \ldots, n\}$ is a set of vertices and A is a set of arcs. With each arc $(r, s) \in A$ is associated a nonnegative length L_{rs} which is assumed to be constant over time. Moreover, let $[0, T]$ be the time horizon over which a vehicle route (or a set of vehicle routes) may be completed.
According to Ichoua et al (2003), for any arc $(r, s) \in A$ the time horizon is partitioned into H_{rs} time slots $[T_{rsh}, T_{rs(h+1)}]$ $(h = 0, \ldots, H_{rs} - 1)$. During each time slot h the speed is assumed to be equal to a constant ν_{rsh} on arc $(r, s) \in A$. Given these speeds, the arc travel time functions $\tau_{rs}(t)$ can be computed by using the *IGP* Algorithm [2]:

Algorithm 1 Computing $\tau_{rs}(t)$ according to the *IGP* model

```
Let t ∈ [T_rsh, T_rs(h+1)[.
k ← h
d ← L_rs
t' ← t + d/ν_rsk
while t' > T_rs(k+1) do
    d ← d − ν_rsk(T_rs(k+1) − t)
    t ← T_rs(k+1)
    t' ← t + d/ν_rs(k+1)
    k ← k + 1
end while
    return t' − t.
```

The main idea of this model is that when the vehicle traverses an arc, speed is not a constant over the entire length but it changes when the boundary between two consecutive time periods is crossed. Firstly, we prove some properties of the IGP model. For the sake of simplicity, from now on, we omit the arc indices rs when it is clear which arc (r, s) we are referring to.

Theorem 1. *Given an arc $(r, s) \in A$ and a start time t, the parameters of the IGP model satisfy the following equation:*

$$L = (T_{p+1} - t)\nu_p + \sum_{\ell=p+1}^{q-1} (T_{\ell+1} - T_\ell)\nu_\ell + [t + \tau(t) - T_q]\nu_q, \tag{1}$$

where $[T_p, T_{p+1}]$ and $[T_q, T_{q+1}]$ are the time intervals in which the start time t and the arrival time $t + \tau(t)$ fall, respectively.

Proof. The length traversed by a vehicle in a time interval $[t_1, t_2]$ is equal to the integral of its speed function, between t_1 and t_2. In particular, for any arc $(r, s) \in A$ and any start time t, it turns out that:

$$L = \int_t^{t+\tau(t)} v(t)dt. \tag{2}$$

Since the IGP model assumes a stepwise speed function, the integral can be expressed as:

$$\int_t^{T_{p+1}} \nu_p dt + \sum_{\ell=p+1}^{q-1} \int_{T_\ell}^{T_{\ell+1}} \nu_\ell dt + \int_{T_q}^{t+\tau(t)} \nu_q dt. \tag{3}$$

Hence the thesis is proved. □

Given any arc $(r, s) \in A$, the travel time $\tau(t)$ generated by the IGP Algorithm is a continuous piecewise linear function. Let t_k $(k = 0, \dots, K - 1)$ be the breakpoints at which its slope changes. The following property establishes a relationship between the breakpoints t_k of $\tau(t)$ and the breakpoints T_h of the speed function for the IGP model.

Theorem 2. *Given an arc $(r, s) \in A$, for any travel time breakpoint t_k, one of the following conditions holds:*

1. *a travel speed change occurs at t_k, i.e. there exists a time period $h \in \{0, \dots, H - 1\}$ such that $T_h = t_k$;*

2. *a travel speed change occurs at* $t_k + \tau(t_k)$*, i.e. there exists a time period* $h \in \{0, \dots, H-1\}$ *such that* $T_h = t_k + \tau(t_k)$*.*

