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Citation style: Dawidowski Łukasz. (2017). The quasilinear parabolic Kirchhoff equation. "Open Mathematics" (Vol. 15, iss. 1, (2017), s. 382-392), doi 10.1515/math-2017-0036

Open Math. 2017; 15: 382–392 **DE GRUYTER** OPEN

Open Mathematics **Open Access**

Research Article

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The quasilinear parabolic Kirchhoff equation

DOI 10.1515/math-2017-0036
Received May 11, 2015; accepted November 3, 2015.

Abstract: In this paper the existence of solution of a quasilinear generalized Kirchhoff equation with initial – boundary conditions of Dirichlet type will be studied using the Leray – Schauder principle.

Keywords: Kirchhoff equation, Quasi–linear parabolic equation

MSC: 35K55, 35K15

1 Introduction

G. Kirchhoff in [1] proposed the hyperbolic integro-differential equation in order to describe small, transversal vibrations of an elastic string of length l (at rest) when the longitudinal motion can be considered negligible with respect to the transversal one.

In their papers M. Gobbino [2] and M. Nakao [3] considered some generalized degenerate Kirchhoff equations. M. Gobbino studied the equation:

$$u_t - (1 + \|\nabla u\|^2_{L^2(\Omega)})\Delta u = 0,$$

but his method does not use fixed point theorems and can not be applied to the problem considered in this article. In another papers M. Ghisi and M. Gobbino [4, 5] showed certain connections between the above equation and equation of hyperbolic type containing term u_{tt}. However M. Nakao proved the existence of solutions of the equation of hyperbolic type. We will investigate a quasilinear parabolic generalization of the Kirchhoff equation.

The proof of the existence of solution of problem considered in this paper, which is indeed quasilinear (i.e. the derivative of solution is a part of coefficient of the main part), can not be carried out using most classical methods. This paper is devoted to this proof.

Consider the Dirichlet problem for quasilinear generalized degenerate Kirchhoff equation

$$u_t - (1 + \|\nabla u\|^2_{L^2(\Omega)})\Delta u + g(u, x) = 0 \tag{1}$$

with initial condition

$$u(0, x) = u_0(x), x \in \Omega, \tag{2}$$

and boundary condition of the Dirichlet type

$$u|_{\partial\Omega} = 0. \tag{3}$$

We will assume that $u_0 \in H^2(\Omega)$ and $\Omega \subseteq \mathbb{R}^N$ is a domain of the class C^2.

The following conditions will be imposed on the nonlinear function $g: \mathbb{R} \times \Omega \to \mathbb{R}$ throughout the paper:

(A1) There exists a function $d: \Omega \to \mathbb{R}$, such that $\int_\Omega d(x)\, dx = d < \infty$ and a constant $c > 0$, that

$$-g(u, x)u \le cu^2 + d(x).$$

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(A2) There exists a constant $\bar{c} > 0$, such that

$$|g(u,x)| \le \bar{c}(1 + |u|^q),$$

with certain exponent $q \le \frac{N+2}{N}$.

(A3) There exist constants $c_1, c_2 > 0$ and exponents $s_1 \in (0, \frac{4}{N-2})$, $s_2 \in (0, \frac{N+4}{N-2})$ such that:

$$\left|\frac{\partial g}{\partial u}\right| \le c_1(1 + |u|^{s_1}) \qquad \text{and} \qquad \left|\frac{\partial g}{\partial x_i}\right| \le c_2(1 + |u|^{s_2}).$$

In dimension $N = 2$ we assume only that $s_1, s_2 > 0$.

(A4) The function g is locally Lipschitz continuous with respect to the first variable, i.e. there exist constants $L > 0$, $q_1, q_2 \in (0, \frac{N}{N-4})$ (or if $N \le 4$, then $q_1, q_2 \in \mathbb{R}$), such that

$$|g(u_1, x) - g(u_2, x)| \le L|u_1 - u_2|(1 + |u_1|^{q_1} + |u_2|^{q_2}).$$

(A5) $g(0, x) = 0$ for all $x \in \Omega$.

Remark 1.1. *Instead of assuming (A2), (A4) and the first part of (A3) (i.e. there exist constant $c_1 > 0$ and $s_1 \in (0, \frac{4}{N-2})$ such that $\frac{\partial g}{\partial u} \le c_1(1 + |u|^{s_1})$) we can assume that:*
There exist constant $L_1 > 0$ and exponents $r_1, r_2 \in (0, \frac{2}{N})$ such that

$$|g(u_1, x) - g(u_2, x)| \le L_1|u_1 - u_2|(1 + |u_1|^{r_1} + |u_2|^{r_2}). \tag{4}$$

Putting $u_2 = 0$ to (4) *and using (A5) we obtain:*

$$|g(u_1, x)| = |g(u_1, x) - g(0, x)| \le L_1|u_1|(1 + |u_1|^{r_1}).$$

When we note that index $r_1 + 1$ is no greater than q we observe that assumption (A3) holds. Similarly, as a consequence of (4)

$$\frac{|g(u_1, x) - g(u_2, x)|}{|u_1 - u_2|} \le L_2(1 + |u_1|^{r_1} + |u_2|^{r_2}).$$

Taking the limit with $u_2 \to u_1$ we obtain that $\left|\frac{\partial g}{\partial u}\right| \le L_2(1 + |u_1|^{r_1} + |u_1|^{r_2})$ and when we put $s_1 := \max(r_1, r_2)$, then (A4) holds.

