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A primer on triangle functions II

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Abstract. In Saminger-Platz and Sempi (Aequationes Math 76:201–240, 2008) we presented an overview of concepts, facts and results on triangle functions based on the notions of t-norm, copula, (generalized) convolution, semicopula, quasi-copula. Here, we continue our presentation. In particular, we treat the concept of duality and study a few important cases of functional equations and inequalities for triangle functions like, e.g., convolution, Cauchy's equation, dominance, and Jensen convexity.

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1. Introduction

In this note we continue the presentation of concepts, facts and results on triangle functions that we started in [32]. We briefly recall that a *triangle function* is a binary operation on Δ^+ that is commutative, associative, and increasing in each place and has ε_0 as identity where Δ^+ denotes the set of all *distance distribution functions*, (briefly a d.d.f.), i.e., the set of increasing functions F from the extended reals $\overline{\mathbb{R}}$ into $[0, 1]$, being left–continuous on $\mathbb{R}$, and fulfilling $F(0) = 0$ and $F(\infty) = 1$. Moreover, ε_0 is a particular d.d.f. defined, for all $x \in \overline{\mathbb{R}}$, by

$$\varepsilon_0(x) := \begin{cases} 0, & x \leq 0, \\ 1, & \text{otherwise.} \end{cases}$$

Triangle functions have either been treated in their most general form, i.e., as special continuous semigroups on Δ^+ or have been studied by inspecting their important subclasses. Such important subclasses have been defined by recourse to t-norms, copulas, semicopulas, and quasi-copulas as well as (generalized) convolutions. These concepts, facts and results have been the main focus of the first part of this primer [32]. The reader is also referred to this first part [32] for the notations and the symbols.

In this second part, we treat a number of topics which were omitted in the first one, namely:

- the important concept of duality, which is studied in Sect. 3;
- some functional equations satisfied by triangle functions; in Sect. 4 we characterize convolution, while Cauchy's functional equation is the object of Sect. 5;
- functional inequalities related to dominance in Sects. 6 and 7, and to Jensen convexity in Sect. 8.

2. Quasi-inverses

Quasi-inverses of distribution functions are needed both in the theory of probabilistic metric (and normed) spaces and in dealing with the concept of duality (see next section). Quasi-inverses allow us to define an "inverse" even for d.f.'s that are not invertible in the usual sense. They have been used in different contexts and by several authors (see, e.g., [12,14,15,17,25,26]). We follow here the presentation of [11] reported also in [35], but keep in mind the detailed analysis of [12]; in both of these papers the setting is actually more general in that it was given for increasing functions φ from the closed interval $[p, q]$ with $-\infty \leq p < q \leq +\infty$ into $[r, s]$ ($-\infty \leq r < s \leq +\infty$) such that $\varphi(p) = r$ and $\varphi(q) = s$.

Notice that, given a distribution function $F : \overline{\mathbb{R}} \to [0, 1]$ and y in $]0, 1[$, one, and only one of the following statements holds:

(QI.1) there exists a unique $x \in \overline{\mathbb{R}}$ such that $F(x) = y$;
(QI.2) there is an interval $I \subset \overline{\mathbb{R}}$ such that $F(t) = y$ for every $t \in I$;
(QI.3) there is a unique $x_0 \in \overline{\mathbb{R}}$ such that $\ell^- F(x_0) < y < \ell^+ F(x_0)$.

The formal definition runs as follows.

Definition 2.1. A *quasi-inverse* of a d.f. F is a function $F^{(q)} : [0, 1] \to \overline{\mathbb{R}}$ is defined by

$$F^{(q)}(0) := -\infty \qquad F^{(q)}(1) := +\infty$$

and, if y is in $]0, 1[$, by $F^{(q)}(y) := x$, if (QI.1) holds, by $F^{(q)}(y) := x_0$, if (QI.3) holds, while, when (QI.2) holds, one chooses a value $t \in I$ for $F^{(q)}(y)$. If there is a point $y \in]0, 1[$ for which (QI.2) holds, then F will have infinitely many quasi-inverses.

Graphically, a quasi-inverse $F^{(q)}$ of F may be obtained from the graph of F through the following steps:

- at a point of discontinuity x_0 of F, if any, add to the graph of F the vertical segment of endpoints $(x_0, \ell^- F(x_0))$ and $(x_0, \ell^+ F(x_0))$;
- reflect the graph thus modified in the diagonal of the first and third quadrant;

- if the reflected graph has a vertical segment, which corresponds to case (QI.2) above, then choose the ordinate t of exactly one of the points of this vertical segment as the value of $F^{(q)}$.

The following result holds.

Theorem 2.1. ([11]) *Let F be a d.f., $F \in \Delta$, and let $F^{(q)}$ be a quasi-inverse of F. Then*

(a) *$F^{(q)}$ is uniquely determined at its points of continuity.*

(b) *The following conditions are equivalent:*

 (b1) *F is strictly increasing;*

 (b2) *$F^{(q)}$ is continuous on $]0, 1[$;*

 (b3) *$F^{(q)}$ is unique.*

 If any of these conditions holds, then $F^{(q)}(F(t)) = t$ for every $t \in \overline{\mathbb{R}}$. If F is a strictly increasing bijection then the inverse function F^{-1} of F exists.

(c) *The following conditions are equivalent:*

 (c1) *F is continuous;*

 (c2) *$F^{(q)}$ is strictly increasing.*

 If any of these conditions holds, then $F(F^{(q)}(t)) = t$ for every $t \in [0, 1]$; the inverse $\left(F^{(q)}\right)^{-1}$ of $F^{(q)}$ exists if, and only if, $F^{(q)}$ is continuous on $]0, 1[$, in which case $F = \left(F^{(q)}\right)^{-1}$.

(d) *F has a unique quasi-inverse that is left-continuous on $]0, 1[$ and a unique quasi-inverse that is right-continuous on $]0, 1[$; these are denoted respectively by $F^{\wedge}$ and $F^{\vee}$ and are defined by*

$$F^{\wedge}(t) := \begin{cases} \inf\{x \mid F(x) \geq t\}, & t < 1, \\ +\infty, & t = 1, \end{cases}$$

 and

$$F^{\vee}(t) := \begin{cases} -\infty, & t = 0, \\ \sup\{x \mid F(x) \leq t\}, & t > 0. \end{cases}$$

(e) *$F(F^{(q)}(x)) \leq x$ for every $x \in \mathbb{R}$.*

(f) *For every $x \in \overline{\mathbb{R}}$,*

$$F^{\wedge}(F(x)) \leq x \qquad \text{and} \qquad F^{\vee}(F(x)) \geq x.$$

(g) *For every increasing function $G : [0, 1] \to \overline{\mathbb{R}}$ there exists a d.f. $F \in \Delta$ such that $G = F^{(q)}$.*

(h) *At every point x of continuity for F one has $(F^{(q)})^{(q)}(x) = F(x)$.*

Notice that if F is taken, as is often done, to be right-continuous, then one has $F(F^{(q)}(x)) \geq x$ for every $x \in \mathbb{R}$.

Given two d.d.f.'s F_1 and F_2, we shall write $F_1 \sim F_2$ if their graphs, completed, if needed, as above, coincide. Clearly, $\sim$ is an equivalence relation.

Notice that $F_1 \sim F_2$ if, and only if, they have the same points of continuity and coincide at these points.

3. Duality

Duality for triangle functions deals essentially with the following situation. One has a triangle function τ acting on pairs (F, G) of d.d.f.'s, and one wishes, if possible, to express the left-continuous quasi-inverse of $\tau(F, G)$ as a function φ acting on the pair $(F^\wedge, G^\wedge)$ of the quasi-inverses of F and G, viz.

$$(\tau(F, G))^\wedge = \varphi(F^\wedge, G^\wedge)$$

or, equivalently,

$$\tau(F, G) = (\varphi(F^\wedge, G^\wedge))^\wedge .$$

The concept of duality was introduced with a greater generality than that of our presentation by Frank and Schweizer [11]. It must be stressed that the literature does not deal with duality for all the types of triangle function considered in [32] but only with few of them. We provide here the proofs of results dealing with those cases, since these do not appear in the monograph [35].

The most important instance of duality is that concerning the triangle functions

$$\tau_{T,L}(F, G)(x) := \sup \{T(F(u), G(v)) \mid L(u, v) = x\} ,$$

where T is a left-continuous t-norm, L belongs to $\mathfrak{L}$ (see [35] and [32, Definition 3.5]) and is commutative and associative; we recall that in this case $\tau_{T,L}$ is a triangle function ([35] and [32, Theorem 7.11]). However, since in order to consider a quasi-inverse, one only needs to have a d.d.f., it will be enough to consider the mappings $\tau_{S,L}$ where S is a left-continuous semicopula.

Let ∇^+ be the set of left-continuous quasi-inverses of d.d.f.'s,

$$\nabla^+ := \{F^\wedge \mid F \in \Delta^+\}.$$

Definition 3.1. For every left-continuous semicopula $S \in \mathcal{S}$ and for every $L \in \mathfrak{L}$ define a function $\tau^\wedge_{S,L} : \nabla^+ \times \nabla^+ \to \overline{\mathbb{R}}_+^{[0,1]}$ by

$$\tau^\wedge_{S,L}(F^\wedge, G^\wedge)(x) := \inf \{L (F^\wedge(u), G^\wedge(v)) \mid S(u, v) = x\} . \qquad (3.1)$$

When L is the sum, we write $\tau^\wedge_S$.

Lemma 3.1. *Let $L \in \mathfrak{L}$ satisfy condition* (LS) *of Definition 3.5 in [32], let S be a left-continuous semicopula, and let F and G belong to Δ^+. Then, for every $t \in [0, 1]$,*

$$(\tau_{S,L}(F, G))^\wedge (t) \leq \tau^\wedge_{S,L}(F^\wedge, G^\wedge)(t). \qquad (3.2)$$

Proof. It is known ([32, Lemma 7.1]) that $\tau_{S,L}(F,G)$ is a d.d.f. for all F and G in Δ^+. Assume, if possible, that (3.2) does not hold; then there exist $t_1 \in \,]0,1[$ and $x_1 \in \mathbb{R}_+$ such that

$$\tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})(t_1) < x_1 < \left(\tau_{S,L}(F,G)\right)^{\wedge}(t_1). \tag{3.3}$$

By the definition of $\tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})$, there exists a pair (u_1, v_1) such that

$$S(u_1, v_1) = t_1 \quad \text{and} \quad L\left(F^{\wedge}(u_1), G^{\wedge}(v_1)\right) < x_1.$$

Since $t_1 < 1$ one has both $u_1 < 1$ and $v_1 < 1$ and since L satisfies condition (LS), there exist y_1 and z_1 in $]0,+\infty[$ such that

$$F^{\wedge}(u_1) < y_1, \quad G^{\wedge}(v_1) < z_1 \quad \text{and} \quad L(y_1, z_1) = x_1;$$

this implies $u_1 \le F(y_1)$ and $v_1 \le G(z_1)$. Hence

$$t_1 = S(u_1, v_1) \le S(F(y_1), G(z_1)) \quad \text{and} \quad L(y_1, z_1) = x_1;$$

thus the definition of $\tau_{S,L}$ yields $\tau_{S,L}(F,G)(x_1) \ge t_1$. It follows from Theorem 2.1 (f) that

$$\left(\tau_{S,L}(F,G)\right)^{\wedge}(t_1) \le \left(\left(\tau_{S,L}(F,G)\right)^{\wedge}\right)\left(\tau_{S,L}(F,G)(x_1)\right) \le x_1,$$

which contradicts (3.3). □

In the other direction one has the following result.