Proof. Let $[T_p, T_{p+1}]$ and $[T_q, T_{q+1}]$ be the time intervals in which t_k and $t_k + \tau(t_k)$ fall, respectively. We prove the thesis by contradiction. Therefore we suppose that there exists a breakpoint t_k with $k \in \{0, \dots, K-1\}$ and a $\Delta > 0$, such that :

$$[t_k - \Delta, t_k + \Delta] \subseteq [T_p, T_{p+1}] \tag{4}$$

and

$$[t_k - \Delta + \tau(t_k - \Delta), t_k + \Delta + \tau(t_k + \Delta)] \subseteq [T_q, T_{q+1}]. \tag{5}$$

By writing (1) for the time instants $(t_k - \Delta)$ and t_k, we obtain:

$$L = (T_{p+1} - t_k + \Delta)\nu_p + \sum_{\ell=p+1}^{q-1} (T_{\ell+1} - T_\ell)\nu_\ell + (t_k - \Delta + \tau(t_k - \Delta) - T_q)\nu_q, \tag{6}$$

$$L = (T_{p+1} - t_k)\nu_p + \sum_{\ell=p+1}^{q-1} (T_{\ell+1} - T_\ell)\nu_\ell + (t_k + \tau(t_k) - T_q)\nu_q. \tag{7}$$

By subtracting (6) from (7), we obtain:

$$\frac{\nu_p}{\nu_q} - 1 = \frac{(\tau(t_k) - \tau(t_k - \Delta))}{\Delta}. \tag{8}$$

Similarly, we determine (9) for the time instants t_k and $(t_k+\Delta)$ by subtracting (7) from (1) rewritten for the time instant $(t_k + \Delta)$:

$$\frac{\nu_p}{\nu_q} - 1 = \frac{(\tau(t_k + \Delta) - \tau(t_k))}{\Delta}. \tag{9}$$

By hypothesis t_k is a breakpoint for $\tau(t)$, that is:

$$\frac{\tau(t_k) - \tau(t_k - \Delta)}{\Delta} \neq \frac{\tau(t_k + \Delta) - \tau(t_k)}{\Delta}. \tag{10}$$

Since (8) and (9) contradict (10), then the thesis is proved. □

This theorem implies that the speed function is constant piecewise with at most $2K$ breakpoints T_h included into the set $\{t_k, k = 0, \ldots, K-1\} \cup \{t_k + \tau(t_k), k = 0, \ldots, K-1\}$.

From Theorem 2 the following Corollary follows.

Corollary 3. *Given an arc $(r, s) \in A$, if for each time interval $[T_{h-1}, T_h]$ no speed change occurs in $[T_{h-1} + \tau(T_{h-1}), T_h + \tau(T_h)]$ with $h = 0, \ldots, H-1$, then it results that:*

$$\{t_k, k = 0, \ldots, K-1\} \subseteq \Omega \equiv \{T_h, h = 1, \ldots, H-1\}.$$

It is worth noting that the analytical representation of $\tau(t)$ is not explicitly given by the IGP model. The Corollary 3 states when the travel time function $\tau(t)$ can be analytically defined as the set of line segments connecting the points $(T_h, \tau(T_h))$, with $h = 0, \ldots, H-1$.

We now prove that any continuous piecewise linear travel time model can be generated by an appropriate Ichoua et al (2003) model. In particular, we demonstrate that, for each arc, the model parameters are obtained into two steps. In the first step the Algorithm 2 determines a set of speed breakpoints satisfying the hypothesis of Corollary 3 . In the second step the speed constant values and the arc length are obtained by solving a system of linear equations.

We start by demonstrating the correctness of Algorithm 2 through the Lemma 4. We denote $\Gamma(t)$ and $\Gamma^{-1}(t)$ the arrival time function of $\tau(t)$ and its inverse function, respectively. It is worth noting that if the $\tau(t)$ satysfies the FIFO property then $\Gamma(t)$ and $\Gamma^{-1}(t)$ are nondecreasing functions. The positive monotony of $\Gamma(t)$ and $\Gamma^{-1}(t)$ assures a finite number of iterations of the *while loops*.

