Title: Nonsingular bilinear forms on direct sums of ideals

Author: Beata Rothkegel

Citation style: Rothkegel Beata. (2015). Nonsingular bilinear forms on direct sums of ideals. "Mathematica Slovaca" (2013, no. 4, s. 707-724), doi 10.2478/s12175-013-0130-5

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0130-5
Math. Slovaca **63** (2013), No. 4, 707–724

NONSINGULAR BILINEAR FORMS ON DIRECT SUMS OF IDEALS

Beata Rothkegel

(*Communicated by Stanislav Jakubec*)

ABSTRACT. In the paper we formulate a criterion for the nonsingularity of a bilinear form on a direct sum of finitely many invertible ideals of a domain. We classify these forms up to isometry and, in the case of a Dedekind domain, up to similarity.

1. Introduction

The theory of bilinear forms over commutative rings is a natural generalization of the theory of bilinear forms over fields. In both of these theories (in particular in the construction of the Witt ring) the notion of a nonsingular bilinear form plays an important role.

Let R be a commutative ring and M be a finitely generated projective R-module. A symmetric bilinear form $\alpha\colon M \times M \to R$ is said to be *nonsingular* if the adjoint homomorphism $\widehat{\alpha}\colon M \to M^* = \operatorname{Hom}_R(M, R)$ defined by

$$\widehat{\alpha}(m)(n) = \alpha(m, n) \qquad \text{for all} \quad m, n \in M,$$

is an isomorphism of the module M and the module $\operatorname{Hom}_R(M, R)$ of all linear functionals $f\colon M \to R$. When the form α is nonsingular, the bilinear space (M, α) is said to be *nonsingular* or an *inner product space* over R.

Similarly as in the case of a bilinear space over a field, if the module M is free, then α is nonsingular if and only if its matrix in any basis of M is invertible. But unlike a space over a field, in general the module M has not a basis. Therefore

2010 Mathematics Subject Classification: Primary 11E81.
Keywords: nonsingular bilinear form, isometry, similarity.

we should formulate a necessary and sufficient condition for the nonsingularity of α for any finitely generated projective module M, not necessarily free.

In [2] such a condition is given for an invertible fractional ideal of a domain. In our paper, in Section 2, we prove a criterion for the nonsingularity of α for a direct sum $I_1 \oplus \cdots \oplus I_n$ of $n \geq 1$ invertible ideals $I_1, \ldots, I_n$. For example, every finitely generated projective module M of rank n over a one-dimensional noetherian domain has such a form (cf. [7: Chapter I, Prop. 3.4, 3.5]; $M \cong I \oplus R^{n-1}$ for some invertible ideal I of the ring R).

In Section 3 we classify all nonsingular bilinear forms on a module $I_1 \oplus \cdots \oplus I_n$ up to isometry. Assuming R is a Dedekind domain, in Section 4 we classify these forms up to similarity.

Throughout the paper R^* denotes the group of invertible elements of the ring R.

2. Nonsingularity

In the paper [2] the following theorem is proved.

Theorem 2.1. ([2: Thm. 2.5]) *Let R be a domain and K its field of fractions. Let I be a fractional ideal in K. The ideal I admits a nonsingular bilinear form if and only if I^2 is a principal ideal.*

The next theorem describes all nonsingular bilinear forms on I.

Theorem 2.2. ([2: Thm. 3.1]) *Let R be a domain and K its field of fractions. Let I be a fractional ideal in K and assume $I^2 = pR$ for some $p \in K$, $p \neq 0$. If α is a nonsingular bilinear form on I, there exists a unique element $u \in R^*$ such that for all $x, y \in I$ we have*

$$\alpha\,(x, y) = \frac{u}{p}xy. \tag{1}$$

Conversely, if $u \in R^*$, then the map $\alpha\colon I \times I \to R$ defined by (1) is a nonsingular bilinear form on I.

We describe all nonsingular bilinear forms on a direct sum of finitely many fractional ideals.

We use the following lemma.

Lemma 2.3. ([7: Chapter I, Prop. 3.5]) *Let R be a domain and let I be an ideal in R. If I is invertible, then it is a finitely generated projective R-module of rank 1 and conversely, each finitely generated projective R-module of rank 1 is isomorphic to some invertible ideal of R.*

Proposition 2.4. *Let R be a domain and K its field of fractions. Let M be a direct sum of finitely many fractional ideals in K. Then M is a finitely generated projective R-module of rank $n \geq 1$ if and only if there exist invertible ideals $I_1, \dots, I_n$ of the ring R such that*

$$M \cong I_1 \oplus \cdots \oplus I_n.$$

Proof.

($\Longleftarrow$) This implication is obvious. Since every ideal I_j, $j = 1, \dots, n$, is a finitely generated projective R-module of rank 1, the module M is finitely generated, projective and

$$\operatorname{rank} M = \operatorname{rank} I_1 + \cdots + \operatorname{rank} I_n = n.$$

($\Longrightarrow$) Let

$$M = J_1 \oplus \cdots \oplus J_k$$

for some fractional ideals $J_1, \dots, J_k$ in the field K. We prove that $k = n$.