Lemma 3.2. *Let $L \in \mathfrak{L}$ satisfy the assumptions of Lemma 3.1, let S be a left-continuous semicopula, and let F and G be two d.d.f.'s, $F, G \in \Delta^+$. If $\left(\tau_{S,L}(F,G)\right)^{\wedge}$ is continuous at $t_0 \in \,]0,1[$, then*

$$\left(\tau_{S,L}(F,G)\right)^{\wedge}(t_0) \ge \tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})(t_0). \tag{3.4}$$

Proof. Assume, if possible, that (3.4) does not hold, namely

$$\left(\tau_{S,L}(F,G)\right)^{\wedge}(t_0) < \tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})(t_0).$$

In order to reach a contradiction, it will be shown that $\left(\tau_{S,L}(F,G)\right)^{\wedge}$ is not continuous at t_0. To this end, let x belong to the open interval

$$\left]\left(\tau_{S,L}(F,G)\right)^{\wedge}(t_0), \tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})(t_0)\right[. \tag{3.5}$$

Thus

$$\tau_{S,L}(F,G)(x) \ge t_0.$$

If this inequality were strict, then there would exist y and z in $\overline{\mathbb{R}}_+$ such that $L(y,z) = x$ and $S(F(y), G(z)) > t_0$. But then

$$\begin{aligned} \tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})(t_0) &\le \left(\tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})\right)\left(S(F(y), G(z))\right) \\ &\le L\left(F^{\wedge} F(y), G^{\wedge} G(z)\right) \le L(y,z) = x, \end{aligned}$$

which is a contradiction. Therefore, $\tau_{S,L}(F,G)(x) = t_0$; this means that $\tau_{S,L}(F,G)$ is constant on the interval (3.5), and that, as a consequence, $\tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})$ is not continuous at t_0. □

In general, $(\tau_{S,L}(F,G))^\wedge(x_0)$ and $\tau^\wedge_{S,L}(F^\wedge,G^\wedge)(x_0)$ may differ at a point x_0 where $(\tau_{S,L}(F,G))^\wedge$ is not continuous. However, Lemmata 3.1 and 3.2 together yield the next result.

Theorem 3.3. *Let $L \in \mathfrak{L}$ satisfy the assumptions of Lemma 3.1 and let S be a left-continuous semicopula. Then, for all F and G in Δ^+,*

$$(\tau_{S,L}(F,G))^\wedge \sim \tau^\wedge_{S,L}(F^\wedge,G^\wedge).$$

Corollary 3.4. *If F_1, F_2, G_1 and G_2 are d.d.f.'s such that $F_1 \sim F_2$ and $G_1 \sim G_2$, then $\tau_{S,L}(F_1,G_1) \sim \tau_{S,L}(F_2,G_2)$.*

Proof. Since $F_1^\wedge = F_2^\wedge$ and $G_1^\wedge = G_2^\wedge$, Theorem 3.3 yields

$$(\tau_{S,L}(F_1,G_1))^\wedge \sim \tau^\wedge_{S,L}(F_1^\wedge,G_1^\wedge) = \tau^\wedge_{S,L}(F_2^\wedge,G_2^\wedge) \sim (\tau_{S,L}(F_2,G_2))^\wedge,$$

which proves the assertion. □

Remark 3.1. Let $\mathcal{F}$ be any well-formed formula consisting of elements of Δ^+ and ∇^+, of the operations $^\wedge$, $^\vee$, S and L, and of the relations $\sim$, $=$, $\leq$ and $\geq$. The *dual* of $\mathcal{F}$ is the well-formed formula obtained from $\mathcal{F}$ by interchanging the elements of the pairs Δ^+ and ∇^+, $^\wedge$ and $^\vee$, T and L and by reversing all the inequalities (in particular, hence, by interchanging sup and inf). If $\mathcal{F}$ is true, then so is its dual; and the proof of the dual of $\mathcal{F}$ is the dual of the proof of $\mathcal{F}$. As a consequence, any result consisting of two dual statements may be established by proving just one of the two statements.

The next result provides a sufficient condition that ensures that the equivalence sign $\sim$ in Theorem 3.3 may be replaced by the equality one.

Theorem 3.5. *Let $L \in \mathfrak{L}$ satisfy the assumptions of Lemma 3.1. If $L \in \mathfrak{L}$ is continuous on $\overline{\mathbb{R}}_+^2$, and if the semicopula S is continuous, then, for all F and G in Δ^+,*

$$(\tau_{S,L}(F,G))^\wedge = \tau^\wedge_{S,L}(F^\wedge,G^\wedge). \tag{3.6}$$

Proof. Because of Lemma 3.1, it suffices to establish inequality (3.4). Without loss of generality, one may assume $(\tau_{S,L}(F,G))^\wedge(t_0) < 1$. Choose any x_0 such that

$$(\tau_{S,L}(F,G))^\wedge(t_0) < x_0 < 1. \tag{3.7}$$

It is enough to prove that

$$\tau^\wedge_{S,L}(F^\wedge,G^\wedge)(t_0) \leq x_0, \tag{3.8}$$

from which (3.4) follows immediately.

It follows from (3.7) that $\tau_{S,L}(F,G)(x_0) \geq t_0$; thus, for every $n \in \mathbb{N}$, there exist y_n and z_n in $\overline{\mathbb{R}}_+$ such that

$$L(y_n,z_n) = x_0 \quad \text{and} \quad S(F(y_n),G(z_n)) > t_0 - \frac{1}{n} \tag{3.9}$$

Now (y_n) has a convergent subsequence $(y_{n(k)})$ and the corresponding subsequence $(z_{n(k)})$ has a convergent subsequence $(z_j)_{j\in\mathbb{N}}$. One may choose the sequences $(y_j)_{j\in\mathbb{N}}$ and $(z_j)_{j\in\mathbb{N}}$ to be monotone. Set

$$y_0 := \lim_{j\to+\infty} y_j \quad \text{and} \quad z_0 := \lim_{j\to+\infty} z_j.$$

Because of the continuity of L,

$$L(y_0, z_0) = L\left(\lim_{j\to+\infty} y_j, \lim_{j\to+\infty} z_j\right) = \lim_{j\to+\infty} L(y_j, z_j) = x_0.$$

Thus, by (3.7), $y_0 < 1$ and $z_0 < 1$. Since F and G are isotonic, the limits of both of the sequences $(F(y_j))_{j\in\mathbb{N}}$ and $(G(z_j))_{j\in\mathbb{N}}$ exist. Now the continuity of T and (3.9) yield

$$\begin{aligned} t_0 &= \lim_{j\to+\infty}\left(t_0 - \frac{1}{j}\right) \le \lim_{j\to+\infty} S(F(y_j), G(z_j)) \\ &= S\left(\lim_{j\to+\infty} F(y_j), \lim_{j\to+\infty} G(z_j)\right) = S\left(\ell^+F(y_0), \ell^+G(z_0)\right). \end{aligned}$$

For every $\varepsilon >$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$L(y_0 + \delta, z_0 + \delta) < L(y_0, z_0) + \varepsilon = x_0 + \varepsilon.$$

Therefore

$$\begin{aligned} \tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})(t_0) &\le \tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})\left(S\left(\ell^+F(y_0), \ell^+G(z_0)\right)\right) \\ &= \inf\left\{L\left(F^{\wedge}(u), G^{\wedge}(v)\right) \mid S(u,v) = S\left(\ell^+F(y_0), \ell^+G(z_0)\right)\right\} \\ &\le L\left(F^{\wedge}\left(\ell^+F(y_0)\right), G^{\wedge}\left(\ell^+G(z_0)\right)\right) \\ &\le L\left(F^{\wedge}(F(y_0+\delta), G^{\wedge}(G(z_0+\delta)\right) \\ &\le L(y_0+\delta, z_0+\delta) < x_0 + \varepsilon, \end{aligned}$$

which establishes (3.8) and, hence, the assertion. □

The following theorem will have an important consequence (see below Theorem 3.7).

Theorem 3.6. *Let S be a left-continuous semicopula and let $L \in \mathfrak{L}$ satisfy condition* (LS) *of Definition 3.5 of [32]. If $S(u_1, v_1) < S(u_2, v_2)$, whenever $u_1 < u_2$ and $v_1 < v_2$, then*

$$\forall\ F, G \in \Delta^+ \qquad (\tau_{S,L}(F,G))^{\wedge} = \tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge}). \tag{3.10}$$

Similarly,

$$\forall\ F^{\wedge}, G^{\wedge} \in \nabla^+ \qquad \left(\tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})\right)^{\wedge} = \tau_{S,L}(F,G). \tag{3.11}$$

or, respectively,

$$\tau_{S,L}(F,G) = \left(\tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge})\right)^{\wedge}, \tag{3.12}$$

and

$$\tau^{\wedge}_{S,L}(F^{\wedge}, G^{\wedge}) = (\tau_{S,L}(F,G))^{\wedge}. \tag{3.13}$$

Proof. In accordance with Remark 3.1, it suffices to prove one of the two statements, for instance, equation (3.10). To this end, all that is needed is the proof that $\tau_{S,L}(F,G)$ is left-continuous on $\mathbb{R}_+$. □

In a special case, the last result leads to an elegant and very useful expression.

Theorem 3.7. *If* $T = M = \mathrm{Min}$, *then*

$$\forall\ F, G \in \Delta^+ \qquad \tau_{M,L}(F,G) = L(F^{\wedge}, G^{\wedge})^{\wedge}. \tag{3.14}$$

In particular, if $L = \mathrm{Sum}$,

$$\forall\ F, G \in \Delta^+ \qquad \tau_M(F,G) = (F^{\wedge} + G^{\wedge})^{\wedge}. \tag{3.15}$$

4. Functional equations: convolution

An interesting characterization of convolution was given in [4] as the unique solution of a functional equation satisfied by a function τ that belongs to a class of functions larger than that of triangle functions. In this section we give the details and the proofs in the case of triangle functions. The functional equation in question is the following

$$\tau\left(\frac{F+G}{2}, H\right) = \frac{1}{2}\,\tau(F,H) + \frac{1}{2}\,\tau(G,H), \tag{4.1}$$

where F, G and H are distance distribution functions. This equation bears a marked analogy to the classical Jensen functional equation.