Lemma 4. *Given a continuous piecewise linear travel time function $\tau(t)$ satisfying the FIFO property, the Algorithm 2 computes a set of speed breakpoints T_h, such that no speed change occurs in the time interval $[\Gamma(T_{h-1}), \Gamma(T_h)]$ with $h = 1, \ldots, H-1$.*

Proof. We start by observing that the Algorithm 2 determines a set of speed breakpoints $\{T_h, h = 1, \ldots, H-1\}$ where

$$t_0 = T_0 < T_h \leq T_{H-1} = \Gamma(t_{K-1}). \quad (11)$$

Algorithm 2 Determine a set of speed breakpoints Ω given the set of time breakpoints $\{t_0, \ldots, t_{K-1}\}$

```
Ω=∅
for all t ∈ {t_0, ..., t_{K-1}} do
    if t ∉ Ω then
        Ω ← t
        t' ← t
        while (t' ≤ t_{K-1}) ∧ (Γ(t) ∉ Ω) do
            Ω ← Γ(t')
            t' ← Γ(t')
        end while
        t' ← t
        while (t' ≥ Γ(t_0) ∧ (Γ^{-1}(t') ∉ Ω) do
            Ω ← Γ^{-1}(t')
            t' ← Γ^{-1}(t')
        end while
    end if
end for
return Ω
```

Moreover for each determined speed breakpoint T_h one of the following condition holds, with $h = 0, \ldots, H-1$:

$$T_h \leq t_{K-1} \Rightarrow \Gamma(T_h) \in \{T_0, \ldots, T_{H-1}\}; \tag{12}$$

$$T_h \geq \Gamma(T_0) \Rightarrow \Gamma^{-1}(T_h) \in \{T_0, \ldots, T_{H-1}\}. \tag{13}$$

In order to prove the thesis by contradiction, we suppose that the Algorithm 2 determines three distinct speed breakpoint T^a, T^b and T^c such that no speed change occurs in $[T^a, T^b]$ but $\Gamma(T^a) < T^c < \Gamma(T^b)$.

Since $\Gamma^{-1}(t)$ is a no-decreasing function, we can assert that $\Gamma^{-1}(T^c) \notin \{T_h, h = 1, \ldots, H-1\}$. From the conditions (13) this means that :

$$T^c < \Gamma(T_0). \tag{14}$$

Since the travel speed is constant after T_{H-1}, from (11) and the positive monotony of the time arrival function $\Gamma(t)$ it results that $T^a < T^b \leq t_{K-1}$. Since the breakpoints T^a and T^b satisfies the condition (12), then $\Gamma(T^a)$ and $\Gamma(T^b)$ are speed breakpoints. Finally, the condition (13) requires that $\Gamma(T_0) \leq \Gamma(T^a) \leq T'$, which contradicts the inequality (14). □

Theorem 5. *Given a continuous piecewise linear travel time function $\tau(t)$ satisfying the FIFO property, there always exist a constant length $L \geq 0$ and a constant stepwise function $v(t)$, such that $\tau(t)$ can be obtained as an output of the IGP Algorithm when L and $v(t)$ are provided as inputs.*

Proof. The thesis is proved if we determine $L > 0$ and a constant stepwise function $v(t)$ such that:

$$L = \int_t^{t+\tau(t)} v(t)dt. \tag{15}$$

We consider as time partition for $v(t)$, induced by the set Ω output from the Algorithm 2.

the travel speed function can be casted as:

$$v(t) = \nu_h \quad t \in [T_h, T_{h+1}[, h = 0, \ldots, H-1,$$

where ν_h are unknown. By writing Equation (1) for start times $t = T_h$ $(h = 0, \ldots, H-1)$, we get:

$$L = (T_{h+1} - T_h)\nu_h + \sum_{\ell=h+1}^{q_h-1} (T_{\ell+1} - T_\ell)\nu_\ell + (T_h + \tau(T_h) - T_{q_h})\nu_{q_h}, \tag{16}$$

