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Variation of constant formulas for fractional difference equations

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In this paper, we establish variation of constant formulas for both Caputo and Riemann-Liouville fractional difference equations. The main technique is the $\mathscr{Z}$-transform. As an application, we prove a lower bound on the separation between two different solutions of a class of nonlinear scalar fractional difference equations.

Key words: fractional difference equation, variation of constant, separation of solutions

1. Introduction

Recently, the theory of fractional calculus became very popular and its development is still very fast (see e.g. [22, 25] and the references therein). In the literature, one can find results on theoretical problems as well as practical applications. In the classical framework of differential or difference equations a powerful tool for analyzing properties of dynamical systems is the so-called variation of constant formula which expresses the solution of a nonlinear equation by the solution of a linear approximation and an implicit term involving the nonlinearity

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(see [10]). The Laplace transform method has been utilized to derive a variation of constant formula for linear fractional differential equations in [14].

This paper is devoted to study linear discrete-time fractional systems. In the discrete-time framework four main types of fractional differences are considered: forward/backward Caputo and forward/backward Riemann-Liouville operators (see e.g. [1, 3, 5]). For linear discrete time-invariant fractional systems the stability problem is studied in [4, 15]. In this paper we use the $\mathscr{Z}$-transform to establish variation of constant formulas for Caputo and Riemann-Liouville fractional difference equations in Section 2. In Section 3 we use the variation of constant formula to show a separation result for solutions of scalar fractional difference equations.

A reader who is familiar with fractional difference equations may very well skip the next paragraph, in which we recall notation to keep the paper self-contained. Denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N} := \mathbb{Z}_{\geqslant 0}$ the set $\{0,1,2,\dots\}$ of natural numbers including 0, and by $\mathbb{Z}_{\leqslant 0} := \{0,-1,-2,\dots\}$ the set of non-positive integers. For $a \in \mathbb{R}$ we denote by $\mathbb{N}_a := a + \mathbb{N}$ the set $\{a, a+1, \dots\}$. By $\Gamma\colon \mathbb{R}\setminus\mathbb{Z}_{\leqslant 0} \to \mathbb{R}$ we denote the Euler gamma function defined by

$$\Gamma(\alpha) := \lim_{n\to\infty} \frac{n^{\alpha} n!}{\alpha(\alpha+1)\cdots(\alpha+n)} \qquad (\alpha \in \mathbb{R}\setminus\mathbb{Z}_{\leqslant 0}). \tag{1}$$

Note that (see [12])

$$\Gamma(\alpha) = \begin{cases} \displaystyle\int_0^{\infty} x^{\alpha-1} e^{-x} dx & \text{if } \alpha > 0, \\ \displaystyle\frac{\Gamma(\alpha+1)}{\alpha} & \text{if } \alpha < 0 \text{ and } \alpha \in \mathbb{R}\setminus\mathbb{Z}_{\leqslant 0}. \end{cases} \tag{2}$$

For $s \in \mathbb{R}$ with $s+1$, $s+1-\alpha \notin \mathbb{Z}_{\leqslant 0}$, the falling factorial power $(s)^{(\alpha)}$ is defined by

$$(s)^{(\alpha)} := \frac{\Gamma(s+1)}{\Gamma(s+1-\alpha)}. \tag{3}$$

By $\lceil x \rceil := \min\{k \in \mathbb{Z}\colon k \geqslant x\}$ we denote the least integer greater or equal to x and by $\lfloor x \rfloor := \max\{k \in \mathbb{Z}\colon k \leqslant x\}$ the greatest integer less or equal to x. Binomial coefficients $\binom{r}{m}$ can be defined for any $r, m \in \mathbb{C}$ as described in [12, Section 5.5, formula (5.90)]. For $r \in \mathbb{R}$ and $m \in \mathbb{Z}$ the binomial coefficient satisfies [12, Section 5.1, formula (5.1)]

$$\binom{r}{m} = \begin{cases} \displaystyle\frac{r(r-1)\cdots(r-m+1)}{m!} & \text{if } m \in \mathbb{Z}_{\geqslant 1}, \\ 1 & \text{if } m = 0, \\ 0 & \text{if } m \in \mathbb{Z}_{\leqslant -1}. \end{cases}$$

For $a \in \mathbb{R}$, $\nu \in \mathbb{R}_{\geqslant 0}$ and a function $x\colon \mathbb{N}_a \to \mathbb{R}^d$, the ν-th delta fractional sum $\Delta_a^{-\nu}x\colon \mathbb{N}_{a+\nu} \to \mathbb{R}^d$ of x is defined as

$$(\Delta_a^{-\nu}x)(t) := \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} (t-k-1)^{(\nu-1)} x(k) \qquad (t \in \mathbb{N}_{a+\nu}).$$

We write $\Delta^{-\nu}x$ instead of $\Delta_0^{-\nu}x$.

