Non-commutative mechanics and Exotic Galilean symmetry

L. Martina*
Dipartimento di Fisica - Università del Salento
and
Sezione INFN di Lecce. Via Arnesano, CP. 193
I-73 100 LECCE (Italy)

January 18, 2011

Abstract

In order to derive a large set of Hamiltonian dynamical systems, but with only first order Lagrangian, we resort to the formulation in terms of Lagrange-Souriau 2-form formalism. A wide class of systems derived in different phenomenological contexts are covered. The non-commutativity of the particle position coordinates are a natural consequence. Some explicit examples are considered.

*e-mail: Luigi.Martina@le.infn.it

1 Introduction

Recently interest in dynamical systems with non commuting (not necessarily non canonical) variables stems from Condensed Matter Physics [1], Optics [2] and String Theory [3]. The applications include a Bloch electrons [4], the Anomalous Hall Effect [5], the Spin Hall Effect [6], and the Optical Hall effect [7]. For example,

a) The semiclassical equations of motion in the $\epsilon_n(\vec{k})$ energy band of a Bloch electron in a crystal solid read

$$\dot{\vec{r}} = \frac{\partial \epsilon_n(\vec{k})}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Theta}(\vec{k}), \quad \dot{\vec{k}} = -e\vec{E} - e\dot{\vec{r}} \times \vec{B}(\vec{r}). \tag{1.1}$$

The purely momentum-dependent $\Theta_i(\vec{k}) = \epsilon_{ijl}\partial_{\vec{k}_j}\mathcal{A}_l(\vec{k})$ is the curvature associated to the so-called Berry connection $\mathcal{A}$.

b) The semiclassical equations that describe the spin-Hall effect into semiconductors near the degenerate point $\vec{k} = 0$ of the valence band read

$$\dot{\vec{r}} = \frac{\partial E_s(\vec{k})}{\partial \vec{k}} + \dot{\vec{k}} \times \vec{\Theta}_s, \qquad \dot{\vec{k}} = -e\vec{E}, \tag{1.2}$$

where $s = \pm\frac{1}{2}, \pm\frac{3}{2}$ is the spin helicity of the holes, the energy is $E_s(\vec{k}) = \frac{\hbar^2}{2m}\left(A - Bs^2\right)k^2$ and the Berry curvature due to the lattice structure is $\vec{\Theta}_s = s\left(2s^2 - \frac{7}{2}\right)\frac{\vec{k}}{k^3}$. The trajectories followed by opposite helicity holes separate during the motion, providing a tool for *spintronic*.

c) The *optical Magnus* and the *optical Hall* effects [2, 7, 8] are described by the approximate equations

$$\dot{\vec{r}} \approx \vec{p} - \frac{s}{\omega}\,\mathrm{grad}(\frac{1}{n}) \times \vec{p}, \qquad \dot{\vec{p}} \approx -n^3\omega^2\mathrm{grad}(\frac{1}{n}), \tag{1.3}$$

where s denotes the photon's spin, parametrizing a term which deviates the light's trajectory from the predictions of ordinary geometrical optics.

d) The motion of a Bogoliubov quasiparticle in a superfluid vortex [9] is described by

$$M\dot{\vec{q}} + \vec{F} \times \vec{r} = -\frac{\partial h}{\partial \vec{r}}, \qquad M\dot{\vec{r}} = \frac{\partial h}{\partial \vec{q}}, \tag{1.4}$$

where the *effective mass* $M_{ij} = \delta_{ij} - \frac{\partial A_i}{\partial q^j}$ and *magnetic field* $F_{ij} = \frac{\partial A_i}{\partial r^j} - \frac{\partial A_j}{\partial r^i}$ are defined in terms of a gauge potential $\vec{A} = \vec{A}(\vec{q}, \vec{r})$, and $h = H(\vec{q}, \vec{r})$ is some Hamiltonian.

e) The string theory inspired dynamical system [10]

$$\dot{x}_i = \frac{p_i}{m} + \Theta_{ij}\frac{\partial V}{\partial x_j}, \qquad \dot{p}_i = -m\frac{\partial V}{\partial x_j} + m\Theta_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j}, \tag{1.5}$$

for the Kepler potential $V \propto r^{-1}$ in a weak non-commutative background Θ predicts a perihelion point precession in the planetary motions, providing us with a measurable quantity to test cosmological models.

In all these models, the *momentum* satisfies a first order ODE in terms of a (possibly of the Lorentz-type) force. But the same structure appears in the equations for the position, which are distinguished by an unusual *velocity relation*, as a result from the Berry phase contributions, associated with the environment in which the particle moves, or by a postulated fundamental *area scale*. Thus, the momentum and the velocity may not be proportional.

In the absence of any magnetic field, for example, equations (1) reduce to those used to explain the AHE observed in ferric materials [5]: the effect comes, entirely, from the *anomalous velocity term*. Let us emphasize that such a behavior has been advocated a long time ago [11]. It has been shown indeed that the Lorentz-equation follows from general principles, but the momentum-velocity relation is model-dependent. Relations like $\vec{p} = m\dot{\vec{r}}$ are indeed mere Ansätze, and it is not required by any first principle.