Constructing a solution of (1) the Leray – Schauder Principle will be used (see e.g. [6], p. 189). We recall it here for completeness of the presentation:

Proposition 1.2 (Leray – Schauder Principle)**.** *Consider a transformation $y = T(x, k)$ where x, y belong to a Banach space X and k is a real parameter which varies in a bounded interval, say $a \le k \le b$. Assume that*

(a) *$T(x, k)$ is defined for all $x \in X$ and $a \le k \le b$,*
(b) *for any fixed k, $T(x, k)$ is continuous as a function of x, i.e. for any $x_0 \in X$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|T(x, k) - T(x_0, k)\| < \varepsilon$ if $\|x - x_0\| < \delta$,*
(c) *for x varying in bounded set in X, $T(x, k)$ is uniformly continuous in k, i.e. for any bounded set $X_0 \subseteq X$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in X_0$, $|k_1 - k_2| < \delta$, $a \le k_1, k_2 \le b$, then $\|T(x, k_1) - T(x, k_2)\| < \varepsilon$,*
(d) *for any fixed k, $T(x, k)$ is a compact transformation, i.e. it maps bounded subsets of X into compact subsets of X,*
(e) *there exists a (finite) constant M such that every possible solution X of $x - T(x, k) = 0$ ($x \in X$, $a \le k \le b$) satisfies: $\|x\| < M$,*
(f) *the equation $x - T(x, a) = y$ has the unique solution for any $y \in X$.*

Then there exists a solution of the equation $x - T(x, b) = 0$.

Assumption (f) means that Leray-Schauder degree

$$deg_{LS}(I - T(\cdot, a); B(0, M); 0) \neq 0$$

with the constant M which comes from assumption (e). The more standard version of Leray-Schauder Principle, called also Leray-Schauder continuation theorem, can be found e.g. in [7], p. 351 (Theorem 13.3.7).

2 Main theorem

Let us fix arbitrary $T > 0$.

We introduce an operator $F : \mathcal{L}^\infty([0, T], H_0^1(\Omega)) \times [0, 1] \to \mathcal{L}^\infty([0, T], H_0^1(\Omega))$ in such a way, that for every function $v \in \mathcal{L}^\infty([0, T], H_0^1(\Omega))$ and $\alpha \in [0, 1]$, $u = F(v, \alpha)$ is a solution of the equation

$$u_t - (1 + \alpha\|\nabla v\|^2_{L^2(\Omega)})\Delta u + g(u, x) = 0, \tag{5}$$

with initial – boundary conditions:

$$u(0, x) = u_0(x), x \in \Omega,$$
$$u|_{\partial\Omega} = 0.$$

We will search a fixed point of the operator $F(\cdot, 1)$ in $\mathcal{L}^\infty([0, T], H_0^1(\Omega))$.

The existence of the solution of the problem (1) is equivalent to the existence of the fixed point of operator $F(\cdot, 1)$ in $\mathcal{L}^\infty([0, T], H_0^1(\Omega))$.

We have:

Theorem 2.1. *Under the assumptions (A1), (A2), (A3), (A4) and (A5) there exists a solution of the problem (1) with initial-boundary conditions (2), (3) in the space* $\mathcal{L}^\infty([0, T], H_0^1(\Omega))$.

It can be seen that, using standard theory (see e.g. [8], chapter 3 for details), for $\alpha = 0$ the equation $u - F(u, 0) = y$ has a unique solution for any $y \in \mathcal{L}^\infty([0, T], H_0^1(\Omega))$; equivalently, the semilinear heat equation

$$u_t - \Delta u + g(u, x) = 0$$

with Dirichlet boundary condition has a unique solution.

The proof of the theorem will be given in a few steps. We start with obtaining certain *a priori* estimates.

3 Some lemmas

First it can be mentioned that when $u_0 \in H^2(\Omega)$ then, using the method of Tanabe and Sobolevski (see [9], page 438), the solution of the problem (5) varies in the space $H^2(\Omega)$.