Definition 4.1. Let Φ be a binary associative function on $\mathbb{R}$ that can be represented in the form

$$\Phi(x,y) = \varphi^{-1}\,\left(\varphi(x) + \varphi(y)\right), \tag{4.2}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and such that $\varphi(0) = 0$. The triangle function τ is said to be *in step with* Φ if, for all a and b in $\mathbb{R}_+$,

$$\tau(\varepsilon_a, \varepsilon_b) = \varepsilon_{\Phi(a,b)}. \tag{4.3}$$

Notice that the convolution $*$ is in step with addition.

Lemma 4.1. *For every pair* F *and* G *of d.d.f.'s, one has, for all* $x \in \mathbb{R}$,

$$\sigma_{\Pi,\Phi}(F,G)(x) = \left[\left(F \circ \varphi^{-1}\right) * \left(G \circ \varphi^{-1}\right)\right] \circ \varphi(x). \tag{4.4}$$

Proof. Since $F \circ \varphi^{-1}$ and $G \circ \varphi^{-1}$ are in Δ^+, one has

$$\begin{aligned}\sigma_{\Pi,\Phi}(F,G)(x) &= \int\limits_{\{(u,v):\varphi(u)+\varphi(v)<\varphi(x)\}} d\,(F(u)\cdot G(v)) \\ &= \int\limits_{\{(s,t):s+t<\varphi(x)\}} d\left(\left(F\circ\varphi^{-1}\right)(s)\cdot\left(G\circ\varphi^{-1}\right)(t)\right) \\ &= \left[\left(F\circ\varphi^{-1}\right)*\left(G\circ\varphi^{-1}\right)\right]\circ\varphi(x).\end{aligned}$$

□

Lemma 4.2. *The triangle function $\sigma_{\Pi,\Phi}$ is continuous and in step with Φ.*

Proof. The continuity of $\sigma_{\Pi,\Phi}$ was established in [4, Corollary 1].

For all a and b in $\mathbb{R}_+$, one has

$$\begin{aligned}\sigma_{\Pi,\Phi}(\varepsilon_a,\varepsilon_b) &= \left[\left(\varepsilon_a\circ\varphi^{-1}\right)*\left(\varepsilon_b\circ\varphi^{-1}\right)\right]\circ\varphi = \left[\varepsilon_{\varphi(a)}*\varepsilon_{\varphi(b)}\right]\circ\varphi \\ &= \varepsilon_{\phi(a)+\phi(b)}\circ\varphi = \varepsilon_{\varphi^{-1}(\phi(a)+\phi(b))} = \varepsilon_{\Phi(a,b)}.\end{aligned}$$

□

It is immediate to prove the following.

Lemma 4.3. *The triangle function $\sigma_{\Pi,\Phi}$ is a solution of (4.1).*

The next result contains the main step in the announced characterization.

Lemma 4.4. *For a continuous triangle function τ the following statements are equivalent:*

(a) *it satisfies (4.1);*

(b) *for every pair of sequences $(F_j)_{j\in\mathbb{N}}$ and $(G_k)_{k\in\mathbb{N}}$ in Δ^+,*

$$\tau\left(\sum_{j\in\mathbb{N}}\frac{F_j}{2^j},\sum_{k\in\mathbb{N}}\frac{G_k}{2^k}\right) = \sum_{j,k\in\mathbb{N}}\frac{\tau(F_j,G_k)}{2^{j+k}}. \tag{4.5}$$

(c) *for all F, G and H in Δ^+ and for every number $\alpha\in[0,1]$*

$$\tau(\alpha\,F+(1-\alpha)\,G,H) = \alpha\,\tau(F,H)+(1-\alpha)\,\tau(G,H);$$

(d) *for every pair of sequences $(F_j)_{j\in\mathbb{N}}$ and $(G_k)_{k\in\mathbb{N}}$ in Δ^+, and for all sequences of positive real numbers $(\lambda_j)_{j\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$ with $\sum_{j\in\mathbb{N}}\lambda_j = \sum_{k\in\mathbb{N}}\mu_k = 1$,*

$$\tau\left(\sum_{j\in\mathbb{N}}\lambda_j\,F_j,\sum_{k\in\mathbb{N}}\mu_k\,G_k\right) = \sum_{j,k\in\mathbb{N}}\lambda_j\,\mu_k\,\tau(F_j,G_k). \tag{4.6}$$

Proof. (a) $\Longrightarrow$ (b) An induction argument, the commutativity of τ and a bit of patient work show that

$$\tau\left(\sum_{j=1}^{n}\frac{F_j}{2^j}+\frac{\varepsilon_\infty}{2^n},\sum_{k=1}^{n}\frac{G_k}{2^k}+\frac{\varepsilon_\infty}{2^n}\right)=\sum_{j,k=1}^{n}\frac{\tau\left(F_j,G_k\right)}{2^{j+k}}+\left[1-\left(1-\frac{1}{2^n}\right)^2\right]\varepsilon_\infty,$$

for all d.d.f.'s F_1, $F_2,\ldots$, F_n, G_1, $G_2,\ldots$, G_n. Because of the continuity of τ, letting n go to $+\infty$ yields (4.5).

(b) $\Longrightarrow$ (c) Consider the binary expansion $\alpha=\sum_{j\in\mathbb{N}}\alpha_j/2^j$, with α_j in $\{0,1\}$ for every $j\in\mathbb{N}$; then $1-\alpha=\sum_{j\in\mathbb{N}}(1-\alpha_j)/2^j$; here, again, $1-\alpha_j$ is in $\{0,1\}$. Then

$$\begin{aligned}\tau(\alpha\,F+(1-\alpha)\,G,H)&=\tau\left(\sum_{j\in\mathbb{N}}\frac{\alpha_j\,F+(1-\alpha_j)\,G}{2^j},H\right)\\&=\sum_{j\in\mathbb{N}}\frac{\tau(\alpha_j\,F+(1-\alpha_j)\,G,H)}{2^j}\\&=\sum_{j\in\mathbb{N}}\frac{\alpha_j\,\tau(F,H)+(1-\alpha_j)\,\tau(G,H)}{2^j}\\&=\alpha\,\tau(F,H)+(1-\alpha)\,\tau(G,H).\end{aligned}$$

(c) $\Longrightarrow$ (d) This is established by an induction argument similar to that of the first step in the present proof, and by recourse to the continuity of τ.

(d) $\Longrightarrow$ (a) It suffices to take in (4.6), $G_k=H$ for every $k\in\mathbb{N}$, $\lambda_j=1/2^j$, for every $j\in\mathbb{N}$, $F_1=F$, and $F_n=G$ for $n\geq 2$. □

Theorem 4.5. *The following statements are equivalent for a continuous triangle function τ:*

(a) $\tau=\sigma_{\Pi,\Phi}$;
(b) τ *satisfies* (4.1) *and is in step with* Φ.

Proof. The implication (a) $\Longrightarrow$ (b) follows from Lemmata 4.3 and 4.2.

(b) $\Longrightarrow$ (a) Let (a_j) and (b_k) be isotonic sequences of positive numbers and let (α_j) and (β_k) be sequences of positive numbers such that $\sum_{j\in\mathbb{N}}\alpha_j=$

$\sum_{k\in\mathbb{N}} \beta_k = 1$. Then, Lemma 4.4 and (4.3) imply

$$\begin{aligned}\tau\left(\sum_{j\in\mathbb{N}} \alpha_j\,\varepsilon_{a_j}, \sum_{k\in\mathbb{N}} \beta_k\,\varepsilon_{b_k}\right) &= \sum_{j,k\in\mathbb{N}} \alpha_j\,\beta_k\,\tau(\varepsilon_{a_j}, \varepsilon_{b_k}) \\ &= \sum_{j,k\in\mathbb{N}} \alpha_j\,\beta_k\,\varepsilon_{\Phi(a_j,b_k)} = \sum_{j,k\in\mathbb{N}} \alpha_j\,\beta_k\,\sigma_{\Pi,\Phi}(\varepsilon_{a_j}, \varepsilon_{b_k}) \\ &= \sigma_{\Pi,\Phi}\left(\sum_{j\in\mathbb{N}} \alpha_j\,\varepsilon_{a_j}, \sum_{k\in\mathbb{N}} \beta_k\,\varepsilon_{b_k}\right).\end{aligned}$$

Therefore, τ equals $\sigma_{\Pi,\Phi}$ on the set $E \times E$, where

$$E := \left\{\sum_{j\in\mathbb{N}} \alpha_j\,\varepsilon_{a_j} \;\middle|\; \sum_{j\in\mathbb{N}} \alpha_j = 1, \alpha_j \geq 0, a_j \geq 0, a_j < a_{j+1}\ (j \in \mathbb{N})\right\}.$$

Since E is dense in Δ^+, when this space is endowed with the topology of weak convergence, and since τ is continuous, τ equals $\sigma_{\Pi,\Phi}$ on the whole space $\Delta^+ \times \Delta^+$. □

5. Functional equations: Cauchy's equation

Basing himself on the work of Powers [19], Riedel [22,23] made a detailed study of Cauchy's functional equation on Δ^+. In this section we report his results. It is necessary to consider first Cauchy's equation for a t-norm T. A function $\theta : [0,1] \to [0,1]$ is a solution of Cauchy's equation for T if, and only if, it satisfies, for all x and y in $[0,1]$,

$$\theta\left(T(x,y)\right) = T\left(\theta(x), \theta(y)\right). \tag{5.1}$$

Related results are also obtained in [9,30].

The constant solutions $\theta(x) = 0$ and $\theta(x) = 1$, for every $x \in [0,1]$, will be called *trivial*.

We recall that a continuous t-norm T is *Archimedean* if there exists a continuous strictly decreasing function $g : [0,1] \to \overline{\mathbb{R}}_+$ with $g(1) = 0$, such that T is represented in the form

$$T(x,y) = g^{(-1)}\left(g(x) + g(y)\right), \tag{5.2}$$

where the function g is an *(inner) additive generator* of T and

$$g^{(-1)}(x) := g^{-1}\left(\min\{x, g(0)\}\right)$$

is its quasi–inverse; if g and h generate the same t-norm T, then they are related by the relationship $g(x) = k \cdot h(x)$ $(x \in [0,1])$.

When T is Archimedean the continuous solutions of (5.1) are given in the following theorem, which is proved by relying on the theory of the usual Cauchy's functional equation (see [1,2]).

