On the stability of the Motzkin representation of closed convex sets

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Abstract

A set is called Motzkin decomposable when it can be expressed as the Minkowski sum of a compact convex set with a closed convex cone. This paper analyzes the continuity properties of the set-valued mapping associating to each couple (C, D) formed by a compact convex set C and a closed convex cone D its Minkowski sum $C + D$. The continuity properties of other related mappings are also analyzed.

Key words: Motzkin decomposition, Minkowski sum, stability

1 Introduction

We say that a nonempty set $F \subset \mathbb{R}^n$ is decomposable in Motzkin's sense (*M-decomposable* in short) if there exist a compact convex set C and a closed convex cone D such that $F = C + D$. Then we say that $C + D$ is a *Motzkin representation* (or *decomposition*) of F with compact and conic components C and D, respectively. Any M-decomposable set F has a unique conic component $D = 0^+F$ (the recession cone of F) and $F = F + \{0_n\}$ is the unique Motzkin decomposition of F whenever F is bounded. The classical Motzkin Theorem [17] asserts that any polyhedral convex set is M-decomposable. This class of closed convex sets has been characterized in different ways in [7], [8] and [9]. For instance, a closed convex set $F \subset \mathbb{R}^n$ is M-decomposable iff $F \cap (\operatorname{lin} F)^\perp$ is M-decomposable iff the *Pareto-like set* of F,

$$M(F) := \left\{ x \in F \cap (\operatorname{lin} F)^\perp : \left(x - \left(0^+F\right) \cap (\operatorname{lin} F)^\perp \right) \cap F = \{x\} \right\},$$

is bounded (here $\operatorname{lin} F := (0^+F) \cap (-0^+F)$ denotes the lineality space of F and $(\operatorname{lin} F)^\perp$ its orthogonal complement). In that case, [7, Theorem 19] shows that

$$F = \operatorname{cl}\operatorname{conv} M(F) + 0^+F = \operatorname{cl} M(F) + 0^+F, \tag{1}$$

although the last equation is not explicit in the statement). If F contains no line, $M(F) = \{x \in F : (x - 0^+F) \cap F = \{x\}\}$ is the efficient set of F relative to the cone 0^+F and $C(F) := \operatorname{cl}\operatorname{conv} M(F)$ is the *smallest compact component* of F (which does not exist when F contains lines).

M-decomposable sets with uncertain compact component arise, for instance, in Data Envelopment Analysis (DEA), whose purpose is the comparison of the efficiency of a set of decision making units (e.g., firms, factories, branches or schools) or technologies in order to obtain certain outputs from the available inputs. When the number of decision making units (DMUs) to be compared is $p \in \mathbb{N}$, the efficiency ratios are usually computed via Linear Programming from a set of the form $C + \mathbb{R}^n_+$, where C is the convex hull of $\{x_1, ..., x_p\} \subset \mathbb{R}^n$ and each x_j depends on the inputs and outputs of the j-th DMU (e.g., see [5]). Analogously, in the case of chemical processes which are controlled by means of certain parameters (pressure, temperature, concentrations, etc.), the efficiency ratios of the virtual technologies are computed via Linear Semi-Infinite (or Bilevel) Programming ([13]) from a set of the form $C + \mathbb{R}^n_+$ where C is the convex hull of certain infinite compact set $X \subset \mathbb{R}^n$. In all practical applications, the set $\{x_1, ..., x_p\}$ (or its infinite counterpart X) is uncertain, i.e., the compact convex set C is subject to perturbations whereas the closed convex cone D remains fixed, so that the efficiency ratios are also uncertain and their stability behavior depend on the stability behavior of $C + \mathbb{R}^n_+$ under sufficiently small perturbations of C.

The main objective of this paper is the study of the stability properties of the sum of a compact convex set with a closed convex cone when one of these two sets, or both, are subject to small perturbations that preserve the mentioned properties. This problem can be seen as a particular case of the following one: studying the stability of the feasible set for the different types of representations of closed convex sets. In fact, representing a given closed convex set $F \subset \mathbb{R}^n$ consists of choosing an element θ (called *nominal parameter*) in certain set Θ (called *parameter space*) whose elements are the results of all admissible perturbations of θ (due, e.g., to the inaccuracy of the data). It is assumed the existence of a set-valued mapping $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$, called *feasible set mapping*, associating to each perturbation of the nominal data θ the corresponding perturbation of F. Obviously, we must have, in particular, $\mathcal{F}(\theta) = F$. The *domain* of $\mathcal{F}$ is $\operatorname{dom} \mathcal{F} := \{\theta_1 \in \Theta : \mathcal{F}(\theta_1) \neq \emptyset\}$. A closed convex set F_1 has a unique representation in Θ when $\mathcal{F}^{-1}(F_1)$ is singleton. The question to be answered, from the stability perspective, is whether the effect on the feasible set of small perturbations of the data are necessarily small too, so that we also assume that Θ is endowed with some topology. The stability results on $\mathcal{F}$ either characterize the topological interior of $\operatorname{dom}\mathcal{F}$ (i.e., the elements of Θ that represent nonempty closed convex sets under sufficiently small perturbations) or provide conditions for the continuity (in some sense) of $\mathcal{F}$ at $\theta \in \operatorname{dom}\mathcal{F}$. The completeness of some neighborhood of θ is a desirable feature of the selected type of representation as far as this property connects the continuity properties of $\mathcal{F}$ at θ with the metric regularity of $\mathcal{F}^{-1}$ at (x, θ), with $x \in \mathcal{F}(\theta)$. Now we describe briefly three types of representations of closed convex sets, namely: linear, conic and Motzkin representations (the one we are interested in).

Let F be the solution set of the linear system $\theta = \{a_t' x \geq b_t, t \in T\}$ (the given *linear representation* of F). Then Θ is the class of all linear systems obtained by perturbing the coefficients of θ (i.e., the functions $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$) maintaining the number of variables and constraints, n and T. In other words, $\Theta = (\mathbb{R}^n \times \mathbb{R})^T$ and, given $\theta_1 = \left\{(a_t^1)' x \geq b_t^1, t \in T\right\} \in \Theta$,

$$\mathcal{F}(\theta_1) := \left\{x \in \mathbb{R}^n : \left(a_t^1\right)' x \geq b_t^1, t \in T\right\}.$$

Observe that, given an arbitrary set of positive numbers $\{\lambda_t, t \in T\}$, $\theta_1 = \{\lambda_t a_t' x \geq \lambda_t b_t, t \in T\}$ is another linear representation of F in Θ, so that there are infinitely many linear representations of F. The most common way to measure the size of the perturbations appeals to the *metric of the uniform convergence* introduced in [12]: given $\theta_i = \left\{(a_t^i)' x \geq b_t^i, t \in T\right\} \in \Theta$, $i = 1, 2$,

$$\rho(\theta_1, \theta_2) := \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t^2 \\ b_t^2 \end{pmatrix} \right\|_\infty, \tag{2}$$

where $\|\cdot\|_\infty$ stands for the Chebyshev norm in $\mathbb{R}^{n+1}$ (actually ρ is an infinite-valued metric iff T is infinite). The neighborhood of θ formed by the finite perturbations of the nominal parameter θ, $\{\theta_1 \in \Theta : \rho(\theta_1, \theta) < +\infty\}$, is a complete metric space.

