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Ranking Functions Induced by Weighted Average of Fuzzy Numbers

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Abstract. In this paper we present two definitions of possibilistic weighted average of fuzzy numbers, and by them we introduce two different rankings on the set of real fuzzy numbers. The two methods are dependent on several parameters. In the first case, the parameter is constant and the results generalize what Carlsson and Fuller have obtained in (2001)[2]. In the second case, the parameter is a function, not fixed *a priori* by the decision maker, but it depends on the position of the interval on the real axe. In all the two cases we call the parameter *degree of risk*, which takes into account of a risk-tendency or aversion of the decision maker.

Keywords: fuzzy numbers, average value, possibilistic weighted average, ranking methods.

1. Introduction

In most real situation, one is forced to take decision on the basis of ill-defined variables and imprecise data. The theory of fuzzy sets is a natural tool to model this situation as fuzzy numbers well represent imprecise quantities. In decision-making problems we have the necessity to optimise some procedure and so it is necessary to have ranking of the quantities involved. Many authors have studied different definitions of ranking on the set of fuzzy numbers F, (Bortolan and Degani (1985)). Most of these are based on the definition of an *evaluation function* (*F-evaluation function*), which maps fuzzy numbers in the real line. The order on F is induced by the real number total order. As fuzzy numbers are intervals in which boundaries are blurred, the difficulty in ranking them arise from the problem created in ranking real intervals. When the supports of fuzzy numbers are disjointed, there are no problems and all the methods lead to the same solution. But the decision is not evident when the intersection between supports is not empty. In these cases, it seems that the solution depends on subjective elements depending of the nature of the problem and the decision-maker.

Following the idea of Campos and Gonzalez (1989) we start with the introduction of two different *evaluation functions* on the set of real intervals, we will call *I-evaluation functions*. They contain a parameter we call *degree of risk*, which takes into account of a risk-tendency or aversion of the decision maker, that is constant in the first case, a function in the second, and then, using the definition of a fuzzy numbers by its α-cuts, we introduce two *average values* (AV) as *F-evaluation functions*. Using these two, we obtain several results. One is to generalise the results of Carlsson and Fuller. They, in a paper of 2001, starting from a particular AV, have introduced the notations of *lower and upper possibilistic mean value* and consequently defined *the interval-valued possibilistic mean, crisp possibilistic mean value* and *crisp variance* of a continuous possibility distribution.

The two new AV are consistent with the extension principle and with the well-known definitions of expectation and variance in probability theory. Another result is the introduction of a definition of ranking functions on the set of fuzzy numbers F.

2. Two Different Types of Average Value

Following the idea introduced by Campos and Gonzalez (1989), (see also Gonzalez (1990), Campos and Gonzalez (1994)) we begin with the introduction of an *I-evaluation function* on real intervals, to define an *F-evaluation function* on fuzzy numbers.

We define one *I-evaluation function* a real function φ: $I \to R$, where $I = \{A = [a_1, a_2] : a_1, a_2 \in R,\ a_2 \geq a_1\}$.

The imagine of φ is denoted by $\varphi(A) = \varphi([a_1, a_2]) = \varphi(a_1, a_2)$.
In general, it is not possible to request, a priori, any property on $\varphi(.)$, but it seems reasonable to consider functions which are increasing in both variables and have some regularity properties (i.e., $\varphi \in C^{(1)}$).

We consider two families of such functions, which are based on a parameter, called *degree of risk*, (the risk-tendency or aversion) of the decision maker. They have the very interesting property to be sensitive to the uncertainty associated to the use of real intervals instead of the reals.

Definition 1 Campos and Gonzalez (1989) *The family* $\{\varphi_\lambda\}_{\lambda \in [0,1]}$ *of linear functions*

$$\varphi_\lambda(A) = \varphi_\lambda(a_1, a_2) = \lambda a_2 + (1 - \lambda)a_1, \quad \lambda \in [0,1] \tag{1}$$

The function $\varphi_\lambda(a_1, a_2)$ is a convex combination of the interval extremes and the coefficient λ, we call degree of risk of the decision maker, is constant.

Definition 2 Facchinetti and Ghiselli Ricci (2001) *The family* $\{\varphi_\rho\}_{\rho \in \Gamma}$ *of not linear functions*

$$\varphi_\rho(a_1, a_2) = a_1 + \rho(a_1, a_2)(a_2 - a_1) \tag{2}$$

and Γ *is the set of class* $C^{(1)}$ *functions,* $\rho(a_1, a_2)$: $D \to [0, 1]$, $D = \{(a_1, a_2) : a_1, a_2 \in R,$ $a_2 > a_1 \geq 0\}$, *such that*:

a) $\rho(a_1, a_2)$ *is strictly decreasing in the first variable,*
b) $\rho(a_1, a_2)$ *is strictly increasing in the second variable,*
c) $\rho(a_1, a_2) \to 0$ *as* $a_1 \to a_2$,
d) $\varphi_\rho(a_1, a_2)$ *is increasing in* a_1,

It is clear that $\varphi_\rho(a_1, a_2)$ is a convex combination of the interval extremes and the coefficient $\rho(a_1, a_2)$, we could interpret *degree of risk* of the decision maker, is not constant, but depends on the interval position in the real axe.

We emphasize that in this new model ρ depends on a_1 and a_2 and this leads to a more realistic choice by the decision maker. For example, let the intervals $I_1 = [2, 3]$ and $I_2 = [200, 201]$, represents the possible outputs of two different investments: the spread is the same, but if we compare it with 2 and 200, it appear unquestionable that the uncertainty associated to the evaluation of I_2 is much lower than the one related to I_1. The fact that ρ depends on a_1 and a_2 let us to express different degrees of risk associated to I_1 and I_2, while the assumption of ρ constant completely ignores this fact.

Some comments on the hypothesis a)-d). We explicitly point out the conditions a) and b) as we are not interested, in this case, to let ρ to be constant, moreover they assure that ρ, fixing a_1, increase as a_2 increases, and symmetrically that, fixing a_2, decrease as the value of a_1 tends to a_2. The condition c) is a boundary condition: when the interval width is very close to zero the degree of risk goes to zero, as the uncertainty goes to zero. The condition d) translate the monotonicity of the evaluation when a_1 tends to a_2.

As consequences we have that a) and b) force $\rho(a_1, a_2)$ to be valued in the open interval]0,1[. Furthermore the condition a) and the regularity of ρ imply $\varphi_\rho(a_1, a_2)$ is strictly increasing in a_2.

Starting from $\varphi_\lambda(a_1, a_2)$ and $\varphi_\rho(a_1, a_2)$ we induce two evaluation functions on the set of fuzzy numbers F.

Definition 3 *The fuzzy set $\tilde{A}$ is a fuzzy number iff*:

1) $\forall\ \alpha \in [0,1]$, $A_\alpha = \{x \in R : \mu_A(x) \geq \alpha\} = \left[a_1^\alpha, a_2^\alpha\right]$ *is a convex set.*
2) $\mu_A(.)$ *is an upper-semicontinuous function.*
3) *Supp(A)* $= \{x \in R : \mu_A(x) > 0\}$ *is a bounded set in R.*

Now we define an *I-evaluation functions* φ on the family of set A^α, called α-cuts of $\tilde{A}$, and, by it, we define an *F-evaluation function*, we call the *Average Value* of $\tilde{A}$.

Definition 4 (Campos and Gonzalez (1989), Gonzalez (1990)) *We call Average Value* (AV) *of the fuzzy number $\tilde{A}$, made by an adolitive measure S on* [0, 1] *the value*

$$M_\varphi(S, \tilde{A}) = \int_0^1 \varphi(A^\alpha)dS \tag{3}$$

If φ is respectively $\varphi_\lambda(.)$ and $\varphi_\rho(.)$, on the set of fuzzy number F we obtain the following two different *Average Value*:

$$M_\lambda(S, \tilde{A}) = \int_0^1 \varphi_\lambda(A^\alpha)dS \quad \text{and} \quad M_\rho(S, \tilde{A}) = \int_0^1 \varphi_\rho(A^\alpha)dS$$

or more explicitly

$$M_\lambda(S,\tilde{A}) = \int_0^1 \left(\lambda a_2^\alpha + (1-\lambda)a_1^\alpha\right)dS \ \text{ and } \ \overline{M}_\rho(S,\tilde{A}) = \int_0^1 \left(a_1^\alpha + \rho(a_1^\alpha, a_2^\alpha)(a_2^\alpha - a_1^\alpha)\right)dS \tag{4}$$

The difference between the two AV in (4) is that the first is linear in A^α, and the second not. The lost of linearity, as always happens, will create a lot of difficulties. We will treat these problems and propose some solutions in section 3. The interest of the second case is due to the fact that the decision maker has not a fixed *degree of risk*. It changes with the fuzzy number $\tilde{A}$.

It is possible to write the first AV in (4) in a more useful way for the next section:

$$M_\lambda(S,\tilde{A}) = \lambda M^*(S,\tilde{A}) + (1-\lambda)M_*(S,\tilde{A}) \tag{5}$$

with

$$M_*(S,\tilde{A}) = \int_0^1 a_1^\alpha dS \ \text{ and } \ M^*(S,\tilde{A}) = \int_0^1 a_2^\alpha dS \tag{6}$$

In the case in which S is the normalized Stieltjes measure generated by the function $s(\alpha) = \alpha^r$, $\forall\ r > 0 : S(]a, b]) = b^r - a^r$, $\forall\ a, b \in [0, 1]$ we obtain

$$M_*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_1^\alpha d\alpha \ \text{ and } \ M^*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_2^\alpha d\alpha \tag{7}$$

and so the two AV introduced in (4) become

$$M_\lambda(r,\tilde{A}) = r\int_0^1 \alpha^{r-1}\varphi_\lambda(A^\alpha)d\alpha \ \text{ and } \ M_\rho(r,\tilde{A}) = r\int_0^1 \alpha^{r-1}\varphi_\rho(A^\alpha)d\alpha \tag{8}$$

The choice of r is connected with these types of preferences:

- $r > 1$, S gives more weight to the high values of $\alpha \in [0, 1]$
- $r < 1$, S gives more weight to the low values of $\alpha \in [0, 1]$
- $r = 1$ we obtain a linear preference and S gives equal weight to all values of $\alpha \in [0, 1]$.

The last case produces a particular case of a Stieltjes measure, which is a Lebesgue measure L: $L(]a, b]) = b - a$, $\forall\ a, b \in [0, 1]$.