where $[T_{q_h}, T_{q_h+1}[$ represents the time interval in which arrival time $T_h + \tau(T_h)$ falls. Let $a_{h\ell}$ be the time spent on the arc during period $[T_\ell, T_{\ell+1}]$ ($\ell = 0, \ldots, H-1$) if the start travel time is T_h. Equations (16) constitute a square linear system:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,H-1} \\ 0 & a_{22} & \cdots & a_{2,H-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{H-1,H-1} \end{bmatrix} * \begin{bmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{H-1} \end{bmatrix} = \begin{bmatrix} L \\ L \\ L \\ \vdots \\ L \end{bmatrix}, \tag{17}$$

where ν_h ($h = 0, \ldots, H-1$) and L are the unknowns. For any $L > 0$ this system has a unique solution since matrix A is in inferior triangular form with not null diagonal elements. As stated by Corollary 2, by applying the IGP model to L and $v(t)$ we obtain a travel time function with no further time breakpoint guarantees that by applying the IGP model with L and □

The contribution of Theorem 3 is twofold. Firstly, it shows that it is always possible to define the *IGP* constant stepwise function $v(t)$ over the time partition induced by $T_h \in \{t_k, k = 0, \ldots, K-1\} \cup \{\tau(t_k), k = 0, \ldots, K-1\}$:

$$v(t) = \nu_h \quad t \in [T_h, T_{h+1}[, \tag{18}$$

with $h = 0, \ldots, H-1$. Secondly, given a constant value $L > 0$, the constant values of $v(t)$ are univocally determined by solving the square linear system (17), where $a_{h,\ell}$ corresponds to the travel time spent in interval $[T_\ell, T_{\ell+1}[$, when the start travel time is T_h, with $h = 0, \ldots, H-1$ and $\ell = 0, \ldots, H-1$.

Finally, we prove that if the continuous piecewise linear model satisfies the FIFO property, then the *IGP* model parameters are nonnegative.

Theorem 6. *If the continuous piecewise linear travel time function $\tau(t)$ satisfies the FIFO property, then parameters ν_h, are nonnegative for any $L > 0$.*

Proof. The thesis is proved if we demonstrate that, for any given $L > 0$, the solution of (17) is a strictly positive vector, i.e. $\nu_h > 0$, with $h = 0, \ldots, H-1$. Matrix A is in inferior triangular form with not null diagonal elements. Since the coefficient $a_{h,\ell}$ represents the time spent on the arc during the period

$[T_\ell, T_{\ell+1}]$ if the start travel time is T_h, then:

$$a_{h,\ell} = \begin{cases} \geq 0 & if \quad h < \ell \\ > 0 & if \quad h = \ell \ , \\ 0 & if \quad h > \ell \end{cases} \tag{19}$$

with $h = 0, \ldots, H-1$ and $\ell = 0, \ldots, H-1$. Since $\tau(t)$ satisfies the FIFO property, then :

$$0 \leq a_{0,\ell} \leq \cdots \leq a_{\ell-1,\ell} < a_{\ell,\ell}, \tag{20}$$

with $\ell = 0, \ldots, H-1$. System (17) can be solved by executing a sequence of H pivot operations. In particular during the *q-th* iteration, we choose as pivot the diagonal element (ℓ, ℓ), with $\ell = H - q$ and $q = 1, \ldots, H$. Let us denote with L_{hq} the *h-th* component of the right hand side value of (17) after q pivot operations, with $q = 0, \ldots, H$ and $h = 0, \ldots, H-1$. It is worth noting that the solution of (17) is a strictly positive vector, only if, during the q-th iteration, the minimum ratio test on the ℓ*-th* column is satisfied only at the ℓ*-th* row, with $\ell = H - q$ and $q = 1, \ldots, H$. Since the matrix A is in superior triangular form, this implies that:

$$L_{\ell,q-1}/a_{\ell,\ell} = \min_{h=0,\ldots,\ell} (L_{h,q-1}/a_{h,\ell} : a_{h,\ell} > 0), \tag{21}$$

with $\ell = H - q$ and $q = 1, \ldots, H$. From (20), it results that the thesis is proved if, given $L > 0$, then:

$$0 < L_{\ell,q-1} \leq L_{\ell-1,q-1} \leq \cdots \leq L_{0,q-1}, \tag{22}$$

with $\ell = H - q$ and $q = 1, \ldots, H$. We demonstrate (22) by induction on q.
Case $q = 1$. Since in (22) $\ell = (H-1)$ and $q - 1 = 0$, it results that

$$0 < L = L_{H-1,0} = L_{H-2,0} = \cdots = L_{0,0}.$$

Case $q > 1$. We suppose that (22) holds for q. From (20) and (22) it results that

$$\begin{aligned} 0 &< L_{\ell-1,q-1} - L_{\ell,q-1}(a_{\ell-1,\ell}/a_{\ell,\ell}) \leq \\ &\leq L_{\ell-2,q-1} - L_{\ell,q-1}(a_{\ell-2,\ell}/a_{\ell,\ell}) \leq \ldots \\ &\cdots \leq L_{0,q-1} - L_{\ell,q-1}(a_{0,\ell}/a_{\ell,\ell}), \end{aligned} \tag{23}$$

with $\ell = H - q$. At the q-th iteration the pivot is $a_{\ell,\ell}$ with $\ell = H - q$ and it results that:

$$L_{\ell-h,q} = L_{\ell-h,q-1} - L_{\ell,q-1}(a_{\ell-h,\ell}/a_{\ell,\ell}), \tag{24}$$

with $h = 1, \dots, \ell$. From (24) and (23), it results that (22) holds for $(q+1)$, that is:

$$0 < L_{\ell,q} \leq L_{\ell-1,q} \leq \cdots \leq L_{0,q}, \tag{25}$$

with $\ell = H - q - 1$.

□

3. A Numerical Example

We provide a numerical example to illustrate the previous properties. The arc lenght L is equal to 3. Figure 1(a) describes a continuous piecewise linear travel time function $\tau(t)$, whose breakpoints t_k are reported in Table 1 along with the corresponding $\tau(t_k)$ travel times.

k	t_k	$\tau(t_k)$
0	0	2
1	4	2
2	5	1.5

Table 1: Values of t_k and $\tau(t_k)$ for the numerical example

h	T_h	$\Gamma(T_h)$	$\Gamma^{-1}(T_h)$
0	0	2	-
1	1	3	-
2	2	4	0
3	3	5	1
4	4	6	2
5	5	6.5	3
6	6	-	4
7	6.5	-	5

Table 2: Values of T_h determined by Algorithm 2 for the numerical example

The Algorithm 2 determines the speed breakpoints reported in Table 3. We compute the constant stepwise IGP speed function $v(t)$ by solving the following linear system:

$$\begin{cases} \nu_0+\nu_1 & = 3 \\ \nu_1+\nu_2 & = 3 \\ \nu_2+\nu_3 & = 3 \\ \nu_3+\nu_4 & = 3 \\ \nu_4+\nu_5 & = 3 \\ \nu_5+0.5\nu_6 & = 3 \\ 0.5\nu_6+\nu_7 & = 3 \\ 1.5\nu_7 & = 3 \end{cases} \tag{26}$$

It is worth noting that the sum of the variables'coefficients in the $h-th$ equation is equal to $\tau(T_h)$. In particular, it is equal to $\tau(0) = 2$ in the first equation, $\tau(1) = 2$ in the second equation, etc. The speed values come up to be: $\nu_0 = 1, \nu_1 = 2, \nu_2 = 1, \nu_3 = 1, \nu_4 = 2, \nu_5 = 2, \nu_6 = 2, \nu_7 = 2$. See Figure 1(b) for a graphical representation.