The Caputo forward difference ${}_{\mathrm{C}}\Delta_a^{\alpha}x\colon \mathbb{N}_{a+1-\alpha} \to \mathbb{R}^d$ of x of order α is defined as the composition ${}_{\mathrm{C}}\Delta_a^{\alpha} := \Delta_a^{-(1-\alpha)} \circ \Delta$ of the $(1-\alpha)$-th delta fractional sum with the classical difference operator $t \mapsto \Delta x(t) := x(t+1) - x(t)$, i.e.

$$({}_{\mathrm{C}}\Delta_a^{\alpha}x)(t) := (\Delta_a^{-(1-\alpha)}\Delta x)(t) \qquad (t \in \mathbb{N}_{a+1-\alpha}).$$

The Riemann-Liouville forward difference ${}_{\text{R-L}}\Delta_a^{\alpha}x\colon \mathbb{N}_{a+1-\alpha} \to \mathbb{R}^d$ of x of order α is defined as ${}_{\text{R-L}}\Delta_a^{\alpha} := \Delta \circ \Delta_a^{-(1-\alpha)}$, i.e.

$$({}_{\text{R-L}}\Delta_a^{\alpha}x)(t) := (\Delta\Delta_a^{-(1-\alpha)}x)(t) \qquad (t \in \mathbb{N}_{a+1-\alpha}).$$

Similarly, as for the fractional sum, if $a = 0$ we simply write ${}_{\mathrm{C}}\Delta^{\alpha}x$ and ${}_{\text{R-L}}\Delta^{\alpha}x$.

Let $\alpha \in (0,1)$. Consider a linear fractional difference equation of the form

$$(\Delta^{\alpha}x)(n+1-\alpha) = Ax(n) + f(n) \qquad (n \in \mathbb{N}), \tag{4}$$

where $x\colon \mathbb{N} \to \mathbb{R}^d$, Δ^{α} is either the Caputo ${}_{\mathrm{C}}\Delta^{\alpha}$ or Riemann-Liouville ${}_{\text{R-L}}\Delta^{\alpha}$ forward difference operator of order α, $f\colon \mathbb{N} \to \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$. For an initial value $x_0 \in \mathbb{R}^d$, (4) has a unique solution $x\colon \mathbb{N} \to \mathbb{R}^d$ which satisfies the initial condition $x(0) = x_0$. We denote x by $\varphi_{\mathrm{C}}(\cdot, x_0)$ or $\varphi_{\text{R-L}}(\cdot, x_0)$, respectively. If $f \equiv 0$, (4) is called homogeneous, and its solutions can be expressed with discrete-time Mittag-Leffler functions. In the literature, different types of discrete-time Mittag-Leffler functions are defined [17,21,24]. In [17], for $\beta \in \mathbb{C}$, two functions $E_{(\alpha,\beta)}$ and $\mathscr{E}_{(\alpha,\beta)}$ are defined by

$$E_{(\alpha,\beta)}(A,n) = \sum_{k=0}^{\infty} A^k \binom{n-k+k\alpha+\beta-1}{n-k} \qquad (n \in \mathbb{Z}), \tag{5}$$

and

$$\mathscr{E}_{(\alpha,\beta)}(A,z) = \sum_{k=0}^{\infty} A^k \frac{(z+(k-1)(\alpha-1))^{(k\alpha)}(z+k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k+\beta)} \qquad (z \in \mathbb{C}).$$

These are two different functions, however,

$$E_{(\alpha,1)}(A,n) = \mathscr{E}_{(\alpha,1)}(A,n+1-\alpha) \qquad (n \in \mathbb{N}),$$

since for $\beta = 1$, by setting $z = n-1+\alpha$,

$$\begin{aligned}\frac{(z+(k-1)(\alpha-1)^{(k\alpha)}(z+k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k+\beta)} &= \frac{(z+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(\alpha k+1)}\\ &= \frac{\Gamma(z+k\alpha-k-\alpha+2)}{\Gamma(z-k-\alpha+2)\Gamma(\alpha k+1)}\\ &= \frac{\Gamma(n+k\alpha-k+1)}{\Gamma(n-k+1)\Gamma(\alpha k+1)},\end{aligned}$$

and

$$\binom{n-k+k\alpha+\beta-1}{n-k} = \frac{\Gamma(n-k+k\alpha+1)}{\Gamma(n-k+1)\Gamma(k\alpha+\beta)} = \frac{\Gamma(n-k+k\alpha+1)}{\Gamma(n-k+1)\Gamma(k\alpha+1)}.$$

