Voronoi cells via linear inequality systems

M.A. Goberna [1]

Dep. of Statistics and Operations Research, Universidad de Alicante, San Vicente del Raspeig, 03080, Spain; email: mgoberna@ua.es

M.M.L. Rodríguez

Dep. of Statistics and Operations Research, Universidad de Alicante, San Vicente del Raspeig, 03080, Spain; email: marga.rodriguez@ua.es

V.N. Vera de Serio

Faculty of Economics, I.C.B., Universidad Nacional de Cuyo, Centro Universitario-M5502JMA, Mendoza, Argentina; email: vvera@fcemail.uncu.edu.ar

Abstract

The theory and methods of linear algebra are a useful alternative to those of convex geometry in the framework of Voronoi cells and diagrams, which constitute basic tools of computational geometry. As shown by Voigt and Weis in 2010, the Voronoi cells of a given set of sites T, which provide a tesselation of the space called Voronoi diagram when T is finite, are solution sets of linear inequality systems indexed by T. This paper exploits systematically this fact in order to obtain geometrical information on Voronoi cells from sets associated with T (convex and conical hulls, tangent cones and the characteristic cones of their linear representations). The particular cases of T being a curve, a closed convex set and a discrete set are analyzed in detail. We also include conclusions on Voronoi diagrams of arbitrary sets.

Key words: Voronoi cells, Voronoi diagrams, linear inequality systems, locally polyhedral systems, quasipolyhedral sets.
Mathematics Subject Classification 2010: 51M20, 52C22, 15A39.

[1] Corresponding author; fax: 34 965903667; Tel: 34 965903533.

1 Introduction

Let T be a subset of $\mathbb{R}^n$, whose elements are called *Voronoi sites,* and containing at least two elements. Each site s has a *Voronoi cell* (also called a *Dirichlet-Voronoi cell*), $V_T(s)$, consisting of all points closer to s than to any other site. In formal terms, the *Voronoi cell* of $s \in T$ with respect to (w.r.t.) T is

$$V_T(s) := \{x \in \mathbb{R}^n : d(x,s) \leq d(x,t),\ t \in T\},$$

where d denotes the Euclidean distance on $\mathbb{R}^n$, and the *Voronoi diagram* of T is $\operatorname{Vor}(T) = \{V_T(t), t \in T\}$. We analyze the properties of $V_T(s)$, where $s \in T \subset \mathbb{R}^n$, exploiting the fact (already observed by I. Voigt and S. Weis, in [20]) that $V_T(s)$ is the solution set of a linear inequality system indexed by T. Indeed, since

$$d(x,s) \leq d(x,t) \Leftrightarrow \|x-s\|^2 \leq \|x-t\|^2, \text{ for all } t \in T,$$

after some algebra, we get that $V_T(s)$ is the solution set of

$$\upsilon_T(s) := \left\{(t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2}, t \in T\right\}. \tag{1}$$

This is an ordinary linear system when T is finite and a linear semi-infinite system otherwise. The first attempt to get geometrical information on the solution sets of linear semi-infinite systems (as their dimension or their facial structure) was due to U. Eckhardt ([4], [5]), whose general theory was revised and completed in [9], where different desirable properties were characterized for a particular class of linear systems. A series of works published in the 1990s analyze other types of systems with the same purpose (see, e.g., [10] and references therein). Moreover, the theory of linear semi-infinite systems is a well-known tool of optimization with infinitely many linear constraints; it applies in statistics ([2]), variational inequalities ([18]), convex geometry ([12]), and moment problems ([16]). This paper presents a new application to computational geometry. We characterize different geometrical and topological properties of $V_T(s)$ in terms of sets associated with s and T (convex and conical hulls, tangent cones and the characteristic cones of their linear representations), and we apply these properties to Voronoi diagrams of arbitrary sets.

Descartes and Dirichlet already used Voronoi diagrams with T finite, in $\mathbb{R}^2$ and $\mathbb{R}^3$, in 1644 and 1850, respectively, whereas Voronoi studied them, in arbitrary $\mathbb{R}^n$, in 1908. Voronoi diagrams w.r.t. finite sets, which are formed by polyhedral convex sets, are widely applied in computational geometry, operations research, data compression, economics, marketing, geophysics, meteorology, forest management, condensed matter physics, computational chemistry, robot navigation, etc. (see, e.g., [17]).

Voronoi cells and diagrams have been generalized in different ways, e.g., replacing: d with other distances in $\mathbb{R}^n$, the space $\mathbb{R}^n$ with different metric spaces, the set T with a family of pairwise disjoint subsets of $\mathbb{R}^n$, and the finite set $T \subset \mathbb{R}^n$ with infinite sets. In this paper we will consider the space $\mathbb{R}^n$ with the euclidean distance and T as possibly infinite, paying attention to the particular cases of T being a curve, a closed convex set and a discrete set.

The first paper on Voronoi diagrams for infinite sites in $\mathbb{R}^n$, oriented towards crystallography, was published by Delaunay, who asserted in 1934 ([3, General Lemma]) that Vor (T) is a tesselation of $\mathbb{R}^n$ formed by convex hulls (polytopes) of centers of closed balls with no points of T in their interiors (called *empty balls* by Delaunay), but having exactly $n+1$ points of T on their boundaries. The rigorous proof of this statement requires the assumption that T is a *Delaunay set*, i.e., that there exist scalars $\alpha > \beta > 0$ such that each closed ball of radius β contains at most one point of T and every closed ball of radius α contains at least one point of T (see [6] and [11] for more details). This definition is related with two well-known optimization problems arising in location decision making, where the elements of T and $\mathbb{R}^n$ are interpreted as *facilities* and *locations*, respectively): the *largest empty ball problem* consists of finding the worst location to access the nearest facility (or the best location, when dealing with obnoxious facilities), whereas the *bottleneck problem* consists of finding the minimum distance between pairs of facilities, in formal terms:

$$(P_1) \operatorname*{Max}_{x\in\mathbb{R}^n} \operatorname*{Min}_{t\in T} d(x,t) \quad \text{and} \quad (P_2) \operatorname*{Min}_{t_1\neq t_2\in T} d(t_1,t_2).$$

Geometrically, (P_1) consists of finding the biggest closed ball whose interior contains no point of T, and (P_1) is bounded (i.e., its optimal value, $v(P_1)$, is finite) if and only if there exits $\alpha > 0$ such that every closed ball of radius α contains at least one point of T. Alternatively, (P_2) is always a bounded problem and, if $v(P_2)$ is positive, then, taking $0 < \beta < \dfrac{v(P_2)}{2}$, any closed ball of radius β contains at most one element of T. Moreover, if T is closed and (P_1) is solvable, then the optimal solutions of (P_1) belong to the union of the boundaries of the Voronoi cells w.r.t. T, so it is important to identify the boundary of $V_T(t)$. In [19] T is assumed to be "discrete" in the sense that T has no accumulation points (e.g., any Delaunay set) and it is shown that $V_T(s)$ is a polyhedral convex set if and only if the convex conical hull of $T-s$ is finitely generated and, then, it is bounded (i.e., a polytope) if and only if s belongs to the interior of the convex hull of T. In [20] the same questions are considered for discrete and non-discrete sets. The next two examples show that Voronoi cells w.r.t. non-discrete sets arise in a natural way in practice.

Distance is obviously a most important issue for customers when choosing a service. When several suppliers provide similar services, a natural question arises: which service can each customer reach faster? Consider a service which can be delivered at any point of a given curve $T \subset \mathbb{R}^2$ (e.g., T could be formed by the networks of rivers or highways crossing certain region or country). Thus, a customer resident at $x \in \mathbb{R}^2$ will be served at the point $s \in T$ (e.g., he or she will take the water for irrigation or enter the highway at s) provided $d(x,s) \leq d(x,t)$ for all $t \in T$, i.e., when $x \in V_T(s)$.

On the other hand, in many real applications, there are finitely many sites, but some of them are uncertain. Consider, e.g., a bank that operates at present in a certain region through branches placed at the points $t_i, i = 1, ..., m$, which is planning to open a new branch at a point t_0 to be decided. A preliminary decision on the location of t_0 has been taken: it will be installed at some point of some (possibly infinite) set $T' \subset \mathbb{R}^2$, e.g., the city center of a certain town. The future Voronoi cell of $s \in \{t_1, ..., t_m\}$ obviously depends on the final location of the new service point t_0. In the spirit of robust (or pessimistic) optimization, we can define the *robust cell* of s as the set of points which will be served by s whichever is the final location of the new branch, i.e., the set $V_T(s)$, with $T = \{t_1, ..., t_m\} \cup T'$.

This paper is organized as follows: Section 2 provides a brief review of the theory of linear inequalities to be used later. Section 3 deals with a variant of the inverse problem tackled in [14], with $V_T(s)$ replacing $\operatorname{Vor}(T)$, showing that any closed convex set (any polyhedral convex set) containing s is the Voronoi cell w.r.t. a suitable closed set (finite set, respectively) $T \subset \mathbb{R}^n$ such that $s \in T$. Section 4 characterizes different geometrical and topological properties of $V_T(s)$ in terms of sets associated with s and T as the tangent cone of T at s and the characteristic cone of the linear representation of $V_T(s)$. Section 5 analyzes in detail the particular cases of T being a curve, a closed convex set and a discrete set. Specific results for the previous applications are Proposition 16 and Corollary 19. Finally, Section 6 draws some conclusions concerning Voronoi diagrams of arbitrary sets.

2 Preliminaries

Throughout the paper we use the following notation. The scalar product of $x, y \in \mathbb{R}^n$ is denoted either by $x'y$ or by $\langle x, y \rangle$, the Euclidean norm of x by $\|x\|$, the canonical basis by $\{e_1, ..., e_n\}$, the zero vector by 0_n, the open unit ball by B_n, and the unit sphere by $\mathbb{S}^{n-1}$.