Theorem 5.1. *Let T be an Archimedean continuous t-norm and let g be one of its (inner) additive generators. Then the following statements are equivalent:*

(a) *θ is a continuous solution of (5.1);*

(b) *there is a constant $k \geq 1$ such that*

$$\theta(x) = g^{(-1)}\left(k \cdot g(x)\right).$$

A mapping $\varphi : \Delta^+ \to \Delta^+$ is a solution of the Cauchy's functional equation for a triangle function τ, if, and only if, for all d.d.f.'s F and G,

$$\varphi\left(\tau(F, G)\right) = \tau\left(\varphi(F), \varphi(G)\right). \tag{5.3}$$

If τ has no non-trivial idempotents, namely if $\tau(F, F) = F$ implies either $F = \varepsilon_0$ or $F = \varepsilon_\infty$, then ε_0 and ε_∞ are the only constant solutions of (5.3). In general, if φ is a solution of (5.3), then φ maps idempotents to idempotents, and preserves n-th τ-powers, i.e. $\varphi\left(H_\tau^n\right) = \left(\varphi(H)\right)_\tau^n$, for every $H \in \Delta^+$ and for every $n \in \mathbb{N}$. We recall that, for a given sequence $(F_j)_{j\in\mathbb{N}}$ in Δ^+, one defines

$$\tau^2(F_1, F_2) := \tau(F_1, F_2)$$

and, for $n > 2$,

$$\tau^n(F_1, F_2, \ldots, F_n) := \tau\left(\tau^{n-1}(F_1, F_2, \ldots, F_{n-1}), F_n\right)$$

so that the nth τ-power of $H \in \Delta^+$ is defined by $H_\tau^n := \tau^n(H, H, \ldots, H)$. Moreover, it is easily seen that, for every triangle function τ, the following are solutions of (5.3):

(a) the identity map id_{Δ^+};

(b) the τ-power function $\varphi(F) := F_\tau^n$ $(n \in \mathbb{N};\ F \in \Delta^+)$;

(c) if τ has a non-trivial idempotent $H \in \Delta^+$, $\tau(H, H) = H$, the mapping φ_H defined, for all $F \in \Delta^+$, by $\varphi_H(F) = \tau(F, H)$.

To the best of the authors' knowledge, the general solution of (5.3) is known for a proper subset of sup-continuous triangle functions only [24].

Definition 5.1. A mapping $\varphi : \Delta^+ \to \Delta^+$ is said to be *sup-continuous* if, for every family $\{F_\iota \mid \iota \in I\}$ of d.d.f.'s,

$$\sup_{\iota\in I} \varphi\left(F_\iota\right) = \varphi\left(\sup_{\iota\in I} F_\iota\right),$$

while a triangle function τ is *sup-continuous* if, for every family $\{F_\iota \mid \iota \in I\}$ of d.d.f.'s, and for every $G \in \Delta^+$,

$$\sup_{\iota\in I} \tau(F_\iota, G) = \tau\left(\sup_{\iota\in I} F_\iota, G\right).$$

Tardiff (see [38, Sect. 6.2]) proved that τ_T is sup-continuous whenever T is a continuous t-norm and that convolution is not sup-continuous.

In looking for solutions of the Cauchy's equation, the following lemma allows us to restrict our attention to a subset of Δ^+ in the presence of sup-continuity.

Lemma 5.2. [22, Lemma 4.4] *When both $\varphi : \Delta^+ \to \Delta^+$ and the triangle function τ are sup-continuous, φ is a solution of the Cauchy's equation (5.3) if, and only if*

$$\varphi\left(\tau\left(\delta_{a,b}, \delta_{c,d}\right)\right) = \tau\left(\varphi\left(\delta_{a,b}\right), \varphi\left(\delta_{c,d}\right)\right), \tag{5.4}$$

for all a and c in $\overline{\mathbb{R}}_+$, and for all b and d in $[0,1]$, where

$$\delta_{a,b}(x) := \begin{cases} 0, & x \in [0,a]\,, \\ b, & x \in \left]a, +\infty\right[\,, \\ 1, & x = +\infty. \end{cases}$$

Definition 5.2. Given a lattice L, an *order automorphism* is a mapping $\varphi : L \to L$ such that both φ and φ^{-1} are order-preserving.

When Δ^+ is regarded as a lattice, one has the following result, proved in greater generality by Powers.

Theorem 5.3. [19, Theorem 3.4] *For a mapping $\varphi : \Delta^+ \to \Delta^+$ the following are equivalent*

(a) *φ is an order-automorphism of Δ^+;*
(b) *there are an order-automorphism θ of $[0,1]$ and an order-automorphism γ of $\overline{\mathbb{R}}_+$ such that $\varphi = \theta \circ id_{\Delta^+} \circ \gamma$, or, equivalently, such that, for every $F \in \Delta^+$,*

$$\varphi(F) = \theta \circ F \circ \gamma. \tag{5.5}$$

The previous theorem immediately allows us to solve the Cauchy's equation (5.3) in the important case of triangle functions of the type τ_T.

Theorem 5.4. [22, Theorem 5.3] *If T is a strict Archimedean t-norm with an additive generator g, i.e., $g(0) = +\infty$, then, for an order automorphism φ of Δ^+, the following statements are equivalent:*

(a) *φ is a solution of the Cauchy's equation for τ_T;*
(b) *there exist two strictly positive numbers k and l such that, for every $x \in \overline{\mathbb{R}}_+$ and for every $F \in \Delta^+$,*

$$\left(\varphi(F)\right)(x) = g^{-1}\left(k \cdot g\left(F(l \cdot x)\right)\right).$$

If T is a continuous, non-strict Archimedean t-norm with an additive generator g, i.e., $g(0) < +\infty$, then, for an order automorphism φ of Δ^+, the following statements are equivalent:

(a) *φ is a solution of the Cauchy's equation for τ_T;*

(b) *there exists a strictly positive number l such that, for every $x \in \overline{\mathbb{R}}_+$ and for every $F \in \Delta^+$,*

$$(\varphi(F))(x) = F(l \cdot x).$$

Finally, we note that Riedel ([23]) has extended the above results to the case when the triangle function τ of equation (5.3) is of the form $\tau = \tau_{T,L}$ where T is an Archimedean t-norm with a generator g, viz.

$$T(u,v) = g^{(-1)}(g(u) + g(v))$$

and L is of the form

$$L(x,y) = f^{-1}(f(x) + f(y));$$

here f is a continuous, strictly increasing map from $\overline{\mathbb{R}}_+$ onto $\overline{\mathbb{R}}_+$ such that $f(0) = 0$.

We also mention that Alsina and Schweizer ([6], but see also [18]), relying on the theory of duality (see Sect. 3), proved that τ_M is the unique solution of the functional equation

$$\tau\left(F\left(\frac{id_{\overline{\mathbb{R}}_+}}{a}\right), F\left(\frac{id_{\overline{\mathbb{R}}_+}}{b}\right)\right) = F\left(\frac{id_{\overline{\mathbb{R}}_+}}{a+b}\right),$$

where F is in Δ^+, a and b are strictly positive real numbers and $id_{\overline{\mathbb{R}}_+}$ is the identity function on $\overline{\mathbb{R}}_+$.

6. Functional inequalities: dominance

The concept of dominance plays a crucial role in the definition of products of probabilistic metric (and normed) spaces (see [16,38,39]). It was introduced for triangle functions in [39] but was soon extended to binary operations on a partially ordered set ([35]). We state the definition in this more general setting.

Definition 6.1. Let $(X, \leq)$ be a partially ordered set and let f and g be two binary operations on X. Then f dominates g, written $f \gg g$, if, for all x, y, u, and v in X,

$$f(g(x,y), g(u,v)) \geq g(f(x,u), f(y,v)). \tag{6.1}$$

When the operations share the same neutral element, dominance between them induces their order. Associativity and symmetry, resp. bisymmetry, of the operation ensure its self-dominance (see also [27,35,39]). Therefore, it immediately follows that dominance is an antisymmetric and reflexive relation on the set of triangle functions as well as on the set of all t-norms. The question of whether it is also transitive on the set of t-norms has been open for several years. Examples of the transitivity of dominance in special families lead to the conjecture that this would indeed be the case in general [8,13,33,37]).

However, it was proved by Sarkoci [34] that the dominance relation is not transitive on the class of continuous t-norms and therefore also not on the class of t-norms, in general. Sarkoci's counterexample was found among continuous ordinal sums of t-norms and was based on the violation of the necessary conditions for dominance between such t-norms (compare also [28,31]). The example of non-transitivity of dominance given in [8, Example 4.2.3] deals with functions that are associative, commutative, but not increasing.

Since triangle functions of the type Π_T are pointwise induced by some left-continuous t-norm, we can immediately state the following result.

Corollary 6.1. *The dominance relation is reflexive and antisymmetric, but not transitive on the set of triangle functions.*

It is remarkable that although dominance has been introduced for operations on partially ordered sets, it had been discussed mainly for triangle functions and t-norms. With the introduction of dominance for aggregation functions in [29], the discussion of dominance in a more general setting has been revived. In this section we prove several general results on dominance between operations on Δ^+. Since there is no full characterization of such operations but there are several classes known, we then turn to dominance between members of such classes and, in particular, dominance between triangle functions included in these classes in Sect. 7.

6.1. General results on dominance

Since triangle functions are increasing in each argument and associative as well as commutative the following results follow immediately.

Proposition 6.2. [39] *For an arbitrary triangle function τ, $\tau \gg \tau$ and $\Pi_M \gg \tau$.*

Moreover, Π_M dominates any operation on Δ^+ which is increasing in each argument.

Corollary 6.3. *The triangle function Π_M dominates all n-ary operations α on Δ^+ that are increasing in each place.*

Proof. Let α be an n-ary operation on Δ^+ which is increasing in each argument, let $F_1, \ldots, F_n, G_1, \ldots, G_n$ be in Δ^+ and choose $x \in \overline{\mathbb{R}}$; then

$$\alpha(F_1, \ldots, F_n)(x) \geq \alpha\left(\min\{F_1, G_1\}, \ldots, \min\{F_n, G_n\}\right)(x),$$
$$\alpha(G_1, \ldots, G_n)(x) \geq \alpha\left(\min\{F_1, G_1\}, \ldots, \min\{F_n, G_n\}\right)(x).$$

Therefore,

$$\begin{aligned} &\alpha\left(\min\{F_1, G_1\}, \ldots, \min\{F_n, G_n\}\right)(x) \\ &\quad \leq \min\left\{\alpha(F_1, \ldots, F_n)(x), \alpha(G_1, \ldots, G_n)(x)\right\} \\ &\quad = \Pi_M\left(\alpha(F_1, \ldots, F_n), \alpha(G_1, \ldots, G_n)\right)(x). \end{aligned}$$

□

Proposition 6.4. *Let α, α_i, α_j, α_k be binary operations on Δ^+ that are increasing in each place. If for some $i, j, k \in \mathbb{N}$, α_i, α_j, α_k dominate α, then the mapping $\beta_{i,j,k} \colon (\Delta^+)^4 \to \Delta^+$ defined, for all $x \in \overline{\mathbb{R}}$, by*

$$\beta_{i,j,k}(F_1, F_2, F_3, F_4)(x) := \alpha_i(\alpha_j(F_1, F_2), \alpha_k(F_3, F_4))(x)$$

is increasing in each place and dominates α.