Similarly, $E_{(\alpha,\alpha)}(A,n) = \mathscr{E}_{(\alpha,\alpha)}(A,n+1-\alpha)$ for $n \in \mathbb{N}$, since for $\beta = \alpha$, by setting $z = n-1+\alpha$,

$$\begin{aligned}\frac{(z+(k-1)(\alpha-1))^{(k\alpha)}(z+k(\alpha-1))^{(\alpha-1)}}{\Gamma(\alpha k+\alpha)} &= \frac{\Gamma(z+k\alpha-k-\alpha+2)}{\Gamma(z-k-\alpha+2)}\frac{\Gamma(z+k\alpha-k+1)}{\Gamma(z+k\alpha-k-\alpha+2)}\frac{1}{\Gamma(\alpha k+\alpha)}\\ &= \frac{\Gamma(n-k+k\alpha+\alpha)}{\Gamma(n-k+1)\Gamma(\alpha k+\alpha)}\\ &= \binom{n-k+k\alpha+\alpha-1}{n-k}.\end{aligned}$$

The next remark provides formulas for solutions of homogeneous Caputo and Riemann-Liouville equations in terms of discrete-time Mittag-Leffler functions.

Remark 1 *(a) The solution of the linear homogeneous Caputo difference equation*

$$({}_{\mathrm{C}}\Delta^{\alpha}x)(n+1-\alpha) = Ax(n), \quad x(0) = x_0 \in \mathbb{R}^d,$$

is given by

$$\varphi_{\mathrm{C}}(n,x_0) = E_{(\alpha)}(A,n)x_0 \qquad (n \in \mathbb{N}), \tag{6}$$

with the discrete-time Mittag-Leffler function

$$E_{(\alpha)}(A,n) := E_{(\alpha,1)}(A,n) = \sum_{k=0}^{\infty} A^k \binom{n-k+k\alpha}{n-k} \qquad (n \in \mathbb{N}). \tag{7}$$

See e.g. [2].

(b) The solution of the linear homogeneous Riemann-Liouville difference equation

$$({}_{\text{R-L}}\Delta^{\alpha}x)(n+1-\alpha) = Ax(n), \quad x(0) = x_0 \in \mathbb{R}^d,$$

is given by

$$\varphi_{\text{R-L}}(n,x_0) = E_{(\alpha,\alpha)}(A,n)x_0 \qquad (n \in \mathbb{N}), \tag{8}$$

with the discrete-time Mittag-Leffler function

$$E_{(\alpha,\alpha)}(A,n) = \sum_{k=0}^{\infty} A^k \binom{n-k+(k+1)\alpha-1}{n-k} \qquad (n \in \mathbb{N}). \tag{9}$$

Instead of giving a direct proof, we refer to our main Theorem 1 which implies (6) and (8) for the special case $f \equiv 0$.

Note that the sums in the right-hand sides of (5), (7) and (9) for $n \in \mathbb{Z}$ are taken over only finitely many summands, since $\binom{r}{m} = 0$ if $r \in \mathbb{R}$ and $m \in \mathbb{Z}_{\leqslant -1}$, therefore

$$\varphi_{\text{C}}(n,x_0) = \sum_{k=0}^{n} A^k \binom{n-k+k\alpha}{n-k} x_0 = \sum_{k=0}^{n} A^k (-1)^{n-k} \binom{-k\alpha-1}{n-k} x_0$$

and

$$\varphi_{\text{R-L}}(n,x_0) = \sum_{k=0}^{n} A^k \binom{n-k+(k+1)\alpha-1}{n-k} x_0 = \sum_{k=0}^{n} A^k(-1)^{n-k} \binom{-k\alpha-\alpha}{n-k} x_0.$$

In the last step we used the following identity for binomial coefficients [12, p. 174]

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k} \qquad (r \in \mathbb{R}, k \in \mathbb{Z}). \tag{10}$$

2. Variation of constant formula

The next theorem presents variation of constant formulas for Caputo and Riemann-Liouville fractional difference equations.

Theorem 1 *(a) The solution of the linear Caputo difference equation*

$$({}_{\text{C}}\Delta^{\alpha}x)(n+1-\alpha) = Ax(n) + f(n) \qquad (n \in \mathbb{N}),$$

with initial condition $x(0) = x_0 \in \mathbb{R}^d$, *is given by*

$$\varphi_{\mathrm{C}}(n,x_0) = E_{(\alpha)}(A,n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(A,n-k-1)f(k) \qquad (n \in \mathbb{N}). \tag{11}$$