It easy to see that, for particular choice of λ and S, the (5) coincides with other comparison indexes (Adamo (1980), Tsumura et al (1981), Yager (1981)), (cfr. Campos-Gonzalez 1989). In particular if $r = 1$, $\tilde{A}$ is a triangular fuzzy number defined by (a_1, a_3, a_2), for which $\mu(a_3) = 1$, S is the Lebesgue measure, the (5) is equivalent to the method of convex combination between the pessimistic and optimistic choice, introduced by Facchinetti, Ghiselli Ricci and Muzzioli in 1998.

3. Weighted Average Value and Variance of Fuzzy Numbers with Constant Weights

Dubois and Prade in a paper of 1987 have defined the *mean value* of a fuzzy number $E(\tilde{A})$ as an interval whose bounds are *upper* $E_*(\tilde{A})$ and *lower* $E^*(\tilde{A})$ *expectation*, that is,

$$E(\tilde{A}) = \left[E_*(\tilde{A}), E^*(\tilde{A})\right] \text{ with } E_*(\tilde{A}) = \int_{-\infty}^{+\infty} xdF^*(x) \text{ and } E^*(\tilde{A}) = \int_{-\infty}^{+\infty} xdF_*(x)$$

with $F^*(x) = \sup\{\mu_A(t) : t \leq x\}$ and $F_*(x) = \inf\{1 - \mu_A(t) : t > x\}$.

The last two integrals can be written as Choquet integrals with respect the possibility Π and necessity measure N associated to the fuzzy number $\tilde{A}$:

$$E_*(\tilde{A}) = \int x \, \mathrm{d}\Pi \quad \text{and} \quad E^*(\tilde{A}) = \int x \, \mathrm{dN}.$$

If we calculate $E_*(\tilde{A})$ and $E^*(\tilde{A})$, when $\tilde{A}$ is a fuzzy number with continuous and strictly increasing membership function before the modal values $[m_1, m_2]$, and strictly decreasing after the modal values, using definition 4, it is easy to note that:

$$E_*(\tilde{A}) = M_*(1, \tilde{A}) = \int_0^1 a_1^{\alpha} d\alpha = \int_0^1 a_1^{\alpha} dL \quad \text{and}$$

$$E^*(\tilde{A}) = M^*(1, \tilde{A}) = \int_0^1 a_2^{\alpha} d\alpha = \int_0^1 a_2^{\alpha} dL.$$

By analogy we define respectively *S-upper* ad *S-lower* expectations, the quantity in (6) and define the *S-mean value* of a fuzzy number $\tilde{A}$ the interval

$$\overline{M}(S, \tilde{A}) = \left[M_*(S, \tilde{A}),\ M^*(S, \tilde{A})\right].$$

If S is a normalized Stieltjes measure, Gonzalez (1990) shows that $M_*(S, \tilde{A}) = E_*(\tilde{A}^s)$ and $M^*(S, A) = E^*(\tilde{A}^s)$, where $\tilde{A}^s$ is a fuzzy number with membership function $\mu_{A^s} = s \circ \mu_A$, and so

$$\overline{M}(S, \tilde{A}) = \left[M_*(S, A),\ M^*(S, A)\right] = \left[E_*(A^s),\ E^*(A^s)\right]$$
$$= \left[M_*(1, \tilde{A}^s),\ M^*(1, \tilde{A}^s)\right]$$

If S is generated by the function $s(\alpha) = \alpha^r$, we may indicate $\tilde{A}^s = \tilde{A}^r$. Its membership function is $\mu_{A^r} = x^r \circ \mu_A = \mu_A^r$. The last function was defined *operation of concentration* if $r > 1$ and *dilution* if $r < 1$ by Zadeh (1973).

These last results give us the possibility to reformulate the *S-mean value* in terms of possibility and necessity measures. In fact it is easy to show that

$$M_*(S,\tilde{A}) = \int_{-\infty}^{+\infty} x dF_S^*(x) \quad \text{and} \quad M^*(S,\tilde{A}) = \int_{-\infty}^{+\infty} x dF_*^S(x)$$

where

$$F_S^*(x) = \begin{cases} s \circ \mu_A(x) & \text{if } x < m_1 \\ 1 & \text{if } x \geq m_1 \end{cases} \qquad F_*^S(x) = \begin{cases} 1 - s \circ \mu_A(x) & \text{if } x \geq m_2 \\ 0 & \text{if } x < m_2 \end{cases}$$

therefore

$$M_\lambda(S,\tilde{A}) = \int_0^1 \varphi_\lambda(A^\alpha) dS = \lambda \int x \, \mathrm{d}\Pi^s + (1-\lambda) \int x \, \mathrm{d}\mathrm{N}^s$$

where Π^s and N^s are respectively the possibility and necessity measures associated with the fuzzy number $\tilde{A}^s$ with membership function $\mu_{A^s} = s \circ \mu_A$.

Using the previous notations, the next results generalise what Carlsson and Fuller (2001) have introduced. Their definitions are particular cases in which $s(x) = x^r$ with $r = 2$ and $\lambda = \frac{1}{2}$.

Definition 6 *We call respectively lower and upper possibilistic mean values of order r of a fuzzy number $\tilde{A}$ the quantities defined in* (7):

$$M_*(r,\tilde{A}) = r \int_0^1 \alpha^{r-1} a_1^\alpha d\alpha, \qquad M^*(r,\tilde{A}) = r \int_0^1 \alpha^{r-1} a_2^\alpha d\alpha,$$

Following Carlsson and Fuller (2001), we define a crisp possibilistic weighted mean of $\tilde{A}$ of order r with constant weights their convex combination:

$$M_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda) M_*(r,\tilde{A}) = \lambda \int x \, \mathrm{d}\Pi^s + (1-\lambda) \int x \, \mathrm{d}\mathrm{N}^s. \quad (9)$$

The last definition is the *Average Value* introduced by Campos Gonzalez (1989).

In the same paper, it is possible to find the proof of the following

THEOREM 1 *$\overline{M}_\lambda(r, \tilde{A})$ is a linear functional on the space of the fuzzy number F, that is: if $\oplus$ denotes the sum on F, then $\forall\ \tilde{A}, \tilde{B} \in F$, $\forall\ \delta \in R$*

$$\overline{M}_\lambda(r, \tilde{A} \oplus \tilde{B}) = \overline{M}_\lambda(r,\tilde{A}) + \overline{M}_\lambda(r,\tilde{B})$$

$$\overline{M}_\lambda(r, \delta\tilde{A}) = \delta \overline{M}_\lambda(r,\tilde{A})$$

Example 1. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number it easy to see that

$$M_*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_1^\alpha d\alpha = a_3 - \frac{a_3 - a_1}{r+1} \quad M^*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_2^\alpha d\alpha = a_3 + \frac{a_2 - a_3}{r+1}$$

$$\overline{M}_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda)M_*(r,\tilde{A}) = a_3 + \frac{\lambda(a_2 - a_3) - (1-\lambda)(a_3 - a_1)}{(r+1)}$$

If the triangular fuzzy number $\tilde{A}$ is symmetric, that is $a_2 - a_3 = a_3 - a_1$

$$\overline{M}_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda)M_*(r,\tilde{A}) = a_3 + \frac{(a_2 - a_3)(2\lambda - 1)}{(r+1)}$$

and if $\lambda = \frac{1}{2}$, $\overline{M}_\lambda(r,\tilde{A}) = a_3$. That is, the *crisp possibilistic weighted mean of* $\tilde{A}$ *of order r with constant weights* of a symmetric triangular fuzzy number is the central value, with membership function equal to one, if and only if the weight is $\lambda = \frac{1}{2}$.

Example 2. If $\tilde{A} = (m_1, m_2, \gamma, \beta)$ is a trapezoidal fuzzy number with $[m_1, m_2]$ flat graphic and left-width $\gamma > 0$ and right-width $\beta > 0$,

$$M_*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_1^\alpha d\alpha = m_1 - \frac{\gamma}{r+1}, \quad M^*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_2^\alpha d\alpha$$
$$= m_2 + \frac{\beta}{r+1}$$

and consequently

$$\overline{M}_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda)M_*(r,\tilde{A}) = \lambda m_2 + (1-\lambda)m_1 + \frac{(\lambda\beta - (1-\lambda)\gamma)}{(r+1)}$$

If $\beta = \gamma$ and $\lambda = \frac{1}{2}$, $\overline{M}(r,\tilde{A}) = \frac{m_1+m_2}{2}$. That is, the *crisp possibilistic weighted mean of* $\tilde{A}$ *of order r with constant weights* of a simmetric trapezoidal fuzzy number is the central value of the flat part, if and only if the weight is $\lambda = \frac{1}{2}$.

In Gonzalez (1989), there is a simple, but interesting, calculus for a trapezoidal fuzzy number $\tilde{A}$, useful in the next remark.

As $\varphi_\lambda(A^\alpha) = \lambda m_2 + (1-\lambda)m_1 + (1-\alpha)(\lambda\beta - (1-\lambda)\gamma)$,

$$M_\lambda(S,\tilde{A}) = \int_0^1 \varphi_\lambda(A^\alpha)dS = \lambda m_2 + (1-\lambda)m_1 + k(s)(\lambda\beta - (1-\lambda)\gamma) = \varphi_\lambda(A^{1-k(s)})$$

where $k(s) = \int_0^1 s(\alpha)d\alpha$.

The last result shows that the *crisp possibilistic weighted mean of order r with constant weights of* $\tilde{A}$ is an *I-evaluation* not of the support of $\tilde{A}$, but of a particular α-cut of $\tilde{A}$ with $\alpha = \alpha_s = 1 - \int_0^1 s(\alpha)d\alpha$.

Remark. The vision of the *Average Value* introduced by Gonzalez, or the *crisp possibilistic weighted mean of order r with constant weights of* $\tilde{A}$, as an *I-evaluation function*, let to understand the meaning of these quantity. In the general case, fixed λ and *S*, the *Average Value* is a value of the interval $\overline{M}(S, \tilde{A})$. In this particular case, not only we may assure that $\overline{M}(S, \tilde{A})$ is a subset of the support of $\tilde{A}$, but we may define it as the projection of the α-cut of $\tilde{A}$, with $\alpha = \alpha_s$, on the support of $\tilde{A}$. This idea let us to reduce the uncertainty in our decision.