Lemma 3.1. *There exists* $A \in \mathbb{R}$ *that for all* $t \in (0, T)$ *this estimate holds:*

$$\int_\Omega u^2\, dx \leq e^{2At}\left(\int_\Omega u_0^2\, dx + \frac{d}{A}\right) - \frac{d}{A}.$$

Proof. Multiplying the equation (5) by u and integrating over Ω we obtain:

$$\int_\Omega u \cdot u_t\, dx - (1 + \alpha\|\nabla v\|^2_{L^2(\Omega)}) \int_\Omega \Delta u \cdot u\, dx + \int_\Omega g(u, x)u\, dx = 0.$$

Then from $(A1)$:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx \le (1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_{\Omega}\Delta u\cdot u + \int_{\Omega}(cu^2+d(x))\,dx.$$

Integrating first right hand side component by parts we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx \le -(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_{\Omega}|\nabla u|^2\,dx + c\int_{\Omega}u^2\,dx + d.$$

Using the Poincaré inequality $\int_{\Omega}|u|^2\,dx \le p\int_{\Omega}|\nabla u|^2\,dx$, we have next:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx \le -\frac{1}{p}(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_{\Omega}|u|^2\,dx + c\int_{\Omega}u^2\,dx + d,$$

so that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx \le \left[c-\frac{1}{p}(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\right]\int_{\Omega}|u|^2\,dx + d.$$

Choosing $A = c - \frac{1}{p}$ the estimates holds

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx \le A\int_{\Omega}|u|^2\,dx + d.$$

Finally using Gronwall inequality (see [10], p. 35):

$$\int_{\Omega}u^2\,dx \le e^{2At}\left(\int_{\Omega}u_0^2\,dx + \frac{d}{A}\right) - \frac{d}{A},$$

for $t \in (0,T)$. □

Lemma 3.2. *There exist constants $B, D \in \mathbb{R}$ such that:*

$$\int_{\Omega}|\nabla u|^2\,dx \le e^{Bt}\left(\int_{\Omega}|\nabla u_0|^2\,dx - \frac{D}{B}\right) + \frac{D}{B}.$$

Proof. Multiplying equation (5) by Δu and integrating over Ω:

$$\int_{\Omega}\Delta u\cdot u_t\,dx - (1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_{\Omega}\Delta u\cdot\Delta u\,dx + \int_{\Omega}g(u,x)\cdot\Delta u\,dx.$$

Integrating by parts:

$$\int_{\Omega}\Delta u\cdot u_t\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla u|^2\,dx.$$

Then using Cauchy inequality with $\varepsilon = \frac{1}{2}$ and $(A2)$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla u|^2\,dx &= -(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_{\Omega}\Delta u\cdot\Delta u\,dx + \int_{\Omega}g(u,x)\cdot\Delta u\,dx \le \\ &\le -(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_{\Omega}\Delta u\cdot\Delta u\,dx + \frac{1}{2}\int_{\Omega}(g(u,x))^2\,dx + \frac{1}{2}\int_{\Omega}(\Delta u)^2\,dx \le \\ &\le -\left(\frac{1}{2}+\alpha\|\nabla v\|^2_{L^2(\Omega)}\right)\int_{\Omega}\Delta u\cdot\Delta u\,dx + \frac{1}{2}\bar{c}^2\int_{\Omega}(1+|u|^q)^2\,dx.\end{aligned}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx \le -\left(\frac{1}{2}+\alpha\|\nabla v\|^2_{L^2(\Omega)}\right)\int_\Omega \Delta u\cdot\Delta u\,dx + \frac{1}{2}\bar{c}^2\int_\Omega (1+|u|^q)^2\,dx \le$$

$$\le -\left(\frac{1}{2}+\alpha\|\nabla v\|^2_{L^2(\Omega)}\right)\int_\Omega |\Delta u|^2\;dx + \bar{c}^2\int_\Omega (1+|u|^{2q})\,dx =$$

$$= -\left(\frac{1}{2}+\alpha\|\nabla v\|^2_{L^2(\Omega)}\right)\int_\Omega |\Delta u|^2\;dx + \bar{c}^2|\Omega| + \bar{c}^2\int_\Omega |u|^{2q}\,dx.$$

Since $q < \frac{N+2}{N}$ when $\theta = \frac{N}{4}\left(1-\frac{1}{q}\right)$, the following estimate holds:

$$\|\phi\|^{2q}_{L^{2q}(\Omega)} \le c_2\|\phi\|^{2q(1-\theta)}_{L^2(\Omega)}\cdot\|\phi\|^{2q\theta}_{H^2(\Omega)} \quad \text{and} \quad 2q\theta < 1.$$

Consequently, our resulting estimate has the form:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx \le -\left(\frac{1}{2}+\alpha\|\nabla v\|^2_{L^2(\Omega)}\right)\int_\Omega (\Delta u)^2\,dx + \bar{c}^2|\Omega| + \bar{c}^2c_2\|u\|^{2q(1-\theta)}_{L^2(\Omega)}\cdot\|u\|^{2q\theta}_{H^2(\Omega)}$$