Given $X \subset \mathbb{R}^n$, $\operatorname{int} X$, $\operatorname{cl} X$, $\operatorname{acc} X$, and $\operatorname{bd} X$ denote the *interior*, the *closure*, the set of *accumulation* points, and the *boundary* of X, respectively. We also denote by $\operatorname{conv} X$, and $\operatorname{cone} X = \mathbb{R}_+ \operatorname{conv} X$, the *convex hull* of X, and the *convex conical hull* of X, respectively. The *orthogonal complement* of a linear subspace X is $X^\perp := \{y \in \mathbb{R}^n : \langle x, y \rangle = 0, \ x \in X\}$. If X is a convex set,

$\dim X$, 0^+X, and $\operatorname{lin} X := (0^+X) \cap (-0^+X)$ denote the *dimension*, the *recession cone* and the *lineality space* of X, respectively. A convex cone is pointed when it contains no lines. If X is a convex cone, its corresponding *pointed cone* is $X \cap (\operatorname{lin} X)^{\perp}$ and its (negative) *polar cone* is $X^{\circ} := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 0,\ x \in X\}$. Given a convex set X and $x \in X$, the (convex) *cone of feasible directions* at x is

$$D(X; x) = \{y \in \mathbb{R}^n : \exists \mu > 0 \text{ such that } x + \mu y \in X\}$$

and the *normal cone* to X at x is $N_X(x) = D(X; x)^{\circ}$.

If $\emptyset \neq X \subset \mathbb{R}^n$ and $\operatorname{acc} X = \emptyset$, then X is closed, countable and any $x \in X$ is *isolated* in X, i.e., there exists $\rho > 0$ such that $X \cap (x + \rho B_n) = \{x\}$. In particular, given a nonempty closed set $X \subset \mathbb{R}^n$, $\operatorname{acc} X = \emptyset$ if and only if it is formed by isolated points.

In order to study the properties of Voronoi cells it is sufficient to consider closed sets of sites because $V_T(t) = V_{\operatorname{cl} T}(t)$ for all $t \in T$. Concerning Voronoi diagrams, given an arbitrary $T \subset \mathbb{R}^n$, we are interested in the Voronoi cells of sites on the boundary of T because $V_T(s) = \{s\}$ whenever $s \in \operatorname{int} T$. Moreover, we have $\operatorname{int} V_T(t_1) \cap \operatorname{int} V_T(t_2) = \emptyset$ if $t_1 \neq t_2$, $t_1, t_2 \in T$, because $d(x, t_1) = d(x, t_2)$ defines a hyperplane, but we can have $\bigcup_{t \in T} V_T(t) \neq \mathbb{R}^n$ when T is infinite (observe that $V_T(t) \cap T = \{t\}$ for all $t \in T$ and, in particular, $V_T(t) = \{t\}$ for all $t \in T$, when T is dense in an open set, so that $\bigcup_{t \in T} V_T(t) = T$ in that case). In fact, according to the next result, $\operatorname{Vor}(T)$ is a tesselation of $\mathbb{R}^n$ if and only if T is closed. Thus, $\operatorname{Vor}(T) \subset \operatorname{Vor}(\operatorname{cl} T)$ for all $T \subset \mathbb{R}^n$, with $\operatorname{Vor}(T) = \operatorname{Vor}(\operatorname{cl} T)$ if and only if T is closed.

Proposition 1 $\operatorname{Vor}(T)$ *is a tesselation of* $\mathbb{R}^n$ *if and only if* T *is closed.*

Proof: We need to show that $\bigcup_{t \in T} V_T(t) = \mathbb{R}^n$ if and only if $T \subset \mathbb{R}^n$ is closed. Suppose that $\bigcup_{t \in T} V_T(t) = \mathbb{R}^n$ and that $x \in \operatorname{cl} T$. Let $\{t_k\}_k$ be a sequence in T such that $t_k \to x$, i.e., $d(t_k, x) \to 0$. Now, $x \in V_T(s)$ for certain $s \in T$. Since $0 \leq d(s, x) \leq d(t_k, x)$ for all $k \in \mathbb{N}$, we have $d(s, x) = 0$, i.e., $x = s \in T$. So, T is closed.
Conversely, let $T \subset \mathbb{R}^n$ be a closed set, $x \in \mathbb{R}^n$, and $f(t) := d(x, t)$, $t \in T$. The closedness of T assures the existence of some $s \in T$ such that f attains its minimum at this s. Then $x \in V_T(s)$. Thus, $x \in \bigcup_{t \in T} V_T(t)$. $\square$

From the linear representation (1), $V_T(s)$ is a nonempty closed convex set, and it is a polyhedral convex set whenever T is finite. By using the notation

$T-s=\{t-s : t\in T\}$, in particular we have

$$\begin{aligned} V_{T-s}(0_n) &= \left\{x\in\mathbb{R}^n : (t-s)'x \leq \tfrac{\|t-s\|^2}{2},\ t\in T\right\} \\ &= \left\{x\in\mathbb{R}^n : (t-s)'(x+s) \leq \tfrac{\|t\|^2-\|s\|^2}{2},\ t\in T\right\}, \end{aligned}$$

because

$$\frac{\|t-s\|^2}{2} = \frac{\|t\|^2-\|s\|^2}{2} - (t-s)'s, \tag{2}$$

so that $x\in V_{T-s}(0_n)$ if and only if $x+s\in V_T(s)$. Hence,

$$V_T(s) = s + V_{T-s}(0_n). \tag{3}$$

We also associate with $s\in T$ the *cone of tangential directions* for T at s, denoted by $C_T(s)$, formed by those vectors $d\in\mathbb{R}^n$ for which there exist sequences $\{d_k\}\subset\mathbb{R}^n$ and $\{\alpha_k\}\subset\mathbb{R}_{++}$ such that $d_k\to d$, $\alpha_k\to 0$, and $t_k=s+\alpha_k d_k\in T$. Equivalently, $d\in C_T(s)$ if and only if there exist sequences $\{t_k\}\subset T$ and $\{\lambda_k\}\subset\mathbb{R}_{++}$ such that $t_k\to s$, $\lambda_k\to+\infty$, and $\lambda_k(t_k-s)\to d$. The cone $C_T(s)$ is closed ([13]) and $0_n\in C_T(s)\subset \operatorname{cl}\operatorname{cone}(T-s)$. Observe that $C_T(s)$ is possibly nonconvex (consider, e.g., $T=\{t\in\mathbb{R}^n : t_1...t_n=0\}$, for which $C_T(0_n)=T$), and

$$C_T(s) = C_{T-s}(0_n). \tag{4}$$

Now we recall some concepts and results about linear systems to be used in the next sections (the proofs can be found in [10] and references therein). Let $\sigma=\{a_t'x\leq b_t,\ t\in T\}$ be a linear system with solution set $F\subset\mathbb{R}^n$, $F\neq\emptyset$. The *characteristic cone* of σ is

$$\operatorname{cone}\{(a_t,b_t), t\in T; (0_n,1)\},$$

whose closure, denoted by $C(F)$, is the so-called *conic representation* of F in the sense of [7], that it is the closed convex cone

$$C(F) = \left\{(a,b)\in\mathbb{R}^{n+1} : a'x\leq b,\ x\in F\right\}.$$

In particular, the characteristic cone of $\upsilon_T(s)$ is

$$K_T(s) := \operatorname{cone}\left\{\left(t-s, \frac{\|t\|^2-\|s\|^2}{2}\right), t\in T; (0_n,1)\right\}.$$

Observe that this cone does not change if we remove from $\upsilon_T(s)$ the trivial inequality $0_n'x\leq 0$ whereas its closure, $\operatorname{cl}K_T(s)$, does not change if we replace, in $\upsilon_T(s)$, T by $\operatorname{cl}T$. Unfortunately, there is no simple counterpart of (3) and (4) for $K_T(s)$ and $K_{T-s}(0_n)$ (e.g., taking $T = \mathbb{R}$ and $s = 1$, $K_T(1) = \{x \in \mathbb{R}^2 : x_2 > x_1\} \cup \{0_2\}$ is not related by inclusion with $K_{T-1}(0) = (\mathbb{R} \times \mathbb{R}_{++}) \cup \{0_2\}$). Nonetheless it is easy to see that $K_T(s)$ is closed if and only if $K_{T-s}(0_n)$ is closed (result implicitly used in [20]). In fact we only need to observe that $a'x \leq b$ is a consequence of the system $\upsilon_T(s)$ if and only if $a'x \leq b - a's$ is a consequence of $\upsilon_{T-s}(0_n)$, or, equivalently, $(a,b) \in \operatorname{cl} K_T(s)$ if and only if $(a, b - a's) \in \operatorname{cl} K_{T-s}(0_n)$. Finally, some elementary algebra shows that $(a,b) \in K_T(s)$ if and only if $(a, b - a's) \in K_{T-s}(0_n)$, because of (2).

Clearly F is also the solution set of the system $\{a'x \leq b, (a,b) \in C(F)\}$. There exists a dual relationship between F and $C(F)$, e.g., $\dim F + \dim \operatorname{lin} C(F) = n$ and, given another nonempty closed convex set G, $F \subset G \Leftrightarrow C(G) \subset C(F)$. Consequently,

$$(0_n, 1) + \varepsilon \operatorname{cl} B_{n+1} \subset C(F) \Rightarrow F \subset \frac{1}{\varepsilon} \operatorname{cl} B_n \tag{5}$$

for all $\varepsilon \in \,]0,1[$. In fact, $\frac{1}{\varepsilon} \operatorname{cl} B_n$ is the solution set of $\left\{a'x \leq \frac{1}{\varepsilon}, a \in \mathbb{S}^{n-1}\right\}$, whose characteristic cone,

$$\operatorname{cone}\left\{\left(a, \frac{1}{\varepsilon}\right), a \in \mathbb{S}^{n-1}; (0_n, 1)\right\} = \operatorname{cone}\left\{(\varepsilon a, 1), a \in \mathbb{S}^{n-1}\right\},$$

is contained in $\operatorname{cone}((0_n, 1) + \varepsilon \operatorname{cl} B_{n+1}) \subset C(F)$. In that case, F is bounded, i.e., $0^+F = \{0_n\}$, where

$$0^+F = \{a_t'x \leq 0, \ t \in T\}. \tag{6}$$

Given $x \in F$, the *set of active indices* at x is $T(x) := \{t \in T : a_t'x = b_t\}$ and the *active cone* at x is

$$A(x) := \operatorname{cone}\{a_t, \ t \in T(x)\}. \tag{7}$$

It follows that $A(x) \subseteq D(F;x)^\circ$ for all $x \in F$. Finally, σ is *locally polyhedral* (*LOP*) if $A(x)^\circ = D(F;x)$ for all $x \in F$. In this particular case, given $\overline{x} \in F$, the normal cone to F at $\overline{x}$, $N_F(\overline{x})$, coincides with $D(F;\overline{x})^\circ = A(\overline{x})$ and this is a polyhedral convex cone ([10, Theorem 5.6 (i)]). Alternatively, when the characteristic cone of σ is closed, the equality $N_F(\overline{x}) = A(\overline{x})$ is also satisfied ([10, Theorem 5.3]). If σ is *LOP*, then F is *quasipolyhedral* (i.e., the intersection of F with any polytope is either empty or a polytope) and, conversely, σ is *LOP* when F is a full dimensional quasipolyhedral set and σ satisfies $\dim A(x) > 0$ for all $x \in \operatorname{bd} F$ ([10, Theorems 5.6(ii) and 5.5(ii), respectively]). Some authors call the quasipolyhedral sets generalized polyhedral sets (observe that the polyhedral convex sets are quasipolyhedral).