Proof. Since α_i, α_j, and α_k are increasing in each place, so is also $\beta_{i,j,k}$. Let F_i and G_i $(i = 1, \ldots, 4)$ be in Δ^+. Then

$$\begin{aligned}
&\beta_{i,j,k}(\alpha(F_1, G_1), \alpha(F_2, G_2), \alpha(F_3, G_3), \alpha(F_4, G_4)) \\
&\quad = \alpha_i(\alpha_j(\alpha(F_1, G_1), \alpha(F_2, G_2)), \alpha_k(\alpha(F_3, G_3), \alpha(F_4, G_4))) \\
&\quad \geq \alpha_i(\alpha(\alpha_j(F_1, F_2), \alpha_j(G_1, G_2)), \alpha(\alpha_k(F_3, F_4), \alpha_k(G_3, G_4))) \\
&\quad \geq \alpha(\alpha_i(\alpha_j(F_1, F_2), \alpha_k(F_3, F_4)), \alpha_i(\alpha_j(G_1, G_2), \alpha_k(G_3, G_4))) \\
&\quad = \alpha(\beta_{i,j,k}(F_1, F_2, F_3, F_4), \beta_{i,j,k}(G_1, G_2, G_3, G_4)).
\end{aligned}$$

□

This rather general proposition allows us to derive several results on dominance for binary operations on Δ^+ by considering special α_i's or by restricting them to subclasses of $(\Delta^+)^4$.

Proposition 6.5. *Let α, α_i, α_j and α_k be binary operations on Δ^+ that are increasing in each place and define the following operations on Δ^+:*

$$\begin{aligned}
&\beta^{(2)}_{i,j,k} : (\Delta^+)^2 \to \Delta^+, && \beta^{(2)}_{i,j,k}(F_1, F_2) && := \alpha_i(\alpha_j(F_1, F_2), \alpha_k(F_1, F_2)); \\
&\beta^{(3,1)}_{i,k} : (\Delta^+)^3 \to \Delta^+, && \beta^{(3,1)}_{i,k}(F_1, F_3, F_4) && := \alpha_i(F_1, \alpha_k(F_3, F_4)); \\
&\beta^{(3,2)}_{i,j} : (\Delta^+)^3 \to \Delta^+, && \beta^{(3,2)}_{i,j}(F_1, F_2, F_3) && := \alpha_i(\alpha_j(F_1, F_2), F_3).
\end{aligned}$$

If α_i, α_j and α_k dominate α, then $\beta^{(2)}_{i,j,k}$ dominates α. If α_i and α_k, resp. α_j, dominates α, then $\beta^{(3,1)}_{i,k}$, resp. $\beta^{(3,2)}_{i,j}$, dominates α.

Proof. Let $\alpha, \alpha_i, \alpha_j, \alpha_k$ be binary operations on Δ^+ that are increasing in each place. Then, for all F_1 and F_2 in Δ^+, $\beta^{(2)}_{i,j,k}(F_1, F_2) = \beta_{i,j,k}(F_1, F_2, F_1, F_2)$ so that, because of Proposition 6.4, $\beta^{(2)}_{i,j,k}$ dominates α whenever α_i, α_j and α_k dominate α. Moreover, for all F_1, F_2, F_3 and F_4 in Δ^+,

$$\beta^{(3,1)}_{i,k}(F_1, F_3, F_4) = \alpha_i(\Pi_M(F_1, \varepsilon_0), \alpha_k(F_3, F_4));$$

$$\beta^{(3,2)}_{i,j}(F_1, F_2, F_3) = \alpha_i(\alpha_j(F_1, F_2), \Pi_M(F_3, \varepsilon_0)).$$

Since Π_M dominates all the operations on Δ^+ that are increasing in each place, it follows that $\beta^{(3,1)}_{i,k}$, resp. $\beta^{(3,2)}_{i,j}$, dominates α whenever α_i and α_k, resp. α_j, dominate α. □

It is immediate that, by recursion, the previous result can be extended to n-ary operations recursively defined by some binary operations α_i.

Corollary 6.6. *Let* α, $\alpha_1, \ldots, \alpha_n$ $(n \in \mathbb{N})$ *be some binary operations on* Δ^+ *that are increasing in each place. Then, for all* $F_1, \ldots, F_{n+1}$ *in* Δ^+*, the operations* $\beta^{(n+1)}, \beta_{(n+1)} \colon (\Delta^+)^{n+1} \to \Delta^+$ *defined by*

$$\beta^{(n+1)}(F_1, \ldots, F_{n+1}) := \alpha_n(\alpha_{n-1}(\alpha_{n-2}(\ldots, \ldots), F_n), F_{n+1});$$
$$\beta_{(n+1)}(F_1, \ldots, F_{n+1}) := \alpha_n(F_1, \alpha_{n-1}(F_2, \alpha_{n-2}(\ldots, \ldots)))$$

dominate α *whenever every* α_i $(i = 1, \ldots, n)$ *dominates* α.

6.2. Triangle inequalities in probabilistic metric spaces

In probabilistic metric (=PM) spaces a function $\mathcal{F}$ assigns to each pair of elements p and q in a non-empty set X a d.d.f.. Then, for $x > 0$, the value $\mathcal{F}(p, q)(x)$ is interpreted as the probability that the distance between p and q is less than x. The *triangle inequality* takes, for all $p, q, r \in X$, the following form

$$\mathcal{F}(p, r) \geq \tau(\mathcal{F}(p, q), \mathcal{F}(p, q))$$

where τ denotes a triangle function associated with the given PM space. Different PM spaces will be characterized by the choice of the triangle function τ. For more details and special classes of PM spaces, we recommend [35].

Several proposals for products of PM spaces may be found in the literature (see, e.g., [3,7,10,16,38,39]). We prove a more general result that covers the approaches mentioned before for the case of finite products.

Theorem 6.7. *Consider a (finite) family of PM spaces* $(X_i, \mathcal{F}_i, \tau_i)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, *let* α *be an* n*-ary operation on* Δ^+ *which is increasing in each place, and define* $\overrightarrow{\mathcal{F}}$ *on* $X := \prod_{i=1}^n X_i$*, for all* $\overrightarrow{p}$, $\overrightarrow{q}$, $\overrightarrow{r}$ *in* X*, by*

$$\overrightarrow{\mathcal{F}}(\overrightarrow{p}, \overrightarrow{q}) = \alpha(\mathcal{F}_1(p_1, q_1), \ldots, \mathcal{F}_n(p_n, q_n)). \tag{6.2}$$

If there exists a triangle function τ *such that* α *dominates* τ, $\tau \ll \alpha$, *and that* $\tau \leq \tau_i$ *for every* $i = 1, \ldots, n$, *then* $\overrightarrow{\mathcal{F}}$ *satisfies the triangle inequality on* X *with respect to* τ, *so that* $(X, \overrightarrow{\mathcal{F}}, \tau)$ *is a probabilistic metric space..*

Proof. Because α dominates τ and is increasing in each argument, we can conclude that

$$\begin{aligned}
&\tau(\overrightarrow{\mathcal{F}}(\overrightarrow{p}, \overrightarrow{q}), \overrightarrow{\mathcal{F}}(\overrightarrow{q}, \overrightarrow{r})) \\
&\quad = \tau(\alpha(\mathcal{F}_1(p_1, q_1), \ldots, \mathcal{F}_n(p_n, q_n)), \alpha(\mathcal{F}_1(q_1, r_1), \ldots, \mathcal{F}_n(q_n, r_n))) \\
&\quad \leq \alpha(\tau(\mathcal{F}_1(p_1, q_1), \mathcal{F}_1(q_1, r_1)), \ldots, \tau(\mathcal{F}_n(p_n, q_n), \mathcal{F}_n(q_n, r_n))) \\
&\quad \leq \alpha(\tau_1(\mathcal{F}_1(p_1, q_1), \mathcal{F}_1(q_1, r_1)), \ldots, \tau_n(\mathcal{F}_n(p_n, q_n), \mathcal{F}_n(q_n, r_n))) \\
&\quad \leq \alpha(\mathcal{F}_1(p_1, r_1), \ldots, \mathcal{F}_n(p_n, r_n)) = \overrightarrow{\mathcal{F}}(\overrightarrow{p}, \overrightarrow{r}),
\end{aligned}$$

which concludes the proof. □

Since every triangle function τ dominates itself, it follows that also every τ-power dominates τ, since in this case $\tau^n = \beta^{(n)}$ with $\alpha = \tau$ as described by Corollary 6.6. Therefore, any finite τ-*product* of probabilistic metric spaces $(X_i, \mathcal{F}_i, \tau_i)$ where $\tau_i \geq \tau$ is a probabilistic metric space under τ (compare also [3]).

In addition, since every weighted mean, and, hence, in particular, the arithmetic mean AM dominates W (see [20,21,29]), Π_{AM} dominates Π_W as well as $\tau_{W,L}$ for arbitrary $L \in \mathfrak{L}$ (see also Propositions 7.1 and 7.7 below). Further, because of Corollary 6.6, for every $n \in \mathbb{N}$, $\beta_{(n)}$ applied to Π_{AM} also dominates $\tau_{W,L}$. In this case one explicitly has for $\beta_{(n)}$ and for all $F_i \in \Delta^+$

$$\beta_{(n)}(F_1, F_2 \ldots, F_n) = \sum_{i=1}^{n} \frac{1}{2^i} F_i$$

such that every (finite) Σ-*product* of Menger spaces with respect to τ_W is again a Menger space with respect to τ_W (compare also [3]).

Similar considerations hold for probabilistic normed spaces (see [36]).

7. Functional inequalities: dominance in special classes

Some of the results we prove in this section are new while others extend those by Tardiff [38, Theorem 6.11] proved for the special case of $L = +$ only. We shall often use the following class of d.d.f.'s: for $a \in [0, 1]$, define $\gamma_a : \overline{\mathbb{R}} \to [0, 1]$ by

$$\gamma_a(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ a, & \text{if } 0 < x < \infty, \\ 1, & \text{if } x = \infty. \end{cases} \tag{7.1}$$

Each γ_a belongs to Δ^+. Notice that $\varepsilon_0 = \gamma_1$ and $\varepsilon_\infty = \gamma_0$. Moreover, the set $\gamma^+ := \{\gamma_a \mid a \in [0, 1]\}$ is linearly ordered, i.e., $\gamma_a \leq \gamma_b$ whenever $a \leq b$, so that $(\gamma^+, \leq, \gamma_0, \gamma_1)$ forms a sub-chain of the lattice $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$.

7.1. Pointwise induced operations

Let us first turn to pointwise induced operations Π_A on Δ^+, in particular to pointwise induced triangle functions. Recall that $\Pi_A(F, G)(x) = A(F(x), G(x))$ for all $x \in \overline{\mathbb{R}}$.

