Linear and Nonlinear Free Surface Flows in Electrohydrodynamics

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I, Mat Hunt confirm that the work presented in this thesis is his own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

This thesis examines free surface flows in electrohydrodynamics under forcing in the form of a moving pressure distribution or topography. The ideas from examining free surface flows with forcing and those ideas and methods coming from examining solitary waves within electrohydrodynamics are combined to study free surface flows under forcing in electrohydrodynamics. Chapter 1 gives a brief introduction to the ideas and work that have gone into investigating free surface flows and solitary waves in general and gives an idea of what will happen in the thesis. Chapter 2 formulates the general problem for the full nonlinear case and then examines the linear solution for both a moving pressure distribution and topography and presents profiles of the free surfaces and then shows that the solutions are nonuniform by examining the deep water case. Chapter 3 introduces the scaling for the weakly nonlinear problem and produces an equation which there is no nonuniformity and the amplitude of the free surface is finite. The case when the Bond number is around a $1/3$ is also examined. Stokes analysis is performed to look for Wilton ripples. Chapter 4 examines conducting fluids adhering to an upper surface, the basic equations are set up and then the dispersion relation is derived to examine the existence of linear waves for certain values of the wavenumber k. A set of weakly nonlinear equations are examined and then solved numerically with examples of periodic profiles presented. A Stokes analysis is carried out for small amplitudes to look for Wilton ripples. An analysis is carried out for the approximation of long wavelength but finite depth, where the wave amplitude is the depth of the fluid. Chapter 5 considers surface flows and generalised the results from chapters 2 and 3 from two dimensions to three, linear free surface profiles are calculated and plotted and the weakly nonlinear equation is derived for the cases where the Bond number is close

to $1/3$ and not close to $1/3$ giving a 5th order (Kadomstev-Petviashvili) (KP) equation.

Chapter 6 is the set of conclusions and avenues of future research.

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Chapter 1

Introduction

The mathematical study of solitary waves began with the paper of Korteweg and de-Vries when they derived their famous equation:

$$\frac{\partial \eta}{\partial t} + \eta \frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0 \tag{1.1}$$

This equation was for shallow water and small amplitude along with surface tension. Since this paper, a number of modifications have been made to the modelling to try and include as many physical effects as possible. One of the most important extensions to the modelling is the modelling of interfacial waves where one of the mediums is shallow compared to the other regions. This was carried out by Benjamin and Ono ([28],[29]) without surface tension and then by Benjamin ([30]) with surface tension. With more phenomena modelled, the solutions become more and more complicated, the solitary waves in ([30]) having slowly dying oscillatory tales which was completely different to previous solitary waves discovered which were waves of elevation ([2]). The type of interfacial wave which will be considered for this thesis will be for fluids which are electrically conducting. These have various technological applications, which include coating and curtain coating processes[1] to obtain the coating patterns and hence product quality.

When trying to model waves, it is always necessary to make some simplifications. These are usually the assumption of small amplitude and/or the absence of disturbances. It is possible to include these in the modelling as topography and as an external pressure

[1]Another potential application has been the application of electrohydrodynamics to aerodynamics

distribution. These will result in what are known as *forced* equations and the resulting waves are not solitary waves in the technical sense. These forced waves will be the primary focus of this thesis. There has been much work on flow in the presence of disturbances such as topography and moving pressure distribution ([1],[2],[4],[17],[21]). Linear and nonlinear theories can be developed, the linear theory being a perturbation from a steady state. If the size of the disturbance (either pressure distribution or height of the topography) is ε, then as $\varepsilon \to 0$, all the nonlinear solution should converge to the linear ones. This is usually the case but the solutions rely on other parameters such as the Bond number, electric Bond number and the capillary number. This leads to non-uniformities as the parameters approach critical values (which can be obtained from the linear analysis) and manifest in having unbounded amplitudes for linear free surface profiles. An example of this was shown in ([1]) with the infinite depth case when the capillary number tended to $1/4$. The equivalent condition for the inclusion of electric field will be derived in chapter 2. When deriving the KdV equation one has a term (which can be scaled away) of $(B - 1/3)$ where B is the Bond number and if B is close to $1/3$, a different analysis is needed. This leads to a fifth order equation Korteweg de Vries equation. It was first investigated in the plasma physics literature [19] and then it was proposed for water waves by Kawahara in [24] and then formally derived using asymptotic analysis in [13] by Hunter and Vanden-Broeck.

The electric field complicates matters when integrating it into the hydrodynamics. The approach which is taken is to use the Maxwell stress tensor ([6],[7]) which is then made a part of the general stress tensor in the Navier-Stokes equations. This in general constitutes a coupled nonlinear problem which must be solved numerically. This is not really noticed in the main equations but has to be taken into account in the interfacial stresses, where the Maxwell stresses do change the usual hydrodynamics stresses. However, the coupled systems of equations may be reduced by making some assumptions. The disturbances are small enough to perturb from a uniform flow (as in the linear case). Another simplification of the problem is to look for travelling waves which are stationary given a suitable frame of reference. The electric field acts in the plane and extends to a constant at infinity. It is assumed that there are no charges inside the fluid and none on the interface either.

There has been a great deal of work done in the area of solitary waves in many settings, namely in the forced and unforced cases. Work has also been done on solitary waves with electric fields without any forcing, but as yet there has been no work done on forced waves in shallow water with long wavelengths and electric fields. This is the problem that this piece of work will address. The method of investigation will combine the analysis for forced waves without an external electric field with the methods for examining wave with electric fields and no forcing. The scope of this body of work will cover the basic extension of the KdV equation with electric fields and a moving pressure distribution and will then move to the case where the Bond number B is around $1/3$. The analysis will cover both linear and weakly nonlinear cases in two and three dimensions, generalising the KdV, Kawahara and KP equations.

Chapter 2

Formulation and Linear Theory

2.1 Formulation

The problem of consideration is the flow down a channel of a perfectly conducting inviscid and incompressible fluid (referred to as fluid 1) over a fixed surface. The configuration is assumed to be two-dimensional (three-dimensional flows will be considered in Chapter 5). Cartesian co-ordinates (x, y) are introduced, with y pointing vertically upwards and denote the equation of the fixed surface by $y = f(x)$. The velocity vector is given as $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + 0\mathbf{k}$. The incompressibility condition implies that $\nabla \cdot \mathbf{u} = 0$. The other key assumption is that the fluid is irrotational, which means that $\nabla \times \mathbf{u} = \mathbf{0}$, which implies the existence of a scalar function, φ, called the *velocity potential*, defined by $\mathbf{u} = \nabla\varphi$. Combining the notions of the velocity potential and incompressibility shows that:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{2.1}$$

There is an interface with the incompressible fluid with another fluid which (known as fluid 2) is a perfect dielectric, and in fluid 2 there is an electric field $\mathbf{E}$, as can be seen in figure 2.1. For simplicity it is assumed that fluid 2 is a perfect dielectric. On the surface $y = f(x)$, as there is no fluid flow across the boundary, the impermeable boundary condition is used:

$$\frac{D}{Dt}(y - f(x)) = 0. \tag{2.2}$$

The boundary condition is then

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial x}\frac{df}{dx} \tag{2.3}$$

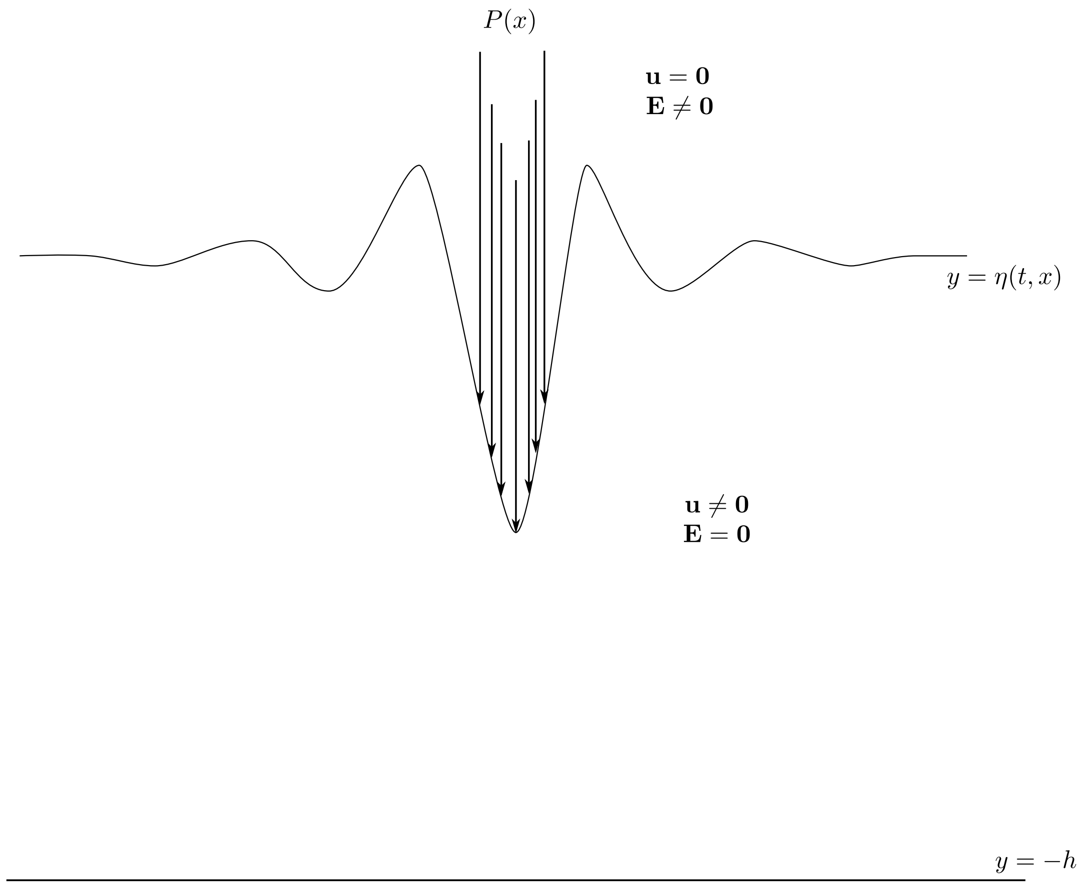


Figure 2.1: Set-up of Wave Problem

For simplicity, it is assumed that the electric field is not moving and so there is no magnetic field to be concerned about. Maxwell's equations then imply $\nabla \times \mathbf{E} = \mathbf{0}$. It is then possible to introduce a potential, V, such that $\mathbf{E} = -\nabla V$. Furthermore since there are no charges within the fluid, $\nabla \cdot \mathbf{E} = 0$ and the potential V satisfies Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{2.4}$$

The reference point $V = 0$ is chosen on the fixed surface $y = f(x)$. Since the lower fluid is a perfect conductor, $V = 0$ everywhere in it. A vertical electric field is imposed by requiring $V \sim -E_0 y$ as $y \to \infty$, with E_0 constant. The equations of motion are:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\rho \mathbf{g} + \nabla \cdot T \tag{2.5}$$

Here $\mathbf{g} = g\mathbf{j}$ is the acceleration of gravity and T is the stress tensor made up of the hydrostatic pressure and of the Faraday tensor:

$$T_{ij} = -p\delta_{ij} + \Sigma_{ij} \tag{2.6}$$

where δ_{ij} is the Kronecker symbol and

$$\Sigma_{ij} = \epsilon_p \left(E_i E_j - \frac{1}{2}\delta_{ij} E_k E_k \right) \tag{2.7}$$

where ϵ_p is the dielectric constant. Computing the divergence of the Faraday tensor gives

$$\begin{aligned}
(\nabla \cdot T)_j &= \partial_i T_{ij} \\
&= -\delta_{ij}\partial_i p + \partial_i \Sigma_{ij} \\
&= -\delta_{ij}\partial_i p + \epsilon_p \partial_i \left(E_i E_j - \frac{1}{2}\delta_{ij} E_k E_k \right) \\
&= -\partial_j p + \epsilon_p \left(\partial_i (E_i E_j) - \frac{1}{2}\partial_j (E_k E_k) \right) \\
&= -\partial_j p + \epsilon_p \left(\partial_i \left[\frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} \right] - \frac{1}{2}\partial_j \left[\frac{\partial V}{\partial x_k} \right]^2 \right) \\
&= -\partial_j p + \epsilon_p \left(\partial_i \partial_j V \partial_i V + \partial_j V \sum_i \partial_i^2 V - \frac{1}{2}\partial_j \left[\frac{\partial V}{\partial x_k} \right]^2 \right) \\
&= -\partial_j p + \epsilon_p \left(\frac{1}{2}\partial_j \left[\frac{\partial V}{\partial x_i} \right]^2 - \frac{1}{2}\partial_j \left[\frac{\partial V}{\partial x_k} \right]^2 \right) \\
&= -\partial_j p.
\end{aligned}$$

Therefore the electric field does not come into the equations of motion (2.5) and the only coupling between the potential functions φ and V comes through the boundary conditions. The only other boundary condition in which such coupling can occur is the Bernoulli equation. The equation of the interface between the two fluids is denoted by $y = \eta(t, x)$. The function $\eta(t, x)$ is unknown and has to be found as part of the solution. For this reason the interface is referred to as the free surface. Defining $F := \eta(t, x) - y$, the kinematic boundary condition on the free surface can be expressed as

$$\frac{DF}{Dt} = 0 \tag{2.8}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}.\nabla \tag{2.9}$$

is the material derivative. The condition (2.8) expresses the fact that particles which are on the free surface remain on the free surface. Expanding (2.8) shows that:

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v. \tag{2.10}$$

The main result for free surfaces is the Young-Laplace equation which states that

$$[\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]_1^2 = \sigma \nabla \cdot \hat{\mathbf{n}} \tag{2.11}$$

where σ denotes the surface tension. The notation $[\ldots]_1^2$ denotes the difference of the values of a quantity across the interface. The vector normal to the free surface is given by $\mathbf{n} = \nabla F = (\partial_x \eta, -1)$. Therefore the unit normal to the free surface is then given by:

$$\hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|} = \frac{(\partial_x \eta, -1)}{(1 + (\partial_x \eta)^2)^{1/2}} \tag{2.12}$$

A simple calculation shows that the divergence of the unit normal vector is given by

$$\begin{aligned}
\nabla \cdot \hat{\mathbf{n}} &= \frac{\partial}{\partial x}\left(\frac{\partial_x \eta}{(1 + (\partial_x \eta)^2)^{1/2}}\right) \\
&= \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{1/2}} + \partial_x \eta \partial_x (1 + (\partial_x \eta)^2)^{1/2} \\
&= \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{1/2}} - \partial_x \eta \left[(1 + (\partial_x \eta)^2)^{-3/2} \partial_x \eta \partial_x^2 \eta\right] \\
&= \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{1/2}} - \frac{(\partial_x \eta)^2 \partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{3/2}} \\
&= \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{3/2}} \left[1 + (\partial_x \eta)^2 - (\partial_x \eta)^2\right] \\
&= \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{3/2}}.
\end{aligned} \tag{2.13}$$

The geometric interpretation is that $\nabla \cdot \hat{\mathbf{n}}$ is the mean curvature of the free surface. Using (2.6) to obtain

$$\begin{aligned}
\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}} &= \hat{n}_i T_{ij} \hat{n}_j \\
&= \hat{n}_i (-p\delta_{ij} + \Sigma_{ij}) \hat{n}_j \\
&= -p\hat{n}_i \hat{n}_i + \hat{n}_i \Sigma_{ij} \hat{n}_j \\
&= -p + \epsilon_p (\hat{n}_1^2 \Sigma_{11} + 2\hat{n}_1 \hat{n}_2 \Sigma_{12} + \hat{n}_2^2 \Sigma_{22})
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}\hat{n}_1 &= \frac{\partial_x \eta}{(1+(\partial_x \eta)^2)^{1/2}} \\ \hat{n}_2 &= -\frac{1}{(1+(\partial_x \eta)^2)^{1/2}} \\ \Sigma_{11} &= \frac{\epsilon_p}{2}((\partial_x V)^2 - (\partial_y V)^2) && (2.15) \\ \Sigma_{12} &= \epsilon_p \partial_x V \partial_y V && (2.16) \\ \Sigma_{22} &= -\frac{\epsilon_p}{2}((\partial_x V)^2 - (\partial_y V)^2). && (2.17)\end{aligned}$$

Inserting (2.13) and (2.14) into (2.11) gives

$$[p]_1^2 + \left[\frac{(\partial_x \eta)^2}{1+(\partial_x \eta)^2}\Sigma_{11}\right]_1^2 - 2\left[\frac{\partial_x \eta}{1+(\partial_x \eta)^2}\Sigma_{12}\right]_1^2 + \left[\frac{1}{1+(\partial_x \eta)^2}\Sigma_{22}\right]_1^2 = \sigma \frac{\partial_x^2 \eta}{(1+(\partial_x \eta)^2)^{3/2}}. \tag{2.18}$$

As before the brackets mean to take the difference of the quantity at each side of the interface. The electric potential on one side of the interface is zero, which is why the brackets can be dropped. This can be substituted into the unsteady Bernoulli equation to obtain a boundary condition for the free surface, via the pressure term. The unsteady Bernoulli equation is:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\nabla \varphi|^2 + \frac{p}{\rho} + g\eta = r(t) \tag{2.19}$$

It is usual to incorporate $r(t)$ into φ via the transformation $\varphi \mapsto \varphi + \int r(s)ds$, which will turn the function of time on the right hand side of (2.19) into a constant.

By analogy with problems without electric fields, the solutions are expected to be qualitatively independent of the type of disturbance used (submerged object, surface piercing object, pressure distribution, etc). Therefore attention is restricted to one type of disturbance, namely a pressure distribution and the bottom surface is taken to be flat (i.e. $f(x) = -h$ where h is the constant depth). Problems with $f(x)$ not being constant are considered in Section 1.6. More precisely, write

$$p_2 = P(t, x) \tag{2.20}$$

where $P(t, x)$ is a given function with a bounded support. Using (2.18) to obtain

$$p_1 = P(t,x) - \frac{1}{1+(\partial_x \eta)^2}\left[(\partial_x \eta)^2 \Sigma_{11} + 2\partial_x \eta \Sigma_{12} + \Sigma_{22}\right] + \frac{\sigma \partial_x^2 \eta}{(1+(\partial_x \eta)^2)^{3/2}} \tag{2.21}$$

This expression for p_1 is then inserted in (2.19) evaluated on the lower side of the interface. This gives

$$\begin{aligned}\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{P(t,x)}{\rho} &- \frac{1}{\rho}\frac{1}{1+(\partial_x\eta)^2}((\partial_x\eta)^2\Sigma_{11} \\ &+ 2\partial_x\eta\Sigma_{12} + \Sigma_{22}) + g\eta = \frac{\sigma}{\rho}\frac{\partial_x^2\eta}{(1+(\partial_x\eta)^2)^{3/2}} + C\end{aligned} \tag{2.22}$$

where Σ_{11}, Σ_{12} and Σ_{22} are defined by (2.15-2.17) with V denoting the electric potential in the upper fluid. Across an interface the tangential part of the electric field is continuous, so:

$$[\mathbf{E}\cdot\hat{\mathbf{t}}]_2^1 = 0 \tag{2.23}$$

where $\hat{\mathbf{t}}$ is the unit tangent vector to the interface $y = \eta(t,x)$. In a perfect conductor, there is no electric field, and so $\mathbf{E}$ vanishes in fluid 1, so in terms of V and η the continuity equation (2.23) becomes

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial y} = 0.$$

Similarly Σ_{11}, Σ_{12} and Σ_{22} vanish in the conducting lower fluid.

In summary the basic nonlinear equations are

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{on} \;\; y > \eta(t,x) \tag{2.24}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{on} \;\; -h \leqslant y \leqslant \eta(t,x) \tag{2.25}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial y} \quad \text{on} \;\; y = \eta(t,x) \tag{2.26}$$

$$\begin{aligned}\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{P(t,x)}{\rho} &- \frac{1}{\rho}\frac{1}{1+(\partial_x\eta)^2}((\partial_x\eta)^2\Sigma_{11} + 2\partial_x\eta\Sigma_{12} + \Sigma_{22}) \\ &+ g\eta = \frac{\sigma}{\rho}\frac{\partial_x^2\eta}{(1+(\partial_x\eta)^2)^{3/2}} + C \quad \text{on} \quad y = \eta(t,x)\end{aligned} \tag{2.27}$$

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial y} = 0 \quad \text{on} \quad y = \eta(t,x) \tag{2.28}$$

$$\frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \quad y = -h \tag{2.29}$$

$$\frac{\partial V}{\partial y} \to -E_0 \quad y \to \infty \tag{2.30}$$

In order to calculate the constant C, use the conditions where there is no moving pressure distribution and have a constant flow of fluid at a speed U in the x direction and the electric field is constant in the y direction. For simplicity let the $\mathbf{E}$ field be $\mathbf{E} = -E_0\mathbf{j}$. These conditions will be in general satisfied as $|x| \to \infty$. The free surface will then be a horizontal line at $y = 0$ and the bed is at a depth $y = -h$. The velocity potential will be $\varphi = Ux$ and as the surface is flat, the potential of the undisturbed flow will be independent of x, hence $\partial_y^2 V = 0$, which integrates to $V = ay + b$. The potential at $y = 0$ is zero which shows that $b = 0$, computing $\partial_y V$ and letting $y \to \infty$ shows that $a = -E_0$. Inserting the values for the undisturbed case will give the value of the constant C, doing this shows that:

$$C = \frac{U^2}{2} + \frac{\epsilon_p E_0^2}{\rho} \tag{2.31}$$

2.2 Non-dimensionalisation

The nondimensionalisation is carried out according to:

$$\begin{gathered} x = L\hat{x}, \quad y = L\hat{y}, \quad \eta = L\hat{\eta}, \quad t = \sqrt{\frac{\rho L^3}{\sigma}}\hat{t} \\ \varphi = \sqrt{\frac{\sigma L}{\rho}}\hat{\varphi}, \quad V = E_0 L\hat{V}, \quad P(t,x) = \frac{\sigma}{L}\hat{p}, \end{gathered} \tag{2.32}$$

where L is a typical length scale. In the lower fluid (which is conducting), $\mathbf{E} = 0$, so $V =$ constant, so expanding the boundary conditions (2.23) out shows that:

$$\mathbf{E} \cdot \hat{\mathbf{t}} \;=\; 0 \tag{2.33}$$

The electric field is scaled as $\mathbf{E} = E_0\hat{\mathbf{E}}$, then the equation (2.23) is scaled as:

$$\hat{\mathbf{E}} \cdot \hat{\mathbf{t}} \;=\; 0 \tag{2.34}$$

The Faraday tensor can then be scaled as:

$$\Sigma_{ij} = \epsilon_p E_0^2 M_{ij} \tag{2.35}$$

The Bernoulli equation is then scaled as:

$$\begin{aligned} \partial_{\hat{t}}\hat{\varphi} + \frac{1}{2}((\partial_{\hat{x}}\hat{\varphi})^2 + (\partial_{\hat{y}}\hat{\varphi})^2) &+ \hat{P}(\hat{t},\hat{x}) + B\hat{\eta} \\ &+ \frac{E_b}{1+(\partial_{\hat{x}}\hat{\eta})^2}((\partial_{\hat{x}}\hat{\eta})^2 M_{11} - 2\partial_{\hat{x}}\hat{\eta} M_{12} + M_{22}) \\ &= \frac{\partial_{\hat{x}}\hat{\eta}}{(1+(\partial_{\hat{x}}\hat{\eta})^2)^{3/2}} + K \end{aligned} \tag{2.36}$$

where

$$E_b = \epsilon_p E_0^2 L/\sigma, \quad B = gL^2\rho_1/\sigma \tag{2.37}$$

The equation (2.30) is scaled as:

$$\frac{\partial \hat{V}}{\partial \hat{y}} \to -1 \quad \hat{y} \to \infty \tag{2.38}$$

Equation (2.29) is scaled as:

$$\frac{\partial \hat{\varphi}}{\partial \hat{y}} = 0 \quad \hat{y} = -\hat{h} \tag{2.39}$$

The other equations remain the same but with hats. To compute K, take an undisturbed solution with $\mathbf{u} = \hat{P} = \hat{\eta} = 0$ and take $V = -\hat{y}$, these values show that $K = E_b/2$. The next question is to examine the stability of the undisturbed solution. For brevity the hats will be dropped. The level $y = 0$ will be associated with the undisturbed level of the fluid when $\hat{P} = 0$. Equations (2.24) to (2.30) will be known as the basic system, and will be referred to in this chapter. For brevity, the nondimensionalised basic system will be referred to as the nondimensional system.

2.3 Linear Theory

The distribution of pressure is moving at a constant velocity, U and a frame of reference is chosen where the pressure distribution is stationary and hence the flow is stationary. An exact solution of the nondimensionalised system is given by $\eta = 0$, $V = -y$ and $\varphi = Ux$ when $P = 0$. The linear theory is progressed by setting the pressure distribution $P(x)$ to be small. Therefore seek a solution which is a perturbation about the exact solution in the following way:

$$P = \varepsilon p_1 + o(\varepsilon) \tag{2.40}$$

$$\eta = \varepsilon \eta_1 + o(\varepsilon) \tag{2.41}$$

$$V = -y + \varepsilon V_1 + o(\varepsilon) \tag{2.42}$$

$$\varphi = Ux + \varepsilon \varphi_1 + o(\varepsilon) \tag{2.43}$$

Inserting equations (2.40)-(2.43) into the nondimensional system shows that the perturbed quantities must satisfy the following:

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0 \quad \text{on} \quad y > 0 \tag{2.44}$$

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2} = 0 \quad \text{on} \quad -h \leq y \leq 0 \tag{2.45}$$

$$U\frac{\partial \eta_1}{\partial x} = \frac{\partial \varphi_1}{\partial y} \quad \text{on} \quad y = 0 \tag{2.46}$$

$$U\frac{\partial \varphi_1}{\partial x} + B\eta_1 + p_1(x) + E_b\frac{\partial V_1}{\partial y} = \frac{\partial^2 \eta_1}{\partial x^2} \quad \text{on} \quad y = 0 \tag{2.47}$$

$$-\frac{\partial \eta_1}{\partial x} + \frac{\partial V_1}{\partial x} = 0 \quad \text{on} \quad y = 0 \tag{2.48}$$

$$\frac{\partial \varphi_1}{\partial y} = 0 \quad \text{on} \quad y = -h \tag{2.49}$$

$$\frac{\partial V_1}{\partial y} \to 0 \quad y \to \infty. \tag{2.50}$$

In order to solve the above set of equations an extra term, $\mu\varphi_1$ has to be added to (2.47), μ and this is called *Rayleigh viscosity* which will allow the solution to be computed numerically for values of U which are above the minimum in the dispersion relation. The solution will then be obtained in the limit as $\mu \to 0$. Such an approach was used before by Rayleigh in [31] for problems without electrical fields. So (2.47) will have to be replaced by:

$$U\frac{\partial \varphi_1}{\partial x} + B\eta_1 + p_1(x) + E_b\frac{\partial V_1}{\partial y} - \frac{\partial^2 \eta_1}{\partial x^2} + \mu\varphi_1 = 0 \quad \text{on} \quad y = 0. \tag{2.51}$$

Writing φ_1 as a Fourier transform:

$$\varphi_1 = \int_{\mathbb{R}} F(k, y)e^{ikx}dk \tag{2.52}$$

and inserting this into the equation (2.45) shows that:

$$\frac{\partial^2 F}{\partial y^2} - k^2 F = 0 \tag{2.53}$$

$$\frac{\partial F}{\partial y} = 0 \quad \text{on} \quad y = -h. \tag{2.54}$$

The general solution of equation (2.53) is $F = A(k)e^{ky} + B(k)e^{-ky}$, the boundary condition equation (2.54) shows that $B(k) = A(k)e^{2kh}$, which gives the solution as:

$$F(k, y) = A(k)\cosh k(y + h) \tag{2.55}$$

on absorbing the function of k into A. The velocity potential can be written as:

$$\varphi_1 = \int_{\mathbb{R}} A(k)e^{ikx}\cosh k(y + h)dk. \tag{2.56}$$

Write the perturbed potential as a Fourier transform:

$$V_1 = \int_{\mathbb{R}} G(k, y)e^{ikx}dk \tag{2.57}$$

and inserting it into (2.44) and using the boundary condition (2.50) shows that:

$$\frac{\partial^2 G}{\partial y^2} - k^2 G = 0 \tag{2.58}$$

$$\lim_{y\to\infty} \frac{\partial G}{\partial y} = 0. \tag{2.59}$$

The general solution of equation (2.58) is $G = C(k)e^{ky} + D(k)e^{-ky}$, using the boundary condition at infinity (equation 2.59) shows that $C = 0$ and forces k to be replaced by $|k|$ and the solution becomes $G = D(k)e^{-|k|y}$. The solution for the potential becomes:

$$V_1 = \int_{\mathbb{R}} D(k)e^{-|k|y}e^{ikx}dk. \tag{2.60}$$

The distributed pressure may also be written as a Fourier transform:

$$p_1(x) = \int_{\mathbb{R}} E(k)e^{ikx}dk. \tag{2.61}$$

The pressure is an input to the problem, so it is specified beforehand. For ease of computation functions like $\exp(-x^2)$ and $\text{sech}(x)$ will be used because their Fourier transform is well known. Let $E(k)$ be the Fourier transform of the pressure distribution. Write the free surface in terms of a Fourier transform.

$$\eta_1 = \int_{\mathbb{R}} H(k)e^{ikx}dk. \tag{2.62}$$

Using (2.46) shows that:

$$\int_{\mathbb{R}} ikUH(k)e^{ikx} - kA(k)e^{ikx}\sinh(kh)dk = 0 \tag{2.63}$$

this shows

$$H(k) = -\frac{i}{U}A(k)\sinh(kh). \tag{2.64}$$

Using (2.48) gives:

$$\int_{\mathbb{R}} -\frac{k}{U}A(k)e^{ikx}\sinh(kh) + ikD(k)e^{ikx}dk = 0 \tag{2.65}$$

which implies:

$$D(k) = -\frac{i}{U}A(k)\sinh(kh) \tag{2.66}$$

In summary:

$$\varphi_1 = \int_{\mathbb{R}} A(k)e^{ikx}\cosh k(y+h)dk \tag{2.67}$$

$$V_1 = \int_{\mathbb{R}} -\frac{i}{U}A(k)e^{-|k|y}e^{ikx}\sinh(kh)dk \tag{2.68}$$

$$\eta_1 = \int_{\mathbb{R}} -\frac{i}{U}A(k)e^{ikx}\sinh(kh)dk. \tag{2.69}$$

The next task is to relate $A(k)$ to $E(k)$, this is done by differentiating (2.51) with respect to x and inserting the expressions (2.67) - (2.69). Differentiating (2.51) shows that:

$$U\frac{\partial^2\varphi_1}{\partial x^2} + B\frac{\partial\eta_1}{\partial x} + \frac{\partial p_1}{\partial x} + E_b\frac{\partial^2 V_1}{\partial x\partial y} - \frac{\partial^3\eta_1}{\partial x^3} + \mu\frac{\partial\varphi_1}{\partial x} = 0 \tag{2.70}$$

inserting all the above shows that:

$$A(k) = -\frac{iE(k)}{-kU\cosh(kh) + (BU^{-1} - E_bU^{-1}|k| + k^2U^{-1})\sinh(kh) + i\mu\cosh(kh)} \tag{2.71}$$

This can be written in the form:

$$A(k) = -\frac{iE(k)}{-kU\cosh(kh) + f(k)\sinh(kh) + i\mu\cosh(kh)}, \tag{2.72}$$

where:

$$f(k) = BU^{-1} - E_bU^{-1}|k| + k^2U^{-1} \tag{2.73}$$

To compute the linear dispersion relation, write:

$$\varphi_1 = Ae^{ikx}\cosh k(y+h) \tag{2.74}$$
$$V_1 = -\frac{i}{U}Ae^{-|k|y}e^{ikx}\sinh(kh) \tag{2.75}$$
$$\eta_1 = -\frac{i}{U}Ae^{ikx}\sinh(kh) \tag{2.76}$$

Upon taking $y = \mu = p_1 = 0$ and inserting into (2.70) shows that:

$$U^2(-k\cosh(kh)) + (B - E_b|k| + k^2)\sinh(kh) = 0, \tag{2.77}$$

which gives the dispersion relation($k \geqslant 0$) as:

$$U^2 = \left(\frac{B}{k} - E_b + k\right)\tanh(kh). \tag{2.78}$$

Plotting U^2 against kh there will be one of two situations, 1) the graph will be monotonically increasing or 2) the graph will have a unique minimum which will happen at the point U_{min}. Computing the free surface for values of U above U_{min} will result in two singularities which is why the Rayleigh viscosity is added to the equation. It is useful to divide both sides by Bh to obtain the relation:

$$F^2 = \left(\frac{1}{kh} - \alpha + \beta kh\right)\tanh(kh), \tag{2.79}$$

where $F^2 = U^2/Bh$, $\alpha = E_b/Bh$ and $\beta = 1/Bh^2$. Now the variable is kh and there are two free constants to play with. It is simple to show that (2.79) always has a minimum, the idea is to use Rolle's theorem. Clearly $F^2 \to \infty$ as $kh \to \infty$. The gradient of F^2 is:

$$\frac{dF^2}{d(kh)} = \left[-\frac{1}{(kh)^2} + \beta\right]\tanh(kh) + \left[\frac{1}{kh} - \alpha + \beta kh\right]\text{sech}^2(kh) \tag{2.80}$$

If it is possible to show that at $kh = 0$, F^2 is a decreasing function then it is possible to use Rolle's theorem to show the existence of a minimum. For this the power series of both $\tanh$ and sech are required:

$$\tanh x = x - \frac{x^3}{3} + o(x^3)$$
$$\text{sech}\, x = 1 - \frac{x^2}{2} + o(x^2)$$

Writing the derivative in series form:

$$\begin{aligned}\frac{dF^2}{dkh} &= \left[-\frac{1}{(kh)^2}+\beta\right]\left[kh-\frac{(kh)^3}{3}\right]+\left[\frac{1}{kh}-\alpha+\beta kh\right][1-(kh)^2] \\ &= -\frac{1}{kh}+\beta kh+\frac{kh}{3}+\frac{1}{kh}-\alpha-kh+\beta kh+o(kh) \\ &= -\alpha+\left(2\beta-\frac{2}{3}\right)kh+o(kh).\end{aligned}$$

Taking the limit as $kh \to 0$ is well defined, so $dF^2/dkh \to -\alpha$ as $kh \to 0$. This shows that F^2 is a decreasing function at $kh = 0$ but $F^2 \to \infty$ as $kh \to \infty$. So this shows that there must be a turning point on the interval $(0, \infty)$. If $\alpha = 0$ and F still has a chance of being decreasing, if $\beta < 1/3$ then for small values of $kh > 0$, the derivative will still be negative. What was not proven was that the turning point (which would be a minimum) is always above the F axis. Indeed, in some cases it lies *below* the F axis indicating a region of instability. The next point to examine is where $F^2 = 0$ for $kh > 0$ as this will show when the dispersion relation will cross the F axis and the flow will be unstable. As $\tanh$ is an increasing function, there are no points $kh > 0$ such that $\tanh(kh) = 0$, so the possibility for F^2 to have zeros is if there exists a kh for which:

$$\frac{1}{kh}-\alpha+\beta kh=0 \tag{2.81}$$

This is the quadratic, $\beta(kh)^2 - \alpha kh + 1 = 0$, the solutions to this are:

$$kh=\frac{\alpha\pm\sqrt{\alpha^2-4\beta}}{2\beta}. \tag{2.82}$$

These solutions will be real if $\alpha^2 \geqslant 4\beta$, or in original variables $E_b^2 \geqslant 4B$.

In order to examine the free surface, take the profile for the pressure to be $p_1 = \exp(-x^2)$ because the Fourier transform is known to be $E(k) = \sqrt{\pi}\exp(-k^2/4)$ and so finally, the free surface is given by:

$$\eta_1(x)=\int_{\mathbb{R}}\frac{\sqrt{\pi}e^{-\frac{k^2}{4}}e^{ikx}\tanh(kh)}{-kU+(BU^{-1}-E_bU^{-1}|k|+k^2U^{-1})\tanh(kh)+i\mu}dk. \tag{2.83}$$

Typical profiles for values of U above and below the minimum U_{min} are shown in figures 2.3 and 2.2 with $h = 3, E_b = 2, B = 1.5$ respectively. The profile of figure 2.2 is symmetric respect to $x = 0$ and has decaying oscillations in the far field. The profile of figure 2.3 is characterised by a train of waves of constant amplitude as $x \to \pm\infty$. Solution for $U < U_{min}$ are obtained by taking $\mu = 0$ in (2.83) and when $U > U_{min} = 0.67$, it is necessary to take $\mu > 0$

The damping of the waves is dependent upon the value of μ (see figure 2.4). As μ decreases, so does the amount of damping until it gets to a point where the solution itself becomes unstable. The free surface profile changes dramatically depending upon the value of U, there is a cross over when the value of U reaches the minimum of the dispersion relation from both above and below. It should be noted that as $\mu \to 0$ the problem is ill-posed because the dispersion relation will have zeros which will make the integral in the expression of the free surface diverge. As the value of μ is decreased, the poles get closer to the y-axis and therefore cause problems when trying to numerically evaluate the integral which can be seen by the examination of figures (2.4 (a) and 2.4 (b)) where the height of the peaks are seemingly random in height where higher values of μ makes the peaks decrease in height in a monotonic way. So this numerical experiment shows that there is a minimum which beyond which it is impossible to get meaningful results.

2.3.1 Induced Charge on the Interface

Although surface charge was not part of the model, there will be some *induced* charge on the surface, the expression for the induced charge is given by:

$$\Sigma_Q = \epsilon_p \hat{\mathbf{n}} \cdot \nabla V \tag{2.84}$$

In order to find the correct scaling for Σ_Q, non-dimensionalise the RHS of equation (2.84) to obtain:

$$\begin{aligned} \Sigma_Q &= \epsilon_p \hat{\mathbf{n}} \cdot \nabla V \\ &= \epsilon_p(-\partial_x \eta, 1) \cdot (\partial_x V, \partial_y V) \\ &= \epsilon_p(-\partial_{\hat{x}} \hat{\eta}, 1) \cdot (E_0 \partial_{\hat{x}} \hat{V}, E_0 \partial_{\hat{y}} \hat{V}) \\ &= \epsilon_p E_0 \hat{\mathbf{n}} \cdot \hat{\nabla} \hat{V}. \end{aligned}$$

So the following scaling is adopted for the surface charge:

$$\Sigma_Q = \epsilon_p E_0 \hat{\Sigma}_Q \tag{2.85}$$

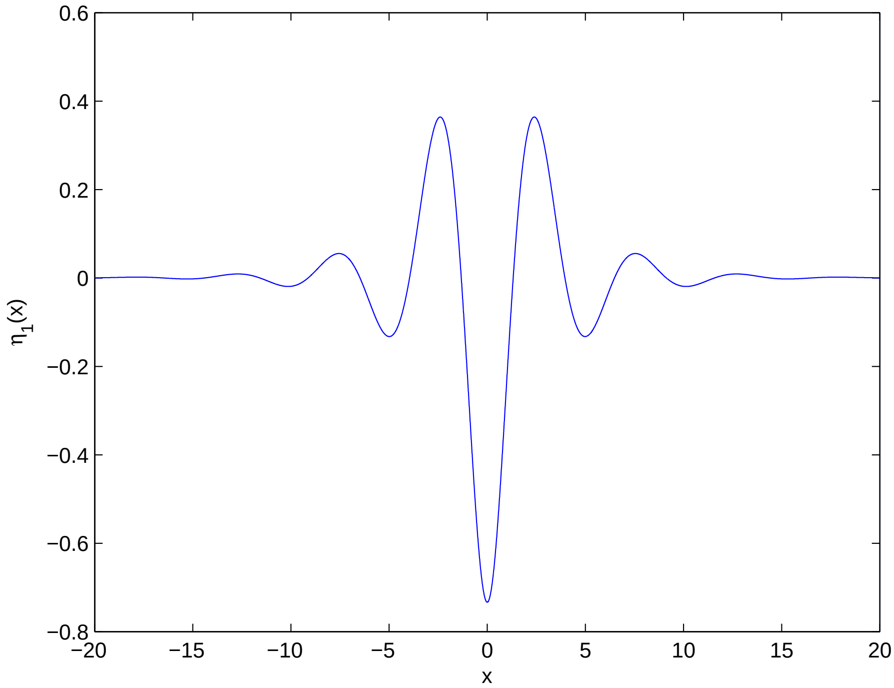


Figure 2.2: Free Surface Profile, $\eta_1(x), U = 0.6325 < U_{min} = 0.67$

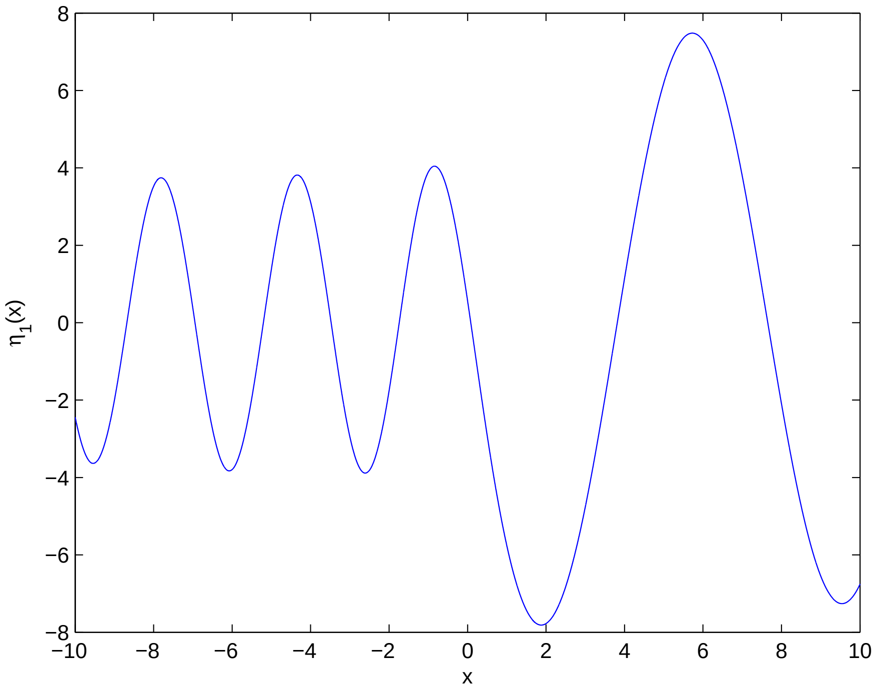
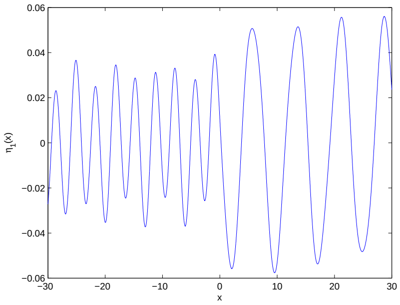
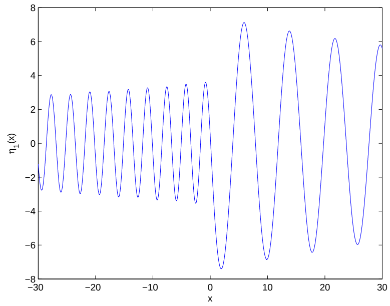


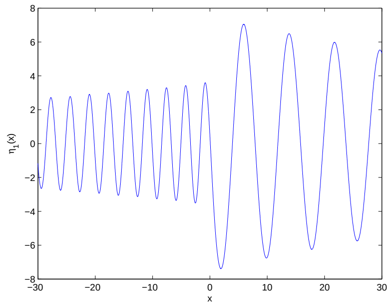
Figure 2.3: Free Surface Profile, $\eta_1(x), U = 0.8 > U_{min} = 0.67$



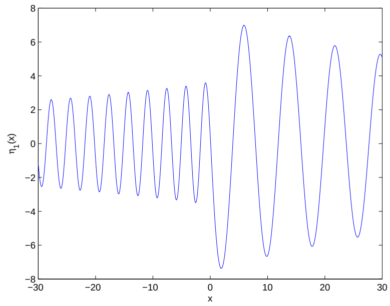
(a) $\mu = 0.002$



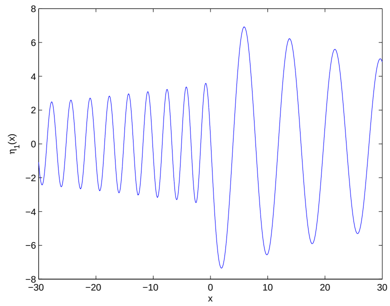
(b) $\mu = 0.011$



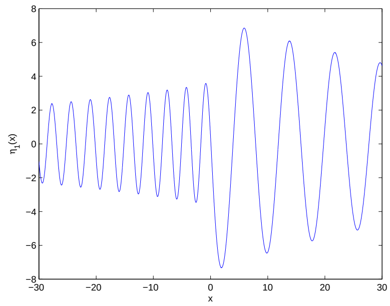
(c) $\mu = 0.013$



(d) $\mu = 0.015$



(e) $\mu = 0.017$



(f) $\mu = 0.019$

Figure 2.4: Different values of μ in (a)- (f),$h = 3, E_b = 2, B = 1.5, U = 0.8$

So the scaled equation becomes:

$$\hat{\Sigma}_Q = \hat{\mathbf{n}} \cdot \hat{\nabla} \hat{V} \tag{2.86}$$

The induced charge from the exact solution, $\eta = p = 0$, $\varphi = Ux$ and $V = -E_0 y$ can be computed to obtain $\Sigma_Q = -\epsilon_p E_0$. The induced charge on the interface can be computed, inserting the expansions (2.41) and (2.42) into (2.86) to obtain:

$$\begin{aligned}\Sigma_Q &= \hat{\mathbf{n}}\cdot\nabla V \\ &= (-\varepsilon\partial_x\eta_1, 1)\cdot(\varepsilon\partial_x V, -1_\varepsilon\partial_y V_1) \\ &= -1+\varepsilon\partial_y V_1 + o(\varepsilon).\end{aligned}$$

At $O(1)$, the previous result of the trivial exact solution is obtained and the $O(\varepsilon)$ term is the next approximation. So:

$$\Sigma_Q^{(1)} = \frac{\partial V}{\partial y}\bigg|_{y=0} \qquad (2.87)$$

It is a simple matter to compute this and is given by:

$$\Sigma_Q^{(1)} = -\int_{\mathbb{R}} \frac{|k|E(k)e^{ikx}\tanh hk}{-U^2k+(B-E_b|k|+k^2)\tanh hk} \qquad (2.88)$$

Note that the Fourier symbol of the induced charge is given by:

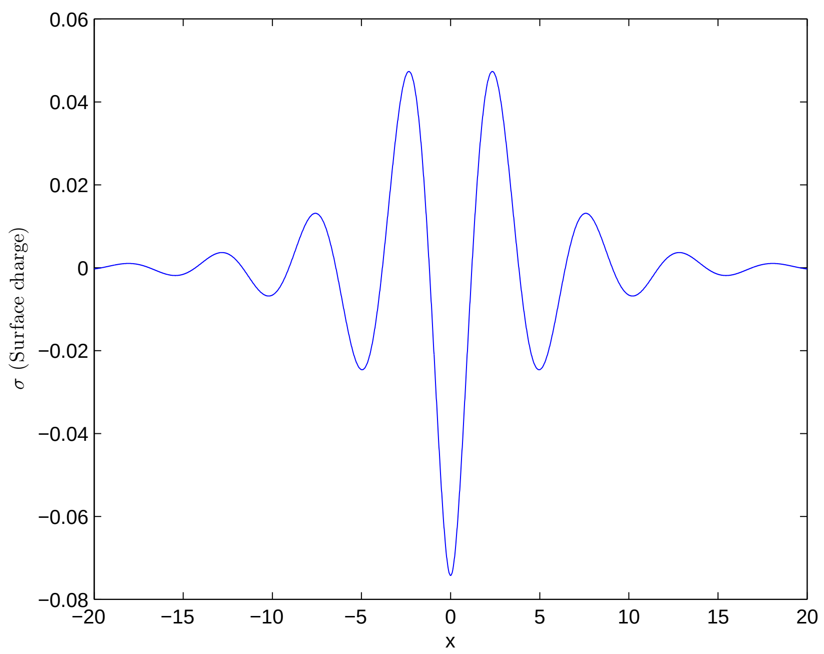


Figure 2.5: Induced Surface Charge

$$\hat{\Sigma}_Q = |k|\hat{\eta} \qquad (2.89)$$

The operator with the Fourier symbol is $\partial_x\mathscr{H}$. This suggests that the same expression for the induced charge will be valid for the weakly nonlinear theory also.