In the conic representation perspective, the nominal parameter θ is a given closed convex cone in $\mathbb{R}^{n+1}$ such that $(0_n, -1) \in \theta$ and

$$F = \{x \in \mathbb{R}^n : a'x \geq b \ \forall (a, b) \in \theta\},$$

with $F \neq \emptyset$ if and only if $(0_n, 1) \notin \theta$. In that case, F is bounded if and only if $(0_n, 1)$ is an interior point of θ ([10]). It is worth observing that, if $F \neq \emptyset$, θ coincides with the *reference cone* of F, i.e.,

$$\theta = \left\{(a, b) \in \mathbb{R}^{n+1} : a'x \geq b \ \forall x \in F\right\},$$

so that there is a unique *conic representation* for every nonempty closed convex set. Now Θ is formed by the class of all closed convex cones in $\mathbb{R}^{n+1}$ containing the vector $(0_n, -1)$ and, given $\theta_1 \in \Theta$,

$$\mathcal{F}(\theta_1) := \{x \in \mathbb{R}^n : a'x \geq b \; \forall (a,b) \in \theta_1\}.$$

Obviously, the Hausdorff metric d_H in $\mathbb{R}^{n+1}$ defined in (5) is an inconvenient measure for the size of the perturbations of θ because $d_H(\theta_1, \theta) = +\infty$ for all $\theta_1 \in \Theta$ such that $\theta_1 \neq \theta$. One of the ways to avoid this drawback is to replace $d_H(\theta_1, \theta)$ by $d_H(\psi(\theta_1), \psi(\theta))$, where ψ is the *truncation mapping* associating to each closed convex cone its intersection with the unit closed ball (it could be another compact neighborhood of the origin), i.e., $\psi(\theta_1) := \theta_1 \cap B_{n+1}$, $\theta_1 \in \Theta$. Thus, a suitable metric on Θ is

$$\rho(\theta_1, \theta_2) := d_H(\psi(\theta_1), \psi(\theta_2)) = d_H(\theta_1 \cap B_{n+1}, \theta_2 \cap B_{n+1}) \leq 1, \;\; \theta_1, \theta_2 \in \Theta. \tag{3}$$

A simple modification of Lemma 4 shows that $\langle \Theta, \rho \rangle$ is complete.

This paper is focussed on the stability of the feasible set mapping for *Motzkin representations* of those closed convex sets which are M-decomposable, i.e., the nominal parameter is a couple $\theta = (C, D)$ such that C is a compact convex set, D is a closed convex cone, and $F = C + D$, whereas Θ is the cartesian product of the space of nonempty compact convex sets Δ_1 and the space of closed convex cones Δ_2, endowed with a suitable metric (the product of Hausdorff metrics, denoted by $d_H \times d_H$, is an inconvenient metric for Θ because $(d_H \times d_H)(\theta_1, \theta) = +\infty$ if $\theta_1 = (C_1, D_1) \in \Theta$ satisfies $D_1 \neq D$, i.e., any perturbation of the conic component has infinite size). So, we consider the product of $\langle \Delta_1, d_H \rangle$ and $\langle \Delta_2, d_H \circ (\psi, \psi) \rangle$, say $\langle \Theta, \rho \rangle$. In other words, the distance between $\theta_1 = (C_1, D_1)$ and $\theta_2 = (C_2, D_2)$ is

$$\rho(\theta_1, \theta_2) := \max \{d_H(C_1, C_2), d_H(D_1 \cap B_n, D_2 \cap B_n)\}, \tag{4}$$

so that the topology induced by ρ on Θ is the product of the topologies induced by the Hausdorff metric (5) on Δ_1 and the Hausdorff-like metric (3) on Δ_2 (the induced topology on Δ_2 coincides with the so-called bounded Hausdorff topology; see, e.g., [14]), respectively. Our feasible set mapping is $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ such that

$$\mathcal{F}(\theta_1) := C_1 + D_1,$$

for any $\theta_1 = (C_1, D_1) \in \Theta$. Observe that the Motzkin representation of $F = C + D$ is not unique when $D \neq \{0_n\}$ (i.e., when F is unbounded) because $\mathcal{F}(\operatorname{conv}\{C \cup (C + \lambda d)\}, D) = \mathcal{F}(C, D)$ for every $d \in D \diagdown \{0_n\}$ and $\lambda > 0$. Obviously, the problem of analyzing the stability of $\mathcal{F}$ can also be seen as a particular case of the more general one of studying the continuity properties of the sum of closed convex sets.

The secondary purpose of this paper is the stability analysis of the set-valued mappings $\mathcal{M}, \mathcal{C} : (\Theta, \rho) \rightrightarrows \mathbb{R}^n$ such that $\mathcal{M}(\theta_1) = M(\mathcal{F}(\theta_1))$ and $\mathcal{C}(\theta_1) = C(\mathcal{F}(\theta_1)) := \operatorname{cl}\operatorname{conv}\mathcal{M}(\theta_1)$ for all $\theta_1 \in \Theta$.

The paper is organized as follows. Section 2 contains the necessary notation, antecedents, and auxiliary results to be used later. Section 3 proves the completeness of the parameter spaces, Section 4 analyzes the stability of $\mathcal{F}$ and Section 5 studies the stability of $\mathcal{M}$ and $\mathcal{C}$.

2 Preliminaries

Throughout the paper we use the following notation and concepts.

Let X be a metrizable space. For any $A \subset X$, $\operatorname{int} A$, $\operatorname{cl} A$, and $\operatorname{bd} A$ denote the *interior*, the *closure*, and the *boundary* of A, respectively. We denote by $CL(X)$ the class of all nonempty closed subsets of X and by 2^X the class of all closed subsets of X, i.e., $2^X := CL(X) \cup \{\emptyset\}$. Let us recall the continuity concepts for set-valued mappings we use in the sequel. Let $\mathcal{N} : X \rightrightarrows \mathbb{R}^n$.