Fix $j \in \{1, \dots, k\}$. There exists an element $0 \neq d_j \in R$ such that

$$d_j J_j \lhd R.$$

The map $\psi_j \colon J_j \to d_j J_j$ defined by

$$\psi_j(x) = d_j x \qquad \text{for all} \quad x \in J_j$$

is an isomorphism of R-modules. Then

$$M \cong d_1 J_1 \oplus \cdots \oplus d_k J_k.$$

Let $\mathfrak{m}$ be a maximal ideal in the ring R and $M_{\mathfrak{m}}$ be the localisation of the module M at $\mathfrak{m}$. Then

$$M_{\mathfrak{m}} \cong (d_1 J_1)_{\mathfrak{m}} \oplus \cdots \oplus (d_k J_k)_{\mathfrak{m}}.$$

Since $d_j J_j$ is a finitely generated projective R-module, the ideal $(d_j J_j)_{\mathfrak{m}}$ is a finitely generated projective $R_{\mathfrak{m}}$-module. Therefore $(d_j J_j)_{\mathfrak{m}}$ is a free module (cf. [3: Chapter I, 2.4 Cor.]), so $(d_j J_j)_{\mathfrak{m}}$ is a principal ideal, i.e.

$$\operatorname{rank}_{\mathfrak{m}} (d_j J_j)_{\mathfrak{m}} = 1.$$

Then $\operatorname{rank}(d_j J_j) = 1$, so

$$n = \operatorname{rank} M = \operatorname{rank}(d_1 J_1) + \cdots + \operatorname{rank}(d_k J_k) = k.$$

Finally, we have

$$M \cong d_1 J_1 \oplus \cdots \oplus d_n J_n,$$

where $d_1J_1, \dots, d_nJ_n$ are finitely generated projective R-modules of rank 1. Therefore by Lemma 2.3 there exist invertible ideals I_j, $j = 1, \dots, n$, of the ring R such that $d_jJ_j \cong I_j$, so

$$M \cong I_1 \oplus \cdots \oplus I_n.$$

□

We give a necessary condition for the existence of a nonsingular bilinear form on $I_1 \oplus \cdots \oplus I_n$. In order to do that, we use the following properties and theorem of Steinitz.

Lemma 2.5. ([7: Chapter I, Lemma 3.1, Prop. 3.5]) *Let R be a domain and let I be an invertible ideal in R. Then*

(1) $I^{-1} \cong I^*$,

(2) $I \otimes_R J \cong IJ$ *for any fractional ideal* J,

(3) $I \otimes_R I^* \cong R$.

Theorem 2.6 (Steinitz). ([4: I.1.6]) *Let R be a domain and let K be its field of fractions. If $\mathfrak{a}_1, \dots, \mathfrak{a}_k$, $\mathfrak{b}_1, \dots, \mathfrak{b}_l$ are nonzero ideals in R and the R-modules $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ and $\mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_l$ are isomorphic, then there is an element $c \in K$ such that*

$$\mathfrak{a}_1 \cdots \mathfrak{a}_k = c\mathfrak{b}_1 \cdots \mathfrak{b}_l.$$

Theorem 2.7. *Let R be a domain and let $I_1, \dots, I_n$ be invertible ideals in R. If the module $I_1 \oplus \cdots \oplus I_n$ admits a nonsingular bilinear form, then $(I_1 \cdots I_n)^2$ is a principal ideal.*

P r o o f. Let $\alpha\colon \bigoplus_{j=1}^{n} I_j \times \bigoplus_{j=1}^{n} I_j \to R$ be a nonsingular bilinear form on the module $\bigoplus_{j=1}^{n} I_j = I_1 \oplus \cdots \oplus I_n$. Then the map $\widehat{\alpha}\colon \bigoplus_{j=1}^{n} I_j \to \Big(\bigoplus_{j=1}^{n} I_j\Big)^*$ is an isomorphism of R-modules, so

$$\bigoplus_{j=1}^{n} I_j \cong \Big(\bigoplus_{j=1}^{n} I_j\Big)^*.$$

But

$$\Big(\bigoplus_{j=1}^{n} I_j\Big)^* \cong \bigoplus_{j=1}^{n} I_j^*$$

and by Lemma 2.5 $I_j^* \cong I_j^{-1}$. Therefore

$$\bigoplus_{j=1}^{n} I_j \cong \bigoplus_{j=1}^{n} I_j^{-1}.$$

Obviously

$$(I_1 \cdots I_n) \otimes_R \bigoplus_{j=1}^{n} I_j \cong (I_1 \cdots I_n) \otimes_R \bigoplus_{j=1}^{n} I_j^{-1},$$

so

$$\bigoplus_{j=1}^{n} \left((I_1 \cdots I_n) \otimes_R I_j \right) \cong \bigoplus_{j=1}^{n} \left((I_1 \cdots I_n) \otimes_R I_j^{-1} \right). \tag{2}$$

By Lemma 2.5

$$(I_1 \cdots I_j \cdots I_n) \otimes_R I_j \cong I_1 \cdots I_j^2 \cdots I_n.$$

Moreover,

$$(I_1 \cdots I_j \cdots I_n) \cong (I_1 \cdots I_{j-1} \cdot I_{j+1} \cdots I_n) \otimes_R I_j,$$

so

$$\begin{aligned} (I_1 \cdots I_n) \otimes_R I_j^{-1} &\cong (I_1 \cdots I_{j-1} \cdot I_{j+1} \cdots I_n) \otimes_R I_j \otimes_R I_j^{-1} \\ &\cong (I_1 \cdots I_{j-1} \cdot I_{j+1} \cdots I_n) \otimes_R R \\ &\cong I_1 \cdots I_{j-1} \cdot I_{j+1} \cdots I_n. \end{aligned}$$

Therefore from (2) it follows that

$$\bigoplus_{j=1}^{n} \left(I_1 \cdots I_j^2 \cdots I_n \right) \cong \bigoplus_{j=1}^{n} \left(I_1 \cdots I_{j-1} \cdot I_{j+1} \cdots I_n \right).$$