2.3.2 Blow Up of Profile

The linear theory whilst being able to give perfectly reasonable results for the majority of parameters, does not give a perfectly uniform profile for all parameter values and therefore will introduce a nonuniformity into the asymptotic expansion. To see how this happens fix a set of parameters (B, E_b) and gradually take the limit as the uniform speed U reaches the minimum of the dispersion relation from below as can be seen in figure 2.6. Similar results are found for $U > U_{min}$ but it is the necessary to take

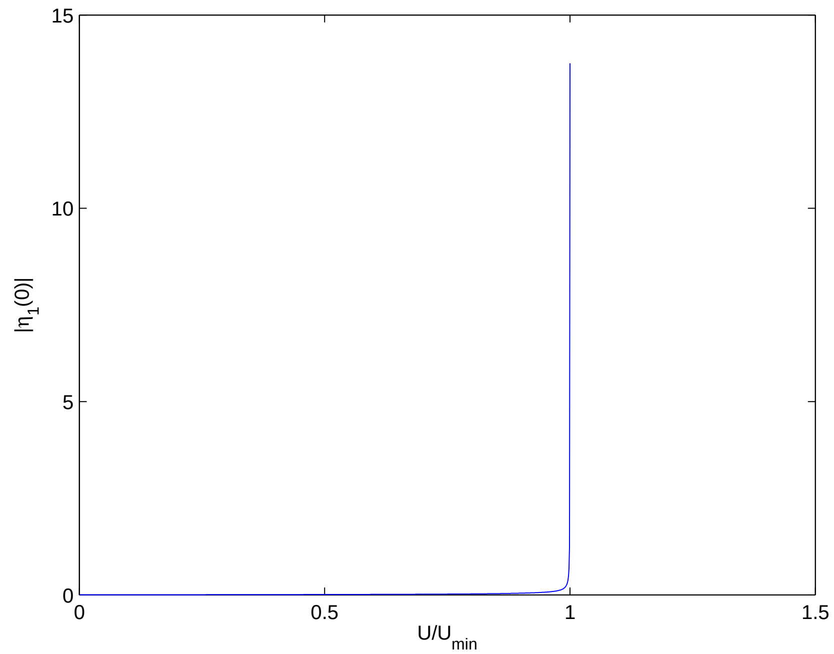


Figure 2.6: Value of $\eta_1(0)$ with $B = 1.2$, $E_b = 1.5$ and $h = 1$

μ different from zero. Therefore a different approximation must be sought for a completely uniform description of the wave, for this one can either go to the full nonlinear equations or one can take a weakly nonlinear approximation as is done in chapter two.

2.3.3 The Free Surface in the Far Field

It is possible to analytically compute the shape of the free surface in the far field by the use of Cauchy's residue theorem of complex analysis provided a quarter-circle contour (figure 2.7) is chosen. The analysis is a generalisation of that developed by Rayleigh [31] in the absence of electric fields. For symmetric distributions of pressure $(p(-x) =$

$p(x)$), then the free surface is given by:

$$\eta_1(x) = 2Re\int_0^\infty \frac{\sqrt{\pi}e^{-\frac{k^2}{4}}e^{ikx}\tanh(kh)}{-kU + (BU^{-1} - E_bU^{-1}k + k^2U^{-1})\tanh(kh) + i\mu}dk, \quad (2.90)$$

where $U > U_{min}$, this is why μi was added to the denominator. Depending on the sign of x, the relevant contours are given in figure 2.7. The complex contour integral is then:

$$\eta_1(x) = \oint_\gamma \frac{E(z)e^{ixz}\tanh(hz)}{-Uz + U^{-1}(B - E_bz + z^2)\tanh(hz) + i\mu}dz \quad (2.91)$$

where μ is small. The poles in γ correspond to the zeros of the denominator. The free surface in the far field will be calculated by taking the limit as $\mu \to 0$. The contours of figure 2.7 are chosen so that there is no contribution from the quarter-circles as the radius tends to infinity. The residues correspond to the zeros of the denominator. The

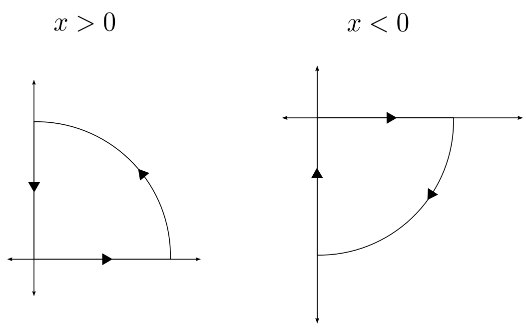


Figure 2.7: Poles of $D(z)$

zeros of the denominator with $\mu \neq 0$ can be obtained from the denominator with $\mu = 0$, since μ is small. Denote the denominator with $\mu = 0$ by $D(z)$. The denominator with $\mu = 0$ has two real zeros a_1, a_2, and as μ is small then the zeros of $D(z)$ for $\mu \neq 0$ will be a small perturbation from the zeros of $D(z)$ with $\mu = 0$. Assume that without loss of generality $0 < a_1 < a_2$, the zeros a_1 and a_2 are the wavenumbers of waves travelling at the phase velocity U. Write the zeros of $D(z)$ with $\mu \neq 0$ as $a_{1,2} + \varepsilon(\mu)$. The denominator in (2.91) can be written as $D(z) = z(-U + U^{-1}c^2(z))$ where

$$c^2(z) = \left(\frac{B}{z} - E_b + z\right)\tanh(hz) \quad (2.92)$$

Then using Taylor's series

$$\begin{aligned}
D(a_{1,2}+\varepsilon(\mu)) &= D(a_{1,2})+\varepsilon(\mu)D'(a_{1,2}) \\
&= 0-U+\frac{U^2}{U}+\frac{\varepsilon(\mu)}{U}(2Uc'(a_{1,2}))+\mu i \\
&= 2\varepsilon(\mu)c'(a_{1,2})+\mu i \\
&= 0
\end{aligned}$$

which shows that:

$$\varepsilon(\mu) = -\frac{i\mu}{2c'(a_{1,2})}. \tag{2.93}$$

So the roots to $D(z)$ with $\mu \neq 0$ are of the form:

$$z = a_1 - \frac{i\mu}{2c'(a_1)} + o(\mu) \tag{2.94}$$

$$z = a_2 - \frac{i\mu}{2c'(a_2)} + o(\mu) \tag{2.95}$$

Assume without loss of generality that $a_1 < a_2$, $c'(a_1) < 0$ and $c'(a_2) > 0$. It then follows that the zero (2.94) must lie in the first contour $(x > 0)$ and the root (2.95) must lie in the contour for $x < 0$. To calculate the residues using the well known expression:

$$\text{Res}\left\{\frac{f(z)}{g(z)};a\right\} = \frac{f(a)}{g'(a)}. \tag{2.96}$$

The residue is given by:

$$\begin{aligned}
\text{Res}\left\{\frac{f(z)}{g(z)};a\right\} &= \frac{E(a_1-i\mu/2c'(a_1))e^{i(a_1-i\mu/2c'(a_1))x}\tanh h(a_1-i\mu/2c'(a_1))}{D'(a_1-i\mu/2c'(a_1))} \\
&= \frac{E(a_1)e^{ia_1x}\tanh ha_1}{D'(a_1)} \quad \text{in the limit as } \mu \to 0 \\
&= \frac{E(a_1)e^{ia_1x}\tanh ha_1}{2a_1c'(a_1)}
\end{aligned}$$

The contribution from the positive imaginary axis tend to zero as $x \to +\infty$ because of the factor $\exp(ixz)$ in (2.91). It should be noted that there are also singularities in the integrand of (2.91) on the positive imaginary axis. The countour in Figure (2.7) needs then to be deformed by including small half circles to avoid each of these singularities. The contributions from these half circles (which are half the residues at the singularities) tend also to zero as x tends tends to infinity because of the factor $exp(ixz)$ in

(2.91). The approximation to the free surface is:

$$\begin{aligned}\eta_1(x) &\approx Re\left\{2\pi i\frac{E(a_1)e^{ia_1x}\tanh ha_1}{2a_1c'(a_1)}\right\}\\ &= \frac{\pi E(a_1)\tanh ha_1}{a_1c'(a_1)}\sin(a_1x) \quad x>0\end{aligned}$$

Similarly:

$$\eta_1(x) = -\frac{\pi E(a_1)\tanh ha_1}{a_1c'(a_1)}\sin(a_1x) \quad x<0 \tag{2.97}$$

Note $c'(a_1) \to 0$ and $c'(a_2) \to 0$ as $U \to U_{min}$, therefore linear theory predicts that waves of unbounded amplitude as $U \to U_{min}$.

2.4 Ignoring the Surface Tension

This section examines the question of the fluid under the influence of the electric field in the absence of surface tension. Unfortunately this can't be obtained as a simple limit from previous work as the velocity potential was scaled using the surface tension and them formally it would be taking the limit as $B \to \infty$ which clearly isn't possible. In order to carry out the analysis, set $\sigma = 0$ in equation (2.22) and all other equations remain unchanged. The scaling is now given by:

$$x = h\tilde{x}, \quad y = h\tilde{y}, \quad \eta = h\tilde{\eta}, \quad t = \sqrt{\frac{h}{g}}\tilde{t}, \quad \varphi = \sqrt{gh^3}\tilde{\varphi}, \quad V = E_0h\tilde{V}. \tag{2.98}$$

The external pressure distribution, εp is scaled in the same way as before: $p = \rho U^2\hat{p}$. The unsteady Bernoulli equation becomes (dropping the hats):

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\|\nabla\varphi\|^2+\eta+\varepsilon p-E_b\left\{\frac{(\partial_x\eta)^2}{1+(\partial_x\eta)^2}M_{11}-\frac{2\partial_x\eta}{1+(\partial_x\eta)^2}M_{21}+\frac{1}{1+(\partial_x\eta)^2}M_{11}\right\} = K \tag{2.99}$$

Where E_b in this case is:

$$E_b = \frac{\epsilon_p E_0^2}{gh} \tag{2.100}$$

The K is the Bernoulli constant and is not important for a linear analysis.

2.4.1 Linear Theory

To compute the expression for the linear waves, write:

$$\begin{aligned}\varphi &= Ux + \varepsilon\varphi_1 \\ V &= -y + \varepsilon V_1 \\ \eta &= \varepsilon\eta_1.\end{aligned}$$

Inserting the above equations into the unsteady Bernoulli equation and differentiating with respect to x shows:

$$U\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial\eta_1}{\partial x} + \frac{\partial p}{\partial x} + E_b\frac{\partial^2 V_1}{\partial x\partial y} = 0 \tag{2.101}$$

The set of equations is now:

$$\begin{aligned}
\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} &= 0 \quad \text{on} \quad y > 0 && (2.102)\\
\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial y^2} &= 0 \quad \text{on} \quad -1 \le y \le 0 && (2.103)\\
U\frac{\partial\eta_1}{\partial x} &= \frac{\partial\varphi_1}{\partial y} \quad \text{on} \quad y = 0 && (2.104)\\
U\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial\eta_1}{\partial x} + \frac{\partial p}{\partial x} + E_b\frac{\partial^2 V_1}{\partial x\partial y} &= 0 \quad \text{on} \quad y = 0 && (2.105)\\
-\frac{\partial\eta_1}{\partial x} + \frac{\partial V_1}{\partial x} &= 0 \quad \text{on} \quad y = 0 && (2.106)\\
\frac{\partial\varphi_1}{\partial y} &= 0 \quad \text{on} \quad y = -1 && (2.107)\\
\frac{\partial V_1}{\partial y} &\to 0 \quad y \to \infty. && (2.108)
\end{aligned}$$

To solve equation (2.102), write:

$$V_1 = \int_{\mathbb{R}} A(k,y)e^{ikx}dk, \tag{2.109}$$

then the following holds:

$$\int_{\mathbb{R}} (-k^2A + \partial_y^2 A)e^{ikx}dk \tag{2.110}$$

which implies $\partial_y^2 A - k^2 A = 0$, the solution to this equation is $A(k,y) = \alpha_1(k)e^{ky} + \alpha_2(k)e^{-ky}$, the boundary condition equation (2.108) which means that:

$$k(\alpha_1 e^{ky} - \alpha_2 e^{-ky}) \to 0$$

As $y \to \infty$. Hence:

$$A(k,y) = \alpha(k,y)e^{-|k|y}. \tag{2.111}$$

Considering the velocity potential, equation (2.103), write:

$$\varphi_1 = \int_{\mathbb{R}} B(k,y)e^{ikx}dk \tag{2.112}$$

Then B satisfies $\partial_y^2 B - k^2 B = 0$. The solution to this equation is $B = \beta_1(k)e^{ky} + \beta_2(k)e^{-ky}$. The boundary condition for this is equation (2.107). This gives the relationship between β_1 and β_2 as $\beta_1 + \beta_2 e^{2k} = 0$, this shows that:

$$B = \beta(k)\cosh[k(y+1)] \tag{2.113}$$

Writing the free surface as:

$$\eta_1 = \int_{\mathbb{R}} C(k)e^{ikx}dk \tag{2.114}$$

Using (2.104), $C(k)$ can be related to $\beta(k)$. Inserting φ_1 and η_1, shows that:

$$\int_{\mathbb{R}} [ikUC(k) - k\beta(k)\sinh k]e^{ikx}dk = 0$$

which shows that:

$$C(k) = -\frac{i\beta}{U}\sinh k. \tag{2.115}$$

Now it is necessary to relate α to β, equation (2.106) is used for this. Inserting η_1 and V_1, shows that:

$$\int_{\mathbb{R}} \left(-\frac{k\beta}{U}\sinh k + ik\alpha \right) e^{ikx}dk = 0$$

which yields:

$$\alpha = -\frac{i\beta}{U}\sinh k \tag{2.116}$$

The solutions become:

$$V_1 = -\frac{i}{U}\int_{\mathbb{R}} \beta e^{-|k|y}e^{ikx}\sinh k dk \tag{2.117}$$

$$\varphi_1 = \int_{\mathbb{R}} \beta e^{ikx}\cosh[k(y+1)]dk \tag{2.118}$$

$$\eta_1 = -\frac{i}{U}\int_{\mathbb{R}} \beta e^{ikx}\sinh k dk \tag{2.119}$$

Write the pressure as:

$$p(x) = \int_{\mathbb{R}} D(k)e^{ikx}dk$$

It is now possible to relate β to D using the Bernoulli equation. The required derivatives are:

$$
\begin{aligned}
\partial_x^2 \varphi_1 &= -\int_{\mathbb{R}} k^2 \beta e^{ikx} \cosh k dk \\
\partial_x \eta_1 &= \frac{1}{U} \int_{\mathbb{R}} k \beta e^{ikx} \sinh k dk \\
\partial_x \partial_y V_1 &= -\frac{1}{U} \int_{\mathbb{R}} |k| k \beta e^{ikx} \sinh k dk \\
\partial_x p &= \int_{\mathbb{R}} ik D e^{ikx} dk.
\end{aligned}
$$

Inserting the above into the Bernoulli equation shows that:

$$
\beta = \frac{-iD(k)}{-kU \cosh k + U^{-1} \sinh k - U^{-1} E_b |k| \sinh k}
$$

The full solutions are then:

$$
V_1 = -\int_{\mathbb{R}} \frac{D(k) e^{ikx} e^{-|k|y} \tanh k}{-kU^2 + (1 - E_b|k|) \tanh k} \quad (2.120)
$$

$$
\varphi_1 = \int_{\mathbb{R}} \frac{-ikD(k) e^{ikx} \cosh[(1+y)k]}{-kU \cosh k + U^{-1} \sinh k - U^{-1} E_b |k| \sinh k} dk \quad (2.121)
$$

$$
\eta_1 = -\int_{\mathbb{R}} \frac{D(k) e^{ikx} \tanh k}{-kU^2 + (1 - E_b|k|) \tanh k} dk \quad (2.122)
$$

The next task is to calculate the dispersion relation, to do this set:

$$
V_1 = \frac{iA}{U} e^{-|k|y} e^{ikx} \sinh k \quad (2.123)
$$

$$
\varphi_1 = A e^{ikx} \cosh(y+1)k \quad (2.124)
$$

$$
\eta_1 = -\frac{iA}{U} e^{ikx} \sinh k, \quad (2.125)
$$

where A is constant independent of k. Inserting the above into the linearised unsteady Bernoulli equation shows that:

$$
U(-kAe^{ikx} \cosh k) + \frac{A}{U} e^{ikx} \sinh k + \frac{E_b A |k|}{U} \sinh k \quad (2.126)
$$

Which reveals the dispersion relation as:

$$
U^2 = (1 - E_b|k|) \frac{\tanh k}{k} \quad (2.127)
$$

The dispersion relation has no minimum as U^2 is unbounded below.

2.5 Solutions in Conducting Fluids of Infinite Depth

Taking the limit as $h \to \infty$ allows for a more in-depth investigation of the problem as it simplifies much of the algebra and therefore simplifies much of the problem. The analysis of this section follows [1], section 4.2.2. The equations are as before but the lower boundary condition is replaced by:

$$\frac{\partial \varphi}{\partial x} \to U, \quad \frac{\partial \varphi}{\partial y} \to 0 \quad \text{as} \quad y \to -\infty \tag{2.128}$$

The expansions used are as before:

$$\begin{aligned} p &= \varepsilon p_1 + o(\varepsilon) \\ \varphi &= Ux + \varepsilon \varphi_1 + o(\varepsilon) \\ \eta &= \varepsilon \eta_1 + o(\varepsilon) \\ V &= -E_0 y + \varepsilon V_1 + o(\varepsilon) \end{aligned}$$

Inserting these expansions into the basic nonlinear equations yields the following linear equations:

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2} = 0, \quad y < 0 \tag{2.129}$$

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0, \quad y > 0 \tag{2.130}$$

$$U\frac{\partial \eta_1}{\partial X} = \frac{\partial \varphi_1}{\partial y}, \quad y = 0 \tag{2.131}$$

$$\frac{\partial V_1}{\partial x} = E_0 \frac{\partial \eta_1}{\partial x}, \quad y = 0 \tag{2.132}$$

$$\frac{\partial \varphi_1}{\partial y} \to 0 \quad y = 0 \tag{2.133}$$

$$\frac{\partial V_1}{\partial y} \to 0, \quad y = 0 \tag{2.134}$$

$$U\frac{\partial \varphi_1}{\partial x} + \frac{p_1}{\rho} + g\eta_1 + \frac{\epsilon_p E_0}{\rho}\frac{\partial V_1}{\partial y} - \frac{\sigma}{\rho}\frac{\partial^2 \eta_1}{\partial x^2} + \mu \varphi_1 = 0, \quad y = 0 \tag{2.135}$$

$$\frac{\partial \varphi_1}{\partial y} \to 0 \quad y \to -\infty \tag{2.136}$$

Where μ is the Rayleigh viscosity. Writing the perturbed velocity potential, φ_1 as:

$$\varphi_1 = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varphi}_1 e^{ikx} dk \tag{2.137}$$

Equation (2.129) becomes:

$$\frac{\partial^2 \hat{\varphi}_1}{\partial y^2} - k^2 \hat{\varphi}_1 = 0, \tag{2.138}$$

Along with the boundary condition equation (2.136). The solution of the equation is:

$$\varphi_1 = \frac{1}{2\pi} \int_{\mathbb{R}} A(k) e^{|k|y} e^{ikx} dk$$

Likewise, writing the perturbed electric potential as:

$$\varphi_1 = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varphi}_1 e^{ikx} dk$$

reduces equation (2.130) to:

$$\frac{\partial^2 \hat{V}_1}{\partial y^2} - k^2 \hat{V}_1 = 0, \tag{2.139}$$

Which shows that the perturbed velocity potential as:

$$V_1 = \frac{1}{2\pi} \int_{\mathbb{R}} B(k) e^{-|k|y} e^{ikx} dk$$

Using equation (2.131) and representing the free surface as:

$$\eta_1 = \frac{1}{2\pi} \int_{\mathbb{R}} C(k) e^{ikx} dk$$

shows that:

$$C(k) = \frac{i}{U} \text{sgn}(k) A(k)$$

Using equation (2.132) shows that:

$$B(k) = -i \frac{E_0}{U} \text{sgn}(k) A(k)$$

Writing the pressure as:

$$p_1 = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{p} e^{ikx} dk$$

It is possible to express $A(k)$ as:

$$A(k) = -\frac{i\hat{p}}{\rho U} \left[k - \frac{g}{U^2} \text{sgn}(k) - \frac{\sigma k^2}{\rho U} \text{sgn}(k) + \frac{\epsilon_p E_0^2}{\rho U} \text{sgn}(k)|k| + i\mu \right]^{-1}.$$

If the pressure function is symmetric, $(p(-x) = p(x))$ the integral for the free surface can be reduced to an integral from 0 to ∞:

$$\eta_1 = \frac{2}{\rho U^2} Re \int_0^\infty \frac{\hat{p} e^{ikx}}{k - \frac{g}{U} - \frac{\sigma k^2}{\rho U^2} + \frac{\epsilon_p E_0^2 k}{\rho U^2} + i\mu} dk \tag{2.140}$$

So the equation for the free surface is the same as equation 4.58 in [1]. Choose a pressure distribution of the form.

$$P(x) = \frac{\rho U^2}{2} \text{sech} \left(\frac{5gx}{U^2} \right) \tag{2.141}$$

The Fourier transform is then:

$$\hat{p} = \frac{\rho U^2}{2} \frac{\pi U^2}{5g} \text{sech} \left(\frac{\pi U^2 k}{10g} \right)$$

So:

$$\eta_1 = \frac{\pi U^2}{5g} Re \int_0^\infty \frac{\text{sech} \left(\frac{\pi U^2 k}{10g} \right) e^{ikx}}{k - \frac{g}{U^2} - \frac{\sigma k^2}{\rho U^2} + \frac{\epsilon_p E_0^2 k}{\rho U^2} + i\mu} dk$$

Now make a change of variable $\nu = U^2 k/g$ to obtain:

$$\eta_1 = \frac{\pi U^2}{5g} Re \int_0^\infty \frac{e^{i\nu g x U^{-2}} \text{sech} \left(\frac{\pi \nu}{10} \right)}{\nu - 1 - \alpha \nu^2 + \beta \nu + i\mu_1} d\nu \tag{2.142}$$

Where:

$$\alpha = \frac{\sigma g}{\rho U^4}, \quad \beta = \frac{\epsilon_p E_0^2}{\rho U^2}.$$

Let $\hat{x} = gxU^{-2}$ and $\bar{\eta} = g\eta U^{-2}$, then the scaled free surface is:

$$\bar{\eta}(\hat{x}) = \frac{\pi \varepsilon}{5} Re \int_0^\infty \frac{e^{i\hat{x}\nu} \text{sech} \left(\frac{\pi \nu}{10} \right)}{\nu - 1 - \alpha \nu^2 + \beta \nu + i\mu_1} d\nu \tag{2.143}$$

If $1 + \beta = 2\sqrt{\alpha}$ then the denominator of (2.143) can be written as

$$\nu - 1 - \alpha \nu^2 + \beta \nu = -(\alpha \nu^2 - 2\sqrt{\alpha} + 1) = -(\sqrt{\alpha}\nu - 1)^2 \tag{2.144}$$

So the free surface will have an unbounded amplitude when $1 + \beta = 2\sqrt{\alpha}$. This is a direct generalisation of equation equation (4.72) in [1].

2.6 Topography

The analysis will now move from examining a moving pressure distribution to topography on the bottom. The topography will be given in the form:

$$y = f(t, x) = -h + L\varepsilon g(t, x), \tag{2.145}$$

where ε is assumed to be very small. The set of fully nonlinear equations is given by:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{on} \quad y > \eta(t, x) \tag{2.146}$$

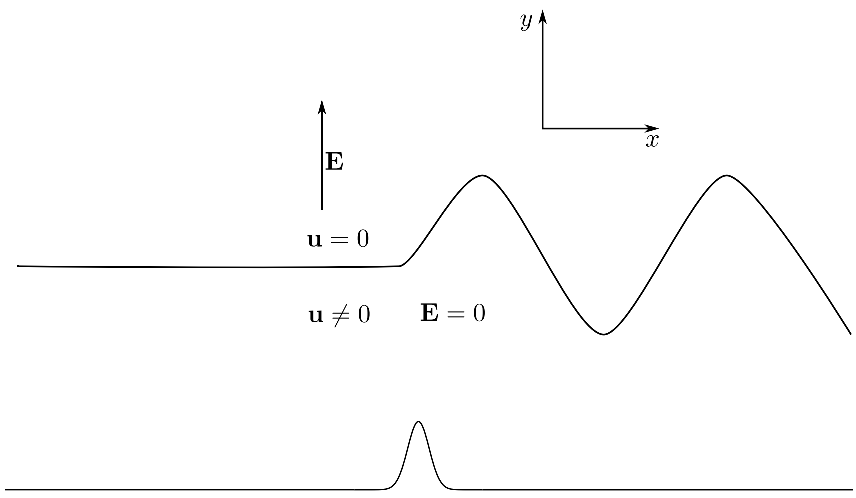


Figure 2.8: Problem Set-Up

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{on} \quad -h \leqslant y \leqslant \eta(t,x) \tag{2.147}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y} \quad \text{on} \;\; y = \eta(t,x) \tag{2.148}$$

$$\begin{aligned} \partial_t \varphi + \frac{1}{2}|\nabla \varphi|^2 \;\; &- \;\; \frac{1}{\rho}\frac{1}{1+(\partial_x \eta)^2}((\partial_x \eta)^2 \Sigma_{11} \\ &- \;\; 2\partial_x \eta \Sigma_{12} + \Sigma_{22}) + g\eta = \frac{\sigma}{\rho}\frac{\partial_x^2 \eta}{(1+(\partial_x \eta)^2)^{3/2}} + C \quad \text{on} \quad y = \eta(t,x) \end{aligned} \tag{2.149}$$

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial y} = 0 \quad \text{on} \quad y = \eta(t,x) \tag{2.150}$$

$$-\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \;\; y = f(t,x) \tag{2.151}$$

$$\frac{\partial V}{\partial y} \to -E_0 \quad y \to \infty \tag{2.152}$$

The process will now be nondimensionalisation and perform a analysis of the linear theory. The nondimensionalisation will be done according to:

$$x = L\hat{x}, \;\; y = L\hat{y}, \;\; \eta = L\hat{\eta}, \;\; t = \sqrt{\frac{\rho L^3}{\sigma}}\hat{t}, \;\; \phi = \sqrt{\frac{\sigma L}{\rho}}\hat{\phi}, \;\; V = E_0 L\hat{V} \quad f = L\hat{f}$$

According to the nondimensionalisation, the equations become:

$$\frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} = 0 \quad \text{on} \;\; \hat{y} > \hat{\eta}(\hat{t}, \hat{x}) \tag{2.153}$$

$$\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\varphi}}{\partial \hat{y}^2} = 0 \quad \text{on} \quad -\hat{h} \leqslant \hat{y} \leqslant \hat{\eta}(\hat{t}, \hat{x}) \tag{2.154}$$

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} + \frac{\partial \hat{\varphi}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{x}} = \frac{\partial \hat{\varphi}}{\partial \hat{y}} \quad \text{on} \;\; \hat{y} = \hat{\eta}(\hat{t}, \hat{x}) \tag{2.155}$$

$$\begin{aligned} \partial_{\hat{t}}\hat{\varphi} + \frac{1}{2}|\nabla\hat{\varphi}|^2 \quad & - \quad \frac{E_b}{1+(\partial_{\hat{x}}\hat{\eta})^2}((\partial_{\hat{x}}\hat{\eta})^2 M_{11} \\ & - \quad 2\partial_{\hat{x}}\hat{\eta} M_{12} + M_{22}) + B\hat{\eta} = \frac{\partial^2_{\hat{x}}\hat{\eta}}{(1+(\partial_{\hat{x}}\hat{\eta})^2)^{3/2}} + K \quad \text{on} \quad \hat{y} = \hat{\eta}(\hat{t}, \hat{x}) \end{aligned} \tag{2.156}$$

$$\frac{\partial \hat{V}}{\partial \hat{x}} + \frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{y}} = 0 \quad \text{on} \quad \hat{y} = \hat{\eta}(\hat{t}, \hat{x}) \tag{2.157}$$

$$-\frac{\partial \hat{f}}{\partial \hat{t}} - \frac{\partial f}{\partial \hat{x}}\frac{\partial \hat{\varphi}}{\partial \hat{x}} + \frac{\partial \hat{\varphi}}{\partial \hat{y}} = 0 \quad \text{on} \;\; \hat{y} = f(\hat{t}, \hat{x}) \tag{2.158}$$

$$\frac{\partial \hat{V}}{\partial \hat{y}} \to -1 \quad \hat{y} \to \infty \tag{2.159}$$

Upon performing a Galilean transformation to a references frame where the topography is stationary. A simple solution of the above equations is $\hat{\eta} = 0$, $\hat{V} = \hat{y}$, $\hat{\varphi} = U\hat{x}$ and $\hat{f} = -\hat{h}$. Drop the hats for convenience and write the following for the linear theory:

$$f(x) = -h + \varepsilon g(x) \tag{2.160}$$

$$\varphi = Ux + \varepsilon\varphi_1 \tag{2.161}$$

$$V = -y + \varepsilon V_1 \tag{2.162}$$

$$\eta = \varepsilon\eta_1 \tag{2.163}$$

The linearised equations then become:

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0 \quad \text{on} \;\; y > 0 \tag{2.164}$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \quad \text{on} \quad -h \leqslant y \leqslant 0 \tag{2.165}$$

$$U\frac{\partial \eta_1}{\partial x} = \frac{\partial \varphi_1}{\partial y} \quad \text{on} \ \ y = 0 \tag{2.166}$$

$$U\frac{\partial \varphi_1}{\partial x} + B\eta_1 + E_b\frac{\partial V_1}{\partial y} = \frac{\partial^2 \eta_1}{\partial x^2} \tag{2.167}$$

$$\frac{\partial V_1}{\partial x} - \frac{\partial \eta_1}{\partial x} = 0 \quad \text{on} \quad y = 0 \tag{2.168}$$

$$-U\frac{dg}{dx} + \frac{\partial \varphi_1}{\partial y} = 0 \quad \text{on} \ \ y = -h \tag{2.169}$$

$$\frac{\partial V_1}{\partial y} \to 0 \quad y \to \infty \tag{2.170}$$

As before the equations (2.164)-(2.170) are solved via Fourier transform, write:

$$\varphi_1 = \int_{\mathbb{R}} A(k, y)e^{ikx}dk \tag{2.171}$$

Then inserting this into (2.164) shows that:

$$\frac{\partial^2 A}{\partial y^2} - k^2 A = 0. \tag{2.172}$$

Which shows that

$$A(k, y) = \alpha(k)e^{ky} + \beta(k)e^{-ky}. \tag{2.173}$$

Write

$$g(x) = \int_{\mathbb{R}} G(k)e^{ikx}dk \tag{2.174}$$

Then inserting (2.171) into (2.169) shows that

$$\begin{aligned} \int_{\mathbb{R}} -U(ik)G(k)e^{ikx}dk + \int_{\mathbb{R}} e^{ikx}\partial_y(\alpha(k)e^{ky} + \beta(k)e^{-ky})dk &= 0 \\ \int_{\mathbb{R}} -U(ik)G(k)e^{ikx}dk + \int_{\mathbb{R}} ke^{ikx}(\alpha(k)e^{-kh} - \beta(k)e^{kh}) &= 0 \\ \int_{\mathbb{R}} k(-UiG(k) + \alpha(k)e^{-kh} - \beta(k)e^{kh})e^{ikx}dk &= 0 \\ \Rightarrow \alpha(k)e^{-kh} &= \beta(k)e^{kh} + UiG(k) \end{aligned}$$

which shows that:

$$\alpha(k) = iUe^{kh}G(k) + \beta(k)e^{2kh} \tag{2.175}$$

which after inserting back into (2.173) shows that:

$$\begin{aligned}
A(k,y) &= [iUe^{kh}G(k) + \beta(k)e^{2kh}]e^{ky} + \beta(k)e^{-ky} \\
&= Ue^{k(y+h)}G(k)i + \beta(k)e^{ky+2kh} + \beta(k)e^{-ky} \\
&= Ue^{k(y+h)}G(k)i + e^{kh}\beta(k)(e^{ky+kh} + e^{-ky-kh}) \\
&= Ue^{k(y+h)}G(k)i + e^{kh}\beta(k)(e^{k(y+h)} + e^{-k(y+h)}) \\
&= Ue^{k(y+h)}G(k)i + 2e^{kh}\beta(k)\cosh(k(y+h))
\end{aligned}$$

The solution for φ_1 is then

$$\varphi_1 = \int_{\mathbb{R}} [Ue^{k(y+h)}G(k)i + 2e^{kh}\beta(k)\cosh(k(y+h))]e^{ikx}dk \tag{2.176}$$

The next point is to solve the equation for the electric potential, write:

$$V_1(x,y) = \int_{\mathbb{R}} C(k,y)e^{ikx}dk \tag{2.177}$$

Then Laplace's equation becomes the following:

$$\frac{\partial^2 C}{\partial y^2} - k^2 C = 0 \tag{2.178}$$

Then the solution to this equation is:

$$C(k,y) = \gamma(k)e^{ky} + \delta(k)e^{-ky}$$

The boundary condition is $\partial V_1/\partial y \to 0$, then this shows that $\gamma(k) = 0$ when $k > 0$ and $\delta(k) = 0$ when $k < 0$. The solution is:

$$V_1 = \int_{\mathbb{R}} \delta(k)e^{-|k|y}e^{ikx}dk \tag{2.179}$$

Now write the free surface as

$$\eta_1 = \int_{\mathbb{R}} D(k)e^{ikx}dk \tag{2.180}$$

Inserting (2.176) and (2.180) into (2.166) yields:

$$\int_{\mathbb{R}} k[Ue^{kh}G(k)i + 2e^{kh}\beta(k)\sinh(kh) - iUD(k)]e^{ikx}dk = 0$$

which in turn shows that:

$$D(k) = e^{kh}G(k) - \frac{2ie^{kh}\beta(k)\sinh(kh)}{U} \tag{2.181}$$

Then,

$$\eta_1 = \int_{\mathbb{R}} \left[e^{kh}G(k) - \frac{2ie^{kh}\beta(k)\sinh(kh)}{U} \right] e^{ikx} dk \tag{2.182}$$

Inserting the values for η_1 and V_1 into (2.168) to show that:

$$\delta(k) = D(k) = e^{kh}G(k) - \frac{2ie^{kh}\beta(k)\sinh(kh)}{U}$$

Then the expressions are:

$$\begin{aligned}
\eta_1 &= \int_{\mathbb{R}} \left[e^{kh}G(k) - \frac{2ie^{kh}\beta(k)\sinh(kh)}{U} \right] e^{ikx} dk \\
\varphi_1 &= \int_{\mathbb{R}} [Ue^{k(y+h)}G(k)i + 2e^{kh}\beta(k)\cosh(k(y+h))]e^{ikx} dk \\
V_1 &= \int_{\mathbb{R}} \left[e^{kh}G(k) - \frac{2ie^{kh}\beta(k)\sinh(kh)}{U} \right] e^{-|k|y}e^{ikx} dk
\end{aligned}$$

Then to find $\beta(k)$ in terms of $G(k)$, we insert everything into (2.167). Inserting the above into (2.167) shows that:

$$\beta(k) = \frac{[kU^2 - B + E_b|k| - k^2]UG(k)i}{-2U^2\cosh(kh) + 2B\sinh(kh) - 2E_b|k|\sinh(kh) + 2k^2\sinh(kh)} \tag{2.183}$$

To compute the dispersion relation, write:

$$\begin{aligned}
V_1 &= \frac{iA}{U}e^{-|k|y}e^{ikx}\sinh(kh) \\
\eta_1 &= -\frac{iA}{U}e^{ikx}\sinh(kh) \\
\phi_1 &= Ae^{-kh}e^{ikx}\cosh(k(y-h))
\end{aligned}$$

Computing the required derivatives:

$$\begin{aligned}
\frac{\partial^2\phi_1}{\partial x^2} &= -k^2Ae^{-kh}e^{ikx}\cosh(k(y-h)) \\
\frac{\partial\eta_1}{\partial x} &= \frac{A}{U}e^{ikx}\sinh(kh) \\
\frac{\partial^2 V_1}{\partial x\partial y} &= -\frac{|k|kA}{U}e^{-|k|y}e^{ikx}\sinh(kh) \\
\frac{\partial^3\eta_1}{\partial x^3} &= -\frac{Ak^3}{U}e^{ikx}\sinh(kh)
\end{aligned}$$

Inserting the above into the (free surface equation2.167) shows that the dispersion relation is:

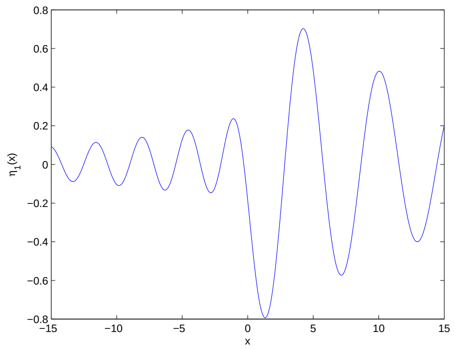
$$U^2 = (k^2 - E_b|k| + B)\frac{\tanh(kh)}{k} \tag{2.184}$$

The free surface is given by:

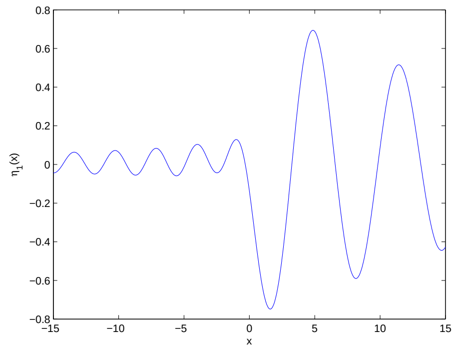
$$\eta_1 = -\int_{\mathbb{R}} \frac{kU^2 e^{ikx}\text{sech }(kh)G(k)}{-kU^2 + (B - E_b|k| + k^2)\tanh(kh)} dk. \tag{2.185}$$

The topography in this case is $g(x) = \text{sech }(x)$. If the topography is symmetric about $x = 0$, then the profile itself will be symmetric as $\exp(ikx) = \cos kx + i \sin kx$ is the sum of an odd and even function and the rest of the integrand is an even function, so integrated over a symmetric domain (which the real line is), the result will be and even function. The following free surface profiles are taken with $B = E_b = h = 2$, the minimum of the free surface profile is at $U = 0.9069$, and the first set are taken when U is above the minimum using Rayleigh viscosity and the second set are taken below the minimum with no Rayleigh viscosity.