3 Computing the sites

In this section we consider the following inverse problem: given a nonempty closed convex set $F \subset \mathbb{R}^n$ and a point $s \in F$, find a set of sites $T \subset \mathbb{R}^n$, $s \in T$, such that $V_T(s) = F$. The constructive proof of the next result provides one solution of this inverse problem, showing that any closed convex set is the Voronoi cell of any of its points for some closed set of sites.

Theorem 2 *Let F be a closed convex set such that $s \in F \subset \mathbb{R}^n$. Then there exists a closed set $T \subset \mathbb{R}^n$, $s \in T$, such that $V_T(s) = F$.*

Proof: Let $s \in F \subset \mathbb{R}^n$; suppose that F is a closed convex set, and let $F' = F - s$. Then $0_n \in F'$ and F' is also a closed convex set. Let $C(F')$ be the conic representation of F'. Obviously, $b \geq 0$ for all $(a, b) \in C(F')$, hence $C(F') \subset \mathbb{R}^n \times \mathbb{R}_+$. Consider the closed set

$$T' := \left\{ t \in \mathbb{R}^n : \left(t, \frac{\|t\|^2}{2} \right) \in C(F') \right\}. \tag{8}$$

By the separation theorem, $V_{T'}(0_n) = F'$ if and only if their conic representations coincide, so we only need to show that

$$\operatorname{cl} \operatorname{cone} \left\{ \left(t, \frac{\|t\|^2}{2} \right), t \in T'; (0_n, 1) \right\} = C(F').$$

In fact, by definition of T', and because $(0_n, 1) \in C(F')$ and this is a closed convex cone, we have

$$\operatorname{cl} \operatorname{cone} \left\{ \left(t, \frac{\|t\|^2}{2} \right), t \in T'; (0_n, 1) \right\} = \operatorname{cl} \operatorname{cone} \{ C(F') \cup \{(0_n, 1)\} \} = C(F'),$$

so that $V_{T'}(0_n) = F'$. Finally, from (3),

$$F - s = F' = V_{T'}(0_n) = V_{T'+s}(s) - s,$$

so that the closed set $T := T' + s$ satisfies $V_T(s) = F$. $\square$

Example 3 *The closed convex set $F = \mathbb{R}^2_-$ is the Voronoi cell $V_{\mathbb{R}^2_+}(0_2)$ because its conic representation is $\mathbb{R}^3_+$. Observe that $\mathbb{R}^2_- = V_{\mathbb{R}^2_+ \setminus U}(0_2)$ for any open set $U \subset \mathbb{R}^2_+$, so that the closed set T in Theorem 2 is not unique. The next result shows that it is impossible to write $F = V_T(0_n)$ for some finite set T.*

Proposition 4 *Let $F \subset \mathbb{R}^n$ be a polyhedral convex set and let $s \in F$. Then there exists a finite set $T \subset \mathbb{R}^n$ such that $s \in T$ and $V_T(s) = F$ if and only if $s \in \operatorname{int} F$.*

Proof: As in the previous proof, we can consider without loss of generality (w.l.o.g.) that F is a polyhedral convex set with $s = 0_n \in F$. First, we assume that $0_n \in \operatorname{int} F$. Let $F = \{x \in \mathbb{R}^n : a_i' x \leq b_i, i = 1, .., m\}$, with $a_i \neq 0_n$, $i = 1, .., m$. We have $b_i > 0$, $i = 1, .., m$, because $0_n \in \operatorname{int} F$. Let $\lambda_i := \frac{2b_i}{\|a_i\|^2} > 0$, $i = 1, .., m$, and consider the finite set

$$T := \{\lambda_i a_i, i = 1, .., m; 0_n\}. \tag{9}$$

Since $\left(\lambda_i a_i, \frac{\|\lambda_i a_i\|^2}{2}\right) = \lambda_i (a_i, b_i)$, $i = 1, .., m$, the conic representations of $V_T(0_n)$ and F coincide, so that $V_T(0_n) = F$.
Conversely, assume that $F = V_T(0_n)$ for a finite set T such that $\{0_n\} \subsetneq T$. Let $T \setminus \{0_n\} = \{t_1, ..., t_m\}$. Then, $F = \left\{x \in \mathbb{R}^n : t_i' x \leq \frac{\|t_i\|^2}{2}, i = 1, .., m\right\}$ and $0_n \in \operatorname{int} F$ because $t_i' 0_n < \frac{\|t_i\|^2}{2}, i = 1, .., m$. $\square$

Observe that the set T in (8) is the orthogonal projection on $\mathbb{R}^n$ of the intersection of the conic representation of F, $C(F)$, with the paraboloid $x_{n+1} = \frac{1}{2}\|(x_1, ..., x_n)\|^2$ whereas the set in (9) is formed by 0_n together with the orthogonal projection on $\mathbb{R}^n$ of the intersections of the latter paraboloid with the rays $\operatorname{cone}\{(a_i, b_i)\}$, $i = 1, .., m$.

4 Global properties of Voronoi cells

In this section we consider an arbitrary set of sites $T \subset \mathbb{R}^n$ and, for a site s, its Voronoi cell $V_T(s)$, i.e., the solution set of the linear system $\upsilon_T(s)$ given in (1). The objective of this section is to get information on $V_T(s)$ from T or from sets associated with T as simple as possible (preferably not involving limits). Thus this section provides results involving T, $C_T(s)$, and $\operatorname{cl} K_T(s)$, in this order. The equivalence (i)$\Leftrightarrow$(iv) in the next proposition is [19, Theorem 3.2.8] when T is discrete and [20, Proposition 4.2] when T is arbitrary.

Proposition 5 *The following statements are equivalent to each other:*
(i) $V_T(s)$ *is bounded.*
(ii) $[\operatorname{cone}(T - s)]^\circ = \{0_n\}$.
(iii) $\operatorname{cone}(T - s) = \mathbb{R}^n$.
(iv) $s \in \operatorname{int} \operatorname{conv} T$.

Proof: From (6),

$$0^+ V_T(s) = \left\{x \in \mathbb{R}^n : (t - s)' x \leq 0, \; t \in T\right\} = [\operatorname{cone}(T - s)]^\circ,$$

so that (i)-(iv) are equivalent to $0^+ V_T(s) = \{0_n\}$. $\square$

Remark 6 *We consider now the problem of obtaining an upper bound for* $\|x - s\|$ *on a bounded Voronoi cell* $V_T(s)$. *According to (3), we can assume w.l.o.g. that* $s = 0_n$. *Since* $0_n \in \operatorname{int}\operatorname{conv} T$, *there exists a finite set* $\{t_1, ..., t_m\} \subset T$, *such that* $0_n \in \operatorname{int}\operatorname{conv}\{t_1, ..., t_m\}$. *One can get the desired bound as the maximum, say* ρ, *of the norm of the extreme points of*

$$G := \left\{ x \in \mathbb{R}^n : t_k{}'x \leq \frac{\|t_k\|^2}{2}, k = 1, ..., m \right\}$$

because $V_T(0_n) \subset G$ *(for large* m *one can appeal to a complete description method for polytopes). Alternatively, using Fourier elimination, it is possible to write the conic representation of* G *as follows:*

$$\begin{aligned} C(G) &= \operatorname{cone}\left\{ \left(t_k, \tfrac{\|t_k\|^2}{2} \right), k = 1, ..., m \right\} \\ &= \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} : c_i'x + d_i x_{n+1} \leq 0, i = 1, ..., p \right\}, \end{aligned} \tag{10}$$

for some $(c_i, d_i) \in \mathbb{R}^{n+1}$, $d_i < 0$, $i = 1, ..., p$, $p \in \mathbb{N}$. *Then the distance from* $(0_n, 1)$ *to the hyperplane* $H_i := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : c_i'x + d_i x_{n+1} = 0\}$ *is*

$$d\left((0_n, 1), H_i\right) = -\frac{d_i}{\sqrt{\|c_i\|^2 + d_i^2}}, i = 1, ..., p.$$

Thus, according to (5),

$$\rho = \left[\min \left\{ -\frac{d_i}{\sqrt{\|c_i\|^2 + d_i^2}}, i = 1, ..., p \right\} \right]^{-1}.$$

Example 7 *Let* $T = \left\{ \left(k - 1, 1 - \frac{1}{k}\right), \left(-k - 1, -1 + \frac{1}{k}\right), k \in \mathbb{N} \right\}$ *(set inspired in [20, Figure 6]). It is easy to see that* $0_2 \in \operatorname{int}\operatorname{conv}\{t_1, t_2, t_3\}$, *where* $t_1 = \left(2 - 1, 1 - \frac{1}{2}\right)$, $t_2 = \left(-2 - 1, -1 + \frac{1}{2}\right)$, *and* $t_3 = \left(10 - 1, 1 - \frac{1}{10}\right)$. *The extreme points of* G *are* $\left(-\frac{21}{8}, \frac{13}{2}\right)$, $\left(\frac{221}{40}, -\frac{49}{5}\right)$, *and* $\left(\frac{547}{40}, -\frac{913}{10}\right)$, *whose norm is* $\rho = 92.32$. *Alternatively, the inequalities in (10) are* $547x_1 - 3652x_2 - 40x_3 \leq 0$, $221x_1 - 392x_2 - 40x_3 \leq 0$, *and* $-21x_1 + 52x_2 - 8x_3 \leq 0$, *so that we get again* $\rho = \left(1.08314 \times 10^{-2}\right)^{-1} = 92.32$.