Definition 7 *We call Variance of order r of a fuzzy number* $\tilde{A}$*, respect the weighted average* $\varphi_\lambda(\tilde{A})$*, the quantity:*

$$\begin{aligned} Var_\lambda(r, \tilde{A}) &= r\int_0^1 \alpha^{r-1}\left[\lambda a_2^\alpha + (1-\lambda)a_1^\alpha - a_1^\alpha\right]^2 d\alpha \\ &\quad + r\int_0^1 \alpha^{r-1}\left[\lambda a_2^\alpha + (1-\lambda)a_1^\alpha - a_2^\alpha\right]^2 d\alpha \\ &= (\lambda^2 + (1-\lambda)^2)r\int_0^1 \alpha^{r-1}\left[a_2^\alpha - a_1^\alpha\right]^2 d\alpha \end{aligned}$$

The variance of order *r* of a fuzzy number $\tilde{A}$, respect the weighted average $\varphi_\lambda(\tilde{A})$, is defined as the expected value of the squared deviations between the weighted average and the endpoints of its α-cuts respect the Stieltjes measure generated by $s(x) = x^r$. The deviation standard of $\tilde{A}$ of order *r* of a fuzzy number $\tilde{A}$, respect the weighted average $\varphi_\lambda(\tilde{A})$, is $\sigma_\lambda(r, \tilde{A}) = \sqrt{Var_\lambda(r, \tilde{A})}$.

Example 3. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number it easy to see that

$$Var_\lambda(r, \tilde{A}) = (\lambda^2 + (1-\lambda)^2)\frac{2}{(r+1)(r+2)}(a_2 - a_1)^2$$

If $\lambda = \frac{1}{2}$ and $r = 2$, $Var_\lambda(r, \tilde{A}) = \frac{1}{12}(a_2 - a_1)^2$. If $\tilde{A}$ is a crisp number $Var_\lambda(r, \tilde{A}) = 0$.

Following the same proof proposed by Carlsson and Fuller (2001) it is easy to show that $Var_\lambda(r, \tilde{A})$ is invariant for shifting by a real value of the fuzzy number. That is if $\tilde{A}$ is shifted by a value $\theta \in R$, and so we obtain a new fuzzy number $\tilde{B}$ which membership function is $\mu_B(x) = \mu_A(x - \theta)$, we have $Var_\lambda(r, \tilde{B}) = Var_\lambda(r, \tilde{A})$.

Definition 8 *We call the Covariance of two Fuzzy numbers of order r respect the weighted average $\varphi_\lambda(\tilde{A})$*

$$Cov_\lambda(r,\tilde{A},\tilde{B}) = (\lambda^2 + (1-\lambda)^2)r\int_0^1 \alpha^{r-1}[a_2^\alpha - a_1^\alpha][b_2^\alpha - b_1^\alpha]d\alpha$$

if $\tilde{A} = (a_1, a_3, a_2)$ and $\tilde{B} = (b_1, b_3, b_2)$ are triangular fuzzy numbers

$$Cov_\lambda(r,\tilde{A},\tilde{B}) = (\lambda^2 + (1-\lambda)^2)\frac{2}{(r+1)(r+2)}(a_2 - a_1)(b_2 - b_1)$$

Even in this case, following the proof of Carlsson and Fuller (2001), it is possible to proof the next two theorems. The first shows that the variance of a linear combination of fuzzy numbers follows the same rule than in probability theory. That is:

THEOREM 2 *Let $\eta, \tau \in R$ and let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. Then*

$$Var_\lambda(r,\eta\tilde{A} \oplus \tau\tilde{B}) = \eta^2 Var_\lambda(r,\tilde{A}) + \tau^2 Var_\lambda(r,\tilde{B}) + 2|\eta\tau|Cov_\lambda(\tilde{A},\tilde{B}),$$

where $\oplus$ denotes the sum on F.

The second puts a relation between the inclusion of two fuzzy numbers and an inequality between the relative variance.

THEOREM 3 *Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers and $\tilde{A} \subset \tilde{B}$ (that is $\mu_A(x) < \mu_B(x), \forall x$). Then*

$$Var_\lambda(r,\tilde{A}) \leq Var_\lambda(r,\tilde{B}).$$

4. Weighted Average Value and Variance of Fuzzy Numbers with Not Constant Weights

In the last section we have generalized the results obtained by Carlsson and Fuller (2001), using the first *I-evaluation function* on A^α. Here we try to go over again the same road using, on F the second one defined in (4), in the case in which the Stieltjes measure S is generated by $s(x) = x^r$. The AV now is:

$$M_\rho(r,\tilde{A}) = r\int_0^1 \alpha^{r-1}\varphi_\rho(A^\alpha)d\alpha = r\int_0^1 \alpha^{r-1}\left(a_1^\alpha + \rho(a_1^\alpha, a_2^\alpha)(a_2^\alpha - a_1^\alpha)\right)d\alpha.$$

This is a linear combination of the extremes of A^α with not constant coefficient. It depends on the extremes of A^α.

This last formulation shows that in this general framework, it is not possible to have an analogous formulation of convex combination of *lower and upper mean value of a Fuzzy* *number* as in (9), as we are not able to evaluate the two integrals without having an explicit formulation of the *degree of risk* $\rho(a_1, a_2)$.

To overcome this difficulty, we remember the meaning we have given in section 1 to the term $\rho(a_1, a_2)$. It is the uncertainty filled with the position of the fuzzy number on the real axis. Now we may suppose that this function is independent by the several levels of α-cuts, that is, it is constant respect the variable α, and assumes the value with $\alpha = 0$. This is equivalent to put

$$\rho(a_1^\alpha, a_2^\alpha) = \rho(a_1^0, a_2^0) = \rho(a_1, a_2). \tag{9}$$

Using this condition, we obtain:

Definition 9 *We call crisp possibilistic weighted mean of $\tilde{A}$ of order r with not constant weights the quantity*

$$\begin{aligned} M_\rho(r, \tilde{A}) &= \rho(a_1, a_2) M^*(r, \tilde{A}) + (1 - \rho(a_1, a_2)) M_*(r, \tilde{A}) \\ &= \rho(a_1, a_2) \int x \, \mathrm{d}\Pi^s + (1 - \rho(a_1, a_2)) \int x \, \mathrm{d}\mathrm{N}^r \end{aligned} \tag{10}$$

where $\rho(a_1, a_2)$ is defined in (2).

Following the same idea of section 2 we may say that even in this case it is an "*I-evaluation function*", on the interval $[M_*(r, \tilde{A}), M^*(r, \tilde{A})]$, with not constant coefficient $\rho(a_1, a_2)$, which is related to $\tilde{A}$. It is the convex combination, with not constant coefficient $\rho(a_1, a_2)$, not of the support extremes of $\tilde{A}$, but of its α-cut with $\alpha = \alpha_r = \frac{r}{r+1}$. Even in this case, the idea to consider an *F-evaluation function* of $\tilde{A}$ as an *I-evaluation function* offers the possibility to understand what evaluation the decision maker is doing. The difference, compared to the linear case, is that of when the decision maker has to evaluate the fuzzy number $\tilde{A}$, decides which is the degree of risk that he assigns to $\tilde{A}$, and then he fixes r, in this way he decides the width of the interval in which he finds the final evaluation.

Because of the hypothesis we have put on $\rho(a_1, a_2)$, we cannot have the linearity of $M_\rho(r, \tilde{A})$, but we have the following

THEOREM 4 *$M_\rho(r, \tilde{A})$ is not, in general, a linear functional on the space of the fuzzy number F. What we can say is: if $\oplus$ denotes the sum on F, then $\forall\ \tilde{A}, \tilde{B} \in F$, $\forall\ \delta \in R$*

$$M_\rho(r, \tilde{A} \oplus \tilde{B}) = \rho_{A \oplus B}[M_\rho(r, \tilde{A}) + M_\rho(r, \tilde{B})]$$

$$M_\lambda(r, \delta\tilde{A}) = \rho_{\delta A}\ \delta\ M_\lambda(r, \tilde{A})$$

where $\rho_{A\oplus B}$ is related with the fuzzy number $A \oplus B$, and $\rho_{\delta A}$ is related to δA.

Example 4. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number we have:

$$M_\rho(r, \tilde{A}) = a_3 + \frac{\rho(a_1, a_2)(a_2 - a_3) - (1 - \rho(a_1, a_2))(a_3 - a_1)}{(r + 1)}$$

It easy to see that there are no possibility in which $\overline{M}_\rho(r, \tilde{A}) = a_3$. If it happens $\frac{\rho(a_1,a_2)(a_2-a_3)-(1-\rho(a_1,a_2))(a_3-a_1)}{(r+1)} = 0$ and this produce that $\rho(a_1, a_2) = \frac{a_3-a_1}{a_2-a_3}$. This function do not verify the hypothesis a), b), c) defined in section 1. In the symmetric case the only possibility to be equal to a_3 is that $\rho(a_1, a_2) = 1/2$, $\forall a_1, a_2$, but it is impossible why we have supposed $\rho(a_1, a_2)$ is not constant.

Example 5. If $\tilde{A} = (m_1, m_2, \gamma, \beta)$ is a trapezoidal fuzzy number with $[m_1, m_2]$ flat graphic and left-width $\gamma > 0$ and right-width $\beta > 0$,

$$M_*(r, \tilde{A}) = r \int_0^1 \alpha^{r-1} a_1^\alpha d\alpha = m_1 - \frac{\gamma}{r + 1} \quad \text{and} \quad M^*(r, \tilde{A}) = r \int_0^1 \alpha^{r-1} a_2^\alpha d\alpha$$
$$= m_2 + \frac{\beta}{r + 1}$$

and consequently

$$M_\rho(r, \tilde{A}) = \rho(a_1, a_2)m_2 + (1 - \rho(a_1, a_2))m_1 + \frac{(\beta\rho(a_1, a_2) - (1 - \rho(a_1, a_2))\gamma)}{(r + 1)}$$

For the same reason of Example 4, it is easy to see that the *crisp possibilistic weighted mean of $\tilde{A}$ of order r with not constant weights* of a trapezoidal fuzzy number is not $\frac{m_1+m_2}{2}$.