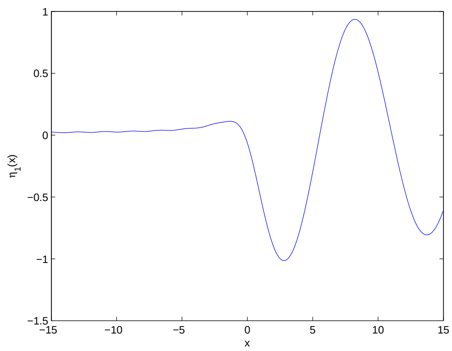
As can be seen the numerical evaluation does indeed show that the profiles in figure 2.10 are symmetric.



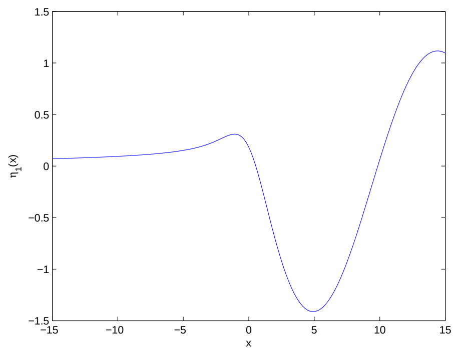
(a) $U = 0.95$



(b) $U = 1.0$

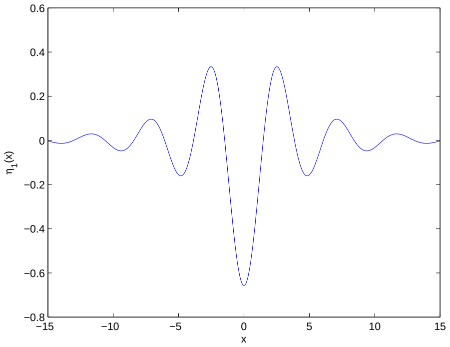


(c) $U = 1.3$

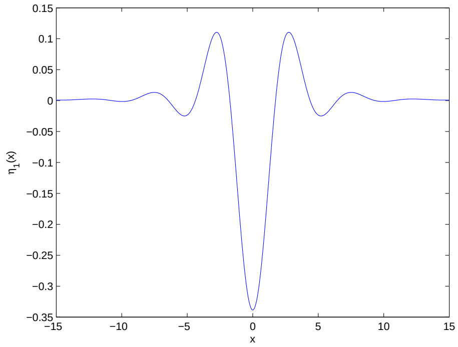


(d) $U = 1.6$

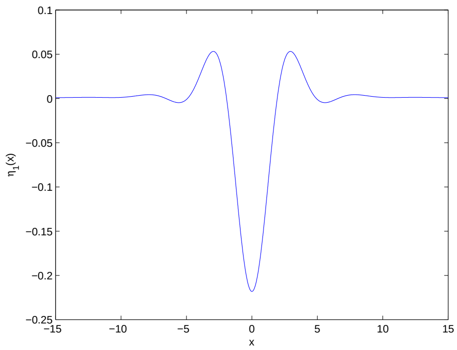
Figure 2.9: Profiles of Topography with $E_b = B = h = 2$



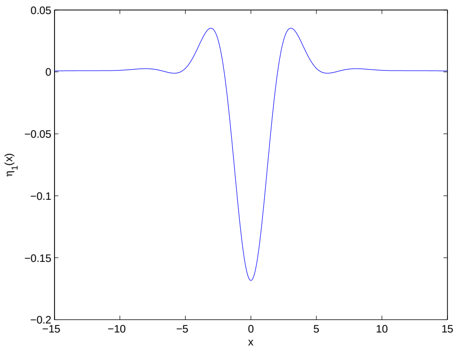
(a) $U = 0.88$



(b) $U = 0.82$



(c) $U = 0.75$



(d) $U = 0.7$

Figure 2.10: Profiles of Topography with $E_b = B = h = 2$

Chapter 3

Weakly Nonlinear Wave Theory

As was shown in the previous section, certain values of parameters can lead to an unbounded free surface, and to overcome this failing of the modelling an extension of the modelling is considered. The idea is to consider the typical horizontal length scale L to be large in comparison to the height of the fluid h, so the ratio $h/L \ll 1$, this is called the *long wavelength approximation*. The small amplitude approximation is still kept, so $a/h \ll 1$, where a is a typical height of the waves involved. Using these approximations it is possible to derive a nonlinear equation that overcomes the problem of blow up that was encountered in the first chapter.

3.1 Moving Pressure Distribution

The basic equations of interest are:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{on } \; y > \eta(t,x) \tag{3.1}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{on } \; -h \leqslant y \leqslant \eta(t,x) \tag{3.2}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial y} \quad \text{on } \; y = \eta(t,x) \tag{3.3}$$

$$\begin{aligned} \partial_t \varphi + \frac{1}{2}|\nabla \varphi|^2 + \frac{P(t,x)}{\rho} \quad &- \quad \frac{1}{\rho}\frac{1}{1+(\partial_x \eta)^2}((\partial_x \eta)^2 \Sigma_{11} + 2\partial_x \eta \Sigma_{12} + \Sigma_{22}) \\ &+ \quad g\eta = \frac{\sigma}{\rho}\frac{\partial_x^2 \eta}{(1+(\partial_x \eta)^2)^{3/2}} + C \quad \text{on } \quad y = \eta(t,x) \end{aligned} \tag{3.4}$$

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial y} = 0 \quad \text{on} \quad y = \eta(t,x) \tag{3.5}$$

$$\frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \;\; y = -h \tag{3.6}$$

$$\frac{\partial V}{\partial y} \to -E_0 \quad y \to \infty \tag{3.7}$$

The governing equations are scaled in the following manner:

$$x = L\hat{x}, \quad t = \frac{L}{c_0}\hat{t}, \quad \eta = a\hat{\eta}, \quad \varphi = \frac{gLa}{c_0}\hat{\varphi}, \quad y^{(1)} = h\hat{y}, \quad y^{(2)} = L\hat{Y}, \quad V = LE_0\hat{V}. \tag{3.8}$$

There are two different scalings for y depending on which region is of interest, the perfect conductor or the dielectric. Define the parameters:

$$\alpha = \frac{a}{h}, \quad \beta = \frac{h^2}{L^2} \tag{3.9}$$

The governing equations scale as:

$$\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}^2} + \frac{1}{\beta}\frac{\partial^2 \hat{\varphi}}{\partial \hat{y}^2} = 0 \tag{3.10}$$

$$\frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{Y}^2} = 0. \tag{3.11}$$

The scaling of the unsteady Bernoulli equation is carried out as follows, for the velocity potential

$$\frac{\partial \varphi}{\partial t} \mapsto ga\frac{\partial \hat{\varphi}}{\partial \hat{t}}, \quad \frac{\partial \varphi}{\partial x} \mapsto \sqrt{\alpha g a}\frac{\partial \hat{\varphi}}{\partial \hat{x}}, \quad \frac{\partial \varphi}{\partial y} \mapsto \sqrt{ga}\sqrt{\frac{\alpha}{\beta}}\frac{\partial \hat{\varphi}}{\partial \hat{y}}. \tag{3.12}$$

The pressure scales as:

$$P(t,x) = \rho g a \hat{P}(\hat{t},\hat{x}). \tag{3.13}$$

The free surface derivative scales as:

$$\frac{\partial \eta}{\partial x} \mapsto \alpha\sqrt{\beta}\frac{\partial \hat{\eta}}{\partial \hat{x}}. \tag{3.14}$$

The components of the Faraday tensor scale as:

$$T_{ij} \mapsto \epsilon_d E_0^2 \hat{T}_{ij}. \tag{3.15}$$

For brevity denote:

$$E = \frac{1}{2}(\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2 - 1)\left[\left(\frac{\partial\hat{V}}{\partial\hat{x}}\right)^2 + \left(\frac{\partial\hat{V}}{\partial\hat{Y}}\right)^2\right] + 2\alpha\sqrt{\beta}\partial_{\hat{x}}\hat{\eta}\frac{\partial\hat{V}}{\partial\hat{x}}\frac{\partial\hat{V}}{\partial\hat{Y}} \tag{3.16}$$

The unsteady Bernoulli equation scales as:

$$\frac{\partial\hat{\varphi}}{\partial\hat{t}}+\frac{1}{2}\left[\alpha\left(\frac{\partial\hat{\varphi}}{\partial\hat{x}}\right)^2+\frac{\alpha}{\beta}\left(\frac{\partial\hat{\varphi}}{\partial\hat{y}}\right)^2\right]+\frac{1}{\alpha}\frac{E_bE}{1+\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2}+\hat{P}(\hat{t},\hat{x})+\hat{\eta} = \frac{\beta B\partial_{\hat{x}}^2\hat{\eta}}{(1+\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2)^{3/2}}+C, \tag{3.17}$$

where the constants are:

$$E_b = \frac{\epsilon_d E_0^2}{\rho g h}, \quad B = \frac{\sigma}{\rho g h^2}. \tag{3.18}$$

The Bernoulli constant can be found be setting $\hat{\varphi} = \hat{\eta} = \hat{P} = 0$ and $\hat{V} = -Y$ to find that $C = -E_b/2\alpha$. The free surface condition scales as:

$$\frac{\partial\hat{\eta}}{\partial\hat{t}} + \alpha\frac{\partial\hat{\varphi}}{\partial\hat{x}}\frac{\partial\hat{\eta}}{\partial\hat{x}} = \frac{1}{\beta}\frac{\partial\hat{\varphi}}{\partial\hat{y}} \quad \text{on} \;\; y = \alpha\hat{\eta}. \tag{3.19}$$

The interfacial conditions for the electric field scale as:

$$\alpha\sqrt{\beta}\frac{\partial\hat{\eta}}{\partial\hat{x}}\frac{\partial\hat{V}}{\partial\hat{x}} - \frac{\partial\hat{V}}{\partial\hat{y}} = 0 \quad \text{on} \;\; y = 0 \tag{3.20}$$

$$\frac{\partial\hat{V}}{\partial\hat{x}} + \alpha\sqrt{\beta}\frac{\partial\hat{\eta}}{\partial\hat{x}}\frac{\partial\hat{V}}{\partial\hat{y}} = 0 \quad \text{on} \;\; y = 0. \tag{3.21}$$

Applying the scaling to the expression for induced charge on the surface shows that:

$$\hat{\Sigma}_Q = \frac{1}{1+\alpha^2(\partial_{\hat{x}}\hat{\eta})^2}\left(\alpha\sqrt{\beta}\frac{\partial\hat{\eta}}{\partial\hat{x}}\frac{\partial\hat{V}}{\partial\hat{x}} + \frac{\partial\hat{V}}{\partial\hat{y}}\right) \tag{3.22}$$

3.2 Canonical Korteweg de Vries (KdV) Scaling

Choose the usual scalings for the KdV equation by setting $\alpha = \beta = \varepsilon \ll 1$, also change the co-ordinates to a moving set of co-ordinates by:

$$T = \varepsilon\hat{t} \tag{3.23}$$

$$X = \hat{x} - \hat{t}, \tag{3.24}$$

which makes the differential operators become:

$$\frac{\partial}{\partial\hat{t}} = \varepsilon\frac{\partial}{\partial T} - \frac{\partial}{\partial X} \tag{3.25}$$

$$\frac{\partial}{\partial\hat{x}} = \frac{\partial}{\partial X} \tag{3.26}$$

The two equations which this transformation of co-ordinates affects are the unsteady Bernoulli equation and the free surface condition. Use the principle of least degeneration for the pressure and scale it as $P = \varepsilon^n p$, where n is to be determined later. They become:

$$\varepsilon\frac{\partial\hat{\eta}}{\partial T} - \frac{\partial\hat{\eta}}{\partial X} + \varepsilon\frac{\partial\hat{\varphi}}{\partial X}\frac{\partial\hat{\eta}}{\partial X} = \frac{1}{\varepsilon}\frac{\partial\hat{\varphi}}{\partial\hat{y}} \tag{3.27}$$

$$\begin{aligned}\frac{\varepsilon B\partial_X^2\hat{\eta}}{(1+\varepsilon^3(\partial_X\hat{\eta})^2)^{3/2}} - \frac{E_b}{2\varepsilon} = \varepsilon\frac{\partial\hat{\varphi}}{\partial T} - \frac{\partial\hat{\varphi}}{\partial X} + \frac{1}{2}\left[\varepsilon\left(\frac{\partial\hat{\varphi}}{\partial X}\right)^2 + \left(\frac{\partial\hat{\varphi}}{\partial\hat{y}}\right)^2\right] + \varepsilon^n p(X) + \hat{\eta} + \\ + \frac{E_b}{\varepsilon}\frac{1}{1+\varepsilon^3(\partial_X\hat{\eta})^2}\left[\frac{1}{2}(\varepsilon^3(\partial_{\hat{X}}\hat{\eta})^2 - 1)\left[\left(\frac{\partial\hat{V}}{\partial\hat{X}}\right)^2 + \left(\frac{\partial\hat{V}}{\partial\hat{Y}}\right)^2\right]\right. \\ \left. + 2\varepsilon^{3/2}\partial_{\hat{X}}\hat{\eta}\frac{\partial\hat{V}}{\partial\hat{X}}\frac{\partial\hat{V}}{\partial\hat{Y}}\right]\end{aligned} \tag{3.28}$$

$$\Sigma_Q = \frac{1}{\varepsilon^2(\partial_{\hat{x}}\hat{\eta})^2}\left(\varepsilon^{\frac{3}{2}}\frac{\partial\hat{\eta}}{\partial\hat{x}}\frac{\partial\hat{V}}{\partial\hat{x}} + \frac{\partial\hat{V}}{\partial\hat{y}}\right) \tag{3.29}$$

Upon dropping the hats, write:

$$\varphi(T,X,y) = \varphi_0(T,X,y) + \varepsilon\varphi_1(T,X,y) \tag{3.30}$$

$$V(T,X,Y) = -Y + \varepsilon^{3/2}V_1(T,X,Y) \tag{3.31}$$

$$\eta(T,X) = \eta_0(T,X) + \varepsilon\eta_1(T,X) \tag{3.32}$$

The asymptotic expansion of V_1 deserves an explanation. Write the expansion for V_1 as $V = -Y + \delta(\varepsilon)V_1$ and insert this into (3.21), showing that:

$$\delta(\varepsilon)\frac{\partial V_1}{\partial X} + \varepsilon^{3/2}\frac{\partial\eta}{\partial X}\left(-1 + \delta(\varepsilon)\frac{\partial V_1}{\partial Y}\right)$$

Which shows that $\delta(\varepsilon) = \varepsilon^{3/2}$. Note that the terms of order $1/\varepsilon$ cancel. The governing equations reduce to the following:

$$\partial_y^2\varphi_0 = 0 \tag{3.33}$$

$$\partial_X^2 V_1 + \partial_Y^2 V_1 = 0 \tag{3.34}$$

On $y = -1$, the boundary condition is $\partial_y\varphi = 0$ which readily reduces to $\partial_y\varphi_0 = 0$ on $y = -1$. The largest term in the free surface boundary is of order $1/\varepsilon$ which means that $\partial_y\varphi_0 = 0$ on $y = \varepsilon\eta_0$, upon using a Taylor series expansion:

$$\partial_y\varphi_0(T,X,\varepsilon\eta_0) = \partial_y\varphi_0(T,X,0) + \varepsilon\eta_0\partial_y^2\varphi_0(T,X,0) = 0 \tag{3.35}$$

So $\partial_y \varphi_0 = 0$ on $y = 0$. The final boundary condition to find is from the unsteady Bernoulli equation, it is a simple matter to see that:

$$-\partial_X \varphi_0 + \eta_0 = 0 \tag{3.36}$$

The interface condition shows that

$$\varepsilon^{3/2} \partial_X V_1 + \varepsilon^{3/2} \partial_X \eta_0 (1 + \varepsilon \partial_Y V_1) \quad \text{on} \quad y = 0 \tag{3.37}$$

which shows that:

$$\partial_X V_1 + \partial_X \eta_0 = 0 \quad \text{on} \quad y = 0 \tag{3.38}$$

This is now a system which can be solved. The solution of the governing equation for φ_0 shows that $\varphi_0 = A(T, X) + B(T, X)y$, the boundary conditions show that $B(T, X) = 0$, and so $\varphi_0 = \varphi_0(T, X)$. Inserting this into the other boundary conditions shows that:

$$\begin{aligned} \partial_X \varphi_0 &= \eta_0 & (3.39) \\ \partial_X V_1 &= -\partial_X \eta_0 & (3.40) \end{aligned}$$

Now consider the $O(\varepsilon)$ case. The governing equation for φ becomes:

$$\frac{\partial^2 \varphi_0}{\partial X^2} + \frac{\partial^2 \varphi_1}{\partial y^2} = 0 \tag{3.41}$$

The boundary condition at $y = -1$ is $\partial_y \varphi_1(T, X, -1) = 0$, integrating (3.41) once shows that

$$\frac{\partial \varphi_1}{\partial y} = -y \frac{\partial^2 \varphi_0}{\partial X^2} + A(T, X) \tag{3.42}$$

Using the boundary condition at $y = -1$ shows that $0 = \partial_X^2 \varphi_0 + A(T, X)$ and so $A(T, X) = \partial_X^2 \varphi_0$, and hence:

$$\varphi_1 = -\frac{(y+1)^2}{2} \frac{\partial^2 \varphi_0}{\partial X^2} + B(T, X) \tag{3.43}$$

Examining the free surface boundary condition at $O(1)$ shows

$$\begin{aligned} \partial_y \varphi_0 + \varepsilon \partial_y \varphi_1 &= \varepsilon(\partial_X \eta_0 + \varepsilon \partial_X \eta_1) \\ &= \varepsilon \partial_X \eta_0 + o(\varepsilon) \end{aligned}$$

So the $O(1)$ free boundary equation is:

$$\frac{\partial \varphi_1}{\partial y} - \frac{\partial \eta_0}{\partial X} = 0 \quad \text{on} \quad y = 0 \tag{3.44}$$

The leading order terms for the unsteady Bernoulli equation are:

$$\frac{E_b}{\varepsilon}\left[-\frac{1}{2} - \varepsilon^{3/2}\frac{\partial V_1}{\partial Y}\right] \tag{3.45}$$

On examining the $O(\varepsilon)$ terms in the equation it makes sense to scale E_b as $E_b = \hat{E}_b\sqrt{\varepsilon}$ to get the electrical field term into the equation, likewise appealing to the principle of least degeneracy, choose $n = 1$ to keep as many terms in the equation. The $O(\varepsilon)$ unsteady Bernoulli equation becomes:

$$\frac{\partial \varphi_0}{\partial T} - \frac{\partial \varphi_1}{\partial X} + \frac{1}{2}\left(\frac{\partial \varphi_0}{\partial X}\right)^2 + p(X) + \eta_1 - \hat{E}_b\frac{\partial V_1}{\partial Y} = B\frac{\partial^2 \eta_0}{\partial X^2} \tag{3.46}$$

It is possible to obtain an expression for η_1 by using (3.44) to obtain at $y = 0$,

$$\eta_1 = B\frac{\partial^2 \eta_0}{\partial X^2} - \frac{\partial \varphi_0}{\partial T} - \frac{1}{2}\frac{\partial^3 \varphi_0}{\partial X^3} + \frac{\partial B}{\partial X} - \frac{1}{2}\left(\frac{\partial \varphi_0}{\partial X}\right)^2 - p(X) + \hat{E}_b\frac{\partial V_1}{\partial Y} \tag{3.47}$$

The idea is to obtain an expression for η_0, and in order to do this the next order in the perturbation must be considered. The free surface condition (3.27) has the $O(\varepsilon^2)$ expansion as:

$$\frac{\partial \eta_0}{\partial T} - \frac{\partial \eta_1}{\partial X} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X} = -\eta_0\frac{\partial^2 \varphi_0}{\partial X^2} + \frac{\partial \varphi_2}{\partial y}\bigg|_{y=0} \quad \text{on} \quad y = 0 \tag{3.48}$$

The governing equation at $O(\varepsilon^2)$ becomes:

$$\frac{\partial^2 \varphi_1}{\partial X^2} + \frac{\partial^2 \varphi_2}{\partial y^2} = 0 \tag{3.49}$$

Hence:

$$\frac{\partial \varphi_2}{\partial y} = \frac{(y+1)^3}{6}\frac{\partial^4 \varphi_0}{\partial X^4} - y\frac{\partial^2 B}{\partial X^2} + C(T, X) \tag{3.50}$$

The boundary condition at $y = -1$ is the usual $\partial_y \varphi_2 = 0$, this gives $C(T, X)$ to be $C(T, X) = \partial_X^2 B$, the solution is now:

$$\frac{\partial \varphi_2}{\partial y} = \frac{(y+1)^3}{6}\frac{\partial^4 \varphi_0}{\partial X^4} - (y+1)\frac{\partial^2 B}{\partial X^2} \tag{3.51}$$

Inserting (3.51) into (3.50) shows that:

$$\frac{\partial \eta_0}{\partial T} - \frac{\partial \eta_1}{\partial X} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X} = -\eta_0 \frac{\partial^2 \varphi_0}{\partial X^2} + \frac{1}{6}\frac{\partial^4 \varphi_0}{\partial X^4} - \frac{\partial^2 B}{\partial X^2} \tag{3.52}$$

Inserting (3.47) into (3.52) and rearranging shows that:

$$2\frac{\partial \eta_0}{\partial T} + 3\eta_0 \frac{\partial \eta_0}{\partial X} + \left(\frac{1}{3} - B\right)\frac{\partial^3 \eta_0}{\partial X^3} - \hat{E}_b \frac{\partial^2 V_1}{\partial X \partial Y} + \frac{dP}{dX} = 0 \tag{3.53}$$

The next task is to find V_1, the appendix on Hilbert transforms states that:

$$\partial_y f(x_0, y_0) = \mathscr{H}(\partial_x f)(x_0, y_0) \tag{3.54}$$

Applying this to the case in hand shows that:

$$\partial_Y V_1 = \mathscr{H}(\partial_X V_1) \tag{3.55}$$

and hence that:

$$\partial_X \partial_Y V_1 = \mathscr{H}(\partial_X^2 V_1). \tag{3.56}$$

The term in the Hilbert transform can be related to η_0 by the use of (3.40) which shows that:

$$\frac{\partial^2 V_1}{\partial X^2}(T, X, 0) = -\frac{\partial^2 \eta_0}{\partial X^2} \tag{3.57}$$

Inserting this into the original equation:

$$\partial_X \partial_Y V_1(T, X_0, 0) = \mathscr{H}(\partial_X^2 \eta_0) \tag{3.58}$$

The integral transform $\mathscr{H}(f)$ is known as the *Hilbert transform* of f. The equation is then:

$$2\frac{\partial \eta_0}{\partial T} + 3\eta_0 \frac{\partial \eta_0}{\partial X} + \left(\frac{1}{3} - B\right)\frac{\partial^3 \eta_0}{\partial X^3} + \hat{E}_b \mathscr{H}(\partial_X^2 \eta_0) + \frac{dP}{dX} = 0 \tag{3.59}$$

Converting back into the original co-ordinates and putting the dimensions back into the equation shows that:

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{3}{2}\frac{c_0}{h}\eta \frac{\partial \eta}{\partial x} + \left(\frac{1}{3} - B\right)\frac{h^2 c_0}{2}\frac{\partial^3 \eta}{\partial x^3} + \frac{E_b h}{2}\frac{\partial^2}{\partial x^2}\mathscr{H}(\eta) + \frac{c_0}{2\rho g} p'(x) = 0 \tag{3.60}$$

When $E_b = 0$ and $p'(x) = 0$ equation (3.60) reduces to the classical Korteweg de Vries equation. The electric field introduces a nonlocal term in the Hilbert transform similar to the Benjamin Ono equation ([28],[29]) which describes an interfacial wave, and the relevant equation in the literature is called the Benjamin equation [30]. The analysis has been expended to include a forcing term (in this case a moving pressure distribution). However when $B = 1/3$, the dispersive term disappears and the equation has shock wave solutions. The idea for the next section will be to derive a generalisation of the equation in [16] to include a forcing term. A different scaling is required to obtain the equation, moreover the Bond number B will also have to be expanded. This analysis is carried out in the following section.

3.3 Scaling around $B = 1/3$

Clearly from (3.60) when $B = 1/3$ the dispersion term disappears and the equation reduces to modified Burgers equation which can contain shock wave solution. The task then is to obtain a dispersive equation that will be valid for B close to $1/3$. In this derivation, take the scaling to be $\alpha = \varepsilon^2$ and $\beta = \varepsilon$, also make the transformation of co-ordinates as:

$$T = \varepsilon^2 \hat{t} \tag{3.61}$$

$$X = \hat{x} - \hat{t} \tag{3.62}$$

The Parameter B in the governing equation is expanded as:

$$B = \frac{1}{3} + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots \tag{3.63}$$

The analysis follows very similarly to the previous case. The variables are expanded in the same way:

$$\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + o(\varepsilon^2) \tag{3.64}$$

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + o(\varepsilon^3) \tag{3.65}$$

$$V_1 = Y + \varepsilon^{3/2} V_1 + o(\varepsilon^{3/2}) \tag{3.66}$$

The equations that will be used for this case are:

$$\varepsilon^2\frac{\partial\hat{\eta}}{\partial T}-\frac{\partial\hat{\eta}}{\partial X}+\varepsilon^2\frac{\partial\hat{\varphi}}{\partial X}\frac{\partial\hat{\eta}}{\partial X}=\frac{1}{\varepsilon}\frac{\partial\hat{\varphi}}{\partial\hat{y}} \tag{3.67}$$

$$\begin{aligned}\frac{\varepsilon B\partial_X^2\hat{\eta}}{(1+\varepsilon^3(\partial_X\hat{\eta})^2)^{3/2}}-\frac{E_b}{2\varepsilon^2}&=\varepsilon^2\frac{\partial\hat{\varphi}}{\partial T}-\frac{\partial\hat{\varphi}}{\partial X}+\frac{1}{2}\left[\varepsilon^2\left(\frac{\partial\hat{\varphi}}{\partial X}\right)^2+\varepsilon\left(\frac{\partial\hat{\varphi}}{\partial\hat{y}}\right)^2\right]+\varepsilon^n p(X)+\hat{\eta}+\\&+\frac{A}{\varepsilon^2}\frac{1}{1+\varepsilon^5(\partial_X\hat{\eta})^2}\left[\frac{1}{2}(\varepsilon^5(\partial_{\hat{X}}\hat{\eta})^2-1)\left[\left(\frac{\partial\hat{V}}{\partial\hat{X}}\right)^2+\left(\frac{\partial\hat{V}}{\partial\hat{Y}}\right)^2\right]\right.\\&\left.+2\varepsilon^{5/2}\partial_{\hat{X}}\hat{\eta}\frac{\partial\hat{V}}{\partial\hat{X}}\frac{\partial\hat{V}}{\partial\hat{Y}}\right]\end{aligned} \tag{3.68}$$

$$\frac{\partial^2\hat{\varphi}}{\partial X^2}+\frac{1}{\varepsilon}\frac{\partial^2\hat{\varphi}}{\partial\hat{y}^2}=0 \tag{3.69}$$

$$\frac{\partial^2\hat{V}}{\partial X^2}+\frac{\partial^2\hat{V}}{\partial\hat{y}^2}=0 \tag{3.70}$$

$$\frac{\partial\hat{V}}{\partial X}+\varepsilon^{5/2}\frac{\partial\hat{\eta}}{\partial X}\frac{\partial\hat{V}}{\partial Y}=0 \tag{3.71}$$

From here on in, the hats are dropped. To first order the following holds:

$$\eta_0 = \frac{\partial\varphi_0}{\partial X} \tag{3.72}$$

$$\varphi_0 = \varphi_0(T,X) \tag{3.73}$$

$$\frac{\partial^2 V_1}{\partial X^2}+\frac{\partial^2 V_1}{\partial Y^2} = 0 \tag{3.74}$$

$$\varphi_1 = -\frac{(y+1)^2}{2}\frac{\partial^2\varphi_0}{\partial X^2}+A(T,X) \tag{3.75}$$

Where the function $A(T,X)$ is found later. Examining the unsteady Bernoulli at $O(\varepsilon)$ shows that:

$$-\partial_X\varphi_1+\eta_1=\frac{1}{3}\partial_X^2\eta_0 \quad \text{on} \quad y=0 \tag{3.76}$$

Inserting the expression for φ_1 and re-arranging gives η_1

$$\eta_1=-\frac{1}{6}\partial_X^3\varphi_0+\partial_X A \tag{3.77}$$

Moving on to the $O(\varepsilon^2)$ terms. The governing equation for φ shows that:

$$\partial_y^2\varphi_2=-\partial_X^2\varphi_1 \tag{3.78}$$

Along with the boundary condition $\partial_y\varphi_2=0$ on $y=-1$. Integrating once and using the boundary condition shows that:

$$\partial_y\varphi_2=\frac{(y+1)^3}{6}\partial_X^4\varphi_0-(y+1)\partial_X^2 A(T,X) \tag{3.79}$$

Integrating once more shows that:

$$\varphi_2 = \frac{(y+1)^4}{24}\partial_X^4\varphi_0 - \frac{(y+1)^2}{2}\partial_X^2 A(T,X) + B(T,X) \tag{3.80}$$

The unsteady Bernoulli equation has terms of $O(\varepsilon^2)$ but there has to be some scaling by the principle of least degeneration. Take $n = 2$ to include the pressure term, and scale A multiplying the electrical term as $E_b = \varepsilon^{5/2}\hat{E}_b$ to include terms from the Faraday tensor. The $O(\varepsilon^2)$ part of the unsteady Bernoulli equation is:

$$\partial_T\varphi_0 - \partial_X\varphi_2 + \frac{1}{2}(\partial_X\varphi_0)^2 + \eta_2 + p(X) + \hat{E}_b\partial_Y V_1 = B_1\partial_X^2\eta_0 + \frac{1}{3}\partial_X^2\eta_1 \quad \text{on} \quad y = 0 \tag{3.81}$$

The free surface equation has an $O(\varepsilon^2)$ term. Extracting the $O(\varepsilon^2)$ terms of the LHS of this equation shows:

$$\text{LHS}(\varepsilon^2) = \partial_T\varphi_0 - \partial_X\eta_2 + \partial_X\varphi_0\partial_X\eta_0 \tag{3.82}$$

The $O(\varepsilon^2)$ terms on the RHS are somewhat more complicated to derive. As there is a $1/\varepsilon$ term multiplying the derivative on the RHS, the expansion has to go to $O(\varepsilon^3)$.

$$\begin{aligned} \frac{1}{\varepsilon}\partial_y\varphi(T,X,\varepsilon^2\eta) &= \frac{1}{\varepsilon}\partial_y\varphi(T,X,0) + \varepsilon\eta\partial_y^2\varphi \\ &= \frac{1}{\varepsilon}(\partial_y\varphi_1 + \varepsilon\partial_y\varphi_2 + \varepsilon^2\partial_y\varphi_3) + \varepsilon^2(\eta_0 + \varepsilon\eta_1)(\partial_y^2\varphi_1 + \varepsilon\partial_y^2\varphi_2) \end{aligned}$$

The ε^2 free boundary equation becomes:

$$\partial_T\varphi_0 - \partial_X\eta_2 + \partial_X\varphi_0\partial_X\eta_0 = \partial_y\varphi_3 + \eta_0\partial_y^2\varphi_1 \quad \text{on} \quad y = 0 \tag{3.83}$$

Going to $O(\varepsilon^3)$ in the governing equation shows that:

$$\frac{\partial^2\varphi_3}{\partial y^2} = -\frac{\partial^2\varphi_2}{\partial X^2} \tag{3.84}$$

Integrating once and using the usual boundary condition at $y = -1$, gives:

$$\frac{\partial\varphi_3}{\partial y} = -\frac{(y+1)^5}{120}\frac{\partial^6\varphi_0}{\partial X^6} + \frac{(y+1)^3}{6}\frac{\partial^4 B}{\partial X^4} - (y+1)\frac{\partial^2 D}{\partial X^2} \tag{3.85}$$

Differentiating the unsteady Bernoulli equation yields:

$$\partial_T\partial_X\varphi_0 - \partial_X^2\varphi_2 + \partial_X\varphi_0\partial_X^2\varphi_0 + \partial_X\eta_2 + p'(X) + \hat{E}_b\partial_X\partial_Y V_1 = B_1\partial_X^3\eta_0 + \frac{1}{3}\partial_X^3\eta_1 \tag{3.86}$$

Inserting everything into (3.86) shows that:

$$2\partial_T\eta_0 + \frac{1}{45}\partial_X^5\eta_0 + 3\eta_0\partial_X\eta_0 + \hat{E}_b\partial_X\partial_Y V_1 + p'(X) - B_1\partial_X^3\eta_0 = 0 \qquad (3.87)$$

As before it is possible to transform the electrical term as a Hilbert transform of the free surface, so:

$$2\partial_T\eta_0 + \frac{1}{45}\partial_X^5\eta_0 + 3\eta_0\partial_X\eta_0 + \hat{E}_b\partial_X^2\mathscr{H}(\eta_0) + p'(X) - B_1\partial_X^3\eta_0 = 0 \qquad (3.88)$$

In terms of the original variables, the equation is

$$\partial_t\eta + c_0\partial_x\eta + \frac{1}{90}c_0h^4\partial_x^5\eta + \frac{3}{2}\frac{c_0}{h}\eta\partial_x\eta - \frac{1}{2}\left(B - \frac{1}{3}\right)c_0h^2\partial_x^3\eta + \frac{1}{2}E_bhc_0\partial_x^2\mathscr{H}(\eta) + c_0\frac{p'}{2\rho g} = 0 \qquad (3.89)$$

The Induced surface charge can be expanded as:

$$\begin{aligned}
\Sigma_Q &= \frac{1}{1+\varepsilon^2(\partial_{\hat{x}}\hat{\eta})^2}\left(\varepsilon^{\frac{3}{2}}\frac{\partial\hat{\eta}}{\partial\hat{x}}\frac{\partial\hat{V}}{\partial\hat{x}} + \frac{\partial\hat{V}}{\partial\hat{y}}\right) \\
&= \left.\frac{\partial V}{\partial y}\right|_{y=0} + o(\varepsilon^{\frac{3}{2}}) \\
&= -1 + \varepsilon^{\frac{3}{2}}\left.\frac{\partial V_1}{\partial y}\right|_{y=0} + o(\varepsilon^{\frac{3}{2}}) \\
&= -1 + \varepsilon^{\frac{3}{2}}\mathscr{H}\left.\left(\frac{\partial V_1}{\partial X}\right)\right|_{y=0} + o(\varepsilon^{\frac{3}{2}}) \\
&= -1 + \varepsilon^{\frac{3}{2}}\mathscr{H}\left(\frac{\partial\eta_0}{\partial X}\right) + o(\varepsilon^{\frac{3}{2}})
\end{aligned}$$

Which aligns with linear theory. To compute the linear dispersion relation, ignore the nonlinear terms, set $p' = 0$ and write $\eta(t,x) = \alpha\cos(kx - \omega t)$ to obtain:

$$\omega = c_0k + \frac{1}{90}c_0h^4k^5 + \frac{1}{2}\left(B - \frac{1}{3}\right)c_0h^2k^3 - \frac{E_bhk^2c_0\text{sgn}(k)}{2} \qquad (3.90)$$

Write $\omega = kc$ to obtain:

$$\frac{c}{c_0} = 1 + \frac{1}{90}(hk)^4 + \frac{1}{2}\left(B - \frac{1}{3}\right)(hk)^2 - \frac{E_bhk\text{sgn}(k)}{2} \qquad (3.91)$$

The dispersion relation (equation 3.91) is illustrated in figure 3.3 where the values of $F = c/c_0$ versus kh for $B = 0.35$ and $E_b = 0.2$. To find the shape of the free surface

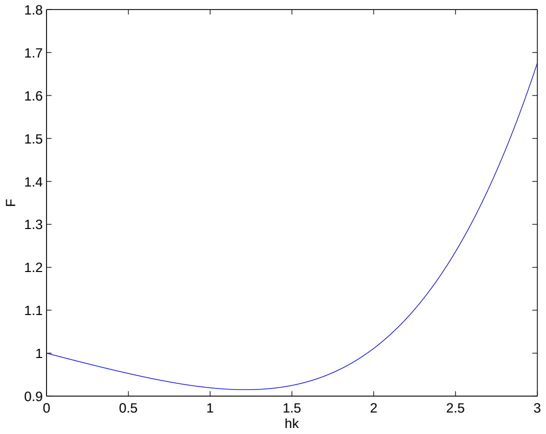


Figure 3.1: $B = 0.35, \quad E_b = 0.2$

in the linear theory, the nonlinear terms are ignored and the resulting ODE is solved by Fourier transform. Write $\eta = f(x - ct)$, inserting this into (3.89), shows that:

$$-c\partial_x f + c_0\partial_x f + \frac{1}{90}c_0 h^4 \partial_x^5 f - \frac{c_0}{2}\left(B - \frac{1}{3}\right)h^2\partial_x^3 f + \frac{1}{2}E_b h c_0 \partial_x^2 \mathscr{H}(f) + c_0\frac{p'}{2\rho g} = 0 \tag{3.92}$$

This equation can be immediately be integrated w.r.t. x to yield:

$$\left(1 - \frac{c}{c_0}\right)f + \frac{1}{90}h^4\partial_x^4 f - \frac{1}{2}\left(B - \frac{1}{3}\right)h^2\partial_x^2 f + \frac{1}{2}E_b h \partial_x \mathscr{H}(f) + \frac{p}{2\rho g} = 0 \tag{3.93}$$

Now write:

$$f = \int_{\mathbb{R}} \alpha(k)e^{ikx}dk, \quad p = \int_{\mathbb{R}} \beta(k)e^{ikx}dk \tag{3.94}$$

Inserting the Fourier expansions in the above equations shows that the integrand must satisfy:

$$\left(1 - \frac{c}{c_0}\right)\alpha + \frac{1}{90}h^4k^4\alpha + \frac{1}{2}\left(B - \frac{1}{3}\right)h^2k^2\alpha - \frac{E_b h\alpha|k|}{2} + \frac{\beta}{2\rho g} = 0 \tag{3.95}$$

Solving this for α shows that:

$$\alpha(k) = -\frac{\beta(k)}{2\rho g}\left[1 - \frac{c}{c_0} + \frac{h^4k^4}{90} + \frac{1}{2}\left(B - \frac{1}{3}\right)h^2k^2 - \frac{E_b h|k|}{2}\right]^{-1} \tag{3.96}$$

3.4 Integrating the Weakly Nonlinear Equation

In this section solutions of equation (3.89) are found numerically. To find travelling wave solutions substitute $\eta = f(x - ct)$ in (3.89) to obtain:

$$\left(1-\frac{c}{c_0}\right)f' + \frac{1}{90}h^4 f^{(5)} + \frac{3}{2h}ff' - \frac{1}{2}\left(B - \frac{1}{3}\right)hf^{(3)} + \frac{E_b h}{2}\mathscr{H}(f'') + \frac{p'}{2\rho g} = 0 \quad (3.97)$$

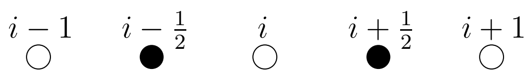
The equation to solve is given by:

$$\left(1-\frac{c}{c_0}\right)f + \frac{1}{90}h^4 \frac{d^4 f}{dx^4} + \frac{3}{4h}f^2 - \frac{1}{2}\left(B - \frac{1}{3}\right)h^2\frac{d^2 f}{dx^2} + \frac{E_b h}{2}\mathscr{H}\left(\frac{df}{dx}\right) + \frac{p}{2\rho g} = 0 \quad (3.98)$$

This is done via finite differences, the finite differences for the derivatives are the following:

$$\begin{aligned} f^{(4)}(x) &= \frac{f(x+2\delta x) - 4f(x+\delta x) + 6f(x) - 4f(x-\delta x) + f(x-2\delta x)}{(\delta x)^4} \\ f^{(2)}(x) &= \frac{f(x+\delta x) - 2f(x) + f(x-\delta x)}{(\delta x)^2} \end{aligned}$$

The way forward will be to take the midpoints for the main finite difference (black dots) and the Hilbert transform will be evaluated at all the integer points (white dots). At the end of the algorithm, the solution will be translated back to the integer points, by taking the average of the two neighbouring points.



The trapezium rule is used to compute the integral:

$$\int_a^b f(x)dx \approx f(x_0)\frac{\delta x}{2} + f(x_N)\frac{\delta x}{2} + \sum_{n=1}^{N-1} f(x_n)\delta x \quad (3.99)$$

The Hilbert transform is then written as:

$$\mathscr{H}\left(\frac{df}{dx}\right) = \sum_{n=0}^{N-1} \frac{f'(x_{n+\frac{1}{2}})}{(x_{n+\frac{1}{2}} - x_i)}\delta x \quad (3.100)$$

The derivatives are estimated as:

$$f'(x_{n+\frac{1}{2}}) = \frac{f(x_{n+1}) - f(x_n)}{\delta x} \quad (3.101)$$

Inserting this in the definition of the trapizium rule, shows that:

$$\mathscr{H}\left(\frac{df}{dx}\right) = \sum_{n=1}^{N-1} \frac{f(x_{n+1}) - f(x_n)}{(x_{n+\frac{1}{2}} - x_i)} \tag{3.102}$$

An alternative way of computing the Hilbert transform is via its Fourier transform, note:

$$\widehat{\mathscr{H}(f)}(\xi) = -i\text{sgn}(\xi)\hat{f} \tag{3.103}$$

This has been proven to work quite well.