(b) The solution of the linear Riemann-Liouville difference equation

$$({}_{\text{R-L}}\Delta^{\alpha}x)(n+1-\alpha) = Ax(n) + f(n) \qquad (n \in \mathbb{N}),$$

with initial condition $x(0) = x_0 \in \mathbb{R}^d$, *is given by*

$$\varphi_{\text{R-L}}(n,x_0) = E_{(\alpha,\alpha)}(A,n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(A,n-k-1)f(k) \qquad (n \in \mathbb{N}). \tag{12}$$

In order to prepare the proof of Theorem 1, we summarize some results about the $\mathscr{Z}$-transform of a sequence $x\colon \mathbb{N} \to \mathbb{R}$, which is defined by

$$\mathscr{Z}[x](z) = \sum_{i=0}^{\infty} x(i)z^{-i} \qquad (z \in \mathbb{C}, |z| > R),$$

for $R = \limsup_{i\to\infty} |x(i)|^{1/i}$, see e.g. [10, Chapter 6] and [13]. The $\mathscr{Z}$-transform of $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$ valued sequences is defined component-wise.

The next lemma is devoted to the $\mathscr{Z}$-transform of discrete-time Mittag-Leffler functions and fractional differences.

Lemma 6 *Let* $A \in \mathbb{R}^{d\times d}$, $x\colon \mathbb{N} \to \mathbb{R}$. *Then*

(i) $$\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot)\right](z) = \left(\frac{z}{z-1}\right)^{\beta} \left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1},$$

(ii) $$\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot-1)\right](z) = \frac{1}{z}\left(\frac{z}{z-1}\right)^{\beta} \left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1},$$

(iii) $$\mathscr{Z}\left[({}_{\mathrm{C}}\Delta^{\alpha}x)(\cdot+1-\alpha)\right] = z\left(\frac{z}{z-1}\right)^{-\alpha} \left[\mathscr{Z}[x](z) - \frac{z}{z-1}x(0)\right],$$

(iv) $$\mathscr{Z}\left[({}_{\text{R-L}}\Delta^{\alpha}x)(\cdot+1-\alpha)\right] = z\left(\frac{z}{z-1}\right)^{-\alpha} \mathscr{Z}[x](z) - zx(0).$$

Proof. (i) The proof is similar to [20, Proposition 2]. By the definition of the $\mathscr{Z}$-transform, we have

$$\begin{aligned}\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot)\right](z) &= \sum_{n=0}^{\infty} E_{(\alpha,\beta)}(A,n)\frac{1}{z^n}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A^k(-1)^{n-k}\binom{-k\alpha-\beta}{n-k}\frac{1}{z^n}\\ &= \sum_{k=0}^{\infty} A^k \sum_{n=0}^{\infty}(-1)^{n-k}\binom{-k\alpha-\beta}{n-k}\frac{1}{z^n}.\end{aligned}$$

With $s=n-k$, we get

$$\begin{aligned}\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot)\right](z) &= \sum_{k=0}^{\infty} A^k \sum_{s=0}^{\infty}(-1)^{s}\binom{-k\alpha-\beta}{s}\frac{1}{z^{s+k}}\\ &= \sum_{k=0}^{\infty}\left(\frac{1}{z}A\right)^k \sum_{s=0}^{\infty}(-1)^{s}\binom{-k\alpha-\beta}{s}\frac{1}{z^{s}}\\ &= \sum_{k=0}^{\infty}\left(\frac{1}{z}A\right)^k \left(1-\frac{1}{z}\right)^{-k\alpha-\beta}\\ &= \sum_{k=0}^{\infty}\left(\frac{1}{z}A\right)^k \left(\frac{z}{z-1}\right)^{k\alpha+\beta}.\end{aligned}$$

Hence, we obtain

$$\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot)\right](z) = \left(\frac{z}{z-1}\right)^{\beta}\left(I-\frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}A\right)^{-1}.$$

(ii) By the definition of the $\mathscr{Z}$-transform, we have

$$\begin{aligned}\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot-1)\right](z) &= \sum_{n=0}^{\infty} E_{(\alpha,\beta)}(A,n-1)\frac{1}{z^n}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A^k(-1)^{n-1-k}\binom{-k\alpha-\beta}{n-1-k}\frac{1}{z^n}\\ &= \sum_{k=0}^{\infty} A^k \sum_{n=0}^{\infty}(-1)^{n-1-k}\binom{-k\alpha-\beta}{n-1-k}\frac{1}{z^n}.\end{aligned}$$

With $s = n-1-k$, we get

$$\begin{aligned}\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot-1)\right](z) &= \sum_{k=0}^{\infty} A^k \sum_{s=0}^{\infty}(-1)^s \binom{-k\alpha-\beta}{s}\frac{1}{z^{s+k+1}} \\ &= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}A\right)^k \sum_{s=0}^{\infty}(-1)^s \binom{-k\alpha-\beta}{s}\frac{1}{z^s} \\ &= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}A\right)^k \left(1-\frac{1}{z}\right)^{-k\alpha-\beta} \\ &= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}A\right)^k \left(\frac{z}{z-1}\right)^{k\alpha+\beta}.\end{aligned}$$