Because the norms $\|\cdot\|_{H^2(\Omega)}$ and $\int_\Omega \cdot^2\,dx + \int_\Omega(\Delta\cdot)^2\,dx$ are equivalent on the domain of $(-\Delta)$ (for more details see e.g. [11]):

$$\left(\frac{1}{2}+\alpha\|\nabla v\|^2_{L^2(\Omega)}\right)\int_\Omega (\Delta\phi)^2\,dx \ge h_1\|\phi\|^2_{H^2(\Omega)} - h_2\int_\Omega \phi^2\,dx,$$

for some constants $h_1, h_2 > 0$. Thus

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx \le -\left[h_1\|u\|^2_{H^2(\Omega)} - h_2\int_\Omega u^2\,dx\right]\bar{c}^2|\Omega| + \bar{c}^2c_2\|u\|^{2q(1-\theta)}_{L^2(\Omega)}\cdot\|u\|^{2q\theta}_{H^2(\Omega)} =$$

$$= -h_1\|u\|^2_{H^2(\Omega)} + \bar{c}^2c_2\|u\|^{2q(1-\theta)}_{L^2(\Omega)}\cdot\|u\|^{2q\theta}_{H^2(\Omega)} + \bar{c}^2|\Omega| + h_2\int_\Omega u^2\,dx.$$

Due to lemma 3.1, $\sup_{t\in[0,T]}\int_\Omega u^2\,dx \le e = e(c,d,T)$, so that the estimate holds

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx \le -h_1\|u\|^2_{H^2(\Omega)} + \bar{c}^2c_2\|u\|^{2q(1-\theta)}_{L^2(\Omega)}\cdot\|u\|^{2q\theta}_{H^2(\Omega)} + \bar{c}^2|\Omega| + h_2e.$$

Using the Young inequality with ϵ_1

$$\bar{c}^2c_2\|u\|^{2q(1-\theta)}_{L^2(\Omega)}\cdot\|u\|^{2q\theta}_{H^2(\Omega)} \le \epsilon_1h_1\|u\|^2_{H^2(\Omega)} + a\|u\|^P_{L^2(\Omega)},$$

with a positive constant $a = a(\bar{c}, c_2, \epsilon_1)$, and the exponent $P = \frac{1}{1-q\theta}$ is chosen in such a way that Young inequality holds. Then:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx \le -(\epsilon_1+1)\,h_1\|u\|^2_{H^2(\Omega)} + a\|u\|^P_{L^2(\Omega)} + \bar{c}^2|\Omega| + h_2e.$$

Since there exists a constant $\check{c} > 0$ such that $\|u\|^2_{H^2(\Omega)} \ge \check{c}\int_\Omega |\nabla u|^2\,dx$ and $\|u\|^2_{L^2(\Omega)} \le b < \infty$ then:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx \le -(\epsilon_1+1)\,h_1\check{c}\int_\Omega |\nabla u|^2\,dx + ab^P + \bar{c}^2|\Omega| + h_2e.$$

Therefore

$$\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + 2(\epsilon_1+1)\,h_1\check{c}\int_\Omega |\nabla u|^2\,dx \le 2ab^P + 2\bar{c}^2|\Omega| + 2h_2e$$

and the right side is a constant. Using Gronwall inequality (see [10], p. 35), denoting $B = 2(\epsilon_1 + 1)\, h_1 \check{c}$ and $D = 2ab^P + 2\bar{c}^2|\Omega| + 2h_2 e$, we obtain:

$$\int_\Omega |\nabla u|^2\, dx \le e^{Bt} \left(\int_\Omega |\nabla u_0|^2\, dx - \frac{D}{B} \right) + \frac{D}{B}.$$

□

Remark 3.3. *The two previous lemmas provide us an* a priori *estimate of the solution u of (5) in the space $\mathcal{L}^\infty([0,T], H^1(\Omega))$.*

Let us take a constant $M > 0$ such that:

$$\|u\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))} = \sup_{t\in[0,T]} \|u(t)\|_{H^1(\Omega)} < M$$

The lemmas show also that, if u is a fixed point of the operator F, its norm $\|u\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))}$ will be bounded by M, since the constants in both lemmas are independent of u and α.