Definition 7 *We call Variance of order r of a Fuzzy number $\tilde{A}$ respect the weighted average $\varphi_\rho(\tilde{A})$ with not constant weights*

$$Var_\rho(r, \tilde{A}) = r \int_0^1 \alpha^{r-1} \left[\rho(a_1, a_2)a_2^\alpha + (1 - \rho(a_1, a_2))a_1^\alpha - a_1^\alpha\right]^2 d\alpha$$
$$+ r \int_0^1 \alpha^{r-1} \left[\rho(a_1, a_2)a_2^\alpha + (1 - \rho(a_1, a_2))a_1^\alpha - a_2^\alpha\right]^2 d\alpha$$
$$= (\rho^2(a_1, a_2) + (1 - \rho(a_1, a_2))^2) r \int_0^1 \alpha^{r-1} [a_2^\alpha - a_1^\alpha]^2 d\alpha$$

The deviation standard of $\tilde{A}$ respect the weighted average $\varphi_\rho(\tilde{A})$ with not constant weights is $\sigma_\rho(A) = \sqrt{Var_\rho(r, \tilde{A})}$.

Example 6. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number, it is easy to see that

$$Var_\rho(r, \tilde{A}) = (\rho^2(a_1, a_2) + (1 - \rho(a_1, a_2))^2)\frac{2}{(r+1)(r+2)}(a_2 - a_1)^2$$

If $\tilde{A}$ is a crisp number $Var_\rho(r, \tilde{A}) = 0$.
If $\tilde{A}$ is shifted by a value $\theta \in R$, and so we obtain a new fuzzy number $\tilde{B}$ which membership function is $\mu_B(x) = \mu_A(x - \theta)$, because of the hypothesis on $\rho(a_1, a_2)$, we cannot have information about the relation between $Var_\rho(r, \tilde{A})$ and $Var_\rho(r, \tilde{B})$.

Definition 8 *We call the Covariance of two Fuzzy numbers, $\tilde{A}$ and $\tilde{B}$, of order r respect the weighted average $\varphi_\rho(\tilde{A})$ with not constant weights:*

$$Cov_\rho(\tilde{A}, \tilde{B}) = (\rho(a_1, a_2)\rho(b_1, b_2) + (1 - \rho(a_1, a_2))(1 - \rho(b_1, b_2))) \times r\int_0^1 \alpha^{r-1}[a_2^\alpha - a_1^\alpha]\left[b_2^\alpha - b_1^\alpha\right]d\alpha$$

if $\tilde{A} = (a_1, a_3, a_2)$ and $\tilde{B} = (b_1, b_3, b_2)$ are triangular fuzzy numbers,

$$Cov_\rho(\tilde{A}, \tilde{B}) = (\rho(a_1, a_2)\rho(b_1, b_2) + (1 - \rho(a_1, a_2))(1 - \rho(b_1, b_2)))\frac{2}{(r+1)(r+2)}(a_2 - a_1)(b_2 - b_1)$$

Because of the hypothesis on $\rho(a_1, a_2)$, we cannot have the proof of the analogous theorem in Carlsson and Fuller (2001) we obtain the variance of a linear combination of fuzzy numbers.

THEOREM 5 *Let η, $\tau \in R$ and let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. Then*

$$\begin{aligned} Var_\rho(r, \eta\tilde{A} \oplus \tau\tilde{B}) = {} & \eta^2 \frac{\rho^2_{\eta A\oplus\tau B} + (1 - \rho_{\eta A\oplus\tau B})^2}{\rho_A^2 + (1 - \rho_A)^2} Var_\rho(r, \tilde{A}) \\ & + \tau^2 \frac{\rho^2_{\eta A\oplus\tau B} + (1 - \rho_{\eta A\oplus\tau B})^2}{\rho_B^2 + (1 - \rho_B)^2} Var_\rho(r, \tilde{B}) \\ & + 2|\eta\tau| \frac{\rho^2_{\eta A\oplus\tau B} + (1 - \rho_{\eta A\oplus\tau B})^2}{\rho_A\rho_B + (1 - \rho_A)(1 - \rho_B)} Cov_\rho(\tilde{A}, \tilde{B}) \end{aligned} \tag{11}$$

where $\rho_{\eta A\oplus\tau B}$ is the degree of risk associated with the fuzzy number $\eta\tilde{A} \oplus \tau\tilde{B}$, ρ_A and ρ_B are respectively the degrees of risk associated with the fuzzy number $\tilde{A}$ and $\tilde{B}$, the addition $\oplus$ and the multiplication by a scalar are the usual definitions of sum and multiplication by scalar in fuzzy number set.

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers and $\tilde{A} \subset \tilde{B}$ (that is $\mu_A(x) < \mu_B(x)\ \forall x$). Then $\rho_A < \rho_B$, but we cannot say anything about the relation between $Var_\rho(r, \tilde{A})$ and $Var_\rho(r, \tilde{B})$.

5. Ranking Functions of Fuzzy Numbers

The definition of *crisp possibilistic weighted mean of* $\tilde{A}$ *of order r with constant weights* is an *F-evaluation function* and induces a natural ranking on fuzzy numbers. Let $\tilde{A}$, $\tilde{B}$ be in F and the *F-evaluation function* $M_\lambda(r, \tilde{A})$ defined in (5), we have

Definition 9 (Campos and Gonzalez (1989), Gonzalez (1990)) *We say that* $\tilde{B}$ *is* $M_\lambda(r)$ – *preferred to* $\tilde{A}$*, in symbols*

$$\tilde{A} \prec_{M_\lambda} \tilde{B} \quad \text{iff} \quad M_\lambda(r, \tilde{A}) < M_\lambda(r, \tilde{B})$$

This is a crisp preorder on R and an order relation on the quotient set generated by the equivalence relation $\tilde{A} \approx_{M_\lambda} \tilde{B}$ if and only if $M_\lambda(r, \tilde{A}) = M_\lambda(r, \tilde{B})$.

If $\tilde{A}$, $\tilde{B}$ are triangular fuzzy numbers, $\tilde{A} = (a_1, a_3, a_2)$ and $\tilde{B} = (b_1, b_3, b_2)$

$$\overline{M}_\lambda(r, \tilde{A}) - \overline{M}_\lambda(r, \tilde{B}) = \frac{r(a_3 - b_3) + \lambda(a_2 - b_2) + (1 - \lambda)(a_1 - b_1)}{r + 1}$$

It is easy to see that if $\tilde{A} = \tilde{B}$ then $\overline{M}_\lambda(r, \tilde{A}) = \overline{M}_\lambda(r, \tilde{B})$, but the converse is not true. We consider now, how the definition of *crisp possibilistic weighted mean of* $\tilde{A}$ *of order r with not constant weights* given in (10), induces a natural ranking on fuzzy numbers.

Let $\tilde{A}$, $\tilde{B}$ be in F and the *F-evaluation function* $\overline{M}_\rho(r,.)$. We have

$$\begin{aligned} M_\rho(r, \tilde{A}) &= \rho(a_1, a_2) M^*(r, \tilde{A}) + (1 - \rho(a_1, a_2)) M_*(r, \tilde{A}) \\ &= M_*(r, \tilde{A}) + \rho(a_1, a_2)(M^*(r, \tilde{A}) - M_*(r, \tilde{A})) \end{aligned} \tag{12}$$

Looking to the last formula, we may notice a similarity between it and (2). It is the second type of "*I-evaluation Function*", on the interval $[M_*(r, \tilde{A}), M^*(r, \tilde{A})]$, with not constant coefficient $\rho(a_1, a_2)$, which is related to $\tilde{A}$.

It is possible to use $\overline{M}_\rho(r, \tilde{A})$ to obtain a ranking function on F.

Definition 10 *We state that* $\tilde{B}$ *is* $\overline{M}_\rho(r)$ – *preferred to* $\tilde{A}$*, in symbols*

$$\tilde{A} \prec_{\overline{M}_\rho} \tilde{B} \quad \text{iff} \quad \overline{M}_\rho(r, \tilde{A}) < \overline{M}_\rho(r, \tilde{B})$$

This is a crisp preorder on R and an order relation on the quotient set generated by the equivalence relation $\tilde{A} \approx_{\overline{M}\rho} \tilde{B}$ *if and only if* $\overline{M}_\rho(r, \tilde{A}) = \overline{M}_\rho(r, \tilde{B})$*.*

If $\tilde{A}$ *is a triangular fuzzy number,* $\tilde{A} = (a_1, a_3, a_2)$*,*

$$\begin{aligned} \overline{M}_\rho(r, \tilde{A}) = \overline{M}_\rho(r, a_1, a_2, a_3) &= \frac{r}{r+1} a_3 + \frac{1}{r+1}[a_1 + \rho(a_1, a_2)(a_2 - a_1)] \\ &= \frac{r}{r+1} a_3 + \frac{1}{r+1} \varphi_\rho(a_1, a_2). \end{aligned}$$

As $\varphi_\rho(a_1, a_2)$ is increasing in the two variables, for r fixed, $\overline{M}_\rho(r, a_1, a_2, a_3)$ is increasing in all the variables if it is increasing in a_3. Classically no property is requested for ranking functions; nevertheless that is the increasing monotonicity in all its variables. The reason of this request is very natural. A fuzzy number is more preferable as it runs along the positive direction of the real axis.

6. Conclusion

We have introduced two types of evaluation functions on intervals and by them we have proposed ranking functions, mean values, variance and covariance of fuzzy numbers in a general framework. It is interesting to note that, in this field of research, many interesting results are present for linear evaluation functions, but few authors have tried to extend them in the not linear case. This paper is one of the first attempts in this direction. The introduction of the indexes, r, λ and ρ, let the possibility with the first to privilege the part of the fuzzy number one decide to choose, with the two others, to put in evidence the decision-maker risk tendency or aversion, that may be constant or not. In the first case it is fixed "ex ante" and cannot be changed, in the second it depends on the circumstances, which may be affected by the moment in which the decision has to be kept, but even by the importance that the decision maker gives to the results produced by the choice. We think that, in real applications, the second approach is more realistic.

Another interesting field of application of these results is in the defuzzification step. We may think to use the several average values, here introduced, at the final step of Fuzzy Expert Systems (FES). But the problem we meet in this type of application is that the output of a FES is not always a fuzzy number, but only a fuzzy set not convex. In these cases the approach of α-cuts is, at the moment, impossible. We are working in this direction to overcome this difficulty.

Notes

1. Tel. 0039-59-2056779.
2. We desire to thank Professor Fuller who looked the section 2 of this paper in 2001 and encouraged to present our results for printing.