Write $f(x + n\delta x) = f_{i+n}$, for convenience and write the ODE as:

$$a_1 f + a_2 \frac{d^4 f}{dx^4} + a_3 f^2 - a_4 \frac{d^2 f}{dx^2} + a_5 \mathscr{H}\left(\frac{df}{dx}\right) + a_6 p = 0 \tag{3.104}$$

The obtained equations are then:

$$\begin{aligned} 0 &= a_1 f_i + \frac{a_2}{(\delta x)^4}(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}) + a_3 f_i^2 \\ &+ \frac{a_4}{(\delta x)^2}(f_{i+1} - 2f_i + f_{i-1}) + a_6 p_i \\ &+ a_5 \sum_{n=1}^{N-1} \frac{f(x_{n+1}) - f(x_n)}{(x_{n+\frac{1}{2}} - x_i)} \end{aligned}$$

This is in a form which can be solved via Newton's method. The generic form of Newton's method for a system $F_i(f_1, \ldots, f_{N-1}) = 0$, $i = 1, \ldots, N-1$ is that of an iteration of the form:

$$J(F, x)(x_{n+1} - x_n) = -F(x) \tag{3.105}$$

Where $J(F, x)$ is the Jacobian of F. A resulting typical free surface profile is shown in figure (3.2) The pressure was chosen to be $p(x) = \exp(-x^2)$ with $B = E_b = 0.2$ and $h = 1$. It is quantitatively similar to that in figure 2.2.

It is now shown that the blow up of linear theory is removed by introducing nonlinearity. Results are first presented based on the linearised "KdV", equations (3.94) and (3.96). Values of $|f(0)|$ versus the Froude number $F = c/c_0$ for $B = E_b = 0.2$ are presented in figure 3.4. In figure 3.3 the corresponding result for the complete weakly nonlinear KdV (equation 3.89) are presented. They show that the unbounded displacements have been removed by introducing nonlinearity.

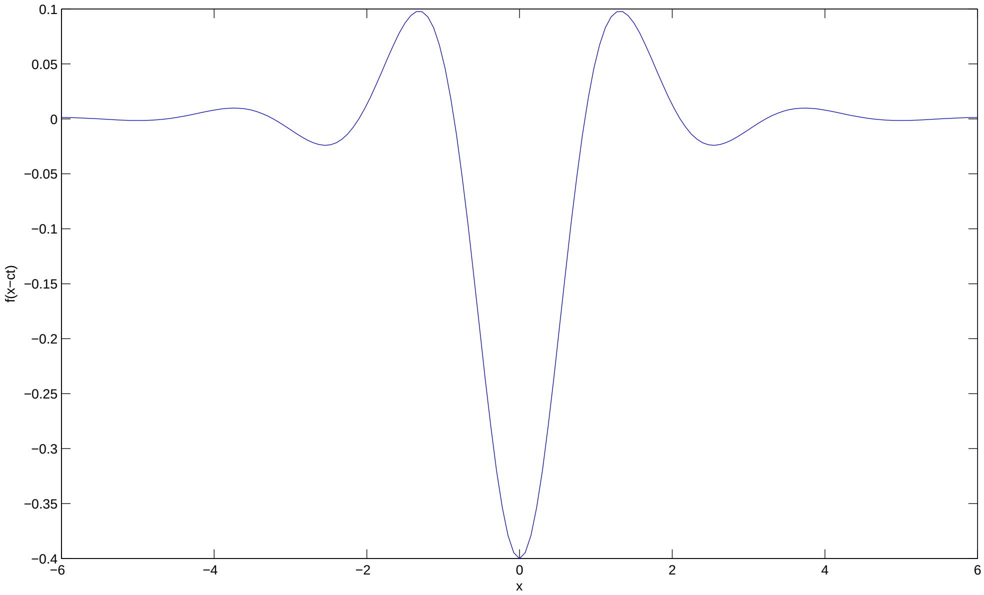


Figure 3.2: Wave Profile, $B = E_b = 0.2, \quad h = 1$

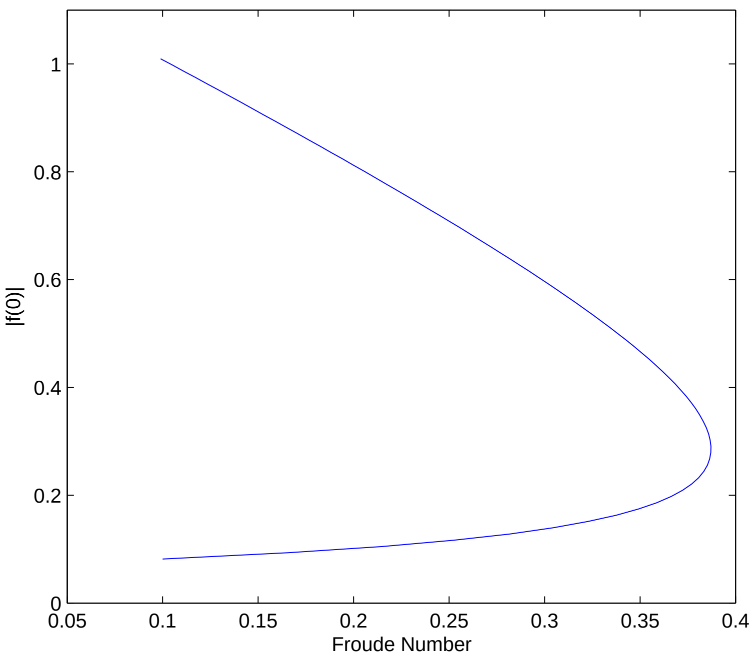


Figure 3.3: Weakly nonlinear

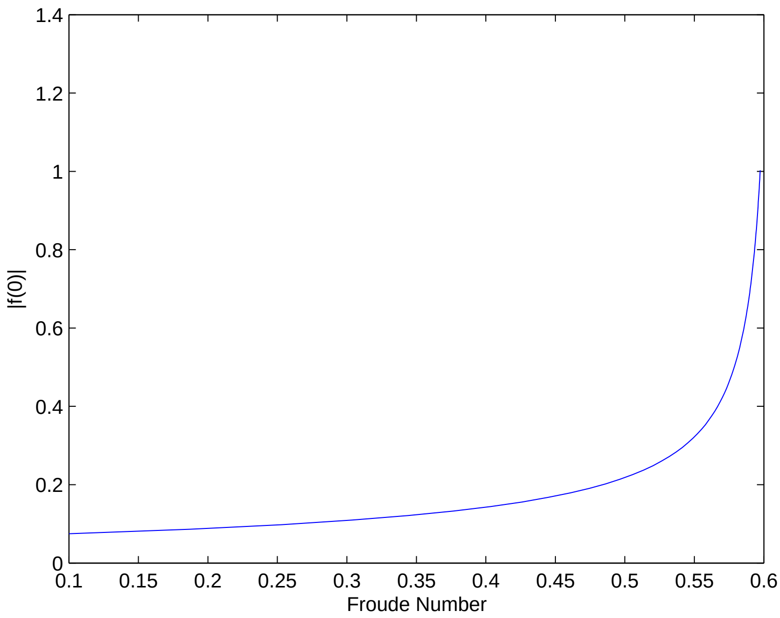
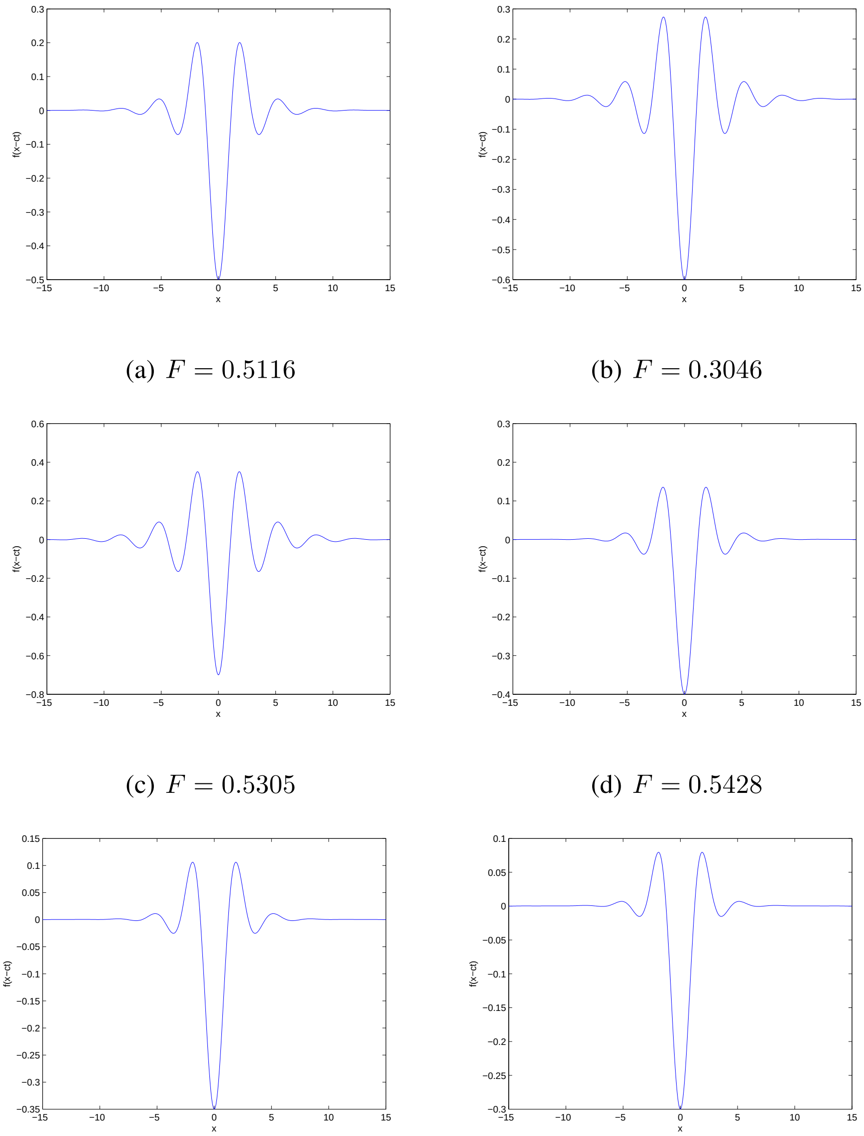


Figure 3.4: Blow Up in the Linearised Theory

3.5 Weakly Nonlinear Profiles

From examination of figure 3.3, we can examine some of the profiles with large displacements. The profiles of different amplitudes share very similar characteristics, they are shown in figure 3.5. The profiles in figure (3.5) are responses to localised forcing.



(a) $F = 0.5116$ (b) $F = 0.3046$

(c) $F = 0.5305$ (d) $F = 0.5428$

(e) $F = 0.4557$ (f) $F = 0.4217$

Figure 3.5: Profiles of Solitary Waves with $B = E_b = 0.2$

3.6 Topography

In this section the linear analysis of section 1.6 is extended by developing a weakly nonlinear theory. A frame of reference where the flow is steady and in which the topography moves from right to left with velocity U and

$$y = f(t, x) = -h + g(t, x) \tag{3.106}$$

The condition on the bottom can be expressed as:

$$\frac{D}{Dt}(y - f(t, x)) = 0 \Rightarrow \frac{\partial \varphi}{\partial y} = \frac{\partial g}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial g}{\partial x} \tag{3.107}$$

Nondimensionising the boundary condition, the function g is scaled as $g = \nu\hat{g}$, the equation becomes:

$$\frac{L^2}{h^2}\frac{a}{\nu}\frac{\partial\hat{\varphi}}{\partial\hat{y}} = \frac{\partial\hat{g}}{\partial\hat{t}} + \frac{a}{h}\frac{\partial\hat{\varphi}}{\partial\hat{x}}\frac{\partial\hat{g}}{\partial\hat{x}} \tag{3.108}$$

Using the scaling around $B = 1/3$ of section 2.3 and dropping the hats, shows that:

$$\frac{1}{\varepsilon}\frac{a}{\nu}\frac{\partial\varphi}{\partial y} = \frac{\partial g}{\partial t} + \varepsilon^2\frac{\partial\varphi}{\partial x}\frac{\partial g}{\partial x} \tag{3.109}$$

Choosing the scaling $a/\nu = \varepsilon^{-2}$ and now specifying that $g(t,x) = g(x - Ut)$, where:

$$x - Ut = L\hat{x} - \frac{UL}{c_0}\hat{t} = L\left(\hat{x} - \frac{U}{c_0}\hat{t}\right) = L\left(\hat{x} - F\hat{t}\right) \tag{3.110}$$

so $\hat{g}(\hat{t},\hat{x}) = \hat{g}(\hat{x} - F\hat{t})$. Equation (3.109) becomes:

$$\frac{\partial\varphi}{\partial y} = \varepsilon^3 F\frac{dg}{dX} + \varepsilon^5\frac{\partial\varphi}{\partial X}\frac{dg}{dX} \tag{3.111}$$

This will enter the boundary condition for φ_3, however this method of analysis is only valid around $F = 1$, so write $F = 1 + \varepsilon F_1$ so the boundary condition for φ_3 becomes:

$$\frac{\partial\varphi_3}{\partial y} = -g'(X). \tag{3.112}$$

Then:

$$\frac{\partial\varphi_3}{\partial y} = -\frac{(y+1)^5}{120}\frac{\partial^6\varphi_0}{\partial X^6} + \frac{(y+1)^3}{6}\frac{\partial^4 B}{\partial X^4} - (y+1)\frac{\partial^2 D}{\partial X^2} - g'(X) \tag{3.113}$$

Equation (3.113) replaces equation (3.85), inserting this into (3.86) shows that:

$$2\partial_T\partial_X\varphi_0 + \frac{1}{45}\partial_X^6\varphi_0 + 3\partial_X\varphi_0\partial_X^2\varphi_0 - g' - B_1\partial_X^3\eta_0 + \hat{E}_b\partial_X\partial_Y V_1 = 0 \tag{3.114}$$

As previous, it is possible to use the Hilbert transform to get rid of electric potential term to obtain the equation:

$$2\partial_T\eta_0 + \frac{1}{45}\partial_X^5\eta_0 + 3\eta_0\partial_X\eta_0 - g' - B_1\partial_X^3\eta_0 + \hat{E}_b\mathscr{H}(\partial_X^2\eta_0) = 0 \tag{3.115}$$

To convert back into dimensional variables note that:

$$\frac{\partial\eta_0}{\partial T} = \frac{1}{\varepsilon^2}\left(\frac{\partial}{\partial\hat{t}} + \frac{\partial}{\partial\hat{x}}\right), \quad \frac{\partial}{\partial\hat{t}} = \frac{L}{c_0}\frac{\partial}{\partial t} \tag{3.116}$$

So the $\partial\eta/\partial t$ term becomes

$$\frac{1}{a}\varepsilon^{-2}\frac{L}{c_0}\frac{\partial\eta}{\partial t} = \frac{h}{a}\varepsilon^{-2}\frac{L}{h}\frac{1}{c_0}\frac{\partial\eta}{\partial t} = \frac{\varepsilon^{-4.5}}{c_0}\frac{\partial\eta}{\partial t} \tag{3.117}$$

The topography term becomes:

$$\frac{d\hat{g}}{dX} = \frac{d\hat{g}}{d\hat{x}} = \frac{L}{\nu}\frac{dg}{dx} = \frac{L}{h}\frac{a}{\nu}\frac{h}{a}\frac{dg}{dx} = \varepsilon^{-4.5}\frac{dg}{dx}. \tag{3.118}$$

The final equation is then:

$$\partial_t\eta + c_0\partial_x\eta + \frac{1}{90}c_0h^4\partial_x^5\eta + \frac{3}{2}\frac{c_0}{h}\eta\partial_x\eta - \frac{1}{2}\left(B - \frac{1}{3}\right)c_0h^2\partial_x^3\eta + \frac{1}{2}E_bhc_0\partial_x^2\mathscr{H}(\eta) = \frac{c_0}{2}g' \tag{3.119}$$

3.7 Solitary Waves

The results of figure 3.3 show that two different solution can exist for the same values of F. The one of smaller amplitude is a perturbation of the uniform stream. the one with the higher amplitude is a perturbation of solitary waves with decaying tails, this can be shown numerically. The resulting curve is shown in figure 3.6. Profiles of the solitary waves are qualitatively similar to those in figure 3.5.

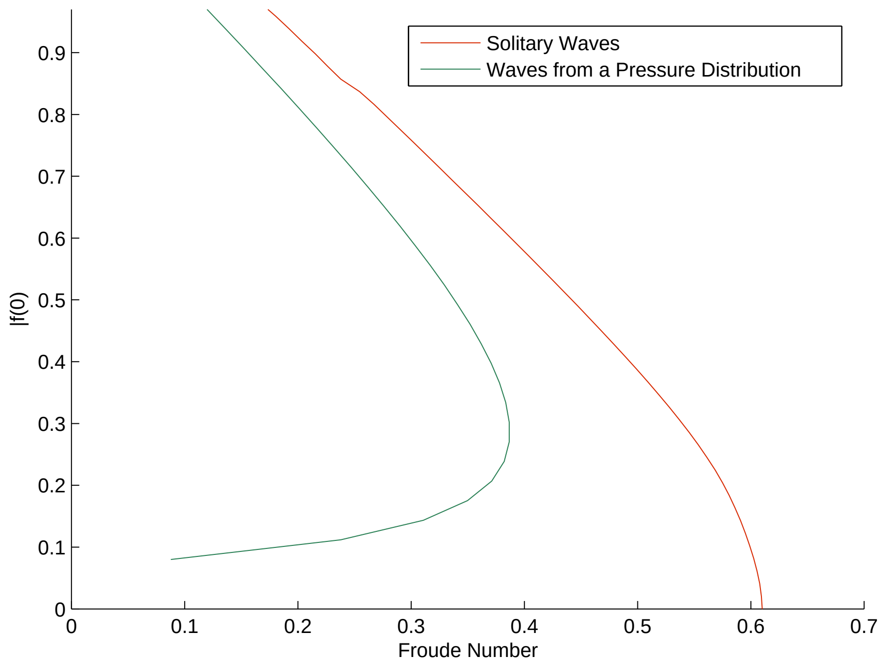


Figure 3.6: Weakly nonlinear

3.8 Stokes Expansion

Properties of periodic waves of the fifth order Benjamin-Ono equation are now studied, setting $p = 0$ in equation (3.89)

$$(1-F)f' + \frac{h^4}{90}f''''' + \frac{3}{2h}ff' - \frac{1}{2}\left(B - \frac{1}{3}\right)h^2 f''' + \frac{E_b h}{2}\mathscr{H}(f'') = 0 \quad (3.120)$$

Write:

$$f = \varepsilon f_1 + \varepsilon^2 f_2 + o(\varepsilon^3) \quad (3.121)$$

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + o(\varepsilon^3) \quad (3.122)$$

The $O(\varepsilon)$ equation is:

$$(1-F_0)f_1' + \frac{h^4}{90}f_1''''' - \frac{1}{2}\left(B - \frac{1}{3}\right)h^2 f_1''' + \frac{E_b h}{2}\mathscr{H}(f_1'') = 0 \quad (3.123)$$

Now look for a symmetric wave of the form:

$$f_1 = \sum_{n \geqslant 1} a_n \cos(nkx) \quad (3.124)$$

This results in:

$$a_n \left[F_0 - 1 - \frac{1}{90}(hk)^4 n^4 - \frac{(hkn)^2}{2}\left(B - \frac{1}{3}\right) + \frac{E_b kh}{2}\right] = 0 \quad (3.125)$$

Taking without loss of generality $a_1 \neq 0$, shows that:

$$F_0 = 1 + \frac{(hk)^4}{90} + \frac{(hk)^2}{2}\left(B - \frac{1}{3}\right) - \frac{E_b hk}{2}, \quad (3.126)$$

which is the usual dispersion relation. There are two possibilities: the first possibility is that,

$$F_0 - 1 - \frac{1}{90}(hk)^4 n^4 - \frac{(hkn)^2}{2}\left(B - \frac{1}{3}\right) + \frac{E_b kh}{2} \neq 0. \quad (3.127)$$

The second possibility is:

$$F_0 - 1 - \frac{1}{90}(hk)^4 m^4 - \frac{(hkm)^2}{2}\left(B - \frac{1}{3}\right) + \frac{E_b kh}{2} = 0. \quad (3.128)$$

for some integer m, this allows that possibility for a_m to be different from zero. Using the dispersion relation (3.126), (3.127) can be simplified to:

$$\frac{kh(n-1)}{2}\left[-\frac{(hk)^3}{45}(1+n+n^2+n^3) - hk(n+1)\left(B - \frac{1}{3}\right) + E_b\right] \neq 0 \quad (3.129)$$

Assuming that (3.129) holds then, $a_n = 0$ for $n \geqslant 2$ and:

$$f(x) = a_1 \cos kx \tag{3.130}$$

Now define:

$$f(x) = \sum_{n \geqslant 1} \alpha_n(\varepsilon) \cos nkx, \tag{3.131}$$

where $\alpha_n = \varepsilon\alpha_n^{(1)} + \varepsilon^2\alpha_n^{(2)} + \cdots$, then if $a = \alpha_1$ then define $\varepsilon = ak$ and this shows that $a_1 = k^{-1}$. Now suppose there is an $m \in \mathbb{N}$ such that $a_m \neq 0$, The second order equation for this is:

$$(1 - F_0)f_2' + \frac{h^4}{90}f_2''''' - \frac{1}{2}\left(B - \frac{1}{3}\right)h^2 f_2''' + \frac{E_b h}{2}\mathscr{H}(f_2'') = F_1 \sin kx + \frac{3}{4kh}\sin 2kx \tag{3.132}$$

As before, let:

$$f_2 = \sum_{n \geqslant 2} b_n \cos(nkx) \tag{3.133}$$

and substitute into (3.132) to obtain two equations,

$$\left[(F_0 - 1) - \frac{(hk)^4}{90} - \frac{(hk)^2}{2}\left(B - \frac{1}{3}\right) + \frac{E_b hk}{2}\right] kb_1 = F_1 \tag{3.134}$$

$$\left[2(F_0 - 1)k - \frac{2^5 h^4 k^5}{90} - \frac{2^3 h^2 k^3}{2} + 2E_b hk^2\right] b_2 = \frac{3}{4hk} \tag{3.135}$$

and $b_n = 0$ for $n \geqslant 3$. So the second of these equations gives a unique definition for b_2.

$$b_2 = \left[2(F_0 - 1)k - \frac{2^5 h^4 k^5}{90} - \frac{2^3 h^2 k^3}{2} + 2E_b hk^2\right]^{-1} \frac{3}{4hk} \tag{3.136}$$

From the definition of ε, $b_1 = 0$ and this shows that $F_1 = 0$, so this defines the second expansion uniquely. To calculate the higher order term for the Froude number, for this let:

$$F = F_0 + \varepsilon^2 F_2 + o(\varepsilon^2) \tag{3.137}$$

$$f = \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + o(\varepsilon^3) \tag{3.138}$$

The task is to compute the $O(\varepsilon^3)$ equation, as before write:

$$f_3 = \sum_{n \geqslant 2} c_n \cos nkx \tag{3.139}$$

The sum starts from $n = 2$ due to the definition of ε. So examining the inhomogeneous part of the equation and setting the coefficient of the $\sin kx$ to 0 will yield the required information. The inhomogeneous term is:

$$\begin{aligned} -\frac{3}{2h}(f_1 f_2' + f_2 f_1') - F_2 f_1' &= \frac{b_2}{k}(-2k\cos kx \sin 2kx - k\cos 2kx \sin kx - F_2 k \sin kx \\ &= -b_2(3\sin 3kx + \sin kx) - kF_2 \sin kx \end{aligned}$$

So:

$$F_2 = -\frac{b_2}{k} \tag{3.140}$$

Suppose now that (3.129) does not hold, for $n = m$. Then (3.129) implies::

$$E_b = \frac{(hk)^3}{45}(1 + m + m^2 + m^3) + hk(m+1)\left(B - \frac{1}{3}\right) \tag{3.141}$$

where a_m is arbitrary. The analysis will be presented for $m = 2$:

$$E_b = \frac{(hk)^3}{3} + 3hk\left(B - \frac{1}{3}\right) \tag{3.142}$$

The general solution will be:

$$f_1 = a_1 \cos kx + a_2 \cos 2kx \tag{3.143}$$

Write:

$$f = \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \tag{3.144}$$

$$F = F_0 + \varepsilon F_1 \tag{3.145}$$

Then the equation shows that:

$$(1 - F_0) f_2' + \frac{h^4}{90} f_2''''' - \frac{1}{2}\left(B - \frac{1}{3}\right) h^2 f_2''' + \frac{E_b h}{2}\mathscr{H}(f_2'') = F_1 f_1' - \frac{3}{2h} f_1 f_1' \tag{3.146}$$

The RHS of (3.146), is given by:

$$\begin{aligned} RHS &= F_1(-a_1 k \sin kx - 2ka_2 \sin 2kx) - \\ &\quad - \frac{3}{2h}(a_1 \cos kx + a_2 \cos 2kx)(-a_1 k \sin kx - 2ka_2 \sin 2kx) \\ &= F_1(-a_1 k \sin kx - 2ka_2 \sin 2kx) + \frac{3}{2h}\left(\frac{a_1^2 k}{2}\sin 2kx + ka_2^2 \sin 4kx\right) + \\ &\quad + \frac{3}{2h}\left(2ka_1 a_2 \cos kx \sin 2kx + a_1 a_2 k \cos 2kx \sin kx\right) \\ &= \left(\frac{3ka_1 a_2}{2} - F_1 a_1 k\right)\sin kx + \left(\frac{3a_1^2}{4h} - 2ka_2 F_1\right)\sin 2kx + \\ &\quad + \frac{9ka_1 a_2}{2h}\sin 3kx + \frac{3ka_2^2}{2h}\sin 4kx \end{aligned}$$

Now write f_2 as:

$$f_2 = \sum_{n \geqslant 1} b_n \cos(nkx) \tag{3.147}$$

The LHS of (3.146) becomes:

$$\begin{aligned} LHS &= (1 - F_0) \sum_{n \geqslant 1} -(nk) b_n \sin(nkx) + \frac{h^4}{90} \sum_{n \geqslant 1} -(nk)^5 b_n \sin(nkx) - \\ &\quad - \frac{h^2}{2} \left(B - \frac{1}{3} \right) \sum_{n \geqslant 1} (nk)^2 b_n \sin(nkx) + \frac{E_b h}{2} \sum_{n \geqslant 1} (nk)^2 b_n \sin(nkx) \end{aligned}$$

Taking $n = 1$ shows that:

$$\left[(F_0 - 1 - \frac{(hk)^4}{90} - \frac{(hk)^2}{2} \left(B - \frac{1}{3} \right) + \frac{E_b hk}{2} \right] kb_1 = \frac{3ka_1 a_2}{2} - F_1 a_1 k \tag{3.148}$$

However it was an assumption that the bracketed term in equation (3.148) is zero and this implies that:

$$F_1 = \frac{3a_2}{2} \tag{3.149}$$

Likewise the same argument can be used for $n = 2$ which shows that:

$$2ka_2 F_1 = \frac{3a_1^2}{4h} \tag{3.150}$$

Conbining this equation with (3.149) shows that:

$$a_2 = \pm \frac{a_1}{2\sqrt{kh}} \tag{3.151}$$

and

$$f_1 = a_1 \left(\cos kx \pm \frac{1}{2\sqrt{kh}} \cos 2kx \right) + \cdots , \tag{3.152}$$

showing the existence of Wilton ripples. A typical profile corresponds to the plus sign in (3.152) is shown in figure 3.7 Resonance is an important phenomena in physics and is the explanation for Wilton ripples (first calculated by Wilton). Wilton ripples occur when there is second harmonic resonance present, when a mode interacts with another mode of half the original wavelength causing the pattern in figure (3.7).

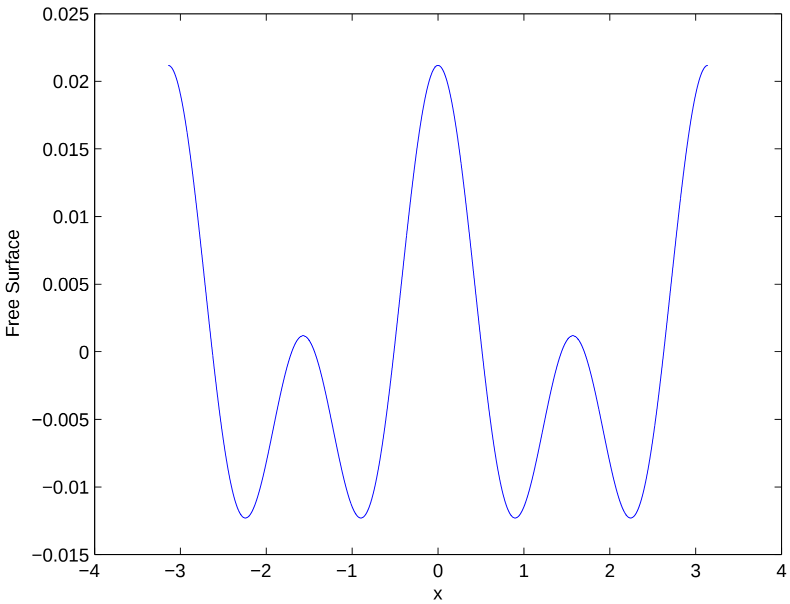


Figure 3.7: Wilton Ripples with $k = 2$, $a_1 = 0.01$, $h = 0.1$

Chapter 4

Insulating Fluids Adhering to an Upper Surface

The problem under consideration in this chapter is the motion the free surface of a fluid adhering to an upper surface. Without an electric field this problem is unstable due to gravity, however as it will be seen the fluid can be stabilised by the addition of a horizontal electric field. The cases of shallow water and shallow water with small amplitude will be studied. A relaxation of the criterion in chapter 1 is also considered where the requirement of small amplitude for the waves is no longer required.

4.1 Governing Equations

The set up of the problem is as follows: There are three different mediums involved in

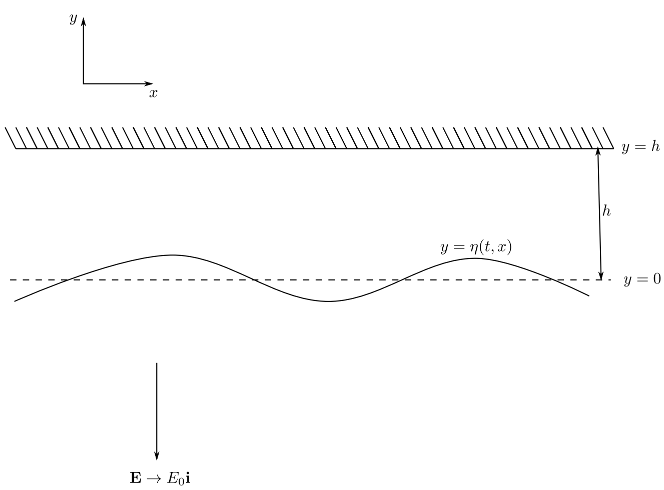


Figure 4.1: Problem Set-Up

the model, a solid dielectric, a fluid dielectric and a gas dielectric constants denoted by ϵ_s, ϵ_f and ϵ_g respectively. There are no charges involved, so the governing equations for the potentials are:

$$\nabla^2 V^s = 0 \tag{4.1}$$

$$\nabla^2 V^f = 0 \tag{4.2}$$

$$\nabla^2 V^g = 0. \tag{4.3}$$

In the fluid there is no circulation and is incompressible which indicates the existence of a velocity potential φ for the velocity vector field which satisfies the following equation:

$$\nabla^2 \varphi = 0, \tag{4.4}$$

where $\mathbf{u} = \nabla\varphi$.

4.2 Boundary Conditions

The next point to address are the boundary conditions. There are two boundaries and essentially two asymptotic conditions. The boundary at $y = h$ is the boundary between the solid and the fluid, it is assumed that the normal component of the electric displacement[1] is continuous across the boundary, so that shows that:

$$\epsilon_s E_y^s(x,h) = \epsilon_f E_y^f(x,h). \tag{4.5}$$

Which in terms of the electric potential, $\mathbf{E} = -\nabla V$ shows that:

$$\epsilon_s \frac{\partial V^s}{\partial y}(x,h) = \epsilon_f \frac{\partial V^f}{\partial y}(x,h). \tag{4.6}$$

The tangential component of the electric field is continuous across the boundary, this condition reduces to:

$$V^s(x,h) = V^f(x,h). \tag{4.7}$$

The boundary condition for the velocity potential is simply $v(x,h) = 0$, which in terms of the velocity potential is:

$$\frac{\partial \varphi}{\partial y}(x,h) = 0. \tag{4.8}$$

[1]Which in this case is just $\epsilon\mathbf{E}$

The other boundary is the fluid/gas boundary at $y = \eta(t, x)$, the normal for the surface is given by:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{1 + (\partial_x \eta)^2}}(-\partial_x \eta, 1) \tag{4.9}$$

and the relevant boundary condition for the electric displacement is:

$$\hat{\mathbf{n}} \cdot (\epsilon_f \mathbf{E}^s) = \hat{\mathbf{n}} \cdot (\epsilon_g \mathbf{E}^g) . \tag{4.10}$$

In terms of the electric potential this is:

$$\epsilon_f \left(-\frac{\partial \eta}{\partial x} \frac{\partial V^f}{\partial x} + \frac{\partial V^f}{\partial y} \right) = \epsilon_g \left(-\frac{\partial \eta}{\partial x} \frac{\partial V^g}{\partial x} + \frac{\partial V^g}{\partial y} \right) . \tag{4.11}$$

The other boundary condition regarding the electric field is that the tangential components are the same, which says that:

$$\hat{\mathbf{t}} \cdot \mathbf{E}^s = \hat{\mathbf{t}} \cdot \mathbf{E}^g . \tag{4.12}$$

In terms of the electric potential this is:

$$\frac{\partial V^f}{\partial x} + \frac{\partial \eta}{\partial x} \frac{\partial V^f}{\partial y} = \frac{\partial V^g}{\partial x} + \frac{\partial \eta}{\partial x} \frac{\partial V^g}{\partial y} . \tag{4.13}$$

The fluid is on an upper surface the Young-Laplace equation must be examined again as there will be a difference in signs. There is a jump in the magnitude of pressure *towards* the fluid which contains the centre of curvature, so the pressure difference is positive on the side containing the centre of curvature. The Young-Laplace equation is given by:

$$[\hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{n}}]_g^f = \sigma \nabla \cdot \hat{\mathbf{n}}$$

The boundary condition for the velocity potential is given by the Bernoulli equation:

$$\begin{aligned} \partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2 + g\eta \quad &- \quad \frac{1}{\rho} \frac{1}{1 + (\partial_x \eta)^2} ((\partial_x \eta)^2 [\Sigma_{11}]_g^f + 2\partial_x \eta [\Sigma_{12}]_g^f + [\Sigma_{22}]_g^f) \\ &= \quad -\frac{\sigma}{\rho} \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{3/2}} + C \end{aligned} \tag{4.14}$$

The last boundary condition at $y = \eta(t, x)$ is the free surface condition for the fluid which is:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y} . \tag{4.15}$$

The two asymptotic conditions are given as:

$$V^g \sim -E_0 x \quad y \to -\infty \tag{4.16}$$

and

$$V^s \sim -E_0 x \quad y \to \infty \tag{4.17}$$

The continuity of electric displacement also shows that:

$$V^f = -E_0 x + \tilde{V}^f \tag{4.18}$$

Note that it has been implicitly assumed that there is no charge accruing on the interface as was the case on chapters 2 and 3.

4.3 Nondimensionalisation

In order to proceed further it is necessary to nondimensionalise the model. Choose the following scalings:

$$x = h\hat{x}, \quad y = h\hat{y}, \quad \eta = h\hat{\eta}, \quad t = \sqrt{\frac{\rho h^3}{\sigma}}\hat{t} \quad \varphi = \sqrt{\frac{\sigma h}{\rho}}\hat{\varphi}, \quad V = E_0 h\hat{V} \tag{4.19}$$

The governing equations scale simply as:

$$\hat{\nabla}^2 \hat{V}^s = 0 \tag{4.20}$$

$$\hat{\nabla}^2 \hat{V}^f = 0 \tag{4.21}$$

$$\hat{\nabla}^2 \hat{V}^g = 0 \tag{4.22}$$

$$\hat{\nabla}^2 \hat{\varphi} = 0. \tag{4.23}$$

Some notation to begin with, denote $\epsilon_{s,f} = \epsilon_s/\epsilon_f$. Examining the boundary conditions at $y = h$ to begin with. The continuity of the electric displacement is given by:

$$\epsilon_{s,f}\frac{\partial \hat{V}^s}{\partial \hat{y}}(\hat{x}, 1) = \frac{\partial \hat{V}^f}{\partial \hat{y}}(\hat{x}, 1) \quad \hat{V}^s(x, 1) = \hat{V}^f(x, 1) \tag{4.24}$$

and the velocity potential scales as:

$$\frac{\partial \hat{\varphi}}{\partial \hat{y}}(\hat{x}, 1) = 0. \tag{4.25}$$

Moving on to the boundary conditions at the free surface, the free surface condition scales very simply as:

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} + \frac{\partial \hat{\varphi}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{x}} = \frac{\partial \hat{\varphi}}{\partial \hat{y}}. \tag{4.26}$$

The scaling for the Bernoulli equation is rather involved, in order to do this define the following:

$$\begin{aligned} M_{11} &= \frac{1}{2}\left(\left(\frac{\partial \hat{V}}{\partial \hat{x}}\right)^2 - \left(\frac{\partial \hat{V}}{\partial \hat{y}}\right)^2\right) \\ M_{12} &= \frac{\partial \hat{V}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{y}} \\ M_{22} &= -\frac{1}{2}\left(\left(\frac{\partial \hat{V}}{\partial \hat{x}}\right)^2 - \left(\frac{\partial \hat{V}}{\partial \hat{y}}\right)^2\right) \end{aligned}$$

The scaling becomes:

$$\frac{\partial \hat{\varphi}}{\partial \hat{t}} + \frac{1}{2}|\hat{\nabla}\hat{\varphi}|^2 + B\hat{\eta} + \frac{E_b Q}{1+(\partial_{\hat{x}}\hat{\eta})^2} = -\frac{\partial_{\hat{x}}^2\hat{\eta}}{(1+(\partial_{\hat{x}}\hat{\eta})^2)^{3/2}} + K, \tag{4.27}$$

where

$$B = \frac{g\rho h^2}{\sigma}, \quad E_b = \frac{E_0^2 \epsilon_g h}{\sigma} \tag{4.28}$$

and the quantity Q is defined as:

$$Q = (\partial_{\hat{x}}\hat{\eta})^2(\epsilon_{f,g}M_{11}^f - M_{11}^g) + 2\partial_{\hat{x}}\hat{\eta}(\epsilon_{f,g}M_{12}^f - M_{12}^g) + \epsilon_{f,g}M_{22}^f - M_{22}^g. \tag{4.29}$$

The continuity of the normal component of the electric displacement at the free surface scales as:

$$\epsilon_{f,g}\left(-\frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}^f}{\partial \hat{x}} + \frac{\partial \hat{V}^f}{\partial \hat{y}}\right) = \left(-\frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}^g}{\partial \hat{x}} + \frac{\partial \hat{V}^g}{\partial \hat{y}}\right) \tag{4.30}$$

and the tangential component scales as:

$$\frac{\partial \hat{V}^f}{\partial \hat{x}} + \frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}^f}{\partial \hat{y}} = \frac{\partial \hat{V}^g}{\partial \hat{x}} + \frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}^g}{\partial \hat{y}} \tag{4.31}$$

The last part of the scaling is the asymptotic conditions which simply scale as:

$$\hat{V}^s \sim -\hat{x} \quad \hat{y} \to \infty \tag{4.32}$$

$$\hat{V}^g \sim -\hat{x} \quad \hat{y} \to -\infty. \tag{4.33}$$

4.4 Linear Theory

The first stage in the process is to perform a linear analysis, for this write (upon dropping hats for convenience):

$$V^s = -x + \varepsilon V_1^s + o(\varepsilon) \quad (4.34)$$

$$V^f = -x + \varepsilon V_1^f + o(\varepsilon) \quad (4.35)$$

$$V^g = -x + \varepsilon V_1^g + o(\varepsilon) \quad (4.36)$$

$$\varphi = \varepsilon \varphi_1^s + o(\varepsilon) \quad (4.37)$$

$$\eta = \varepsilon \eta_1. \quad (4.38)$$

The form for V^f comes from the electric displacement continuity boundary condition on the free surface and the solid interface. The linearised equations then take the form:

$$\nabla^2 V_1^s = 0 \quad 1 \leqslant y \leqslant \infty \quad (4.39)$$

$$\nabla^2 V_1^f = 0 \quad 0 \leqslant y \leqslant 1 \quad (4.40)$$

$$\nabla^2 V_1^g = 0 \quad -\infty \leqslant y \leqslant 0 \quad (4.41)$$

$$\nabla^2 \varphi_1 = 0 \quad 0 \leqslant y \leqslant 1. \quad (4.42)$$

The linearised boundary conditions become:

$$\frac{\partial \eta_1}{\partial t} = \frac{\partial \varphi_1}{\partial y} \quad y = 0 \quad (4.43)$$

$$\frac{\partial \varphi_1}{\partial y} = 0 \quad y = 1 \quad (4.44)$$

$$\epsilon_{s,f} \frac{\partial V_1^s}{\partial y} = \frac{\partial V_1^f}{\partial y} \quad y = 1 \quad (4.45)$$

$$\frac{\partial^2 \varphi_1}{\partial t \partial x} + B\frac{\partial \eta_1}{\partial x} + \epsilon_{f,g} E_b \frac{\partial^2 V_1^f}{\partial x^2} - E_b \frac{\partial^2 V_1^g}{\partial x^2} + \frac{\partial^3 \eta_1}{\partial x^3} = 0 \quad y = 0 \quad (4.46)$$

$$(\epsilon_{f,g} - 1)\frac{\partial \eta_1}{\partial x} + \epsilon_{f,g}\frac{\partial V_1^f}{\partial y} - \frac{\partial V_1^g}{\partial y} = 0 \quad y = 0 \quad (4.47)$$

$$V_1^s = V_1^f \quad y = 1 \quad (4.48)$$

$$V_1^f = V_1^g \quad y = 0 \quad (4.49)$$

$$\frac{\partial V_1^s}{\partial x} \to 0 \quad y \to \infty \quad (4.50)$$

$$\frac{\partial V_1^g}{\partial x} \to 0 \quad y \to -\infty. \quad (4.51)$$

The system may now be solved, Writing the variables as:

$$V_1 = \int_{\mathbb{R}} \hat{V}_1 e^{ikx-i\omega t} dk, \quad \varphi_1 = \int_{\mathbb{R}} \hat{\varphi}_1 e^{ikx-i\omega t} dk, \quad \eta = \int_{\mathbb{R}} \hat{\eta}_1 e^{ikx-i\omega t} dk. \tag{4.52}$$

The governing equations are then:

$$\frac{\partial^2 \hat{V}_1^s}{\partial y^2} - k^2 \hat{V}_1^s = 0 \tag{4.53}$$

$$\frac{\partial^2 \hat{V}_1^f}{\partial y^2} - k^2 \hat{V}_1^f = 0 \tag{4.54}$$

$$\frac{\partial^2 \hat{V}_1^g}{\partial y^2} - k^2 \hat{V}_1^g = 0 \tag{4.55}$$

$$\frac{\partial^2 \hat{\varphi}_1}{\partial y^2} - k^2 \hat{\varphi}_1 = 0. \tag{4.56}$$

The two limiting conditions for V_1^s and V_1^g show that they can be written as:

$$V_1^s = \int_{\mathbb{R}} C_1(k) e^{-|k|y} e^{ikx-i\omega t} dk \quad V_1^g = \int_{\mathbb{R}} C_2(k) e^{|k|y} e^{ikx-i\omega t} dk. \tag{4.57}$$

The electric potential and the velocity potential are seen to be:

$$\varphi_1 = \int_{\mathbb{R}} (A_2(k) e^{ky} + B_2(k) e^{-ky}) e^{ikx-i\omega t} dk \tag{4.58}$$

$$V_1^f = \int_{\mathbb{R}} (A_1(k) e^{ky} + B_1(k) e^{-ky}) e^{ikx-i\omega t} dk \tag{4.59}$$

Using (4.48) shows that:

$$C_1 e^{-|k|} = A_1 e^k + B_1 e^{-k} \tag{4.60}$$

The other boundary condition (4.45) on $y = 1$ shows that:

$$-\frac{\epsilon_{s,f} |k| e^{-|k|}}{k} C_1 = A_1 e^k - B_1 e^{-k} \tag{4.61}$$

These equations can be solved to find A_1 and B_1 in terms of C_1 to obtain:

$$A_1 = \frac{1}{2} \left(e^{-|k|-k} - \frac{\epsilon_{s,f} |k| e^{-|k|-k}}{k} \right) C_1 \tag{4.62}$$

$$B_1 = \frac{1}{2} \left(e^{-|k|+k} + \frac{\epsilon_{s,f} |k| e^{-|k|+k}}{k} \right) C_1 \tag{4.63}$$

For simplicity, write:

$$A_1 = \alpha C_1, \quad B_1 = \beta C_1 \tag{4.64}$$

Using the boundary conditions at $y = 1$, for example (4.49) then:

$$C_2 = (\alpha + \beta)C_1 \tag{4.65}$$

The other boundary condition (4.47) shows that:

$$C_1 = \frac{i(\epsilon_{f,g} - 1)k\hat{\eta}_1}{(\alpha + \beta)|k| - \epsilon_{f,g}k(\alpha - \beta)} = i\gamma\hat{\eta}_1 \tag{4.66}$$

So now all the electric potentials have been found as a function of the free surface, the next task is to relate this to the velocity potential. Using (4.44) it can be shown that:

$$A_2 = B_2 e^{-2k}. \tag{4.67}$$

Which makes the velocity potential become:

$$\varphi_1 = 2\int_{\mathbb{R}} B_2 e^{-k} e^{ikx - i\omega t} \cosh k(y-1) dk \tag{4.68}$$

Now using the free surface condition (4.43) shows that:

$$B_2 = \frac{i\omega\hat{\eta}_1}{2ke^{-k}\sinh k} \tag{4.69}$$

The potentials are now:

$$\begin{aligned}
\varphi_1 &= \int_{\mathbb{R}} \frac{i\omega\hat{\eta}_1}{k} e^{ikx-i\omega t} \frac{\cosh k(y-1)}{\sinh k} dk \\
V_1^f &= \frac{i}{2}\int_{\mathbb{R}} (\alpha\gamma e^{-ky} + \beta\gamma e^{ky})\hat{\eta}_1 e^{ikx-i\omega t} dk \\
V_1^g &= \frac{i}{2}\int_{\mathbb{R}} (\alpha + \beta)\gamma\hat{\eta}_1 e^{|k|y} e^{ikx-i\omega t} dk \\
V_1^s &= i\int_{\mathbb{R}} \gamma\hat{\eta}_1 e^{-|k|y} e^{ikx-i\omega t} dk \\
\eta_1 &= \int_{\mathbb{R}} \hat{\eta}_1 e^{ikx-i\omega t} dk.
\end{aligned}$$

Define:

$$\delta = \frac{\coth k}{k}. \tag{4.70}$$

Inserting these expressions into the Bernoulli equation (4.46) shows that:

$$i\omega^2 k\delta\hat{\eta}_1 + ikB\hat{\eta}_1 - iE_b\epsilon_{f,g}k^2(\alpha\gamma + \beta\gamma)\hat{\eta}_1 + iE_b k^2(\alpha\gamma + \beta\gamma)\hat{\eta}_1 - ik^3\hat{\eta}_1 \tag{4.71}$$

The dispersion relation for the system is then:

$$\omega^2 = \frac{k^2 - B + (\epsilon_{f,g} - 1)E_b k(\alpha + \beta)\gamma}{\delta} \tag{4.72}$$

for $k > 0$, the following holds:

$$\begin{aligned}
\alpha &= \frac{1}{2}(1-\epsilon_{s,f})e^{-2k} \\
\beta &= \frac{1}{2}(1+\epsilon_{s,f}) \\
\gamma &= \frac{\epsilon_{f,g}-1}{(1+\epsilon_{s,f})(1+\epsilon_{f,g})+(1-\epsilon_{s,f})(1-\epsilon_{f,g})e^{-2k}}
\end{aligned}$$

Then define $c(k)$ as the following: $\omega = kc(k)$, then:

$$c^2(k) = \left[k^2 - B + \frac{E_b(\epsilon_{f,g}-1)^2 k(1+\epsilon_{s,f}+(1-\epsilon_{s,f})e^{-2k})}{(1+\epsilon_{s,f})(1+\epsilon_{f,g})+(1-\epsilon_{s,f})(1-\epsilon_{f,g})e^{-2k}}\right]\frac{\tanh k}{k}. \quad (4.73)$$

For $k = 0$ the value of $c^2(k) < 0$, showing that linear waves are not possible.