More recently, similar features has been found in 2D systems for the so called *exotic* particles, i.e., ones which are associated with the kinematical two-parameter central extension of the planar Galilei group [12, 13]. These models are, once again, instrumental in describing the Fractional Quantum Hall Effect. Those systems support a sort of *duality* between the magnetic field, $\vec{B}(\vec{r})$, and its analog in momentum space, $\vec{\Theta}(\vec{p})$, called sometimes a dual magnetic field. By consequence the *Poisson bracket* of the *position* variables they may no longer vanish, in analogy with the components of the momentum in the presence of a magnetic field. In the relativistic case, similar, but more elaborated constructions have been proposed [14, 15], which however we do not discuss here.

Even the question of consistency of the above effective models with the general principles of mechanics is legitimate, since they are mainly derived by some semiclassical de-quantization procedure, which does not necessarily fit into the framework of classical mechanics. Thus, our primary goal below is to prove that no new mechanics has to be invented : all these models fit perfectly into Souriau's presymplectic framework of Classical Mechanics [16]. The second aim is to formulate the hamiltonian theory in that context, exploiting the supplementary structure encoded into the second central extension of the Galilei group.

2 The Lagrange-Souriau 2-form

The modern geometrical formulation [16] of the calculus of variations, originated by Lagrange and continued by Cartan [17], consists in mapping the Lagrangian function $L = L(\vec{x}, \vec{v}, t)$, defined on the *evolution space* $TQ \times \mathbb{R} \to \mathbb{R}$, into the so-called Cartan 1-form λ, defined by the crucial property

$$\int L(\gamma(t), \dot{\gamma}, t)dt = \int_{\widetilde{\gamma}} \lambda \quad \text{with} \quad \lambda = \frac{\partial L}{\partial p_i} dx^i + \left(L - \frac{\partial L}{\partial p_i} p_i \right) dt. \tag{2.1}$$

where $\widetilde{\gamma} = (\gamma(t), \dot{\gamma}, t)$ is the lifted world-line in the tangent bundle of the evolution space $TQ \times \mathbb{R}$. The exterior derivative of the Cartan form provides us with a closed *Lagrange-Souriau* (LS) 2-form $\sigma = d\lambda$, which in general cannot be separated canonically into a *symplectic* and a *Hamiltonian part* [16]. However, the associated Euler-Lagrange equations are still expressed by determining the kernel of σ, i.e. by the equation $\sigma(\dot{\widetilde{\gamma}}, \cdot) = 0$. If the kernel is one-dimensional, the variational problem is called regular. Otherwise the singular case requires to resort to the symplectic reduction techniques [16, 17, 18, 19]. Thus more general procedures have to be adopted to build such a system and clarify their Hamiltonian structure.

Conversely, again following Souriau [16], a generalized mechanical system is given by a 2-form σ, which is closed $d\sigma = 0$ and with constant rank d. Then, its kernel defines an integrable foliation with d-dimensional leaves, which can be viewed as generalized solutions of the variational problem. Moreover, by the Poincaré lemma, $d\sigma = 0$ implies the local existence of a Cartan 1-form λ. Rewriting it as $\lambda = a_\alpha d\xi^\alpha$, one can plainly define the first-order Lagrangian function on the evolution space as

$$\mathcal{L} = a_\alpha \dot{\xi}^\alpha \quad \text{such that} \quad \int \mathcal{L} dt = \int \lambda. \tag{2.2}$$

Thus, the above-mentioned models do not have a usual Lagrange function defined on the tangent bundle. Put in another way, the position does not satisfy a second-order Newton equation. The general question of the existence of a locally defined Lagrangian has been discussed [20, 25].

Moreover, if the LS 2-form σ can be split as

$$\sigma = \omega - dH \wedge dt, \tag{2.3}$$

where ω is a closed and regular 2-form on the *phase space* TQ and H is a Hamiltonian function on $TQ \times \mathbb{R}$, than the equations of motion read

$$\omega(\dot{\gamma}) = dH. \tag{2.4}$$

Assuming that ω is regular, the inverse $(\omega^{\alpha\beta})$ of the symplectic matrix $\omega = \omega_{\alpha\beta}$ (i.e. $\omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}$) yields the Hamilton equations $\dot{\xi_i} = \{\xi_i, H\}$, through the Poisson brackets defined by

$$\{f, g\} = \omega^{\alpha\beta}\partial_\alpha f \partial_\beta g. \tag{2.5}$$

In the case of singularity, as said above, symplectic reductions are needed.

Thus, in Souriau's framework, one can state the inverse problem of the calculus of variations for given equations for a one-particle system in presence of a position-dependent force field $\vec{E}$ only, rewriting them as the set of 1-forms on $TQ \times \mathbb{R}$

$$\alpha_1 = d\vec{r} - \vec{p}\, dt \qquad \alpha_2 = m\, d\vec{p} - \vec{E}\, dt, \tag{2.6}$$

and view both of them as hyperplanes in the evolution space, defined by the kernel of the one-forms α_i. Notice that we have introduced the kinetic momentum $\vec{p} = m\vec{v}$, and we will refer to it in the sequel. Then, the simultaneous solutions correspond to the intersection of these hyperplanes, described by the kernel $\sigma(\delta y, \cdot) = 0$ of exterior product $\sigma = \alpha_1 \wedge \alpha_2$, where $\wedge$ warrants the antisymmetry. In presence of an electromagnetic field $\vec{E}(\vec{r}, \vec{p}, t), \vec{B}(\vec{r}, \vec{p}, t)$ acting on a charge e, Souriau [16] generalized the previous simplest 2-form to