$\mathcal{N}$ is *closed* at x if for any $y \in \mathbb{R}^n$ and any two sequences, $\{x_k\} \subset X$ and $\{y_k\} \subset \mathbb{R}^n$ such that $\lim_k x_k = x$, $y_k \in \mathcal{N}(x_k)$, $k = 1, 2, ...$, and $\lim_k y_k = y$, one gets $\overline{y} \in \mathcal{N}(x)$.

$\mathcal{N}$ is *lower semicontinuous* in Berge-Kuratowski sense (lsc) at x if for each open set U such that $U \cap \mathcal{N}(x) \neq \emptyset$ there exists an open set V, $x \in V \subset X$, such that $U \cap \mathcal{N}(x_1) \neq \emptyset$ for every $x_1 \in V$.

$\mathcal{N}$ is *upper semicontinuous* in Berge-Kuratowski sense (usc) at x if for each open set U such that $\mathcal{N}(x) \subset U$ there exists an open set V, $x \in V \subset X$, such that $\mathcal{N}(x_1) \subset U$ for every $x_1 \in V$. This stability property is considered too strong in most frameworks (see, e.g., [18]).

Finally, we say that $\mathcal{N}$ is *closed (lsc, usc)* on X when it is closed (lsc, usc) at x for all $x \in X$.

Given $A \subset X = \mathbb{R}^p$, we denote by $\operatorname{rint} A$, $\operatorname{conv} A$, and $\operatorname{cone} A = \mathbb{R}_+ \operatorname{conv} A$, the *relative interior*, the *relative boundary*, the *convex hull* of A, and the *convex conical hull* of A, respectively. The scalar product of $x, y \in \mathbb{R}^p$ is denoted either by $x'y$ or by $\langle x, y \rangle$, the Euclidean norm of x by $\|x\|$, the canonical basis by $\{e_1, ..., e_p\}$, the zero vector by 0_p, the closed unit ball by B_p, and the Hausdorff distance between two closed sets A, B by

$$d_H(A, B) = \inf \{\eta \in \mathbb{R}_+ : A \subset B + \eta B_p \text{ and } B \subset +\eta B_p\}. \tag{5}$$

Let $\{A_k\}$ be a sequence of nonempty sets in $\mathbb{R}^p$. We denote by $\liminf_k A_k$ ($\limsup_k A_k$) the set formed by all the possible limits (cluster points, respectively) of sequences $\{x_k\}$ such that $x_k \in A_k$ for all $k \in \mathbb{N}$ (we usually write $\lim_k x_k = x$, or even $x_k \to x$, instead of $\lim_{k\to\infty} x_k = x$). When these two limit sets are non-empty and coincide, then it is said that $\{A_k\}$ converges in the *Painlevé-Kuratowski sense* to the set

$$\lim_k A_k := \liminf_k A_k = \limsup_k A_k.$$

Then we write $A_k \to^{PK} \lim_k A_k$. If $\{A_k\}$ is a sequence of closed sets in $\mathbb{R}^p$ such that $\lim_k d_H(A_k, A) = 0$ (in short $A_k \to^H A$), then $A_k \to^{PK} A$ and the converse statement holds when there exists $\mu > 0$ such that $A, A_k \subset \mu B_p$ for all $k \in \mathbb{N}$ ([18]).

Now we summarize the antecedents on the stability of the feasible set for linear and conic representations.

Concerning linear representations, it is easy to prove that $\langle \Theta, \rho \rangle$, with ρ defined as in (2), is a complete metric space. Moreover, it is known (see, e.g., [11]) that $\mathcal{F}$ is closed whereas $\mathcal{F}$ is lsc at $\theta = \{a_t' x \geq b_t, t \in T\} \in \operatorname{dom} \mathcal{F}$ iff $\theta \in \operatorname{int} \operatorname{dom} \mathcal{F}$ iff $0_{n+1} \notin \operatorname{cl} \operatorname{conv} \{(a_t, b_t), t \in T\}$ (a condition involving the data). The usc property of $\mathcal{F}$ was characterized later, in [3], by means of a condition on the set $\{(a_t, b_t), t \in T\}$ which is usually difficult to be checked. A sufficient condition for the usc property of $\mathcal{F}$ at θ is that $\mathcal{F}(\theta)$ is either a compact set or the whole space $\mathbb{R}^n$. When T is a compact Hausdorff topological space and Θ is formed by the linear inequality systems in $\mathbb{R}^n$ whose coefficients are continuous functions of the index t on T (which trivially holds when T is finite by considering the discrete topology on T), then Θ is a Banach space, $\mathcal{F}$ is closed, the lsc property of $\mathcal{F}$ at $\theta \in \operatorname{dom} \mathcal{F}$ is characterized as before, and $\mathcal{F}$ is usc at $\theta \in \operatorname{dom} \mathcal{F}$ iff $\mathcal{F}(\theta)$ is either a compact set or the whole space $\mathbb{R}^n$ ([2], [6]).

Concerning the stability of the feasible set mapping for conic representations, it is just known that $\operatorname{dom} \mathcal{F}$ is open and $\operatorname{bd} \operatorname{dom} \mathcal{F}$ is the class of inconsistent linear systems whose finite subsystems are consistent ([16]). Thus $\Theta \diagdown \operatorname{cl} \operatorname{dom} \mathcal{F}$ is formed by the strongly inconsistent linear systems (those systems containing some inconsistent subsystem). The continuity properties of $\mathcal{F}$ in this framework have not yet been explored.

The set-valued mapping $\mathcal{M} : (\Theta, \rho) \rightrightarrows \mathbb{R}^n$ is related with the efficient set mapping in multiobjective optimization, whose stability properties have been widely analyzed in the literature for linear (and nonlinear) representations of the feasible set F (see, e.g., [19], [20], [4], and references therein). F.i., in the case of linear representations, under suitable conditions, generic lower semicontinuity of the mapping $\mathcal{M}$ has been proven in ([20]). In our setting we have to exploit the special structure of the vector optimization problems, coming from the ordering cone $D = 0^+ F$, which, f.i., keeps the sets $M(F)$ always bounded, a fact which is not true in general.

Now we consider the parameter space of the Motzkin representations, $\langle \Theta, \rho \rangle$, where ρ is the product of d_H and $d_H \circ (\psi, \psi)$, with $\psi(D_1) = D_1 \cap B_n$ for any closed convex cone D_1. Next we apply two known results on the sum of closed sets in $\mathbb{R}^n$ to obtain consequences for the stability of feasible set mapping $\mathcal{F}$ in terms of the continuity of the *associated single-valued mapping* $\widetilde{\mathcal{F}} : \Theta \to CL(\mathbb{R}^n)$ such that $\widetilde{\mathcal{F}}(C_1, D_1) := \{C_1 + D_1\}$. Unfortunately, in these results $\mathcal{F}$ ranges on $\langle \Theta, d_H \times d_H \rangle$, where Θ is the union of disjoint open and closed sets which are formed by parameters sharing the same compact component, in such a way that the distance between two parameters is finite iff they belong to the same element of the partition.