By theorem of Steinitz there exists an element $c \in K$ such that

$$(I_1 \cdots I_n)^{n+1} = c \, (I_1 \cdots I_n)^{n-1}.$$

Hence

$$(I_1 \cdots I_n)^{n+1} \cong (I_1 \cdots I_n)^{n-1}.$$

Of course

$$(I_1 \cdots I_n)^{-(n-1)} \otimes_R (I_1 \cdots I_n)^{n+1} \cong (I_1 \cdots I_n)^{-(n-1)} \otimes_R (I_1 \cdots I_n)^{n-1},$$

so by Lemma 2.5

$$(I_1 \cdots I_n)^2 \cong R.$$

Therefore $(I_1 \cdots I_n)^2$ is a free R-module, i.e. it is a principal ideal. □

Now we describe any symmetric, not necessarily nonsingular, bilinear form on $\bigoplus_{j=1}^{n} I_j$.

For every $j \in \{1, \ldots, n\}$ let us denote

$$S_{jj} := (I_1 \cdots I_{j-1})^2 \cdot (I_{j+1} \cdots I_n)^2$$

(if $n = 1$, then $S_{11} = R$). Moreover, for $j, k \in \{1, \dots, n\}$, $j \neq k$, put

$$S_{jk} := (I_1 \cdots I_{j-1})^2 \cdot I_j \cdot (I_{j+1} \cdots I_{k-1})^2 \cdot I_k \cdot (I_{k+1} \cdots I_n)^2 .$$

Proposition 2.8. *Let R be a domain and $I_1, \dots, I_n$ be ideals in R such that $(I_1 \cdots I_n)^2 = pR$ for some $0 \neq p \in R$. A map $\alpha\colon \bigoplus_{j=1}^{n} I_j \times \bigoplus_{j=1}^{n} I_j \to R$ is a symmetric bilinear form on $\bigoplus_{j=1}^{n} I_j$ if and only if there exist uniquely determined elements $a_{jk} \in S_{jk}$, $a_{jk} = a_{kj}$, $j, k \in \{1, \dots, n\}$ such that*

$$\alpha\left((x_1, \dots, x_n), (y_1, \dots, y_n)\right) = \sum_{j,k=1}^{n} \frac{a_{jk}}{p} x_j y_k$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \bigoplus_{j=1}^{n} I_j$.

Proof.

($\Longleftarrow$) The bilinearity of α is obvious. It suffices to notice that for $j = 1, \dots, n$ we have

$$a_{jj} \in S_{jj} \implies a_{jj} \underbrace{x_j y_j}_{\in I_j^2} \in (I_1 \cdots I_n)^2 = pR \implies \frac{a_{jj}}{p} x_j y_j \in R$$

and for $j, k = 1, \dots, n$, $j \neq k$

$$a_{jk} \in S_{jk} \implies a_{jk} \underbrace{x_j y_k}_{\in I_j I_k} \in (I_1 \cdots I_n)^2 = pR \implies \frac{a_{jk}}{p} x_j y_k \in R,$$

so

$$\alpha\left((x_1, \dots, x_n), (y_1, \dots, y_n)\right) \in R$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \bigoplus_{j=1}^{n} I_j$.

($\Longrightarrow$) Fix $j, k \in \{1, \dots, n\}$, $j \neq k$. We prove that there exists a uniquely determined $a_{jk} \in S_{jk}$ such that

$$\alpha\left((0, \dots, x_j, \dots, 0), (0, \dots, y_k, \dots, 0)\right) = \frac{a_{jk}}{p} x_j y_k$$

for all $x_j \in I_j$, $y_k \in I_k$.

Let $x_j \in I_j$, $y_k \in I_k$ and K be the field of fractions of the ring R. Then

$$\alpha\left((0, \dots, x_j, \dots, 0), \cdot\right)|_{I_k} \in I_k^*.$$

By [2: Lemma 2.3] there exists an element $c_j \in K$ such that

$$\alpha\left((0, \dots, x_j, \dots, 0), (0, \dots, y_k, \dots, 0)\right) = c_j y_k$$

for (all) $y_k \in I_k$. Similarly,

$$\alpha\left((0,\ldots,y_k,\ldots,0),\cdot\right)|_{I_j} \in I_j^*,$$

so there exists $c_k \in K$ such that

$$\alpha\left((0,\ldots,y_k,\ldots,0),(0,\ldots,x_j,\ldots,0)\right) = c_k x_j.$$

for (all) $x_j \in I_j$. Since α is symmetric, $c_j y_k = c_k x_j$. Hence for all $x_j \in I_j \setminus \{0\}$, $y_k \in I_k \setminus \{0\}$ the ratio

$$\frac{c_j}{x_j} = \frac{c_k}{y_k} =: d_{jk} \in K$$

is a constans. Moreover,

$$d_{jk} x_j y_k = c_j y_k = \alpha\left((0,\ldots,x_j,\ldots,0),(0,\ldots,y_k,\ldots,0)\right) \in R.$$

The elements $x_j y_k$ generate the ideal $I_j I_k$, so

$$d_{jk} p \in d_{jk}\left(I_1 \cdots I_n\right)^2 \subseteq d_{jk} I_j I_k \subseteq R.$$

Denoting $a_{jk} := d_{jk} p$, we finally get

$$\alpha\left((0,\ldots,x_j,\ldots,0),(0,\ldots,y_k,\ldots,0)\right) = \frac{a_{jk}}{p} x_j y_k$$

for all $x_j \in I_j$, $y_k \in I_k$.