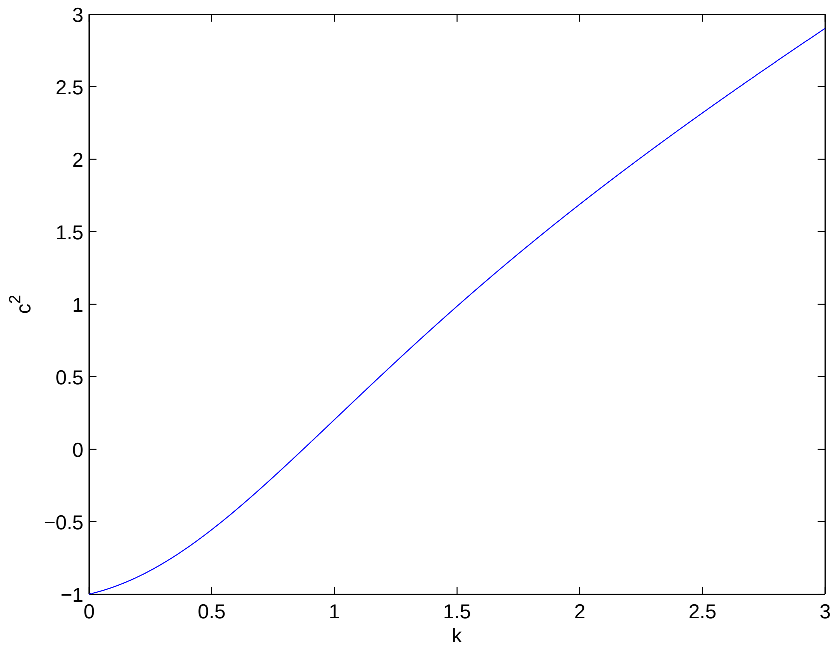


Figure 4.2: $B = 1$, $\epsilon_{s,f} = 0.5$, $\epsilon_{f,g} = 3$, $E_b = 50$

It is possible to expand (4.72) to obtain the linear part of the weakly nonlinear equation, for this the following expansions are used:

$$\begin{aligned}
\delta &= k^2 + o(k^2) \\
\alpha &= \frac{1}{2}(1-\epsilon_{s,f}\text{sgn}\,(k)) + o(1) \\
\beta &= \frac{1}{2}(1+\epsilon_{s,f}\text{sgn}\,(k)) + o(1) \\
\gamma &= \frac{(\epsilon_{f,g}-1)\text{sgn}\,(k)}{1+\epsilon_{s,f}\epsilon_{f,g}} + o(1)
\end{aligned}$$

Inserting the above expansions into (4.72) shows that:

$$\omega^2 = -Bk^2 + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}|k|k^2 + k^4 \tag{4.74}$$

Swapping the operator symbols for the operators themselves:

$$\omega \to i\partial_t \quad k \to -i\partial_x \quad |k| \to -\partial_x \mathscr{H} \tag{4.75}$$

The linear PDE which satisfies this relation is:

$$\frac{\partial^2 \bar{\eta}}{\partial t^2} + B\frac{\partial^2 \bar{\eta}}{\partial x^2} + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}\mathscr{H}\left(\frac{\partial^3 \bar{\eta}}{\partial x^3}\right) + \frac{\partial^4 \bar{\eta}}{\partial x^4} = 0 \tag{4.76}$$

4.5 Weakly Nonlinear Theory

It is useful to combine some boundary conditions to obtain another boundary condition for the solid/fluid interface which will simplify much of the analysis. To do this write the electric potentials as:

$$V^s = -x + \tilde{V}^s \tag{4.77}$$

Then $\tilde{V}^s$ also satisfies Laplace's equations, so it is possible to write:

$$\partial_x \tilde{V}^s(x,h) = -\mathscr{H}(\partial_y \tilde{V}^s(x,h)) \tag{4.78}$$

$$\partial_y \tilde{V}^s(x,h) = \mathscr{H}(\partial_x \tilde{V}^s(x,h)) \tag{4.79}$$

The boundary conditions for the equation are now:

$$\begin{aligned} \epsilon_g \partial_y \tilde{V}^s(x,h) &= \epsilon_f \partial_y \tilde{V}^s(x,h) \\ \partial_x \tilde{V}^s(x,h) &= \partial_x \tilde{V}^s(x,h) \end{aligned}$$

Then it is possible to transfer the above equations to $\tilde{V}^f(x,h)$, from the normal boundary condition:

$$\epsilon_{s,f}\partial_y \tilde{V}^s(x,h) = \partial_y \tilde{V}^f(x,h)$$

So, multiplying (4.79) through by $\epsilon_{s,f}$, shows that:

$$\partial_y \tilde{V}^f(x,h) = \epsilon_{s,f}\mathscr{H}(\partial_x \tilde{V}^s(x,h))$$

Upon using the tangent boundary condition, it is possible to write:

$$\partial_y \tilde{V}^f(x,h) = \epsilon_{s,f}\mathscr{H}(\partial_x \tilde{V}^f(x,h)) \tag{4.80}$$

This is a global boundary condition for the solid/liquid interface. The reduced potentials $\tilde{V}$ satisfy the same governing equation as the main potentials V, therefore the problem can be restated in terms of the reduced potentials, and this is what is done. For ease of calculation, set the interface of the solid/fluid interface at $y = 0$ and the average level of the free surface to be at $y = -h$. Use the following scaling:

$$x = L\hat{x}, \quad y^s = L\hat{y}, \quad y^g = LY \quad y^f = h\xi \quad \eta = h\hat{\eta} \tag{4.81}$$

$$t = \sqrt{\frac{\rho L^3}{\sigma}}\hat{t} \quad \varphi = \sqrt{\frac{\sigma L}{\rho}}\hat{\varphi}, \quad V = E_0 L\hat{V} \tag{4.82}$$

where L is a typical horizontal length scale and denote $\varepsilon = h/L$. From here on in drop tildes and hats. The equations become:

$$\frac{\partial^2 V^s}{\partial x^2} + \frac{\partial^2 V^s}{\partial y^2} = 0 \quad 0 \leqslant y \leqslant \infty \tag{4.83}$$

$$\frac{\partial^2 V^f}{\partial x^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 V^f}{\partial \xi^2} = 0 \quad -1 + \eta \leqslant \xi \leqslant 0 \tag{4.84}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \varphi}{\partial \xi^2} = 0 \quad -1 + \eta \leqslant \xi \leqslant 0 \tag{4.85}$$

$$\frac{\partial^2 V^g}{\partial x^2} + \frac{\partial^2 V^g}{\partial Y^2} = 0 \quad -\infty \leqslant Y \leqslant -1 + \varepsilon\eta \tag{4.86}$$

The boundary conditions for the electric potential become:

$$\epsilon_{s,f}\frac{\partial V^s}{\partial \hat{y}}(x,0) = \frac{1}{\varepsilon}\frac{\partial V^f}{\partial \xi}(x,0), \quad \frac{\partial V^s}{\partial x}(x,0) = \frac{\partial V^f}{\partial x}(x,0) \tag{4.87}$$

The scaled velocity potential satisfies:

$$\frac{\partial \varphi}{\partial \xi}(x,0) = 0 \tag{4.88}$$

Moving on to the free surface, the free surface equation:

$$\varepsilon\frac{\partial \eta}{\partial t} + \varepsilon\frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{1}{\varepsilon}\frac{\partial \varphi}{\partial \xi} \quad \xi = -1 + \eta \tag{4.89}$$

The normal electric potential boundary conditions become:

$$\epsilon_{f,g}\left(-\varepsilon\frac{\partial \eta}{\partial x}\frac{\partial V^f}{\partial x} + \frac{1}{\varepsilon}\frac{\partial V^f}{\partial \xi}\right) = -\varepsilon\frac{\partial \eta}{\partial x}\frac{\partial V^g}{\partial x} + \frac{\partial V^g}{\partial Y} \tag{4.90}$$

The tangential equation scales as:

$$\frac{\partial V^f}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V^f}{\partial \xi} = \frac{\partial V^g}{\partial x} + \varepsilon\frac{\partial \eta}{\partial x}\frac{\partial V^g}{\partial Y} \tag{4.91}$$

The Bernoulli equations becomes:

$$\frac{\partial \varphi}{\partial t}+\frac{1}{2}\left(\left(\frac{\partial \varphi}{\partial x}\right)^2+\frac{1}{\varepsilon^2}\left(\frac{\partial \varphi}{\partial \xi}\right)^2\right)+B\varepsilon\eta+\frac{E_b Q}{1+\varepsilon^2(\partial_x\eta)^2}=-\frac{\varepsilon\partial_x^2\eta}{(1+\varepsilon^2(\partial_x\eta)^2)^{3/2}}+K \tag{4.92}$$

Where Q is given by:

$$Q=\varepsilon^2(\partial_x\eta)^2(\epsilon_{f,g}M_{11}^f-M_{11}^g)-2\varepsilon\partial_x\eta(\epsilon_{f,g}M_{12}^f-M_{12}^g)+\epsilon_{f,g}M_{22}^f-M_{22}^g. \tag{4.93}$$

The components of the Faraday tensor for the fluid are given by:

$$\begin{aligned}
M_{11}^f &= \frac{1}{2}\left(\left(\frac{\partial V^f}{\partial x}\right)^2-\frac{1}{\varepsilon^2}\left(\frac{\partial V^f}{\partial \xi}\right)^2\right)\\
M_{12}^f &= \frac{1}{\varepsilon}\frac{\partial V^f}{\partial x}\frac{\partial V^f}{\partial \xi}\\
M_{22}^f &= -\frac{1}{2}\left(\left(\frac{\partial V^f}{\partial x}\right)^2-\frac{1}{\varepsilon^2}\left(\frac{\partial V^f}{\partial \xi}\right)^2\right)
\end{aligned}$$

The quantities are expanded as follows:

$$\begin{aligned}
\varphi(t,x,\xi) &= \varphi_0(t,x,\xi)+\varepsilon^2\varphi_1(t,x,\xi)+o(\varepsilon^2)\\
\eta(t,x,\xi) &= \eta_0(t,x)+\varepsilon^2\eta_1(t,x)+o(\varepsilon^2)\\
V^s(x,\hat{y}) &= -x+\varepsilon V_1^s(x,y)+o(\varepsilon)\\
V^f(x,\xi) &= -x+\varepsilon V_1^f(x,\xi)+o(\varepsilon)\\
V^g(x,\hat{Y}) &= -x+\varepsilon V_1^g(x,Y)+o(\varepsilon)
\end{aligned}$$

The global boundary condition is only valid for the perturbed quantity:

$$\frac{1}{\varepsilon}\frac{\partial \varepsilon V_1^f}{\partial \xi}(x,0)=\mathscr{H}\left(\frac{\partial \varepsilon V_1^f}{\partial x}\right)(x,0) \tag{4.94}$$

The equation for φ_0 is given by:

$$\frac{\partial^2\varphi_0}{\partial \xi^2}=0 \tag{4.95}$$

Integrating twice shows that $\varphi_0=A(t,x)+\xi B(t,x)$. Upon using the boundary condition $\partial_\xi\varphi_0(x,0)=0$, shows that $\varphi_0=\varphi_0(t,x)$. The equation for φ_1 is:

$$\frac{\partial^2\varphi_0}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial \xi^2}=0 \tag{4.96}$$

Integrating once shows that:

$$\frac{\partial \varphi_1}{\partial \xi} = -\xi \frac{\partial^2 \varphi_0}{\partial x^2} + A(t,x)$$

Using the boundary condition at $\xi = 0$ shows that:

$$\varphi_1 = -\frac{\xi^2}{2}\frac{\partial^2 \varphi_0}{\partial x^2} + A(t,x) \tag{4.97}$$

Inserting this into the free surface equation shows that:

$$\varepsilon\frac{\partial \eta_0}{\partial t} + \varepsilon\frac{\partial \eta_0}{\partial x}\frac{\partial \varphi_0}{\partial x} = -\frac{1}{\varepsilon}\left(\varepsilon^2\left(\eta_0\frac{\partial^2 \eta_0}{\partial x^2}\right)\right)$$

After some manipulations the equation becomes:

$$\frac{\partial \eta_0}{\partial t} + \frac{\partial}{\partial x}(u\eta_0) = 0 \tag{4.98}$$

The equation for V_1^f is given as:

$$\frac{\partial^2 V_1^f}{\partial \xi^2} = 0 \tag{4.99}$$

Integrating once shows that:

$$\frac{\partial V_1^f}{\partial Y} = A_1(x). \tag{4.100}$$

So $\partial_Y V_1^f$ is purely a function of x and is independent of ξ, the boundary condition can be expanded as:

$$\frac{1}{\varepsilon}\partial_\xi(\varepsilon V_1^f + \varepsilon^2 V_2 + \cdots) = \epsilon_{s,f}\mathscr{H}(\varepsilon\partial_x V_1^f + \varepsilon^2 V_2^f + \cdots) \tag{4.101}$$

The boundary conditions are then given by:

$$\frac{\partial V_1^f}{\partial \xi} = 0, \quad \frac{\partial V_2^f}{\partial \xi} = \epsilon_{s,f}\mathscr{H}(\partial_x V_1^f), \tag{4.102}$$

and so $A_1 = 0$, inserting the expansion for V_1^f into the governing equation shows that:

$$\frac{\partial^2 V_2^f}{\partial \xi^2} = 0 \tag{4.103}$$

Integrating once shows that:

$$\frac{\partial V_2^f}{\partial \xi} = A_2(x) \tag{4.104}$$

Using the boundary condition this is:

$$\frac{\partial V_2^f}{\partial \xi} = -\epsilon_{s,f}\mathscr{H}(\partial_x V_1^f), \tag{4.105}$$

This can then be inserted into the boundary condition for the normal component of the electric field to show that:

$$\epsilon_{f,g}\partial_x\eta_0 + \epsilon_{s,f}\epsilon_{f,g}\mathscr{H}(\partial_x V_1^f) = -\partial_x\eta_0 + \partial_Y V_1^g \tag{4.106}$$

Now write the following:

$$\eta_0 = \int_{\mathbb{R}} \hat{\eta}_0 e^{ikx} dk, \quad V_1^f = \int_{\mathbb{R}} \hat{V}_1^f e^{ikx} dk, \quad V_1^g = \int_{\mathbb{R}} \hat{V}_1^g e^{ikx} dk \tag{4.107}$$

Inserting this into the governing equation for V_1^g, shows that:

$$\frac{\partial^2 V_1^g}{\partial Y^2} - k^2 V_1^g = 0 \tag{4.108}$$

along with the boundary that $V_1^g \to 0$ as $Y \to -\infty$, the solution to this equation is given by:

$$\hat{V}_1^g = Ae^{|k|Y} \tag{4.109}$$

This can then substituted into the boundary condition for the normal component of the electric field to get:

$$ik(\epsilon_{f,g} - 1)\hat{\eta}_0 - k\text{sgn}(k)\epsilon_{s,f}\epsilon_{f,g}\hat{V}_1^f = |k|A \tag{4.110}$$

The tangential boundary conditions are:

$$\frac{\partial V_1^f}{\partial x} = \frac{\partial V_1^g}{\partial x}, \tag{4.111}$$

showing that:

$$\hat{V}_1^f = A \tag{4.112}$$

which then shows that:

$$ik(\epsilon_{f,g} - 1)\hat{\eta}_0 - k\text{sgn}(k)\epsilon_{s,f}\epsilon_{f,g}\hat{V}_1^f = |k|\hat{V}_1^f \tag{4.113}$$

Which implies:

$$\hat{V}_1^f = \frac{i\text{sgn}(k)(\epsilon_{f,g} - 1)\hat{\eta}_0}{1 + \epsilon_{s,f}\epsilon_{f,g}} \tag{4.114}$$

This then translates as:

$$V_1^f = \frac{(\epsilon_{f,g} - 1)\mathscr{H}(\eta_0)}{1 + \epsilon_{s,f}\epsilon_{f,g}} \tag{4.115}$$

The value $\partial_Y V_1^g(x,0)$ is required for the boundary condition, and hence:

$$V_1^g = \int_{\mathbb{R}} \frac{i(\epsilon_{f,g}+1)\text{sgn}(k)\hat{\eta}_0}{1+\epsilon_{s,f}\epsilon_{f,g}} e^{|k|Y} e^{ikx} dk \tag{4.116}$$

So that value on the free surface is given when $Y = 0$, and inserting this into the above expression shows that:

$$\begin{aligned} V_1^g &= \int_{\mathbb{R}} \frac{i(\epsilon_{f,g}-1)\text{sgn}(k)\hat{\eta}_0}{1+\epsilon_{s,f}\epsilon_{f,g}} e^{ikx} dk \\ &= \frac{(\epsilon_{f,g}-1)}{1+\epsilon_{s,f}\epsilon_{f,g}} \int_{\mathbb{R}} i\text{sgn}(k) e^{ikx}\hat{\eta}_0 dk \\ &= \frac{(\epsilon_{f,g}-1)}{1+\epsilon_{s,f}\epsilon_{f,g}} \int_{\mathbb{R}} \mathscr{H}(e^{ikx})\hat{\eta}_0 dk \\ &= \mathscr{H}\left(\frac{(\epsilon_{f,g}-1)}{1+\epsilon_{s,f}\epsilon_{f,g}} \int_{\mathbb{R}} \hat{\eta}_0 e^{ikx} dk\right) \\ &= \frac{(\epsilon_{f,g}-1)\mathscr{H}(\eta_0)}{1+\epsilon_{s,f}\epsilon_{f,g}} \end{aligned}$$

So on the free surface the two potentials V_1^f and V_1^f are the same to order ε. The next equation to examine is the Bernoulli equation, the equation required in the $O(1)$ terms. Differentiation of the Bernoulli equation shows that:

$$\frac{\partial^2\varphi_0}{\partial t\partial x} + \frac{\partial\varphi_0}{\partial x}\frac{\partial^2\varphi_0}{\partial x^2} + B\frac{\partial\eta_0}{\partial x} + E_b\frac{\partial Q}{\partial x} = -\frac{\partial^3\eta}{\partial x^3} \tag{4.117}$$

There are no $O(1)$ term in Q, so in order to include the electric terms, E_b must be scaled as $E_b = \varepsilon^{-1}\hat{E}_b$, these will include two terms in the M_{22} terms, they are:

$$Q = -\frac{\epsilon_{f,g}}{2}(1-2\varepsilon\partial_x V_1^f) + \frac{1}{2}(1-2\varepsilon\partial_x V_1^f) + o(\varepsilon) \tag{4.118}$$

then:

$$\frac{\partial Q}{\partial x} = (\epsilon_{f,g}-1)\varepsilon\partial_x^2 V_1^f = \frac{(\epsilon_{f,g}-1)^2\varepsilon}{1+\epsilon_{s,f}\epsilon_{f,g}}\mathscr{H}(\partial_x^2\eta_0) \tag{4.119}$$

Upon using $u = \partial_x\varphi_0$, the equation becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + B\frac{\partial\eta_0}{\partial x} + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}\mathscr{H}\left(\frac{\partial^2\eta_0}{\partial x^2}\right) = -\frac{\partial^3\eta_0}{\partial x^3} \tag{4.120}$$

For the linear theory, expand the variables in the following way.

$$\eta_0 = -1 + \bar{\eta} \quad u_0 = \bar{u}$$

The linearised equations become

$$\begin{aligned} \frac{\partial \bar{\eta}}{\partial t} - \frac{\partial \bar{u}}{\partial x} &= 0 \\ \frac{\partial \bar{u}}{\partial t} + B\frac{\partial \bar{\eta}}{\partial x} + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}\mathscr{H}\left(\frac{\partial^2 \bar{\eta}}{\partial x^2}\right) &= -\frac{\partial^3 \bar{\eta}}{\partial x^3} \end{aligned}$$

Upon cross differentiation, the above reduces to one equation:

$$\frac{\partial^2 \bar{\eta}}{\partial t^2} + B\frac{\partial^2 \bar{\eta}}{\partial x^2} + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}\mathscr{H}\left(\frac{\partial^3 \bar{\eta}}{\partial x^3}\right) + \frac{\partial^4 \bar{\eta}}{\partial x^4} = 0 \tag{4.121}$$

This matches up with equation (4.76). To compute the dispersion relation, write $\bar{\eta} = a\cos(kx-\omega t)$, this yields a dispersion relation:

$$\omega^2 = -Bk^2 + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}|k|k^2 + k^4 \tag{4.122}$$

Which upon writing $\omega = kc(k)$ becomes:

$$c^2 = -B + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}|k| + k^2 \tag{4.123}$$

Now look for travelling wave solutions of the above system, write $\eta(t,x) = f(x-ct)$, $u = g(x-ct)$ and $\xi = x - ct$ to obtain the system:

$$-cf' + (g(f-1))' = 0 \tag{4.124}$$

$$-cg' + \left(\frac{1}{2}g^2\right)' + Bf' + \frac{\hat{E}_b(\epsilon_{f,g}-1)^2}{1+\epsilon_{s,f}\epsilon_{f,g}}\mathscr{H}(f'') = f'''. \tag{4.125}$$

The system is solved for $\partial_x \eta$ and u. To solve these equations numerically, choose a domain x_i, where $x_0 = 0$ and $X_N = \lambda/2$. Then let the equations be satisfied at every point in the domain. Define the mid-point of the interval as:

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$$

for $1 \leqslant i < N$. The Hilbert transform is given by the following for periodic function:

$$\mathscr{H}(\partial_x^2 \eta(x)) = -PV\int_{-\frac{1}{2}}^{\frac{1}{2}} \cot(\pi(x-u))\partial_u^2 \eta du \tag{4.126}$$

In order to get the integral over the wavelength λ, let $u = \lambda^{-1}v$ and $x = \lambda^{-1}y$, and so:

$$\begin{aligned}
\mathscr{H}(\partial_x^2\eta(x)) &= -PV\int_{-\frac{1}{2}}^{\frac{1}{2}} \cot(\pi(x-u))\partial_u^2\eta du \\
&= -PV\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \cot\left(\pi\left(\frac{y}{\lambda}-\frac{v}{\lambda}\right)\right)\lambda^2\partial_v^2\eta\frac{dv}{\lambda} \\
&= -\lambda PV\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \cot\left(\frac{\pi}{\lambda}(y-v)\right)\partial_v^2\eta dv \\
&= -\lambda PV\int_{-\frac{\lambda}{2}}^{0} \cot\left(\frac{\pi}{\lambda}(y-v)\right)\partial_v^2\eta dv - \\
&\quad -\lambda PV\int_{0}^{\frac{\lambda}{2}} \cot\left(\frac{\pi}{\lambda}(y-v)\right)\partial_v^2\eta dv \\
&= -\lambda PV\int_{0}^{\frac{\lambda}{2}} \cot\left(\frac{\pi}{\lambda}(y+v)\right)\partial_v^2\eta dv - \\
&\quad -\lambda PV\int_{0}^{\frac{\lambda}{2}} \cot\left(\frac{\pi}{\lambda}(y-v)\right)\partial_v^2\eta dv
\end{aligned}$$

The Hilbert transform is computed via the trapezium rule over the half points, $x_{i+\frac{1}{2}}$. The MATLAB function *grad* was used to compute the gradients at the half points and the value of $\partial_x\eta$ was averaged over the two neighbouring points and then the function grad applied. There are a number of additional constraints, as the wave is assumed to be periodic and so the gradients are zero at each end of the domain: $\partial_x\eta(1) = \partial_x\eta(N) = 0$. The height of the wave is given by $\eta(N) - \eta(1) = h$, where h is specified at the beginning of the code and η is obtained from integration of $\partial_x\eta$. The two other constraints are that the average over the wavelength of the velocity is the wavelength:

$$\frac{1}{\lambda}\int_0^{\frac{\lambda}{2}} u dx = \frac{\lambda}{2}$$

This equation sets the speed of the wave. The other condition is obtained by averaging over the wavelength of η to give the average height of the wave, z,

$$\frac{1}{\lambda}\int_0^{\frac{\lambda}{2}} \eta dx = \frac{z}{2}$$

The only two other unknowns that have to be calculated are the first point of the free surface $\eta(1)$ and the wave speed, c. Equation (4.124) is evaluated at the points $x_i, i = 1, \ldots, N-1$ and (4.125) is evaluated at the points $x_i, i = 2, \ldots, N-1$. This

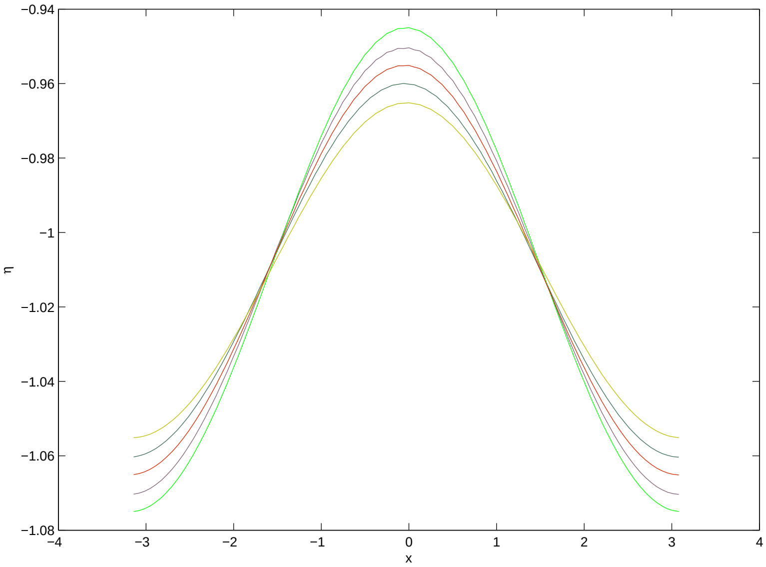


Figure 4.3: Wave profiles from $h = 0.09$ to $h = 0.13$

analysis in this section has been about deriving an equation for a thin layer of fluid where the amplitude of the free surface is of the same order the depth of the fluid. The resulting equations turned out to be a set of coupled weakly nonlinear partial differential equations. A numerical scheme was proposed and profiles were produced for a range of amplitudes shown in figure (4.3).

4.6 Stokes Expansion Analysis

Denote the following:

$$\bar{u} = u + c \tag{4.127}$$

$$\eta_0 = \eta \tag{4.128}$$

$$\frac{\partial}{\partial t} \equiv 0 \tag{4.129}$$

inserting these into (4.98) and (4.120) shows that:

$$c\frac{\partial \eta}{\partial x} + \eta\frac{\partial u}{\partial x} + u\frac{\partial \eta}{\partial x} = 0 \tag{4.130}$$

$$(u+c)\frac{\partial u}{\partial x} + B\frac{\partial \eta}{\partial x} + e_b\mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \frac{\partial^3 \eta}{\partial x^2} = 0 \tag{4.131}$$

where:

$$e_b = \frac{\hat{E}_b(\epsilon_{f,g} - 1)^2}{2(1 + \epsilon_{s,f}\epsilon_{f,g})}.$$

Now expand the variables as:

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \tag{4.132}$$

$$\eta = -1 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \cdots \tag{4.133}$$

$$c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 \tag{4.134}$$

The $O(\varepsilon)$ equations are:

$$c_0 \frac{\partial \eta_1}{\partial x} - \frac{\partial u_1}{\partial x} = 0 \tag{4.135}$$

$$c_0 \frac{\partial u_1}{\partial x} + B\frac{\partial \eta_1}{\partial x} + e_b \mathscr{H}\left(\frac{\partial^2 \eta_1}{\partial x^2}\right) + \frac{\partial^3 \eta_0}{\partial x^3} = 0 \tag{4.136}$$

The $O(\varepsilon^2)$ equations are:

$$c_0 \frac{\partial \eta_2}{\partial x} - \frac{\partial u_2}{\partial x} = -c_1 \frac{\partial \eta_1}{\partial x} - \eta_1 \frac{\partial u_1}{\partial x} - u_1 \frac{\partial \eta_1}{\partial x} \tag{4.137}$$

$$c_0 \frac{\partial u_2}{\partial x} + B\frac{\partial \eta_2}{\partial x} + e_b \mathscr{H}\left(\frac{\partial^2 \eta_2}{\partial x^2}\right) + \frac{\partial^3 \eta_2}{\partial x^3} = -(c_1 + u_1)\frac{\partial u_1}{\partial x} \tag{4.138}$$

Working with (4.135) and (4.136) for now, it can be easily seen that they can be combined into one equation of the form:

$$(c_0^2 + B)\frac{\partial \eta_1}{\partial x} + e_b \mathscr{H}\left(\frac{\partial^2 \eta_1}{\partial x^2}\right) + \frac{\partial^3 \eta_1}{\partial x^3} = 0 \tag{4.139}$$

Assuming that the wave is symmetric, the free surface can be expanded as a Fourier cosine series:

$$\eta_1(x) = \sum_{n \geqslant 1} a_n \cos(knx) \quad u_1(x) = \sum_{n \geqslant 1} b_n \cos(knx) \tag{4.140}$$

Where $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength. Inserting (4.140) into (4.139) shows that the coefficients must satisfy:

$$ka_n\left[-(c_0^2 + B)n + e_b k n^2 + k^2 n^3\right] = 0 \tag{4.141}$$

Taking $a_1 \neq 0$ without loss of generality, the term for $n = 1$ has to be that:

$$-(c_0^2 + B) + e_b k + k^2 = 0 \tag{4.142}$$

Or that:

$$c_0^2 + B = e_b k + k^2 \tag{4.143}$$

Inserting (4.143) into (4.141) shows that:

$$k^2 n(n-1)a_n(e_b + (n+1)k) = 0 \quad n \geqslant 2 \tag{4.144}$$

So if $a_m \neq 0$ for $m \geqslant 2$ for example, this would suggest that

$$(m+1)k = -e_b$$

Which obviously cannot be true if $e_b > 0$, and so is it possible to conclude that $a_n = b_n = 0$ for $n \geqslant 2$ and the solutions are:

$$\eta_1 = a_1 \cos(kx) \tag{4.145}$$

$$u_1 = c_0 a_1 \cos(kx) \tag{4.146}$$

Now let:

$$\eta = -1 + \sum_{n \geqslant 1} \alpha_n(\varepsilon) \cos(nkx) \tag{4.147}$$

and suppose:

$$\alpha_n(\varepsilon) = \varepsilon \alpha_n^{(1)} + \varepsilon^2 \alpha_n^{(2)} + \cdots \tag{4.148}$$

Then if:

$$\varepsilon := \alpha_1 k \tag{4.149}$$

Then:

$$\frac{\varepsilon}{k} = \varepsilon \alpha_1^{(1)} + \varepsilon^2 \alpha_1^{(2)} + \cdots \tag{4.150}$$

This demonstrates that $\alpha_1^{(n)} = 0$ for $n \geqslant 2$ and that $\alpha_1^{(1)} = a_1 = k^{-1}$. The equations for $O(\varepsilon^2)$ are:

$$\begin{aligned} c_0 \frac{\partial \eta_2}{\partial x} - \frac{\partial u_2}{\partial x} &= c_1 \sin kx + \frac{c_0}{k} \sin(2kx) \\ c_0 \frac{\partial u_2}{\partial x} + B \frac{\partial \eta_2}{\partial x} + e_b \mathscr{H}\left(\frac{\partial^2 \eta_2}{\partial x^2}\right) + \frac{\partial^3 \eta_2}{\partial x^3} &= \frac{c_0^2}{2k} \sin(2kx) + c_1 c_0 \sin kx \end{aligned}$$

Write:

$$\eta_2 = \sum_{n \geqslant 2} a_n^{(2)} \cos(nkx), \quad u_2 = \sum_{n \geqslant 1} b_n^{(2)} \cos(nkx) \tag{4.151}$$

Inserting (4.151) into the $O(\varepsilon^2)$ equations and examining the $n = 1$ terms shows that $kb_1^{(2)} = c_1$ and $-c_0kb_1^{(2)} = c_0c_1$ showing that $b_1^{(2)} = c_1 = 0$. Computing the the the rest shows that:

$$a_2^{(2)} = \frac{3(-B + e_bk + k^2)}{4k^2(e_bk + 3k^2 - B)} \tag{4.152}$$

$$b_2^{(2)} = \frac{\sqrt{-B + e_bk + k^2}}{2k^2} + \frac{(-B + e_bk + k^2)^{\frac{3}{2}}}{4k^2(e_bk + 3k^2 - B)} \tag{4.153}$$

and $a_2^{(n)} = b_2^{(n)} = 0$ for $n \geqslant 3$. The $O(\varepsilon^3)$ equations are:

$$\begin{aligned} c_0\frac{\partial \eta_3}{\partial x} - \frac{\partial u_3}{\partial x} &= \left(c_2 + \frac{1}{2}(a_2^{(2)}c_0 + b_2^{(2)})\right)\sin kx + \frac{3}{2}(a_2^{(2)}c_0 + \beta)\sin(3k \\ c_0\frac{\partial u_3}{\partial x} + B\frac{\partial \eta_3}{\partial x} + e_b\mathscr{H}\left(\frac{\partial^2\eta_3}{\partial x^2}\right) + \frac{\partial^3\eta_3}{\partial x^3} &= \frac{3}{2}c_0b_2^{(2)}\sin(3kx) + \left(\frac{c_0b_2^{(2)}}{2} + c_0c_2\right)\sin kx \end{aligned}$$

For convenience write:

$$\begin{aligned} c_0\frac{\partial \eta_3}{\partial x} - \frac{\partial u_3}{\partial x} &= \nu_1\sin kx + \nu_2\sin(3kx) \\ c_0\frac{\partial u_3}{\partial x} + B\frac{\partial \eta_3}{\partial x} + e_b\mathscr{H}\left(\frac{\partial^2\eta_3}{\partial x^2}\right) + \frac{\partial^3\eta_3}{\partial x^3} &= \mu_1\sin kx + \mu_2\sin(3kx) \end{aligned}$$

Then the two equations can be combined into one equation:

$$(c_0^2+B)\partial_x\eta_3+e_b\mathscr{H}(\partial_x^2\eta_3)+\partial_x^3\eta_3 = (\mu_1+c_0\nu_1)\sin kx+(\mu_2+c_0\nu_2)\sin 3kx \tag{4.154}$$

Then write:

$$\eta_3 = \sum_{n\geqslant 1} a_n^{(3)}\cos(nkx) \tag{4.155}$$

Then inserting (4.155) into (4.154), shows that:

$$\sum_{n\geqslant 1}\left[-(c_0^2 + B)kn + e_bK^2n^2 + n^3k^3\right]a_n^{(3)}\sin(nkx) = (\mu_1+c_0\nu_1)\sin kx+(\mu_2+c_0\nu_2)\sin 3kx \tag{4.156}$$

Taking $n = 1$ shows that:

$$k\left[-(c_0^2 + B) + e_bk + k^2\right]a_1^{(3)} = \mu_1 + c_0\nu_1 \tag{4.157}$$

However, it was shown that the bracketed term in (4.157) is zero and hence $\mu_1 + c_0\nu_1 = 0$ which means that c_2 must satisfy:

$$c_2 = -\frac{1}{4}\left(a_2^{(2)} + \frac{b_2^{(2)}}{c_0} + B\right) \tag{4.158}$$

The asymptotic solution is defined by η_1, η_2, η_3, c_0 and c_2.

4.7 Finite Amplitude and Long Wavelength

In this section the case in chapters 2 and 3 are revisited, where an equation for the free surface was derived under the assumption of shallow water and small amplitude using asymptotic analysis. Here a system of equations are derived with the long wavelength and finite amplitude which is of the same order as the depth of the fluid. The main modelling assumptions are:

1. the flow is irrotational.
2. The fluid is incompressible.
3. The wavelength λ is very long, $h/\lambda \ll 1$.
4. The amplitude of the wave is of the same order as the average height of the fluid, $a/h = O(1)$.