Consider the set-valued mapping $\mathcal{S} : 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n} \rightrightarrows \mathbb{R}^n$ such that $\mathcal{S}(A, B) = \operatorname{cl}(A + B)$ for each pair A, B of closed sets in $\mathbb{R}^n$, whose associated single-valued mapping is $\widetilde{\mathcal{S}} : 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ such that $\widetilde{\mathcal{S}}(A, B) = \{\operatorname{cl}(A + B)\}$. It is known that its restriction $\widetilde{\mathcal{S}} : \langle CL(\mathbb{R}^n) \times CL(\mathbb{R}^n), d_H \times d_H \rangle \to \langle CL(\mathbb{R}^n), d_H \rangle$ is continuous (see, e.g., [1, Exercise 3.2.12]). Since $\widetilde{\mathcal{S}} = \widetilde{\mathcal{F}}$ on Θ, we conclude that $\widetilde{\mathcal{F}} : \langle \Theta, d_H \times d_H \rangle \to \langle CL(\mathbb{R}^n), d_H \rangle$ is continuous too.

On the other hand, since $\mathcal{S}$ has closed images, $\widetilde{\mathcal{S}}$ is continuous relative to the Vietoris topology on $CL(\mathbb{R}^n) \times CL(\mathbb{R}^n)$ iff $\mathcal{S} : 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n} \rightrightarrows \mathbb{R}^n$ is both lsc and usc ([1, Theorem 6.2.9]). Thus, the restriction of $\mathcal{S}$ to Θ, $\mathcal{F} : \langle \Theta, d_H \times d_H \rangle \rightrightarrows \mathbb{R}^n$, is lsc and usc iff $\widetilde{\mathcal{F}}$ is continuous relative to the Vietoris topology on Θ.

3 Completeness of the parameter space

From now on we consider the parameter space $\langle \Theta, \rho \rangle$, with ρ defined as in (4). The next example, to be used later, shows the existence of sequences of M-decomposable sets that converge in Painlevé-Kuratowski sense to another M-decomposable set whereas their respective Motzkin representations may converge or not, in the metric ρ.

Example 1 *Consider the sequence of M-decomposable sets*

$$F_k = \left\{ x \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 \geq 0, \ -\frac{1}{k}x_1 + x_2 \geq -\frac{1}{k} \right\}, k \in \mathbb{N}.$$

It can be realized that

$$F_k \to^{PK} \left\{ x \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 \geq 0 \right\} = F$$

(another M-decomposable set) even though $\{F_k\}$ *is not a Cauchy sequence relative to* ρ*. Observe that the smallest compact component of* F_k *is constant,* $C_k = [0,1] \times \{0\}$*, with* $C_k \to^H C := [0,1] \times \{0\} \neq \overline{C} := \{0_2\}$*, the latter set being the smallest compact component of* F*. Concerning the respective conic components,* $D_k = \left\{ x \in \mathbb{R}^2 : x_1 \geq 0, \ -\frac{1}{k}x_1 + x_2 \geq 0 \right\}$ *for all* $k \in \mathbb{N}$ *and* $D =$

$\mathbb{R}^2_+$, *we have* $D_k \cap B_2 \to^H D \cap B_2$. *Finally, observe that* $(C_k, D_k) \to^\rho (C, D) \neq \left(\overline{C}, D\right)$, *with* $C + D = \overline{C} + D = F$.

In order to prove that (Θ, ρ) is a complete metric space we need the following lemmas.

Lemma 2 *Let* $\emptyset \neq D \subset \mathbb{R}^n$. *Then* D *is a closed convex cone if and only if there exists a compact convex set* $Y \subset \mathbb{R}^n$ *formed by radii of* B_n *such that* $D = \operatorname{cone} Y$.

Proof. If D is a closed convex cone, then $Y := D \cap B_n$ satisfies the required conditions.

Assume that $D = \operatorname{cone} Y$, where Y is a compact convex union of radii of B_n and $D = \operatorname{cone} Y$. We must prove that D is closed.

Let $x = \lim_k x_k$, with $x_k \in D$ for all $k \in \mathbb{N}$. If $x = 0_n$, we are done because $0_n \in Y \subset D$. Thus, the set $\{k \in \mathbb{N} : x_k = 0_n\}$ is finite and we can assume w.l.o.g. that $x, x_k \neq 0_n$ for all $k \in \mathbb{N}$. Then, given $k \in \mathbb{N}$, we have $\frac{x_k}{\|x_k\|} \in Y$ because this is an extreme point of the radium of B_n in the direction of x_k. By compactness of Y, we can assume w.l.o.g. the existence of $y \in Y$ such that $\frac{x_k}{\|x_k\|} \to y$. Then, since $\|x_k\| \to \|x\|$, $x = \lim_k x_k = \|x\| \, y \in D$. □

Lemma 3 $\langle \Delta_1, d_H \rangle$ *is a complete metric space.*

Proof. Since $\mathbb{R}^n$ is complete for the Euclidean metric, the hyperspace $\langle CL(\mathbb{R}^n), d_H \rangle$ is complete too. So, it is sufficient to prove that Δ_1 is a closed subset of $CL(\mathbb{R}^n)$.

Let $\{C_k\}$ be a sequence of compact convex sets such that $C_k \to^H C$. We must show that C is a compact convex set.

$\{C_k\}$ is a Cauchy sequence it is convergent. Let $k_0 \in \mathbb{N}$ be such that $d_H(C_k, C_{k_0}) \leq 1$ for all $k \geq k_0$. We have $C_k \subset C_{k_0} + B_n$ for all $k \geq k_0$. Let $\mu \geq 1$ be such that $\bigcup_{k=1,\ldots,k_0} (C_k + B_n) \subset \mu B_n$. Thus, the sequence of compact convex sets $\{C_k\}$ is contained in μB_n and we can apply Blaschke's convergence theorem: since the compact convex sets contained in a given closed ball of $\mathbb{R}^n$ form a compact metric space for d_H (see, e.g., [21, §4.6]), $\{C_k\}$ contains a convergent subsequence whose limit, necessarily C, is a compact convex set too. □

Lemma 4 $\langle \Delta_2, d_H \circ (\psi, \psi) \rangle$ *is a complete metric space.*

Proof. It is sufficient to prove that $\psi(\Delta_2)$ is a closed subset of the complete metric hyperspace $\langle CL(B_n), d_H \rangle$ (observe that B_n is a closed, and so complete, subset of $\mathbb{R}^n$).