The uniqueness of a_{jk} follows from the cancelation property in a domain. Since α is symmetric, $a_{jk} = a_{kj}$. It suffices to prove that $a_{jk} \in S_{jk}$.

Because

$$\frac{a_{jk}}{p} x_j y_k \in R \qquad \text{for all} \quad x_j \in I_j,\ y_k \in I_k,$$

so

$$a_{jk} x_j y_k \in pR = \left(I_1 \cdots I_n\right)^2 \qquad \text{for all} \quad x_j \in I_j,\ y_k \in I_k.$$

Hence

$$a_{jk} I_j I_k \subseteq \left(I_1 \cdots I_n\right)^2 = \left(I_1 \cdots I_{j-1}\right)^2 \cdot I_j^2 \cdot \left(I_{j+1} \cdots I_{k-1}\right)^2 \cdot I_k^2 \cdot \left(I_{k+1} \cdots I_n\right)^2.$$

Multiplying by $I_j^{-1} \cdot I_k^{-1}$ we obtain

$$a_{jk} R \subseteq S_{jk},$$

i.e. $a_{jk} \in S_{jk}$.

In an analogous way we prove that for every $j \in \{1,\ldots,n\}$ there exists a uniquely determined $a_{jj} \in S_{jj}$ such that

$$\alpha\left((0,\ldots,x_j,\ldots,0),(0,\ldots,y_j,\ldots,0)\right) = \frac{a_{jj}}{p} x_j y_j$$

for all $x_j, y_j \in I_j$.

The bilinearity of α gives the thesis. □

We formulate a necessary and sufficient condition for the nonsingularity of α.

Theorem 2.9. *Let R be a domain and $I_1, \ldots, I_n$ be ideals in R such that $(I_1 \cdots I_n)^2 = pR$ for some $0 \neq p \in R$. Let $\alpha\colon \bigoplus_{j=1}^{n} I_j \times \bigoplus_{j=1}^{n} I_j \to R$ be a symmetric bilinear form on $\bigoplus_{j=1}^{n} I_j$ defined by the formula*

$$\alpha\left(\left(x_1, \ldots, x_n\right), \left(y_1, \ldots, y_n\right)\right) = \sum_{j,k=1}^{n} \frac{a_{jk}}{p} x_j y_k,$$

where $a_{jk} = a_{kj} \in S_{jk}$, $j, k \in \{1, \ldots, n\}$. The form α is nonsingular if and only if

$$\det\left(a_{jk}\right)_{1 \leq j,\, k \leq n} = p^{n-1} \cdot u$$

for some invertible element $u \in R^*$.

P r o o f. We know that

$$\det\left(a_{jk}\right) = \sum_{(k_1, \ldots, k_n)} \pm a_{1k_1} \cdots a_{nk_n},$$

where $\{k_1, \ldots, k_n\} = \{1, \ldots, n\}$. Hence

$$\det\left(a_{jk}\right) \in \sum_{(k_1, \ldots, k_n)} S_{1k_1} \cdots S_{nk_n}.$$

But

$$S_{1k_1} \cdots S_{nk_n} = \left(I_1 \cdots I_n\right)^{2(n-1)} = p^{n-1} \cdot R,$$

so

$$\det\left(a_{jk}\right) = p^{n-1} \cdot u \tag{3}$$

for some element $u \in R$.

Let $\mathfrak{m}$ be a maximal ideal in R and $\left(\bigoplus_{j=1}^{n} \left(I_j\right)_{\mathfrak{m}}, \alpha_{\mathfrak{m}}\right)$ be the localisation of the space $\left(\bigoplus_{j=1}^{n} I_j, \alpha\right)$ at $\mathfrak{m}$. The ideals $\left(I_1\right)_{\mathfrak{m}}, \ldots, \left(I_n\right)_{\mathfrak{m}}$ are finitely generated projective $R_{\mathfrak{m}}$-modules. From [3: Chapter I, 2.4 Cor.] it follows that $\left(I_1\right)_{\mathfrak{m}}, \ldots, \left(I_n\right)_{\mathfrak{m}}$ are free modules, so they are principal ideals. Let

$$\left(I_1\right)_{\mathfrak{m}} = g_1 R_{\mathfrak{m}}, \ldots, \left(I_n\right)_{\mathfrak{m}} = g_n R_{\mathfrak{m}}$$

for some $g_1, \ldots, g_n \in R_{\mathfrak{m}}$. Then

$$g_1^2 \cdots g_n^2 R_{\mathfrak{m}} = \left(\left(I_1\right)_{\mathfrak{m}} \cdots \left(I_n\right)_{\mathfrak{m}}\right)^2 = pR_{\mathfrak{m}},$$

so

$$g_1^2 \cdots g_n^2 = pv \tag{4}$$

for some invertible element $v \in R_{\mathfrak{m}}^*$. Observe that the form $\alpha_{\mathfrak{m}}$ has the following matrix

$$A = \frac{1}{p} \cdot \begin{pmatrix} a_{11} \cdot g_1^2 & a_{12} \cdot g_1 g_2 & \dots & a_{1n} \cdot g_1 g_n \\ a_{21} \cdot g_2 g_1 & a_{22} \cdot g_2^2 & \dots & a_{2n} \cdot g_2 g_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} \cdot g_n g_1 & a_{n2} \cdot g_n g_2 & \dots & a_{nn} \cdot g_n^2 \end{pmatrix}$$

in the basis

$$\mathcal{B} = ((g_1, 0, \dots, 0), \dots, (0, \dots, g_j, \dots, 0), \dots, (0, \dots, 0, g_n))$$

of the free $R_{\mathfrak{m}}$-module $\bigoplus_{j=1}^{n} (I_j)_{\mathfrak{m}}$.