Hence, we obtain

$$\mathscr{Z}\left[E_{(\alpha,\beta)}(A,\cdot-1)\right](z) = \frac{1}{z}\left(\frac{z}{z-1}\right)^{\beta}\left(I-\frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}A\right)^{-1}.$$

(iii) This is [18, Corollary 9].
(iv) This is [19, Proposition 8]. □

Proof. [Proof of Theorem 1](a) Applying the $\mathscr{Z}$-transform to equation (4) with the Caputo forward difference operator, we get

$$\begin{aligned} z\left(\frac{z}{z-1}\right)^{-\alpha}&\left[\mathscr{Z}\left[\varphi_C(\cdot,x_0)\right](z) - \frac{z}{z-1}x_0\right] \\ &= A\mathscr{Z}\left[\varphi_C(\cdot,x_0)\right](z) + \mathscr{Z}[f](z).\end{aligned}$$

Using Lemma 6(i), we obtain

$$\begin{aligned}\mathscr{Z}\left[\varphi_C(\cdot,x_0)\right](z) &= \mathscr{Z}\left[E_{(\alpha)}(A,\cdot)(z)x_0\right] \\ &\quad + \left(z\left(\frac{z}{z-1}\right)^{-\alpha}I-A\right)^{-1}\mathscr{Z}[f](z).\end{aligned}$$

For notational clarity, we write $\mathscr{Z}^{-1}[z\mapsto w(z)] := \mathscr{Z}^{-1}[w]$ for applying the inverse of the $\mathscr{Z}$-transform to a function $w(\cdot)$, and get

$$\begin{aligned}\varphi_C(n,x_0) &= E_{(\alpha)}(A,n)x_0 \\ &\quad + \mathscr{Z}^{-1}\left[z\mapsto\left(z\left(\frac{z}{z-1}\right)^{-\alpha}I-A\right)^{-1}\mathscr{Z}[f](z)\right](n) \qquad (n\in\mathbb{N}).\end{aligned}$$

Using

$$\mathscr{Z}^{-1}\left[z \mapsto \left(z\left(\frac{z}{z-1}\right)^{-\alpha} I - A\right)^{-1}\right](n)$$

$$= \mathscr{Z}^{-1}\left[z \mapsto \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1}\right](n) \qquad (n \in \mathbb{N}),$$

and the abbreviation $g(\cdot) := E_{(\alpha,\alpha)}(A, \cdot - 1)$, we have from Lemma 6(ii),

$$\mathscr{Z}[g](z) = \mathscr{Z}\left[E_{(\alpha,\alpha)}(A, \cdot - 1)\right](z) = \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1}.$$

Hence, we get

$$\begin{aligned}
\varphi_{\mathrm{C}}(n, x_0) &= E_{(\alpha)}(A, n)x_0 + \mathscr{Z}^{-1}\left[z \mapsto \mathscr{Z}[g](z)\mathscr{Z}[f](z)\right](n) \\
&= E_{(\alpha)}(A, n)x_0 + (g * f)(n) \\
&= E_{(\alpha)}(A, n)x_0 + \sum_{k=0}^{n} g(n-k) f(k) \\
&= E_{(\alpha)}(A, n)x_0 + \sum_{k=0}^{n} E_{(\alpha,\alpha)}(A, n-k-1) f(k) \qquad (n \in \mathbb{N}).
\end{aligned}$$

By definition of the discrete-time Mittag-Leffler function and since $\binom{r}{m} = 0$ if $r \in \mathbb{R}$ and $m \in \mathbb{Z}_{\leqslant -1}$, we have $E_{(\alpha,\alpha)}(A, -1) = 0$, and therefore

$$\varphi_{\mathrm{C}}(n, x_0) = E_{(\alpha)}(A, n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(A, n-k-1) f(k) \qquad (n \in \mathbb{N}).$$

(b) Applying the $\mathscr{Z}$-transform to equation (4) with the Riemann-Liouville forward difference operator, we get

$$\begin{aligned}
z\left(\frac{z}{z-1}\right)^{-\alpha} &\mathscr{Z}\left[\varphi_{\text{R-L}}(\cdot, x_0)\right](z) - zx_0 \\
&= A\mathscr{Z}\left[\varphi_{\text{R-L}}(\cdot, x_0)\right](z) + \mathscr{Z}[f](z).
\end{aligned}$$