$$\sigma \;\; = \;\; \left(md\vec{v} - e\vec{E}dt\right) \wedge (d\vec{r} - \vec{v}dt) + e\vec{B} \cdot d\vec{r} \times d\vec{r}, \tag{2.7}$$

where we have defined $(d\vec{r} \times d\vec{r})_k = \frac{1}{2}\epsilon_{kij} dx_i \wedge dx_j$. Then, the usual equations of motion of a charged particle in the electromagnetic field are seen to arise as the kernel of σ, together with the regularity condition $d\sigma = 0$ leading to the the homogeneous Maxwell equations. These formulas can be readily generalized to the multi-particles case.

Now, in the same spirit we would like to write down a Lagrangian 2-form for a particle of mass m, which is subjected both to the electromagnetic field $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$, and to a peculiar environment, the local characteristics of which may depend on the momentum.

3 The (2+1)-dim Duval - Horváthy model

The simplest example of such a situation is the *exotic* mechanical model in 2 space dimensions proposed in [13], defined by a form generalizing (2.7) which can be split as in (2.3): precisely we introduce

$$\sigma = dp_i \wedge dx_i + \frac{1}{2}\theta\, \epsilon_{ij}\, dp_i \wedge dp_j + eB\, dx_1 \wedge dx_2 + d\left(\frac{\vec{p}^{\,2}}{2m} + eV\right) \wedge dt, \quad (3.1)$$

where $B(\vec{r})$ is the magnetic field, perpendicular to the plane, $V(\vec{r})$ the electric potential (both of them assumed to be time-independent for simplicity) and θ is a constant, called *the non-commutative parameter*, for reasons clarified below. The resulting equations of motion read [13]

$$m^* \dot{x}_i = p_i - em\theta\, \epsilon_{ij} E_j, \qquad \dot{p}_i = eE_i + eB\, \epsilon_{ij} \dot{x}_j, \quad (3.2)$$

where we have introduced the *effective mass* $m^* = m\,(1 - e\,\theta\, B)$. The novel physical features are: i) the *anomalous velocity* term $-em\theta\, \epsilon_{ij} E_j$, so that $\dot{x}_i$ and p_i are not in general parallel, ii) the interplay between the exotic structure and the magnetic field, yielding the effective mass m^* in (3).

The *LS* 2-form (3.1) can obtained by exterior derivation from a *Cartan* 1-form on the evolution space $\mathbb{R}^2 \times \mathbb{R}$, namely from

$$\lambda = (p_i - A_i)dx_i - \frac{\vec{p}^{\,2}}{2m}\, dt + \frac{\theta}{2}\, \epsilon_{ij}\, p_i\, dp_j \quad (3.3)$$

defining the action functional as in (2.2). Accordingly, for $m^* \neq 0$, the associated Poisson brackets are

$$\{x_1, x_2\} = \frac{m}{m^*}\, \theta, \quad \{x_i, p_j\} = \frac{m}{m^*}\, \delta_{ij}, \quad \{p_1, p_2\} = \frac{m}{m^*}\, eB. \quad (3.4)$$

and they automatically satisfy the Jacobi identity. When effective mass vanishes, i.e. when the magnetic field takes the critical value $B_{crit} = \frac{1}{e\theta}$, the system becomes singular. Then symplectic reduction procedure leads to a two-dimensional system characterized by the remarkable Poisson structure $\{x_1, x_2\} = \theta$, reminiscent of the "Chern-Simons mechanics" [22]. Thus, the symplectic plane plays, simultaneously, the role of both configuration and phase spaces. The only motions are those following the *Hall law.* Moreover, in the quantization of the reduced system, not only the position operators no longer commute, but the quantized equation of motions yields the *Laughlin* wave functions [23], which are the ground states in the Fractional Quantum Hall Effect (FQHE). Thus one can claim that the classical counterpart of the *anyons* are in fact the *exotic* particles in the system (3.2). In the review article [24] several examples of 2-dimensional models, which generalize the form (3.1) and the equations (3.2) have been discussed. Here let us conclude that the Poisson structure (3.4) can be obtained by applying the Lie-algebraic Kirillov-Kostant-Souriau method for constructing dynamical systems to the (2+1) 2-fold centrally extended Galilei group [25, 26]. The two cohomological parameters are the mass and a second, "exotic", parameter, identified with a constant Berry curvature in the context of the condensed matter physics, or as a non relativistic limit of spin (see [24] and references therein). The latter is highlighted by the non-commutativity of Galilean boost generators,

$$[K_1, K_2] = \imath\theta m^2. \tag{3.5}$$

4 The general model in (3+1)-dim

After the above digression on two-dimensional models, let us look for further generalizations [27] - [31] of the 2-form (2.7) with *momentum* dependent fields. Straightforward algebraic considerations lead to define on the space - momentum - time evolution space the manifestly anti-symmetric covariant 2-tensor on the evolution space of