Let $\{D_k\}$ be a sequence of closed convex cones such that $(C_k, D_k \cap B_n) \to^H (C, Y)$ in $\langle CL(B_n), d_H \rangle$. We must show that Y is the intersection of some closed convex cone D with B_n. Observe that $0_n \in Y$ because $0_n \in D_k \cap B_n$ for all $k \in \mathbb{N}$ and $D_k \cap B_n \to^{PK} Y$.

Since the sequence of compact convex sets $\{D_k \cap B_n\}$ is contained in the closed ball B_n, again by Blaschke's theorem, we can assert that $D_k \cap B_n \to^H Y$, and Y is a compact convex set.

Now we prove that the convex cone $D := \operatorname{cone} Y = \mathbb{R}_+ Y$ is closed, i.e., according to Lemma 2, that Y is a union of radii of B_n. Let $x \in Y$ and $\lambda > 0$ be such that $y := \lambda x \in B_n$. If $x = 0_n$, then $y = 0_n \in Y$ because $0_n \in D_k \cap B_n$ for all $k \in \mathbb{N}$ and $D_k \cap B_n \to^{PK} Y$. Alternatively, if $x \neq 0_n$, then we can write $x = \lim_k x_k$, with $x_k \in D_k \cap B_n$ and $x_k \neq 0_n$ for all $k \in \mathbb{N}$ large enough. Let $y_k := \frac{\lambda \|x\|}{\|x_k\|} x_k$, for all $k \geq k_0$. Since $y_k \in D_k \cap B_n$ for all $k \geq k_0$ and $y_k \to y$, y belongs to the Painlevé-Kuratowski limit of $\{D_k \cap B_n\}$. Hence $\lambda x = y \in Y$. Thus, D is a closed convex cone.

Finally, we show that $Y = D \cap B_n$. The inclusion $Y \subset D \cap B_n$ holds by the definition of D. Conversely, let $d \in D \cap B_n$. We have $D = \operatorname{cone}\{y \in Y : \|y\| = 1\}$ because Y is a union of radii of B_n. Then, there exists $y \in Y$ such that $d \in [0_n, y] \subset Y$. So, $D \cap B_n \subset Y$. □

Theorem 5 (Θ, ρ) *is a complete metric space.*

Proof. It is straightforward consequence of Lemmas 3 and 4 as far as the product of complete metric spaces is a complete metric space. □

4 Stability of $\mathcal{F}$

The next result shows that the feasible set mapping of Motzkin representations is highly stable.

Theorem 6 *The set-valued mapping $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ is closed and lsc. Moreover, $\mathcal{F}$ is usc at $\theta \in \Theta$ if and only if $\mathcal{F}(\theta)$ is either a compact set or the whole space $\mathbb{R}^n$.*

Proof. First, we prove that $\mathcal{F}$ is closed.

Let $\{\theta_k\} \subset \Theta$ and $\{y_k\} \subset \mathbb{R}^n$ be such that $y_k \in \mathcal{F}(\theta_k)$ for all $k \in \mathbb{N}$, $\theta_k \to^\rho \theta \in \Theta$ and $y_k \to y$. let $\theta_k = (C_k, D_k)$, for all $k \in \mathbb{N}$, and $\theta = (C, D)$. For every $k \in \mathbb{N}$ we can write $y_k = c_k + d_k$, where $(c_k, d_k) \in C_k \times D_k$.

Since $C_k \to^H C$ and the latter set is compact, the sequence $\{c_k\}$ is bounded and we can assume w.l.o.g. that $c_k \to c \in C$ (recall that $C_k \to^{PK} C$). Then $d_k \to d := y - c$. If $d = 0_n$, $y = c \in C \subset \mathcal{F}(\theta)$ and we are done. Thus we assume $d \neq 0_n$ and we can also suppose w.l.o.g. that $d_k \neq 0_n$ for all $k \in \mathbb{N}$. Since $\frac{d_k}{\|d_k\|} \to \frac{d}{\|d\|}$, $\frac{d_k}{\|d_k\|} \in D_k \cap B_n$ for all $k \in \mathbb{N}$, and $D_k \cap B_n \to^{PK} D \cap B_n$, we get $\frac{d}{\|d\|} \in D$ and $d \in D$ as well. This means that $y = c + d \in C + D = \mathcal{F}(\theta)$ and so $\mathcal{F}$ is closed at θ.

Now we prove that $\mathcal{F}$ is lsc.

Let $\theta = (C, D) \in \Theta$ and let W be an open set in $\mathbb{R}^n$ such that $\mathcal{F}(\theta) \cap W \neq \emptyset$. Let $(c, d) \in C \times D$ such that $c + d \in W$. Let W', W'' be open sets in $\mathbb{R}^n$ such that $c \in W'$, $d \in W''$ and $W' + W'' \subset W$ (selecting $\mu > 0$ such that $c + d + 2\mu B_n \subset W$, we can take $W' = c + \mu B_n$ and $W'' = d + \mu B_n$). Let $0 < \varepsilon_1 < 1$ be such that $c + \varepsilon_1 B_n \subset W'$ and $d + \varepsilon_1 B_n \subset W''$.

For any compact convex set C_1 such that $d_H(C_1, C) < \varepsilon_1$, we have $C \subset C_1 + \varepsilon_1 B_n$. Let $c = c_1 + \varepsilon_1 u$, with $c_1 \in C_1$ and $u \in B_n$. Then

$$c_1 = c - \varepsilon_1 u \in c + \varepsilon_1 B_n \subset W', c_1 \in C_1. \tag{6}$$

If $D = \{0_n\}$ and D_1 is a closed convex cone such that $d_H(D_1 \cap B_n, D \cap B_n) < \varepsilon_1 < 1$, then $D_1 \cap B_n = \{0_n\}$ (otherwise $D_1 \cap B_n$ contains at least one radius of B_n). So, if $\theta_1 = (C_1, D_1) \in \Theta$ satisfies $\rho(\theta_1, \theta) < \varepsilon_1$, then $\theta_1 = (C_1, \{0_n\})$, with $d_H(C_1, C) < \varepsilon_1$. Thus, (6) and $0_n \in \varepsilon_1 B_n \subset W''$ ($d = 0_n$ because $D = \{0_n\}$) yield

$$c_1 \in C_1 \cap W' \subset C_1 \cap (W' + W'') \subset \mathcal{F}(\theta_1) \cap W,$$

so that $\mathcal{F}(\theta_1) \cap W \neq \emptyset$.