The following result is [20, Corollary 3.5] proved in a different way and it gives a necessary condition, in terms of T, for $V_T(s)$ to be a polyhedral convex set. This condition is of no use when $n = 2$ because any closed convex cone in $\mathbb{R}^2$ is polyhedral, i.e., finitely generated. Moreover, it is not sufficient because, by Proposition 5, $\operatorname{cone}(T - s) = \mathbb{R}^n$ for any bounded Voronoi cell, even in case that $V_T(s)$ is not a polytope.

Proposition 8 *If* $V_T(s)$ *is a polyhedral convex set, then* $\operatorname{cl}\operatorname{cone}(T - s)$ *is finitely generated.*

Proof: Let $V_T(s)$ be a polyhedral convex set. From (3), $V_{T-s}(0_n)$ is also a polyhedral convex set. Then, we can write

$$\operatorname{cl} K_{T-s}(0_n) = \operatorname{cone}\left\{\left(z^i, \alpha_i\right), i = 1, ..., m; (0_n, 1)\right\},$$

for some $z^i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}, i = 1, ..., m$ (see Table 1, Eq. 2 below). Hence, given $t \in T$, it holds that $\left(t - s, \frac{\|t-s\|^2}{2}\right) \in \operatorname{cl} K_{T-s}(0_n)$, consequently the orthogonal projection on the hyperplane $x_{n+1} = 0$ yields $t - s \in \operatorname{cone}\{z^i, i = 1, ..., m\}$. Thus, $T - s \subset \operatorname{cone}\{z^i, i = 1, ..., m\}$. Taking conical convex hulls and closures on the latter inclusion we get

$$\operatorname{cl}\operatorname{cone}(T - s) \subset \operatorname{cone}\left\{z^i, i = 1, ..., m\right\}. \tag{11}$$

On the other hand, given $i \in \{1, ..., m\}$, $(z^i, \alpha_i) \in \operatorname{cl} K_{T-s}(0_n)$ means that we can write

$$\left(z^i, \alpha_i\right) = \lim_k \left\{\sum_{t \in T_k} \lambda_t^k \left(t - s, \frac{\|t - s\|^2}{2}\right) + \mu^k (0_n, 1)\right\}, \tag{12}$$

for some $T_k \subset T$, T_k finite, $\lambda_t^k \in \mathbb{R}_+$ for all $t \in T_k$, and $\mu^k \in \mathbb{R}_+$, $k \in \mathbb{N}$. From (12) we get $z^i = \lim_k \left\{\sum_{t \in T_K} \lambda_t^k (t - s)\right\} \in \operatorname{cl}\operatorname{cone}(T - s)$. Hence, the reverse inclusion of (11) also holds, so that $\operatorname{cl}\operatorname{cone}(T - s)$ is finitely generated. $\square$

Example 9 *We cannot replace* $\operatorname{cl}\operatorname{cone}(T - s)$ *by just* $\operatorname{cone}(T - s)$ *in Proposition 8. In fact, consider the closed set*

$$T = \left\{(t_1, t_2) \in \mathbb{R}_+^2 : t_1^2 \leq t_2 \leq \sqrt{t_1}\right\}.$$

Then $V_T(0_2) = \mathbb{R}_-^2$ *is a polyhedral convex set although* $\operatorname{cone} T = \{0_2\} \cup \mathbb{R}_{++}^2$ *is not finitely generated. Nevertheless,* $\operatorname{cl}\operatorname{cone} T = \mathbb{R}_+^2$ *is finitely generated.*

The next two propositions give conditions, in terms of the tangential cone $C_T(s)$, for $V_T(s) = \{s\}$ and for $s \in \operatorname{int} V_T(s)$.

Proposition 10 *Let* $C_T(s)$ *be the tangential cone of* T *at* $s \in T$. *Then:*
(i) $V_T(s) \subset s + (\operatorname{cone} C_T(s))^\circ$.
(ii) If $C_T(s) \cap (-C_T(s)) \neq \{0_n\}$, *then* $\dim V_T(s) \leq n - 1$.
(iii) If $\operatorname{cone} C_T(s) = \mathbb{R}^n$, *then* $V_T(s) = \{s\}$.
(iv) If $V_T(s) = \{s\}$, *then there is no hyperplane supporting* T *at* s.

Proof. (i) Let $d \in C_T(s)$ and let $\{t_k\} \subset T$ and $\{\lambda_k\} \subset \mathbb{R}_{++}$ be such that $t_k \to s$, $\lambda_k \to +\infty$, and $\lambda_k (t_k - s) \to d$. We have $\|t_k - s\| \to 0$ and $\|\lambda_k (t_k - s)\| \to \|d\|$, so that $\lambda_k \|t_k - s\|^2 \to 0$.
Let $x \in V_T(s)$ and $k \in \mathbb{N}$. From (3), we have $x - s \in V_{T-s}(0_n)$. Then, $(t_k - s)'(x - s) \leq \frac{\|t_k - s\|^2}{2}$ and so $\lambda_k (t_k - s)'(x - s) \leq \frac{\lambda_k \|t_k - s\|^2}{2}$. Taking limits as $k \to \infty$ we get $d'(x - s) \leq 0$. Hence, $x - s \in (\operatorname{cone} C_T(s))^\circ$. Therefore

$V_T(s) \subset s + (\operatorname{cone} C_T(s))^{\circ}$.
(ii) Let $d \in C_T(s) \cap (-C_T(s))$, $d \neq 0_n$. Then, $\pm d \in C_T(s)$ and so

$$(\operatorname{cone} C_T(s))^{\circ} \subset \{x \in \mathbb{R}^n : d'x = 0\}.$$

By virtue of (i), we obtain $V_T(s) \subset s + \{x \in \mathbb{R}^n : d'x = 0\}$.
(iii) From (i) we get $V_T(s) \subset s + (\mathbb{R}^n)^{\circ} = \{s\}$.
(iv) Let $a \in \mathbb{R}^n \backslash \{0_n\}$ be such that $a'(t-s) \leq 0$ for all $t \in T$. Then $(t-s)' a \leq \frac{\|t-s\|^2}{2}$ for any $t \in T$, which yields $0_n \neq a \in V_{T-s}(0_n)$. Thus, by (3), we have $s \neq s + a \in V_T(s)$. $\square$

Remark 11 *When $C_T(s) \cap (-C_T(s)) \neq \{0_n\}$, it is easy to obtain a linear representation of $V_T(s)$ in $n-1$ variables. In fact, let $0_n \neq d \in C_T(s) \cap (-C_T(s))$. We can assume w.l.o.g. that $d_n \neq 0$. Since $V_T(s) - s \subset \{x \in \mathbb{R}^n : d'x = 0\}$, we have that $x \in V_T(s)$ if and only if $\sum_{i=1}^{n} d_i(x_i - s_i) = 0$ and $(x_1, .., x_{n-1})$ is a solution of the linear semi-infinite system*

$$\left\{ \sum_{i=1}^{n-1} \left[(t_i - s_i) - \frac{d_i}{d_n}(t_n - s_n) \right] (x_i - s_i) \leq \frac{1}{2} \sum_{i=1}^{n} (t_i - s_i)^2, \; t = (t_1, \ldots, t_n) \in T \right\}.$$

The next examples show that the inclusion in the statement (i) of Proposition 10 can be strict whereas the converse statements of (iii) and (iv) can fail.

Example 12 *Let $T = \left\{ t \in \mathbb{R}^2 : |t_2| \leq |t_1|^{\frac{3}{2}} \right\}$. Since the derivative at 0 of the function $h(x) := |x|^{\frac{3}{2}}$ is 0, the tangential directions for T at 0_2 are the multiples of $(1,0)$, i.e., $C_T(0_2) = \operatorname{span}\{(1,0)\}$. Thus, $\operatorname{cone} C_T(0_2) = C_T(0_2)$ is a proper linear subspace of $\mathbb{R}^2$. On the other hand, given $b > 0$, the distance from $(0,b)$ to T is less than its distance to 0_2. In fact, $\{(x, h(x)) \in \mathbb{R}^2 : 0 < |x| < r\} \subset (0,b) + bB_2$, where $r < b$ is the unique positive root of $x^6 + 2x^5 + x^4 - 4b^2x^3 = 0$ (equivalently $x(x+1)^2 - 4b^2 = 0$). Thus, any point of the line $x_1 = 0$ is closer to the graph of h than it is to 0_2, and the same is true for any point in the epigraph of h different of 0_2. Hence, $\operatorname{cone} C_T(0_2) = \operatorname{span}\{(1,0)\}$ and $V_T(0_2) = \{0_2\}$, so that $V_T(0_2) \subsetneq (\operatorname{cone} C_T(0_2))^{\circ} = \operatorname{span}\{(0,1)\}$, and $\operatorname{cone} C_T(0_2) \neq \mathbb{R}^2$. This implies that there is no hyperplane supporting T at 0_2.*

Example 13 *If $T = \{0_n, \pm e_1, ..., \pm e_n\}$, then there is no hyperplane supporting T at 0_n, but $V_T(0_n) \neq \{0_n\}$. In this case, $C_T(0_n) = \{0_n\}$.*

Proposition 14 *The following statements are equivalent:*
(i) $s \in \operatorname{int} V_T(s)$.
(ii) s is an isolated point of T.
(iii) $C_T(s) = \{0_n\}$.