Finally a third *a priori* estimation in $H^2(\Omega)$ will be shown:

Lemma 3.4. *There exists a constant $M_1 > 0$ such that*

$$\|\Delta u\|_{L^2(\Omega)} \le M_1 < \infty,$$

Proof. By applying the Laplace operator Δ to (5), multiplying the result by Δu and integrating over Ω we obtain:

$$\int_\Omega \Delta u \cdot \Delta u_t\, dx - (1 + \alpha\|\nabla v\|^2_{L^2(\Omega)}) \int_\Omega \Delta u \cdot \Delta^2 u\, dx + \int_\Omega g(u,x)\Delta^2 u\, dx = 0.$$

Integrating by parts and using (A5):

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2\, dx + (1 + \alpha\|\nabla v\|^2_{L^2(\Omega)}) \int_\Omega |\nabla \Delta u|^2\, dx = \int_\Omega \nabla g(u,x)\,(\nabla \Delta u)\, dx.$$

Then thanks to the Cauchy inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2\, dx + \left(\frac{1}{2} + \alpha\|\nabla v\|^2_{L^2(\Omega)}\right) \int_\Omega |\nabla \Delta u|^2\, dx \le \frac{1}{2}\int_\Omega |\nabla g(u,x)|^2\, dx. \tag{6}$$

Now, using assumption (A3), we will estimate last integral:

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla g(u,x)|^2\, dx &\le \int_\Omega \sum_{i=1}^N \left(\frac{\partial g}{\partial u}\right)^2 \left(\frac{\partial u}{\partial x_i}\right)^2 dx + \int_\Omega \sum_{i=1}^N \left(\frac{\partial g}{\partial x_i}\right)^2 dx \le \\
&\le \sum_{i=1}^N \sqrt{\int_\Omega \left(\frac{\partial g}{\partial u}\right)^4 dx} \cdot \sqrt{\int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^4 dx} + \sum_{i=1}^N \int_\Omega \left(\frac{\partial g}{\partial x_i}\right)^2 dx \le \\
&\le N \cdot \sqrt{c_1} \cdot \sqrt{\int_\Omega \left(1 + |u|^{4s_1}\right) dx} \cdot \|u\|^2_{W^{1,4}(\Omega)} + N \cdot c_2 \cdot \int_\Omega \left(1 + |u|^{2s_2}\right) dx \le \\
&\le \left(\sqrt{c_1} N \sqrt{|\Omega|} + \sqrt{c_1} N \|u\|^{2s_1}_{L^{4s_1}(\Omega)}\right) \cdot \|u\|^2_{W^{1,4}(\Omega)} + c_2 N|\Omega| + c_2 N \|u\|^{2s_2}_{L^{2s_2}(\Omega)}
\end{aligned} \tag{7}$$

Next the norms $\|\cdot\|_{L^{4s_1}(\Omega)}$, $\|\cdot\|_{W^{1,4}(\Omega)}$ and $\|\cdot\|_{L^{2s_2}(\Omega)}$ are estimated using Gagliardo–Nierenberg inequality. Since $s_1 \in (0, \frac{4}{N-2})$, $s_2 \in (0, \frac{N+4}{N-2})$ it is possible to find constants $c_3, c_4, c_5 > 0$, and powers $\theta_1, \theta_2, \theta_3 \in (0,1)$, such that:

$$\|u\|_{L^{4s_1}(\Omega)} \le c_3 \|u\|^{\theta_1}_{H^3(\Omega)} \|u\|^{1-\theta_1}_{H^1(\Omega)},$$

$$\|u\|_{W^{1,4}(\Omega)} \le c_4 \|u\|^{\theta_2}_{H^3(\Omega)} \|u\|^{1-\theta_2}_{H^1(\Omega)},$$
$$2s_1\theta_1 + 2\theta_2 < 2,$$
$$\|u\|_{L^{2s_2}(\Omega)} \le c_5 \|u\|^{\theta_3}_{H^3(\Omega)} \|u\|^{1-\theta_3}_{H^1(\Omega)},$$
$$2s_2\theta_3 < 2.$$

Due to Remark 3.3, $\|u\|_{H^1(\Omega)} < M$, inequality (7) will take the form:

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla g(u,x)|^2\, dx &\le \left(\sqrt{c_1}N\sqrt{|\Omega|} + \sqrt{c_1}N\|u\|^{2s_1}_{L^{4s_1}(\Omega)}\right)\cdot \|u\|^2_{W^{1,4}(\Omega)} + \\ &+ c_2N|\Omega| + c_2N\|u\|^{2s_2}_{L^{2s_2}(\Omega)} \le \\ &\le \left(\sqrt{c_1}N\sqrt{|\Omega|} + \sqrt{c_1}Nc_3M^{2s_1(1-\theta_1)}\|u\|^{2s_1\theta_1}_{H^3(\Omega)}\right)\cdot \\ &\cdot c_4M^{2(1-\theta_2)}\|u\|^{2\theta_2}_{H^3(\Omega)} + \\ &+ c_2N|\Omega| + c_2Nc_5M^{2s_2(1-\theta_3)}\|u\|^{2s_2\theta_3}_{H^3(\Omega)}.\end{aligned} \tag{8}$$