$$\begin{aligned}\sigma \;&=\; [(1-\mu_i)\,dp_i - e\,E_i\,dt]\wedge(dr_i - g_i\,dt) + \tfrac{1}{2}\,e\,B_k\,\epsilon_{kij}\,dr_i\wedge dr_j + \\ &\quad \tfrac{1}{2}\,\kappa_k\,\epsilon_{kij}\,dp_i\wedge dp_j + q_k\,\epsilon_{kij}\,dr_i\wedge dp_j,\end{aligned} \tag{4.1}$$

where we have put into evidence the Lorentz contribution to the Lagrangian, the quantities $\vec{g}$, $\vec{\kappa}$ and $\vec{q}$ are 3-vectors and $Q = \mathrm{diag}\,(\mu_i)$ is a 3×3 matrix, respectively. They depend on all independent variables and have to be determined in such a way that σ is closed and has constant rank. By the expression $1-\mu_i$ we would like to distinguish among the *bare* mass, normalized to 1, and the possible variable contributions. It is interesting to note that, with respect to the expressions adopted in [16] and [13], we have to introduce the 1-form $dr_i - g_i\,dt$, which defines a new conjugate momentum.

The equations of motion can be written as the kernel of the LS 2-form $\sigma\,(\delta y,\cdot) = 0$ for a tangent vector $\delta y = (\delta\vec{r}, \delta\vec{p}, \delta t)$. Specifically one obtains the equations

$$e\,\delta\vec{r}\cdot\vec{E} - (1-Q)\,\delta\vec{p}\cdot\vec{g} \;=\; 0, \tag{4.2}$$

$$(1-Q-Q_A)\,\delta\vec{p} \;=\; e\left(\vec{E}\,\delta t + \,\delta\vec{r}\times\vec{B}\right), \tag{4.3}$$

$$(1-Q)\;(\delta\vec{r} - \vec{g}\,\delta t) \;=\; -\delta\vec{r}\times\vec{q}\, - \delta\vec{p}\times\vec{\kappa}\,, \tag{4.4}$$

where we have introduced the anti-symmetric matrix $(Q_A)_{ij} = \epsilon_{ijk}\, q_k$. From the equation (4.3) we can solve with the respect of the vector $\delta\vec{p}$, if the mass matrix $M = \mathbf{1} - Q - Q_A$ is invertible, i.e. if $\det(M) \neq 0$. Under such an hypothesis together with $\det(m - Q) \neq 0$, one replaces $\delta\vec{p}$ in the equation (4.4), finding an equation for the position tangent vector $\delta\vec{r}$ of the form

$$M^*\,\delta\vec{r} = \Big((1-Q)\,\vec{g} + e\,\mathcal{K}M^{-1}\cdot\vec{E}\Big)\,\delta t, \tag{4.5}$$

where the effective mass matrix is given by

$$M^\star = M + \left(2Q_A - e\,\Theta M^{-1}\mathcal{B}\right)\ , \quad \Theta_{ij} = \epsilon_{ijk}\kappa_k\ ,\ \mathcal{B}_{ij} = \epsilon_{ijk}B_k. \tag{4.6}$$

Both M^* and the eq. (4.5) generalize of the expressions obtained in [13] leading to (3.2). Singularities in the motion can arise, both from the vanishing of $\det(M)$ and degeneracies of the first factor in (4.6), that is at $\det(M^*) = 0$. However, if this is not the case, one can solve (4.5) w.r.t $\delta\vec{r}$ and show that

1. equation (4.2) is identically satisfied independently from the specific choice for the vector $\vec{g}$,

2. the equation (4.3) becomes

$$M^*\delta\vec{p} = \frac{e}{\mathrm{Det}\,(M)}\left(R\,\vec{E} - \vec{g}^{\,T}N\vec{B}\right)\delta t, \tag{4.7}$$

   where matrices R and N have an involved dependency on m, $\vec{B}$, $\vec{\kappa}$, Q and Q_A to be spelled here.

Thus we have a system of simultaneous first order differential equations: (4.5) for the velocity of the particle $\frac{\delta\vec{r}}{\delta t}$ and (4.7) for the momentum variation $\frac{\delta\vec{p}}{\delta t}$. Notice that these two equations simplify to the equations (8) and (9) of the [13], when Q and Q_A go to 0 and $\vec{g} \equiv \vec{p}$.

Now, the closure condition in the evolution space $d\sigma = 0$ (called also "Maxwell Principle" by Souriau [16]) of the Lagrangian-Souriau form σ in (4.1) leads to the coefficients of the independent 3-forms. It is quite natural to assume the following limitations on the involved functions: $\partial_{p_i}E_j = \partial_{p_i}B_j = 0$. Thus we are lead to the equations

$$\partial_{r_j}B_j = 0, \qquad \varepsilon_{kij}\partial_{r_i}E_j = -\partial_t B_k, \tag{4.8}$$

$$\partial_{p_j}\kappa_j = 0, \qquad \varepsilon_{kij}\partial_{p_i}\left[(1-\mu_j)\,g_j\right] = \partial_t\kappa_k, \tag{4.9}$$

$$\partial_t\mu_i = \partial_{r_i}\left[(1-\mu_i)\,g_i\right], \qquad \tfrac{1}{2}\varepsilon_{kij}\,\partial_{r_i}\,\left[(1-\mu_j)\,g_j\right] = \partial_t q_k, \tag{4.10}$$

$$\partial_{r_i}\,\mu_j = \varepsilon_{ijk}\partial_{r_j}q_k, \quad \partial_{r_i}\kappa_j = \varepsilon_{ijk}\partial_{p_k}\,\mu_i + \partial_{p_i}\,q_j - \delta_{ij}\partial_{p_k}\,q_k, \tag{4.11}$$

with the residual closure relations (without summation over repeated indices)

$$\partial_{r_j}\left[\left(1-\mu_i\right)g_i\right]+\partial_{r_i}\left[\left(1-\mu_j\right)g_j\right]=0, \qquad i\neq j=1,2,3. \tag{4.12}$$