Proof: [(i)⇒(ii)] Assume that $s \in \operatorname{int} V_T(s)$. Then there exists $\varepsilon > 0$ such that $s + \varepsilon B_n \subset V_T(s)$. Since $V_T(s) \cap T = \{s\}$, $(s + \varepsilon B_n) \cap (T \setminus \{s\}) = \emptyset$ and s is an isolated point of T.
[(ii)⇒(i)] Assume that s is an isolated point of T and let $(s + \varepsilon B_n) \cap (T \setminus \{s\}) = \emptyset$, with $\varepsilon > 0$. Then $s + \frac{\varepsilon}{2} B_n \subset V_T(s)$.
[(ii)⇒(iii)] It is trivial.
[(iii)⇒(ii)] If s is not an isolated point of T, there exists a sequence $\{t_k\} \subset T \setminus \{s\}$ such that $t_k \to s$. Assuming that $\dfrac{(t_k - s)}{\|t_k - s\|} \to d$, we have $d \in C_T(s)$, with $\|d\| = 1$. $\square$

Table 1 characterizes some desirable geometric properties of $V_T(s)$ in terms of the corresponding properties of $\operatorname{cl} K_T(s)$ (the proofs can be found in [10, Theorem 5.13], [9], [8], and [7, Proposition 18]). "Eq." stands for "Equivalence Property". Observe that the only set associated with T providing a characterization of $V_T(s) = \{s\}$ is $\operatorname{cl} K_T(s)$ (see Eq. 1).

Eq.	$V_T(s)$	$\operatorname{cl} K_T(s)$
1	$\{s\}$	halfspace
2	polyhedral convex set	polyhedral
3	linear subspace	its pointed cone is cone $\{(0_n, 1)\}$
4	full-dimensional closed convex set	pointed cone
5	compact convex set	$(0_n, 1)$ in its interior
6	polytope	polyhedral and $(0_n, 1)$ in its interior
7	sum of compact convex set with linear subspace	$(0_n, 1)$ in its relative interior
8	sum of compact convex set with closed convex cone	there exist two closed convex cones $C \subset \mathbb{R}^{n+1}$ and $L \subset \mathbb{R}^n$ such that $K_T(s) = C \cap (L \times \mathbb{R})$, $(0_n, -1) \notin C$ and $(0_n, 1) \in \operatorname{int} C$
9	$k-$simplex, i.e., the convex hull of $k+1$ affinely independent points	$(0_n, 1) \in \operatorname{int} K_T(s)$, $\dim \operatorname{lin} \operatorname{cl} K_T(s) = n - k$, and the pointed cone of $\operatorname{cl} K_T(s)$ has $k+1$ extreme rays
10	$k-$sandwich, i.e., the convex hull of the union of a pair of parallel affine manifolds of dimension $k-1$	$\operatorname{cl} K_T(s) = C + W$, where C is a pointed closed convex cone and W is a linear subspace such that $\dim C = 2$, $\dim W = n - k$, $(0_n, 1) \in \operatorname{rint} C$, and $C \cap (W + \operatorname{span}\{(0_n, 1)\}) = \operatorname{cone}\{(0_n, 1)\}$
11	parallelotope, i.e., the sum of n segments whose directions form a basis of $\mathbb{R}^n$	$\operatorname{cl} K_T(s) = \operatorname{cone}\left\{\binom{a_i}{\alpha_i}, i \in I; -\binom{a_i}{\beta_i}, i \in I; \binom{0_n}{1}\right\}$, with $I = \{1, \ldots, n\}$, $\{a_i,\ i \in I\}$ linearly independent, and $\alpha_i < \beta_i$ for $i \in I$

Table 1

The expression of $\operatorname{cl} K_T(s)$, $s \in T$,

$$\operatorname{cl} K_T(s) = \operatorname{cl} \operatorname{cone}\left\{\left(t - s, \frac{\|t\|^2 - \|s\|^2}{2}\right) : t \in T;\ (0_n, 1)\right\},$$

can be simplified by elimination of either the generator $(0_n, 1)$ or the operator "cl" (observe that, when $K_T(s)$ is closed, $y \in \operatorname{bd} V_T(s)$ if and only if there exists $t \in T \setminus \{s\}$ such that $d(y, t) = d(y, s)$ by [9, Theorem 4.1]). The next result shows that these eliminations are possible when T is unbounded and when $T \setminus \{s\}$ is closed, respectively.

Proposition 15 *Given $s \in T$, the following statements are true:*

(i) If T is unbounded, then

$$\operatorname{cl} K_T(s) = \operatorname{cl}\operatorname{cone}\left\{\left(t-s, \frac{\|t\|^2 - \|s\|^2}{2}\right) : t \in T\right\}.$$

(ii) If $K_T(s)$ is closed, then s is isolated in T. The converse statement holds when T is closed.

Proof. We denote $T' := T \setminus \{s\}$.

(i) Let $\{t_k\} \subset T'$ be such that $\|t_k\| \to +\infty$. Then,

$$(0_n, 1) \in \operatorname{cl}\operatorname{cone}\left\{\left(t-s, \frac{\|t\|^2 - \|s\|^2}{2}\right) : t \in T\right\}$$

because $2\|t_k - s\|^{-2}\left(t_k - s, \frac{\|t_k\|^2 - \|s\|^2}{2}\right) \to (0_n, 1)$.

(ii) Assume that $K_T(s)$ is closed. By [10, Theorem 5.3], this condition guarantees the equality between the normal cone and the active cone at any point. Then, since the unique active index of $\upsilon_T(s)$ at s is the same s, the cone of feasible directions $D(V_T(s); s)$ satisfies

$$\operatorname{cl} D(V_T(s); s) = D(V_T(s); s)^{\circ\circ} = A(s)^{\circ} = \{0_n\}^{\circ} = \mathbb{R}^n.$$

Hence $D(V_T(s); s) = \mathbb{R}^n$ and, so, $s \in \operatorname{int} V_T(s)$. The conclusion follows from Proposition 14.
For the converse, recall that $K_T(s)$ is closed if and only if $K_{T-s}(0_n)$ is closed. Since s is an isolated point of the set T if and only if 0_n is an isolated point of the set $T-s$, and this last set is closed whenever T is closed, we will assume w.l.o.g. that $s = 0_n$. Hence we only need to show that the cone

$$\begin{aligned} K &= \operatorname{cone}\left\{\left(t, \frac{\|t\|^2}{2}\right),\ t \in T'; (0_n, 1)\right\} \\ &= \operatorname{cone}\left\{\frac{2}{\|t\|^2}\left(t, \frac{\|t\|^2}{2}\right),\ t \in T'; (0_n, 1)\right\} \subset \mathbb{R}^{n+1} \end{aligned}$$

is closed. Let

$$B := \left\{\frac{2}{\|t\|^2}\left(t, \frac{\|t\|^2}{2}\right),\ t \in T'; (0_n, 1)\right\},$$

a subset of the hyperplane $x_{n+1} = 1$. B is bounded because 0_n is isolated in T, and clearly $0_{n+1} \notin \operatorname{conv} B$. Proving that B is closed, we can conclude that $K = \mathbb{R}_+ \operatorname{conv} B$ is closed as well. Let $\{t_k\}$ be any sequence in T' such that $\left\{\frac{2}{\|t_k\|^2}\left(t_k, \frac{\|t_k\|^2}{2}\right)\right\}$ converges to some (z, z_{n+1}) in $\mathbb{R}^{n+1}$. It follows immediately that $z_{n+1} = 1$. If $\{t_k\}$ is unbounded, then $z = 0_n$ and so $(z, z_{n+1}) = (0_n, 1) \in B$. In the case that $\{t_k\}$ is bounded, we may assume w.l.o.g. that it converges to some non null $\bar{t} \in T$, because T is closed and 0_n is isolated in T. Then

$$(z, z_{n+1}) = \lim_{k\to\infty} \frac{2}{\|t_k\|^2}\left(t_k, \frac{\|t_k\|^2}{2}\right) = \frac{2}{\|\bar{t}\|^2}\left(\bar{t}, \frac{\|\bar{t}\|^2}{2}\right) \in B.$$

Therefore B is closed, which completes the proof. □

Taking $n = 1$, $T = \{0\} \cup]1, +\infty[$ and $s = 0$ (isolated in T), we have a non closed characteristic cone $K_T(s)$.

Table 2 illustrates each type of Voronoi cell listed in Table 1. In all cases $s = 0_n$.

Eq.	T	$V_T(0_n) = \left\{x \in \mathbb{R}^n : t'x \leq \frac{\|t\|^2}{2},\ t \in T\right\}$
1	$\mathbb{R}^2$	$\{0_2\}$
2	$\mathbb{R}^2_+$	$\mathbb{R}^2_-$
3	$\operatorname{span}\{(1,0)\}$	$\{x \in \mathbb{R}^2 : x_1 = 0\}$
4	$\mathbb{R}^2_+$	$\mathbb{R}^2_-$
5	$\{0_2\} \cup \{t \in \mathbb{R}^2 : \|t\| = 1\}$	$\frac{1}{2} \operatorname{cl} B_2$
6	$\{0_2\} \cup (\{-1, 1\} \times (\mathbb{Z} \setminus \{0\}))$	$\operatorname{conv}\{(\pm 1, 0), (0, \pm 1)\}$
7	$\{0_3\} \cup \{(t_1, t_2, 0) \in \mathbb{R}^3 : \|(t_1, t_2)\| = 1\}$	$\frac{1}{2} \operatorname{cl} B_2 \times \{0\} + \operatorname{span}\{(0_2, 1)\}$
8	$\{0_2\} \cup (\{\pm 1\} \times \mathbb{R}_+)$	$\{x \in \mathbb{R}^2 : 2\|x_1\| + x_2^2 \leq 1\} + \operatorname{cone}\{(0, -1)\}$
9	$\{0_2, (0, -1)\} \cup \{(2z + 1, 1), z \in \mathbb{Z}\}$	$\operatorname{conv}\left\{\left(\pm\frac{3}{2}, -\frac{1}{2}\right), (0, 1)\right\}$
10	$\mathbb{Z} \times \{0\}$	$\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{R}$
11	$\{0_2\} \cup (\{-1, 1\} \times (\mathbb{Z} \setminus \{0\}))$	$\operatorname{conv}\{(\pm 1, 0), (0, \pm 1)\}$

Table 2

5 Voronoi cells of special sets

In this section we consider the special cases in which the set of sites T is a curve, or a closed convex set, or a discrete set.