Using Lemma 6(i), we obtain

$$\mathscr{Z}\left[\varphi_{\text{R-L}}(\cdot,x_0)\right](z) = \mathscr{Z}\left[E_{(\alpha,\alpha)}(A,\cdot)(z)x_0\right] + \left(z\left(\frac{z}{z-1}\right)^{-\alpha} I - A\right)^{-1} \mathscr{Z}[f](z).$$

Applying the inverse of the $\mathscr{Z}$-transform yields

$$\varphi_{\text{R-L}}(n,x_0) = E_{(\alpha,\alpha)}(A,n)x_0 + \mathscr{Z}^{-1}\left[z \mapsto \left(z\left(\frac{z}{z-1}\right)^{-\alpha} I - A\right)^{-1} \mathscr{Z}[f](z)\right](n) \qquad (n \in \mathbb{N}).$$

Using

$$\mathscr{Z}^{-1}\left[z \mapsto \left(z\left(\frac{z}{z-1}\right)^{-\alpha} I - A\right)^{-1}\right] = \mathscr{Z}^{-1}\left[z \mapsto \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1}\right]$$

and the abbreviation $g(\cdot) := E_{(\alpha,\alpha)}(A,\cdot-1)$, we have from Lemma 6(ii),

$$\mathscr{Z}[g](z) = \mathscr{Z}\left[E_{(\alpha,\alpha)}(A,\cdot-1)\right](z) = \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1}.$$

Hence, we get

$$\begin{aligned}
\varphi_{\text{R-L}}(n,x_0) &= E_{(\alpha,\alpha)}(A,n)x_0 + \mathscr{Z}^{-1}\left[z \mapsto \mathscr{Z}[g](z)\mathscr{Z}[f](z)\right](n) \\
&= E_{(\alpha,\alpha)}(A,n)x_0 + (g * f)(n) \\
&= E_{(\alpha,\alpha)}(A,n)x_0 + \sum_{k=0}^{n} g(n-k)f(k) \\
&= E_{(\alpha,\alpha)}(A,n)x_0 + \sum_{k=0}^{n} E_{(\alpha,\alpha)}(A,n-k-1)f(k) \qquad (n \in \mathbb{N}).
\end{aligned}$$

By definition of the discrete-time Mittag-Leffler function, $E_{(\alpha,\alpha)}(A,-1) = 0$, and therefore

$$\varphi_{\text{R-L}}(n,x_0) = E_{(\alpha,\alpha)}(A,n)x_0 + \sum_{k=0}^{n-1} E_{\alpha,\alpha}(A,n-k-1)f(k) \qquad (n \in \mathbb{N}). \qquad \square$$

Theorem 1 can be applied to a nonlinear equation yielding an implicit solution representation by the variation of constant formula. Let $x\colon \mathbb{N} \to \mathbb{R}^d$ be a solution of the nonlinear fractional difference equation

$$(\Delta^\alpha x)(n+1-\alpha) = Ax(n) + g(x(n)) \qquad (n \in \mathbb{N}),$$

where Δ^α is either the Caputo ${}_{\mathrm{C}}\Delta^\alpha$ or Riemann-Liouville ${}_{\mathrm{R\text{-}L}}\Delta^\alpha$ forward difference operator of order α, $f\colon \mathbb{R}^d \to \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$. Then x is also a solution of the (nonautonomous) linear fractional difference equation (4) with

$$f\colon \mathbb{N} \to \mathbb{R}^d, \quad n \mapsto g(x(n)).$$

By Theorem 1, x satisfies the implicit equation

$$x(n) = E_{(\alpha,\beta)}(A,n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(A,n-k-1)g(x(k)) \qquad (n \in \mathbb{N}) \tag{13}$$

with $\beta = 1$ or $\beta = \alpha$, respectively.

3. Scalar solution separation

Consider scalar nonlinear fractional difference equations of the form

$$(\Delta^\alpha x)(n+1-\alpha) = \lambda x(n) + f(x(n)) \qquad (n \in \mathbb{N}), \tag{14}$$

where $x\colon \mathbb{N} \to \mathbb{R}$, Δ^α is either the Caputo ${}_{\mathrm{C}}\Delta^\alpha$ or Riemann-Liouville ${}_{\mathrm{R\text{-}L}}\Delta^\alpha$ forward difference operator of a real order $\alpha \in (0,1)$, $\lambda > 0$, and $f\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, i.e. there is a constant $L > 0$ such that

$$|f(x) - f(y)| \leqslant L|x-y| \qquad (x,y \in \mathbb{R}). \tag{15}$$

Solutions of initial value problems (14), $x(0) \in \mathbb{R}$, exist on $\mathbb{N}$ (see e.g. [26, Section 3]).

The next theorem presents a lower bound on the separation between two solutions.

Theorem 2 *Consider equation* (14) *and assume that* f *satisfies* (15) *with* $L \in [0,\lambda)$.

