Motzkin decomposition of closed convex sets via truncation

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Abstract

A nonempty set F is called Motzkin decomposable when it can be expressed as the Minkowski sum of a compact convex set C with a closed convex cone D. In that case, the sets C and D are called compact and conic components of F. This paper provides new characterizations of the Motzkin decomposable sets involving truncations of F (i.e., intersections of F with closed halfspaces), when F contains no lines, and truncations of the intersection $\widehat{F}$ of F with the orthogonal complement of the lineality of F, otherwise. In particular, it is shown that a nonempty closed convex set F is Motzkin decomposable if and only if there exists a hyperplane H parallel to the lineality of F such that one of the truncations of $\widehat{F}$ induced by H is compact whereas the other one is a union of closed halflines emanating from H. Thus, any Motzkin decomposable set F can be expressed as $F = C + D$, where the compact component C is a truncation of $\widehat{F}$. These Motzkin decompositions are said to be of type T when F contains no lines, i.e., when C is a truncation of F. The minimality of this type of decompositions is also discussed.

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1 Introduction

A nonempty set $F \subset \mathbb{R}^n$ is called *Motzkin decomposable* (M-decomposable in short) if there exist a compact convex set C and a closed convex cone D such that $F = C + D$. Then we say that the pair (C, D) is a *Motzkin decomposition* of F with *compact and conic components* C and D, respectively. This paper is mainly focussed on those Motzkin decompositions of F such that the compact component is a *truncation* of F (i.e., the intersection of F with some closed halfspace), which are called of *type* T (MT-decomposition in short).

The classical Motzkin Theorem [9] asserts that any polyhedral convex set is M-decomposable. For this reason, Bair ([1], [2]) called these sets *generalized convex polyhedral* (unfortunately, the same name has been given by other authors to those sets whose non-empty intersection with polytopes are polytopes, which are also called *quasipolyhedral* or *boundedly polyhedral*). In the same vein, a function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is called *Motzkin decomposable* (M-decomposable in short) when its epigraph is M-decomposable. If f is M-decomposable, it is convex and lower semicontinuous (lsc in short) [4] and so any local minimum of f is a global minimum of f. The main property of the M-decomposable functions in the optimization framework is that they achieve their minima when they are bounded from below on $\mathbb{R}^n$.

Any M-decomposable set F is closed, as it is the sum of a compact set with a closed set. Moreover F has a unique conic component $D = 0^+F$ (the recession cone of F) but multiple compact components when F is unbounded. Five different characterizations of the M-decomposable sets have been given in [3] and two more in [4], where calculus rules for M-decomposable sets and functions have been developed. The most relevant of these characterizations involve the intersection $\widehat{F}$ of F with the orthogonal complement of the lineality of F, with $\widehat{F} = F$ whenever F contains no lines. In the latter case, there exists a unique compact component of F, say C_1, such that $C_1 \subset C$ for any compact component C of F; such a set C_1 is called the *minimal* (or the *smallest*) *compact component* of F (the M-minimal component in short). The M-minimal component of an M-decomposable set F without lines has been characterized in different ways in [3] and [4].

We associate with any hyperplane H such that $F \cap H \neq \emptyset$, which is called the *slice* of F induced by H, the truncations of F induced by H, $F \cap H^+$ and $F \cap H^-$, where H^+ and H^- denote the closed halfspaces whose common boundary is H. If $F = C + 0^+F$, with C being a compact truncation of F, we say that $(C, 0^+F)$ is a Motzkin decomposition of F of *type T*. When a compact component of F, say C_2, is a truncation of F and $C_2 \subset C$ for any compact component C of F of the same type, then C_2 is called the *minimal compact component* of F *of type T* (MT-minimal component in short). Two questions arise in connection with the MT-minimal components:

i) Does any Motzkin decomposable set without lines admit a minimal Motzkin decomposition of type T?
ii) If F admits a minimal Motzkin decomposition of type T, does the MT-minimal component of F coincide with the M-minimal component of F?

In this paper we provide a negative answer for the first question and a positive one for the second one. If F is a compact convex set, then any supporting hyperplane to F provides, by truncation, the unique (type T) Motzkin decomposition of F, $(F, \{0_n\})$, so that F is the MT-minimal component of F. Thus, we analyze in this paper the M-decomposability of unbounded closed convex sets. The paper is organized as follows. Section 2 recalls the basic characterization of the M-decomposability of F in terms of the boundedness of the set of extreme points of $\widehat{F}$ ([3, Theorem 11]), which provides alternative proofs of classical results due to Bair [2] and new results on M-decomposable sets and functions. Section 3 characterizes the compact truncations and slices of closed convex sets whereas Section 4 provides new geometric characterizations of the M-decomposable sets in terms of the existence of a hyperplane H whose associated truncations for $\widehat{F}$ satisfy certain conditions, e.g., that one of them is compact whereas the other one is the union of halflines emanating from H (or, equivalently, its extreme points are contained in H). Finally, Section 5 characterizes those M-decomposable sets without lines that have a minimal Motzkin decomposition of type T.

Throughout the paper we use the following notation. For any $X \subset \mathbb{R}^n$, we denote by $\operatorname{int} X$, $\operatorname{cl} X$, $\operatorname{bd} X$, $\operatorname{rint} X$, $\operatorname{span} X$, $\operatorname{conv} X$, and $\operatorname{cone} X = \mathbb{R}_+ \operatorname{conv} X$, the *interior*, the *closure*, the *boundary,* the *relative interior*, the linear subspace spanned by X, the *convex hull* of X, and the *convex conical hull* of X, respectively. If X is a nonempty convex set, $\dim X$ denotes the dimension of X.

The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $x'y$, the Euclidean norm of x by $\|x\|$, the zero vector by 0_n, the closed unit ball by B_n, and the unit sphere by S^{n-1}. The *orthogonal complement* of a linear subspace X is $X^\perp := \{y \in \mathbb{R}^n : x'y = 0 \ \forall x \in X\}$. Given a convex cone X, its *dual cone* is $X^\circ := \{y \in \mathbb{R}^n : x'y \geq 0 \ \forall x \in X\}$. If X is a convex set, $\operatorname{extr} X$, 0^+X and $\operatorname{lin} X := (0^+X) \cap (-0^+X)$ denote the *set of extreme points*, the *recession cone* and the *lineality space* of X, respectively.

Given $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we denote by $\operatorname{epi} f$ and $\operatorname{dom} f$ its *epigraph* and its *domain*, respectively. Given $\alpha \in \mathbb{R}$, $\max\{f, \alpha\}$ is said to be the *truncation* of f by α (observe that $\operatorname{epi} \max\{f, \alpha\}$ is a truncation of $\operatorname{epi} f$).

Any set $X \subset \mathbb{R}^n$ is represented in a unique way by its *indicator function*

$$\delta_X(x) := \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{otherwise.} \end{cases}$$

The indicator function δ_X is M-decomposable if and only if X is M-decomposable.