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Ranking Functions Induced by Weighted Average of Fuzzy Numbers

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Abstract. In this paper we present two definitions of possibilistic weighted average of fuzzy numbers, and by them we introduce two different rankings on the set of real fuzzy numbers. The two methods are dependent on several parameters. In the first case, the parameter is constant and the results generalize what Carlsson and Fuller have obtained in (2001)[2]. In the second case, the parameter is a function, not fixed *a priori* by the decision maker, but it depends on the position of the interval on the real axe. In all the two cases we call the parameter *degree of risk*, which takes into account of a risk-tendency or aversion of the decision maker.

Keywords: fuzzy numbers, average value, possibilistic weighted average, ranking methods.

1. Introduction

In most real situation, one is forced to take decision on the basis of ill-defined variables and imprecise data. The theory of fuzzy sets is a natural tool to model this situation as fuzzy numbers well represent imprecise quantities. In decision-making problems we have the necessity to optimise some procedure and so it is necessary to have ranking of the quantities involved. Many authors have studied different definitions of ranking on the set of fuzzy numbers F, (Bortolan and Degani (1985)). Most of these are based on the definition of an *evaluation function* (*F-evaluation function*), which maps fuzzy numbers in the real line. The order on F is induced by the real number total order. As fuzzy numbers are intervals in which boundaries are blurred, the difficulty in ranking them arise from the problem created in ranking real intervals. When the supports of fuzzy numbers are disjointed, there are no problems and all the methods lead to the same solution. But the decision is not evident when the intersection between supports is not empty. In these cases, it seems that the solution depends on subjective elements depending of the nature of the problem and the decision-maker.

Following the idea of Campos and Gonzalez (1989) we start with the introduction of two different *evaluation functions* on the set of real intervals, we will call *I-evaluation functions*. They contain a parameter we call *degree of risk*, which takes into account of a risk-tendency or aversion of the decision maker, that is constant in the first case, a function in the second, and then, using the definition of a fuzzy numbers by its α-cuts, we introduce two *average values* (AV) as *F-evaluation functions*. Using these two, we obtain several results. One is to generalise the results of Carlsson and Fuller. They, in a paper of 2001, starting from a particular AV, have introduced the notations of *lower and upper possibilistic mean value* and consequently defined *the interval-valued possibilistic mean, crisp possibilistic mean value* and *crisp variance* of a continuous possibility distribution.

The two new AV are consistent with the extension principle and with the well-known definitions of expectation and variance in probability theory. Another result is the introduction of a definition of ranking functions on the set of fuzzy numbers *F*.

2. Two Different Types of Average Value

Following the idea introduced by Campos and Gonzalez (1989), (see also Gonzalez (1990), Campos and Gonzalez (1994)) we begin with the introduction of an *I-evaluation function* on real intervals, to define an *F-evaluation function* on fuzzy numbers.

We define one *I-evaluation function* a real function φ: $I \rightarrow R$, where $I = \{A = [a_1, a_2] : a_1, a_2 \in R,\ a_2 \geq a_1\}$.

The imagine of φ is denoted by $\varphi(A) = \varphi([a_1, a_2]) = \varphi(a_1, a_2)$.
In general, it is not possible to request, a priori, any property on $\varphi(.)$, but it seems reasonable to consider functions which are increasing in both variables and have some regularity properties (i.e., $\varphi \in C^{(1)}$).

We consider two families of such functions, which are based on a parameter, called *degree of risk*, (the risk-tendency or aversion) of the decision maker. They have the very interesting property to be sensitive to the uncertainty associated to the use of real intervals instead of the reals.

Definition 1 Campos and Gonzalez (1989) *The family* $\{\varphi_\lambda\}_{\lambda \in [0,1]}$ *of linear functions*

$$\varphi_\lambda(A) = \varphi_\lambda(a_1, a_2) = \lambda a_2 + (1 - \lambda)a_1, \ \ \lambda \in [0,1] \tag{1}$$

The function $\varphi_\lambda(a_1, a_2)$ is a convex combination of the interval extremes and the coefficient λ, we call degree of risk of the decision maker, is constant.

Definition 2 Facchinetti and Ghiselli Ricci (2001) *The family* $\{\varphi_\rho\}_{\rho \in \Gamma}$ *of not linear functions*

$$\varphi_\rho(a_1, a_2) = a_1 + \rho(a_1, a_2)(a_2 - a_1) \tag{2}$$

and Γ *is the set of class* $C^{(1)}$ *functions,* $\rho(a_1, a_2)$: $D \rightarrow [0, 1]$, $D = \{(a_1, a_2) : a_1, a_2 \in R, a_2 > a_1 \geq 0\}$, *such that*:

a) $\rho(a_1, a_2)$ *is strictly decreasing in the first variable,*
b) $\rho(a_1, a_2)$ *is strictly increasing in the second variable,*
c) $\rho(a_1, a_2) \rightarrow 0$ *as* $a_1 \rightarrow a_2$,
d) $\varphi_\rho(a_1, a_2)$ *is increasing in* a_1,

It is clear that $\varphi_\rho(a_1, a_2)$ is a convex combination of the interval extremes and the coefficient $\rho(a_1, a_2)$, we could interpret *degree of risk* of the decision maker, is not constant, but depends on the interval position in the real axe.

We emphasize that in this new model ρ depends on a_1 and a_2 and this leads to a more realistic choice by the decision maker. For example, let the intervals $I_1 = [2, 3]$ and $I_2 = [200, 201]$, represents the possible outputs of two different investments: the spread is the same, but if we compare it with 2 and 200, it appear unquestionable that the uncertainty associated to the evaluation of I_2 is much lower than the one related to I_1. The fact that ρ depends on a_1 and a_2 let us to express different degrees of risk associated to I_1 and I_2, while the assumption of ρ constant completely ignores this fact.

Some comments on the hypothesis a)-d). We explicitly point out the conditions a) and b) as we are not interested, in this case, to let ρ to be constant, moreover they assure that ρ, fixing a_1, increase as a_2 increases, and symmetrically that, fixing a_2, decrease as the value of a_1 tends to a_2. The condition c) is a boundary condition: when the interval width is very close to zero the degree of risk goes to zero, as the uncertainty goes to zero. The condition d) translate the monotonicity of the evaluation when a_1 tends to a_2.

As consequences we have that a) and b) force $\rho(a_1, a_2)$ to be valued in the open interval]0,1[. Furthermore the condition a) and the regularity of ρ imply $\varphi_\rho(a_1, a_2)$ is strictly increasing in a_2.

Starting from $\varphi_\lambda(a_1, a_2)$ and $\varphi_\rho(a_1, a_2)$ we induce two evaluation functions on the set of fuzzy numbers *F*.

Definition 3 *The fuzzy set $\tilde{A}$ is a fuzzy number iff*:

1) $\forall\ \alpha \in [0,1]$, $A_\alpha = \{x \in R : \mu_A(x) \geq \alpha\} = \left[a_1^\alpha, a_2^\alpha\right]$ *is a convex set.*
2) $\mu_A(.)$ *is an upper-semicontinuous function.*
3) *Supp(A)* $= \{x \in R : \mu_A(x) > 0\}$ *is a bounded set in R.*

Now we define an *I-evaluation functions* φ on the family of set A^α, called α-cuts of $\tilde{A}$, and, by it, we define an *F-evaluation function*, we call the *Average Value* of $\tilde{A}$.

Definition 4 (Campos and Gonzalez (1989), Gonzalez (1990)) *We call Average Value* (AV) *of the fuzzy number $\tilde{A}$, made by an adolitive measure S on* [0, 1] *the value*

$$M_\varphi(S, \tilde{A}) = \int_0^1 \varphi(A^\alpha)dS \tag{3}$$

If φ is respectively $\varphi_\lambda(.)$ and $\varphi_\rho(.)$, on the set of fuzzy number *F* we obtain the following two different *Average Value*:

$$M_\lambda(S, \tilde{A}) = \int_0^1 \varphi_\lambda(A^\alpha)dS \quad \text{and} \quad M_\rho(S, \tilde{A}) = \int_0^1 \varphi_\rho(A^\alpha)dS$$

or more explicitly

$$M_\lambda(S,\tilde{A}) = \int_0^1 \left(\lambda a_2^\alpha + (1-\lambda)a_1^\alpha\right) dS \quad \text{and} \quad \overline{M}_\rho(S,\tilde{A}) = \int_0^1 \left(a_1^\alpha + \rho(a_1^\alpha, a_2^\alpha)(a_2^\alpha - a_1^\alpha)\right) dS \tag{4}$$

The difference between the two AV in (4) is that the first is linear in A^α, and the second not. The lost of linearity, as always happens, will create a lot of difficulties. We will treat these problems and propose some solutions in section 3. The interest of the second case is due to the fact that the decision maker has not a fixed *degree of risk*. It changes with the fuzzy number $\tilde{A}$.

It is possible to write the first AV in (4) in a more useful way for the next section:

$$M_\lambda(S,\tilde{A}) = \lambda M^*(S,\tilde{A}) + (1-\lambda)M_*(S,\tilde{A}) \tag{5}$$

with

$$M_*(S,\tilde{A}) = \int_0^1 a_1^\alpha dS \quad \text{and} \quad M^*(S,\tilde{A}) = \int_0^1 a_2^\alpha dS \tag{6}$$

In the case in which S is the normalized Stieltjes measure generated by the function $s(\alpha) = \alpha^r$, $\forall\, r > 0 : S(]a, b]) = b^r - a^r$, $\forall\, a, b \in [0, 1]$ we obtain

$$M_*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_1^\alpha d\alpha \quad \text{and} \quad M^*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_2^\alpha d\alpha \tag{7}$$

and so the two AV introduced in (4) become

$$M_\lambda(r,\tilde{A}) = r\int_0^1 \alpha^{r-1}\varphi_\lambda(A^\alpha)d\alpha \quad \text{and} \quad M_\rho(r,\tilde{A}) = r\int_0^1 \alpha^{r-1}\varphi_\rho(A^\alpha)d\alpha \tag{8}$$

The choice of r is connected with these types of preferences:

- $r > 1$, S gives more weight to the high values of $\alpha \in [0, 1]$
- $r < 1$, S gives more weight to the low values of $\alpha \in [0, 1]$
- $r = 1$ we obtain a linear preference and S gives equal weight to all values of $\alpha \in [0, 1]$.

The last case produces a particular case of a Stieltjes measure, which is a Lebesgue measure L: $L(]a, b]) = b - a$, $\forall\, a, b \in [0, 1]$.