One can observe that the homogeneous Maxwell equations (4.8) are the only restrictions on the electromagnetic fields $\left(\vec{E},\vec{B}\right)$. Equations (4.9) are the analogs of the previous relations in the velocity (momentum) space for the vector-field $\vec{\kappa}$, which measures the extent to which the spatial coordinates fail to commute in three dimensions, as we will see in the Hamiltonian formalism. For such a reason, sometimes $\vec{\kappa}$ is called dual magnetic field.

If $\vec{\kappa}$ is non trivial in time, then a change in the velocity dependence is induced for the mass flow $\left(\mathbf{1}-Q\right)\vec{g}$, as prescribed by the second equation in (4.9). In its turn, equations (4.10) say how the particle mass may change in time. This seems to be a quite unusual situation, but we cannot discard it at the moment. On the other hand, the first set of three equations in (4.10) has the form of independent continuity equations, leading to the global conservation law for the total mass, i.e. $\partial_t\left(\sum_i\mu_i\right)+\partial_{r_i}\left[\left(1-\mu_i\right)g_i\right]=0$, which however holds separately in different directions. Due to $\vec{q}$, also the skew-symmetric contributions to the mass matrix M may change on time, but they generate modification of the mass flux in space, accordingly to the second set of equations in (4.10).

The equations in (4.11) are more difficult to interpret: they provide consistency relations for both the space and the velocity dependency among the mass matrix elements and the dual magnetic field. Putting such expressions into the equation of motion in the form (4.3)-(4.4), in a pure axiomatic way one re-obtains the equations found in the context of the semiclassical motion of electronic wave-packets in [4].

For a particle in a *flat* evolution space, i.e. for $\mu_i\equiv 0$ and momentum $p_i=g_i$, one easily concludes that $\vec{\kappa}$ and $\vec{q}$ have to be constants. The analysis of the closure relations (4.10)-(4.11) leads to the expression $\kappa_i=\sum_{j\neq i}\left(x_j\partial_{p_j}q_i-x_i\partial_{p_j}q_j\right)+\chi_i$,where the q_i's and χ_i 's depend only on the velocities and moreover the divergenceless condition $\partial_{p_j}\chi_j=0$ has to be satisfied. A remarkable example for its phenomenological implications is provided by the monopole field in momentum space $\vec{\kappa}=\theta\frac{\vec{p}}{|\vec{p}|^3}$, which is indeed the only possibility consistent with the spherical symmetry and the canonical relations $\{r_i,p_j\}=\delta_{ij}$ [29]. Its expression appears to be consistent, at least qualitatively, with the data reported in ([5]) and in Spin Hall Effects [6], [30].

Limiting ourselves to two spacial dimensions and setting $\kappa_3=\text{const}$, a free particle admits in fact the "exotic" two-parameter centrally extended Galilei group as symmetry [13]. In a more general situation one deals with a momentum-dependent non-commutativity field $\kappa_3 = \kappa(\vec{p})$, like it was considered by [32] with $\kappa = -\frac{\theta}{1+\theta p^2}\varepsilon_{ij}p_i r_j$ (and $\mu_i \equiv 0$), or $\kappa^{\alpha\beta} = \frac{s}{2}\frac{p_\alpha \epsilon^{\alpha\beta\gamma}}{(p^2)^{3/2}}$ for the planar relativistic model for a spinning particle (again a sort of monopole in relativistic momentum space) [33, 15].

Finally, in the singular submanifold of the phase space defined by $M^* = 0$, we need to look at the proper restrictions on vector-fields $\delta\vec{p}$ and $\delta\vec{r}$, in order to avoid motion with infinite velocities. In particular in two dimensions, assuming that the only non vanishing components of the magnetic, of the dual magnetic fields and of the $\vec{q}$ are the ones orthogonal to the plane of motion, with $\det(1-Q) \neq 0$, the singularity mass manifold is described by

$$B_{crit} = \frac{q^2 + (m-\mu_1)(m-\mu_2)}{e\kappa}. \tag{4.13}$$

For sake of simplicity, here we assume all the above quantities as constants on space-velocity variables. The equations of motion (4.2) - (4.4) and the closure relations are satisfied by the following constraints for the mass flow and electric field components

$$(1-\mu_i)\, g_i = -\frac{e\kappa\,(qE_i - \epsilon_{ij}E_j\,(1-\mu_i))}{q^2 + (1-\mu_1)(1-\mu_2)} = \frac{(qE_i - \epsilon_{ij}E_j\,(1-\mu_i))}{B_{crit}}, \tag{4.14}$$

$$(1-\mu_2)\,\partial_{r_1}E_1 + (\mu_1 - 1)\,\partial_{r_2}E_2 + q\,(\partial_{r_1}E_2 + \partial_{r_2}E_1) = 0. \tag{4.15}$$

Notice that the last relation comes from the closure condition on the mass terms (4.12), not from the Faraday law (4.8), which is identically satisfied. Since the same equations provides the time evolution of the mass terms and of κ, the only left by above assumptions, the electric field can depends at most linearly on space coordinates, compatibly with (4.15). Finally, the equation (4.14) generalizes the *Hall law* discussed in the previous Section.