2 Motzkin sets and functions revisited

Given a closed convex set F such that $\emptyset \neq F \subset \mathbb{R}^n$, we denote $Q(F) := \operatorname{cl}\operatorname{conv}\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$. So, if F contains no lines, $Q(F) = \operatorname{cl}\operatorname{conv}\operatorname{extr} F$. The next result characterizes the Motzkin decomposability of F in terms of the boundedness of $Q(F)$. We illustrate the importance of this characterization for the analysis of Motzkin decomposable sets and functions with several immediate applications.

Theorem 1 *([4, Theorem 11]) Let F be a closed convex set, $\emptyset \neq F \subset \mathbb{R}^n$. Then the following statements hold:*
(i) F is Motzkin decomposable if and only if $\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ *is bounded. In that case, $Q(F)$ is a compact component of F.*
(ii) If F is a Motzkin decomposable set without lines, then $Q(F)$ is the M-minimal component of F.

An immediate consequence of Theorem 1 in the Motzkin decomposition framework is that the intersection of an arbitrary family of compact components of F is a compact component too, whereas the counterpart of this intersection property for the subfamily of compact components of F which are truncations of F fails (see Example 20 in Section 3 below, where F is a convex polyhedral set). Nevertheless, we get the following characterization of the hyperplanes inducing a Motzkin decomposition of type T.

Corollary 2 *A hyperplane H induces a Motzkin decomposition of type T of a set F if and only if $F \cap H^+$ is compact and* $\operatorname{extr} F \subset H^+$, *where H^+ denotes one of the closed halfspaces determined by H.*

Proof: For the "only if" part, take H such that $F \cap H^+$ is compact and $F = F \cap H^+ + 0^+F$. Then it is easy to see that $\operatorname{extr} F \subset F \cap H^+ \subset H^+$. For the "if" part, note that $\operatorname{extr} F \subset H^+$ entails $\operatorname{extr} F \subset F \cap H^+$, so that $\operatorname{extr} F$ is bounded. Invoking Theorem 1, we have

$$F = Q(F) + 0^+F \subset F \cap H^+ + 0^+F \subset F,$$

implying that $F = F \cap H^+ + 0^+F$, and this a Motzkin decomposition of F of type T. $\square$

Corollary 3 *([2, 2.4]) Any face of a Motzkin decomposable set is Motzkin decomposable too.*

Proof: Let G be a face of an M-decomposable set F. Obviously, $\operatorname{lin} G \subset \operatorname{lin} F$. We prove now that $\operatorname{lin} F \subset \operatorname{lin} G$. Let $u \in \operatorname{lin} F$ and take an arbitrary $x \in G$. Given $\lambda \geq 0$, we have $x = \frac{1}{2}\left[(x + \lambda u) + (x - \lambda u)\right]$, with $x \pm \lambda u \in F$. Thus $x \pm \lambda u \in G$ for all $\lambda \geq 0$, i.e., $u \in \operatorname{lin} G$. Denote $L := \operatorname{lin} F = \operatorname{lin} G$.

Since $G \cap L^\perp$ is a face of $F \cap L^\perp$, $\operatorname{extr}\left(G \cap L^\perp\right) \subset \operatorname{extr}\left(F \cap L^\perp\right)$ and the conclusion follows from Theorem 1. $\square$

The truncations and slices of an M-decomposable set are not necessarily M-decomposable: if F is the "ice-cream cone" with axis $(0,0,1)$, i.e. $F = \left\{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} \leq x_3\right\}$, and H^+ is one of the closed halfspaces determined by a vertical hyperplane $H \subset \mathbb{R}^3$, then $\operatorname{extr}(F \cap H^+) = \operatorname{extr}(F \cap H) = \{0_3\}$ when $0_3 \in H$, whereas $\operatorname{extr}(F \cap H) \subset \operatorname{extr}(F \cap H^+)$, both sets being unbounded because $\operatorname{extr}(F \cap H)$ is a hyperbola, otherwise. Nevertheless, if H is a hyperplane supporting an M-decomposable set F, the corresponding slice is M-decomposable by Corollary 3. Concerning functions, although Example 20 in [4] shows that the sublevel sets of the M-decomposable functions are not necessarily M-decomposable, Corollary 3 will allow us to show that the optimal set of any unconstrained optimization problem with M-decomposable objective function inherits this desirable property.

Corollary 4 *If $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is Motzkin decomposable and bounded from below, then the set of global minima of f is Motzkin decomposable.*

Proof: The set of global minima of f is $f^{-1}(\alpha)$, where $\alpha := \inf\{f(x) : x \in \mathbb{R}^n\}$. Since f is lsc and convex [4], $f^{-1}(\alpha) = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is a nonempty closed convex set. The hyperplane $H := \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = \alpha\}$ supports $\operatorname{epi} f$ at any point (x, α) such that $x \in f^{-1}(\alpha)$. Then, by Corollary 3, $\operatorname{epi} f \cap H = f^{-1}(\alpha) \times \{\alpha\}$ is M-decomposable. Hence $f^{-1}(\alpha)$ is M-decomposable too. $\square$

In general, the restriction of an M-decomposable function to a hyperplane is not M-decomposable. For instance, if $f(x) = \|x\|$ and H is a hyperplane in $\mathbb{R}^2$, then

$$(f\mid_H)(x) := \begin{cases} \|x\|, \ x \in H, \\ +\infty, \text{ otherwise,} \end{cases}$$

is M-decomposable if and only if $0_2 \in H$. From Corollary 3, if F is M-decomposable and G is a face of F, then δ_G is M-decomposable because $\operatorname{epi} \delta_G$

is a face of $\operatorname{epi}\delta_F$. This observation suggests the next result.

Corollary 5 *If $f:\mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is Motzkin decomposable and H is a supporting hyperplane to* $\operatorname{dom} f$, *then $f\mid_H$ is Motzkin decomposable.*

Proof: Since $H\times\mathbb{R}$ is a supporting hyperplane to the M-decomposable set $\operatorname{epi} f$, $\operatorname{epi}(f\mid_H) = \operatorname{epi} f \cap (H\times\mathbb{R})$ is M-decomposable too. The conclusion follows from Corollary 3. □

Proposition 6 *Let A be a closed convex set and let B and $A+B$ be Motzkin decomposable sets such that $0^+B \subset \operatorname{lin} A$. Then A is Motzkin decomposable.*