It easy to see that, for particular choice of λ and S, the (5) coincides with other comparison indexes (Adamo (1980), Tsumura et al (1981), Yager (1981)), (cfr. Campos-Gonzalez 1989). In particular if $r = 1$, $\tilde{A}$ is a triangular fuzzy number defined by (a_1, a_3, a_2), for which $\mu(a_3) = 1$, S is the Lebesgue measure, the (5) is equivalent to the method of convex combination between the pessimistic and optimistic choice, introduced by Facchinetti, Ghiselli Ricci and Muzzioli in 1998.

3. Weighted Average Value and Variance of Fuzzy Numbers with Constant Weights

Dubois and Prade in a paper of 1987 have defined the *mean value* of a fuzzy number $E(\tilde{A})$ as an interval whose bounds are *upper* $E_*(\tilde{A})$ and *lower* $E^*(\tilde{A})$ *expectation*, that is,

$$E(\tilde{A}) = \left[E_*(\tilde{A}), E^*(\tilde{A})\right] \text{ with } E_*(\tilde{A}) = \int_{-\infty}^{+\infty} x dF^*(x) \text{ and } E^*(\tilde{A}) = \int_{-\infty}^{+\infty} x dF_*(x)$$

with $F^*(x) = \sup\{\mu_A(t) : t \leq x\}$ and $F_*(x) = \inf\{1 - \mu_A(t) : t > x\}$.

The last two integrals can be written as Choquet integrals with respect the possibility Π and necessity measure N associated to the fuzzy number $\tilde{A}$:

$$E_*(\tilde{A}) = \int x \, \mathrm{d}\Pi \text{ and } E^*(\tilde{A}) = \int x \, \mathrm{dN}.$$

If we calculate $E_*(\tilde{A})$ and $E^*(\tilde{A})$, when $\tilde{A}$ is a fuzzy number with continuous and strictly increasing membership function before the modal values $[m_1, m_2]$, and strictly decreasing after the modal values, using definition 4, it is easy to note that:

$$E_*(\tilde{A}) = M_*(1, \tilde{A}) = \int_0^1 a_1^\alpha d\alpha = \int_0^1 a_1^\alpha dL \quad \text{and}$$

$$E^*(\tilde{A}) = M^*(1, \tilde{A}) = \int_0^1 a_2^\alpha d\alpha = \int_0^1 a_2^\alpha dL.$$

By analogy we define respectively *S-upper* ad *S-lower* expectations, the quantity in (6) and define the *S-mean value* of a fuzzy number $\tilde{A}$ the interval

$$\overline{M}(S, \tilde{A}) = \left[M_*(S, \tilde{A}), \ M^*(S, \tilde{A})\right].$$

If S is a normalized Stieltjes measure, Gonzalez (1990) shows that $M_*(S, \tilde{A}) = E_*(\tilde{A}^s)$ and $M^*(S, A) = E^*(\tilde{A}^s)$, where $\tilde{A}^s$ is a fuzzy number with membership function $\mu_{A^s} = s \circ \mu_A$, and so

$$\overline{M}(S, \tilde{A}) = \left[M_*(S, A), \ M^*(S, A)\right] = \left[E_*(A^s), \ E^*(A^s)\right]$$
$$= \left[M_*(1, \tilde{A}^s), \ M^*(1, \tilde{A}^s)\right]$$

If S is generated by the function $s(\alpha) = \alpha^r$, we may indicate $\tilde{A}^s = \tilde{A}^r$. Its membership function is $\mu_{A^r} = x^r \circ \mu_A = \mu_A^r$. The last function was defined *operation of concentration* if $r > 1$ and *dilution* if $r < 1$ by Zadeh (1973).

These last results give us the possibility to reformulate the *S-mean value* in terms of possibility and necessity measures. In fact it is easy to show that

$$M_*(S,\tilde{A}) = \int_{-\infty}^{+\infty} x dF_S^*(x) \quad \text{and} \quad M^*(S,\tilde{A}) = \int_{-\infty}^{+\infty} x dF_*^S(x)$$

where

$$F_S^*(x) = \begin{cases} s \circ \mu_A(x) & \text{if } x < m_1 \\ 1 & \text{if } x \geq m_1 \end{cases} \qquad F_*^S(x) = \begin{cases} 1 - s \circ \mu_A(x) & \text{if } x \geq m_2 \\ 0 & \text{if } x < m_2 \end{cases}$$

therefore

$$M_\lambda(S,\tilde{A}) = \int_0^1 \varphi_\lambda(A^\alpha) dS = \lambda \int x \, \mathrm{d}\Pi^s + (1-\lambda) \int x \, \mathrm{d}\mathrm{N}^s$$

where Π^s and N^s are respectively the possibility and necessity measures associated with the fuzzy number $\tilde{A}^s$ with membership function $\mu_{A^s} = s \circ \mu_A$.

Using the previous notations, the next results generalise what Carlsson and Fuller (2001) have introduced. Their definitions are particular cases in which $s(x) = x^r$ with $r = 2$ and $\lambda = \frac{1}{2}$.

Definition 6 *We call respectively lower and upper possibilistic mean values of order r of a fuzzy number $\tilde{A}$ the quantities defined in* (7):

$$M_*(r,\tilde{A}) = r \int_0^1 \alpha^{r-1} a_1^\alpha d\alpha, \qquad M^*(r,\tilde{A}) = r \int_0^1 \alpha^{r-1} a_2^\alpha d\alpha,$$

Following Carlsson and Fuller (2001), we define a crisp possibilistic weighted mean of $\tilde{A}$ of order r with constant weights their convex combination:

$$M_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda) M_*(r,\tilde{A}) = \lambda \int x \, \mathrm{d}\Pi^s + (1-\lambda) \int x \, \mathrm{d}\mathrm{N}^s. \quad (9)$$

The last definition is the *Average Value* introduced by Campos Gonzalez (1989).

In the same paper, it is possible to find the proof of the following

THEOREM 1 *$\overline{M}_\lambda(r, \tilde{A})$ is a linear functional on the space of the fuzzy number F, that is: if $\oplus$ denotes the sum on F, then $\forall\ \tilde{A}, \tilde{B} \in F$, $\forall\ \delta \in R$*

$$\overline{M}_\lambda(r, \tilde{A} \oplus \tilde{B}) = \overline{M}_\lambda(r,\tilde{A}) + \overline{M}_\lambda(r,\tilde{B})$$

$$\overline{M}_\lambda(r, \delta\tilde{A}) = \delta \overline{M}_\lambda(r,\tilde{A})$$

Example 1. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number it easy to see that

$$M_*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_1^\alpha d\alpha = a_3 - \frac{a_3 - a_1}{r+1} \quad M^*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_2^\alpha d\alpha = a_3 + \frac{a_2 - a_3}{r+1}$$

$$\overline{M}_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda)M_*(r,\tilde{A}) = a_3 + \frac{\lambda(a_2 - a_3) - (1-\lambda)(a_3 - a_1)}{(r+1)}$$

If the triangular fuzzy number $\tilde{A}$ is symmetric, that is $a_2 - a_3 = a_3 - a_1$

$$\overline{M}_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda)M_*(r,\tilde{A}) = a_3 + \frac{(a_2 - a_3)(2\lambda - 1)}{(r+1)}$$

and if $\lambda = \frac{1}{2}$, $\overline{M}_\lambda(r,\tilde{A}) = a_3$. That is, the *crisp possibilistic weighted mean of $\tilde{A}$ of order r with constant weights* of a symmetric triangular fuzzy number is the central value, with membership function equal to one, if and only if the weight is $\lambda = \frac{1}{2}$.

Example 2. If $\tilde{A} = (m_1, m_2, \gamma, \beta)$ is a trapezoidal fuzzy number with $[m_1, m_2]$ flat graphic and left-width $\gamma > 0$ and right-width $\beta > 0$,

$$M_*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_1^\alpha d\alpha = m_1 - \frac{\gamma}{r+1}, \quad M^*(r,\tilde{A}) = r\int_0^1 \alpha^{r-1} a_2^\alpha d\alpha$$
$$= m_2 + \frac{\beta}{r+1}$$

and consequently

$$\overline{M}_\lambda(r,\tilde{A}) = \lambda M^*(r,\tilde{A}) + (1-\lambda)M_*(r,\tilde{A}) = \lambda m_2 + (1-\lambda)m_1 + \frac{(\lambda\beta - (1-\lambda)\gamma)}{(r+1)}$$

If $\beta = \gamma$ and $\lambda = \frac{1}{2}$, $\overline{M}(r,\tilde{A}) = \frac{m_1+m_2}{2}$. That is, the *crisp possibilistic weighted mean of $\tilde{A}$ of order r with constant weights* of a simmetric trapezoidal fuzzy number is the central value of the flat part, if and only if the weight is $\lambda = \frac{1}{2}$.

In Gonzalez (1989), there is a simple, but interesting, calculus for a trapezoidal fuzzy number $\tilde{A}$, useful in the next remark.

As $\varphi_\lambda(A^\alpha) = \lambda m_2 + (1-\lambda)m_1 + (1-\alpha)(\lambda\beta - (1-\lambda)\gamma)$,

$$M_\lambda(S,\tilde{A}) = \int_0^1 \varphi_\lambda(A^\alpha)dS = \lambda m_2 + (1-\lambda)m_1 + k(s)(\lambda\beta - (1-\lambda)\gamma) = \varphi_\lambda(A^{1-k(s)})$$

where $k(s) = \int_0^1 s(\alpha)d\alpha$.

The last result shows that the *crisp possibilistic weighted mean of order r with constant weights of* $\tilde{A}$ is an *I-evaluation* not of the support of $\tilde{A}$, but of a particular α-cut of $\tilde{A}$ with $\alpha = \alpha_s = 1 - \int_0^1 s(\alpha)d\alpha$.

Remark. The vision of the *Average Value* introduced by Gonzalez, or the *crisp possibilistic weighted mean of order r with constant weights of* $\tilde{A}$, as an *I-evaluation function*, let to understand the meaning of these quantity. In the general case, fixed λ and *S*, the *Average Value* is a value of the interval $\overline{M}(S, \tilde{A})$. In this particular case, not only we may assure that $\overline{M}(S, \tilde{A})$ is a subset of the support of $\tilde{A}$, but we may define it as the projection of the α-cut of $\tilde{A}$, with $\alpha = \alpha_s$, on the support of $\tilde{A}$. This idea let us to reduce the uncertainty in our decision.