($\Longrightarrow$) We show that u is invertible.

Since by the assumption α is nonsingular, from [1: (1.4) Prop.] it follows that $\alpha_{\mathfrak{m}}$ is nonsingular. Hence there exists an invertible element $\nu \in R_{\mathfrak{m}}^*$ such that

$$\nu = \det A = \frac{1}{p^n} \cdot g_1^2 g_2^2 \cdots g_n^2 \cdot \det (a_{jk}).$$

Therefore by (3) and (4)

$$\nu = v \cdot u,$$

so $u = \nu \cdot v^{-1} \in R_{\mathfrak{m}}^*$. Hence u is invertible in R.

($\Longleftarrow$) By the assumption $u \in R^*$, so $u \in R_{\mathfrak{m}}^*$. Therefore $\det A = v \cdot u \in R_{\mathfrak{m}}^*$, so $\alpha_{\mathfrak{m}}$ is nonsingular. From [1: (1.4) Prop.] it follows that the form α is nonsingular. □

Example. Let R be a domain and $I_1, \dots, I_n$ be ideals in R such that $I_1^2 = q_1 R, \dots, I_n^2 = q_n R$ for some $q_1, \dots, q_n \in R \setminus \{0\}$. For every $j \in \{1, \dots, n\}$ let $\alpha_j \colon I_j \times I_j \to R$ be a symmetric bilinear form on the ideal I_j. By Proposition 2.8 for every $j \in \{1, \dots, n\}$ there exists a unique element $a_j \in R$ such that

$$\alpha_j (x, y) = \frac{a_j}{q_j} xy \qquad \text{for all} \quad x, y \in I_j.$$

Let

$$\left(\bigoplus_{j=1}^{n} I_j, \alpha \right) = (I_1, \alpha_1) \perp \cdots \perp (I_n, \alpha_n)$$

be an orthogonal direct sum of the spaces $(I_1, \alpha_1), \dots, (I_n, \alpha_n)$. Then

$$\begin{aligned}\alpha\left(\left(x_1,\ldots,x_n\right),\left(y_1,\ldots,y_n\right)\right) &= \alpha_1\left(x_1,y_1\right)+\alpha_2\left(x_2,y_2\right)+\cdots+\alpha_n\left(x_n,y_n\right)\\ &= \frac{a_1}{q_1}x_1y_1+\frac{a_2}{q_2}x_2y_2+\cdots+\frac{a_n}{q_n}x_ny_n\\ &= \sum_{j=1}^{n}\frac{a_jb_j}{p}x_jy_j\end{aligned}$$

for all $\left(x_1,\ldots,x_n\right),\left(y_1,\ldots,y_n\right) \in \bigoplus_{j=1}^{n} I_j$, where $p := q_1q_2\cdots q_n$, $b_j := q_1\cdots \cdots q_{j-1}q_{j+1}\ldots q_n$, $j=1,\ldots,n$. We show the geometrically obvious fact that the space $\left(\bigoplus_{j=1}^{n} I_j,\alpha\right)$ is nonsingular if and only if the space (I_j,α_j) is nonsingular for every $j\in\{1,\ldots,n\}$.

Observe that

$$\left(I_1\cdots I_n\right)^2 = q_1\cdots q_nR = pR.$$

Moreover, for $j\in\{1,\ldots,n\}$ we have

$$a_{jj} := a_jb_j \in b_jR = \left(I_1\cdots I_{j-1}\right)^2\cdot\left(I_{j+1}\cdots I_n\right)^2 = S_{jj}$$

and for $j,k\in\{1,\ldots,n\}$, $j\neq k$,

$$a_{jk} := 0 \in S_{jk}.$$

By Theorem 2.9

$$\begin{aligned}\alpha \text{ is nonsingular} &\iff \det\left(a_{jk}\right) = p^{n-1}\cdot u \quad \text{for some} \quad u\in R^*\\ &\iff \det\begin{pmatrix} a_1b_1 & 0 & \ldots & 0\\ 0 & a_2b_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & a_nb_n\end{pmatrix} = p^{n-1}\cdot u\\ &\qquad \text{for some} \quad u\in R^*\\ &\iff a_1a_2\cdots a_n = u \quad \text{for some } u\in R^*\\ &\iff a_j\in R^* \quad \text{for every } j\in\{1,\ldots,n\}\\ &\iff \alpha_j \quad \text{is nonsingular for every } j\in\{1,\ldots,n\}.\end{aligned}$$

3. Isometry

Now we classify nonsingular bilinear forms on $\bigoplus_{j=1}^{n} I_j$ up to isometry. For $k, r \in \{1, \ldots, n\}$, $k \neq r$, let us denote

$$T_{kr} := (I_1 \cdots I_n)^2 \cdot I_k^{-1} \cdot I_r .$$

We describe all automorphisms of the module $\bigoplus_{j=1}^{n} I_j$.

Proposition 3.1. *Let R be a domain and $I_1, \ldots, I_n$ be ideals in R such that $(I_1 \cdots I_n)^2 = pR$ for some $0 \neq p \in R$. Assume that $\bigoplus_{j=1}^{n} I_j$ admits a nonsingular bilinear form. A map $\varphi \colon \bigoplus_{j=1}^{n} I_j \to \bigoplus_{j=1}^{n} I_j$ is an automorphism of the module $\bigoplus_{j=1}^{n} I_j$ if and only if there exists a matrix*

$$C = \frac{1}{p} \cdot (c_{kr})_{1 \leq k, r \leq n}, \qquad c_{rr} \in pR, \quad c_{kr} \in T_{kr}, \quad k, r \in \{1, \ldots, n\}, \quad k \neq r,$$

such that $\det C$ *is an invertible element in R and*

$$\varphi(y_1, \ldots, y_n) = (y_1, \ldots, y_n) \cdot C \qquad \text{for all} \quad (y_1, \ldots, y_n) \in \bigoplus_{j=1}^{n} I_j .$$

P r o o f.