Proof: Let A be a closed convex set and let B and $F := A+B$ be M-decomposable sets. We have $B = C_1 + D_1$ and $F = C_2 + D_2$ for some compact convex sets C_1 and C_2 and some closed convex cones D_1 and D_2. Denote $L := \operatorname{lin} A$. Let a be an exposed point of $A\cap L^\perp$ and $p\in\mathbb{R}^n$ be such that a is the unique minimizer of $x\mapsto p'x$ on $A\cap L^\perp$. We can assume w.l.o.g. that $p\in L^\perp$. Then the set of minimizers of $x\mapsto p'x$ on A is $\{a\}+L$. We also have $p\in L^\perp \subset (0^+B)^\circ = D_1^\circ$, so that the infimum of $x\mapsto p'x$ on B is achieved at some point $b\in C_1$. Clearly, $a+b$ is a minimizer of $x\mapsto p'x$ on

$$\left(A\cap L^\perp\right) + L + B = A + B = C_2 + D_2.$$

It follows that p belongs to the dual cone D_2° of D_2. We have $a+b = c+d$ for some $c\in C_2$ and $d\in D_2$. Since $p\in D_2^\circ$,

$$p'(a+b) = p'c + p'd \geq p'c \geq p'(a+b);$$

hence $p'd = 0$. As a consequence, for every $\lambda \geq 0$ the point $c+\lambda d$ is a minimizer of $x\mapsto p'x$ on $C_2 + D_2$. This implies that $d\in L$, because the set of minimizers is contained in $\{a\}+L+B$. Hence $a = c-b+d \in C_2 - C_1 + L$. We have thus proved that the set of exposed points of $A\cap L^\perp$ is contained in the compact set $(C_2 - C_1 + L)\cap L^\perp$. By Straszewicz's Theorem [10, Theorem 18.6], $\operatorname{extr}\left(A\cap L^\perp\right) \subset (C_2 - C_1 + L)\cap L^\perp$. According to Theorem 1, we conclude that A is M-decomposable. □

The assumption $0^+B \subset \operatorname{lin} A$ in Proposition 6 is not superfluous: consider $A = \{x\in\mathbb{R}^2 : x_2 \geq x_1^2\}$ and $B = \{x\in\mathbb{R}^2 : x_1 \geq 0\}$.

We finish this section by showing that an interesting result by Bair on M-decomposable sets can be obtained in a straightforward way from Proposition 6 and a lemma (see Lemma 8 below), which is a consequence of the so-called Lexicographical Separation Theorem. To this aim, we introduce the necessary notation: for $x = (x_1,...,x_n), (y_1,...,y_n) \in \mathbb{R}^n$, by $x <_L y$ we mean that $x \neq y$ and for $k = \min\{i\in\{1,...,n\} : x_i \neq y_i\}$ we have $x_k < y_k$; we write $x \leq_L y$ if $x <_L y$ or $x = y$.

Theorem 7 *Lexicographical Separation Theorem ([7, p. 258], [8, Theorem 1.1]) Let C be a convex subset of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n \setminus C$. Then there exists an $n \times n$ matrix M such that $Mx <_L Mx_0$ for all $x \in C$.*

Lemma 8 *Let $A, B \subset \mathbb{R}^n$. If A is convex, B is compact and $A + B$ is closed, then A is closed.*

Proof: Let $\{a_k\}$ be a sequence in A converging to $a \in \mathbb{R}^n$. Suppose that $a \notin A$. Then, by the Lexicographical Separation Theorem, there exists an $n \times n$ matrix M such that $Mx <_L Ma$ for all $x \in A$. Take a lexicographical maximum b over B of the mapping $x \longmapsto Mx$, that is, $My \leq_L Mb$ for all $y \in B$. The existence of such a lexicographical maximum follows by successively applying Weierstrass Theorem n times. We thus have $M(x + y) <_L M(a + b)$ for all $x \in A$ and $y \in B$. Hence $a + b \notin A + B$, which, as this point is the limit of the sequence $\{a_k + b\}$, contradicts the closedness assumption on $A + B$. Thus a must belong to A, and therefore A is closed. □

Corollary 9 *([1, Proposition 1]) Let A and B be convex sets such that B is bounded and $A + B$ is M-decomposable. Then A is M-decomposable too.* □

3 Compact truncations

We associate with $a \in \mathbb{R}^n \setminus \{0_n\}$ and $\alpha \in \mathbb{R}$ the hyperplane $H_{a,\alpha} := \{x \in \mathbb{R}^n : a'x = \alpha\}$ and the corresponding closed halfspaces $H_{a,\alpha}^+ := \{x \in \mathbb{R}^n : a'x \geq \alpha\}$ and $H_{a,\alpha}^- := \{x \in \mathbb{R}^n : a'x \leq \alpha\}$. In this section we consider as given a closed convex set F such that $F \cap H_{a,\alpha} \neq \emptyset$ and analyze the boundedness of the truncations and the slice induced by $H_{a,\alpha}$, $F \cap H_{a,\alpha}^-$, $F \cap H_{a,\alpha}^+$, and $F \cap H_{a,\alpha}$.

Observe that the truncations of F that are not slices have the the same dimension as F, i.e.,

$$\dim \left(F \cap H_{a,\alpha}^+\right) < \dim F \Rightarrow F \cap H_{a,\alpha}^+ = F \cap H_{a,\alpha}. \tag{1}$$

For proving it, assume the contrary, that is, the existence of $\overline{x} \in F \cap H_{a,\alpha}^+$ such that $a'\overline{x} > \alpha$. Since $F = \operatorname{cl} \operatorname{rint} F$, there exists $\widehat{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $(\widehat{x} + \varepsilon B_n) \cap \operatorname{aff} F \subset F$ and $a'x > \alpha$ for all $x \in \widehat{x} + \varepsilon B_n$. Then $(\widehat{x} + \varepsilon B_n) \cap \operatorname{aff} F \subset F \cap H_{a,\alpha}^+$, so that $\dim \left(F \cap H_{a,\alpha}^+\right) = \dim F$.

On the other hand, the truncations of F that are slices are exposed faces of F. The proof is immediate whenever the slice corresponds to the same couple (a, α), i.e., $F \cap H_{a,\alpha}^+ = F \cap H_{a,\alpha}$, because $F \cap H_{a,\alpha}$ is the set of maximizers of the function $x \longmapsto a'x$ over the set F, so by definition it is an exposed face. Assume now that $F \cap H_{a,\alpha}^+ = F \cap H_{b,\beta}$, with $(a, \alpha) \neq (b, \beta)$. Then $H_{b,\beta}$ is a supporting hyperplane of F. Suppose it is not. Then the sets $F \cap \operatorname{int} H_{b,\beta}^+$ and $F \cap \operatorname{int} H_{b,\beta}^-$

are nonempty, which implies that $\operatorname{conv}\left(\left(F \cap \operatorname{int} H_{b,\beta}^{+}\right) \cup \left(F \cap \operatorname{int} H_{b,\beta}^{-}\right)\right)$ intersects $F \cap H_{b,\beta} = F \cap H_{a,\alpha}^{+}$, but this is absurd because

$$\begin{aligned}\operatorname{conv}\left(\left(F \cap \operatorname{int} H_{b,\beta}^{+}\right) \cup \left(F \cap \operatorname{int} H_{b,\beta}^{-}\right)\right) &= \operatorname{conv}\left(F \setminus \left(F \cap H_{b,\beta}\right)\right) \\ &= \operatorname{conv}\left(F \setminus \left(F \cap H_{a,\alpha}^{+}\right)\right) \\ &= F \cap \operatorname{int} H_{a,\alpha}^{-}.\end{aligned}$$

Lemma 10 *Let $\emptyset \neq C \subset \mathbb{R}^n$ be a closed convex cone and $d \in \mathbb{R}^n \setminus \{0_n\}$. Then $d \in \operatorname{int} C^{\circ}$ if and only if for every $c \in C \setminus \{0_n\}$ it holds $d'c > 0$. Thus, C is pointed if and only if $\operatorname{int} C^{\circ} \neq \emptyset$.*

Proof: Assume that $d \in \operatorname{int} C^{\circ}$ and there exists $c \in C \setminus \{0_n\}$ such that $d'c = 0$. We have $d - \frac{c}{k} \in C^{\circ}$ for k sufficiently large. Then $\left(d - \frac{c}{k}\right)' c = 0 - \frac{\|c\|^2}{k} < 0$, which is a contradiction.