Definition 7 *We call Variance of order r of a fuzzy number* $\tilde{A}$*, respect the weighted average* $\varphi_\lambda(\tilde{A})$*, the quantity:*

$$\begin{aligned} Var_\lambda(r, \tilde{A}) &= r\int_0^1 \alpha^{r-1}\left[\lambda a_2^\alpha + (1-\lambda)a_1^\alpha - a_1^\alpha\right]^2 d\alpha \\ &\quad + r\int_0^1 \alpha^{r-1}\left[\lambda a_2^\alpha + (1-\lambda)a_1^\alpha - a_2^\alpha\right]^2 d\alpha \\ &= (\lambda^2 + (1-\lambda)^2) r\int_0^1 \alpha^{r-1}\left[a_2^\alpha - a_1^\alpha\right]^2 d\alpha \end{aligned}$$

The variance of order *r* of a fuzzy number $\tilde{A}$, respect the weighted average $\varphi_\lambda(\tilde{A})$, is defined as the expected value of the squared deviations between the weighted average and the endpoints of its α-cuts respect the Stieltjes measure generated by $s(x) = x^r$. The deviation standard of $\tilde{A}$ of order *r* of a fuzzy number $\tilde{A}$, respect the weighted average $\varphi_\lambda(\tilde{A})$, is $\sigma_\lambda(r, \tilde{A}) = \sqrt{Var_\lambda(r, \tilde{A})}$.

Example 3. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number it easy to see that

$$Var_\lambda(r, \tilde{A}) = (\lambda^2 + (1-\lambda)^2)\frac{2}{(r+1)(r+2)}(a_2 - a_1)^2$$

If $\lambda = \frac{1}{2}$ and $r = 2$, $Var_\lambda(r, \tilde{A}) = \frac{1}{12}(a_2 - a_1)^2$. If $\tilde{A}$ is a crisp number $Var_\lambda(r, \tilde{A}) = 0$.

Following the same proof proposed by Carlsson and Fuller (2001) it is easy to show that $Var_\lambda(r, \tilde{A})$ is invariant for shifting by a real value of the fuzzy number. That is if $\tilde{A}$ is shifted by a value $\theta \in R$, and so we obtain a new fuzzy number $\tilde{B}$ which membership function is $\mu_B(x) = \mu_A(x - \theta)$, we have $Var_\lambda(r, \tilde{B}) = Var_\lambda(r, \tilde{A})$.

Definition 8 *We call the Covariance of two Fuzzy numbers of order r respect the weighted average $\varphi_\lambda(\tilde{A})$*

$$Cov_\lambda(r,\tilde{A},\tilde{B}) = (\lambda^2 + (1-\lambda)^2)r\int_0^1 \alpha^{r-1}[a_2^\alpha - a_1^\alpha][b_2^\alpha - b_1^\alpha]d\alpha$$

if $\tilde{A} = (a_1, a_3, a_2)$ and $\tilde{B} = (b_1, b_3, b_2)$ are triangular fuzzy numbers

$$Cov_\lambda(r,\tilde{A},\tilde{B}) = (\lambda^2 + (1-\lambda)^2)\frac{2}{(r+1)(r+2)}(a_2 - a_1)(b_2 - b_1)$$

Even in this case, following the proof of Carlsson and Fuller (2001), it is possible to proof the next two theorems. The first shows that the variance of a linear combination of fuzzy numbers follows the same rule than in probability theory. That is:

THEOREM 2 *Let $\eta, \tau \in R$ and let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. Then*

$$Var_\lambda(r,\eta\tilde{A} \oplus \tau\tilde{B}) = \eta^2 Var_\lambda(r,\tilde{A}) + \tau^2 Var_\lambda(r,\tilde{B}) + 2|\eta\tau|Cov_\lambda(\tilde{A},\tilde{B}),$$

where $\oplus$ denotes the sum on F.

The second puts a relation between the inclusion of two fuzzy numbers and an inequality between the relative variance.

THEOREM 3 *Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers and $\tilde{A} \subset \tilde{B}$ (that is $\mu_A(x) < \mu_B(x)$, $\forall x$). Then*

$$Var_\lambda(r,\tilde{A}) \leq Var_\lambda(r,\tilde{B}).$$

4. Weighted Average Value and Variance of Fuzzy Numbers with Not Constant Weights

In the last section we have generalized the results obtained by Carlsson and Fuller (2001), using the first *I-evaluation function* on A^α. Here we try to go over again the same road using, on F the second one defined in (4), in the case in which the Stieltjes measure S is generated by $s(x) = x^r$. The AV now is:

$$M_\rho(r,\tilde{A}) = r\int_0^1 \alpha^{r-1}\varphi_\rho(A^\alpha)d\alpha = r\int_0^1 \alpha^{r-1}\big(a_1^\alpha + \rho(a_1^\alpha, a_2^\alpha)(a_2^\alpha - a_1^\alpha)\big)d\alpha.$$

This is a linear combination of the extremes of A^α with not constant coefficient. It depends on the extremes of A^α.

This last formulation shows that in this general framework, it is not possible to have an analogous formulation of convex combination of *lower and upper mean value of a Fuzzy* *number* as in (9), as we are not able to evaluate the two integrals without having an explicit formulation of the *degree of risk* $\rho(a_1, a_2)$.

To overcome this difficulty, we remember the meaning we have given in section 1 to the term $\rho(a_1, a_2)$. It is the uncertainty filled with the position of the fuzzy number on the real axis. Now we may suppose that this function is independent by the several levels of α-cuts, that is, it is constant respect the variable α, and assumes the value with $\alpha = 0$. This is equivalent to put

$$\rho(a_1^\alpha, a_2^\alpha) = \rho(a_1^0, a_2^0) = \rho(a_1, a_2). \tag{9}$$

Using this condition, we obtain:

Definition 9 *We call crisp possibilistic weighted mean of $\tilde{A}$ of order r with not constant weights the quantity*

$$\begin{aligned} M_\rho(r, \tilde{A}) &= \rho(a_1, a_2) M^*(r, \tilde{A}) + (1 - \rho(a_1, a_2)) M_*(r, \tilde{A}) \\ &= \rho(a_1, a_2) \int x \ \mathrm{d}\Pi^s + (1 - \rho(a_1, a_2)) \int x \ \mathrm{d}\mathrm{N}^r \end{aligned} \tag{10}$$

where $\rho(a_1, a_2)$ is defined in (2).

Following the same idea of section 2 we may say that even in this case it is an "*I-evaluation function*", on the interval $[M_*(r, \tilde{A}), M^*(r, \tilde{A})]$, with not constant coefficient $\rho(a_1, a_2)$, which is related to $\tilde{A}$. It is the convex combination, with not constant coefficient $\rho(a_1, a_2)$, not of the support extremes of $\tilde{A}$, but of its α-cut with $\alpha = \alpha_r = \frac{r}{r+1}$. Even in this case, the idea to consider an *F-evaluation function* of $\tilde{A}$ as an *I-evaluation function* offers the possibility to understand what evaluation the decision maker is doing. The difference, compared to the linear case, is that of when the decision maker has to evaluate the fuzzy number $\tilde{A}$, decides which is the degree of risk that he assigns to $\tilde{A}$, and then he fixes r, in this way he decides the width of the interval in which he finds the final evaluation.

Because of the hypothesis we have put on $\rho(a_1, a_2)$, we cannot have the linearity of $M_\rho(r, \tilde{A})$, but we have the following

THEOREM 4 *$M_\rho(r, \tilde{A})$ is not, in general, a linear functional on the space of the fuzzy number F. What we can say is: if $\oplus$ denotes the sum on F, then $\forall\ \tilde{A}, \tilde{B} \in F$, $\forall\ \delta \in R$*

$$M_\rho(r, \tilde{A} \oplus \tilde{B}) = \rho_{A \oplus B}[M_\rho(r, \tilde{A}) + M_\rho(r, \tilde{B})]$$

$$M_\lambda(r, \delta\tilde{A}) = \rho_{\delta A}\ \delta\ M_\lambda(r, \tilde{A})$$

where $\rho_{A\oplus B}$ is related with the fuzzy number $A \oplus B$, and $\rho_{\delta A}$ is related to δA.

Example 4. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number we have:

$$M_\rho(r, \tilde{A}) = a_3 + \frac{\rho(a_1, a_2)(a_2 - a_3) - (1 - \rho(a_1, a_2))(a_3 - a_1)}{(r+1)}$$

It easy to see that there are no possibility in which $\overline{M}_\rho(r, \tilde{A}) = a_3$. If it happens $\frac{\rho(a_1, a_2)(a_2 - a_3) - (1 - \rho(a_1, a_2))(a_3 - a_1)}{(r+1)} = 0$ and this produce that $\rho(a_1, a_2) = \frac{a_3 - a_1}{a_2 - a_3}$. This function do not verify the hypothesis a), b), c) defined in section 1. In the symmetric case the only possibility to be equal to a_3 is that $\rho(a_1, a_2) = 1/2$, $\forall a_1, a_2$, but it is impossible why we have supposed $\rho(a_1, a_2)$ is not constant.

Example 5. If $\tilde{A} = (m_1, m_2, \gamma, \beta)$ is a trapezoidal fuzzy number with $[m_1, m_2]$ flat graphic and left-width $\gamma > 0$ and right-width $\beta > 0$,

$$M_*(r, \tilde{A}) = r \int_0^1 \alpha^{r-1} a_1^\alpha d\alpha = m_1 - \frac{\gamma}{r+1} \quad \text{and} \quad M^*(r, \tilde{A}) = r \int_0^1 \alpha^{r-1} a_2^\alpha d\alpha$$
$$= m_2 + \frac{\beta}{r+1}$$

and consequently

$$M_\rho(r, \tilde{A}) = \rho(a_1, a_2) m_2 + (1 - \rho(a_1, a_2)) m_1 + \frac{(\beta\rho(a_1, a_2) - (1 - \rho(a_1, a_2))\gamma)}{(r+1)}$$

For the same reason of Example 4, it is easy to see that the *crisp possibilistic weighted mean of $\tilde{A}$ of order r with not constant weights* of a trapezoidal fuzzy number is not $\frac{m_1 + m_2}{2}$.