Now we assume that D contains at least one ray. Let $y \in D \cap W''$, $y \neq 0_n$. Then $\frac{y}{\|y\|} \in D \cap B_n \cap \frac{1}{\|y\|} W''$, so that $D \cap B_n \cap \frac{1}{\|y\|} W'' \neq \emptyset$. The previous argument, with $D \cap B_n$ and $\frac{1}{\|y\|} W''$ replacing C and W', respectively, shows the existence of $\varepsilon_2 > 0$ such that $C_1 \cap \frac{1}{\|y\|} W'' \neq \emptyset$, for any compact convex set C_1 such that $d_H(C_1, D \cap B_n) < \varepsilon_2$. Therefore we have $D_1 \cap B_n \cap \frac{1}{\|y\|} W'' \neq \emptyset$, and so $D_1 \cap W'' \neq \emptyset$, when D_1 is a closed convex cone satisfying $d_H(D_1 \cap B_n, D \cap B_n) < \varepsilon_2$. In this event, there exists d_1 such that

$$d_1 \in W'' \text{ and } d_1 \in D_1. \tag{7}$$

If $\rho(\theta_1, \theta) < \min\{\varepsilon_1, \varepsilon_2\}$, from (6) and (7) we get

$$c_1 + d_1 \in (C_1 + D_1) \cap (W' + W'') \subset \mathcal{F}(\theta_1) \cap W,$$

so that we have again $\mathcal{F}(\theta_1) \cap W \neq \emptyset$ for θ_1 sufficiently close to θ.

Finally, we characterize the upper semicontinuity of $\mathcal{F}$.

If $\mathcal{F}(\theta) = \mathbb{R}^n$, then $\mathcal{F}$ is trivially usc at θ.

Let $\mathcal{F}(\theta)$ be bounded and let $W \subset \mathbb{R}^n$ be an open set such that $\mathcal{F}(\theta) \subset W$. The boundedness assumption means that $\theta = (C, \{0_n\})$, where C is a compact convex set. Then $\rho(\theta_1, \theta) < 1$, with $\theta_1 = (C_1, D_1)$, entails $D_1 = \{0_n\}$.

Because C is a compact set which does not intersect the closed set $\mathbb{R}^n \diagdown W$, we can choose a scalar $0 < \varepsilon < 1$ such that

$$\varepsilon < \inf\{\|c - x\| : c \in C, x \in \mathbb{R}^n \diagdown W\}.$$

Then, if $\theta_1 = (C_1, D_1)$ satisfies $\rho(\theta_1, \theta) < \varepsilon$, we have $D_1 = \{0_n\}$ and $d_H(C_1, C) < \varepsilon$, so that $C_1 \subset C + \varepsilon B_n \subset W$. Thus $\mathcal{F}(\theta_1) = C_1 \subset W$ and $\mathcal{F}$ is usc at θ.

Now we assume that $\mathcal{F}(\theta)$ is an unbounded set different from $\mathbb{R}^n$. Let $\theta = (C, D) \in \Theta$, with $D \neq \{0_n\}$. Consider the open set $W := \mathcal{F}(\theta) + \operatorname{int} B_n = C + \operatorname{int} B_n + D$. Obviously, $\mathcal{F}(\theta) \subset W$.

For each $\varepsilon > 0$ there exists a vector $y \notin D$ such that $\|y\| = 1$ and $d(y, D \cap B_n) < \varepsilon$. Let $\theta_1 = (C_1, D_1)$, where $C_1 := C$ and $D_1 := \operatorname{cl}\operatorname{cone}\{D \cup \{y\}\}$. Then $D \subset D_1$ and $\rho(\theta_1, \theta) = d_H(D_1 \cap B_n, D \cap B_n) < \varepsilon$. Moreover, $y \in D_1 = 0^+\mathcal{F}(\theta_1)$. If $\mathcal{F}(\theta_1) \subset W$, then, by $W \subset \mathcal{F}(\theta) + B_n$, we get $y \in 0^+\mathcal{F}(\theta_1) \subset 0^+(\mathcal{F}(\theta) + B_n) = 0^+\mathcal{F}(\theta) = D$ (contradiction). Hence, $\mathcal{F}$ cannot be usc at θ. The proof is complete. $\square$

In the DEA motivating example in Section 1, only the compact component depends on the observed data, whereas the conic component remains fixed. The opposite situation is also conceivable. Next we show that Theorem 6 still holds in both situations.

Corollary 7 *Let C be a nonempty compact set and let $\mathcal{F}_2 : \langle \Delta_2, d_H \circ (\psi, \psi) \rangle \rightrightarrows \mathbb{R}^n$ be such that $\mathcal{F}_2(D_1) = C + D_1$ for all $D_1 \in \Delta_2$. Then $\mathcal{F}_2$ is closed and lsc. Moreover, $\mathcal{F}_2$ is usc at $D \in \Delta_2$ if and only if $\mathcal{F}_2(D) = C + D$ is either a compact set or the whole space $\mathbb{R}^n$.*

Proof. As an immediate consequence of Theorem 6, $\mathcal{F}_2$ is closed and lsc. Moreover, it is usc at $D \in \Theta$ if $C + D$ is either a compact set or the whole space $\mathbb{R}^n$. This condition is also necessary because in the last part of the proof of Theorem 6, assuming that F is an unbounded set different from $\mathbb{R}^n$, we have shown that small perturbations of the conic component provoke an abrupt growth of $\mathcal{F}_2$. $\square$

Corollary 8 *Let D be a closed convex set and let $\mathcal{F}_1 : \langle \Delta_1, d_H \rangle \rightrightarrows \mathbb{R}^n$ be such that $\mathcal{F}_1(C_1) = C_1 + D$ for all $C_1 \in \Delta_1$. Then, $\mathcal{F}_1$ is closed and lsc. Moreover,*

$\mathcal{F}_1$ is usc at $C \in \Delta_1$ if and only if $\mathcal{F}_1(C) = C + D$ is either a compact set or the whole space $\mathbb{R}^n$.

Proof. The first part is as in Corollary 7, but now we must show that small perturbations of the compact component provoke an abrupt growth of $\mathcal{F}_1$.