Choosing $\theta = \max(2s_1\theta_1 + 2\theta_2, 2s_2\theta_3)$ we find that $\theta < 2$. Then, defining

$$const = c_2N|\Omega|$$

and

$$k = \max(\sqrt{c_1}N\sqrt{|\Omega|}c_4M^{2(1-\theta_2)}, \sqrt{c_1}Nc_3M^{2s_1(1-\theta_1)}c_4M^{2(1-\theta_2)}, c_2Nc_5M^{2s_2(1-\theta_3)}),$$

we obtain

$$\frac{1}{2}\int_\Omega |\nabla g(u,x)|^2\, dx \le const + k\|u\|^\theta_{H^3(\Omega)}.$$

Estimate (6) will be extended to:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2\, dx + \left(\frac{1}{2} + \alpha\|\nabla v\|^2_{L^2(\Omega)}\right)\int_\Omega |\nabla\Delta u|^2\, dx \le const + k\|u\|^\theta_{H^3(\Omega)}.$$

Now, since the norms $\|\cdot\|_{H^3(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)} + \|\nabla\cdot\|_{L^2(\Omega)} + \|\Delta\cdot\|_{L^2(\Omega)} + \|\nabla\Delta\cdot\|_{L^2(\Omega)}$ are equivalent, and due to remark 3.3, we have:

$$\exists_{\bar{c}>0}\ \|u\|_{H^3(\Omega)} \le \bar{c}\left(M + \|\nabla\Delta u\|_{L^2(\Omega)}\right).$$

Since $\theta < 2$, we can find a constant $\hat{c} < \frac{1}{2}$ such that:

$$k\|u\|^\theta_{H^3(\Omega)} \le \hat{c}\|\nabla\Delta u\|^2_{L^2(\Omega)} + const = \hat{c}\int_\Omega |\nabla\Delta u|^2\, dx + const$$

Then we have:

$$\frac{d}{dt}\int_\Omega |\Delta u|^2\, dx + \alpha\|\nabla v\|^2_{L^2(\Omega)}\int_\Omega \nabla\Delta u\, dx \le const,$$

Now, using the Calderon–Zygmund type inequality (see [12], pp. 186-187):

$$\exists_{\tilde{c}>0}\ \forall_{\phi\in D(-\Delta^{3/2})}\ \int_\Omega |\Delta\phi|^2\, dx \le \tilde{c}\int_\Omega |\nabla\Delta\phi|^2\, dx.$$

we can find positive constants h_1, h_0 such that:

$$\frac{d}{dt}\int_\Omega |\Delta u|^2\, dx + h_1\int_\Omega |\Delta u|^2\, dx \le h_2,$$

Using the Gronwall inequality (see [10], p. 35) we finally obtain:

$$\int_\Omega |\Delta u|^2\,dx \le e^{h_1 t}\left(\int_\Omega (\Delta u_0)^2\,dx - \frac{h_2}{h_1}\right) + \frac{h_2}{h_1}$$

for $t \in (0,T)$. Then defining $M_1 = \sup_{t\in(0,T)}\left[e^{h_1 t}\left(\int_\Omega (\Delta u_0)^2\,dx - \frac{h_2}{h_1}\right) + \frac{h_2}{h_1}\right]$ the proof will be completed.

□

Remark 3.5. *Above lemmas show that every eventual solution in the sense of Theorem 2.1 has to be an element of the space* $\mathcal{L}^\infty([0,T], H^2(\Omega)\cap H_0^1(\Omega))$.

4 Proof of the main theorem

This section is devoted to the proof of the Theorem 2.1. Three conditions from the Leray – Schauder Principle: continuities (b), (c) and compactness (d) will be verified.

It has to be proved that the operator $F(\cdot,\alpha)$ is compact, i.e. it maps bounded subsets of $\mathcal{L}^\infty([0,T], H_0^1(\Omega))$ into compact subsets of $\mathcal{L}^\infty([0,T], H_0^1(\Omega))$. Let us fix $\alpha \in [0,1]$. If the bounded subset A of $\mathcal{L}^\infty([0,T], H_0^1(\Omega))$ is taken, then $F(A,\alpha)$ is bounded in the space $\mathcal{L}^\infty([0,T], H^2(\Omega)\cap H_0^1(\Omega))$. Using the equation (5), we can write:

$$u_t = (1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\Delta u - g(u,x)$$

and because an element $v \in A$, $\Delta u \in L^2(\Omega)$ and thanks to (A2) the function $g \in L^2(\Omega)$, we can deduce that $u_t \in L^2(\Omega)$. Additionally, the embedding $H^2(\Omega) \subseteq H^1(\Omega)$ is compact and $H^1(\Omega) \subseteq L^2(\Omega)$ is continuous. Using Aubin lemma (see e.g. [13] and [14]) the set of values of operator $F(A,\alpha)$ is compact in $\mathcal{L}^\infty([0,T], H_0^1(\Omega))$.