Assume that $d'c > 0$ for every $c \in C \setminus \{0_n\}$ and suppose that $d \notin \operatorname{int} C^{\circ}$, i.e., $d \in \operatorname{bd} C^{\circ}$. There exists a sequence $\{d_k\}_{k \geq 1}$ such that $d_k \to d$ and $d_k \notin C^{\circ}$, $k = 1, 2, \ldots$. Then we can find $c_k \in C$ such that $\|c_k\| = 1$ and $d_k' c_k < 0$, $k = 1, 2, \ldots$ W.l.o.g. we may assume that $c_k \to c \in C$, with $\|c\| = 1$. Therefore, after taking the limit we get the contradiction $d'c \leq 0$.

Now we assume that C is pointed. Since $(\operatorname{span} C^{\circ})^{\perp} \subset C^{\circ\circ} = C$, $(\operatorname{span} C^{\circ})^{\perp} = \{0_n\}$, that is, $\operatorname{span} C^{\circ} = \mathbb{R}^n$, which is equivalent to $\operatorname{int} C^{\circ} \neq \emptyset$. Conversely, if $\operatorname{int} C^{\circ} \neq \emptyset$, we can take $d \in \operatorname{int} C^{\circ}$; if $\pm c \in C \setminus \{0_n\}$, we have $\pm d'c > 0$ (contradiction). Hence C is pointed. □

Lemma 11 *Let $\emptyset \neq C \subset \mathbb{R}^n$ be a closed, convex, pointed cone and $a \neq 0_n$. Then $C \cap H_{a,0} = \{0_n\}$ if and only if $a \in \operatorname{int} C^{\circ} \cup -\operatorname{int} C^{\circ}$.*

Proof: Let us suppose that $a \in \operatorname{int} C^{\circ} \cup -\operatorname{int} C^{\circ}$ and there exists $d \neq 0_n$ such that $d \in C \cap H_{a,0}$. Then $a'd = 0$, which is a contradiction by Lemma 10.

Now, let $C \cap H_{a,0} = \{0_n\}$ and suppose that $a \notin \operatorname{int} C^{\circ} \cup -\operatorname{int} C^{\circ}$. Then there exists d^{+}, $d^{-} \in C$ different from 0_n such that $a'd^{+} \leq 0$ and $a'd^{-} \geq 0$. The assumption implies that $a'd^{+} < 0$ and $a'd^{-} > 0$. Let $\alpha > 0$ be such that $a'd^{+} + \alpha a'd^{-} = a'\left(d^{+} + \alpha d^{-}\right) = 0$. The vector $d^{+} + \alpha d^{-} \in C$ and is different from 0_n because C is a pointed cone. This is a contradiction. □

Theorem 12 *Let $F \subset \mathbb{R}^n$ be an unbounded, closed, convex set, $a \in \mathbb{R}^n \setminus \{0_n\}$ and $\alpha \in \mathbb{R}$ such that $F \cap H_{a,\alpha} \neq \emptyset$. The following statements are true:*
(i) $F \cap H_{a,\alpha}^{-}$ is compact if and only if $a \in \operatorname{int}\left(0^{+}F\right)^{\circ}$.
(ii) $F \cap H_{a,\alpha}^{+}$ is compact if and only if $a \in -\operatorname{int}\left(0^{+}F\right)^{\circ}$.
(iii) $a \in \operatorname{int}\left(0^{+}F\right)^{\circ} \cup -\operatorname{int}\left(0^{+}F\right)^{\circ}$ if and only if $F \cap H_{a,\alpha}$ is compact and F contains no lines.

Proof: By the Convex Separation Theorem we can write $F = \bigcap_{t\in T} H^{+}_{a_t,\alpha_t}$, for two sets $\{a_t, t \in T\} \subset \mathbb{R}^n$ and $\{\alpha_t, t \in T\} \subset \mathbb{R}$. Obviously,

$$\begin{aligned}(0^+F)^\circ &= \{x \in \mathbb{R}^n : a_t' x \geq 0, t \in T\}^\circ \\ &= (\operatorname{cone}\{a_t, t \in T\})^{\circ\circ} \\ &= \operatorname{cl}\operatorname{cone}\{a_t, t \in T\}.\end{aligned} \tag{2}$$

(i) By [5, Corollary 9.3.1] and (2), $F \cap H^-_{a,\alpha}$ is bounded if and only if the sublevel sets of the linear semi-infinite programming problem

$$\min\{a'x \text{ s.t. } a_t'x \geq b_t, t \in T\}$$

are bounded if and only if $a \in \operatorname{int}\operatorname{cone}\{a_t, t \in T\} = \operatorname{int}(0^+F)^\circ$.

(ii) By (i), $F \cap H^+_{a,\alpha} = F \cap H^-_{-a,-\alpha}$ is bounded if and only if $-a \in \operatorname{int}(0^+F)^\circ$.

(iii) If $a \in \operatorname{int}(0^+F)^\circ \cup -\operatorname{int}(0^+F)^\circ$ then, by (i) and (ii), at least one of the two truncations of F induced by $H_{a,\alpha}$ is bounded. Thus $F \cap H_{a,\alpha}$ is bounded. Moreover, $\operatorname{int}(0^+F)^\circ \neq \emptyset$ means that 0^+F is pointed (by Lemma 10), i.e., that F contains no lines.

Now we assume that $H_{a,\alpha} \cap F$ is a compact set and F contains no lines. Then $H_{a,0} \cap 0^+F = 0^+(H_{a,\alpha} \cap F) = \{0_n\}$; hence, since 0^+F is pointed, from Lemma 11, we get $a \in \operatorname{int}(0^+F)^\circ \cup -\operatorname{int}(0^+F)^\circ$. □

From the argument for proving statement (i), if $a \in \operatorname{int}(0^+F)^\circ$, then $F \cap H^-_{a,\alpha}$ is compact for any $\alpha \in \mathbb{R}$, but the converse does not hold when $F \cap H^-_{a,\alpha} = \emptyset$. In fact, we may have $\operatorname{int}(0^+F)^\circ = \emptyset$ (e.g., take $F = \{x \in \mathbb{R}^2 : x_2 \geq 1\}$, $a = (0,1)$ and $\alpha = -1$).