Proposition 16 *Let $T = f(I)$, where I is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}^n$ is a nonconstant continuous function. The following statements hold for $s \in T$:*
(i) $s \in \operatorname{bd} V_T(s)$, $C_T(s) \neq \{0_n\}$, and $K_T(s)$ is nonclosed.
(ii) If f is differentiable at $z_0 \in \operatorname{int} I$, with $f(z_0) = s$, then $\operatorname{span}\left\{\left(\frac{df}{dz}\right)_{z_0}\right\} \subset C_T(s)$ and

$$V_T(s) \subset \left\{x \in \mathbb{R}^n : \left(\frac{df}{dz}\right)_{z_0}(x - s) = 0\right\}. \tag{13}$$

Thus, $\dim V_T(s) \leq n-1$ whenever $\left(\frac{df}{dz}\right)_{z_0} \neq 0_n$.
(iii) If I is compact, and f is differentiable at $z_0 \in \operatorname{int} I$ with $f^{-1}(s) = \{z_0\}$ and $\left(\frac{df}{dz}\right)_{z_0} \neq 0_n$, then

$$C_T(s) = \operatorname{span}\left\{\left(\frac{df}{dz}\right)_{z_0}\right\}.$$

Proof: (i) Let $z_0 \in I$ be such that $f(z_0) = s$. By the continuity of f at z_0, $s \in \operatorname{acc} T$ (we are assuming that T is not a singleton set). Then, Proposition 14 yields $s \in \operatorname{bd} V_T(s)$ and $C_T(s) \neq \{0_n\}$, whereas Proposition 15(ii) shows that $K_T(s)$ is nonclosed.

(ii) Take a sequence $\{z_k\}_{k\in\mathbb{N}} \subset I$ such that $z_k > z_0$ for all $k \in \mathbb{N}$ and $z_k \to z_0$. Let $t_k := f(z_k) \in T$ and $\lambda_k := \frac{1}{z_k - z_0} > 0$, $k \in \mathbb{N}$. By the differentiability assumption,

$$\left(\frac{df}{dz}\right)_{z_0} = \lim_k \frac{f(z_k) - f(z_0)}{z_k - z_0} = \lim_k \left(\lambda_k (t_k - s)\right),$$

so that $\left(\frac{df}{dz}\right)_{z_0} \in C_T(s)$. The proof of $-\left(\frac{df}{dz}\right)_{z_0} \in C_T(s)$ is similar, just taking $\{z_k\}_{k\in\mathbb{N}} \subset I$ such that $z_k < z_0$ for all $k \in \mathbb{N}$. So, $\operatorname{span}\left\{\left(\frac{df}{dz}\right)_{z_0}\right\} \subset C_T(s)$ and, by statement (i) in Proposition 10, we obtain (13).

(iii) We must show that $C_T(s) \subset \operatorname{span}\left\{\left(\frac{df}{dz}\right)_{z_0}\right\}$. Let $0_n \neq d = \lim_k \left(\lambda_k (t_k - s)\right)$, with $\{t_k\} \subset T$ such that $t_k \to s$ and $\{\lambda_k\} \subset \mathbb{R}_{++}$ such that $\lambda_k \to +\infty$. Let $\{z_k\} \subset I$ be such that $f(z_k) = t_k$ for all $k \in \mathbb{N}$. By the compactness of I, we can assume w.l.o.g. that $\{z_k\}$ is convergent to some $\overline{z} \in I$. The continuity of f yields $\overline{z} \in f^{-1}(s) = \{z_0\}$ and so $\overline{z} = z_0$. Thus, $z_k \to z_0$.
If $z_k = z_0$ for sufficiently large k, then $\lambda_k (t_k - s) = 0_n$ for sufficiently large k, in contradiction with $\lim_k \left(\lambda_k (t_k - s)\right) = d \neq 0_n$. Thus there exists a subsequence $\{z_{k_r}\}$ such that $z_{k_r} \neq z_0$ for all k. We can assume w.l.o.g. that $z_k > z_0$ for all $k \in \mathbb{N}$. Since $\lim_k \left(\lambda_k (t_k - s)\right) = d \neq 0_n$ and

$$\left(\frac{df}{dz}\right)_{z_0} = \lim_k \frac{f(z_k) - f(z_0)}{z_k - z_0} = \lim_k \frac{\lambda_k (t_k - s)}{\lambda_k (z_k - z_0)},$$

there exists $\alpha \in \mathbb{R}_{++}$ such that $\lim_k \lambda_k (z_k - z_0) = \alpha$. Then,

$$d = \alpha \left(\frac{df}{dz}\right)_{z_0} \in \operatorname{span}\left\{\left(\frac{df}{dz}\right)_{z_0}\right\}. \qquad \square$$

Observe that, under the assumptions of Proposition 16(ii), $n = 2$ implies $\dim V_T(s) \leq 1$, and so $V_T(s)$ is a polyhedral convex set. The existence of singularities for the algebraic curves, where $C_T(s)$ is the union of at least two lines, shows the necessity of the assumption $f^{-1}(s) = \{z_0\}$ in statement (iii). The next example shows that the assumption $\left(\frac{df}{dz}\right)_{z_0} \neq 0_n$ is not superfluous in (ii) and (iii).

Example 17 *Let $n = 2$, $I = [-1, 1]$, $m \geq 2$, and $f(z) = (-z^{2m}, 0)$ if $-1 \leq z \leq 0$ and $f(z) = (0, -z^{2m})$ if $0 < z \leq 1$. Then $f \in \mathcal{C}^{2m-1}([-1, 1])$, $T = [0_2, -e_1] \cup [0_2, -e_2]$, $f^{-1}(0_2) = 0$, but $\dim V_T(0_2) = \dim \mathbb{R}^2_+ = 2$ and*

$$C_T(0_n) = \mathbb{R}_+ \{-e_1, -e_2\} \nsubseteq \operatorname{span}\left\{\left(\frac{df}{dz}\right)_0\right\} = \{0_2\}.$$

Now, we consider the case in which T is a closed convex set. In this event, given $s \in T$, $\operatorname{cone}(T - s) = D(T; s)$, whose closure and polar cone are $C_T(s)$ and $N_T(s)$, respectively (all these cones being finitely generated when T is quasipolyhedral).

In general, given an arbitrary set T, we may consider the associated set-valued mapping that assigns to each element of the Euclidean space the set of the nearest points in T. In this way, the Voronoi cells are the preimages of the elements of T by this metric projection. Moreover, when T is a non empty closed convex set, for any $x \in \mathbb{R}^n$, there exists a unique site $s \in T$ such that

$$d(x, s) = \inf_{t \in T} d(x, t),$$

and, therefore, $\operatorname{Vor} T$ is a partition of $\mathbb{R}^n$.

Proposition 18 *If T is a convex set and $s \in T$, then $V_T(s) = s + N_T(s)$. In particular, if T is an affine manifold, then $V_T(s) = s + (T - s)^{\perp}$.*

Proof. We can assume w.l.o.g. $s \in \operatorname{bd} T$ (otherwise, $V_T(s) = \{s\}$ and $N_T(s) = \{0_n\}$). Recall that $N_T(s) := \left\{x \in \mathbb{R}^n : (t - s)' x \leq 0, t \in T\right\}$.
In view of (3), $V_T(s) - s = V_{T-s}(0_n)$, so it is enough to show that $N_T(s) = V_{T-s}(0_n)$. The inclusion $N_T(s) \subset V_{T-s}(0_n)$ is trivial because $(t - s)' x \leq 0 \leq \frac{\|t-s\|^2}{2}$ for all $t \in T$. For the reverse inclusion, take $x \in V_{T-s}(0_n)$ and an arbitrary $t \in T$. By convexity of T, $s + \lambda(t - s) \in T$ for all $\lambda \in]0, 1[$. Then $[\lambda(t - s)]' x \leq \frac{\|\lambda(t-s)\|^2}{2}$ for all $\lambda \in]0, 1[$. Dividing by λ and taking

limits as $\lambda \searrow 0$ we get $(t-s)' x \leq 0$ for all $t \in T$, i.e., $x \in N_T(s)$. Thus, $V_T(s) = s + N_T(s)$. Moreover, $N_T(s) = (T-s)^{\perp}$ whenever T is an affine manifold. $\square$

Corollary 19 *Let D be a closed convex set and $p \notin D$. Let $T = D \cup \{p\}$ and $E = \{t \in D :]p,t[\cap D = \emptyset\} \cup \{p\}$. Then,*

$$V_T(s) = \begin{cases} s + N_D(s), & \text{if } s \notin E, \\ V_E(p), & \text{if } s = p, \\ (s + N_D(s)) \setminus \operatorname{int} V_E(p), & \text{if } s \in E \setminus \{p\}. \end{cases}$$

Proof. Let $s \notin E$ and $\lambda \in]0,1[$ such that $q = s + \lambda(p-s) \in D$. Since $\lambda(p-s)' x = (q-s)' x \leq \frac{\|q-s\|^2}{2}$ for all $x \in V_{D-s}(0_n)$, we have

$$(p-s)' x \leq \frac{\|q-s\|^2}{2\lambda} = \frac{\lambda \|p-s\|^2}{2} < \frac{\|p-s\|^2}{2} \tag{14}$$

for all $x \in V_{D-s}(0_n)$. Hence, $V_{T-s}(0_n) = V_{D-s}(0_n)$. We obtain the conclusion applying (3) and Proposition 18.

Now, take $s = p \in E$ and let $\bar{t} \in D \setminus E$. Then, $]p, \bar{t}[\cap D \neq \emptyset$ and, since D is a closed convex set and $p \notin D$, there exists $\bar{q} \in]p, \bar{t}[\cap E$. Let $\lambda \in]0,1[$ such that $\bar{q} = p + \lambda(\bar{t} - p)$. Then, for all $x \in V_{E-p}(0_n)$, we have that $\lambda(\bar{t}-p)' x = (\bar{q}-p)' x \leq \frac{\|\bar{q}-p\|^2}{2}$ and, as in (14), $(\bar{t}-p)' x < \frac{\|\bar{t}-p\|^2}{2}$. Therefore, every $x \in V_{E-p}(0_n)$ satisfies $(t-p)' x \leq \frac{\|t-p\|^2}{2}$ for all $t \in D \setminus E$, so $V_{E-p}(0_n) = V_{T-p}(0_n)$.