Definition 7 *We call Variance of order r of a Fuzzy number $\tilde{A}$ respect the weighted average $\varphi_\rho(\tilde{A})$ with not constant weights*

$$Var_\rho(r, \tilde{A}) = r \int_0^1 \alpha^{r-1} \left[\rho(a_1, a_2) a_2^\alpha + (1 - \rho(a_1, a_2)) a_1^\alpha - a_1^\alpha\right]^2 d\alpha$$
$$+ r \int_0^1 \alpha^{r-1} \left[\rho(a_1, a_2) a_2^\alpha + (1 - \rho(a_1, a_2)) a_1^\alpha - a_2^\alpha\right]^2 d\alpha$$
$$= (\rho^2(a_1, a_2) + (1 - \rho(a_1, a_2))^2) r \int_0^1 \alpha^{r-1} [a_2^\alpha - a_1^\alpha]^2 d\alpha$$

The deviation standard of $\tilde{A}$ respect the weighted average $\varphi_\rho(\tilde{A})$ with not constant weights is $\sigma_\rho(A) = \sqrt{Var_\rho(r, \tilde{A})}$.

Example 6. If $\tilde{A} = (a_1, a_3, a_2)$ is a triangular fuzzy number, it is easy to see that

$$Var_\rho(r, \tilde{A}) = (\rho^2(a_1, a_2) + (1 - \rho(a_1, a_2))^2)\frac{2}{(r+1)(r+2)}(a_2 - a_1)^2$$

If $\tilde{A}$ is a crisp number $Var_\rho(r, \tilde{A}) = 0$.
If $\tilde{A}$ is shifted by a value $\theta \in R$, and so we obtain a new fuzzy number $\tilde{B}$ which membership function is $\mu_B(x) = \mu_A(x - \theta)$, because of the hypothesis on $\rho(a_1, a_2)$, we cannot have information about the relation between $Var_\rho(r, \tilde{A})$ and $Var_\rho(r, \tilde{B})$.

Definition 8 *We call the Covariance of two Fuzzy numbers, $\tilde{A}$ and $\tilde{B}$, of order r respect the weighted average $\varphi_\rho(\tilde{A})$ with not constant weights:*

$$Cov_\rho(\tilde{A}, \tilde{B}) = (\rho(a_1, a_2)\rho(b_1, b_2) + (1 - \rho(a_1, a_2))(1 - \rho(b_1, b_2))) \times r\int_0^1 \alpha^{r-1}[a_2^\alpha - a_1^\alpha]\left[b_2^\alpha - b_1^\alpha\right]d\alpha$$

if $\tilde{A} = (a_1, a_3, a_2)$ and $\tilde{B} = (b_1, b_3, b_2)$ are triangular fuzzy numbers,

$$Cov_\rho(\tilde{A}, \tilde{B}) = (\rho(a_1, a_2)\rho(b_1, b_2) + (1 - \rho(a_1, a_2))(1 - \rho(b_1, b_2)))\frac{2}{(r+1)(r+2)}(a_2 - a_1)(b_2 - b_1)$$

Because of the hypothesis on $\rho(a_1, a_2)$, we cannot have the proof of the analogous theorem in Carlsson and Fuller (2001) we obtain the variance of a linear combination of fuzzy numbers.

THEOREM 5 *Let η, $\tau \in R$ and let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. Then*

$$\begin{aligned} Var_\rho(r, \eta\tilde{A} \oplus \tau\tilde{B}) = {} & \eta^2 \frac{\rho^2_{\eta A \oplus \tau B} + (1 - \rho_{\eta A \oplus \tau B})^2}{\rho_A^2 + (1 - \rho_A)^2} Var_\rho(r, \tilde{A}) \\ & + \tau^2 \frac{\rho^2_{\eta A \oplus \tau B} + (1 - \rho_{\eta A \oplus \tau B})^2}{\rho_B^2 + (1 - \rho_B)^2} Var_\rho(r, \tilde{B}) \\ & + 2|\eta\tau| \frac{\rho^2_{\eta A \oplus \tau B} + (1 - \rho_{\eta A \oplus \tau B})^2}{\rho_A\rho_B + (1 - \rho_A)(1 - \rho_B)} Cov_\rho(\tilde{A}, \tilde{B}) \end{aligned} \tag{11}$$

where $\rho_{\eta A \oplus \tau B}$ is the degree of risk associated with the fuzzy number $\eta\tilde{A} \oplus \tau\tilde{B}$, ρ_A and ρ_B are respectively the degrees of risk associated with the fuzzy number $\tilde{A}$ and $\tilde{B}$, the addition $\oplus$ and the multiplication by a scalar are the usual definitions of sum and multiplication by scalar in fuzzy number set.

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers and $\tilde{A} \subset \tilde{B}$ (that is $\mu_A(x) < \mu_B(x)\ \forall x$). Then $\rho_A < \rho_B$, but we cannot say anything about the relation between $Var_\rho(r, \tilde{A})$ and $Var_\rho(r, \tilde{B})$.

5. Ranking Functions of Fuzzy Numbers

The definition of *crisp possibilistic weighted mean of* $\tilde{A}$ *of order r with constant weights* is an *F-evaluation function* and induces a natural ranking on fuzzy numbers. Let $\tilde{A}$, $\tilde{B}$ be in F and the *F-evaluation function* $M_\lambda(r, \tilde{A})$ defined in (5), we have

Definition 9 (Campos and Gonzalez (1989), Gonzalez (1990)) *We say that* $\tilde{B}$ *is* $M_\lambda(r)$ – *preferred to* $\tilde{A}$, *in symbols*

$$\tilde{A} \prec_{M_\lambda} \tilde{B} \quad \text{iff} \quad M_\lambda(r, \tilde{A}) < M_\lambda(r, \tilde{B})$$

This is a crisp preorder on R and an order relation on the quotient set generated by the equivalence relation $\tilde{A} \approx_{M_\lambda} \tilde{B}$ if and only if $M_\lambda(r, \tilde{A}) = M_\lambda(r, \tilde{B})$.

If $\tilde{A}$, $\tilde{B}$ are triangular fuzzy numbers, $\tilde{A} = (a_1, a_3, a_2)$ and $\tilde{B} = (b_1, b_3, b_2)$

$$\overline{M}_\lambda(r, \tilde{A}) - \overline{M}_\lambda(r, \tilde{B}) = \frac{r(a_3 - b_3) + \lambda(a_2 - b_2) + (1 - \lambda)(a_1 - b_1)}{r + 1}$$

It is easy to see that if $\tilde{A} = \tilde{B}$ then $\overline{M}_\lambda(r, \tilde{A}) = \overline{M}_\lambda(r, \tilde{B})$, but the converse is not true. We consider now, how the definition of *crisp possibilistic weighted mean of* $\tilde{A}$ *of order r with not constant weights* given in (10), induces a natural ranking on fuzzy numbers.

Let $\tilde{A}$, $\tilde{B}$ be in F and the *F-evaluation function* $\overline{M}_\rho(r,.)$. We have

$$\begin{aligned} M_\rho(r, \tilde{A}) &= \rho(a_1, a_2) M^*(r, \tilde{A}) + (1 - \rho(a_1, a_2)) M_*(r, \tilde{A}) \\ &= M_*(r, \tilde{A}) + \rho(a_1, a_2)(M^*(r, \tilde{A}) - M_*(r, \tilde{A})) \end{aligned} \tag{12}$$

Looking to the last formula, we may notice a similarity between it and (2). It is the second type of "*I-evaluation Function*", on the interval $[M_*(r, \tilde{A}), M^*(r, \tilde{A})]$, with not constant coefficient $\rho(a_1, a_2)$, which is related to $\tilde{A}$.

It is possible to use $\overline{M}_\rho(r, \tilde{A})$ to obtain a ranking function on F.

Definition 10 *We state that* $\tilde{B}$ *is* $\overline{M}_\rho(r)$ – *preferred to* $\tilde{A}$, *in symbols*

$$\tilde{A} \prec_{\overline{M}_\rho} \tilde{B} \quad \text{iff} \quad \overline{M}_\rho(r, \tilde{A}) < \overline{M}_\rho(r, \tilde{B})$$

This is a crisp preorder on R and an order relation on the quotient set generated by the equivalence relation $\tilde{A} \approx_{\overline{M}\rho} \tilde{B}$ *if and only if* $\overline{M}_\rho(r, \tilde{A}) = \overline{M}_\rho(r, \tilde{B})$.

If $\tilde{A}$ *is a triangular fuzzy number,* $\tilde{A} = (a_1, a_3, a_2)$,

$$\begin{aligned} \overline{M}_\rho(r, \tilde{A}) = \overline{M}_\rho(r, a_1, a_2, a_3) &= \frac{r}{r+1} a_3 + \frac{1}{r+1}[a_1 + \rho(a_1, a_2)(a_2 - a_1)] \\ &= \frac{r}{r+1} a_3 + \frac{1}{r+1} \varphi_\rho(a_1, a_2). \end{aligned}$$

As $\varphi_\rho(a_1, a_2)$ is increasing in the two variables, for r fixed, $\overline{M}_\rho(r, a_1, a_2, a_3)$ is increasing in all the variables if it is increasing in a_3. Classically no property is requested for ranking functions; nevertheless that is the increasing monotonicity in all its variables. The reason of this request is very natural. A fuzzy number is more preferable as it runs along the positive direction of the real axis.

6. Conclusion

We have introduced two types of evaluation functions on intervals and by them we have proposed ranking functions, mean values, variance and covariance of fuzzy numbers in a general framework. It is interesting to note that, in this field of research, many interesting results are present for linear evaluation functions, but few authors have tried to extend them in the not linear case. This paper is one of the first attempts in this direction. The introduction of the indexes, r, λ and ρ, let the possibility with the first to privilege the part of the fuzzy number one decide to choose, with the two others, to put in evidence the decision-maker risk tendency or aversion, that may be constant or not. In the first case it is fixed "ex ante" and cannot be changed, in the second it depends on the circumstances, which may be affected by the moment in which the decision has to be kept, but even by the importance that the decision maker gives to the results produced by the choice. We think that, in real applications, the second approach is more realistic.

Another interesting field of application of these results is in the defuzzification step. We may think to use the several average values, here introduced, at the final step of Fuzzy Expert Systems (FES). But the problem we meet in this type of application is that the output of a FES is not always a fuzzy number, but only a fuzzy set not convex. In these cases the approach of α-cuts is, at the moment, impossible. We are working in this direction to overcome this difficulty.

Notes

1. Tel. 0039-59-2056779.
2. We desire to thank Professor Fuller who looked the section 2 of this paper in 2001 and encouraged to present our results for printing.

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