4. Linking the travel time models of a road graph and the associated complete graph

Vehicle routing problems are often modelled on a *complete graph* G' in which the vertices represent the customers (and possibly additional facilities, such as a depot) and the arcs model quickest paths between pairs of customers and facilities on the underlying *road graph* G. The main point in the IGP model is that it does not assume a constant speed over the entire length of a link. Rather, the speed changes when the boundary between two consecutive time periods is crossed. This feature guarantees that $FIFO$ property holds. While the hypothesis of instantaneous speed variation over an arc is quite realistic for the arcs of the *road graph* (at least if the corresponding streets are not too long), it is not so intuitive that this assumption may be reasonable for the associated complete graph as well.

In this section, we exploit the relationships between the travel time models of the two graphs. In particular, we show that if the arc travel times of the road graph follow Ichoua et al (2003), then the arcs of the complete graph can be modeled by the same variation law (with suitable parameters). Let $\mathfrak{p}$ be a simple path $\{i = i_0, i_1, \ldots, j = i_m\}$ on the road graph G. We denote with $\mathcal{P}_{ij}$ the set of simple paths on G, connecting customer/facility i to customer/facility j. Let $z(\mathfrak{p}, t)$ be the traversal time of path $\mathfrak{p}$, whenever a vehicle leaves vertex i at time t. We observe that $z(\mathfrak{p}, t)$ is the sum of continuous piecewise linear functions. For example, for $m = 2$:

$$z(\mathfrak{p}, t) = \tau_{i_0 i_1}(t) + \tau_{i_1 i_2}(t + \tau_{i_0 i_1}(t)). \tag{27}$$

Hence $z(\mathfrak{p}, t)$ is continuous piecewise linear itself. On the complete graph G', the time-dependent travel time $\tau'_{ij}(t)$ of arc $(i, j) \in A'$ is given by:

$$\tau'_{ij}(t) = \min_{\mathfrak{p} \in \mathcal{P}_{ij}} z(\mathfrak{p}, t). \tag{28}$$

Since $\mathcal{P}_{ij}$ is a finite set, function $\tau'_{ij}(t)$ is continuous piecewise linear too. Hence, Theorem 3 implies that $\tau'_{ij}(t)$ can be generated by an IGP model, with a suitable choice of parameters L'_{ij} and ν'_{hij} $(h \in 0, \ldots, H'_{ij})$. It is worth noting that this property holds for any choice of $L'_{ij} > 0$. In particular, L'_{ij} can be chosen equal to the length of the shortest path from node i to node j on the road graph.

A straightforward consequence is that the IGP model does not suffer from the drawback pointed out by Fleischmann et al [3] who stated: "A drawback of the models with varying speeds but constant distances is that they do not consider potential changes of the fastest paths themselves due to varying travel times, which imply changes of distances". Another outcome is that the lower bounding procedure proposed by Cordeau et al [5] for the Time-Dependent Traveling Salesman Problem can be applied to the wider class of instances with continuous piecewise linear arc travel times: first, the IGP model's parameters have to be computed by solving a system of linear equations for every arc; then speeds ν_{ijh} are expressed as

$$\nu_{ijh} = \delta_{ijh} b_h u_{ij}, \tag{29}$$

where:

- u_{ij} is the maximum travel speed across arc $(i, j) \in A$ during $[0, T]$, i.e. $u_{ij} = \max_{h=0,\ldots,H-1} \nu_{ijh}$;

- b_h belongs to $[0, 1]$ and is the best (i.e. lightest) congestion factor during interval $[T_h, T_{h+1}]$, i.e. $b_h = \max_{(i,j)\in A} \nu_{ijh}/u_{ij}$;
- δ_{ijh} belongs to $[0, 1]$ and represents the degradation of the congestion factor of arc (i, j) in interval $[T_h, T_{h+1}]$ with respect to the less congested arc in $[T_h, T_{h+1}]$;

finally, a lower bound can be computed by: (a) determining a time-*independent* Traveling Salesman Problem optimal solution w.r.t. maximum travel speeds u_{ij}; (b) evaluating its traversal time w.r.t. to the most favourable congestion factor during each interval h, i.e. $v_{ijh} \leftarrow b_h u_{ij}$.