Theorem 7.1. *Let A_1 and A_2 be two left-continuous binary aggregation functions on $[0, 1]$. Then Π_{A_1} dominates Π_{A_2} if, and only if, A_1 dominates A_2.*

Proof. It is immediate that, whenever A_1 dominates A_2, also Π_{A_1} dominates Π_{A_2}. *Vice versa*, assume that Π_{A_1} dominates Π_{A_2}. Then we wish to show that, for all a, b, c and d in $[0,1]$,

$$A_1(A_2(a,b),A_2(c,d)) \geq A_2(A_1(a,c),A_1(b,d)).$$

Let a, b, c, and d be fixed arbitrarily in $[0,1]$; then, for every $x \in]0,+\infty[$,

$$\begin{aligned} A_1(A_2(a,b),A_2(c,d)) &= A_1(A_2(\gamma_a(x),\gamma_b(x)),A_2(\gamma_c(x),\gamma_d(x))) \\ &= A_1(\Pi_{A_2}(\gamma_a,\gamma_b)(x),\Pi_{A_2}(\gamma_c,\gamma_d)(x)) \\ &= \Pi_{A_1}(\Pi_{A_2}(\gamma_a,\gamma_b),\Pi_{A_2}(\gamma_c,\gamma_d))(x) \\ &\geq \Pi_{A_2}(\Pi_{A_1}(\gamma_a,\gamma_c),\Pi_{A_1}(\gamma_b,\gamma_d))(x) \\ &= A_2(A_1(a,c),A_1(b,d)). \end{aligned}$$

□

Corollary 7.2. *Let T_1 and T_2 be two left-continuous t-norms. Then Π_{T_1} dominates Π_{T_2} if, and only if, T_1 dominates T_2.*

7.2. Operations involving semicopulas

We now invoke mainly known sufficient conditions for operations involving semicopulas. For a detailed discussion on these, we refer to [32, Sect. 7].

Theorem 7.3. *Let S_1 and S_2 be left-continuous semicopulas and let $L_1, L_2 \in \mathfrak{L}$ satisfy condition* (LS). *Then, the following holds:*

(a) *If $S_1 \gg S_2$ and $L_1 \ll L_2$, then $\tau_{S_1,L_1} \gg \tau_{S_2,L_2}$.*
(b) *If L_1 and L_2 fulfil* (LB) *and $\tau_{S_1,L_1} \gg \tau_{S_2,L_2}$, then $S_1 \gg S_2$.*

Proof. Consider two left-continuous semicopulas S_1 and S_2, let L_1 and L_2 in $\mathfrak{L}$ satisfy (LS), and let x be in $\overline{\mathbb{R}}$. Assume first that S_1 dominates S_2 and that L_1 is dominated by L_2, $S_1 \gg S_2$ and $L_1 \ll L_2$.

Define

$$\begin{aligned} &A_{x,12} = \{(u_1,u_2,v_1,v_2) \mid L_1(L_2(u_1,u_2),L_2(v_1,v_2)) = x\}, \\ &A_{x,21} = \{(a_1,b_1,a_2,b_2) \mid L_2(L_1(a_1,b_1),L_1(a_2,b_2)) = x\}, \\ &F\colon \overline{\mathbb{R}}^4 \to [0,1], \quad F(a,b,c,d) := S_2(S_1(F_1(a),G_1(b)),S_1(F_2(c),G_2(d))). \end{aligned}$$

Since L_2 dominates L_1, it follows that, for every $(a_1,b_1,a_2,b_2) \in A_{x,21}$,

$$L_1(L_2(a_1,a_2),L_2(b_1,b_2)) \leq L_2(L_1(a_1,b_1),L_1(a_2,b_2)) = x.$$

If equality holds, then (a_1,a_2,b_1,b_2) belongs also to $A_{x,12}$. Otherwise, set

$$y := L_1(L_2(a_1,a_2),L_2(b_1,b_2)) < L_2(L_1(a_1,b_1),L_1(a_2,b_2)) = x.$$

Since L_1 and L_2 are increasing and continuous in each place (except, possibly, at $(0,\infty)$ and $(\infty,0)$), there exist a_1^*, a_2^*, b_1^*, and b_2^* with $a_1^* > a_1$, $a_2^* > a_2$, $b_1^* > b_1$, and $b_2^* > b_2$, such that

$$\begin{aligned} L_1(L_2(a_1^*,a_2),L_2(b_1,b_2)) &= L_1(L_2(a_1,a_2^*),L_2(b_1,b_2)) \\ &= L_1(L_2(a_1,a_2),L_2(b_1^*,b_2)) \\ &= L_1(L_2(a_1,a_2),L_2(b_1,b_2^*)) = x, \end{aligned}$$

thus all quadruples in the previous expression belong to $A_{x,12}$. Since S_1 and S_2 are increasing in each place, it follows that

$$\begin{aligned} F(a_1^*,b_1,a_2,b_2) &\geq F(a_1,b_1,a_2,b_2), \qquad & F(a_1,b_1^*,a_2,b_2) &\geq F(a_1,b_1,a_2,b_2), \\ F(a_1,b_1,a_2^*,b_2) &\geq F(a_1,b_1,a_2,b_2), \qquad & F(a_1,b_1,a_2,b_2^*) &\geq F(a_1,b_1,a_2,b_2), \end{aligned}$$

and, therefore, for all $(a_1,b_1,a_2,b_2) \in A_{x,21}$ there exists $(u_1,u_2,v_1,v_2) \in A_{x,12}$ such that $F(u_1,u_2,v_1,v_2) \geq F(a_1,b_1,a_2,b_2)$ and, even more, that

$$\sup_{A_{x,12}} F(u_1,v_1,u_2,v_2) \geq \sup_{A_{x,21}} F(a_1,b_1,a_2,b_2).$$

Then, because the semicopulas involved are left-continuous, and because $S_1 \gg S_2$ and $L_1 \ll L_2$, one has, for arbitrary F_1, F_2, G_1 and G_2 in Δ^+,

$$\begin{aligned} &\tau_{S_1,L_1}\left(\tau_{S_2,L_2}(F_1,F_2),\tau_{S_2,L_2}(G_1,G_2)\right)(x) \\ &= \sup_{L_1(u,v)=x} S_1\left(\sup_{L_2(u_1,u_2)=u} S_2(F_1(u_1),F_2(u_2)), \sup_{L_2(v_1,v_2)=v} S_2(G_1(v_1),G_2(v_2))\right) \\ &= \sup_{L_1(L_2(u_1,u_2),L_2(v_1,v_2))=x} S_1(S_2(F_1(u_1),F_2(u_2)),S_2(G_1(v_1),G_2(v_2))) \\ &\geq \sup_{L_1(L_2(u_1,u_2),L_2(v_1,v_2))=x} S_2(S_1(F_1(u_1),G_1(v_1)),S_1(F_2(u_2),G_2(v_2))) \\ &= \sup_{L_1(L_2(u_1,u_2),L_2(v_1,v_2))=x} F(u_1,v_1,u_2,v_2) \\ &\geq \sup_{L_2(L_1(a_1,b_1),L_1(a_2,b_2))=x} F(a_1,b_1,a_2,b_2) \\ &= \sup_{L_2(L_1(a_1,b_1),L_1(a_2,b_2))=x} S_2(S_1(F_1(a_1),G_1(b_1)),S_1(F_2(a_2),G_2(b_2))) \\ &= \sup_{L_2(a,b)=x} S_2\left(\sup_{L_1(a_1,b_1)=a} S_1(F_1(a_1),G_1(b_1)), \sup_{L_1(a_2,b_2)=b} S_1(F_2(a_2),G_2(b_2))\right) \\ &= \tau_{S_2,L_2}(\tau_{S_1,L_1}(F_1,G_1),\tau_{S_1,L_1}(F_2,G_2))(x) \end{aligned}$$

which concludes the first part of the proof.

Next assume that L_1 and L_2 fulfil (LB), i.e., $A_{x,i} = \{(u,v) \mid L_i(u,v) = x\}$ is bounded, for every $x \in [0,+\infty[$, so that $u < \infty$ and $v < \infty$, for all $(u,v) \in A_{x,i}$. As a consequence, for every $x \in [0,+\infty[$, for every left-continuous semicopula S_i and for arbitrary a and b in $[0,1]$,

$$\begin{aligned} \tau_{S_i,L_i}(\gamma_a,\gamma_b)(x) &= \sup_{A_{x,i}} S_i(\gamma_a(u),\gamma_b(v)) = \max\{0, S_i(a,b)\} = S_i(a,b) \\ &= \gamma_{S_i(a,b)}(x). \end{aligned}$$

Further, assume that τ_{S_1,L_1} dominates τ_{S_2,L_2}. Let a, b, c, and d be in $]0,1]$ and let x be in $[0,+\infty]$. Then there exist u_1, u_2, v_1, v_2 in $[0,+\infty]$ with $L_1(u_1,v_1) = x$ and $L_2(u_2,v_2) = x$ and

$$\begin{aligned}
\tau_{S_2,L_2}(\gamma_a,\gamma_b)(u_1) &= S_2(a,b) = \gamma_{S_2(a,b)}(u_1),\\
\tau_{S_2,L_2}(\gamma_c,\gamma_d)(v_1) &= S_2(c,d) = \gamma_{S_2(c,d)}(v_1),\\
\tau_{S_1,L_1}(\gamma_a,\gamma_c)(u_2) &= S_1(a,c) = \gamma_{S_1(a,c)}(u_2),\\
\tau_{S_1,L_1}(\gamma_b,\gamma_d)(v_2) &= S_1(b,d) = \gamma_{S_1(b,d)}(v_2).
\end{aligned}$$

One can then conclude

$$\begin{aligned}
S_1(S_2(a,b),S_2(c,d)) &= \sup_{L_1(u_1,v_1)=x} \{0, S_1(S_2(a,b),S_2(c,d))\}\\
&= \sup_{L_1(u_1,v_1)=x} S_1(\tau_{S_2,L_2}(\gamma_a,\gamma_b)(u_1), \tau_{S_2,L_2}(\gamma_c,\gamma_d)(v_1))\\
&= \tau_{S_1,L_1}(\tau_{S_2,L_2}(\gamma_a,\gamma_b), \tau_{S_2,L_2}(\gamma_c,\gamma_d))(x)\\
&\geq \tau_{S_2,L_2}(\tau_{S_1,L_1}(\gamma_a,\gamma_c), \tau_{S_1,L_1}(\gamma_b,\gamma_d))(x)\\
&= S_2(S_1(a,c),S_1(b,d)).
\end{aligned}$$

If 0 is in $\{a,b,c,d\}$, then

$$S_1(S_2(a,b),S_2(c,d)) = 0 = S_2(S_1(a,c),S_1(b,d)),$$

since 0 is the unique null element of each semicopula; thus dominance is proved. □

The next result is an immediate consequence of the fact that, for an operation $L \in \mathfrak{L}$, conditions (LS) and (L0) imply condition (LB).

Proposition 7.4. *Let T_1 and T_2 be two left-continuous t-norms and let $L_1, L_2 \in \mathfrak{L}$ be commutative, associative, and fulfil conditions* (LS) *and* (L0)*. Then the following holds:*

(a) *If $T_1 \gg T_2$ and $L_1 \ll L_2$, then $\tau_{T_1,L_1} \gg \tau_{T_2,L_2}$.*
(b) *If $\tau_{T_1,L_1} \gg \tau_{T_2,L_2}$, then $T_1 \gg T_2$.*

Notice that the commutativity and the associativity of L imply its bisymmetry, i.e., L dominates itself. Therefore, the following holds.