Let $F = C + D$ be an unbounded set different from $\mathbb{R}^n$, i.e., $\{0_n\} \neq D \neq \mathbb{R}^n$. Then there exists $\overline{d} \in \operatorname{bd} D$. By the supporting hyperplane theorem for closed convex cones, there exists $w \in \mathbb{R}^n \diagdown \{0_n\}$ such that $w'\overline{d} = 0$ and $w'd \leq 0$ for all $d \in D$. Let $\overline{c}$ be a maximizer of $w'x$ on C. Then, $w'(c + d) \leq w'\overline{c}$ for all $c \in C$ and $d \in D$, i.e., $F \subset \{x \in \mathbb{R}^n : w'x \leq w'\overline{c}\}$ and the ray $\left\{\overline{c} + \lambda\overline{d} : \lambda \geq 0\right\} \subset \operatorname{bd} F$.

The closed set $\left\{\overline{c} + \lambda\overline{d} + \frac{1}{\lambda+1}w \in \mathbb{R}^n : \lambda \geq 0\right\}$ does not meet F because

$$w'\left(\overline{c} + \lambda\overline{d} + \frac{1}{\lambda+1}w\right) = w'\overline{c} + \frac{\|w\|^2}{\lambda+1} > w'\overline{c}\ \forall\lambda \geq 0.$$

Thus, $U := \mathbb{R}^n \diagdown \left\{\overline{c} + \lambda\overline{d} + \frac{1}{\lambda+1}w : \lambda \geq 0\right\}$ is an open set such that $F \subset U$.

Let $0 < \varepsilon < 1$. Taking $\lambda = \frac{1}{\varepsilon} - 1 > 0$, we have

$$\overline{c} + \lambda\overline{d} + \frac{1}{\lambda+1}w \in C + D + \varepsilon w = F + \varepsilon w.$$

So, $C+\varepsilon w \in \Delta_1$ satisfies $\mathcal{F}_1(C + \varepsilon w) = F+\varepsilon w \not\subseteq U$ despite of $d_H(C + \varepsilon w, C) = \varepsilon\|w\| \to 0$ as $\varepsilon \searrow 0$. Thus, $\mathcal{F}_1$ is not usc at C. $\square$

5 Stability of $\mathcal{M}$ and $\mathcal{C}$

For the discussion of the stability properties of $\mathcal{M}$ and $\mathcal{C}$ it is convenient to consider the following partition of the parameter space Θ associated with $\theta = (C, D)$:

$$\begin{aligned}
\Theta_1 &:= \{(C_1, D_1) \in \Theta : D_1 = \{0_n\}\}, \\
\Theta_2 &:= \{(C_1, D_1) \in \Theta : D_1 = \mathbb{R}^n\}, \\
\Theta_3 &:= \{(C_1, D_1) \in \Theta \diagdown \Theta_1 : D_1 \text{ contains no line}\}, \\
\Theta_4 &:= \{(C_1, D_1) \in \Theta : D_1 \text{ is a proper linear subspace}\}, \\
\Theta_5 &:= \{(C_1, D_1) \in \Theta \diagdown (\Theta_2 \cup \Theta_4) : D_1 \text{ contains lines}\}.
\end{aligned}$$

Observe that $\theta \in \Theta_1 \cup \Theta_2$ if and only if $\mathcal{F}(\theta)$ is either a compact set or the whole space $\mathbb{R}^n$. Obviously, $\Theta = \cup_{i=1}^{5}\Theta_i$, Θ_i are disjoint sets $i = 1, ..., 5$. Θ_1, Θ_2, Θ_3, and Θ_4 are open subsets of Θ.

Proposition 9 Θ_5 *is a nowhere dense set such that* $\Theta_5 \subset \operatorname{cl}\Theta_3$.

Proof. First we show that $\Theta_5 \subset \operatorname{cl}\Theta_3$. Let us have a point $\theta = (C, D) \in \Theta_5$. Since D contains lines and it is not a subspace, $D \cap (\operatorname{lin} D)^{\perp}$ is a nontrivial pointed cone and therefore its dual cone has a nonempty interior, so that there exists $y \in \mathbb{R}^n \setminus \{0^n\}$ such that $y'd > 0$ for every $d \in D \cap (\operatorname{lin} D)^{\perp}$, $d \neq \{0_n\}$. Define the pointed cone

$$K_{\varepsilon}^{e_n} := \operatorname{cone}\left\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 \leq 1,\ x_n = \varepsilon\right\}. \tag{8}$$

Now, we consider the cone K_{ε}^{y}, which is the same, but with its symmetry axis oriented along y instead of e_n. Let us consider $\theta_{\varepsilon} = (C, D \cap K_{\varepsilon}^{y})$. Obviously, $\theta_{\varepsilon} \in \Theta_3$ and $\rho(\theta, \theta_{\varepsilon}) \to 0$ as $\varepsilon \searrow 0$. Thus, $\theta \in \operatorname{cl}\Theta_3$.

Now we suppose that Θ_5 is a dense subset of some nonempty open set $V \subset \Theta$. Then $V \cap \Theta_i = \emptyset$, $i = 1, ..., 4$. Therefore, $V \subset \Theta_5$. Taking an arbitrary $\theta \in V$, we have $\theta \in \operatorname{cl}\Theta_3$ (because $\theta \in \Theta_5$) and $\theta \notin \operatorname{cl}\Theta_3$ (because $V \cap \Theta_3 = \emptyset$). Thus, Θ_5 is a nowhere dense set. $\square$

Note that if $\theta = (C, D) \in \Theta_4$, then $D = 0^{+}\mathcal{F}(\theta) = \operatorname{lin}\mathcal{F}(\theta) = \operatorname{lin} D$, hence $D \cap (\operatorname{lin} D)^{\perp} = \{0_n\}$. $D \cap K_{\varepsilon}^{y} = \{0_n\}$, where $y \neq \{0_n\}$ is perpendicular to D and $\rho(\theta, \theta_{\varepsilon}) \nrightarrow 0$ $\varepsilon \searrow 0$. The only points which are close to θ in this case are the points which have as a cone a proper linear subspace of the same dimension.