5 Hamiltonian Structure

Comparing the system (4.5)-(4.7) with the previous ones in (3.2), one recognizes the general common structure, due to derivation from the same unifying Lagrange-Souriau approach provided by the 2-form (4.1), even if the peculiarity of the doubled folded central extensions is enjoyed only in the 2-dimensional setting.

The 2-form σ can be obtained as the exterior derivative of the Cartan 1-form,

$$\lambda = \left(\vec{p} + \overrightarrow{\mathcal{A}}\right) \cdot d\vec{r} + \vec{\mathcal{R}} \cdot d\vec{p} - \mathcal{T}\, dt. \tag{5.1}$$

In this formula the field $\overrightarrow{\mathcal{A}}\,(\vec{r},t)$ is the usual electromagnetic potential, such that $\vec{B} = \nabla_{\vec{r}} \times \overrightarrow{\mathcal{A}}$, for which we have postulated only a space - time dependency, eventually resorting to a suitable gauge transformation. Then, the electric field (in fact any space-time dependent force), is given by $\vec{E} = -\nabla_{\vec{r}}\mathcal{T} - \partial_t \overrightarrow{\mathcal{A}}$, where we assume the following decomposition for the scalar function $\mathcal{T}\,(\vec{r},\vec{p},t) = \mathcal{E}\,(\vec{p},t) + \varphi\,(\vec{r},t)$, in order to keep valid the previous restrictions on the dependency of $\vec{E}$ fields. On the other hand, the field $\vec{\mathcal{R}}\,(\vec{r},\vec{p},t)$ defines the dual magnetic field $\vec{\kappa} = \nabla_{\vec{p}} \times \vec{\mathcal{R}}$, the mass flow is given by $(\mathbf{1} - Q)\,\vec{g} = -\nabla_{\vec{p}}\mathcal{T} - \partial_t\vec{\mathcal{R}}$, $\mu_i = \partial_{r_i}\mathcal{R}_i$ and $q_k = \partial_{r_i}\mathcal{R}_j = -\partial_{r_j}\mathcal{R}_i$, where k,i,j are in cyclic order. The last equality imposes a set of constraints on $\vec{\mathcal{R}}$ in such a way the Lagrangian-Souriau 2-form takes exactly the expression (4.1). The above restrictions implies certain second order relations, which assure that also the first set of closure relations in (4.11) are satisfied, namely

$$\partial_{r_i}\partial_{r_j}\mathcal{R}_k = 0 \quad (\,i,j,k \text{ cyclic}), \qquad \partial^2_{r_i}\mathcal{R}_j = -\partial_{r_i}\partial_{r_j}\mathcal{R}_i \quad (i \neq j)\,, \tag{5.2}$$

with no summation over repeated indices in the last equation. The remaining closure relations in (4.11) are identically satisfied, while those in (4.10) require the supplementary constraints on time derivatives

$$\partial_t \left(\partial_{p_i}\mathcal{R}_j + 2\partial_{p_j}\mathcal{R}_i\right) = 0 \qquad i \neq j. \tag{5.3}$$

Due to the special form we assumed on the force and magnetic fields, the above restrictions on $\vec{\mathcal{R}}$ limit its space-time dependency, leaving however the gauge freedom with respect the momentum variables.

Thus, in terms of the potentials, or connections, introduced in (5.1), the equations of motion (4.3)-(4.4) become

$$\begin{aligned} (\delta_{ij} + \Xi_{ij})\,\dot{r}_j + \Theta_{ij}\,\dot{p}_j &= \partial_{v_i}\mathcal{E} + \partial_t\mathcal{R}_i, \\ \mathcal{B}_{ij}\,\dot{r}_j + (\delta_{ij} + \Xi_{ij})\,\dot{p}_j &= -\partial_{r_i}\varphi - \partial_t\mathcal{A}_i, \end{aligned} \tag{5.4}$$

where we have used the matrices (see also eq. (4.6))

$$\Xi_{ij} = \begin{cases} -\partial_{r_j}\mathcal{R}_i, & i \leq j \\ \partial_{r_i}\mathcal{R}_j, & i > j \end{cases} \qquad \mathcal{B}_{ij} = \varepsilon_{ijk}\varepsilon_{jhk}\partial_{r_h}\mathcal{A}_k, \quad \Theta_{ij} = \varepsilon_{ijk}\varepsilon_{jhk}\partial_{p_h}\mathcal{R}_k. \tag{5.5}$$

Differently from system (4.3)-(4.4) on the evolution space, now the dynamical system (5.4) is defined on the tangent manifold of the phase space

$TQ \equiv \{\xi = (\vec{r}, \vec{p})\}$ space. But, if $\partial_t \vec{\mathcal{A}} = \partial_t \vec{\mathcal{R}} \equiv 0$, it is possible to rearrange the form (4.1) as $\sigma = \omega - dH \wedge dt$ by introducing the symplectic 2-form on TQ

$$\omega = (\delta_{i,j} + \Xi_{ij})\, dr_{\,i} \wedge dp_{\,j} + \frac{1}{2} [\mathcal{B}_{ij}\, dr_{\,i} \wedge dr_{\,j} - \Theta_{ij}\, dp_{\,i} \wedge dp_{\,j}] \tag{5.6}$$

and the Hamiltonian function $H \equiv \mathcal{T} = \mathcal{E}(\vec{p}, t) + \varphi(\vec{r}, t)$.