($\Longrightarrow$) Let $\alpha \colon \bigoplus_{j=1}^{n} I_j \times \bigoplus_{j=1}^{n} I_j \to R$ be a nonsingular bilinear form on $\bigoplus_{j=1}^{n} I_j$ defined by

$$\alpha((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{j,k=1}^{n} \frac{a_{jk}}{p} x_j y_k, \qquad a_{jk} \in S_{jk},$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \bigoplus_{j=1}^{n} I_j$. For every $r = 1, \ldots, n$ let $f_r := \pi_r \circ \varphi$, where π_r is a projection map,

$$\pi_r(z_1, \ldots, z_r, \ldots, z_n) = z_r$$

for all $(z_1, \ldots, z_r, \ldots, z_n) \in \bigoplus_{j=1}^{n} I_j$. Of course

$$\varphi(y_1, \ldots, y_n) = (f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n))$$

and the maps f_r, $r = 1, \ldots, n$, are linear functionals on the ideal $\bigoplus_{j=1}^{n} I_j$. Since α is nonsingular, there exist elements $(x_{1r}, \ldots, x_{nr}) \in \bigoplus_{j=1}^{n} I_j$, $r = 1, \ldots, n$, such that

$$\alpha\left((x_{1r}, \ldots, x_{nr}), (y_1, \ldots, y_n)\right) = f_r(y_1, \ldots, y_n)$$

for all $(y_1, \ldots, y_n) \in \bigoplus_{j=1}^{n} I_j$, i.e.

$$\frac{1}{p} \cdot \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{jk} x_{jr} \right) y_k = f_r(y_1, \ldots, y_n)$$

for all $(y_1, \ldots, y_n) \in \bigoplus_{j=1}^{n} I_j$. Denote $c_{kr} := \sum_{j=1}^{n} a_{jk} x_{jr}$. Then

$$\frac{1}{p} \cdot \sum_{k=1}^{n} c_{kr} y_k = f_r(y_1, \ldots, y_n), \qquad r = 1, \ldots, n$$

for all $(y_1, \ldots, y_n) \in \bigoplus_{j=1}^{n} I_j$. Therefore

$$\begin{aligned} \varphi(y_1, \ldots, y_n) &= (f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n)) \\ &= (y_1, \ldots, y_n) \cdot \frac{1}{p} \cdot \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{pmatrix} \\ &= (y_1, \ldots, y_n) \cdot C. \end{aligned}$$

We show that $c_{rr} \in pR$, $c_{kr} \in T_{kr}$ for $k, r \in \{1, \ldots, n\}$, $k \neq r$.

Fix $k, r \in \{1, \ldots, n\}$. From the definition of f_r it follows that

$$\frac{1}{p} \cdot c_{kr} \cdot y_k = f_r(0, \ldots, y_k, \ldots, 0) \in I_r \qquad \text{for all} \quad y_k \in I_k.$$

Hence

$$c_{kr} \cdot y_k \in p \cdot I_r \qquad \text{for all} \quad y_k \in I_k,$$

so

$$c_{kr} \cdot I_k \subseteq p \cdot I_r \tag{5}$$

Assume $k = r$. Then

$$c_{rr} \cdot I_r \subseteq p \cdot I_r.$$

Multiplying by I_r^{-1}, we obtain

$$c_{rr} \cdot R \subseteq p \cdot R,$$

i.e. $c_{rr} \in pR$.

Assume $k \neq r$. By (5)

$$c_{kr} \cdot I_k \subseteq (I_1 \cdots I_n)^2 \cdot I_r.$$

Hence

$$c_{kr} \cdot R \subseteq (I_1 \cdots I_n)^2 \cdot I_k^{-1} \cdot I_r = T_{kr},$$

so $c_{kr} \in T_{kr}$.

We have

$$\det C = \det \frac{1}{p} \cdot (c_{kr}) = \frac{1}{p^n} \cdot \sum_{(r_1,\ldots,r_n)} \pm c_{1r_1} \cdots c_{nr_n},$$

where $\{r_1, \ldots, r_n\} = \{1, \ldots, n\}$. Since $c_{ii} \in pR = (I_1 \cdots I_n)^2$ and $c_{ir_i} \in T_{ir_i}$, $i \neq r_i$, it is easy to observe that

$$c_{1r_1} \cdots c_{nr_n} \in (I_1 \cdots I_n)^{2n} = p^n R.$$

Therefore $\det C \in R$. Since φ is an automorphism, $\det C \in R^*$.

($\Longleftarrow$) Let $k, r \in \{1, \ldots, n\}$ and $y_k \in I_k$. Then

$$\frac{1}{p} c_{kr} y_k \in I_r.$$

Indeed, let $k = r$. Then

$$c_{rr} \in pR \implies \frac{1}{p} c_{rr} \in R \implies \frac{1}{p} c_{rr} y_r \in I_r.$$

If $k \neq r$, then

$$\begin{aligned} & c_{kr} \in T_{kr} = (I_1 \cdots I_n)^2 \cdot I_k^{-1} \cdot I_r \\ \implies & c_{kr} y_k \in (I_1 \cdots I_n)^2 \cdot I_r = p I_r \\ \implies & \frac{1}{p} c_{kr} y_k \in I_r. \end{aligned}$$

Therefore

$$\varphi(y_1, \ldots, y_n) = \left(\frac{1}{p} \cdot \sum_{k=1}^{n} c_{k1} y_k, \ldots, \frac{1}{p} \cdot \sum_{k=1}^{n} c_{kn} y_k \right) \in \bigoplus_{j=1}^{n} I_j. \tag{6}$$

Obviously φ is a homomorphism of R-modules. We prove that φ is bijective.