4.7.1 Basic Set-Up

The relevant equations for the modelling of this are given by the following:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{on} \;\; y > \eta(t,x) \tag{4.159}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{on} \quad -h \leqslant y \leqslant \eta(t,x) \tag{4.160}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial y} \quad \text{on} \;\; y = \eta(t,x) \tag{4.161}$$

$$\begin{aligned} \partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{P(x)}{\rho} \quad &- \quad \frac{1}{\rho}\frac{1}{1+(\partial_x\eta)^2}((\partial_x\eta)^2\Sigma_{11} + 2\partial_x\eta\Sigma_{12} + \Sigma_{22}) \\ &+ \quad g\eta = \frac{\sigma}{\rho}\frac{\partial_x^2\eta}{(1+(\partial_x\eta)^2)^{3/2}} + C \quad \text{on} \quad y = \eta(t,x) \end{aligned} \tag{4.162}$$

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial y} = 0 \quad \text{on} \quad y = \eta(t,x) \tag{4.163}$$

$$\frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \;\; y = -h \tag{4.164}$$

$$\frac{\partial V}{\partial y} \to -E_0 \quad y \to \infty \tag{4.165}$$

Using the same scaling as in chapter 3, the equations are transformed by the following scalings:

$$\begin{gathered} x = \lambda \hat{x}, \quad t = \frac{\lambda}{c_0}\hat{t}, \quad \eta = a\hat{\eta}, \quad \varphi = \frac{g\lambda a}{c_0}\hat{\varphi}, \\ y^{(1)} = h\hat{y}, \quad y^{(2)} = \lambda \hat{Y}, \quad V = \lambda E_0 \hat{V}. \end{gathered} \tag{4.166}$$

where λ is the wavelength and the fluid speeds are given as $u = \partial_x \varphi$ and $v = \partial_y \varphi$. Upon defining

$$\alpha = \frac{a}{h}, \quad \beta = \frac{h^2}{\lambda^2}, \tag{4.167}$$

the scaled equations are:

$$\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}^2} + \frac{1}{\beta}\frac{\partial^2 \hat{\varphi}}{\partial \hat{y}^2} = 0 \quad \text{on} \quad -1 \leqslant \hat{y} \leqslant \alpha \hat{\eta}(\hat{t}, \hat{x}) \tag{4.168}$$

$$\frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{Y}^2} = 0 \quad \text{on} \ \ \hat{Y} > \alpha \hat{\eta}(\hat{t}, \hat{x}) \tag{4.169}$$

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} + \alpha \frac{\partial \hat{\varphi}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{x}} = \frac{1}{\beta}\frac{\partial \hat{\varphi}}{\partial \hat{y}} \quad \text{on} \ \ y = \alpha \hat{\eta}(\hat{t}, \hat{x}). \tag{4.170}$$

$$\begin{aligned} &\frac{\partial \hat{\varphi}}{\partial \hat{t}} + \frac{1}{2}\left[\alpha \left(\frac{\partial \hat{\varphi}}{\partial \hat{x}}\right)^2 + \frac{\alpha}{\beta}\left(\frac{\partial \hat{\varphi}}{\partial \hat{y}}\right)^2\right] + \hat{P}(\hat{x}) + \hat{\eta} + \\ &\quad + \frac{1}{\alpha}\frac{E_b}{1+\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2}\left(\frac{1}{2}(\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2 - 1)\left[\left(\frac{\partial \hat{V}}{\partial \hat{x}}\right)^2 + \left(\frac{\partial \hat{V}}{\partial \hat{Y}}\right)^2\right] + \right. \\ &\quad \left. +2\alpha\sqrt{\beta}\partial_{\hat{x}}\hat{\eta}\frac{\partial \hat{V}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{Y}}\right) = \frac{\beta B \partial^2_{\hat{x}}\hat{\eta}}{(1+\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2)^{3/2}} + C \quad \text{on} \ \ y = \alpha\hat{\eta}(\hat{t}, \hat{x}). \end{aligned} \tag{4.171}$$

$$\frac{\partial \hat{V}}{\partial \hat{x}} + \alpha\sqrt{\beta}\frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{y}} = 0 \quad \text{on} \ \ y = \alpha \hat{\eta}(\hat{t}, \hat{x}). \tag{4.172}$$

$$\frac{\partial \hat{\varphi}}{\partial \hat{y}} = 0 \quad \text{on} \ \ y = -1 \tag{4.173}$$

$$\frac{\partial \hat{V}}{\partial \hat{Y}} \to -1 \quad \hat{Y} \to \infty \tag{4.174}$$

The Bernoulli constant can be found be setting $\hat{\varphi} = \hat{\eta} = \hat{P} = 0$ and $\hat{V} = -Y$ to find that $C = -E_b/2\alpha$. Now to formally study finite amplitude wave the parameter α is set to unity and $\beta \ll 1$, which gives the approximation of long wavelength but still keeping the finite amplitude. In doing so the equations finally become:

$$\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}^2} + \frac{1}{\beta}\frac{\partial^2 \hat{\varphi}}{\partial \hat{y}^2} = 0 \quad \text{on} \quad -1 \leqslant \hat{y} \leqslant \hat{\eta}(\hat{x}) \tag{4.175}$$

$$\frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{Y}^2} = 0 \quad \text{on} \quad \hat{Y} > \hat{\eta}(\hat{x}) \tag{4.176}$$

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} + \frac{\partial \hat{\varphi}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{x}} = \frac{1}{\beta}\frac{\partial \hat{\varphi}}{\partial \hat{y}} \quad \text{on} \quad y = \hat{\eta}. \tag{4.177}$$

$$\begin{aligned}
&\frac{\partial \hat{\varphi}}{\partial \hat{t}} + \frac{1}{2}\left[\left(\frac{\partial \hat{\varphi}}{\partial \hat{x}}\right)^2 + \frac{1}{\beta}\left(\frac{\partial \hat{\varphi}}{\partial \hat{y}}\right)^2\right] + \hat{P}(\hat{x}) + \hat{\eta} + \\
&+ \frac{E_b}{1+\beta(\partial_{\hat{x}}\hat{\eta})^2}\left(\frac{1}{2}(\beta(\partial_{\hat{x}}\hat{\eta})^2 - 1)\left[\left(\frac{\partial \hat{V}}{\partial \hat{x}}\right)^2 + \left(\frac{\partial \hat{V}}{\partial \hat{Y}}\right)^2\right] + \right. \\
&\left. + 2\sqrt{\beta}\partial_{\hat{x}}\hat{\eta}\frac{\partial \hat{V}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{Y}}\right) = \frac{\beta B \partial^2_{\hat{x}}\hat{\eta}}{(1+\beta(\partial_{\hat{x}}\hat{\eta})^2)^{3/2}} - \frac{E_b}{2}
\end{aligned} \tag{4.178}$$

$$\frac{\partial \hat{V}}{\partial \hat{x}} + \sqrt{\beta}\frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{y}} = 0 \quad \text{on} \quad y = \hat{\eta}. \tag{4.179}$$

$$\frac{\partial \hat{\varphi}}{\partial \hat{y}} = 0 \quad \text{on} \quad y = -1 \tag{4.180}$$

$$\frac{\partial \hat{V}}{\partial \hat{Y}} \to -1 \quad \hat{Y} \to \infty \tag{4.181}$$

4.7.2 Deriving the Asymptotic Equations

Upon dropping the hats, write:

$$\begin{aligned}
\varphi &= \varphi_0 + \beta\varphi_1 + o(\beta) \\
V &= -Y + \sqrt{\beta}V_1 + o(\sqrt{\beta})
\end{aligned}$$

Inserting the expansion for φ into (4.175) shows that:

$$\frac{\partial^2\varphi_0}{\partial x^2}+\beta\frac{\partial^2\varphi_1}{\partial x^2}+\frac{1}{\beta}\frac{\partial^2\varphi_0}{\partial y^2}+\frac{\partial^2\varphi_1}{\partial y^2}=0$$

on equating powers of β shows that:

$$\begin{aligned}\frac{\partial^2\varphi_0}{\partial y^2} &= 0\\ \frac{\partial^2\varphi_0}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2} &= 0\end{aligned}$$

The solution for φ_0 is $\varphi_0 = yA(t,x) + B(t,x)$, but the boundary condition $\partial_y\varphi(t,x,-1) = 0$ shows that $A(t,x) = 0$, and so $\varphi_0 = \varphi_0(t,x)$, Solving the equation for φ_1 shows that:

$$\varphi_1 = -\frac{(y+1)^2}{2}\frac{\partial^2\varphi_0}{\partial x^2}+A(t,x) \tag{4.182}$$

Inserting this into (4.177) shows that:

$$\frac{\partial\eta}{\partial t}+\frac{\partial\varphi_0}{\partial x}\frac{\partial\eta}{\partial x}=\frac{1}{\beta}\left(-\beta(\eta+1)\frac{\partial^2\varphi_0}{\partial x^2}\right) \tag{4.183}$$

Which after some rearranging shows that:

$$\boxed{\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial x}\left((\eta+1)u\right)=0} \tag{4.184}$$

The other equation to use is the Bernoulli equation and the $O(1)$ equation is sought, the terms in the equation have the following asymptotic expansions:

$$\begin{aligned}\left(\frac{\partial\varphi}{\partial x}\right)^2 &= \left(\frac{\partial\varphi_0}{\partial x}\right)^2+O(\beta)\\ \frac{1}{\beta}\left(\frac{\partial\varphi}{\partial y}\right)^2 &= O(\beta)\\ \left(\frac{\partial V}{\partial x}\right)^2 &= O(\beta)\\ \left(\frac{\partial V}{\partial Y}\right)^2 &= 1-2\sqrt{\beta}\frac{\partial V_1}{\partial Y}+O(\beta)\\ \frac{E_b}{1+\beta(\partial_x\eta)^2} &= E_b+O(\beta)\end{aligned}$$

The principle of least degeneracy requires to keep as many terms as possible in the asymptotic expansion, and so scale the following:

$$E_b=\frac{\hat{E}_b}{\sqrt{\beta}},\quad B=\frac{\hat{B}}{\beta} \tag{4.185}$$

to obtain the $O(1)$ Bernoulli equation:

$$\frac{\partial \varphi_0}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi_0}{\partial x}\right)^2 + P + \eta + \hat{E}_b \frac{\partial V_1}{\partial Y} = \hat{B}\frac{\partial^2 \eta}{\partial x^2} \tag{4.186}$$

So differentiating this equation shows that:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial \eta}{\partial x} + \hat{E}_b \frac{\partial^2 V_1}{\partial x \partial Y} = \hat{B}\frac{\partial^3 \eta}{\partial x^3} \tag{4.187}$$

Not to relate the electric potential to free surface, use equation (4.179), when on expanding:

$$\sqrt{\beta}\frac{\partial V_1}{\partial x} - \sqrt{\beta}\frac{\partial \eta}{\partial x} = 0 \tag{4.188}$$

Upon using the standard expressions for Hilbert transforms:

$$\frac{\partial V_1}{\partial Y} = \mathscr{H}\left(\frac{\partial V_1}{\partial x}\right) \tag{4.189}$$

So finally the equation becomes:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial \eta}{\partial x} + \hat{E}_b \mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right) = \hat{B}\frac{\partial^3 \eta}{\partial x^3}} \tag{4.190}$$

By moving the fixed boundary from $-h$ to 0, equation (4.184) can be put in the more recognisable form:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta u) = 0 \tag{4.191}$$

Inserting the dimensions back into the equation shows that:

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u(h+\eta)) &= 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} + g\frac{\partial \eta}{\partial x} + ghE_b\mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right) &= gh^2 B\frac{\partial^3 \eta}{\partial x^3} \end{aligned}$$

These equations are the shallow water equivalent of the KdV approximations of Chapter 3.

Chapter 5

3D Waves in a Conducting Fluid

In this chapter the analysis in chapter two is extended from a 1D free surface to a 2D free surface. Sections 5.1 and 5.2 will describe the set-up of the model and the non-dimensionalisation. A moving pressure distribution is considered and the resulting linear and weakly nonlinear waves are calculated.

5.1 Set Up of the Model

The analysis will consist of examining flows which are incompressible ($\nabla \cdot \mathbf{u} = 0$) and irrotational ($\nabla \times \mathbf{u} = \mathbf{0}$), these assumptions imply the existence of a velocity potential φ, such that $\mathbf{u} = \nabla\varphi$, the incompressibility condition implies:

$$\nabla^2\varphi = 0 \tag{5.1}$$

The electric potential V satisfies:

$$\nabla^2 V = 0 \tag{5.2}$$

The next task is to examine the boundary conditions, write the free surface equation as $z = \eta(t, x, y)$, then the free surface condition is written as:

$$\frac{D}{Dt}(z - \eta(t, x, y)) = 0 \Rightarrow \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} + v\frac{\partial\eta}{\partial y} = w \tag{5.3}$$

There is a solid surface at $z = -h$, which is impermeable and so:

$$\frac{\partial\varphi}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{5.4}$$

The Bernoulli equation is given by:

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] + g\eta + \frac{p}{\rho} = C \tag{5.5}$$

The pressure is obtained through the Young-Laplace equation. To compute the normal write $F = \eta(t, x, y) - z$. The unit normal is given by:

$$\hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|} \tag{5.6}$$

So $\nabla F = (\partial_x \eta, \partial_y \eta, -1)$ and therefore:

$$\hat{\mathbf{n}} = \frac{(\partial_x \eta, \partial_y \eta, -1)}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{1}{2}}} \tag{5.7}$$

The Young-Laplace equation is given by:

$$[\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]_2^1 = \sigma \nabla \cdot \hat{\mathbf{n}} \tag{5.8}$$

To compute the RHS:

$$\begin{aligned}
\nabla \cdot \hat{\mathbf{n}} &= \frac{\partial}{\partial x}\left[\frac{\partial_x \eta}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{1}{2}}}\right] + \frac{\partial}{\partial y}\left[\frac{\partial_y \eta}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{1}{2}}}\right] \\
&= \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{1}{2}}} - \frac{\partial_x \eta(\partial_x \eta \partial_x^2 \eta + \partial_y \eta \partial_x \partial_y \eta)}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{3}{2}}} + \\
&\quad + \frac{\partial_y^2 \eta}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{1}{2}}} - \frac{\partial_y \eta(\partial_y \eta \partial_y^2 \eta + \partial_y \eta \partial_x \partial_y \eta)}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{3}{2}}} \\
&= \frac{\partial_x^2 \eta(1 + (\partial_y \eta)^2) + \partial_y^2 \eta(1 + (\partial_x \eta)^2) - 2\partial_x \eta \partial_y \eta \partial_x \partial_y \eta}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{3}{2}}}
\end{aligned}$$

The stress tensor in this case is given by:

$$\mathbf{T} = -p\delta_{ij} + \epsilon\left(E_i E_j - \frac{1}{2}\delta_{ij} E_k E_k\right) \tag{5.9}$$

The pressure terms is easily expanded as $\hat{\mathbf{n}} \cdot (-p\mathbf{I}) \cdot \hat{\mathbf{n}} = -p$, the electric terms is somewhat more complicated, for brevity, write:

$$\Sigma_{ij} = \epsilon\left(E_i E_j - \frac{1}{2}\delta_{ij} E_k E_k\right) \tag{5.10}$$

So the tensor can be expanded as:

$$\begin{aligned}
\hat{n}_i \Sigma_{ij} \hat{n}_j &= \hat{n}_1^2 \Sigma_{11} + \hat{n}_2 \hat{n}_1 \Sigma_{12} + \hat{n}_1 \hat{n}_3 \Sigma_{13} + \\
&\quad + \hat{n}_2 \hat{n}_1 \Sigma_{21} + \hat{n}_2^2 \Sigma_{22} + \hat{n}_2 \hat{n}_3 \Sigma_{23} + \\
&\quad + \hat{n}_1 \hat{n}_3 \Sigma_{31} + \hat{n}_2 \hat{n}_3 \Sigma_{32} + \hat{n}_3^2 \Sigma_{33} \\
&= \hat{n}_1^2 \Sigma_{11} + \hat{n}_2^2 \Sigma_{22} + \hat{n}_3^2 \Sigma_{33} + \\
&\quad + 2\hat{n}_2 \hat{n}_1 \Sigma_{12} + 2\hat{n}_1 \hat{n}_3 \Sigma_{13} + 2\hat{n}_2 \hat{n}_3 \Sigma_{23}
\end{aligned}$$

since the Maxwell tensor is symmetric, this doubles the number of terms of the 1D case but it is hoped that a number of these fall out upon linearisation, so the full explicit equations will not be given here but will be examined in the section on linearisation. For now the terms will just be written as $\hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{n}}$ for brevity, so with this notation the pressure term in the Bernoulli can be written:

$$p = P - \hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{n}} - \sigma \nabla \cdot \hat{\mathbf{n}} \tag{5.11}$$

Therefore the Bernoulli equation is written as:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + g\eta + \frac{P}{\rho} - \frac{1}{\rho}\hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{n}} = \frac{\sigma}{\rho}\nabla \cdot \hat{\mathbf{n}} + C \tag{5.12}$$

where:

$$\nabla \cdot \hat{\mathbf{n}} = \frac{\partial_x^2 \eta (1 + (\partial_y \eta)^2) + \partial_y^2 \eta (1 + (\partial_x \eta)^2) - 2\partial_x \eta \partial_y \eta \partial_x \partial_y \eta}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{3}{2}}} \tag{5.13}$$

One of the boundary conditions for the electric field is a direct analogy to the 1D equation equation (2.12):

$$V(t, x, y, z) \sim -E_0 z \quad \text{as} \quad z \to \infty \tag{5.14}$$

The final boundary condition is that $\left[\hat{\mathbf{t}} \cdot \mathbf{E}\right]_1^2 = 0$, as the normal vector is one dimensional there will be a 2D tangent space, so there are 2 orthogonal tangent vectors. There are 2 vectors which are orthogonal:

$$\begin{aligned} \mathbf{t}_1 &= \mathbf{j} + \partial_y \eta \mathbf{k} \\ \mathbf{t}_2 &= \mathbf{i} + \partial_x \eta \mathbf{k} \end{aligned}$$

The system $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$ is a linear independent system but they are not orthogonal, it is possible to make an orthogonal system via the Gramm-Schmidt process, taking $\mathbf{n}$ and $\mathbf{t}_2$ as the linear system and we apply the Gramm-Schmidt process to $\mathbf{t}_1$ and the following vector is obtained:

$$\mathbf{t}_3 = -\partial_x \eta \partial_y \eta \mathbf{i} + (1 + (\partial_x \eta)^2)\mathbf{j} + \partial_y \eta \mathbf{k} \tag{5.15}$$

So the system $\{\mathbf{n}, \mathbf{t}_2, \mathbf{t}_3\}$ is a linearly independent orthogonal system, the two boundary conditions are then:

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial z} = 0 \tag{5.16}$$

$$-\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial V}{\partial x} + \left(1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right)\frac{\partial V}{\partial y} + \frac{\partial \eta}{\partial y}\frac{\partial V}{\partial z} = 0 \tag{5.17}$$

The basic nonlinear equations are then:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad \text{on} \quad -h \leqslant z \leqslant \eta(t, x, y) \tag{5.18}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{on} \quad z > \eta(t, x, y) \tag{5.19}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \varphi}{\partial z} \quad \text{on} \quad z = \eta(t, x, y) \tag{5.20}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + g\eta + \frac{P}{\rho} - \frac{1}{\rho}\hat{\mathbf{n}}\cdot\Sigma\cdot\hat{\mathbf{n}} = \frac{\sigma}{\rho}\nabla\cdot\hat{\mathbf{n}} + C \tag{5.21}$$

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial z} = 0 \quad \text{on} \quad z = \eta(t, x, y) \tag{5.22}$$

$$-\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial V}{\partial x} + \left(1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right)\frac{\partial V}{\partial y} + \frac{\partial \eta}{\partial y}\frac{\partial V}{\partial z} = 0 \quad \text{on} \quad z = \eta(t, x, y) \tag{5.23}$$

$$\frac{\partial \varphi}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{5.24}$$

$$V \sim -E_0 z \quad \text{as} \quad z \to \infty \tag{5.25}$$

5.2 Nondimensionalisation

The nondimensionalisation is done in the same way as in the 2D case (See section 2.2), the length scalings are given as:

$$x = h\hat{x}, \quad y = h\hat{y}, \quad z = h\hat{z}, \quad \eta = h\hat{\eta} \tag{5.26}$$

The rest of the variables are scaled exactly in the same way as the 2D case:

$$t = \sqrt{\frac{\rho h^3}{\sigma}}\hat{t}, \quad \varphi = \sqrt{\frac{\sigma h}{\rho}}\hat{\varphi}, \quad V = E_0 h\hat{V}, \quad P = \frac{\sigma}{h}\hat{p} \tag{5.27}$$

The governing equations remain the same:

$$\frac{\partial^2\hat{\varphi}}{\partial\hat{x}^2} + \frac{\partial^2\hat{\varphi}}{\partial\hat{y}^2} + \frac{\partial^2\hat{\varphi}}{\partial\hat{z}^2} = 0 \tag{5.28}$$

$$\frac{\partial^2\hat{V}}{\partial\hat{x}^2} + \frac{\partial^2\hat{V}}{\partial\hat{y}^2} + \frac{\partial^2\hat{V}}{\partial\hat{z}^2} = 0 \tag{5.29}$$

As do many of the boundary conditions:

$$\frac{\partial\hat{\eta}}{\partial\hat{t}} + \frac{\partial\hat{\varphi}}{\partial\hat{x}}\frac{\partial\hat{\eta}}{\partial\hat{x}} + \frac{\partial\hat{\varphi}}{\partial\hat{y}}\frac{\partial\hat{\eta}}{\partial\hat{y}} = \frac{\partial\hat{\varphi}}{\partial\hat{z}} \tag{5.30}$$

$$\frac{\partial\hat{V}}{\partial\hat{x}} + \frac{\partial\hat{\eta}}{\partial\hat{x}}\frac{\partial\hat{V}}{\partial\hat{z}} = 0 \tag{5.31}$$

$$-\frac{\partial\hat{\eta}}{\partial\hat{x}}\frac{\partial\hat{\eta}}{\partial\hat{y}}\frac{\partial\hat{V}}{\partial\hat{x}} + \left(1 + \left(\frac{\partial\hat{\eta}}{\partial\hat{x}}\right)^2\right)\frac{\partial\hat{V}}{\partial\hat{y}} + \frac{\partial\hat{\eta}}{\partial\hat{y}}\frac{\partial\hat{V}}{\partial\hat{z}} = 0 \tag{5.32}$$

The boundary condition at the bottom now satisfies:

$$\frac{\partial\hat{\varphi}}{\partial\hat{z}} = 0 \quad \text{on} \quad \hat{z} = -1 \tag{5.33}$$

The asymptotic condition for the electric field satisfies:

$$\hat{V} \sim -\hat{z} \quad \text{as} \quad z \to \infty \tag{5.34}$$

The Bernoulli equation is the most complicated one to nondimensionalise. The scaling of the normal vector is simple as all derivatives of η are dimensionless, so the components of the Maxwell tensor scales as $T_{ij} = \epsilon E_0^2 M_{ij}$. The mean curvature term scales as $\nabla \cdot \mathbf{\hat{n}} = h^{-1}\hat{\nabla} \cdot \hat{\mathbf{\hat{n}}}$, the Bernoulli equation then scales as:

$$\frac{\partial\hat{\varphi}}{\partial\hat{t}} + \frac{1}{2}\left[\left(\frac{\partial\hat{\varphi}}{\partial\hat{x}}\right)^2 + \left(\frac{\partial\hat{\varphi}}{\partial\hat{y}}\right)^2 + \left(\frac{\partial\hat{\varphi}}{\partial\hat{z}}\right)^2\right] + B\hat{\eta} + \hat{p} - E_b\hat{\mathbf{\hat{n}}} \cdot \mathbf{M} \cdot \hat{\mathbf{\hat{n}}} = \hat{\nabla} \cdot \hat{\mathbf{\hat{n}}} + K \tag{5.35}$$

Where K is a dimensionless constant. The other constants are defined as:

$$B = \frac{\rho h^2 g}{\sigma}, \quad E_b = \frac{\epsilon E_0^2 \rho h}{\sigma} \tag{5.36}$$

In order to calculate the constant K, note that there is an exact solution with no pressure distribution, $\hat{\eta} = 0$, $\hat{p}$, $\hat{\varphi} = Ux$ and $\hat{V} = -\hat{z}$. This exact solution can substituted into the Bernoulli equation to find that $K = \frac{1}{2}(U^2 - E_b)$. From here the hats will be dropped for ease of notation.

5.3 Linear Theory I

The theory is now restricted to time-independent cases. As with the 2D theory, a solution is sought as a perturbation of the exact solution:

$$p = \varepsilon p_1 + o(\varepsilon) \tag{5.37}$$

$$\eta = \varepsilon \eta_1 + o(\varepsilon) \tag{5.38}$$

$$\varphi = Ux + \varepsilon \varphi_1 + o(\varepsilon) \tag{5.39}$$

$$V = -z + \varepsilon V_1 + o(\varepsilon) \tag{5.40}$$

The linearised quantities satisfy the following equations.

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2} + \frac{\partial^2 \varphi_1}{\partial z^2} = 0 \quad \text{on} \quad -1 \leqslant z \leqslant 0 \tag{5.41}$$

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} = 0 \quad \text{on} \quad z > 0 \tag{5.42}$$

$$U\frac{\partial \eta_1}{\partial x} = \frac{\partial \varphi_1}{\partial z} \quad \text{on} \quad z = 0 \tag{5.43}$$

$$U\frac{\partial \varphi_1}{\partial x} + B\eta_1 + p_1 + E_b\frac{\partial V_1}{\partial z} = \frac{\partial^2 \eta_1}{\partial x^2} + \frac{\partial^2 \eta_1}{\partial y^2} + K \quad \text{on} \quad z = 0 \tag{5.44}$$

$$\frac{\partial V_1}{\partial x} = \frac{\partial \eta_1}{\partial x} \quad \text{on} \quad z = 0 \tag{5.45}$$

$$\frac{\partial V_1}{\partial y} = \frac{\partial \eta_1}{\partial y} \quad \text{on} \quad z = 0 \tag{5.46}$$

$$\frac{\partial \varphi_1}{\partial z} = 0 \quad \text{on} \quad z = -1 \tag{5.47}$$

$$\lim_{z\to\infty} \frac{\partial V_1}{\partial z} \to 0 \tag{5.48}$$

The linearised Bernoulli equation requires some extra explanation, the main difficulty are the electric terms, the only terms that survives the linearisation process are the terms $\hat{n}_1\hat{n}_3M_{13}$, $\hat{n}_2\hat{n}_3M_{23}$ and $\hat{n}_3^2M_{33}$. The linearised normal vector is given by:

$$\hat{\mathbf{n}} = \left(\varepsilon\frac{\partial \eta}{\partial x}, \varepsilon\frac{\partial \eta}{\partial y}, -1\right) \tag{5.49}$$

The case for $n_3^2 M_{33}$ is given by:

$$\begin{aligned}
\hat{n}_3\hat{n}_3 M_{33} &= (-1)(-1)\left(E_3^2 - \frac{1}{2}(E_1^2+E_2^2+E_3^2)\right) \\
&= \frac{\partial V}{\partial z}\frac{\partial V}{\partial z} - \frac{1}{2}\left(\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right) \\
&= \left(-1+\varepsilon\frac{\partial V_1}{\partial z}\right)^2 - \frac{1}{2}\left(-1+\varepsilon\frac{\partial V_1}{\partial z}\right)^2 + O(\varepsilon^2) \\
&= -\varepsilon\frac{\partial V_1}{\partial z}.
\end{aligned}$$

To solve the equations write the following:

$$V_1 = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{iklx+ily}A(k,l,z)dkdl \tag{5.50}$$

$$\varphi_1 = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{iklx+ily}B(k,l,z)dkdl \tag{5.51}$$

$$\eta_1 = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{iklx+ily}C(k,l)dkdl \tag{5.52}$$

Inserting (5.51) into (5.41), therefore:

$$\frac{\partial^2 B}{\partial z^2} - (k^2+l^2)B = 0 \tag{5.53}$$

along with $\partial_z\varphi_1(x,y,-1) = 0$, shows that:

$$B = \alpha(k,l)\cosh\mu(z+1) \tag{5.54}$$

where $\mu = \sqrt{k^2+l^2}$. Likewise inserting (5.50) into (5.42) and using (5.48) shows that:

$$A = \beta(k,l)e^{-\mu z} \tag{5.55}$$

since $\mu \geqslant 0$. Using (5.43) shows that:

$$ikUC(k,l) = \alpha\mu\sinh\mu. \tag{5.56}$$

Hence:

$$C(k,l) = -\frac{i}{kU}\mu\sinh\mu \tag{5.57}$$

Equations (5.45) and (5.46) are equivalent to saying:

$$V(t,x,y,\eta(t,x,y)) = 0 \tag{5.58}$$

so expanding this shows that:

$$\begin{aligned} V(t,x,y,\eta(t,x,y)) &= -\eta(t,x,y) + \varepsilon V_1(t,x,y,\eta(t,x,y)) + o(\varepsilon) \\ &= -\varepsilon\eta_1(t,x,y) + \varepsilon V_1(t,x,y,0) + o(\varepsilon) \end{aligned}$$

So equation (5.58) becomes:

$$\eta_1(t,x,y) = V_1(t,x,y,0) \tag{5.59}$$

Inserting (5.50) and (5.52) into (5.59) shows that:

$$C(k,l) = \beta(k,l) \tag{5.60}$$

The expressions for φ_1 and V_1 then become:

$$\varphi_1 = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \beta(k,l)e^{i(kx+ly)}\cosh\mu(z+1)dkdl \tag{5.61}$$

$$\eta_1 = -\frac{i}{4\pi^2 U}\int_{\mathbb{R}^2} \frac{\beta(k,l)e^{i(kx+ly)}\mu\sinh\mu}{k}dkdl \tag{5.62}$$

$$V_1 = -\frac{i}{4\pi^2 U}\int_{\mathbb{R}^2} \frac{\beta(k,l)\mu e^{-\mu z}e^{i(kx+ly)}\sinh\mu}{k}dkdl \tag{5.63}$$

Inserting these equations into equation (5.44) and writing the pressure:

$$p = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{p}e^{i(kx+ly)}dkdl \tag{5.64}$$

shows that:

$$U(ik)\beta\cosh\mu + B\left(-\frac{i}{Uk}\beta\mu\sinh\mu\right) + \hat{p} + E_b\left(\frac{i}{Uk}\mu^2\beta\sinh\mu\right) = -\mu^2\left(-\frac{i}{Uk}\beta\mu\sinh\mu\right). \tag{5.65}$$

This leads to:

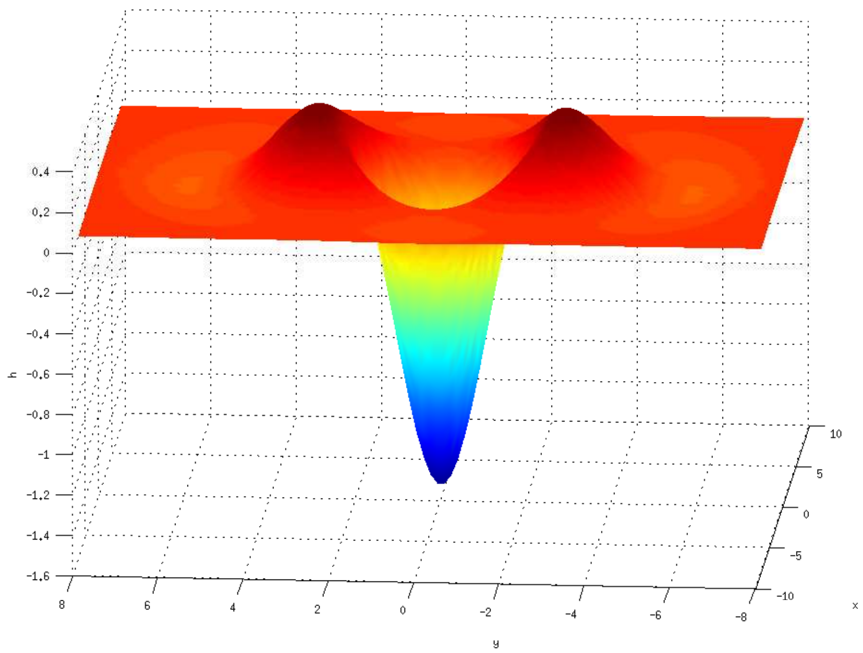
$$\beta\left(ikU\cosh\mu - \frac{Bi}{Uk}\mu\sinh\mu + \frac{iE_b}{Uk}\mu^2\sinh\mu\right) + \hat{p} = \beta\frac{i\mu^3\sinh\mu}{Uk} \tag{5.66}$$

Then $\beta(k,l)$ is given by:

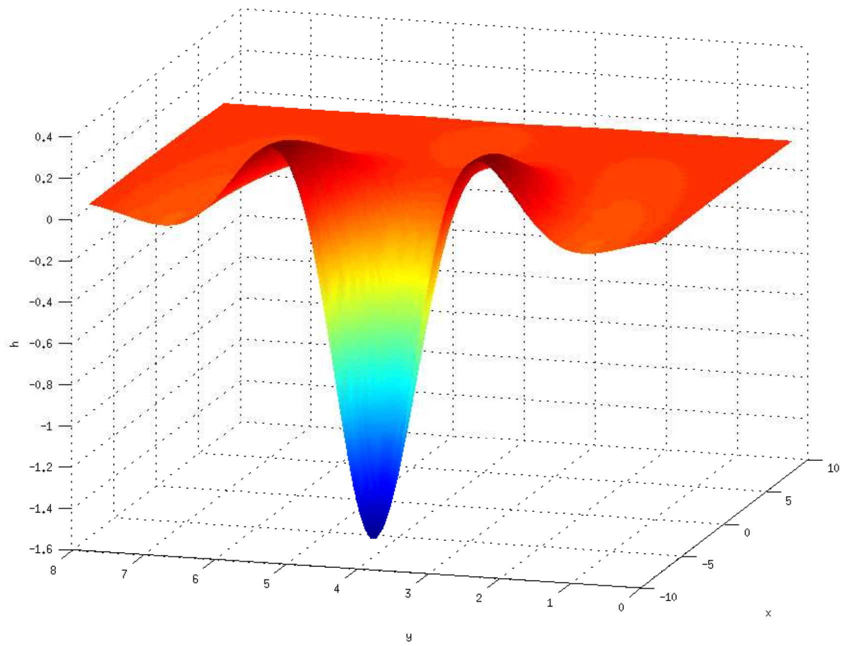
$$\beta = iU\frac{\hat{p}\ \text{sech}\mu}{U^2k^2 - (B\mu - E_b\mu^2 + \mu^3)\tanh\mu} \tag{5.67}$$

Then the free surface becomes:

$$\eta(x,y) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{\mu e^{i(kx+ly)}\hat{p}\tanh\mu}{k^2U^2 - (B\mu - E_b\mu^2 + \mu^3)\tanh\mu}dkdl \tag{5.68}$$



(a) Full Profile



(b) Half Profile

Figure 5.1: Linear Waves Profiles

Equation (5.68) defines the free surface profile. It is assumed here that $U < U_{max}$, so that the denominator of (5.68) is non-zero for k and l. Typical free surface profiles predicted by (5.68) for $E_b = 1.5$, $B = 2$ and $U = 0.5$ in figures (5.1(a)) and (5.1(b)). These profiles of figure (5.1(a)) has decaying oscillatory in the direction and monotonic in the direction perpendicular to the direction of propagation. This is most clearly seen in figure in figure (5.1(b)) where the half of the profile is known.

5.4 Infinite Depth

The difference between the finite depth and the infinite depth is the boundary conditions, these new boundary conditions are given by:

$$\frac{\partial \varphi}{\partial x} \rightarrow U \quad \text{as} \quad z \rightarrow \infty \tag{5.69}$$

$$\frac{\partial \varphi}{\partial z} \rightarrow 0 \quad \text{as} \quad z \rightarrow -\infty \tag{5.70}$$

Using the expansion:

$$\varphi = Ux + \varepsilon\varphi_1 + o(\varepsilon) \tag{5.71}$$

$$\eta = \varepsilon\eta_1 + o(\varepsilon) \tag{5.72}$$

$$V = -E_0 z + \varepsilon V_1 + o(\varepsilon) \tag{5.73}$$

The linearised equations become the following:

$$\nabla^2 \varphi_1 = 0 \tag{5.74}$$

$$\nabla^2 V_1 = 0 \tag{5.75}$$

$$U\frac{\partial \eta_1}{\partial x} = \frac{\partial \varphi_1}{\partial z} \tag{5.76}$$

$$\frac{\partial V_1}{\partial x} = E_0 \frac{\partial \eta_1}{\partial x} \tag{5.77}$$

$$U\frac{\partial \varphi_1}{\partial x} + g\eta_1 + \frac{P}{\rho} + \frac{\epsilon E_0}{\rho}\frac{\partial V_1}{\partial z} = \frac{\sigma}{\rho}\left(\frac{\partial^2 \eta_1}{\partial x^2} + \frac{\partial^2 \eta_1}{\partial y^2}\right) \tag{5.78}$$

$$\frac{\partial \varphi}{\partial x} \rightarrow U \quad \text{as} \quad z \rightarrow -\infty \tag{5.79}$$

$$\frac{\partial \varphi}{\partial z} \rightarrow 0 \quad \text{as} \quad z \rightarrow -\infty \tag{5.80}$$

Upon writing:

$$\varphi_1 = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{\varphi}_1 e^{i(kx+ly)} dkdl \tag{5.81}$$

to obtain:

$$\frac{\partial \hat{\varphi}_1}{\partial z} - \mu^2 \hat{\varphi}_1 = 0 \tag{5.82}$$

which shows that:

$$\hat{\varphi}_1 = Ae^{\mu z} + Be^{-\mu z} \tag{5.83}$$

Which shows that (upon using (5.80)) $B = 0$, applying the same kind of reasoning for V_1 shows that:

$$V_1 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} C(k,l) e^{-\mu z} e^{i(kx+ly)} dkdl \tag{5.84}$$

also writing the free surface as:

$$\eta_1 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} D(k,l) e^{i(kx+ly)} dkdl \tag{5.85}$$

Using (5.77) shows that $C = E_0 D$ and using (5.76) shows that:

$$B = -\frac{iE_0}{kU} \mu A \tag{5.86}$$

The variables are then:

$$\varphi_1 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} A(k,l) e^{\mu z} e^{i(kx+ly)} dkdl \tag{5.87}$$

$$\eta_1 = -\frac{i}{4\pi^2 U} \int_{\mathbb{R}^2} \frac{\mu A}{k} e^{i(kx+ly)} dkdl \tag{5.88}$$

$$V_1 = -\frac{iE_0}{4\pi^2 U} \int_{\mathbb{R}^2} \frac{\mu A}{k} e^{-\mu z} e^{i(kx+ly)} dkdl \tag{5.89}$$

Upon writing the pressure as:

$$P = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{p} e^{i(kx+ly)} dkdl \tag{5.90}$$

Then inserting everything into (5.78) to obtain an expression for A:

$$A = \frac{ikU\hat{p}\rho^{-1}}{k^2U^2 - g\mu + \frac{\epsilon E_0^2}{\rho}\mu^2 - \frac{\sigma}{\rho}\mu^3} \tag{5.91}$$

The expression for the free surface is then:

$$\eta_1 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\mu \hat{p}\rho^{-1}}{k^2U^2 - g\mu + \frac{\epsilon E_0^2}{\rho}\mu^2 - \frac{\sigma}{\rho}\mu^3} e^{i(kx+ly)} dkdl \tag{5.92}$$

Now to make further progress give the pressure as the following:

$$P = \frac{\rho U^2}{2} e^{-\frac{-5g^2(x^2+y^2)}{U^4}} \tag{5.93}$$

Then:

$$\hat{P} = \frac{\rho U^2}{2} e^{-\frac{U^4\mu^2}{20g^2}} \tag{5.94}$$

The free surface equation then becomes:

$$\eta_1 = \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \frac{\mu e^{-\frac{U^4\mu^2}{20g^2}} e^{i(kx+ly)}}{k^2 - \frac{g}{U^2}\mu + \frac{\epsilon E_0^2}{\rho U^2}\mu^2 - \frac{\sigma}{\rho U^2}\mu^3} dkdl \tag{5.95}$$

Make a change of variables in the double integral $\alpha = U^2 g^{-1} k$ and $\beta = U^2 g^{-1} l$ to obtain the expression:

$$\eta_1 = \frac{g}{8\pi^2 U^2} \int_{\mathbb{R}^2} \frac{\nu e^{-\frac{\nu^2}{20}} e^{i(kx+ly)\frac{g}{U^2}}}{\alpha^2 - \nu + \frac{\epsilon E_0^2}{\rho U^2}\nu^2 - \frac{\sigma g}{\rho U^4}\nu^3} d\alpha d\beta \tag{5.96}$$

where $\nu = \sqrt{\alpha^2 + \beta^2}$. Making some final scalings, $\hat{x} = xgU^{-2}$, $\hat{y} = ygU^{-2}$ and $\eta_1 = g\hat{\eta}U^{-2}$ to finally obtain the expression:

$$\hat{\eta}_1(\hat{x}, \hat{y}) = \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \frac{\nu e^{-\frac{\nu^2}{20}} e^{i(\alpha\hat{x}+\beta\hat{y})}}{\alpha^2 - \nu + \mu_1\nu^2 - \mu_2\nu^3} d\alpha d\beta \tag{5.97}$$

where:

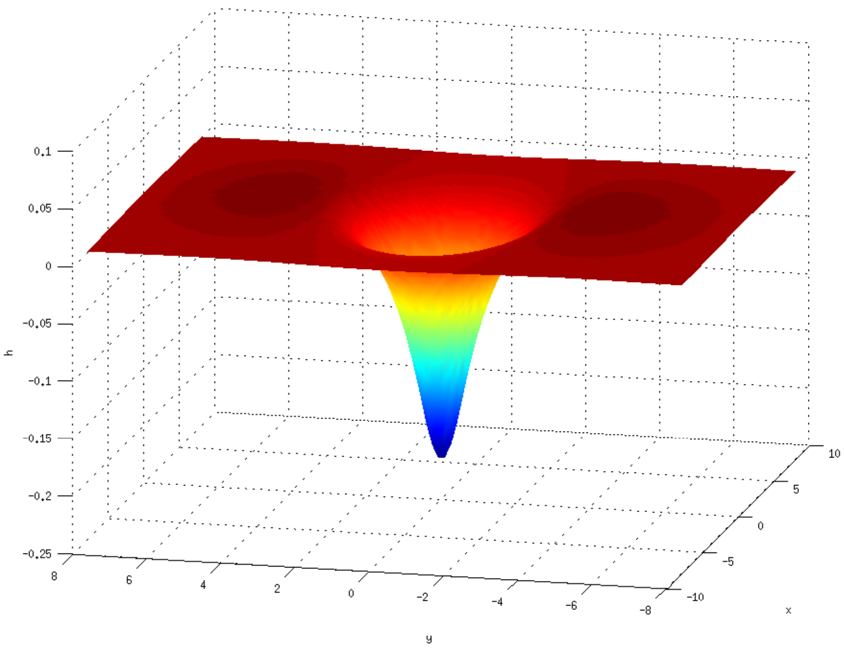
$$\mu_1 = \frac{\epsilon E_0^2}{\rho U^2} \quad \mu_2 = \frac{\sigma g}{\rho U^4} \tag{5.98}$$

Figures (5.2(a)) and (5.2(b)) show the waves for infinite depth for values . As with the one dimensional case for the free surface, the two dimensional linear solution also has a singularity for certain parameter values and it also coincides with the criterion in the one dimensional case. Indeed taking the line $l = 0$ in Fourier space the criterion for blow up is exactly the same, In order to see the blow up phenomena the variable μ_1 was varied whilst the variable μ_2 was kept constant. In figure (5.4), the parameter $\mu_2 = 1$.