Thus, the closure of $\omega = \omega_{\alpha\beta}\, d\xi_\alpha \wedge d\xi_\beta$ is assured by that one of σ, i.e. by the equations (4.8) - (4.12) plus the restriction (5.2). Then, the space TQ becomes a Poisson manifold, with Poisson brackets defined, as in (2.5), by the co-symplectic matrix

$$\omega^{\alpha,\beta} = \left(1 - \tfrac{1}{2}\mathrm{Tr}\left(\Xi^2 + X\,(\mathbf{1} + 2\,\Xi)\,\Theta\right)\right)^{-1} \tag{5.7}$$

$$\left\{ \begin{pmatrix} \Theta + [\Xi, \Theta] & 0 \\ 0 & -\mathcal{B} + [\Xi, \mathcal{B}] \end{pmatrix} + \left[1 - \tfrac{1}{6}\mathrm{Tr}\left(\Xi^2 + \mathcal{B}\,\Theta\right)\right] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \left(\Xi^2 + X\,\Theta\right)^T \\ -\left(\Xi^2 + \mathcal{B}\,\Theta\right) & 0 \end{pmatrix} \right\},$$

non degenerate for $\sqrt{\det(\omega_{\alpha\beta})} = 1 - \frac{1}{2}\mathrm{Tr}\left(\Xi^2 + \mathcal{B}\,(\mathbf{1} + 2\,\Xi)\,\Theta\right) \neq 0$. Such a factor generalizes the denominators present in the Poisson brackets (3.4) or (4.6). Moreover, it crucially appears in the expression of the invariant phase - space volume, ensuring the validity of the Liouville theorem. Finally, notice that the Poisson structure is determined only by gauge invariant quantities and brackets involving position coordinates r_i are in general non-commutative. As it has been elsewhere remarked [24], it is possible to perform a change of variables leading to commutative position variables by a point transformation of the form $r_i \to r'_i = r_i - \mathcal{R}_i(\vec{r}, \vec{p})$. However, the vector field $\vec{\mathcal{R}}$ is defined up to a gauge transformation generated by an arbitrary function on $(\vec{r}, \vec{p})$. Thus, the meaning of the notion of position is unclear in such a context.

For a charge subjected only to a monopole of strength θ in momentum space and to a uniform electric field, the equations of motion obtained (5.8) are readily integrated [30]. The particle suffers a shift $\Delta = \frac{2\theta}{p_0}$ in the direction $\vec{E} \times \vec{p}_0$, being $\vec{p}_0$ the initial linear kinetic momentum. This quantify the discussion in point b) in the Introduction. On the other hand, if the charge of Hamiltonian $H = \frac{|\vec{p}|^2}{2}$ is driven only by a magnetic and by a dual monopole, the equations are

$$\frac{M^*}{|\vec{r}|^3|\vec{p}|^3}\dot{r}_i = p_i - e\theta\frac{r_i}{|\vec{p}||\vec{r}|^3}, \tag{5.8}$$

$$\frac{M^*}{|\vec{r}|^3|\vec{p}|^3}\dot{p}_i = e\varepsilon_{ijk}\frac{p_j r_k}{|\vec{r}|^3}. \tag{5.9}$$

where $M^* = |\vec{r}|^3|\vec{p}|^3 - e\theta\ \vec{r}\cdot\vec{p}$.

A final remark in connection with the coupling to the electromagnetic field adopted in (4.1), sometime said *the minimal addition* and leading to the Hamiltonian $\mathcal{T}$ above. It is quite different from the usual *minimal coupling* procedure and it yields a very different Poisson structure. In the context of the 2-dimensional systems this problem was reviewed in [24]. In particular the two formulations, in that context, were proved to be equivalent under a classical Seiberg-Witten transformation of electromagnetic fields. No results are yet available in the situations discussed in the present paper.

In conclusion a wide set of dynamical systems is derived from the Lagrange-Souriau approach in 3-dimensions. Generalizations to higher number of degrees of freedom seems straightforward. We have shown the conditions to assure their Hamiltonian formulation. From which an analysis for their integrability properties can be pursued more plainly, by resorting to standard methods. Alternatively, one can perform a symmetry analysis directly on the *LS* 2 -form (4.1) accordingly to [25]and [34].

Acknowledgments

The author expresses his indebtedness to P. Horvathy for having been introduced to the problem and for continuous and stimulating discussions. Some of the results were obtained in collaboration with C. Duval, Z. Horváth and P. Stichel. The work has been partially supported by the INFN - Sezione of Lecce under the project LE41.

References

[1] D. Xiao, M. C. Chang and Q. Niu, Rev. Mod. Phys. **82** (2010) , 1959

[2] K. Y. Bliokh, J. Opt. A: Pure Appl. Opt. 11 (2009) 094009.

[3] R. J. Szabo, Phys. Rept. **378** (2003) 207.