Example 13 *Consider the closed convex set*

$$\begin{aligned}F &= \left\{(x,y) : \sqrt{x} + \sqrt{y} \geq 1, x \geq 0, y \geq 0\right\} \\ &= \left\{(x,y) : \sqrt{x} + \sqrt{y} = 1, x \geq 0, y \geq 0\right\} + \mathbb{R}^2_+.\end{aligned}$$

Here $0^+F = \mathbb{R}^2_+ = (0^+F)^\circ$ *is pointed and full dimensional. Moreover, given* $a \in \mathbb{R}^2 \setminus \{0_2\}$ *and* $\alpha \in \mathbb{R}$, $F \cap H_{a,\alpha} \neq \emptyset$ *if and only if* $a \notin \mathbb{R}^2_+ \cup -\mathbb{R}^2_+$ *or* $a \in \mathbb{R}^2_+$ *with* $\frac{a_1 a_2}{a_1 + a_2} \leq \alpha$, *or* $a \in -\mathbb{R}^2_+$ *with* $\frac{a_1 a_2}{a_1 + a_2} \geq \alpha$. *Moreover, assuming that* $F \cap H_{a,\alpha} \neq \emptyset$, $F \cap H^-_{a,\alpha}$ *is compact if and only if the set* $0^+\left(F \cap H^-_{a,\alpha}\right) = 0^+(F) \cap H^-_{a,0} = \mathbb{R}^2_+ \cap H^-_{a,0}$ *reduces to* $\{0_2\}$, *that is, if and only if* $a = (a_1, a_2) \in \mathbb{R}^2_{++}$ *(see Figure 1) and, similarly,* $F \cap H^+_{a,\alpha}$ *is compact if and only if* $a =$

$(a_1, a_2) \in -\mathbb{R}^2_{++}$.

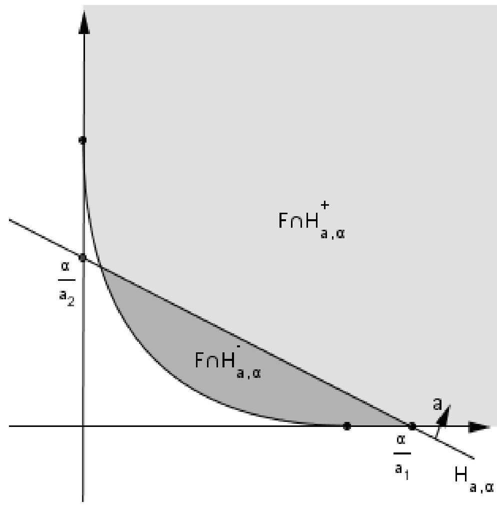

Fig. 1. Truncations induced by $H_{a,\alpha}$ in F.

Observe that F is M-decomposable with M-minimal (MT-minimal) component

$$Q(F) = \text{conv}\left\{(x, y) : \sqrt{x} + \sqrt{y} = 1, x \geq 0, y \geq 0\right\} = F \cap H^-_{(1,1),1}.$$

(see Figure 2.)

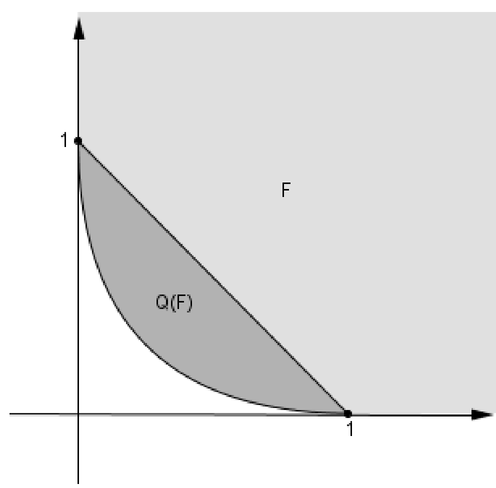

Fig. 2. Minimal compact component of F.

Corollary 14 *Let F be an unbounded, closed convex set without lines and H be a hyperplane such that $F \cap H \neq \emptyset$. Then, $F \cap H$ is compact if and only if at least one of the two truncations of F induced by H is bounded.*

Proof: It is a straightforward consequence of statement (iii) in Theorem 12.□

Corollary 15 *Let F be an unbounded, closed, convex set. Then F contains no lines if and only if there exists a compact truncation of F. In that case, if $F \cap H^-$ is a compact truncation of F induced by a hyperplane H, then $F \cap H_1^-$ is a compact truncation of F for any hyperplane H_1 parallel to H such that $F \cap H_1 \neq \emptyset$.*

Proof: The first part is consequence of statements (i)-(ii) in Theorem 12, recalling that $\operatorname{int}(0^+F)^\circ \neq \emptyset$ iff F contains no lines, and the second part comes from (i), which shows that the compactness of a truncation $F \cap H_{a,\alpha}^-$ is independent of α provided $F \cap H_{a,\alpha} \neq \emptyset$. □

From Corollary 15, if an unbounded, closed, convex set F admits an M-decomposition of type T, F cannot contain lines.

Corollary 16 *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be a convex, lsc, proper function. Then f is inf-compact if and only if $(0_n, 1) \in \operatorname{int}(0^+ \operatorname{epi} f)^\circ$.*

Proof: Let $F := \operatorname{epi} f \subset \mathbb{R}^{n+1}$ and $a := (0_n, 1)$, and let $\alpha \in \mathbb{R}$ be such that $\{x \in \mathbb{R}^n : f(x) \leq \alpha\} \neq \emptyset$. Then, by Theorem 12, $\{(x, y) \in \operatorname{epi} f : y \leq \alpha\}$ is bounded iff $(0_n, 1) \in \operatorname{int}(0^+ \operatorname{epi} f)^\circ$. So, it remains to be shown that $\{(x, y) \in \operatorname{epi} f : y \leq \alpha\}$ is bounded iff $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is bounded. The direct statement is consequence of the continuity of the orthogonal projection of $\mathbb{R}^{n+1}$ on $H := \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$, which projects $\{(x, y) \in \operatorname{epi} f : y \leq \alpha\}$ onto $\{x \in \mathbb{R}^n : f(x) \leq \alpha\} \times \{0\}$. For proving the converse statement, assume that $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is bounded. Then $\beta := \min\{f(x) : f(x) \leq \alpha\} \in \mathbb{R}$ because the lsc function f attains its minimum on the compact set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$, and that minimum cannot be $-\infty$ as f is proper. So, the set

$$\{(x, y) \in \operatorname{epi} f : y \leq \alpha\} \subset \{x \in \mathbb{R}^n : f(x) \leq \alpha\} \times [\beta, \alpha]$$

is bounded too. □

Example 17 *Consider $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ such that*

$$f(x) = \begin{cases} +\infty, & x < 0, \\ \left(1 - \sqrt{x}\right)^2, & 0 \leq x \leq 1, \\ 0, & x > 1. \end{cases}$$

Taking into account that $\operatorname{epi} f$ *is the set F in Example 13, we can write*

$$\operatorname{epi} f = \operatorname{conv}\left\{(x, y) : \sqrt{x} + \sqrt{y} = 1, x \geq 0, y \geq 0\right\} + \mathbb{R}_+^2,$$

so that f is M-decomposable and bounded from below, but its sublevel sets are unbounded because $(0, 1) \notin \operatorname{int}(0^+ \operatorname{epi} f)^\circ = \mathbb{R}_{++}^2$.