(a) Caputo difference equations: solutions of

$$({}_{\mathrm{C}}\Delta^\alpha x)(n+1-\alpha) = \lambda x(n) + f(x(n)) \tag{16}$$

satisfy the estimate

$$|\varphi_C(n,x) - \varphi_C(n,y)| \geqslant E_{(\alpha)}(\lambda - L, n)|x - y| \qquad (x, y \in \mathbb{R}, n \in \mathbb{N}).$$

(b) Riemann-Liouville difference equation: solutions of

$$({}_{\text{R-L}}\Delta^{\alpha}x)(n+1-\alpha) = \lambda x(n) + f(x(n)) \tag{17}$$

satisfy the estimate

$$|\varphi_{\text{R-L}}(n,x) - \varphi_{\text{R-L}}(n,y)| \geqslant E_{(\alpha,\alpha)}(\lambda - L, n)|x - y| \qquad (x, y \in \mathbb{R}, n \in \mathbb{N}).$$

In the proof of the above theorem we will use the following lemma on monotonicity with respect to the initial conditions of scalar equations.

Lemma 7 *Consider equation* (14) *and assume that* f *satisfies* (15) *with* $L \in [0, \lambda)$.

(a) If $x \leqslant y$*, then* $\varphi_C(n,x) \leqslant \varphi_C(n,y)$ *for* $n \in \mathbb{N}$.

(b) If $x \leqslant y$*, then* $\varphi_{\text{R-L}}(n,x) \leqslant \varphi_{\text{R-L}}(n,y)$ *for* $n \in \mathbb{N}$.

Proof. Define $h(x) := Lx + f(x)$. Then equation (14) can be rewritten as

$$(\Delta^{\alpha}x)(n+1-\alpha) = (\lambda - L)x(n) + h(x(n)) \qquad (n \in \mathbb{N}). \tag{18}$$

Moreover, for $x \leqslant y$

$$\begin{aligned} h(y) - h(x) &= Ly + f(y) - (Lx + f(x)) \\ &= f(y) - f(x) + L(y - x) \\ &\geqslant -L(y - x) + L(y - x) \\ &= 0, \end{aligned}$$

i.e., h is monotonically increasing.

(a) By Theorem 1(a) and (13), for $x, y \in \mathbb{R}$,

$$\begin{aligned} &\varphi_C(n,y) - \varphi_C(n,x) \\ &\quad = E_{(\alpha)}(\lambda - L, n)(y - x) \\ &\qquad + \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(\lambda - L, n-k-1)(h(\varphi_C(k,y)) - h(\varphi_C(k,x))) \quad (n \in \mathbb{N}). \end{aligned} \tag{19}$$

By (10) we have for $\alpha > 0$, $\beta \geqslant 0$

$$\begin{aligned}\binom{n-k+k\alpha+\beta-1}{n-k} &= (-1)^{n-k}\binom{-(k\alpha+\beta)}{n-k} \\ &= (-1)^{n-k}\frac{(-(k\alpha+\beta))(-(k\alpha+\beta+1))\cdots(-(k\alpha+\beta+n-k-1))}{1\cdot 2\cdots(n-k)} \\ &= \frac{(k\alpha+\beta)(k\alpha+\beta+1)\cdots(k\alpha+\beta+n-k-1)}{1\cdot 2\cdots(n-k)} > 0.\end{aligned}$$

Substituting into the above inequality $\beta = 0$ and $\beta = 1$ and taking into account that $\lambda - L > 0$, we have $E_{(\alpha)}(\lambda - L, n) > 0$ and $E_{(\alpha,\alpha)}(\lambda - L, n) > 0$ for all $n \in \mathbb{N}$, respectively. Hence, $x \leqslant y$ implies $\varphi_{\mathrm{C}}(n,x) \leqslant \varphi_{\mathrm{C}}(n,y)$ for $n \in \mathbb{N}$.

(b) By Theorem 1(b) and (13), for $x, y \in \mathbb{R}$,

$$\begin{aligned}\varphi_{\text{R-L}}(n,y) - \varphi_{\text{R-L}}(n,x) &= E_{(\alpha,\alpha)}(\lambda - L, n)(y-x) \\ &+ \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(\lambda - L, n-k-1)(h(\varphi_{\text{R-L}}(k,y)) - h(\varphi_{\text{R-L}}(k,x))) \quad (n \in \mathbb{N}). \quad (20)\end{aligned}$$

Since $\lambda - L > 0$, we have $E_{(\alpha,\alpha)}(\lambda - L, n) > 0$ for all $n \in \mathbb{N}$. Hence, $x \leqslant y$ implies $\varphi_{\text{R-L}}(n,x) \leqslant \varphi_{\text{R-L}}(n,y)$ for $n \in \mathbb{N}$. □

We are now in a position to prove Theorem 2.