Fix $(z_1,\dots,z_n)\in\bigoplus_{j=1}^{n} I_j$ and consider the system of n equations

$$\frac{1}{p}\cdot\sum_{k=1}^{n} c_{kr}Y_k = z_r,\qquad r=1,\dots,n. \tag{7}$$

Since the matrix of (7) is equal to C^T and

$$\det C^T=\det C\neq 0,$$

the system (7) has a unique solution $(y_1,\dots,y_n)\in K^n$, where K is the field of fractions of the ring R. For every $k\in\{1,\dots,n\}$ replacing the $k$tk column of the matrix C^T with $(z_1,\dots,z_n)^T$ we obtain

$$y_k=\det\frac{1}{p}\cdot\begin{pmatrix} c_{11} & c_{21} & \dots & pz_1 & \dots & c_{n1}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ c_{1k} & c_{2k} & \dots & pz_k & \dots & c_{nk}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ c_{1n} & c_{2n} & \dots & pz_n & \dots & c_{nn}\end{pmatrix}\cdot(\det C)^{-1}.$$

Using the fact that $z_r\in I_r$, $c_{ii}\in pR$ and $c_{ir_i}\in T_{ir_i}$, $i\neq r_i$, for $r,i,r_i\in\{1,\dots,n\}$, we show that the determinant in the numerator is an element of the ideal I_k. Moreover, $\det C\in R^*$, so $y_k\in I_k$, $k=1,\dots,n$, i.e. $(y_1,\dots,y_n)\in\bigoplus_{j=1}^{n} I_j$.

Finally by (6), for every $(z_1,\dots,z_n)\in\bigoplus_{j=1}^{n} I_j$ there is a unique element $(y_1,\dots,y_n)\in\bigoplus_{j=1}^{n} I_j$ such that

$$\varphi(y_1,\dots,y_n)=(z_1,\dots,z_n),$$

i.e. φ is an automorphism. □

Theorem 3.2. *Let R be a domain and $I_1,\dots,I_n$ be ideals in R such that $(I_1\cdots I_n)^2=pR$ for some $0\neq p\in R$. Let α and β be nonsingular bilinear forms on the module $\bigoplus_{j=1}^{n} I_j$ defined by*

$$\begin{aligned}\alpha((x_1,\dots,x_n),(y_1,\dots,y_n))&=\sum_{j,k=1}^{n}\frac{a_{jk}}{p}x_jy_k,\\ \beta((x_1,\dots,x_n),(y_1,\dots,y_n))&=\sum_{j,k=1}^{n}\frac{b_{jk}}{p}x_jy_k,\end{aligned}\qquad a_{jk},\quad b_{jk}\in S_{jk}.$$

Then the inner product spaces $\left(\bigoplus_{j=1}^{n} I_j, \alpha\right)$ *and* $\left(\bigoplus_{j=1}^{n} I_j, \beta\right)$ *are isometric if and only if there exists a matrix* C *such as in* Proposition 3.1 *and*

$$(a_{jk})_{1\leq j,k\leq n} = C \cdot (b_{jk})_{1\leq j,k\leq n} \cdot C^T.$$

P r o o f. Observe that

$$\left(\bigoplus_{j=1}^{n} I_j, \alpha\right) \cong \left(\bigoplus_{j=1}^{n} I_j, \beta\right)$$

$$\iff \text{there exists an automorphism } \varphi\colon \bigoplus_{j=1}^{n} I_j \to \bigoplus_{j=1}^{n} I_j \quad \text{such that}$$

$$\alpha\left((x_1,\ldots,x_n),(y_1,\ldots,y_n)\right) = \beta\left(\varphi(x_1,\ldots,x_n),\varphi(y_1,\ldots,y_n)\right)$$

$$\text{for all}\quad (x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \bigoplus_{j=1}^{n} I_j$$

$$\iff \text{there exists an automorphism } \varphi\colon \bigoplus_{j=1}^{n} I_j \to \bigoplus_{j=1}^{n} I_j \quad \text{such that}$$

$$\left(x_1,\ldots,x_n\right)\cdot\frac{1}{p}\cdot(a_{jk})\cdot(y_1,\ldots,y_n)^T = \varphi(x_1,\ldots,x_n)\cdot\frac{1}{p}\cdot(b_{jk})\cdot\varphi(y_1,\ldots,y_n)^T$$

$$\text{for all}\quad (x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \bigoplus_{j=1}^{n} I_j$$

$$\iff \text{there exists a matrix } C \text{ such as in Proposition 3.1 and}$$

$$\left(x_1,\ldots,x_n\right)\cdot(a_{jk})\cdot(y_1,\ldots,y_n)^T = (x_1,\ldots,x_n)\cdot C\cdot(b_{jk})\cdot C^T\cdot(y_1,\ldots,y_n)^T$$

$$\text{for all}\quad (x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \bigoplus_{j=1}^{n} I_j$$

$$\iff \text{there exists a matrix } C \text{ such as in Proposition 3.1 and } (a_{jk}) = C\cdot(b_{jk})\cdot C^T.$$

□

4. Similarity

Let (M,α) and (N,β) be inner product spaces over a domain R. We say that (M,α) and (N,β) are *similar*, if there exist metabolic spaces (M_1,α_1) and (N_1,β_1) such that the spaces $(M,\alpha)\perp(M_1,\alpha_1)$ and $(N,\beta)\perp(N_1,\beta_1)$ are isometric.