Finally, for the last case, we consider the interior of $V_T(p) = V_E(p)$. Since D is a closed convex set and $p \notin D$, p is an isolated point of the closed set T and, by Proposition 15(ii), $K_T(p)$ is closed. By the characterization of $\operatorname{bd} V_T(p)$ when $K_T(p)$ is closed, we have that

$$\operatorname{int} V_T(p) = \left\{ x \in \mathbb{R}^n : (t-p)' x < \frac{\|t\|^2 - \|p\|^2}{2}, \, t \in T \setminus \{p\} \right\}. \tag{15}$$

Now, for any $s \in E \setminus \{p\} \subset D$. As $D \subset T$, it is obvious that $V_T(s) \subset V_D(s)$. Moreover, every $x \in V_T(s)$ satisfies the inequality $(s-p)' x \geq \frac{\|s\|^2 - \|p\|^2}{2}$ and, taking into account (15), $x \notin \operatorname{int} V_T(p)$. Conversely, if $x \in V_D(s) \setminus \operatorname{int} V_T(p)$, then there exists $q \in T \setminus \{p\} = D$ such that $(p-q)' x \leq \frac{\|p\|^2 - \|q\|^2}{2}$

and $(q-s)' x \leq \dfrac{\|q\|^2 - \|s\|^2}{2}$. Adding both inequalities, we obtain $(p-s)' x \leq \dfrac{\|p\|^2 - \|s\|^2}{2}$, so $x \in V_T(s)$. $\square$

Corollary 20 *If T is quasipolyhedral and $s \in T$, then $V_T(s) - s$ is a finitely generated cone.*

Proof. Since T is a closed convex set, $V_T(s) = s + N_T(s)$. Moreover, by [1, Corollary 4.1], T admits a LOP representation σ and $N_T(s)$ coincides with $D(T;s)^\circ = A(s)$ that is a polyhedral convex cone ($A(s)$ denotes here the active cone at s for σ). $\square$

Finally, we analyze the special case of $\operatorname{acc} T = \emptyset$. First we will find the accumulation points of the set A, which is the projection onto $\mathbb{S}^n$ of the non-null vectors of the coefficients of the system $\upsilon_{T-s}(0_n)$,

$$A := \left\{ \left\| \left(t-s, \frac{\|t-s\|^2}{2} \right) \right\|^{-1} \left(t-s, \frac{\|t-s\|^2}{2} \right) : t \in T \setminus \{s\} \right\}. \tag{16}$$

Lemma 21 *If* $\operatorname{acc} T = \emptyset$, *then* $\operatorname{acc} A \subset \{(0_n, 1)\}$. *Moreover, the system $\upsilon_{T-s}(0_n)$ is LOP.*

Proof: Assume that A is infinite. The compactness of $\mathbb{S}^n$ implies $\operatorname{acc} A \neq \emptyset$. Suppose that $(a,b) \in \operatorname{acc} A$, and take a sequence

$$\left\{ \left\| \left(t_k - s, \frac{\|t_k - s\|^2}{2} \right) \right\|^{-1} \left(t_k - s, \frac{\|t_k - s\|^2}{2} \right) \right\}$$

of infinite distinct points in A converging to (a,b). Then the t_k's are all different and we may assume w.l.o.g. that $\|t_k - s\| \to \infty$, because $\operatorname{acc} T = \emptyset$. Thus

$$\left\| \left(t_k - s, \frac{\|t_k - s\|^2}{2} \right) \right\|^{-1} (t_k - s) = \frac{2}{\left(4 + \|t_k - s\|^2\right)^{1/2}} \frac{t_k - s}{\|t_k - s\|} \to 0_n,$$

and

$$\left\| \left(t_k - s, \frac{\|t_k - s\|^2}{2} \right) \right\|^{-1} \frac{\|t_k - s\|^2}{2} = \frac{\|t_k - s\|}{\left(4 + \|t_k - s\|^2\right)^{1/2}} \to 1,$$

so that $(a,b) = (0_n, 1)$. Finally, we can conclude that $\upsilon_{T-s}(0_n)$ is *LOP* by [1, Theorem 2.2], because $\upsilon_{T-s}(0_n)$ is consistent while $\upsilon_{T-s}(0_n) \cup \{0_n' x = 1\}$ is not. $\square$

Statements (ii), (iii), and (iv) in Theorem 22 below are [11, Proposition 32.1], [19, Theorems 3.4.9 and 3.4.15] (or [20, Section 3]) and [19, Theorem 3.2.8] (or [20, Proposition 4.2]), respectively. Almost all the proofs are included for two reasons: the sake of completeness and the simplicity of their arguments, which are straightforward consequences of previous results in this paper.

Theorem 22 *Let T be such that* $\operatorname{acc} T = \emptyset$ *and* $s \in T$*. Then the following statements hold:*
(i) $K_T(s)$ *is closed.*
(ii) $V_T(s)$ *is a quasipolyhedral set.*
(iii) $V_T(s)$ *is a polyhedral convex set if and only if* $\operatorname{cone}(T - s)$ *is finitely generated.*
(iv) $V_T(s)$ *is a polytope if and only if* $s \in \operatorname{int}\operatorname{conv} T$.

Proof: Assume that $\operatorname{acc} T = \emptyset$.
(i) $\operatorname{acc} T = \emptyset$ implies that s is isolated in T and that T is closed. Hence Proposition 15 (ii) applies to give the closedness of the cone $K_T(s)$.
(ii) By Lemma 21, $\upsilon_{T-s}(0_n)$ is LOP, which implies that $V_{T-s}(0_n)$ is a quasipolyhedral set, and, by (3), $V_T(s)$ is also quasipolyhedral.
(iii) By (i), the cone $K_{T-s}(0_n)$ is closed (because $\operatorname{acc}(T - s) = \emptyset$), so if $V_T(s)$ is a polyhedral convex set then the proof of Proposition 8 can be modified taking into account that $(z^i, \alpha_i) \in K_{T-s}(0_n)$ (which implies that $z^i \in \operatorname{cone}(T - s)$) to conclude that $\operatorname{cone}(T - s)$ is finitely generated. The converse is [20, Corollary 3.3] (also Theorem 3.4.15 in [19]).
(iv) If $V_T(s)$ is a polytope, then $s \in \operatorname{int}\operatorname{conv} T$ by Proposition 5. Conversely, if $s \in \operatorname{int}\operatorname{conv} T$, then $V_T(s)$ is bounded. Since by (ii), it is quasipolyhedral, we conclude that $V_T(s)$ is a polytope. $\square$

Marchi et al. ([15]) called *p-systems* those consistent systems that satisfy the following property: if the set A in (16) is the projection onto the unit sphere $\mathbb{S}^n$ of the non-null vectors of the coefficients of the system, then $a'x < b$ for all $(a, b) \in \operatorname{acc} A$, and for all feasible points x. By Lemma 21, $\upsilon_{T-s}(0_n)$ satisfies this condition whenever $\operatorname{acc} T = \emptyset$. In that case, the constructive proof of the next result shows the way to obtain a finite subsystem of $\upsilon_T(s)$ representing $V_T(s)$ whenever it is a polytope.

Theorem 23 *Let $T \subset \mathbb{R}^n$ be an infinite set such that* $\operatorname{acc} T = \emptyset$ *and let* $s \in \operatorname{int}\operatorname{conv} T$*. Then, there exists some finite subset T_0 of T such that $V_{T_0}(s) = V_T(s)$. Moreover, if $V_T(s) \subset s + \rho B_n$, $\rho > 0$, then $V_T(s) = V_{T_0}(s)$, where*

$$T_0 = \{t \in T : \|t - s\| \leq 2(1 + n\rho)\}.$$

Proof: Since $s \in \operatorname{int}\operatorname{conv} T$, by Proposition 5, $V_T(s)$ is bounded. Let A be as in (16). By Lemma 21, $\operatorname{acc} A \subset \{(0_n, 1)\}$, so $\upsilon_{T-s}(0_n)$ is a p-system, and Theorem 3.5 in [15] gives the first statement. Let $\rho > 0$ be such that $V_T(s) \subset s + \rho B_n$. Then, applying (3), $V_{T-s}(0_n) \subset \rho B_n$.

First, we show that any scalar δ such that $0 < \delta < \frac{1}{1+n\rho}$ satisfies

$$\mathrm{cl}\left(\delta B_n \times \left]1-\delta, 1+\delta\right[\right) \subset \mathrm{int}\, K_{T-s}\left(0_n\right). \tag{17}$$

Let $\gamma := \frac{1}{1+n\rho}$. If $x \in V_{T-s}\left(0_n\right)$, then $|x_i| \leq \|x\| < \rho$, $i = 1, ..., n$, so that $\sum_{i=1}^{n} |x_i| < n\rho = \frac{1}{\gamma} - 1$. Thus,

$$V_{T-s}\left(0_n\right) \subset \left\{x \in \mathbb{R}^n : \pm x_1 \pm ... \pm x_n \leq \frac{1}{\gamma} - 1\right\}. \tag{18}$$

Taking conic representations in both members of (18), we get the reverse inclusion:

$$\mathrm{cone}\left\{\left(\pm 1, ..., \pm 1, \frac{1}{\gamma} - 1\right), \left(0_n, 1\right)\right\} \subset \mathrm{cl}\, K_{T-s}\left(0_n\right).$$

Then,

$$\begin{aligned} \mathrm{cl}\left(\gamma B_n \times \left]1-\gamma, 1+\gamma\right[\right) &\subset \left[-\gamma, \gamma\right]^n \times \left[1-\gamma, 1+\gamma\right] \\ &\subset \mathrm{cone}\left\{\left(\pm\gamma, ..., \pm\gamma, 1 \pm \gamma\right)\right\} \\ &= \mathrm{cone}\left\{\left(\pm 1, ..., \pm 1, \tfrac{1}{\gamma} \pm 1\right), \left(0_n, 1\right)\right\} \\ &= \mathrm{cone}\left\{\left(\pm 1, ..., \pm 1, \tfrac{1}{\gamma} - 1\right), \left(0_n, 1\right)\right\} \subset \mathrm{cl}\, K_{T-s}\left(0_n\right). \end{aligned} \tag{19}$$