Proof. [Proof of Theorem 2]Assume that $x < y$ and $L \in [0, \lambda)$.

By Lemma 7, equations (19) and (20), and the fact that h is monotonically increasing, we get

$$\varphi_{\mathrm{C}}(n,y) - \varphi_{\mathrm{C}}(n,x) \geqslant E_{(\alpha)}(\lambda - L, n)(y-x) \qquad (n \in \mathbb{N}),$$

and

$$\varphi_{\text{R-L}}(n,y) - \varphi_{\text{R-L}}(n,x) \geqslant E_{(\alpha\alpha)}(\lambda - L, n)(y-x) \qquad (n \in \mathbb{N}),$$

respectively. □

As an application of Theorem 2 to equations (14) with trivial solution, we get that the Lyapunov exponent of non-zero solutions is nonnegative.

Corollary 4 *Consider equation* (14) *with* $\lambda > 0$ *and assume that* f *satisfies* (15) *with* $L \in [0, \lambda)$. *Then for* $x_0 \in \mathbb{R} \setminus \{0\}$ *the nontrivial solutions of the Caputo and Riemann-Liouville difference equations (16) and (17) satisfy*

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\varphi_{\mathrm{C}}(n, x_0)| \geqslant \begin{cases} \lambda - L & \text{if } \lambda - L > 1, \\ 0 & \text{if } 0 < \lambda - L \leqslant 1, \end{cases} \tag{21}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\varphi_{\mathrm{R\text{-}L}}(n, x_0)| \geqslant \begin{cases} \lambda - L & \text{if } \lambda - L > 1, \\ 0 & \text{if } 0 < \lambda - L \leqslant 1, \end{cases} \tag{22}$$

respectively.

Proof. Recall from [5, p. 656] and [12, pp. 165], that for all $\alpha > 0, \beta > 0$,

$$\begin{aligned} &\binom{n-k+k\alpha+\beta-1}{n-k} \\ &\quad = (-1)^{n-k} \binom{-(k\alpha+\beta)}{n-k} \\ &\quad = (-1)^{n-k} \frac{(-(k\alpha+\beta))(-(k\alpha+\beta+1))\cdots(-(k\alpha+\beta+n-k-1))}{1\cdot 2\cdots(n-k)} \\ &\quad = \frac{(k\alpha+\beta)(k\alpha+\beta+1)\cdots(k\alpha+\beta+n-k-1)}{1\cdot 2\cdots(n-k)}. \end{aligned}$$

Hence for $\beta = 1$, we have

$$\binom{n-k+k\alpha}{n-k} \geqslant 1.$$

Choosing $x = x_0, y = 0$, from Theorem 2,

$$\begin{aligned} |\varphi_{\mathrm{C}}(n, x_0)| &\geqslant |E_\alpha(\lambda - L, n)||x_0| \\ &\geqslant \sum_{k=0}^{n} (\lambda - L)^k |x_0|. \end{aligned}$$

It remains to verify, that

$$\lim_{n\to\infty} \frac{1}{n} \ln \sum_{k=n_0}^{n} q^n = \begin{cases} q & \text{if } q > 1, \\ 0 & \text{if } 0 < q \leqslant 1. \end{cases} \tag{23}$$

From the last two inequalities we obtain (21).

For the Riemann-Liouville case, with $n_0 := \left\lceil \dfrac{1-\alpha}{\alpha} \right\rceil$, we have $k\alpha + \alpha \geqslant 1$ for all $k \geqslant n_0$. As a consequence, for $n > n_0$,

$$\binom{n-k+k\alpha+\alpha-1}{n-k} < 1 \qquad (k \in \{0,1,\dots n_0-1\}),$$

and

$$\binom{n-k+k\alpha+\alpha-1}{n-k} \geqslant 1 \qquad (k \in \{n_0, n_0+1,\dots n\}).$$

Therefore

$$\begin{aligned} |\varphi_{\text{R-L}}(n,x_0)| &\geqslant |E_{\alpha,\alpha}(\lambda - L, n)||x_0| \\ &\geqslant \sum_{k=n_0}^{n} (\lambda - L)^k |x_0|. \end{aligned}$$

Combining the last inequality with (23), we obtain (22). □

4. Conclusions

We used the $\mathscr{Z}$-transform to establish variation of constant formulas for Caputo and Riemann-Liouville fractional difference equations. Using this formula we provided a lower bound for the norm of differences between two different solutions of a scalar Caputo or Riemann-Liouville time-varying linear equation. In particular, this result implies that the classical Lyapunov exponent is not an appropriate tool for stability analysis of fractional equations.

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