The spaces (M,α) and (N,β) are similar if and only if their similarity classes $\langle M,\alpha\rangle$, $\langle N,\beta\rangle$ in the Witt ring $W(R)$ of the ring R are equal.

If R is a Dedekind domain and K its field of fractions, then by Knebusch's theorem the natural homomorphism $\phi\colon W(R)\to W(K)$ defined by

$$\phi\left(\langle M,\alpha\rangle\right)=\langle K\otimes_R M,\alpha'\rangle\,,$$

where

$$\alpha'(x\otimes y,z\otimes t)=xz\cdot\alpha(y,t)\qquad\text{for all}\quad x\otimes y,z\otimes t\in K\otimes_R M,$$

is injective (cf. [5: p. 93]). Hence

$$\langle M,\alpha\rangle=\langle N,\beta\rangle\ \text{ in }\ W(R)\iff\langle K\otimes_R M,\alpha'\rangle=\langle K\otimes_R N,\beta'\rangle\ \text{ in }\ W(K)\,.$$

We classify nonsingular bilinear forms on a module $\bigoplus_{j=1}^{n} I_j$ up to similarity in the case when R is a Dedekind domain.

Theorem 4.1. *Let R be a Dedekind domain and K its field of fractions, $\operatorname{char}K\neq 2$. Let $I_1,\dots,I_n$ be ideals in R such that $(I_1\cdots I_n)^2=pR$ for some $0\neq p\in R$. Let α and β be nonsingular bilinear forms on the module $\bigoplus_{j=1}^{n} I_j$ defined by*

$$\alpha\left((x_1,\dots,x_n),(y_1,\dots,y_n)\right)=\sum_{j,k=1}^{n}\frac{a_{jk}}{p}x_jy_k,$$

$$\beta\left((x_1,\dots,x_n),(y_1,\dots,y_n)\right)=\sum_{j,k=1}^{n}\frac{b_{jk}}{p}x_jy_k,$$

$a_{jk},b_{jk}\in S_{jk}$. Then the inner product spaces $\left(\bigoplus_{j=1}^{n} I_j,\alpha\right)$ and $\left(\bigoplus_{j=1}^{n} I_j,\beta\right)$ are similar if and only if there exists a matrix

$$C=(c_{kr})_{1\le k,\,r\le n}\,,\qquad c_{kr}\in K,\quad k,r\in\{1,\dots,n\}\,,$$

such that $\det C$ is an invertible element in R and

$$(a_{jk})_{1\le j,\,k\le n}=C\cdot(b_{jk})_{1\le j,\,k\le n}\cdot C^T.$$

P r o o f. The forms α' and β' have the following matrices

$$(pa_{jk})_{1\leq j,\,k\leq n}\,, \qquad (pb_{jk})_{1\leq j,\,k\leq n}$$

in the basis

$$\mathcal{B} = (1\otimes(p,0,\dots,0)\,,\dots,1\otimes(0,\dots,p,\dots,0)\,,\dots,1\otimes(0,\dots,0,p))$$

of the linear space $K\otimes_R \bigoplus_{j=1}^{n} I_j$ over the field K. Therefore

$$\begin{aligned}
\Big\langle \bigoplus_{j=1}^{n} I_j,\alpha\Big\rangle = \Big\langle \bigoplus_{j=1}^{n} I_j,\beta\Big\rangle \quad &\iff \quad \Big\langle K\otimes_R \bigoplus_{j=1}^{n} I_j,\alpha'\Big\rangle = \Big\langle K\otimes_R \bigoplus_{j=1}^{n} I_j,\beta'\Big\rangle\\
&\iff \quad \Big(K\otimes_R \bigoplus_{j=1}^{n} I_j,\alpha'\Big) \cong \Big(K\otimes_R \bigoplus_{j=1}^{n} I_j,\beta'\Big)\\
&\qquad\ \text{over } K \ \ (\text{by [6: Thm. 13.1.3]; } \operatorname{char} K\neq 2)\\
&\iff \quad \text{there exists a matrix } \ C=(c_{kr})\,,\ c_{kr}\in K,\\
&\qquad\ \text{such that } \ \det C\neq 0 \ \text{ and}\\
&\qquad\ (pa_{jk}) = C\cdot(pb_{jk})\cdot C^T\\
&\iff \quad \text{there exists a matrix } \ C=(c_{kr})\,,\ c_{kr}\in K,\\
&\qquad\ \text{such that } \ \det C\in R^* \ \text{ and}\\
&\qquad\ (a_{jk}) = C\cdot(b_{jk})\cdot C^T.
\end{aligned}$$

The last implication "$\Longrightarrow$" follows from the following observation. Since

$$\det(a_{jk}) = (\det C)^2\cdot\det(b_{jk})\,,$$

by Theorem 2.9

$$p^{n-1}\cdot u = (\det C)^2\cdot p^{n-1}\cdot v$$

for some invertible elements $u,v\in R^*$. Then

$$u = (\det C)^2\cdot v$$

and $(\det C)^2\in R^*$. But R is integrally closed, so $\det C\in R^*$. □

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Received 3. 1. 2011
Accepted 19. 4. 2011

Institute of Mathematics
University of Silesia
Bankowa 14
PL–40-007 Katowice
POLAND
E-mail: brothkegel@ux2.math.us.edu.pl