Proposition 18 *Let F be an unbounded, closed convex set. Then the following statements hold:*
(i) If F contains no lines, there exist compact slices of F. The converse holds when $\operatorname{int} 0^+F \neq \emptyset$.
(ii) If F contains lines, then there exist compact slices of F if and only if F is a Motzkin decomposable set whose conic component is a line.

Proof: (i) The direct statement follows from (iii) in Theorem 12 and Lemma 10. For the converse, we shall prove that if F contains lines and $\operatorname{int} 0^+F \neq \emptyset$, then $F \cap H_{a,\alpha}$ is unbounded. Take $d \in 0^+F \cap (-0^+F) \setminus \{0_n\}$. If $a'd = 0$ then one clearly has $d, -d \in 0^+ (F \cap H_{a,\alpha})$; so, in this case, $F \cap H_{a,\alpha}$ is unbounded. Assume now that $a'd \neq 0$, and take $d_0 \in \operatorname{int} 0^+F$. Then $d_0 + td \in \operatorname{int} 0^+F$ for every $t \in \mathbb{R}$ and, by $a'd \neq 0$, we have $a'(d_0 + t_0 d) = 0$ for some $t_0 \in \mathbb{R}$. If $d_0 + t_0 d = 0$ then $F = \mathbb{R}^n$ and hence $F \cap H_{a,\alpha} = H_{a,\alpha}$. We can thus assume that $d_0 + t_0 d \neq 0$, in which case, since $d_0 + t_0 d \in 0^+ (F \cap H_{a,\alpha})$, the set $F \cap H_{a,\alpha}$ is unbounded.

(ii) Denote $L := \operatorname{lin} F$. Assume that $F = C + L$, where C is a compact convex set and $\dim L = 1$. Then $L^\perp$ is a hyperplane such that $0^+ \left(F \cap L^\perp\right) = L \cap L^\perp = \{0_n\}$, so that the slice $F \cap L^\perp$ is compact.

Now we assume that $H_{a,\alpha}$ is a hyperplane such that $H_{a,\alpha} \cap F$ is compact. Assume that $\dim L > 1$. Then, $\dim (H_{a,0} \cap L) \geq \dim L - 1 > 0$. Thus, $\{0_n\} \subsetneqq H_{a,0} \cap L \subset H_{a,0} \cap 0^+F = 0^+ (H_{a,\alpha} \cap F)$, which contradicts the compactness of $H_{a,\alpha} \cap F$. Hence $\dim L = 1$. Let $u \in L \setminus \{0_n\}$. If $a'u = 0$, then $u \in H_{a,0} \cap L \subset 0^+ (H_{a,\alpha} \cap F)$ (contradiction). Hence $a'u \neq 0$.

According to [10, Theorem 18.5], we can write

$$F = Q(F) + 0^+ \left(F \cap L^\perp\right) + L. \tag{3}$$

Given $d \in 0^+ \left(F \cap L^\perp\right) \subset L^\perp$, take $v := d - \frac{a'd}{a'u} u \in 0^+F + L = 0^+F$. Then, $a'v = 0$, so that $v \in H_{a,0} \cap 0^+F = 0^+ (H_{a,\alpha} \cap F)$. Therefore, $v = 0_n$ and $d = \frac{a'd}{a'u} u \in L \cap L^\perp = \{0_n\}$. Thus,

$$0^+ \left(F \cap L^\perp\right) = \{0_n\}. \tag{4}$$

Thus (3) reduces to $F = Q(F) + \operatorname{span}\{u\}$.

The preceding argument actually shows that $0^+F = 0^+ \left(F \cap L^\perp\right) + L = \{0_n\} + L = L$. Hence, since $\emptyset \neq Q(F) = \operatorname{cl} \operatorname{conv} \operatorname{extr} \left(F \cap L^\perp\right) \subset F \cap L^\perp$, in view of (4) we have $0^+ (Q(F)) \subset 0^+ \left(F \cap L^\perp\right) = \{0_n\}$, so that $Q(F)$ is bounded and therefore F is the sum of the compact convex set $Q(F) = \operatorname{cl} \operatorname{conv} \operatorname{extr} \left(F \cap L^\perp\right)$ with the line $\operatorname{span}\{u\}$. □

Example 19 *The cylinder* $F = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1\}$ *is M-decomposable, with conic component* $\operatorname{span}\{(0,0,1)\}$ *and infinitely many compact components, e.g., the slices induced by hyperplanes which are not parallel to the vertical axis. Thus the condition* $\operatorname{int} 0^+F \neq \emptyset$ *in statement (i) of Proposition 18 is not superfluous. Observe also that the truncations of* F *induced by vertical hyperplanes are unbounded, so that the "only if" statement in Corollary 14 is not true when* F *contains lines.*

If F is an unbounded M-decomposable set containing lines, $\{Q(F) + l : l \in \operatorname{lin} F\}$ is a family of pairwise disjoint compact components of F, so that the intersection of all the compact components of F is empty. Otherwise, according to Theorem 1, the intersection of all the compact components of F is its M-minimal component $Q(F)$. A natural question arises when F an M-decomposable set without lines: does the intersection of all the compact components of F which are truncations of F coincide with $Q(F)$? The next example shows that the answer is negative, even for polyhedral convex sets.

Example 20 *Consider the polyhedral convex set* $F = C + D$, *with*

$$C = \operatorname{conv}\{(0,0,0), (1,0,1), (-1,0,1)\} = Q(F)$$

and

$$D = \operatorname{cone}\{(0,1,1), (0,-1,1)\} = 0^+F.$$

Obviously,

$$\left(0^+F\right)^\circ = \left\{x \in \mathbb{R}^3 \ : \ x_2 + x_3 \geq 0, -x_2 + x_3 \geq 0\right\}.$$

Let us consider an arbitrary halfspace $H_{a,\alpha}^-$ *such that* $F \cap H_{a,\alpha} \neq \emptyset$, $F \cap H_{a,\alpha}^-$ *is a compact set, and* $Q(F) \subset H_{a,\alpha}^-$. *As* $(0,0,1) \in Q(F) \subset F \cap H_{a,\alpha}^-$, *we get* $a_3 = a'(0,0,1) \leq \alpha$. *Moreover,* $a \in \operatorname{int}(0^+F)^\circ$ *by Theorem 12, so that* $a_2 + a_3 > 0$ *and* $-a_2 + a_3 > 0$, *i.e.,* $a_3 > 0$ *and* $|a_2| < a_3 \leq \alpha$. *As*

$$\left(0, \pm\frac{1}{2}, \frac{1}{2}\right) = (0,0,0) + \frac{1}{2}(0, \pm 1, 1) \in C + D = F,$$

$$a'\left(0, \pm\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(a_3 \pm a_2) \leq \frac{1}{2}(a_3 + |a_2|) < \alpha,$$

and $\left(0, \pm\frac{1}{2}, \frac{1}{2}\right) \notin Q(F)$, *we get*

$$\left(0, \pm\frac{1}{2}, \frac{1}{2}\right) \in \left(F \cap H_{a,\alpha}^-\right) \diagdown Q(F).$$

Since $H_{a,\alpha}^-$ *was chosen arbitrarily among those hyperplanes inducing compact components of* F *which are truncations, we have shown that the intersection of this family of truncations of an M-decomposable set without lines* F *may contain strictly its M-minimal component* $Q(F)$.