Now we prove that for any fixed $\alpha \in [0,1]$, the operator $F(v,\alpha)$ is continuous as a function of v, i.e. for any $v_1 \in \mathcal{L}^\infty([0,T], H_0^1(\Omega))$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|F(v_1,\alpha) - F(v_2,\alpha)\| < \varepsilon$ if $\|v_1 - v_2\| < \delta$.

Let us take $\alpha \in [0,1]$ and $v_1, v_2 \in \mathcal{L}^\infty([0,T], H_0^1(\Omega))$. Let $u_1, u_2 \in \mathcal{L}^\infty([0,T], H_0^1(\Omega))$ be the values of the operator F corresponding to v_1 and v_2, i.e.

$$(u_1)_t - (1+\alpha\|\nabla v_1\|^2_{L^2(\Omega)})\Delta u_1 + g(u_1,x) = 0,$$
$$(u_2)_t - (1+\alpha\|\nabla v_2\|^2_{L^2(\Omega)})\Delta u_2 + g(u_2,x) = 0.$$

Subtracting the above equations, defining $u := u_1 - u_2$, using (A4), it can be seen that:

$$u_t - (1+\alpha\|\nabla v_1\|^2_{L^2(\Omega)})\Delta u \le \alpha\cdot(\Delta u_2)\cdot\|v_1 - v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))} + L\cdot u\cdot(1+|u_1|^{q_1}+|u_2|^{q_2}). \quad (9)$$

Then we have that $u(0,x) = \hat{u}_0 = 0$ for all $x \in \Omega$. Analogously as in the lemmas above, we see that for all $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\|v_1 - v_2\|_{\mathcal{L}^\infty([0,T],H_0^1(\Omega))} < \delta \quad \Longrightarrow \quad \|u_1 - u_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))} < \varepsilon.$$

As an example we prove the estimation for $\|u\|_{L^2(\Omega)}$. Similarly, we can prove the estimations for $\|\nabla u\|_{L^2(\Omega)}$, $\|\Delta u\|_{L^2(\Omega)}$.

Lemma 4.1. *If function* $u \in \mathcal{L}^\infty([0,T], H_0^1(\Omega))$ *is a solution of inequality* (9) *with initial condition* $u(0,x) = 0$ *for* $x \in \Omega$*, then for all* $\varepsilon > 0$ *there exists some* $\delta > 0$ *such that if* $\|v_1 - v_2\|_{\mathcal{L}^\infty([0,T],H_0^1(\Omega))} < \delta$ *then*

$$\|u\|_{\mathcal{L}^\infty([0,T],L^2(\Omega))} = \|u_1 - u_2\|_{\mathcal{L}^\infty([0,T],L^2(\Omega))} < \varepsilon.$$

Proof. The proof is similar to the proof of Lemma 3.1. By multiplying the inequality (9) by $|u|$ and integrating over Ω we obtain:

$$\int_\Omega |u|\cdot u_t\,dx-(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_\Omega \Delta u\cdot |u|\,dx\le$$
$$\le \alpha\cdot\|v_1-v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))}\int_\Omega |u|\cdot(\Delta u_2)\,dx$$
$$+L\cdot\int_\Omega u^2\cdot(1+|u_1|^{q_1}+|u_2|^{q_2})\,dx.$$

Then using Cauchy inequality and statement of Lemma 3.4 we obtain:

$$\int_\Omega |u|\cdot(\Delta u_2)\,dx\le\frac{1}{2}\int_\Omega u^2\,dx+\frac{1}{2}\int_\Omega(\Delta u_2)^2\,dx\le\frac{1}{2}\int_\Omega u^2\,dx+\frac{1}{2}M_2.$$

Analogously, using $(A4)$ and the fact that $H^2(\Omega)\subseteq L^{2q_1}(\Omega)\cap L^{2q_2}(\Omega)$:

$$\int_\Omega u^2\cdot(1+|u_1|^{q_1}+|u_2|^{q_2})\,dx\le\frac{1}{2\epsilon}\int_\Omega u^2\,dx+2\epsilon\int_\Omega(1+|u_1|^{2q_1}+|u_2|^{2q_2})\,dx\le$$
$$\le\frac{1}{2\epsilon}\int_\Omega u^2\,dx+2\epsilon\,(|\Omega|+2M_2)\le\frac{1}{2\epsilon}\int_\Omega u^2\,dx+\epsilon C_1.$$

for some $\epsilon>0$. Integrating by parts component $\int_\Omega \Delta u\cdot|u|\,dx$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx+(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_\Omega|\nabla u|^2\,dx\le$$
$$\le\alpha\cdot\|v_1-v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))}\left(\frac{1}{2}\int_\Omega u^2\,dx+\frac{1}{2}M_2\right)$$
$$+\frac{1}{2\epsilon}\cdot L\int_\Omega u^2\,dx+L\cdot\epsilon\cdot C_1.$$