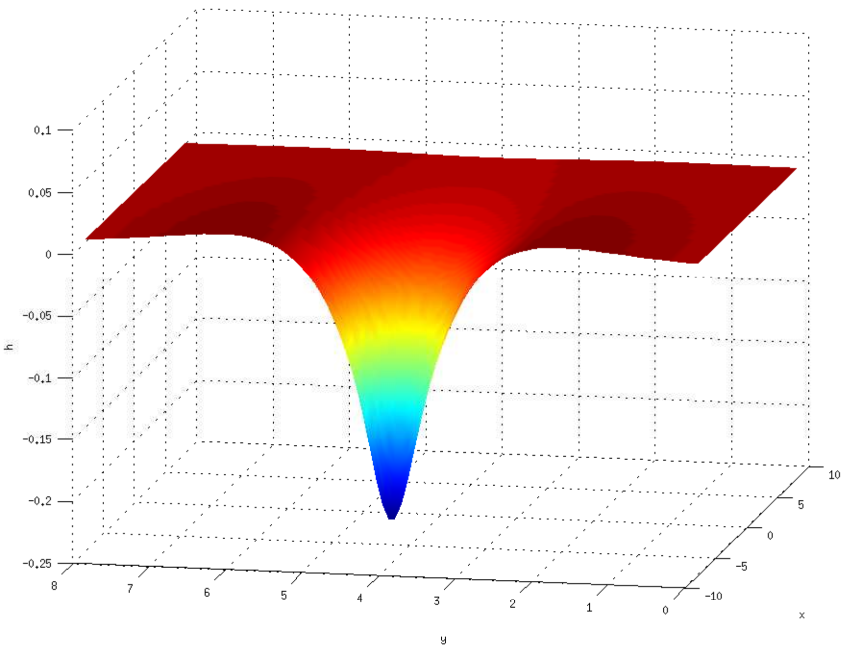
5.5 Weakly Nonlinear Theory I

As was shown in the previous section that there is a blow up of the amplitude of the wave, so this section seeks to overcome that. The way forward is to copy the case in two dimensions to obtain a model which doesn't blow up. The basic nonlinear equations are:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \quad \text{on} \quad -h \leqslant z \leqslant \eta(t, x, y) \tag{5.99}$$



(a) Full Profile



(b) Half Profile

Figure 5.2: Infinite Depth

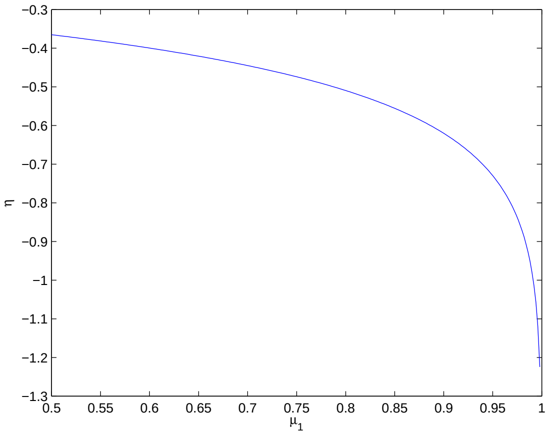


Figure 5.3: Demonstration of blow up phenomena

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{on} \quad z > \eta(t,x,y) \tag{5.100}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \varphi}{\partial z} \quad \text{on} \quad z = \eta(t,x,y) \tag{5.101}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + g\eta + \frac{P}{\rho} - \frac{1}{\rho}\hat{\mathbf{n}}\cdot\Sigma\cdot\hat{\mathbf{n}} = \frac{\sigma}{\rho}\nabla\cdot\hat{\mathbf{n}} + C \tag{5.102}$$

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial z} = 0 \quad \text{on} \quad z = \eta(t,x,y) \tag{5.103}$$

$$-\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial V}{\partial x} + \left(1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right)\frac{\partial V}{\partial y} + \frac{\partial \eta}{\partial y}\frac{\partial V}{\partial z} = 0 \quad \text{on} \quad z = \eta(t,x,y) \tag{5.104}$$

$$\frac{\partial \varphi}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{5.105}$$

$$V \sim -E_0 z \quad \text{as} \quad z \to \infty \tag{5.106}$$

In order to do this, the following scaling is used:

$$x = \lambda\hat{x}, \quad y = \mu\hat{y}, \quad t = \frac{\lambda}{c_0}\hat{t}, \quad \eta = a\hat{\eta} \quad \varphi = \frac{g\lambda a}{c_0}\hat{\varphi} \tag{5.107}$$

$$V = \lambda E_0 \hat{V} \quad z^{(1)} = h\hat{z} \quad z^{(2)} = \lambda \hat{Z}, \tag{5.108}$$

where $c_0 = \sqrt{gh}$, a is the typical amplitude of the waves, and λ is the typical wavelength. Nondimensionasation of the equations. Define the parameters:

$$\alpha = \frac{a}{h}, \quad \beta = \frac{h^2}{\lambda^2}, \quad \gamma = \frac{\lambda^2}{\mu^2} \tag{5.109}$$

The main governing equations scale as:

$$\frac{1}{\beta}\frac{\partial^2 \hat{\varphi}}{\partial z^2} + \gamma \frac{\partial^2 \hat{\varphi}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{\varphi}}{\partial \hat{x}^2} = 0 \tag{5.110}$$

$$\frac{\partial^2 \hat{V}}{\partial \hat{z}^2} + \gamma \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{x}^2} = 0 \tag{5.111}$$

To examine the scaling of the boundary conditions,

$$\frac{\partial \varphi}{\partial t} \mapsto ga\frac{\partial \hat{\varphi}}{\partial \hat{t}}, \quad \frac{\partial \varphi}{\partial x} \mapsto \sqrt{\alpha g a}\frac{\partial \hat{\varphi}}{\partial \hat{x}}, \quad \frac{\partial \varphi}{\partial y} \mapsto \sqrt{\alpha \gamma g a}\frac{\partial \hat{\varphi}}{\partial \hat{y}}, \quad \frac{\partial \varphi}{\partial z} \mapsto \sqrt{ga}\sqrt{\frac{\alpha}{\beta}}\frac{\partial \hat{\varphi}}{\partial \hat{z}}. \tag{5.112}$$

This method is the classical method of deriving the weakly nonlinear equation for the free surface. The following equations are used:

$$\frac{1}{\beta}\frac{\partial^2 \hat{\varphi}}{\partial z^2} + \gamma \frac{\partial^2 \hat{\varphi}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{\varphi}}{\partial \hat{x}^2} = 0 \tag{5.113}$$

$$\frac{\partial^2 \hat{V}}{\partial \hat{z}^2} + \gamma \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{x}^2} = 0 \tag{5.114}$$

The relevant part of the Faraday tensor is given by:

$$\Sigma_{33} = \frac{E_b}{2}\left[\left(\frac{\partial \hat{V}}{\partial \hat{z}}\right)^2 - \left(\frac{\partial \hat{V}}{\partial x}\right)^2 - \gamma\left(\frac{\partial \hat{V}}{\partial y}\right)^2\right] \tag{5.115}$$

The surface tension term scales as:

$$\frac{B\beta\left[\partial_{\hat{x}}^2\hat{\eta}(1+\alpha^2\beta\gamma(\partial_{\hat{y}}\hat{\eta})^2) + \gamma\partial_{\hat{y}}^2\hat{\eta}(1+\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2) - 2\alpha^2\beta\gamma\partial_{\hat{x}}\hat{\eta}\partial_{\hat{y}}\hat{\eta}\partial_{\hat{x}}\partial_{\hat{y}}\hat{\eta}\right]}{(1+\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2+\alpha^2\beta\gamma(\partial_{\hat{y}}\hat{\eta})^2)^{\frac{3}{2}}} \tag{5.116}$$

The free surface equation becomes

$$\frac{1}{\beta}\frac{\partial \hat{\varphi}}{\partial \hat{z}} = \frac{\partial \hat{\eta}}{\partial \hat{t}} + \alpha\frac{\partial \hat{\varphi}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{x}} + \alpha\gamma\frac{\partial \hat{\varphi}}{\partial \hat{y}}\frac{\partial \hat{\eta}}{\partial \hat{y}} \quad \text{on} \quad z = \alpha\hat{\eta} \tag{5.117}$$

The equation for the electric field is given by:

$$\frac{\partial \hat{V}}{\partial \hat{x}} + \alpha\sqrt{\beta}\frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{y}} = 0 \quad \text{on} \;\; z = \alpha\hat{\eta}. \tag{5.118}$$

The Bernoulli equation scales as:

$$
\begin{aligned}
\beta B\hat{\nabla}\cdot\hat{\mathbf{n}} - \frac{E_b}{2\alpha} &= \frac{\partial\varphi}{\partial t} + \eta + p + \frac{E_b}{\alpha}\hat{\mathbf{n}}\cdot\sigma\cdot\hat{\mathbf{n}} + \\
&+ \frac{1}{2}\left[\alpha\left(\frac{\partial\varphi}{\partial\hat{x}}\right)^2 + \alpha\gamma\left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] \qquad (5.119)
\end{aligned}
$$

Now set $\alpha = \beta = \gamma = \varepsilon$ and have the following co-ordinate transformation:

$$T = \varepsilon\hat{t} \qquad (5.120)$$

$$X = \hat{x} - \hat{t} \qquad (5.121)$$

The equations then become (upon dropping hats):

$$\frac{1}{\varepsilon}\frac{\partial^2\varphi}{\partial z^2} + \varepsilon\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial x^2} = 0 \qquad (5.122)$$

$$\frac{\partial^2 V}{\partial z^2} + \varepsilon\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} = 0 \qquad (5.123)$$

$$\Sigma_{33} = \frac{\epsilon E_0^2}{2}\left[\left(\frac{\partial V}{\partial z}\right)^2 - \left(\frac{\partial V}{\partial x}\right)^2 - \varepsilon\left(\frac{\partial V}{\partial y}\right)^2\right] \qquad (5.124)$$

$$\frac{B\varepsilon\left[\partial_X^2\hat{\eta}(1+\varepsilon^3(\partial_y\eta)^2) + \varepsilon\partial_y^2\eta(1+\varepsilon^3(\partial_X\eta)^2) - 2\varepsilon^3\partial_X\eta\partial_y\eta\partial_X\partial_y\eta\right]}{(1+\varepsilon^3(\partial_X\eta)^2+\varepsilon^3(\partial_y\eta)^2)^{\frac{3}{2}}} \qquad (5.125)$$

$$\frac{1}{\varepsilon}\frac{\partial\varphi}{\partial z} = \varepsilon\frac{\partial\eta}{\partial T} - \frac{\partial\eta}{\partial X} + \varepsilon\frac{\partial\varphi}{\partial X}\frac{\partial\eta}{\partial X} + \varepsilon^2\frac{\partial\varphi}{\partial y}\frac{\partial\eta}{\partial y} \quad \text{on} \quad z = \varepsilon\eta \qquad (5.126)$$

$$\frac{\partial V}{\partial X} + \varepsilon^{\frac{3}{2}}\frac{\partial\eta}{\partial X}\frac{\partial V}{\partial z} = 0 \quad \text{on} \quad z = \varepsilon\hat{\eta}. \qquad (5.127)$$

$$
\begin{aligned}
\varepsilon B\hat{\nabla}\cdot\hat{\mathbf{n}} - \frac{E_b}{2\varepsilon} &= \varepsilon\frac{\partial\varphi}{\partial T} - \frac{\partial\varphi}{\partial X} + \eta + p + \frac{E_b}{\varepsilon}\hat{\mathbf{n}}\cdot\sigma\cdot\hat{\mathbf{n}} + \\
&+ \frac{1}{2}\left[\varepsilon\left(\frac{\partial\varphi}{\partial X}\right)^2 + \varepsilon^2\left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] \qquad (5.128)
\end{aligned}
$$

Using the Akylas method, the solution for φ can be written down from previous calculations which is:

$$\varphi = \varphi_0 - \varepsilon\frac{(z+1)^2}{2!}\frac{\partial^2\varphi_0}{\partial X^2} + \varepsilon^2\left[\frac{(z+1)^4}{4!}\frac{\partial^4\varphi_0}{\partial X^4} - \frac{(z+1)^2}{2!}\frac{\partial^2\varphi_0}{\partial y^2}\right] + o(\varepsilon^2) \qquad (5.129)$$

The $O(1)$ terms in the Bernoulli equation show that:

$$\eta_0 = \frac{\partial \varphi_0}{\partial X} \tag{5.130}$$

The $O(\varepsilon)$ terms of Bernoulli's equation are somewhat complicated, the surface tension terms:

$$\varepsilon B \hat{\nabla} \cdot \hat{\mathbf{n}} = B\varepsilon \frac{\partial^2 \eta_0}{\partial X^2} + B\varepsilon^2 \frac{\partial^2 \eta_0}{\partial y^2} + o(\varepsilon^2)$$

In order to include pressure in the $O(\varepsilon)$ term scale it according to $p = \varepsilon P$, the relevant electric term is:

$$n_3^2 \Sigma_{33} = \frac{E_b}{2} \left[\left(-1 + \varepsilon^{\frac{3}{2}} \frac{\partial V_1}{\partial z} \right)^2 \right]$$

In order to include this term, scale the electric Bond number as $E_b \sqrt{\varepsilon} \hat{E}_b$ and the terms in question becomes:

$$-\hat{E}_b \frac{\partial V}{\partial z}$$

So the $O(\varepsilon)$ term of the Bernoulli equation is:

$$\eta_1 = B\partial_X^2 - \partial_T \varphi_0 + \frac{1}{2}\partial_X^3 \varphi_0 - P + \hat{E}_b \partial_z V - \frac{1}{2}(\partial_x \varphi_0)^2 \tag{5.131}$$

The $O(1)$ part of the free surface equation just gives $\partial_X \eta_0 = \partial_X^2 \varphi_0$ which is a repeat of previous information. The $O(\varepsilon)$ part of the equation is:

$$\frac{1}{6}\partial_X^4 \varphi_0 - \partial_y^2 \varphi_0 - \frac{1}{2}\eta_0 \partial_X^2 \varphi_0 = \partial_T \eta_0 - \partial_X \eta_1 + \partial_X \varphi_0 \partial_X \eta_0 \tag{5.132}$$

Inserting the expression for η_1 shows that:

$$\frac{1}{6}\partial_X^4 \varphi_0 - \partial_y^2 \varphi_0 = 2\partial_T \partial_X \varphi_0 + \frac{1}{2}\left(\frac{1}{3} - B\right)\partial_X^4 \varphi_0 + \partial_X^2 P + \frac{3}{2}\partial_X^2(\eta_0^2) + \hat{E}_b \partial_X^2 \partial_Z V = 0 \tag{5.133}$$

In terms of η_0 the equation is:

$$\frac{\partial}{\partial X}\left[\frac{\partial \eta_0}{\partial T} + \frac{1}{2}\left(\frac{1}{3} - B\right)\frac{\partial^3 \eta_0}{\partial X^3} + \frac{\partial P}{\partial X} + \frac{3}{2}\eta_0 \frac{\partial \eta_0}{\partial X} + \frac{\hat{E}_b}{2}\frac{\partial^2 V}{\partial X \partial Z}\right] + \frac{1}{2}\frac{\partial^2 \eta_0}{\partial y^2} = 0 \tag{5.134}$$

Using the standard Hilbert transform identities:

$$\partial_Z V = \mathscr{H}(\partial_X V) \tag{5.135}$$

The boundary condition for the electric field shows that:

$$\frac{\partial V}{\partial z} = \frac{\partial \eta_0}{\partial X}$$

So the final equation can be written as:

$$\frac{\partial}{\partial X}\left[\frac{\partial \eta_0}{\partial T} + \frac{1}{2}\left(\frac{1}{3} - B\right)\frac{\partial^3 \eta_0}{\partial X^3} + \frac{\partial P}{\partial X} + \frac{3}{2}\eta_0\frac{\partial \eta_0}{\partial X} + \frac{\hat{E}_b}{2}\mathscr{H}\left(\frac{\partial^2 \eta_0}{\partial X^2}\right)\right] + \frac{1}{2}\frac{\partial^2 \eta_0}{\partial y^2} = 0 \tag{5.136}$$

In terms of dimensional variables the equation is:

$$\frac{\partial}{\partial x}\left[\frac{1}{c_0}\frac{\partial \eta}{\partial t} + \frac{3}{2h}\eta\frac{\partial \eta}{\partial x} + \frac{h^2}{2}\left(\frac{1}{3} - B\right)\frac{\partial^3 \eta}{\partial x^3} + \frac{1}{\rho g}\frac{\partial p}{\partial x} + \frac{E_b}{2}\mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right)\right] + + \frac{1}{2}\frac{\partial^2 \eta}{\partial y^2} = 0 \tag{5.137}$$

When $E_b = 0$ this equation is the classical KP equation. Here electric fields are included (E_b different from zero) and (5.137) can be described as a Benjamin KP equation. This equation was derived for interfacial waves in [27]. The next stage is to look for a travelling wave solution $\eta(t, x, y) = f(x - ct, y) = f(X, y)$, the equation becomes:

$$\frac{\partial}{\partial X}\left[(1 - F)\frac{\partial f}{\partial X} + \frac{3}{2h}f\frac{\partial f}{\partial X} + \frac{h^2}{2}\left(\frac{1}{3} - B\right)\frac{\partial^3 \eta}{\partial X^3} + \frac{1}{\rho g}\frac{\partial p}{\partial X} + \frac{E_b}{2}\mathscr{H}\left(\frac{\partial^2 \eta}{\partial X^2}\right)\right] + + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} = 0 \tag{5.138}$$

Where $F = c/c_0$ is the Froude number. In order to get an idea of what this wave looks like, look at the linear part of the above equation, to do this write:

$$f(X, y) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\hat{f}(k, l)e^{i(kx+ly)}dkdl, \quad p(X, y) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\hat{p}(k, l)e^{i(kx+ly)}dkdl \tag{5.139}$$

where $\hat{p}$ and p are known to find $\hat{f}$, carrying out the computation shows that:

$$\hat{f} = \frac{k^2\hat{p}}{\rho g}\left[-k^2(1 - F) + \frac{h^2}{2}\left(\frac{1}{3} - B\right)k^4 + \frac{E_b}{2}k^3\text{sgn}(k) - \frac{l^2}{2}\right]^{-1} \tag{5.140}$$

5.6 Linear Theory II- Dispersion Relation

It is possible to directly compute the linear part of (5.137). The method of derivation will be the one in [4], the first stage will be to calculate the dispersion relation and then from there move to the equation itself. To begin, the equations are:

$$\nabla^2 \varphi_1 = 0 \tag{5.141}$$

$$\nabla^2 V_1 = 0 \tag{5.142}$$

$$\frac{\partial \eta_1}{\partial t} = \frac{\partial \varphi_1}{\partial z} \tag{5.143}$$

$$\frac{\partial V_1}{\partial x} = E_0 \frac{\partial \eta_1}{\partial x} \tag{5.144}$$

$$\frac{\partial \varphi_1}{\partial t} + g\eta_1 + \frac{P}{\rho} + \frac{\epsilon E_0}{\rho}\frac{\partial V_1}{\partial z} = \frac{\sigma}{\rho}\left(\frac{\partial^2 \eta_1}{\partial x^2} + \frac{\partial^2 \eta_1}{\partial y^2}\right) \tag{5.145}$$

$$\frac{\partial \varphi_1}{\partial z} = 0 \quad y = -h \tag{5.146}$$

Writing the expressions for φ_1, η_1 and V_1 as:

$$\varphi_1 = \hat{\varphi} e^{i(kx+ly)} \tag{5.147}$$

$$V_1 = \hat{V} e^{i(kx+ly)} \tag{5.148}$$

$$\eta_1 = \hat{\eta} e^{i(kx+ly)} \tag{5.149}$$

Then the solutions are:

$$\varphi_1 = \alpha \sinh \mu(z+h) e^{i(kx+ly)} \tag{5.150}$$

$$V_1 = \beta e^{-\mu z} e^{i(kx+ly)} \tag{5.151}$$

The boundary conditions show that:

$$\beta = E_0 \hat{\eta} \tag{5.152}$$

$$\frac{\partial^2 \hat{\eta}}{\partial t^2} = \mu \sinh \mu \frac{\partial \alpha}{\partial t} \tag{5.153}$$

Inserting everything into equation (5.145) shows that:

$$\frac{\coth h\mu}{\mu}\frac{\partial^2 \hat{\eta}}{\partial t^2} + g\eta_1 - \frac{\epsilon E_0^2}{\rho}\mu\hat{\eta} + \frac{\sigma}{\rho}\mu^2\hat{\eta} = 0 \tag{5.154}$$

Upon writing $\hat{\eta} = \eta_0 e^{-i\omega t}$, the dispersion relation is obtained:

$$\omega^2 = \mu\left(g - \frac{\epsilon E_0^2}{\rho}\mu + \frac{\sigma}{\rho}\mu^2\right)\tanh h\mu \tag{5.155}$$

Upon writing $\omega = c\mu$

$$c^2 = \left(g - \frac{\epsilon E_0^2}{\rho}\mu + \frac{\sigma}{\rho}\mu^2\right)\frac{\tanh h\mu}{\mu} \tag{5.156}$$

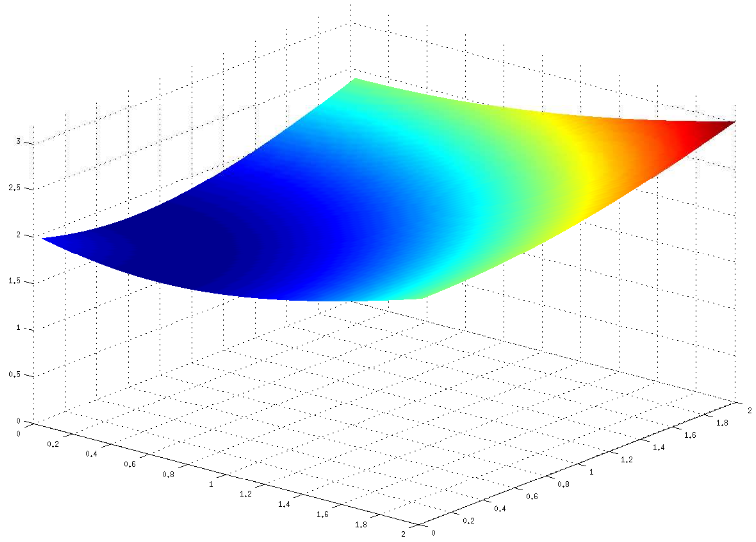


Figure 5.4: Dispersion Relation

Figure 5.6 the pictorial representation of the dispersion relation, equation (5.156) for $\epsilon_d E_0^2/\rho g = 1.5$, $\sigma/\rho g = 2$ and $h = 1$. It shows that there is a minimum and it is above zero showing the existence of linear waves. To obtain a linear equation, expand equation (5.155):

$$\begin{aligned}
\omega^2 &= \mu\left(g - \frac{\epsilon E_0^2}{\rho}\mu + \frac{\sigma}{\rho}\mu^2\right)\tanh h\mu \\
&\approx \mu\left(g - \frac{\epsilon E_0^2}{\rho}\mu + \frac{\sigma}{\rho}\mu^2\right)\left(h\mu - \frac{(h\mu)^3}{3} + \cdots\right) \\
&= gh\mu^2\left(1 - \frac{\epsilon E_0^2}{\rho g}\mu + \frac{\sigma}{\rho g}\mu^2\right)\left(1 - \frac{(h\mu)^2}{3} + \cdots\right)
\end{aligned}$$

Upon taking square roots:

$$\begin{aligned}
\omega &= \sqrt{gh}\mu\left(1 - \frac{\epsilon E_0^2}{\rho g}\mu + \frac{\sigma}{\rho g}\mu^2\right)^{\frac{1}{2}}\left(1 - \frac{(h\mu)^2}{3} + \cdots\right)^{\frac{1}{2}} \\
&= c_0 k\left(1 + \frac{l^2}{k^2}\right)^{\frac{1}{2}}\left(1 - \frac{\epsilon E_0^2}{\rho g}\mu + \frac{\sigma}{\rho g}\mu^2\right)^{\frac{1}{2}}\left(1 - \frac{(hk)^2}{6} + O(l^2)\right) \\
&= c_0\left(k + \frac{l^2}{2k}\right)\left(1 - \frac{\epsilon E_0^2}{\rho g}\mu + \frac{\sigma}{\rho g}\mu^2\right)^{\frac{1}{2}}\left(1 - \frac{(hk)^2}{6} + O(l^2)\right)
\end{aligned}$$

Upon re-arranging:

$$\begin{aligned}
\frac{1}{c_0}\omega k &= \left(k^2+\frac{l^2}{2}\right)\left(1-\frac{\epsilon E_0^2}{\rho g}\mu+\frac{\sigma}{\rho g}\mu^2\right)^{\frac{1}{2}}\left(1-\frac{(hk)^2}{6}+O(l^2)\right)\\
&= \left(k^2+\frac{l^2}{2}\right)\left(1-\frac{\epsilon E_0^2}{\rho g}\mu+\frac{\sigma}{\rho g}k^2\right)\left(1-\frac{(hk)^2}{6}+O(l^2)\right)\\
&= \left(k^2+\frac{l^2}{2}\right)\left(1-\frac{\epsilon E_0^2}{\rho g}|k|\sqrt{1-\frac{l^2}{k^2}}+\frac{\sigma}{\rho g}k^2\right)\left(1-\frac{(hk)^2}{6}+O(l^2)\right)\\
&= \left(k^2+\frac{l^2}{2}\right)\left(1-\frac{\epsilon E_0^2}{\rho g}|k|+\frac{\sigma}{\rho g}k^2\right)\left(1-\frac{(hk)^2}{6}+O(l^2)\right)\\
&= k^2+\frac{l^2}{2}-\frac{\epsilon E_0^2}{2\rho g}|k|k^2+\frac{\sigma}{2\rho g}k^4-\frac{h^2k^4}{6}\\
&= k^2-\frac{\epsilon E_0^2}{2\rho g}|k|k^2+\frac{h^2}{2}\left(\frac{\sigma}{\rho g h^2}-\frac{1}{3}\right)k^4+\frac{l^2}{2}
\end{aligned}$$

The linear PDE associated with this dispersion relation can be written down using:

$$\omega\to i\partial_t \quad k\to -i\partial_x \quad l\to -i\partial_y \quad |k|\to -\partial_x\mathscr{H} \tag{5.157}$$

The PDE is:

$$\frac{\partial}{\partial x}\left(\frac{1}{c_0}\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}-\frac{\epsilon E_0^2}{2\rho g}\mathscr{H}\left(\frac{\partial^2\eta}{\partial x^2}\right)+\frac{h^2}{2}\left(\frac{\sigma}{\rho g h^2}-\frac{1}{3}\right)\frac{\partial^3\eta}{\partial x^3}\right)+\frac{1}{2}\frac{\partial^2\eta}{\partial y^2}=0 \tag{5.158}$$

Equation (5.158) predicts the linear terms of (5.137). It has been shown that it is possible to extract the linear part of the governing equation from the dispersion relation given a particular assumption on the aspect ratio from the transverse and normal directions. This was only done for the canonical KP scaling and not carried over to the 5th order KP equation.

5.7 Weakly Nonlinear Theory II

This section deals with the weakly nonlinear theory around $B=1/3$, for this a different scaling is required:

$$\alpha=\varepsilon^2,\quad \beta=\varepsilon \quad \gamma=\varepsilon^2 \tag{5.159}$$

and expand the Bond number as:

$$B=\frac{1}{3}+\varepsilon B_1+o(\varepsilon) \tag{5.160}$$

Make the transformation:

$$T = \varepsilon^2 \hat{t} \tag{5.161}$$

$$X = \hat{x} - \hat{t}. \tag{5.162}$$

The Bernoulli equation scales very much like the 2D case with the same two parameters B and E_b. The equations become(dropping the hats) the following:

$$\frac{\partial^2 \varphi}{\partial x^2} + \varepsilon^2 \frac{\partial^2 \varphi}{\partial y^2} + \frac{1}{\varepsilon}\frac{\partial^2 \varphi}{\partial z^2} = 0 \quad \text{on} \quad -1 \leqslant z \leqslant \varepsilon^2 \eta(t,x,y) \tag{5.163}$$

$$\frac{\partial^2 V}{\partial x^2} + \varepsilon^2 \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{on} \quad z > \varepsilon^2 \hat{\eta}(t,x,y) \tag{5.164}$$

$$\frac{\partial \eta}{\partial t} + \varepsilon^2 \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} + \varepsilon^4 \frac{\partial \varphi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{1}{\varepsilon}\frac{\partial \varphi}{\partial z} \quad \text{on} \quad z = \varepsilon^2 \eta(t,x,y) \tag{5.165}$$

$$\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V}{\partial z} = 0 \quad \text{on} \quad z = \varepsilon^2 \eta(t,x,y) \tag{5.166}$$

The Bernoulli equation is similar to the 2D case and will be directly generalised to 3D.

$$\begin{aligned} \varepsilon B \hat{\nabla} \cdot \hat{\mathbf{n}} + \frac{E_b}{2\varepsilon^2} &= \varepsilon^2 \frac{\partial \varphi}{\partial T} - \frac{\partial \varphi}{\partial X} + \eta + \varepsilon^n P + \frac{E_b}{\varepsilon^2} \hat{\mathbf{n}} \cdot \sigma \cdot \hat{\mathbf{n}} + \\ &+ \frac{\varepsilon}{2}\left[\varepsilon \left(\frac{\partial \varphi}{\partial X}\right)^2 + \varepsilon^3 \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] \end{aligned} \tag{5.167}$$

To expand the LHS a little bit more:

$$\begin{aligned} \varepsilon B \hat{\nabla} \cdot \hat{\mathbf{n}} &= \varepsilon B (\partial_X^2 \eta + \varepsilon^2 \partial_y^2 \eta) \\ &= \varepsilon \left(\frac{1}{3} + \varepsilon B_1\right) \partial_X^2 \eta + \varepsilon^3 \left(\frac{1}{3} + \varepsilon B_1\right) \partial_y^2 \eta \\ &= \frac{1}{3} \partial_X^2 \eta_0 \varepsilon + \left(\frac{1}{3}\partial_X^2 \eta_1 + B_1 \partial_X^2 \eta_0\right) \varepsilon^2 + o(\varepsilon^2) \end{aligned}$$

To begin solving the equations (5.163)- (5.168),

$$\frac{\partial^2 \varphi_0}{\partial X^2} + \varepsilon \frac{\partial^2 \varphi_1}{\partial X^2} + \varepsilon^2 \frac{\partial^2 \varphi_2}{\partial X^2} + \varepsilon^2 \left(\frac{\partial^2 \varphi_0}{\partial y^2} + \varepsilon \frac{\partial^2 \varphi_1}{\partial y^2}\right) + \frac{1}{\varepsilon}\left(\frac{\partial^2 \varphi_0}{\partial z^2} + \varepsilon \frac{\partial^2 \varphi_1}{\partial z^2} + \varepsilon^2 \frac{\partial^2 \varphi_2}{\partial z^2}\right) = 0, \tag{5.168}$$

the $O(\varepsilon^{-1})$ equation of (5.168) is:

$$\frac{\partial^2 \varphi_0}{\partial z^2} = 0, \quad \frac{\partial \varphi_0}{\partial z}(T, X, y, -1) = 0 \tag{5.169}$$

Which shows that $\varphi_0 = \varphi_0(T, X, y)$, that φ_0 is independent of z. The $O(1)$ equation is then:

$$\frac{\partial^2 \varphi_0}{\partial X^2} + \frac{\partial^2 \varphi_1}{\partial z^2} = 0, \quad \frac{\partial \varphi_1}{\partial z}(T, X, y, -1) = 0 \tag{5.170}$$

The solution to this equation is:

$$\varphi_1 = -\frac{(z+1)^2}{2!}\frac{\partial^2 \varphi_0}{\partial X^2} + A(T, X, y), \tag{5.171}$$

where A is a function of integration, the $O(\varepsilon)$ equation is:

$$\frac{\partial^2 \varphi_1}{\partial X^2} + \frac{\partial^2 \varphi_2}{\partial z^2} = 0, \quad \frac{\partial \varphi_2}{\partial z}(T, X, y, -1) = 0 \tag{5.172}$$

The solution of this equation is:

$$\varphi_2 = \frac{(z+1)^4}{4!}\frac{\partial^4 \varphi_0}{\partial X^4} - \frac{(z+1)^2}{2!}\frac{\partial^2 A}{\partial X^2} + C(T, X, y) \tag{5.173}$$

The solution for φ is then:

$$\begin{aligned} \varphi &= \varphi_0 + \varepsilon\left(-\frac{(z+1)^2}{2!}\frac{\partial^2 \varphi_0}{\partial X^2} + A\right) + \\ &\quad + \varepsilon^2\left(\frac{(z+1)^4}{4!}\frac{\partial^4 \varphi_0}{\partial X^4} - \frac{(z+1)^2}{2!}\frac{\partial^2 A}{\partial X^2} + C\right) \end{aligned} \tag{5.174}$$

To $O(\varepsilon)$ the Bernoulli equation is:

$$\frac{1}{3}\frac{\partial^2 \eta_0}{\partial X^2} = -\frac{\partial \varphi_1}{\partial X} + \eta_1 \tag{5.175}$$

Inserting in the expression for φ_1 and re-arranging shows that:

$$\eta_1 = \frac{1}{3}\frac{\partial^2 \eta_0}{\partial X^2} - \frac{1}{2}\partial_X^3 \varphi_0 + \partial_X A \tag{5.176}$$

Working to $O(\varepsilon^2)$ the free surface boundary condition becomes:

$$\partial_T \eta_0 - \partial_X \eta_2 + \partial_X \varphi_0 \partial_X \eta_0 = -\eta_0 \partial_X^2 \varphi_0 + \partial_z \varphi_3 \tag{5.177}$$

To find φ_3, go to the next term in the governing equation to obtain:

$$\frac{\partial^2 \varphi_2}{\partial X^2} + \frac{\partial^2 \varphi_0}{\partial y^2} + \frac{\partial^2 \varphi_3}{\partial z^2} = 0 \tag{5.178}$$

Inserting the relevant expressions for φ_1 and φ_2 shows that:

$$\frac{\partial \varphi_3}{\partial z} = -\frac{1}{5!}\frac{\partial^6 \varphi_0}{\partial X^6} + \frac{1}{6}\frac{\partial^4 A}{\partial X^4} - \frac{\partial^2 C}{\partial X^2} - \frac{\partial^2 \varphi_0}{\partial y^2} \tag{5.179}$$

In order to include the electrical term in the $O(\varepsilon^2)$ part of the Bernoulli equation, scale the following $E_b = \varepsilon^{3/2}\hat{E}_b$, the Bernoulli equation is then:

$$B_1\partial_X^2\eta_0 + \frac{1}{3}\partial_X^2\eta_1 = \partial_T\varphi_0 - \partial_X\varphi_2 + \eta_2 + P + \frac{1}{2}(\partial_X\varphi_0)^2 - \hat{E}_b\partial_Z V_1 \tag{5.180}$$

The free surface equation is then reduced to:

$$\begin{aligned}\partial_T\eta_0 - \partial_X\eta_2 + \partial_X\varphi_0\partial_X\eta_0 = &-\frac{1}{5!}\partial_X^6\varphi_0 - \eta_0\partial_X^2\varphi_0 - \\ &-\frac{1}{6}\partial_X^4 A + \partial_X^2 C - \partial_y^2\varphi_0\end{aligned} \tag{5.181}$$

Following Akylas, set $A = C = 0$ and inserting everything into (5.181) shows that:

$$\begin{aligned}2\partial_T\eta_0 - B_1\partial_X^3\eta_0 + 3\eta_0\partial_X\eta_0 + \frac{1}{45}\partial_X^5\eta_0 + \\ + \partial_y^2\varphi_0 + \partial_X P + \hat{E}_b\partial_X\partial_z V_1 = 0\end{aligned} \tag{5.182}$$

The next part to examine the term $\partial_X\partial_z V_1$, the way to go about this is to compute the Green's function for V_1, and the problem to solve is the following equation:

$$\frac{\partial^2 V_1}{\partial X^2} + \frac{\partial^2 V_1}{\partial z^2} = 0 \tag{5.183}$$

with the boundary condition $V_1 = \eta_1$ on the plane $z = 0$. The method of solution is via Green's function, the first step is to compute the Green's function for the half space and the way to do this to to use the method of images with the Green's function in free space. The free space Green's function for the Laplace operator is:

$$g = \frac{1}{2\pi}\log r \tag{5.184}$$

In order to obtain the Green's function for the half space, introduce a source at $(X', y', -z')$ of strength $-\delta(X - X')\delta(z + z')$ so the equation for the Green's function is given by:

$$\frac{\partial^2 g}{\partial X^2} + \frac{\partial^2 g}{\partial z^2} = \delta(X - X')\delta(z - z') - \delta(x - x')\delta(z + z') \tag{5.185}$$

This can be thought of as breaking up two expressions $g = g_1 + g_2$ where:

$$\begin{aligned}\frac{\partial^2 g_1}{\partial X^2} + \frac{\partial^2 g_1}{\partial z^2} &= \delta(X - X')\delta(z - z') \\ \frac{\partial^2 g_2}{\partial X^2} + \frac{\partial^2 g_2}{\partial z^2} &= -\delta(X - X')\delta(z + z')\end{aligned} \tag{5.186}$$

Which the solutions can just be written down simply as:

$$g_1 = \frac{1}{4\pi}\log r_- \quad g_2 = -\frac{1}{2\pi}\log r_+ \tag{5.187}$$

where:

$$\begin{aligned} r_- &= \sqrt{(X-X')^2+(z-z')^2} \\ r_+ &= \sqrt{(X-X')^2+(z+z')^2} \end{aligned} \tag{5.188}$$

So the Green's function for the halfspace is given as:

$$g = \frac{1}{2\pi}\log r_- - \frac{1}{2\pi}\log r_+ \tag{5.189}$$

The solution is obtained via Green's second formula:

$$\int_D g\nabla^2 u - u\nabla^2 g d\tau = \int_{\partial D} g(\hat{\mathbf{n}}\cdot\nabla)u - u(\hat{\mathbf{n}}\cdot\nabla)g d\Sigma \tag{5.190}$$

The solution is:

$$V_1(x,y,z) = \int_{\mathbb{R}} \eta_0 \partial_{z'} g\big|_{z'=0} dx' \tag{5.191}$$

So the term in the 2D KdV equation becomes:

$$\partial_X\partial_z V_1\Big|_{z=0} = \int_{\mathbb{R}} \partial_X\left(\eta_0\partial_z\partial_{z'}g\right)\big|_{z=z'=0} dx' \tag{5.192}$$

Where:

$$\partial_z\partial_{z'}g\Big|_{z=z'=0} = -\frac{1}{\pi}\frac{1}{(X-X')^2} \tag{5.193}$$

Which makes V_1 the following:

$$\partial_z V_1\Big|_{z=0} = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\partial_X\eta_0}{X-X'}dX' = \mathscr{H}(\partial_X\eta_0) \tag{5.194}$$

Although it is possible to use the previous Hilbert transform analysis to get the electric term, the method using Green's function shows where the Hilbert transform arises naturally. The equation then becomes:

$$\begin{aligned} 2\partial_T\eta_0 - B_1\partial_X^3\eta_0 + 3\eta_0\partial_X\eta_0 + \frac{1}{45}\partial_X^5\eta_0 + \\ + \partial_y^2\varphi_0 + \partial_X P + \hat{E}_b\mathscr{H}(\partial_X^2\eta_0) = 0 \end{aligned} \tag{5.195}$$

To turn the φ_0 into an η_0, differentiate throughout to obtain:

$$\begin{aligned} \frac{\partial}{\partial X}\Bigg[\frac{\partial\eta_0}{\partial T} + \frac{1}{90}\frac{\partial^5\eta_0}{\partial X^5} + \frac{3}{2}\eta_0\frac{\partial\eta_0}{\partial X} - \frac{B_1}{2}\frac{\partial^3\eta_0}{\partial X^3} + \\ + \frac{\hat{E}_b}{2}\mathscr{H}\left(\frac{\partial^2\eta_0}{\partial X^2}\right) + \frac{1}{2}\frac{\partial P}{\partial X}\Bigg] + \frac{1}{2}\frac{\partial^2\eta_0}{\partial y^2} = 0 \end{aligned} \tag{5.196}$$

In dimensional variables the equation becomes:

$$\frac{\partial}{\partial x}\left[\frac{1}{c_0}\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}+\frac{h^4}{90}\frac{\partial^5\eta}{\partial x^5}+\frac{3}{2h}\eta\frac{\partial \eta}{\partial x}-\frac{h^2}{2}\left(B-\frac{1}{3}\right)\frac{\partial^3\eta}{\partial x^3}+\right.$$
$$\left.+\frac{E_b h}{2}\mathscr{H}\left(\frac{\partial^2\eta}{\partial x^2}\right)+\frac{1}{2\rho g}\frac{\partial p}{\partial x}\right]+\frac{1}{2}\frac{\partial^2\eta}{\partial y^2}=0 \tag{5.197}$$

Write $X = x - ct$ and set $\eta(t, x, y) = f(X, y)$, then the above equation becomes:

$$\frac{\partial}{\partial X}\left[(1-F)\frac{\partial f}{\partial X}+\frac{h^4}{90}\frac{\partial^5 f}{\partial X^5}+\frac{3}{2h}f\frac{\partial f}{\partial X}-\frac{h^2}{2}\left(B-\frac{1}{3}\right)\frac{\partial^3 f}{\partial X^3}+\right.$$
$$\left.+\frac{E_b h}{2}\mathscr{H}\left(\frac{\partial^2 f}{\partial X^2}\right)+\frac{1}{2\rho g}\frac{\partial p}{\partial X}\right]+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}=0 \tag{5.198}$$

Where $F = c/c_0$ is the Froude number and upon writing $f = e^{ikX+ily}$, the dispersion relation becomes:

$$F = 1+\frac{(hk)^4}{90}k^4+\frac{1}{2}\left(B-\frac{1}{3}\right)(kh)^2-\frac{E_b}{2}hk+\frac{1}{2}\frac{(hl)^2}{(hk)^2} \tag{5.199}$$

Ignoring the nonlinear terms and examining only the linear part, the equation can be solved via Fourier transforms by letting:

$$f(X,y)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\hat{f}(k,l)e^{i(kx+ly)}dkdl, \quad p(X,y)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\hat{p}(k,l)e^{i(kx+ly)}dkdl \tag{5.200}$$

The expression for $\hat{f}$ can be found as:

$$\hat{f}=\frac{k^2\hat{p}}{2\rho g}\left[(F-1)k^2-\frac{h^4}{90}k^6-\frac{h^2}{2}\left(B-\frac{1}{3}\right)k^4+\frac{E_b h}{2}k^3\mathrm{sgn}(k)-\frac{l^2}{2}\right]^{-1} \tag{5.201}$$

It is thought that despite the analysis carried out in [35], that the analysis presented here is the first attempt to derive a 5th order KP equation with forcing using asymptotic analysis. It is conjectured that the same equation is obtained when there is are two fluids and an interface at $y = \eta(t, x, y)$ with densities $\rho_{1,2}$ and no electric fields where the electric Bond number E_b is replaced with with the ratio of densities.

Chapter 6

Conclusions

This chapter will be a summary of what was shown in each chapter.

6.1 Chapter 1

This chapter derived the equations that would be used throughout the thesis. A scaling is chosen to nondimensionalise the equations and then a small moving pressure distribution is considered and the system is perturbed around a trivial solution of the full equations to obtain a linearised set of equations which were then solved.

The results were plotted to see the shape of the waves in the interface. A criterion is then derived for the existence of waves and is tested. It is shown that the linear model is not complete in the sense that for certain selection of parameters blow up in the profile occurs which implies that the need to include some nonlinearity into the model. An examination of the free surface of the far field is examined and is shown to by sinusoidal. The amplitude of the sinusoidal waves are unbounded for certain selection of the parameters which implies that a nonlinear solution is required.

The case without surface tension was also considered with an external electric field to see if the situation could support interfacial waves and it was discovered that no linear waves were possible. A full examination of forcing from topography was also examined with expressions obtained for the free surface which correspond to velocities above and below the minimum of the dispersion relation.

6.2 Chapter 2

Using the scaling from the basic KdV approximation, a weakly nonlinear equation was derived and it was seen to be of similar form to the usual KdV equation but with an extra Hilbert transform term which made it into a Benjamin type equation, it was noted that if the Bond number was close to a third then the dispersive term disappeared which means that another perturbation scheme which involved perturbing the Bond number away from a third to obtain a weakly nonlinear equation which is valid for Bond numbers close to a third. The equation was called a 5th order KdV equation.