5. Conclusions

In this paper we have shown that the travel time model proposed by Ichoua et al (2003) is quite general since any continuous piecewise linear travel time model can be generated from it with a suitable choice of its parameters. Then some light has been shed on the relevance of this model on road graphs and the associated complete graphs. As a future research topic, we would suggest the extension of the lower bounding approach [5] to other time-dependent arc and node routing problems.

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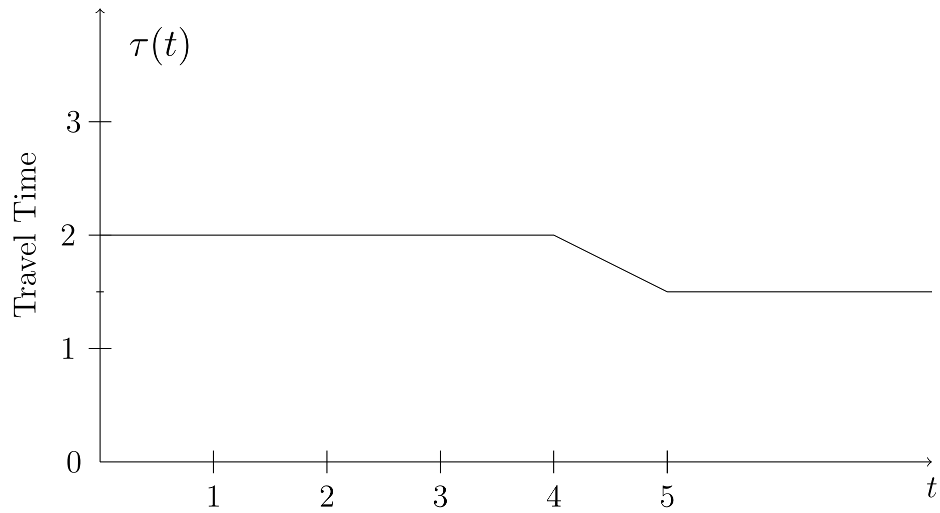

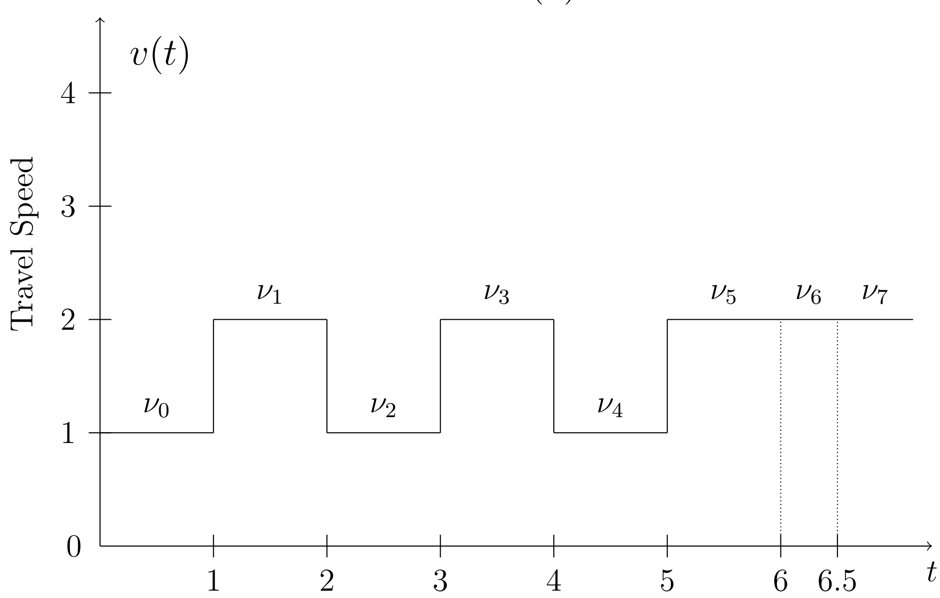

Figure 1: A continous piecewise linear arc travel time function and the associated constant stepwise speed function.