The mappings $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{F}$ coincide on Θ_1 whereas $\mathcal{M}$ and $\mathcal{C}$ are constant (with image $\{0_n\}$) on Θ_2, where $\mathcal{F}$ is also constant (with image $\mathbb{R}^n$). So, $\mathcal{M}$ and $\mathcal{C}$ are closed, lsc, and usc on $\Theta_1 \cup \Theta_2$. The stability properties of $\mathcal{M}$ and $\mathcal{C}$ are non trivial when $\theta \in \Theta_3 \cup \Theta_5 \cup\ \Theta_4$. On the other hand, $\mathcal{M}$ and $\mathcal{C}$ are interesting only if $\theta \in \Theta_3$ because then, for θ_1 in certain neighborhood of θ, $\mathcal{M}(\theta_1)$ and $\mathcal{C}(\theta_1)$ are the efficient set of $\mathcal{F}(\theta_1)$ relative to the cone $0^{+}\mathcal{F}(\theta_1)$ and the smallest compact component of $\mathcal{F}(\theta_1)$, respectively. Thus we focus our attention on the case $\theta \in \Theta_3$. Even in the best situation that $\theta \in \Theta_3$, it is possible that the mappings $\mathcal{M}$ and $\mathcal{C}$ are neither upper semicontinuous, nor closed: consider $\theta = (C, D)$ and $\theta_k = (C_k, D_k)$, $k \in \mathbb{N}$, as in Example 1. Then, $\mathcal{M}(\theta_k) = \mathcal{C}(\theta_k) = C$ for all $k \in \mathbb{N}$ whereas $\mathcal{M}(\theta) = \mathcal{C}(\theta) = \{0_2\} \nsubseteq C + \operatorname{int} B_2$, so that $\mathcal{M}$ and $\mathcal{C}$ are not usc at θ; taking $x_k = (1, 0) \in \mathcal{M}(\theta_k) = \mathcal{C}(\theta_k)$, $k \in \mathbb{N}$, we conclude that $\mathcal{M}$ and $\mathcal{C}$ are not closed at θ because $x_k \to (1, 0) \notin \{0_2\}$. Even more, the next example shows that $\mathcal{F}$ is not necessary closed-valued on Θ_3.

Example 10 *Let $n = 3$ and consider the set $F = C + D$, where*

$$C := \operatorname{conv}\left\{\left\{x \in \mathbb{R}^3 : (x_1 - 1)^2 + x_2^2 \leq 1, x_3 = 1\right\} \cup \{0_3\}\right\} \text{ and } D := \operatorname{cone}\{e_3\}.$$

Obviously,

$$M(F) = \left\{x \in \mathbb{R}^3 : (x_1 - x_3)^2 + x_2^2 = x_3^2, 0 \leq x_3 \leq 1, x_1^2 + x_2^2 \neq 0\right\} \cup \{0_3\}$$

is not closed.

Next we show that both multifunctions are at least lower semicontinuous on Θ_3.

Proposition 11 *$\mathcal{M}$ and $\mathcal{C}$ are lsc at every $\theta \in \Theta_3$.*

Proof. Let $\theta = (C, D) \in \Theta_3$. We have

$$\mathcal{M}(\theta_1) = M(C_1 + D_1) = \{x \in C_1 + D_1 : (x - D_1) \cap (C_1 + D_1) = \{x\}\} \subset C_1$$

for all $\theta_1 \in \Theta_3$.

First, we shall prove that the multivalued mapping $\mathcal{M}$ is lsc at θ. Let us suppose the contrary, i.e., there exists an open set $W \subset \mathbb{R}^n$ such that $W \cap M(C + D) \neq \emptyset$ and a sequence $\{\theta_k\} \subset \Theta$, such that $\theta_k = (C_k, D_k) \to \theta$ and for every $k \in \mathbb{N}$, $W \cap M(C_k + D_k) = \emptyset$ whereby $W \cap \operatorname{cl} M(C_k + D_k) = \emptyset$, as well. W.l.o.g., we may assume that $\{\theta_k\} \subset \Theta_3$, therefore $\operatorname{cl}\operatorname{conv} M(C_k + D_k) \subset C_k$.

Let $x \in W \cap M(C + D) \subset C$. By the Hausdorff convergence $C_k \to^H C$ there exists a sequence $\{x_k\}$, $x_k \in C_k$ for all $k \in \mathbb{N}$, such that $x_k \to x$. By (1), given $k \in \mathbb{N}$, $\mathcal{F}(\theta_k) = \operatorname{cl} M(C_k + D_k) + D_k$ hence $x_k = y_k + d_k$, where $y_k \in \operatorname{cl} M(C_k + D_k) \subset C_k$ and $d_k \in D_k$. By the same convergence, w.l.o.g., we can assume that $y_k \to y \in C$ and therefore $d_k = x_k - y_k$ is convergent too. Let $d_k \to d \in \mathbb{R}^n$. Obviously, $d_k \in (\|d\| + 1) B_n$ for k large enough. Having in mind that

$$d_H(D \cap (\|d\| + 1) B_n, D_k \cap (\|d\| + 1) B_n) \to^H 0,$$

we get that $d \in D$. We have that $d \neq \{0_n\}$, otherwise we have $y_k \to x$, with $x \in W$ and $y_k \notin W$ for all $k \in \mathbb{N}$, which is a contradiction. Then

$$x \neq x - d = y \in (x - D) \cap C \subset (x - D) \cap (C + D),$$

in contradiction with

$$x \in M(C + D) = \{z \in C + D : (z - D) \cap (C + D) = \{z\}\},$$

and this contradiction shows that $\mathcal{M}$ is lsc at θ.

Now, by the well known theorem on the convex hull mapping ([15]), we get that $\operatorname{conv} \mathcal{M}$ is lsc at θ, so that $\mathcal{C} = \operatorname{cl} \operatorname{conv} \mathcal{M}$ is lsc at θ too. $\square$

The next example shows that $\mathcal{M}$ and $\mathcal{C}$ can be highly unstable on $\Theta_2 \cup \Theta_4 \cup \Theta_5$, where $\mathcal{M}(\theta)$ and $\mathcal{C}(\theta)$ cannot be interpreted as the efficient set (w.r.t. its recession cone) and the smallest compact component of $\mathcal{F}(\theta)$, respectively (actually, $\mathcal{F}(\theta)$ has no smallest compact component when $\theta \in \Theta_2 \cup \Theta_4 \cup \Theta_5$).

Example 12 *Let $n \geq 2$ and $\theta = (C, D)$, with*

$$C = \{e_1\} \text{ and } D = \{x \in \mathbb{R}^n : x_n \geq 0\}.$$

Obviously, $\theta \in \Theta_5$ and $\mathcal{M}(\theta) = \mathcal{C}(\theta) = \{0_n\}$. Let us consider the pointed closed convex cones $K_\varepsilon^{e_n}$, $0 < \varepsilon < 1$ ($K_\varepsilon^{e_n}$ was defined in (8)). Taking $\theta_\varepsilon := (C, K_\varepsilon^{e_n}) \in \Theta_3$, we have $\mathcal{M}(\theta_\varepsilon) = \mathcal{C}(\theta_\varepsilon) = \{e_1\}$. Since

$$\rho(\theta, \theta_\varepsilon) = \sqrt{2\left(1 - \sqrt{1 - \varepsilon^2}\right)} \to 0 \text{ as } \varepsilon \searrow 0,$$

$\mathcal{M}$ and $\mathcal{C}$ are neither upper nor lower semicontinuous, nor closed at θ.

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