4 Characterizing Motzkin decomposable sets via truncations

In this section we characterize in two different ways the M-decomposable sets in terms of the existence of certain truncations. Each characterization is first obtained for closed convex sets without lines and then for arbitrary closed convex sets.

We observe that the unbounded truncation arising in an M-decomposition of type T is M-decomposable. Indeed, if

$$F = F \cap H^{+} + 0^{+}F \tag{5}$$

and $F \cap H^{+}$ is compact, then $0^{+}F \subset 0^{+}H^{-}$ and hence one can easily prove that $F \cap H^{-} = F \cap H + 0^{+}F$. Indeed, if $x \in F \cap H^{-}$ then, by (5), one has $x = y + d$ for some $y \in F \cap H^{+}$ and $d \in 0^{+}F$. Clearly, there exists $\lambda \in [0, 1]$ such that $(1 - \lambda)\, x + \lambda y \in F \cap H$. Since $x = (1 - \lambda)\, x + \lambda y + \lambda d$, it turns out that $x \in F \cap H + 0^{+}F$. This proves the inclusion $F \cap H^{-} \subset F \cap H + 0^{+}F$, whereas the reverse one is obvious. We thus have $F \cap H^{-} = F \cap H^{-} \cap H^{+} + 0^{+}F$, which shows that an unbounded truncation $F \cap H^{-}$ admits a decomposition by truncation with the same hyperplane H that generated it.

Lemma 21 *Let $F \subset \mathbb{R}^n$ be an unbounded closed convex set without lines. Then the following statements are equivalent:*
(i) F is Motzkin decomposable.
(ii) For every $a \in (0^{+}F)^{\circ} \setminus \{0_n\}$ there exists $\alpha \in \mathbb{R}$ such that

$$F \cap H^{+}_{a,\alpha} = F \cap H_{a,\alpha} + 0^{+}F. \tag{6}$$

(iii) There exist $a \in \operatorname{int}(0^{+}F)^{\circ}$ and $\alpha \in \mathbb{R}$ such that (6) holds.

Proof: (i) $\Longrightarrow$ (ii) Let $F = C + 0^{+}F$, where $C \subset \mathbb{R}^n$ is a compact convex set and $a \in (0^{+}F)^{\circ} \setminus \{0_n\}$. Take $\alpha := \max_{x \in C} a'x$. Given $z \in F \cap H^{+}_{a,\alpha}$, we can write $z = x + d$, with $x \in C \subset H^{-}_{a,\alpha}$ and $d \in 0^{+}F$. If $x \in H_{a,\alpha}$ then $z \in F \cap H_{a,\alpha} + 0^{+}F$. If $x \notin H_{a,\alpha}$, take $y \in \,]x, z] \cap H_{a,\alpha}$. Obviously, $z = y + \lambda d$, where $0 \leq \lambda < 1$, which proves the inclusion $F \cap H^{+}_{a,\alpha} \subset F \cap H_{a,\alpha} + 0^{+}F$. The reverse inclusion is a consequence of $a \in (0^{+}F)^{\circ}$.

(ii) $\Longrightarrow$ (iii) $\operatorname{int}(0^{+}F)^{\circ} \neq \emptyset$ because $0^{+}F$ is pointed by assumption. Thus a is any element of $\operatorname{int}(0^{+}F)^{\circ}$.

(iii) $\Longrightarrow$ (i) Let $a \in \operatorname{int}(0^{+}F)^{\circ}$ and $\alpha \in \mathbb{R}$ satisfying (6). First we show that the corresponding slice is nonempty. Take $x \in F$ and $d \in (0^{+}F) \setminus \{0_n\}$. By Lemma 10, $a'd > 0$, so that $x + \lambda d \in F \cap H^{+}_{a,\alpha}$ for a sufficiently large λ. Thus $F \cap H^{+}_{a,\alpha} \neq \emptyset$, and the nonemptiness of $F \cap H_{a,\alpha}$ follows from (6). So,

$\emptyset \neq F \cap H_{a,\alpha} \subset F \cap H^{-}_{a,\alpha}$, the latter set being compact by statement (i) in Theorem 12. Denote $C := F \cap H^{-}_{a,\alpha}$. Then, by (6),

$$\begin{aligned} F &= \left(F \cap H^{-}_{a,\alpha}\right) \cup \left(F \cap H^{+}_{a,\alpha}\right) \\ &= C \cup \left(F \cap H_{a,\alpha} + 0^{+}F\right) \\ &\subset C \cup \left(C + 0^{+}F\right) = C + 0^{+}F \subset F, \end{aligned}$$

so that $F = C + 0^{+}F$, where C is a compact convex set. □

In Example 13, $a \in (0^{+}F)^{\circ} \setminus \{0_n\}$ satisfies condition (6) if and only if $\alpha \geq \max\{a_1, a_2\}$. In that case, $F \cap H_{a,\alpha}$ is compact iff $a \in \operatorname{int}(0^{+}F)^{\circ}$.

Corollary 22 *If $F \subset \mathbb{R}^n$ is an unbounded Motzkin decomposable set without lines, then for every $a \in \operatorname{int}(0^{+}F)^{\circ}$ there exists $\alpha \in \mathbb{R}$ such that $F \cap H^{+}_{a,\alpha}$ is Motzkin decomposable with compact component $F \cap H_{a,\alpha}$.*

Proof: Let $a \in \operatorname{int}(0^{+}F)^{\circ}$. By Lemma 21, there exists $\alpha \in \mathbb{R}$ such that (6) holds, with $F \cap H_{a,\alpha}$ compact and nonempty (recall the proof of (iii) $\Longrightarrow$ (i) in Lemma 21). □

Corollary 23 *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be a convex, lsc, proper function such that* $\operatorname{dom} f$ *is bounded. Then, f is Motzkin decomposable if and only if it is bounded on* $\operatorname{dom} f$.