Taking interiors in (19) we get (17) because $\delta < \gamma$ implies

$$\delta B_n \times \left]1-\delta, 1+\delta\right[\subset \gamma B_n \times \left]1-\gamma, 1+\gamma\right[\subset \mathrm{int}\, K_{T-s}\left(0_n\right).$$

Now we will show that $\|t-s\| > \frac{2}{\delta}$, for $t \in T$, implies

$$\frac{2}{\|t-s\|\sqrt{4+\|t-s\|^2}}\left(t-s, \frac{\|t-s\|^2}{2}\right) \in \delta B_n \times \left]1-\delta, 1+\delta\right[. \tag{20}$$

In fact, $\frac{2(t-s)}{\|t-s\|\sqrt{4+\|t-s\|^2}} \in \delta B_n$ because

$$\left\|\frac{2\left(t-s\right)}{\|t-s\|\sqrt{4+\|t-s\|^2}}\right\| = \frac{2}{\sqrt{4+\|t-s\|^2}} < \frac{2}{\sqrt{4+4/\delta^2}} = \frac{\delta}{\sqrt{\delta^2+1}} < \delta.$$

On the other hand,

$$\frac{2}{\|t-s\|\sqrt{4+\|t-s\|^2}}\frac{\|t-s\|^2}{2} = \frac{\|t-s\|}{\sqrt{4+\|t-s\|^2}} < 1 < 1+\delta,$$

and (after some algebra)

$$\frac{\|t-s\|}{\sqrt{4+\|t-s\|^{2}}}>1-\delta \Longleftrightarrow \|t-s\|^{2}>\frac{4\,(1-\delta)^{2}}{\delta\,(2-\delta)}.$$

Then, since $0<\delta<1$, we have $\frac{1}{\delta^{2}}>\frac{(1-\delta)^{2}}{\delta(2-\delta)}$ and so

$$\frac{\|t-s\|}{\sqrt{4+\|t-s\|^{2}}}>1-\delta,$$

which proves that (20) holds. This fact, together with Lemma 3.4 in [15] (applied to the so-called reduced system of $\upsilon_{T-s}\left(0_{n}\right)$), yields that $V_{T-s}\left(0_{n}\right)$ is the feasible set of the subsystem of $\upsilon_{T-s}\left(0_{n}\right)$

$$\left\{(t-s)'\,x\leq\frac{\|t-s\|^{2}}{2},\ t\in T, \|t-s\|\leq\frac{2}{\delta}\right\},$$

whose index set, $\left\{t\in T:\|t-s\|\leq\frac{2}{\delta}\right\}$, is finite because $\operatorname{acc} T=\emptyset$. □

Let us illustrate the previous result with Example 7, where we have found that $\rho=92.32$, so that

$$V_{T}\left(0_{n}\right)=\left\{x\in\mathbb{R}^{n}:t'x\leq\frac{\|t\|^{2}}{2},\ t\in T,\|t\|\leq 371.28\right\}.$$

6 Conclusions on Voronoi diagrams

In Section 2 we have given arguments for focussing our analysis on Voronoi cells w.r.t. closed sets $T\subset\mathbb{R}^{n}$: 1st, $V_{T}(s)=V_{\operatorname{cl}T}(s)$ for any $s\in T$; 2nd, the Voronoi diagram, $\operatorname{Vor}(T)$, is a tesselation of $\mathbb{R}^{n}$ if and only if T is closed. We pay particular attention to curves, closed convex sets and discrete sets. Section 3 shows that, given a closed convex set F and an element $s\in F$, F is the Voronoi cell of the closed set

$$T:=\left\{t\in\mathbb{R}^{n}:\left(t-s,\frac{\|t-s\|^{2}}{2}\right)\in C\left(F-s\right)\right\},$$

where $C\left(F-s\right)$ is the conic representation of $F-s$. Another formula allows to compute a finite set T such that $V_{T}(s)$ coincides with a given polyhedral convex set F such that $s\in\operatorname{int}F$. It is easy to prove that, given a closed convex cone F with apex s, $F=V_{T}(s)$ for $T:=s+(F-s)^{\circ}$, and it is not difficult to find a plane curve T such that $V_{T}(s)=F$ when $F\subset\mathbb{R}^{2}$ is a given one dimensional closed convex set. However, the problem of constructing a discrete set

T, $s \in T$, such that $V_T(s) = F$ for a given quasipolyhedral set F containing s remains open. Concerning the inverse problem for Voronoi diagrams, methods are known for constructing a finite set T such that $\mathrm{Vor}(T) = \mathcal{V}$, where $\mathcal{V}$ is a given tesselation of $\mathbb{R}^n$ formed by polyhedral sets ([14]), but no method is still available for other types of tesselations, e.g., those formed by infinitely many sets.

By assuming that $T \subset \mathbb{R}^n$ is a closed given set, in Sections 4 and 5 we have been able to prove several properties of the Voronoi cells $V_T(s)$ that allow us to draw the following conclusions on Voronoi diagrams $\mathrm{Vor}(T)$:

⧫ $\mathrm{Vor}(T)$ is formed by bounded sets if and only if $T \subset \mathrm{int}\,\mathrm{conv}\,T$. A necessary condition for $\mathrm{Vor}(T)$ to be formed by bounded sets is that no hyperplane supports $\mathrm{conv}\,T$ at the points of T (otherwise, if $a'(x-s) \geq 0$ for all $x \in \mathrm{conv}\,T$, $a \neq 0_n$, then $\{s + \lambda a : \lambda\} \subset V_T(s)$).

⧫ $\mathrm{Vor}(T)$ is formed by neighborhoods of the sites if and only if T is formed by isolated points, e.g., when $\mathrm{acc}\,T = \emptyset$. In this event, $\mathrm{Vor}(T)$ is formed by quasipolyhedral sets, and it is formed by polyhedral convex sets if and only if $\mathrm{cone}(T-t)$ is finitely generated for all $t \in T$ (e.g., when T is contained in some line).

⧫ $\mathrm{Vor}(T)$ is formed by closed convex cones translated to the points of T when T is a closed convex set; in that case, $\mathrm{Vor}(T)$ is a partition of $\mathbb{R}^n$ because $V_T(s)$ is the inverse image of s through the Euclidean projection on T, and this projection is Lipschitz continuous. In the particular case that T is an affine manifold, $V_T(s) = s + (T-T)^{\perp}$ for all $s \in T$, so that $\mathrm{Vor}(T)$ is a bunch of affine manifolds orthogonal to T.

⧫ $\mathrm{Vor}(T)$ is formed by closed convex sets orthogonal to T when T is the image of a continuously differentiable function of a single variable whose derivative does not vanish.

Acknowledgements

1. The authors wish to thank O. Stein for having called their attention to this interesting field of Voronoi diagrams and to an unknown referee for his/her valuable comments and suggestions.

2. This work has been supported by MICINN of Spain, Grant MTM2008-06695-C03-01/03 and SECTyP-UNCuyo, Argentina.

References

[1] E.J. Anderson, M.A. Goberna, M.A. López, Locally polyhedral linear inequality systems, Linear Algebra Appl. 270 (1998) 231-253.

[2] N.G. Bean, M. Fackrell, P. Taylor, Characterization of Matrix-Exponential Distributions, Stochastic Models 24 (2008) 339-363.

[3] B. Delaunay, Sur la sphère vide: À la mémoire de Georges Voronoi (French), Izv. Akad. Nauk SSSR, Otdelenie Matematicheskih i Estestvennyh Nauk 7 (1934) 793-800.

[4] U. Eckhardt, Theorems on the dimension of convex sets, Linear Algebra Appl. 12 (1975) 63-76.

[5] U. Eckhardt, Representation of convex sets, in: A.V. Fiacco, K.O. Kortanek (Eds.) Extremal Methods and Systems Analysis, Springer, New York, 1980, pp. 374-383.

[6] P. Engel, Geometric Crystallography, in: P.M. Gruber, J.M. Wills (Eds.), Handbook of Convex Geometry, vol. B, North-Holland, 1993, pp. 989-1041.

[7] M.A. Goberna, E. González, J.E. Martínez-Legaz, M.I. Todorov, Motzkin decomposition of closed convex sets, Math Anal. Appl. 364 (2010) 209-221.

[8] M.A. Goberna, V. Jornet, M.M.L. Rodríguez, On the characterization of some families of closed convex sets, Beiträge Algebra Geom. 43 (2002) 153-169.

[9] M.A. Goberna, M.A. López, A theory of linear inequality systems, Linear Algebra Appl. 106 (1988) 77-115.

[10] M.A. Goberna, M.A. López, Linear Semi-Infinite Optimization, J. Wiley, 1998.

[11] P.M. Gruber, Convex and Discrete Geometry, Springer-Verlag, 2007.

[12] D. Jaume, R. Puente, Representability of convex sets by analytical linear inequality systems, Linear Algebra Appl. 380 (2004) 135-150.

[13] P.-J. Laurent, Approximation et Optimization, Hermann, Paris, 1972.

[14] G. Liotta, H. Meijer, Voronoi drawings of trees, Comput. Geom. 24 (2003) 147-178.

[15] E. Marchi, R. Puente, V.N. Vera de Serio, Quasipolyhedral sets in linear semi-infinite inequality systems, Linear Algebra Appl. 255 (1997) 157-169.

[16] R. Puente, Cyclic convex bodies and optimization moment problems, Linear Algebra Appl. 426 (2007) 596-609.

[17] A. Okabe, B. Boots, K. Sugihara, S.N. Chiu, Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, 2nd edition, J. Wiley, 2000.

[18] B. Özçam, H. Cheng, A discretization based smoothing method for solving semi-infinite variational inequalities, J. Ind. Manag. Optim. 1 (2005) 219–233.

[19] I. Voigt, Voronoizellen diskreter Punktmengen, Ph.D. thesis, TU Dortmund University, Faculty of Mathematics, Dortmund, 2008.

[20] I. Voigt, S. Weis, Polyhedral Voronoi cells, Beiträge Algebra Geom. 51 (2010) 587-598.