It was shown using the 5th order KdV equation that there was no blow up in the amplitude (figure 3.3). The lower branch (with smaller amplitudes) is a perturbation from a train of waves and the profiles with the higher amplitudes are perturbations from solitary waves. The solitary wave branch was also plotted (figure 3.6) giving numerical evidence that solitary waves exist of all amplitudes. The 5th order KdV equation was modified to include topography rather than a pressure distribution and was seen to include the topography in the usual way.

Periodic solutions of the 5th order KdV equation were examined via Stokes's analysis and the existence of Wilton ripples were shown and plotted for given values.

6.3 Chapter 3

There has been much analysis carried out in the case of two non-interacting fluids in a rectangular tube. Clearly when one of the sides is taken to infinity there resulting flow will be unstable but there is one way to introduce a stabilisation, by introducing a horizontal electric field, [5]. This is what is done in this chapter.

The linear theory is first examined to obtain a dispersion relation, the conclusion of this section of the chapter is that for $k = 0$, $c^2(k) < 0$ A weakly nonlinear model was them put forward by making a rescaling of the linear theory, a set of equations were derived to model with as thin layer and the amplitude of the waves are of the same order as the average depth of the fluid. Profiles of these waves were then produced. Unlike the waves in chapter 2, the weakly nonlinear waves are periodic. A Stokes analysis is performed to examine the requirements for Wilton ripples and was shown that there are no Wilton ripples for this situation.

The final part of this chapter is concerned with cases where the amplitude of the waves are of the same order as the fluid. The analysis shows that such waves are possible and the resulting equations are of the same form (with sign changes in some of the terms)

6.4 Chapter 4

This chapter is a generalisation of the first and second chapters by going to three dimensions. The linear theory is extended and the solutions are plotted for some values of the Bond and electric Bond number. The dispersion relation is derived from two different perspectives and is then used to derive a linear version of weakly nonlinear equation. The infinite depth case was also examined, the surface profiles were also obtained and the same condition was found to hold as in the 2D case for blow up.

The weakly nonlinear case was then looked at, two different cases were initially chosen, when the Bond number, B is away from $1/3$ and when $B \approx 1/3$. In the former case an equation was derived which was very similar to the equation in [27] but with a different set of coefficients on the terms. The scaling when $B \approx 1/3$ was then carried out to obtain a fifth order forced KP equation which is the first in existence. The case $\gamma = 1$ was also examined but this isn't applicable for mediums similar to water but is more appropriate to plasma physics.

6.5 Future Work

Currently the models that I have worked with have no charge on the interface, This isn't physically realistic. According to Maxwell's equations, the charge will migrate to the interface, but if the stress tensor is analysed, one finds that for the inviscid theory:

$$\left[\hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{t}}\right] = 0 \tag{6.1}$$

including a charge shows that [9]:

$$\left[\hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{t}}\right] = q\mathbf{E} \cdot \hat{\mathbf{t}} \tag{6.2}$$

So if charge on the interface is considered than it is obvious that one can only consider viscous flow. The goal of this analysis being to obtain a direct generalisation of the KdV equation which associated forcing and electric terms. One could alternatively perform analysis on the interface of two fluids as in the Benjamin-Ono equation as the equations are exactly the same, the Hilbert transform term modelling the contribution in the upper fluid in the Benjamin-Ono case or the electric field when there is no upper fluid. There are several ways that this problem can be analysed, one is to simply write down The Navier-Stokes equations and scale these thereby examining the equations in the same limit and look for a free surface that way [32]. In this paper a simple analysis was carried out using perturbation theory in the long wavelength and small amplitude approximation to obtain an expression for the Free surface.

Another method that was first covered in ([8]) was to decompose the velocity vector $\mathbf{u}$ as

$$\mathbf{u} = \nabla\varphi + \nabla \times \mathbf{A} \tag{6.3}$$

Then one uses a linear analysis to separate the two quantities $(\varphi, \mathbf{A})$, where $\mathbf{A} = (0, \psi, 0)$ to:

$$\triangle\varphi = 0, \quad \frac{\partial\psi}{\partial t} = \nu\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \tag{6.4}$$

In order to close the system of equations another equation must be added to the system, the conservation of charge which is applied as a boundary condition across the interface.[33]

$$\frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla_\eta q - q\hat{\mathbf{n}}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}] + [\sigma\hat{\mathbf{n}}\cdot\mathbf{E}] = 0 \tag{6.5}$$

Where ∇_η is the covariant derivative on the free surface. Some authors have stated that the effect of viscosity is more important that the nonlinear effect in matching theory to experiment and that is where current analysis is centred.

The goal of the research will be the following:

1. Extend the analysis in [32] to include charge on the interface and look for travelling wave solutions.

2. Formulate electrohydrodynamics in terms of the decomposition (6.3) and look for linear solutions.

3. Add charge to the interface and expand on the model in point 2.

Appendix A

Hilbert Transforms

In this appendix, the properties of the Hilbert transform which are used are derived and calculated. The Hilbert transform is defined as:

$$\mathscr{H}(f)(x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(y)}{y-x} dy \tag{A.1}$$

<u>Lemma</u>

$$\mathscr{H}(f)'(x) = \mathscr{H}(f')(x) \tag{A.2}$$

<u>Proof.</u> Differentiating:

$$\begin{aligned}
\frac{d}{dx}\mathscr{H}(f)(x) &= \frac{1}{\pi}\frac{d}{dx} PV \int_{\mathbb{R}} \frac{f(y)}{y-x} dy \\
&= \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{\partial}{\partial x}\frac{f(y)}{y-x} dy \\
&= \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(y)}{(y-x)^2} dy \\
&= \frac{1}{\pi} \lim_{Y\to\infty} \left[-\frac{f(y)}{y-x} \right]_{-Y}^{Y} + \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f'(y)}{y-x} dy \\
&= \mathscr{H}(f')(x)
\end{aligned}$$

Suppose the function $f(x, y)$ is an analytic function, so the partial derivatives exist and also suppose $\partial_x f, \partial_y f \to 0$ as $\sqrt{x^2+y^2} \to \infty$. Let $g(x, y) = \partial_x f - i\partial_y f$, inserting this into Cauchy Integral formula, evaluating at $z_0 = x_0 + y_0 i$ and using a semi-circular contour of radius R with a semi-circular around $x = x_0$ of radius r. So:

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{\partial_x f(x,y) - i\partial_y f(x,y)}{x - x_0 + (y-y_0)i} dz = 0 \tag{A.3}$$

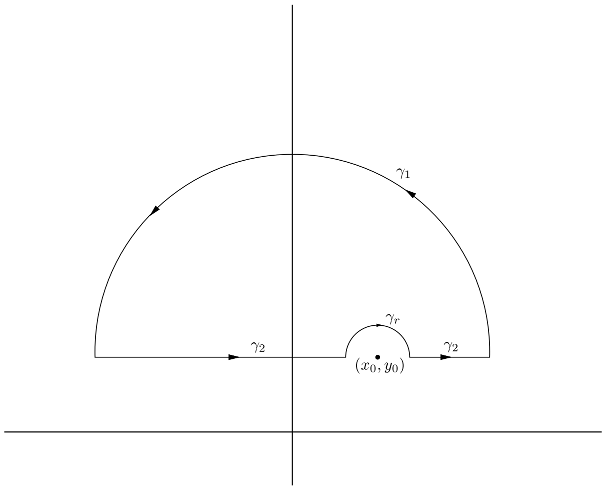


Figure A.1: Contour for the integral

The integral is taken over a semi-circle of radius R in the upper half plane centred at $(0, y_0)$ (figure A.1). Splitting the integral into three parts, the large semicircle (γ_1), the line $y = y_0$ (γ_2) and the small semi-circle γ_r yields:

$$\oint_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_r} \tag{A.4}$$

Estimating the integral over γ_1, shows that as $R \to \infty$ this tends to zero and the integral over γ_2 is left. Now γ_2 can be split up into three pieces, the integral from $-R$ to $x_0 - r$, the semi-circle of radius r and the integral from $x_0 + r$, the integral over the small semi-circle γ_r can be calculated as follows:

$$\frac{1}{2\pi i}\int_{\gamma_r} \frac{\partial_x(x,y) - i\partial_y f(x,y)}{z - z_0} dz \tag{A.5}$$

Making the co-ordinate transformation $z = z_0 + re^{i\theta}$, then $dz = rie^{i\theta}d\theta$, the integral becomes:

$$\begin{aligned}
\lim_{r\to 0}\frac{1}{2\pi i}\int_{\gamma_r}\frac{g(z)}{z-z_0}dz &= \lim_{r\to 0}\frac{1}{2\pi i}\int_{\gamma_r}\frac{\partial_x f(x,y) - i\partial_y f(x,y)}{z - z_0}dz \\
&= \lim_{r\to 0}\frac{1}{2\pi}\int_\pi^0 \frac{g(z_0 + re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta \\
&= \lim_{r\to 0}\frac{1}{2\pi}\int_\pi^0 g(z_0 + re^{i\theta}) d\theta \\
&= -\frac{g(z_0)}{2\pi}\pi \\
&= -\frac{g(z_0)}{2}
\end{aligned}$$

So the result is:

$$\frac{\partial_x f(x_0,y_0) - i\partial_y f(x_0,y_0)}{2} = \frac{1}{2\pi i} PV \int_{\mathbb{R}} \frac{\partial_x f(x,y_0) - i\partial_y f(x,y_0)}{x-x_0} dx \tag{A.6}$$

Which gives:

$$\partial_x f(x_0,y_0) - i\partial_y f(x_0,y_0) = \frac{1}{\pi i} PV \int_{\mathbb{R}} \frac{\partial_x f(x,y_0) - i\partial_y f(x,y_0)}{x-x_0} dx \tag{A.7}$$

To find $\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$, all that is done is to take the real and imaginary parts. So

$$\begin{aligned} \partial_x f(x_0,y_0) &= -\mathscr{H}(\partial_y f(x_0,y_0)) \\ \partial_y f(x_0,y_0) &= \mathscr{H}(\partial_x f(x_0,y_0)) \end{aligned}$$

So defining a function:

$$g(x,y) = \mathscr{H}(f)(x,y) + if(x,y) \tag{A.8}$$

The above equations show that $g(x,y)$ is an analytic function as it obeys the Cauchy-Riemann equations. A useful Hilbert transform is that of e^{ikx}, the definition of the Hilbert transform is:

$$\mathscr{H}(e^{ikx}) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{e^{iky}}{y-x} dy \tag{A.9}$$

To compute this take $k > 0$ and use an upper semi-circular contour with a semi-circular indent at the point $(x,0)$ similar to (A.1), On γ_4, it is possible to estimate the integral:

$$\begin{aligned} \left| \frac{1}{\pi} \int_{\gamma_4} \frac{e^{iky}}{y-x} dy \right| &\leqslant \frac{1}{\pi} \int_{\gamma_4} \left| \frac{e^{iky}}{y-x} dz \right| \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{Re^{-kR\sin\theta}}{|R-|x||} d\theta \\ &\leqslant \frac{1}{\pi} \int_0^{\pi} \frac{Re^{-\frac{2R\theta}{\pi}}}{|R-|x||} d\theta \\ &= \left[-\frac{e^{-\frac{kR\theta}{\pi}}}{2|R-|x||} \right]_0^{\pi} \\ &\to 0 \quad \text{as} \quad R \to \infty \end{aligned}$$

The integral around γ_r becomes:

$$\begin{aligned} \frac{1}{\pi} \int_{\gamma_r} \frac{e^{iky}}{y-x} dy &= i \int_{\pi}^{0} e^{ikx + ikr\cos\theta - rk\sin\theta} d\theta \\ &\to -\frac{i}{\pi} \int_0^{\pi} e^{ikx} d\theta \quad \text{as} \quad r \to 0 \\ &= ie^{ikx} \end{aligned}$$

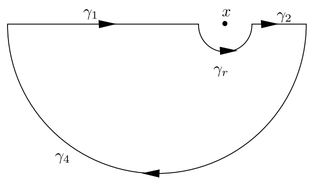


Figure A.2: Contour for the integral

The other two terms become the Hilbert transform and the total sum becomes:

$$\mathscr{H}(e^{ikx}) - ie^{ikx} = 0 \tag{A.10}$$

Showing that for $k > 0$

$$\mathscr{H}(e^{ikx}) = ie^{ikx} \tag{A.11}$$

Now for $k < 0$ the best way is to compute the Hilbert transform of e^{-ikx} for $k > 0$, the contour is now: This changes two things in the analysis, the estimation of the integral around γ_4 and the computation of the integral around γ_r, the integral around γ_4 becomes:

$$\begin{aligned}
\left|\frac{1}{\pi}\int_{\gamma_4}\frac{e^{-iky}}{y-x}dy\right| &\leqslant \frac{1}{\pi}\int_{\gamma_4}\left|\frac{e^{-iky}}{y-x}dz\right| \\
&= \frac{1}{\pi}\int_{2\pi}^{\pi}\frac{Re^{kR\sin\theta}}{|R-|x||}d\theta \\
&= -\frac{1}{\pi}\int_{0}^{\pi}\frac{e^{-Rk\sin\mu}}{|R-|x||}d\mu \quad \theta = \pi + \mu \\
&\leqslant -\frac{1}{\pi}\int_{0}^{\pi}\frac{Re^{-\frac{2R\mu}{\pi}}}{|R-|x||}d\mu \\
&= -\left[-\frac{e^{-\frac{kR\theta}{\pi}}}{2|R-|x||}\right]_0^{\pi} \\
&\to 0 \quad \text{as} \quad R \to \infty
\end{aligned}$$

The integral over γ_r becomes:

$$\begin{aligned}
\frac{1}{\pi}\int_{\gamma_r}\frac{e^{-iky}}{y-x}dy &= i\int_{\pi}^{2\pi} e^{-ikx-ikr\cos\theta+rk\sin\theta}d\theta \\
&\to \frac{i}{\pi}\int_0^{\pi} e^{-ikx}d\theta \quad \text{as} \quad r \to 0 \\
&= ie^{-ikx}
\end{aligned}$$

Hence:

$$\mathscr{H}(e^{ikx}) = i\text{sgn}(k)e^{ikx} \tag{A.12}$$

Lemma For periodic functions the Hilbert transform becomes:

$$\mathscr{H}(f)(x) = -PV\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \cot\left(\frac{\pi(x-u)}{\lambda}\right) f(u)du \tag{A.13}$$

Where λ is the period of the function $f(x)$.

Proof

$$\begin{aligned}
\mathscr{H}(f)(x) &= \frac{1}{\pi}PV\int_{\mathbb{R}} \frac{f(y)}{y-x}dy \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} PV\int_{\frac{(2n+1)\lambda}{2}}^{\frac{(2n+3)\lambda}{2}} \frac{f(y)}{y-x}dy \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} PV\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \frac{f((n+1)\lambda+u)}{(n+1)\lambda+u-x}du \quad y=(n+1)\lambda+u \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} PV\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \frac{f(u)}{(n+1)\lambda+u-x}du \\
&= \frac{1}{\pi}PV\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(u)\sum_{n=-\infty}^{\infty} \frac{1}{(n+1)\lambda+u-x}du
\end{aligned}$$

So the next stage of the calculation to to compute:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+1)\lambda+u-x} = \sum_{n=-\infty}^{\infty} \frac{1}{n\lambda+u-x} \tag{A.14}$$

To find this sum, integrate the function:

$$f(z) = \pi\frac{\cot \pi z}{z\lambda+u-x} \tag{A.15}$$

around the square with vertices at $(\pm 1 \pm i)\left(N+\frac{1}{2}\right)$. Computing the residues, there is a pole at $z=(x-u)\lambda^{-1}$ and a pole at $z=n$ where $n\in\mathbb{Z}$, the residues of these poles are:

$$\pi\cot\left(\frac{\pi(x-u)}{\lambda}\right), \quad \frac{1}{n\lambda+u-x} \tag{A.16}$$

respectively. Standard theory shows that the integral tends towards zero as $N\to\infty$, and so that:

$$\pi\cot\left(\frac{\pi(x-u)}{\lambda}\right) = -\sum_{n=-\infty}^{\infty} \frac{1}{n\lambda+u-x} \tag{A.17}$$

Hence, the Hilbert transform for a periodic function of period λ is given by:

$$\mathscr{H}(f)(x) = -PV\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \cot\left(\frac{\pi(x-u)}{\lambda}\right) f(u)du \tag{A.18}$$

Appendix B

Numerical Solution of the Full Nonlinear Equations

This appendix will outline a solution method for solving the fully nonlinear equations. To start the solution process consider the wave reflected about the y-axis and integrate around a closed loop. Use the arc-length parametrisation, s for the contour, so that if

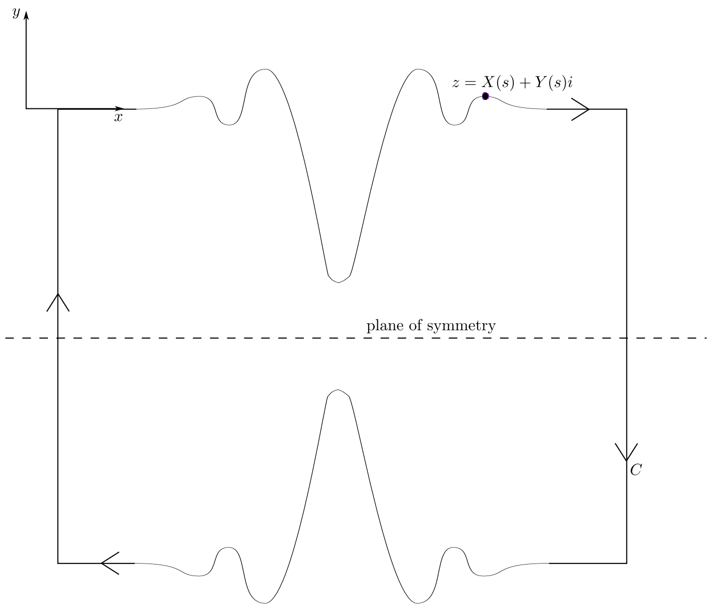


Figure B.1: The Set up for the Solution

the points on the free surface are given by $x = X(s)$ and $y = Y(s)$. Define the height of the free surface to be zero at $\pm\infty$ so:

$$X(s) = \int_{-\infty}^{s} X'(r)dr \tag{B.1}$$

Let α be the height of the wave crests, then:

$$Y(s) = \alpha + \int_{-\infty}^{s} Y'(r)dr \tag{B.2}$$

By definition of the arc length:

$$(X'(s))^2 + (Y'(s))^2 = 1 \tag{B.3}$$

Let ξ be the angle between the x-axis and the tangent vector. The *curvature* of a curve is given by:

$$\kappa = \frac{d\xi}{ds} \tag{B.4}$$

where s is the arc-length. The free surface is given by $\mathbf{r}(s) = X(s)\mathbf{i} + Y(s)\mathbf{j}$, then ξ is given by:

$$\tan\xi = \frac{dY}{dX} = \frac{Y'(s)}{X'(s)} \tag{B.5}$$

So the (plane) curvature is given by:

$$\begin{aligned}
\kappa &= \frac{d\xi}{ds} \\
&= \frac{d}{ds}\tan^{-1}\left(\frac{Y'(s)}{X'(s)}\right) \\
&= \frac{1}{1+\left(\frac{Y'(s)}{X'(s)}\right)^2}\frac{d}{ds}\left(\frac{Y'(s)}{X'(s)}\right) \\
&= (X'(s))^2\left(\frac{Y''(s)}{X'(s)} - \frac{Y'(s)X''(s)}{(X'(s))^2}\right) \\
&= Y''(s)X'(s) - Y'(s)X''(s)
\end{aligned}$$

The height of the free surface is given by the height co-ordinate $Y(s)$, so $\eta = Y(s)$ and the partial derivatives $\partial_x\eta$ are given by:

$$\frac{\partial\eta}{\partial x} = \frac{dY(s)}{dX(s)} = \frac{Y'(s)}{X'(s)}$$

So Bernoulli's equation can be written in the following form:

$$\begin{aligned}
\frac{1}{2}(u^2+v^2) &+ P(x) + BY(s) + \frac{E_b}{1+(Y'(s)/X'(s))^2}((Y'(s)/X'(s))^2 M_{11} - \\
&- 2Y'(s)M_{12}/X'(s) + M_{22}) \\
&= Y''(s)X'(s) - Y'(s)X''(s) + K
\end{aligned} \tag{B.6}$$

The idea is to apply Cauchy's integral formula to obtain a solution. Cauchy's integral formula is:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \tag{B.7}$$

The complex function in question will be $f(z) = u - vi - U$ where u, v are the horizontal and vertical velocities of the fluid respectively and U is the asymptotic velocity of the wave, so that $f(z) \to 0$ as $|z| \to \infty$. By writing $z = X(s) + Y(s)i$ it is possible to write:

$$u(a) - U - v(a)i = \frac{1}{\pi i}\oint_C \frac{u(z) - U - v(z)i}{z-a}dz \tag{B.8}$$

As the singularity is on the contour C, the factor is $1/(\pi i)$ rather than $1/(2\pi i)$. Computing the integral by looking at the four boundaries that make up the contour. The vertical parts of the integral vanish as $|z| \to \infty$ as $u - U - vi \to 0$, so the only parts of the contour integral which contribute are the horizontal parts. Denote $a = X(s) + Y(s)i$ and $z = X(r) + Y(r)i$, then the contour integral becomes:

$$\begin{aligned}
u(s) - U - v(s)i &= \frac{1}{\pi i}\oint_C \frac{u(z) - U - v(z)i}{z-a}dz \\
&= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(u_r - U - v_r i)(X_r' + Y_r' i)}{X_r + Y_r i - X_s - Y_s i}dr + \\
&\quad + \frac{1}{\pi i}\int_{\infty}^{-\infty} \frac{(u_r - U + v_r i)(X_r' - Y_r' i)}{X_r - Y_r i - X_s + 2hi + Y_s i}dr \\
&= \frac{1}{\pi i}\int_{\mathbb{R}} \frac{(u_r - U - v_r)(X_r' + Y_r' i)(X_{rx} - Y_{rs} i)}{X_{rs}^2 + Y_{rs}^2}dr - \\
&\quad - \frac{1}{\pi i}\int_{\mathbb{R}} \frac{(u_r - U + v_r i)(X_r' - Y_r' i)(X_{rs} + (Y_{rs} + 2h)i)}{x_{rs}^2 + (Y_{rs} + 2h)^2}dr
\end{aligned}$$

The last line comes from multiplying the complex conjugate of the denominator. Then to find out $u(s) - U$ and $v(s)$, the real and imaginary parts are taken. So[1]: It is now

[1]For brevity of notation, denote that $X(r) = X_r$ and $X_{rs} = X(r) - X(s)$ etc and use this notation throughout the rest of this appendix.

possible to read off the velocities which are:

$$\begin{aligned}
u_s - U &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{(u_r - U)(X_{rs}Y_r' - Y_{rs}X_r') - v_r(X_{rs}X_r' + Y_{rs}Y_r')}{X_{rs}^2 + Y_{rs}^2}dr - \\
&\quad - \frac{1}{\pi}\int_{\mathbb{R}} \frac{(u_r - U)((Y_{rs} + 2h)X_r' - X_{rs}Y_r') + v_r(X_{rs}X_r' + (Y_{rs} + 2h)Y_r')}{X_{rs}^2 + (Y_{rs} + 2h)^2}dr \\
v_s &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{(u_r - U)(X_{rs}X_r' + Y_{rs}Y_r') + v_r(X_{rs}Y_r' - Y_{rs}X_r')}{X_{rs}^2 + Y_{rs}^2}dr - \\
&\quad - \frac{1}{\pi}\int_{\mathbb{R}} \frac{(u_r - U)(X_{rs}X_r' + (Y_{rs} + 2h)Y_r') - v_r((Y_{rs} + 2h)X_r' - X_{rs}Y_r')}{X_{rs}^2 + (Y_{rs} + 2h)^2}dr
\end{aligned}$$

The same process can be applied to the electric potential but with a different contour and the method of images isn't used. The contour used is The function inserted into

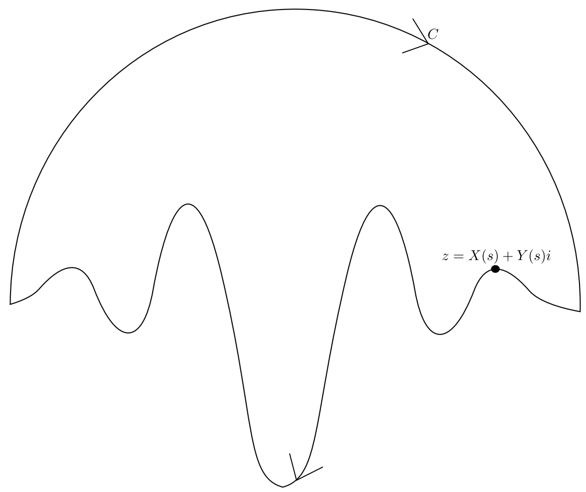


Figure B.2: Contour for the Electric Potential

Cauchy's integral formula is:

$$f(z) = \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}i + E_0 i \qquad \text{(B.9)}$$

so to make $f(z) \to 0$ as $|z| \to \infty$. This means that the semi-circle part of the contour does not contribute to the contour integral, hence:

$$\begin{aligned}
\partial_x V(a) - \partial_y V(a)i - E_0 i &= \frac{1}{\pi i}\oint_C \frac{\partial_x V(z) - \partial_y V(z)i - E_0 i}{z - a}dz \\
&= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(\partial_x V_r - \partial_y V_r + E_0 i)(X_r' + Y_r' i)}{X_r - X_s + (Y_r - Y_s)i}dr \\
&= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(\partial_x V_r - \partial_y V_r + E_0 i)(X_r' + Y_r' i)(X_r - X_s - (Y_r - Y_s)i)}{(X_r - X_s)^2 + (Y_r - Y_s)^2}dr
\end{aligned}$$

Multiplying out and comparing real and imaginary parts shows that

$$\begin{aligned}
\partial_x V_s &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(E_0 X_r' - X_r'\partial_y V_r + Y_r'\partial_x V_r)X_{rs}}{X_{rs}^2 + Y_{rs}^2} - \\
&\quad - \frac{(X_r'\partial_x V_r + Y_r'\partial_y V_r - E_0 Y_r')Y_{rs}}{X_{rs}^2 + Y_{rs}^2} dr \\
\partial_y V_s - E_0 &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(X_r'\partial_x V_r + Y_r'\partial_y V_r - E_0 Y_r')X_{rs}}{X_{rs}^2 + Y_{rs}^2} + \\
&\quad + \frac{(E_0 X_r' - X_r'\partial_y V_r + Y_r'\partial_x V_r)Y_{rs}}{X_{rs}^2 + Y_{rs}^2} dr
\end{aligned}$$

The two integro-differential for the horizontal and vertical velocities are essentially the same equations, so there is only one equation that comes from the velocities, the case is the same for electric potential or the electric field. So there are two other equations that are required for a complete set. The other two equations come from the velocity potential and electric field across the interface. The equation coming from the velocity reads:

$$u\partial_x \eta - v = 0 \Rightarrow Y'(s)u(s) - X'(s)v = 0 \tag{B.10}$$

The equation for the electric potential is

$$\frac{dV}{ds} = 0 \Rightarrow X'(s)\partial_x V + Y'(s)\partial_y V \tag{B.11}$$

The two integro-differential equations along with equations (B.3), (B.6), (B.10) and (B.11) give six equations for six unknowns $(X(s), Y(s), u(s), v(s), \partial_x V(s), \partial_y V(s))$, so there will be a unique solution. The way to go about solving this is to choose a series of N mesh points for the arc-length parameter, s_i, then the set of equations provide a series of $6N$ equations. There are adjustable parameters in the system, there is the constant K in (B.6), but this is just related to the parameter E_b and there is also the height of the crest, α that has to be considered, so there is a system of $6N+2$ equations which need to be solved via the Newton-Raphson method.

B.1 Infinite Depth

A simplification can be made for the case of infinite depth, they can obtained formally when taking the limit as $h \to \infty$. The only equations that this affects are the equations for velocity and they reduce to:

$$\begin{aligned}
u_s - U &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{(u_r - U)(X_{rs}Y'_r - Y_{rs}X'_r) - v_r(X_{rs}X'_r + Y_{rs}Y'_r)}{X_{rs}^2 + Y_{rs}^2} dr - \\
&\quad - \frac{1}{\pi}\int_{\mathbb{R}} \frac{2(u_r - U)X'_r + 2v_r Y'_r}{4Y_{rs}} dr \\
v_s &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{(u_r - U)(X_{rs}X'_r + Y_{rs}Y'_r) + v_r(X_{rs}Y'_r - Y_{rs}X'_r)}{X_{rs}^2 + Y_{rs}^2} dr - \\
&\quad - \frac{1}{\pi}\int_{\mathbb{R}} \frac{2(u_r - U)Y'_r - 2v_r X'_r}{4Y_{rs}} dr
\end{aligned}$$

Appendix C

MATLAB Code

This appendix list the MATLAB code used for the PhD.

C.1 Linear 2D Code

The code used to obtain linear solutions

```
%This program will calculate the inverse Fourier transform which
%will give the shape of the free surface.
%Give the initial constants:
h=2;
mu=0;
U=0.7;
E_b=2;
B=2;
F=0.1;
a=1;
b=0.5;
ep=2.0;
%Start the integration
x=-15:0.01:15;
n=length(x);
y=zeros(1,n);
k=linspace(-15,15,n);
```

```
dk=abs(k(1)-k(2));
for m=1:n
    z=top(k,U,B,E_b,h,mu).*exp(sqrt(-1)*k*x(m));
    y(m)=real(mat_trap(z,dk))/(2*pi);
end
plot(x,y)
xlabel('x');
ylabel('\eta_{1}(x)');
```

Which uses the following programs, which is the trapezium rule:

```
function integral=mat_trap(f,dx)
n=length(f);
%Compute thre integral
integral=0;
for i=1:n-1
    integral=integral+0.5*(f(i)+f(i+1))*dx;
end
```

The code which inputs the actual equation for the wave is:

```
function eta=top(k,U,B,E_b,h,mu)
m=length(k);
    for j=1:m
        if (k(j)==0)
            eta(j)=0;
        else
            eta(j)=k(j)*U^2*exp(k(j)*h)*sqrt(pi)*exp(-0.25*k(j)^2)
                   *(tanh(k(j)*h)-1)./(-k(j)*U^2+(B-E_b*abs(k(j))+
                   k(j)^2)*tanh(k(j)*h)+sqrt(-1)*mu);
        end
    end
```

C.2 Weakly Nonlinear Codes

The way that the weakly nonlinear equations were solved was to write the equation for points as regularly spaced intervals using the usual finite difference equations for derivatives. This leaves a large set of nonlinear simultaneous equations to solve which are done via Newton's method.

```
clear;
L=401; %Defines the number of steps, must always be an integer
x=linspace(-6,6,L); %Defines x
B=0.4; %This is the Bond number.
p=0.25*B*exp(-x.^2); %The pressure distribution
E_b=0.3; %This is the electric Bond number.
f=zeros(1,L+1);
f(1,1:L)=0.01*exp(-x.^2); %have some small initial guess for the
%wave profile
choice=true; %This selects the option to fix the Froude number of
%the minimum of the wave.
Q=0.05; %This is the Froude number (choice=true), or the minimum
%of the profile (choice=false)
f(L+1)=Q; %this is the Foude number, either an exact value or an
%initial guess.
%The way forward is to use Newton's method to reduce the equation
%down to a linear algebra problem and then use Octave
%to solve the linear algrbra problem.

%The first step is to compute the Jacobian matrix
%Do this by diferencing between to close points and dividing
%through by that difference
df=10^-10;
J=zeros(L+1,L+1); %This will be the Jacobian.
```

```
X=zeros(L+1,1);
error=0.5;
while (error>10^-10)
for i=1:L+1
  for j=1:L+1
    T=zeros(1,L+1); %Set all the entries to zero again
    T(j)=df;
    J(i,j)=(g(x,f+T,p,Q,i,choice,B,E_b)-g(x,f-T,p,Q,i,choice,B,E_b))
           /(2*df); %Calculates the (i,j)th element of the Jacobian
  end
end
for m=1:L+1
  X(m)=-g(x,f,p,Q,m,choice,B,E_b);
end
%Now solve the system of equations
sol=J\X;
error=norm(X)
f=f+sol';
end

%Now plot the solution!
plot(x,f(1:L));
xlabel('x');
ylabel('f(x-ct)');
%Write the data to a file
%h=1;
%B=0.2;
%E_b=0.1;
%file_1=fopen('fine_mesh_restults.txt','w');
%fprintf(file_1,'h=%1.4E\n',h);
```

```
%fprintf(file_1,'B=%1.4E\n',B);
%fprintf(file_1,'E_b=%1.4E\n',E_b);
%fprintf(file_1,'F=%1.4E\n',Q);
%fprintf(file_1,'      x            f(x-ct)\n');
%for i=1:L
%    fprintf(file_1,'%1.6E %1.6E\n',x(i),f(i));
%end
%fclose(file_1);
```

Which uses the following function which actually contains the equation to be solved:

```
function r=g(x,f,p,Q,i,flag,B,E_b)
N=length(x);
h=1;
a_1=1-f(N+1);
a_2=h^4/90;
a_3=1.5/h;
a_4=-0.5*(B-1/3)*h^2;
a_5=0.5*E_b*h;
dx=x(2)-x(1);
ff=f(1:N);
f_ghost=[0 0 ff 0 0]; %Add in the ghost cells for f
%Prepare for the Hilbert transform
x_half=zeros(1,N-1);
f_half=zeros(1,N-1);
for k=1:N-1
    x_half(k)=0.5*(x(k)+x(k+1));
    f_half(k)=0.5*(ff(k)+ff(k+1));
end
grad=gradient(f_half,dx);
if (i<N+1)
```

```
    hilbert=trapz(x_half,grad./(x_half-x(i)));
end
%Now comes the formulae for the finite
%differencing part of the algorithm.
if (i<N+1)
    b_1=a_1*f_ghost(i+2);
    b_2=(a_2/dx^4)*(f_ghost(i+4)-4*f_ghost(i+3)+6*f_ghost(i+2)-
        4*f_ghost(i+1)+f_ghost(i)); %Fourth order derivative
    b_3=a_3*f_ghost(i+2)^2; %nonlinear term
    b_4=(a_4/dx^2)*(f_ghost(i+3)-2*f_ghost(i+2)+f_ghost(i+1));
        %Second order derivative
    b_5=a_5*hilbert; %Hilbert transform term
    b_6=p(i); %Pressure term
    r=b_1+b_2+b_3+b_4+b_5+b_6;
else
    if (flag==true)
        r=f(N+1)-Q;
    else
        r=f(1+0.5*(N-1))-Q;
    end
end
```

C.3 Code for Chapter 3

There are two parts to this program:

```
lambda=4*pi; %This is the wavelength.
L=100; %This is the number of points to take
dx=lambda/L;
x_1=-0.5*lambda:dx:0.5*lambda; %This splits the wavelength
%into L pieces from -0.5*lambda to 0.5*lambda
x=x_1(1:L);
```

```
e_s_f=0.6; %This is the ration of electric permativities
%of the solid and fluid
e_f_g=0.2; %This is the ratio of electric permativities
%of fluid and gas
E_b=1.0; %This is the electric Bond number
B=0.9; %This is Bond number
mu=E_b*(e_f_g-1)^2/(1-e_s_f*e_f_g);
mu=2*B;
k=2*pi/lambda;
h=0.3;
c_0=sqrt(-B+k*mu+k^2);
%The first step is to compute the Jacobian matrix
%Do this by diferencing between to close points and dividing
%through by that difference
dW=10^-10;
J=zeros(2*L+1,2*L+1); %This will be the Jacobian.
W=zeros(1,2*L+1);
Z=zeros(2*L+1,1);
W(1,1:L)=-1+h*cos(x)/k;  %Have some non-zero initial state
W(1,L+1:2*L)=c_0+h*c_0*cos(x)/k;  %Have some non-zero initial
%state
W(1,2*L+1)=c_0;
error=10;
while(error>10^-4)
    for i=1:2*L+1
      for j=1:2*L+1
        T=zeros(1,2*L+1); %Set all the entries to zero again
        T(j)=dW;
        J(i,j)=(float_g(x,W+T,i,B,k,lambda,L,h)-
               float_g(x,W-T,i,B,k,lambda,L,h))/(2*dW);
```

```
                %Calculates the (i,j)th element of the Jacobian
        end
      end
      for m=1:2*L+1
        Z(m)=-float_g(x,W,m,B,k,lambda,L,h);
      end
      %Now solve the system of equations
      sol=J\Z;
      W=W+sol';
      error=norm(Z)
end
eta=W(1:L);
plot(x,eta)
xlabel('x');
ylabel('\eta');
c=W(2*L+1)
```

Along with:

```
function r=float_g(x,W,q,B,mu,lambda,L,h)
c=W(2*L+1); %This is the value of eta(0) Initial as a guess.
%c=W(2*L+2);
nu=pi/lambda;
z=-1; %this is the average zero for the wave
%The array W contains the u and eta_x:
eta=W(1:L);
u=W(L+1:2*L);
dx=x(2)-x(1);
%The value p will pick out the the equation used.
if (q>=1) && (q<=L)
    p=0;
```

```
elseif (q>=L+1) && (q<=2*L)
    p=1;
else
    p=2;
end
%The quantity d will pick the d th component of the chosen
%variable.
d=q-p*L;
if (d==0)
    d=L;
end
%Now obtain the values of eta_x and u at the mid points
x_half=zeros(1,L-1);
eta_half=zeros(1,L-1);
for k=1:L-1
    x_half(k)=0.5*(x(k)+x(k+1));
    eta_half(k)=0.5*(eta(k)+eta(k+1));
end
eta_xxx=zeros(1,L);
for i=3:L-2
    eta_xxx(i)=(eta(i+2)-2*eta(i+1)+2*eta(i-1)-eta(i-2))
               /(2*dx^3);
end
a_1=(eta(3)-2*eta(2)+2*eta(L)-eta(L-1))/(2*dx^3);
a_2=(eta(1)-2*eta(L)+2*eta(L-2)-eta(L-3))/(2*dx^3);
a_3=(eta(2)-2*eta(1)+2*eta(L-1)-eta(L-2))/(2*dx^3);
a_4=(eta(4)-2*eta(3)+2*eta(1)-eta(L))/(2*dx^3);
eta_xxx(1)=a_1;
eta_xxx(2)=a_4;
eta_xxx(L-1)=a_3;
```

```
eta_xxx(L)=a_2;
for i=2:L-1
    eta_x(i)=(eta(i+1)-eta(i-1))/(2*dx);
    u_x(i)=(u(i+1)-u(i-1))/(2*dx);
end
eta_x(1)=(eta(2)-eta(L))/(2*dx);
u_x(1)=(u(2)-u(L))/(2*dx);
eta_x(L)=(eta(1)-eta(L-1))/(2*dx);
u_x(L)=(u(1)-u(L-1))/(2*dx);
switch p
    case 0
        if (d==1)
            r=-c*eta_x(d)+u(d)*eta_x(d)+eta(d)*u_x(d);
        elseif (d==L)
            r=-c*eta_x(d)+u(d)*eta_x(d)+eta(d)*u_x(d);
        else
            r=-c*eta_x(d)+u(d)*eta_x(d)+eta(d)*u_x(d);
        end
    case 1
        for i=2:L-1
            eta_xx(i)=(eta(i-1)-2*eta(i)+eta(i+1))/dx^2;
        end
        eta_xx(1)=(eta(2)-2*eta(1)+eta(L))/dx^2;
        eta_xx(L)=(eta(1)-2*eta(L)+eta(L-1))/dx^2;
        for i=1:L-1
            eta_xx_half(i)=0.5*(eta_xx(i)+eta_xx(i+1));
        end
        if (d<L+1)
            hil_1=mat_trap(eta_xx_half.*cot(nu*(x_half-x(d))),dx);
            eta_xx_half_end=0.5*(eta_xx_half(L-1)+eta_xx_half(1));
```

```
            x_half_end=x_half(L-1)+dx;
            hil_2=0.5*dx*(cot(nu*(x_half(L-1)-x(d)))*eta_xx(L-1)+
                  eta_xx_half_end*cot(nu*(x_half_end-x(d))));
            hil_3=0.5*dx*(eta_xx_half_end*cot(nu*(x_half_end-x(d)))+
                  eta_xx_half(1)*cot(nu*(x_half(1)-x(d))));
            hilbert=(hil_1+hil_2+hil_3)/(lambda);
        end
        if (d==1)
            r=trapz(x,u)/lambda-lambda;
        elseif (d==L)
            r=trapz(x,eta)/lambda-z;
        else
            r=-c*u_x(d)+u(d)*u_x(d)+B*eta_x(d)+mu*hilbert+eta_xxx(d)
        end


    case 2
        r=max(eta)-min(eta)-h;
end
```

C.4 Code for 2D Plot

This is the code for the 2D surfaces in chapter 4.

```
%This plots the 2D free surface


x=linspace(-15,15,150); %This is the range of x
y=linspace(-15,15,150); %THis is the range of y
dx=x(2)-x(1); %Calculates the difference
dy=y(2)-y(1);
n=length(x); %Calculates the length of the vectors.
m=length(y);
U=0.5;
```

```
B=2;
E_b=1.5;
F=1;
h=1;
[X,Y]=meshgrid(x,y); %Sets of the surface plot.
%pressure=p(x,y);

eta=zeros(n,m); %This is the free surface

k=-6:0.05:6; %These are the parameters in the
%inverse 2d Fourier transform
l=-6:0.05:6;
[K,L]=meshgrid(k,l);
const=1/(4*pi^2);
nn=length(k);
%Now compute the inverse tranform.
a=1;b=2;xi=0;
%A=JM_eta_2d(k,l,U,B,E_b,xi,pressure);
A=eta_2d(k,l,U,B,E_b,xi);
%A=mexican_hat(k,l);
v=0;
for i=1:n
    for j=1:m
        eta(i,j)=real(inv_fourier_2d(A,x(i),y(j),k,l));
    end
end

surf(X,Y,eta);
shading interp
xlabel('x');ylabel('y');zlabel('\eta');
```

This prgram relies on the following programs:

```
function f=inv_fourier_2d(A,x,y,k,l)
dk=k(2)-k(1);
dl=l(2)-l(1);
n=length(A(1,:)); %This is the l variable
m=length(A(:,1)); %This is the k variable
%Do the k variable first:
v=zeros(n,1);
q=exp(sqrt(-1)*x*k)';
for i=1:n
    u=A(:,i);
    v(i)=mat_trap(u.*q,dk)/(2*pi);
end

%Next compute the inverse transform in the l variable

q=exp(sqrt(-1)*y*l);

f=mat_trap(v'.*q,dl)/(2*pi);
```

Along with:

```
function y=eta_2d(k,l,U,B,E_b,xi)
n=length(k);
m=length(l);

for i=1:n
    for j=1:m
        mu=sqrt(k(i)^2+l(j)^2);
        if (k(i)==0.0) && (l(j)==0.0)
            y(i,j)=2*pi/(U^2+2*B);
```

```
        else
            y(i,j)=pi*mu.*exp(-0.25*mu^2)*tanh(mu)./
            (k(i).^2*U^2-mu.*(B-E_b*mu+mu.^2).*tanh(mu)
            +xi*sqrt(-1));
        end
    end
end
```

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