Corollary 7.5. *Let T_1 and T_2 be two left-continuous t-norms and let $L \in \mathfrak{L}$ be commutative, associative, and fulfil conditions* (LS) *and* (L0)*. Then T_1 dominates T_2 if, and only if, $\tau_{T_1,L}$ dominates $\tau_{T_2,L}$.*

Analogously one can conclude as follows.

Corollary 7.6. *Let T be a left-continuous t-norm and let $L_1, L_2 \in \mathfrak{L}$ be commutative, associative, and fulfil conditions* (LS) *and* (L0)*. If L_2 dominates L_1, then τ_{T,L_1} dominates τ_{T,L_2}.*

Proposition 7.7. *Let A_1 be a left-continuous aggregation function. Let S_2 be a left-continuous semicopula and $L \in \mathfrak{L}$ satisfy condition* (LS)*.*

(a) *If $A_1 \gg S_2$, then $\Pi_{A_1} \gg \tau_{S_2,L}$;*
(b) *if L fulfils* (LB) *and $\Pi_{A_1} \gg \tau_{S_2,L}$, then $A_1 \gg S_2$.*

Proof. (a) For arbitrary $x \in \overline{\mathbb{R}}$, and for all F_1, F_2, G_1 and G_2 in Δ^+, one has

$$\begin{aligned}
&\Pi_{A_1}(\tau_{S_2,L}(F_1, F_2), \tau_{S_2,L}(G_1, G_2))(x) \\
&\quad = A_1 \left(\sup_{L(u_1,u_2)=x} S_2(F_1(u_1), F_2(u_2)), \sup_{L(v_1,v_2)=x} S_2(G_1(v_1), G_2(v_2)) \right) \\
&\quad \geq \sup_{L(u,v)=x} A_1(S_2(F_1(u), F_2(v)), S_2(G_1(u), G_2(v))) \\
&\quad \geq \sup_{L(u,v)=x} S_2(A_1(F_1(u), G_1(u)), A_1(F_2(v), G_2(v))) \\
&\quad = \tau_{S_2,L} \left(\Pi_{A_1}(F_1, G_1), \Pi_{A_1}(F_2, G_2) \right)(x).
\end{aligned}$$

(b) Let L fulfil (LB), let Π_{A_1} dominate $\tau_{S_2,L}$, and let a, b, c and d be in $]0, 1]$. For every $x \in [0, +\infty]$, there exist u and v in $[0, +\infty]$ with $L(u, v) = x$; therefore

$$\begin{aligned}
A_1(S_2(a, b), S_2(c, d)) &= A_1 \left(\tau_{S_2,L}(\gamma_a, \gamma_b)(x), \tau_{S_2,L}(\gamma_c, \gamma_d)(x) \right) \\
&= \Pi_{A_1}(\tau_{S_2,L}(\gamma_a, \gamma_b), \tau_{S_2,L}(\gamma_c, \gamma_d)) \\
&\geq \tau_{S_2,L}(\Pi_{A_1}(\gamma_a, \gamma_c), \Pi_{A_1}(\gamma_b, \gamma_d))(x) \\
&= \sup_{L(u,v)=x} S_2(\Pi_{A_1}(\gamma_a, \gamma_c)(u), \Pi_{A_1}(\gamma_b, \gamma_d)(v)) \\
&= \max\{0, S_2(A_1(a, c), A_1(b, d))\} \\
&= S_2(A_1(a, c), A_1(b, d)).
\end{aligned}$$

If 0 belongs to $\{a, b, c, d\}$, then A_1 trivially dominates S_2. □

In particular, if A_1 and S_2 are left-continuous t-norms and, moreover, if L fulfils (L0), and therefore also (LB), then the following result holds.

Proposition 7.8. *Let T_1 and T_2 be two left-continuous t-norms and let $L \in \mathfrak{L}$ be commutative, associative, and fulfil the conditions* (LS) *and* (L0)*. Then T_1 dominates T_2 if, and only if, Π_{T_1} dominates $\tau_{T_2,L}$.*

7.3. Operations involving co-semicopulas, quasi-copulas, or copulas

Theorem 7.9. *Let S_1^* and S_2^* be two continuous co-semicopulas and let $L_1, L_2 \in \mathfrak{L}$ satisfy* (LS) *and* (L0)*. If S_1^* dominates S_2^*, $S_1^* \gg S_2^*$, and L_1 is dominated by L_2, $L_1 \ll L_2$, then $\tau^*_{S_1^*,L_1} \gg \tau^*_{S_2^*,L_2}$.*

Proof. For an arbitrary $x \in \overline{\mathbb{R}}$, and for arbitrary F_1, F_2, G_1 and G_2 in Δ^+ the following holds, because of the continuity of the co-semicopulas involved and the fact that $S_1^* \gg S_2^*$,

$$\begin{aligned}
&\tau^*_{S_1^*,L_1}\left(\tau^*_{S_2^*,L_2}(F_1,F_2),\tau^*_{S_2^*,L_2}(G_1,G_2)\right)(x)\\
&= \inf_{L_1(u,v)=x} S_1^*\left(\inf_{L_2(u_1,u_2)=u} S_2^*(F_1(u_1),F_2(u_2)), \inf_{L_2(v_1,v_2)=v} S_2^*(G_1(v_1),G_2(v_2))\right)\\
&= \inf_{L_1(L_2(u_1,u_2),L_2(v_1,v_2))=x} S_1^*\left(S_2^*(F_1(u_1),F_2(u_2)),S_2^*(G_1(v_1),G_2(v_2))\right)\\
&\geq \inf_{L_1(L_2(u_1,u_2),L_2(v_1,v_2))=x} S_2^*\left(S_1^*(F_1(u_1),G_1(v_1)),S_1^*(F_2(u_2),G_2(v_2))\right).
\end{aligned}$$

Define

$$\begin{aligned}
A_{x,12} &= \{(u_1,u_2,v_1,v_2) \mid L_1(L_2(u_1,u_2),L_2(v_1,v_2)) = x\};\\
A_{x,21} &= \{(a_1,b_1,a_2,b_2) \mid L_2(L_1(a_1,b_1),L_1(a_2,b_2)) = x\};\\
F\colon \overline{\mathbb{R}}^4 &\to [0,1]\,, F(a,b,c,d) := S_2^*(S_1^*(F_1(a),G_1(b)),S_1^*(F_2(c),G_2(d))).
\end{aligned}$$

Since L_1 is dominated by L_2, for every $(u_1,u_2,v_1,v_2) \in A_{x,12}$ it follows that

$$L_2(L_1(u_1,v_1),L_1(u_2,v_2)) \geq L_1(L_2(u_1,u_2),L_2(v_1,v_2)) = x.$$

If equality holds, then (u_1,v_1,u_2,v_2) belongs to $A_{x,21}$. Otherwise, set

$$y := L_2(L_1(u_1,v_1),L_1(u_2,v_2)) > x.$$

Since L_1 and L_2 are increasing and continuous in each place, there exist u_1^*, u_2^*, v_1^* and v_2^* with $u_1^* < u_1$, $u_2^* < u_2$, $v_1^* < v_1$, or $v_2^* < v_2$, such that

$$\begin{aligned}
L_2(L_1(u_1^*,v_1),L_1(u_2,v_2)) &= L_2(L_1(u_1,v_1^*),L_1(u_2,v_2))\\
&= L_2(L_1(u_1,v_1),L_1(u_2^*,v_2))\\
&= L_2(L_1(u_1,v_1),L_1(u_2,v_2^*)) = x,
\end{aligned}$$

i.e. all the quadruples involved are in $A_{x,21}$. Since S_1^* and S_2^* are increasing in each place, it follows that

$$\begin{aligned}
F(u_1^*,v_1,u_2,v_2) \leq F(u_1,v_1,u_2,v_2), \qquad F(u_1,v_1^*,u_2,v_2) \leq F(u_1,v_1,u_2,v_2),\\
F(u_1,v_1,u_2^*,v_2) \leq F(u_1,v_1,u_2,v_2), \qquad F(u_1,v_1,u_2,v_2^*) \leq F(u_1,v_1,u_2,v_2),
\end{aligned}$$

and, therefore, that

$$\inf_{L_1(L_2(u_1,u_2),L_2(v_1,v_2))=x} F(u_1,v_1,u_2,v_2) \geq \inf_{L_2(L_1(a_1,b_1),L_1(a_2,b_2))=x} F(a_1,b_1,a_2,b_2).$$

But S_1^* and S_2^* are continuous so that

$$\begin{aligned}
&\tau^*_{S_1^*,L_1}(\tau^*_{S_2^*,L_2}(F_1,F_2),\tau^*_{S_2^*,L_2}(G_1,G_2))(x)\\
&\quad\geq \inf_{L_1(L_2(u_1,u_2),L_2(v_1,v_2))=x} F(u_1,v_1,u_2,v_2)\\
&\quad\geq \inf_{L_2(L_1(a_1,b_1),L_1(a_2,b_2))=x} F(a_1,b_1,a_2,b_2)\\
&\quad= \inf_{L_2(L_1(a_1,b_1),L_1(a_2,b_2))=x} S_2^*\left(S_1^*(F_1(a_1),G_1(b_1)),S_1^*(F_2(a_2),G_2(b_2))\right)\\
&\quad= \inf_{L_2(a,b)=x} S_2^*\left(\inf_{L_1(a_1,b_1)=a} S_1^*(F_1(a_1),G_1(b_1)),\inf_{L_1(a_2,b_2)=b} S_1^*(F_2(a_2),G_2(b_2))\right)\\
&\quad= \tau^*_{S_2^*,L_2}\left(\tau^*_{S_1^*,L_1}(F_1,G_1),\tau^*_{S_1^*,L_1}(F_2,G_2)\right)(x),
\end{aligned}$$

which proves the assertion. □

The following corollaries describe particular instances for triangle functions involving t-conorms or, respectively, quasi-copulas.