Using the Poincaré inequality $\int_\Omega|u|^2\,dx\le p\int_\Omega|\nabla u|^2\,dx$, we have next:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx+\frac{1}{p}(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\int_\Omega|u|^2\,dx\le$$
$$\le\alpha\cdot\|v_1-v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))}\left(\frac{1}{2}\int_\Omega u^2\,dx+\frac{1}{2}M_2\right)$$
$$+\frac{1}{2\epsilon}\cdot L\int_\Omega u^2\,dx+L\cdot\epsilon\cdot C_1,$$

so that

$$\frac{d}{dt}\int_\Omega|u|^2\,dx\le\hat{C}\int_\Omega|u|^2\,dx+\hat{C}_1.$$

where

$$\hat{C}=\left[\alpha\cdot\|v_1-v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))}+\frac{L}{\epsilon}-\frac{2}{p}(1+\alpha\|\nabla v\|^2_{L^2(\Omega)})\right]$$

and

$$\hat{C}_1=\alpha\cdot\|v_1-v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))}\cdot M_2+L\cdot\epsilon\cdot C_1.$$

Finally, using the Gronwall inequality (see [10], p. 35):

$$\int_\Omega u^2\,dx \le e^{\hat{C}t}\left(\int_\Omega \hat{u}_0^2\,dx + \frac{\hat{C}_1}{\hat{C}}\right) - \frac{\hat{C}_1}{\hat{C}},$$

for $t \in (0,T)$. Noting that $\hat{u}_0 = 0$, we obtain:

$$\int_\Omega u^2\,dx \le \frac{\hat{C}_1}{\hat{C}} e^{\hat{C}t} - \frac{\hat{C}_1}{\hat{C}},$$

Let us fix $\varepsilon > 0$ and take $\delta > 0$ and $\epsilon > 0$ such that

$$\|v_1 - v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))} < \delta$$

$$\hat{C}_1 = \alpha \cdot \|v_1 - v_2\|_{\mathcal{L}^\infty([0,T],H^1(\Omega))} \cdot M_2 + L \cdot \epsilon \cdot C_1 < \delta$$

$$\frac{\delta}{\hat{C}} \sup_{t\in(0,T)} e^{\hat{C}t} - \frac{\delta}{\hat{C}} < \varepsilon.$$

Then:

$$\int_\Omega u^2\,dx \le \frac{\delta}{\hat{C}} \cdot \sup_{t\in(0,T)} e^{\hat{C}t} - \frac{\delta}{\hat{C}} < \varepsilon. \qquad \square$$

Continuity of the operator $F(v,\alpha)$ with respect to the parameter α will be verified in the similar way. Let $X \subseteq \mathcal{L}^\infty([0,T], H_0^1(\Omega))$ be a bounded subset and $v \in X$. Then there exists a constant $N > 0$ such that

$$\|v\|_{\mathcal{L}^\infty([0,T],H_0^1(\Omega))} \le N, \quad \text{for } v \in X$$

Let us take $\alpha_1, \alpha_2 \in [0,1]$ and assume that u_1, u_2 will be solutions of the problem (5), i.e.

$$(u_1)_t - (1 + \alpha_1\|\nabla v\|^2_{L^2(\Omega)})\Delta u_1 + g(u_1,x) = 0,$$
$$(u_2)_t - (1 + \alpha_2\|\nabla v\|^2_{L^2(\Omega)})\Delta u_2 + g(u_2,x) = 0.$$

Then, after subtracting equations and defining $u = u_1 - u_2$, we receive

$$u_t - (1 + \alpha_2\|\nabla v\|^2_{L^2(\Omega)})\Delta u \le N \cdot (\Delta u_2) \cdot |\alpha_1 - \alpha_2| + L \cdot u \cdot (1 + |u_1|^{q_1} + |u_2|^{q_2}).$$

Because $|\alpha_1 - \alpha_2| < \delta$, the continuity of the operator F with respect to α will be obtained in a similar way as above.

Remark 4.2. *As a result of Theorem 2.1 and Remark 3.5 there exists a solution $u \in \mathcal{L}^\infty([0,T], H^2(\Omega))$ of the problem* (1) *with initial – boundary conditions (2), (3).*

In this paper the existence of a solution of some quasilinear parabolic generalization of the Kirchhoff equation of the form (1) with initial – boundary condition was proved. Using the Leray–Schauder Principle we obtain the solution in the space $\mathcal{L}^\infty([0,T], H^2(\Omega) \cap H_0^1(\Omega))$ for each arbitrary $T > 0$. Its higher regularity will be studied next.

Acknowledgement: The author is grateful to the referee for valuable remarks improving the original version of the paper.

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