[4] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu and J. Zwanziger, Springer Verlag (2003).

[5] Z. Fang , N. Nagaosa , K. S. Takahashi ,A. Asamitsu , R. Mathieu , T. Ogasawara , H. Yamada , M. Kawasaki , Y. Tokura , K. Terakura , *Science* **302** (2003), 92

[6] S. Murakami, N. Nagaosa, and S.-C. Zhang, *Science* **301**, 1348 (2003).

[7] K. Yu. Bliokh, Yu. P. Bliokh, JETP Lett. **79**, 519 (2004); K. Yu. Bliokh and Yu. P. Bliokh, *Phys. Lett.* **A333**, 181 (2004) [physics/0402110]; N.R. Sadykov, *Theor Math. Phys.* **135**, 685 (2003) ; M. Onoda, S. Murakami, N. Nagaosa, *Phys. Rev. Lett.* **93**, 083901 (2004);

[8] C. Duval, Z. Horváth, P. A. Horváthy, *Phys. Rev.* **D74**, 021701 (2006) C. Duval, Z. Horváth, P. A. Horváthy, *Journ. Geom. Phys.* **57**, 925 (2007);

[9] C.-W. Zhang, A. M. Dudarev, Q. Niu, Phys. Rev. Lett. **97**, 040401 (2006).

[10] J. M. Romero and J. D. Vergara, Mod. Phys. Lett. **A 18** (2003) 1673

[11] W. G. Dixon, *Il Nuovo Cimento* **38**, 1616 (1965); J.-M. Souriau, *Ann. Inst. Henri Poincaré*, **20 A**, 315 (1974). In the planar context, see C. Duval and P. A. Horváthy, *Phys. Lett.* **B594**, 402 (2004); *Theor. Math. Phys.* **144**, 899 (2005).

[12] J. Lukierski, P. C. Stichel, W. J. Zakrzewski, *Annals of Physics* (N. Y.) **260**, 224 (1997); J. Lukierski, P.C. Stichel and W.J. Zakrzewski, Ann. Phys. (N.Y.) **306**, 78 (2003)

[13] C. Duval and P. A. Horváthy, *Phys. Lett.* **B 479**, 284 (2000) ; C. Duval and P. A. Horváthy, *Journ. Phys.* **A 34**, 10097 (2001)

[14] A. A. Deriglazov, *JHEP* **03**, 021 (2003)

[15] M. S. Plyushchay, *Phys. Lett.* **B273**, 250 (1991). M. S. Plyushchay, Phys. Lett. B **262** (1991) 71; P. A. Horváthy and M. S. Plyushchay : *JHEP* **06**, 033 (2002); P. A. Horváthy, M. S. Plyushchay and M. Valenzuela, Ann. Phys. **325** 1931-1975 (2010) arXiv:1001.0274.

[16] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod,Paris (1970);

[17] E. Cartan, *Leçons sur les invariants integraux*. Hermann: Paris (1922).

[18] L. D. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).

[19] P. A. Horváthy, *Ann. Phys.* **299**, 128 (2002)

[20] P. A. Horváthy, *Journ. Math. Phys.* **20**, 49 (1979)

[21] D. R. Grigore, *Int. J. Mod. Phys.* **A 7**, 7153 (1992)

[22] G. Dunne R. Jackiw and C. A. Trugenberger, *Phys. Rev.* **D41**, 661 (1990);

[23] R. B. Laughlin,*Phys. Rev. Lett.* **50**, 1395 (1983).

[24] P.A. Horváthy, L. Martina, P. Stichel in "Noncommutative Spaces and Fields", P. Aschieri *et al.* ed.s , SIGMA 6 (2010), 060.

[25] D. R. Grigore, *Journ. Math. Phys.*. **37**, 460 (1996).

[26] A. Ballesteros, M. Gadella, and M. del Olmo,*Journ. Math. Phys.* **33**, 3379 (1992); Y. Brihaye, C. Gonera, S. Giller and P. Kosiński, [`hep-th/9503046` (unpublished)].

[27] C. Duval, Z. Horváth, P. A. Horváthy, L. Martina and P. Stichel, *Mod. Phys. Lett.* **B20**, 373 (2006) ; C. Duval, Z. Horváth, P. A. Horváthy, L. Martina and P. Stichel, *Phys. Rev. Lett* **96**, 099701 (2006).

[28] D. Xiao, J. R. Shi and Q. Niu, Phys. Rev. Lett. **95** (2005) 137204

[29] A. Bérard and H. Mohrbach, *Phys. Rev.* **D 69**, 127701 (2004); P. Gosselin, F. Menas, A. Bérard and H. Mohrbach, Europhys.Lett. 76 (2006) 651-656.

[30] P. A. Horváthy, *Phys.Lett.* **A 359** (2006), 705-706

[31] K. Yu. Bliokh, *Phys. Lett.* **A351** 123 (2006) ;

[32] Snyder H.S., *Phys. Rev.* **71** (1947), 38.

[33] R. Jackiw and V. P. Nair, *Phys. Rev.* **D43**, 1933 (1991)M. S. Plyushchay, *Phys. Lett.* **B248**, 107 (1990);

[34] R. Banerjee *Eur. Phys. J.* **C 47**, 541545 (2006); *Phys. Rev.* **D 75**, 045008 (2006).