Corollary 7.10. *Let T_1^* and T_2^* be continuous t-conorms and let $L_1, L_2 \in \mathfrak{L}$ be commutative, associative and satisfy* (LS) *and* (L0). *If T_1^* dominates T_2^*, $T_1^* \gg T_2^*$, and L_1 is dominated by L_2, $L_1 \ll L_2$, then $\tau^*_{T_1^*,L_1} \gg \tau^*_{T_2^*,L_2}$.*

Corollary 7.11. *Let Q_1 and Q_2 be two symmetric quasi-copulas such that $\overline{Q}_1$ and $\overline{Q}_2$ are associative and let $L_1, L_2 \in \mathfrak{L}$ be commutative, associative and satisfy* (LS) *and* (L0). *If $\overline{Q}_1 \gg \overline{Q}_2$ and $L_1 \ll L_2$, then $\rho_{Q_1,L_1} \gg \rho_{Q_2,L_2}$.*

Proof. It was proved in [32, Corollary 9.2] that under the present assumption, $\rho_{Q_i,L_i} = \tau^*_{\overline{Q_i},L_i}$, thus the result follows immediately. □

Notice that the condition $\overline{Q}_1 \gg \overline{Q}_2$ reads, for all x, y, u, v in $[0,1]$, as

$$\begin{aligned}
\overline{Q}_1(\overline{Q}_2(x,y),\overline{Q}_2(u,v)) &= \overline{Q}_2(x,y)+\overline{Q}_2(u,v)-Q_1(\overline{Q}_2(x,y),\overline{Q}_2(u,v))\\
&= x+y+u+v-Q_2(x,y)-Q_2(u,v)\\
&\quad -Q_1(x+y-Q_2(x,y),u+v-Q_2(u,v))\\
&\geq x+y+u+v-Q_1(x,u)-Q_1(y,v)\\
&\quad -Q_2(x+u-Q_1(x,u),y+v-Q_1(y,v)),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&Q_2(x,y)+Q_2(u,v)-Q_2(x+u-Q_1(x,u),y+v-Q_1(y,v))\\
&\quad\leq Q_1(x,u)+Q_1(y,v)-Q_1(x+y-Q_2(x,y),u+v-Q_2(u,v)).
\end{aligned}$$

Since commutativity and associativity of an operation imply its bisymmetry and, therefore, its self-dominance, the following corollaries follow.

Corollary 7.12. *Let T_1^* and T_2^* be continuous t-conorms and let $L \in \mathfrak{L}$ be commutative, associative and satisfy* (LS) *and* (L0). *If T_1^* dominates T_2^*, $T_1^* \gg T_2^*$, then $\tau^*_{T_1^*,L} \gg \tau^*_{T_2^*,L}$.*

Corollary 7.13. *Let T^* be a continuous t-conorm and let $L_1, L_2 \in \mathfrak{L}$ be commutative, associative and satisfy* (LS) *and* (L0)*. If L_1 is dominated by L_2, $L_1 \ll L_2$, $L_1 \ll L_2$, then $\tau^*_{T^*,L_1} \gg \tau^*_{T^*,L_2}$.*

Corollary 7.14. *Let C_1 and C_2 be two associative copulas (and, thus, also two continuous t-norms). Then $\sigma_{C_1,\max}$ dominates $\sigma_{C_2,\max}$ if, and only if, C_1 dominates C_2.*

Proof. Since, for an associative copula C, the triangle function $\sigma_{C,\max}$ coincides with Π_C (see [32, Lemma 10.4]), the result follows. □

8. Functional inequalities: Jensen convexity

Convexity is an important property for every family of mappings; in the context of triangle functions, one has the following definition.

Definition 8.1. A triangle function τ is said to be *Jensen-convex* if, for all F, G, H, K in Δ^+, one has

$$\tau\left(\frac{F+G}{2}, \frac{H+K}{2}\right) \leq \frac{1}{2}\,\tau(F, H) + \frac{1}{2}\tau(G, K). \tag{8.1}$$

The following lemma is an immediate consequence of the fact that the arithmetic mean dominates W.

Lemma 8.1. *The triangle function Π_W defined by*

$$\Pi_W(F, G)(x) := \max\{F(x) + G(x) - 1, 0\} \qquad (F, G \in \Delta^+)$$

is Jensen-convex.

In [5] inequality (8.1) is studied under slightly different assumptions. Therefore, we give here the details for d.d.f.'s and triangle functions.

Lemma 8.2. *For a continuous triangle function τ the following statements are equivalent:*

(a) *τ is Jensen-convex;*
(b) *for all sequences of d.d.f.'s (F_n) and (G_n), τ satisfies the following inequality*

$$\tau\left(\sum_{j\in\mathbb{N}} \frac{F_j}{2^j}, \sum_{j\in\mathbb{N}} \frac{G_j}{2^j}\right) \leq \sum_{j\in\mathbb{N}} \frac{1}{2^j}\,\tau(F_j, G_j). \tag{8.2}$$

Proof. (b) $\Longrightarrow$ (a) Given F, G, H, K in Δ^+, choose the sequences (F_n) and (G_n) in the following way: $F_1 = F$, $G_1 = H$, while, for $n \geq 2$, $F_n = G$ and

$G_n = K$. Thus

$$\tau\left(\frac{F+G}{2},\frac{H+K}{2}\right) = \tau\left(\frac{F}{2}+G\sum_{n\geq 2}\frac{1}{2^n},\frac{H}{2}+K\sum_{n\geq 2}\frac{1}{2^n}\right)$$
$$\leq \frac{1}{2}\,\tau(F,H)+\tau(G,K)\sum_{n\geq 2}\frac{1}{2^n} = \frac{1}{2}\,\tau(F,H)$$
$$+\frac{1}{2}\,\tau(G,K).$$

(a) $\Longrightarrow$ (b) We prove by induction that, for every $n \geq 2$, the following inequality holds

$$\tau\left(\sum_{j=1}^{n}\frac{F_j}{2^j}+\frac{\varepsilon_\infty}{2^n},\sum_{j=1}^{n}\frac{G_j}{2^j}+\frac{\varepsilon_\infty}{2^n}\right) \leq \sum_{j=1}^{n}\frac{\tau(F_j,G_j)}{2^j}+\frac{\varepsilon_\infty}{2^n}. \tag{8.3}$$

For $n = 2$, (8.1) yields

$$\begin{aligned}
&\tau\left(\frac{F_1}{2}+\frac{F_2}{4}+\frac{\varepsilon_\infty}{4},\frac{G_1}{2}+\frac{G_2}{4}+\frac{\varepsilon_\infty}{4}\right)\\
&= \tau\left(\frac{F_1}{2}+\frac{1}{2}\,\frac{F_2+\varepsilon_\infty}{2},\frac{G_1}{2}+\frac{1}{2}\,\frac{G_2+\varepsilon_\infty}{2}\right)\\
&\leq \frac{1}{2}\,\tau(F_1,G_1)+\frac{1}{2}\,\tau\left(\frac{F_2+\varepsilon_\infty}{2},\frac{G_2+\varepsilon_\infty}{2}\right)\\
&\leq \frac{1}{2}\,\tau(F_1,G_1)+\frac{1}{4}\,\tau(F_2,G_2)+\frac{1}{4}\,\tau(\varepsilon_\infty,\varepsilon_\infty)\\
&= \frac{1}{2}\,\tau(F_1,G_1)+\frac{1}{4}\,\tau(F_2,G_2)+\frac{1}{4}\,\varepsilon_\infty,
\end{aligned}$$

namely (8.3) for $n = 2$. Assume now that (8.3) holds for a natural number $n \geq 2$. Then

$$\begin{aligned}
&\tau\left(\sum_{j=1}^{n+1}\frac{F_j}{2^j}+\frac{\varepsilon_\infty}{2^{n+1}},\sum_{j=1}^{n+1}\frac{F_j}{2^j}+\frac{\varepsilon_\infty}{2^{n+1}}\right)\\
&= \tau\left[\frac{F_1}{2}+\frac{1}{2}\left(\sum_{j=2}^{n+1}\frac{F_j}{2^{j-1}}+\frac{\varepsilon_\infty}{2^n}\right),\frac{G_1}{2}+\frac{1}{2}\left(\sum_{j=2}^{n+1}\frac{G_j}{2^{j-1}}+\frac{\varepsilon_\infty}{2^n}\right)\right]\\
&\leq \frac{\tau(F_1,G_1)}{2}+\frac{1}{2}\,\tau\left(\sum_{j=2}^{n+1}\frac{F_j}{2^{j-1}}+\frac{\varepsilon_\infty}{2^n},\sum_{j=2}^{n+1}\frac{G_j}{2^{j-1}}+\frac{\varepsilon_\infty}{2^n}\right)
\end{aligned}$$

$$\leq \frac{\tau(F_1, G_1)}{2} + \frac{1}{2}\left(\sum_{j=2}^{n+1} \frac{\tau(F_j, G_j)}{2^{j-1}} + \frac{\varepsilon_\infty}{2^n}\right)$$
$$= \sum_{j=1}^{n+1} \frac{\tau(F_j, G_j)}{2^j} + \frac{\varepsilon_\infty}{2^{n+1}},$$

which proves (8.3) in general. The continuity of τ now yields the assertion. □

As in [5] one can now prove that Π_W is the strongest solution of inequality (8.1).

Theorem 8.3. [5, Theorem 1] *For every continuous Jensen-convex triangle function τ, one has $\tau \leq \Pi_W$.*

9. Open questions

As in the first part of this primer ([32]) we end by listing a few open questions.

1. Cauchy's equation (5.3) has been studied only when the triangle function τ is of the form $\tau = \tau_T$ or $\tau = \tau_{T,L}$, with restriction on both the t-norm T and on the function L. What are the solutions of (5.3) when τ belongs to a different family of triangle functions?
2. Cauchy's functional equation is the most fundamental, and, therefore, the most natural functional equation to study, see Chaps. 2–4 in [2]; but, of course, there are many other analogues of important functional equations to be considered.
3. Are there results similar to those of Sect. 8 when Jensen-convexity is replaced by convexity? In other words it is interesting to study the inequality
$$\tau\left(\alpha\, F + (1-\alpha)\, G, \alpha\, H + (1-\alpha)\, K\right) \leq \alpha\, \tau(F, H) + (1-\alpha)\, \tau(G, K)$$
for $\alpha \in]0, 1[$.
4. The question naturally comes to one's mind whether the converse of Theorem 7.9 holds, namely, whether S_1^* dominates S_2^* whenever $\tau^*_{S_1^*, L_1}$ dominates $\tau^*_{S_2^*, L_2}$.
5. The commutativity of addition naturally leads to a different form of inequality (8.1), namely
$$\tau\left(\frac{F+G}{2}, \frac{H+K}{2}\right) \leq \frac{1}{2}\, \min\left\{\tau(F, H) + \tau(G, K), \tau(F, K) + \tau(G, H)\right\}.$$
This seems to be worth investigating.
6. In recent years great attention has been devoted to the study of Schur-convexity. A similar study ought to be started for triangle functions.

10. Conclusion

By extending and streamlining proofs, by making reference to developments intervened after the original results were presented, by collecting the known results, by unifying the notation, the authors have aimed at an up-to-date presentation of triangle functions which are the roots for so many important functions such as, e.g., copulas and triangular norms. It is remarkable that copulas and t-norms somehow started to live a life of their own and have been treated extensively in different contexts. And the authors have been curious to see the impact of the actual results to the roots, triangle functions. And indeed, several new perspectives and results could be found. Moreover, the unified notation has helped to shed a clearer light on some aspects of triangle functions so that additional new results could be achieved. In particular this second part profits from these adopted viewpoints. As an example we simply refer to the large amount of new results related to dominance which is important when constructing Cartesian products of probabilistic metric and normed spaces. The authors, therefore, believe that by this primer, consisting of two parts with different foci, they have provided the background for future investigations on triangle functions and a handy reference for interested readers willing to start research on triangle functions. The list of open questions in both parts indicate that (new) interesting questions on triangle functions are still open.

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