Proof: Let $F := \operatorname{epi} f$. By the assumptions on f, the set F contains no lines and $0^{+}F = \mathbb{R}_{+}(0_n, 1)$. According to Lemma 21, f is M-decomposable iff there exists $\alpha \in \mathbb{R}$ such that $\operatorname{epi} f \cap H^{+}_{(0_n,1),\alpha} = \operatorname{epi} f \cap H_{(0_n,1),\alpha} + \mathbb{R}_{+}(0_n, 1)$ or, equivalently,

$$\operatorname{epi}\max\{f, \alpha\} = \{(x, y) : f(x) \leq \alpha \leq y\}. \tag{7}$$

If f is bounded on $\operatorname{dom} f$ then (7) holds with $\alpha = \sup\{f(x) : x \in \operatorname{dom} f\}$, since in such a case $\operatorname{epi}\max\{f, \alpha\} = \operatorname{dom} f \times [\alpha, +\infty[$. Conversely, assume that (7) holds and let $x \in \operatorname{dom} f$. Then, taking $y \geq \max\{f(x), \alpha\}$, we clearly have $(x, y) \in \operatorname{epi}\max\{f, \alpha\}$, which, by (7), implies that $f(x) \leq \alpha$. We have thus proved that f is bounded above by α on $\operatorname{dom} f$. □

So, according to Corollary 23, the sufficient condition for Motzkin decomposability established by statement (iii) in [4, Theorem 13] is also necessary.

Lemma 24 *Let $C \subset \mathbb{R}^n$ be a nonempty, closed, convex cone. Then*

$$\operatorname{span} C^{\circ} = (\operatorname{lin} C)^{\perp}.$$

Proof: Since $\operatorname{lin} C \subset C$, we have $C^{\circ} \subset (\operatorname{lin} C)^{\circ} = (\operatorname{lin} C)^{\perp}$; hence $\operatorname{span} C^{\circ} \subset$

$(\operatorname{lin} C)^{\perp}$. On the other hand, from $(\operatorname{span} C^{\circ})^{\perp} \subset C$ we deduce that $(\operatorname{span} C^{\circ})^{\perp} \subset \operatorname{lin} C$; therefore $(\operatorname{lin} C)^{\perp} \subset (\operatorname{span} C^{\circ})^{\perp\perp} = \operatorname{span} C^{\circ}$. □

Lemma 25 *Let $F \subset \mathbb{R}^n$ be a nonempty closed, convex set. Then*

$$\left(0^{+}(F \cap (\operatorname{lin} F)^{\perp})\right)^{\circ} = \left(0^{+}F\right)^{\circ} + \operatorname{lin} F.$$

Proof: Denote $L := \operatorname{lin} F$. From $0^{+}(F \cap L^{\perp}) = 0^{+}F \cap 0^{+}(L^{\perp}) = 0^{+}F \cap L^{\perp}$ it follows that $\left(0^{+}(F \cap L^{\perp})\right)^{\circ} = \left(0^{+}F \cap L^{\perp}\right)^{\circ} = \operatorname{cl}\left((0^{+}F)^{\circ} + \left(L^{\perp}\right)^{\circ}\right) = \operatorname{cl}\left((0^{+}F)^{\circ} + L\right) = (0^{+}F)^{\circ} + L$. □

Corollary 26 *Let $F \subset \mathbb{R}^n$ be a nonempty, closed, convex set. Then*

$$\operatorname{int}\left(0^{+}(F \cap (\operatorname{lin} F)^{\perp})\right)^{\circ} = \operatorname{rint}\left(0^{+}F\right)^{\circ} + \operatorname{lin} F. \tag{8}$$

Proof: Let $L := \operatorname{lin} F$, $A := \left(0^{+}(F \cap L^{\perp})\right)^{\circ}$, and $B := (0^{+}F)^{\circ}$. We have $\operatorname{rint}(B + L) = \operatorname{rint} B + L$ because L is a linear subspace such that $B \subset L^{\perp}$. Since $F \cap L^{\perp}$ contains no lines, $0^{+}(F \cap L^{\perp})$ is pointed and, so, $\operatorname{int} A = \operatorname{int}\left(0^{+}(F \cap L^{\perp})\right)^{\circ} \neq \emptyset$. Then, by Lemma 25, we get

$$\operatorname{int} A = \operatorname{rint} A = \operatorname{rint}(B + L) = \operatorname{rint} B + L.$$

Therefore, (8) holds. □

Theorem 27 *Let $F \subset \mathbb{R}^n$ be an unbounded, closed, convex set. Then the following statements are equivalent:*
(i) F is Motzkin decomposable.
(ii) For every $a \in \left((0^{+}F)^{\circ} + \operatorname{lin} F\right) \setminus \{0_n\}$ there exists $\alpha \in \mathbb{R}$ such that

$$F \cap H_{a,\alpha}^{+} = F \cap H_{a,\alpha} + 0^{+}F. \tag{9}$$

(iii) There exist $a \in \operatorname{rint}(0^{+}F)^{\circ} + \operatorname{lin} F$ and $\alpha \in \mathbb{R}$ such that (9) holds.

Proof: Let $L := \operatorname{lin} F$. Recall that F is M-decomposable if and only if $F \cap L^{\perp}$ is M-decomposable ([4, Theorem 6]), and use Lemma 21 with F replaced by this latter set. Concerning statements (ii) and (iii), see Lemma 25 and Corollary 26, respectively. □

From now on, for $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n \setminus \{0_n\}$, we denote by $r_{x,d} := \{x + \lambda d : \lambda \geq 0\}$ the closed halfline emanating from x in the direction of d.

Proposition 28 *Let $F \subset \mathbb{R}^n$ be an unbounded, closed, convex set without lines and $F \cap H^{+}$ be a truncation induced by the hyperplane H. Then, $F \cap H^{+}$ is a union of closed halflines emanating from H if and only if* $\operatorname{extr}(F \cap H^{+}) \subset H$.

Proof: Let $F \cap H^+$ be a union of closed halflines emanating from H. Let $z \in F \cap H^+ \backslash H$ and $x \in H$, $d \in \mathbb{R}^n \backslash \{0_n\}$ be such that $z \in r_{x,d}$. Then we can write $z = x + \lambda d$ for some $\lambda > 0$ ($\lambda \neq 0$ because $z \neq x$). Since $z \in \left[x + \frac{\lambda}{2}d, x + 2\lambda d\right]$, with $x + \frac{\lambda}{2}d \neq x + 2\lambda d$ elements of $r_{x,d} \subset F \cap H^+$, we have $z \notin \operatorname{extr}(F \cap H^+)$.

Now we assume that $\operatorname{extr}(F \cap H^+) \subset H$. By [10, Theorem 18.5],

$$\begin{aligned} F \cap H^+ &= \operatorname{cl}\operatorname{conv}\operatorname{extr}(F \cap H^+) + 0^+(F \cap H^+) \\ &\subset F \cap H + 0^+(F \cap H^+). \end{aligned}$$

Then, any $z \in F \cap H^+ \backslash H$ can be written as $z = x + d$, with $x \in F \cap H$ and $d \in 0^+(F \cap H^+) \setminus \{0_n\}$, so that $z \in r_{x,d} \subset F \cap H^+$. Therefore,

$$F \cap H^+ \backslash H \subset \bigcup_{\substack{x \in F \cap H \\ d \in 0^+(F \cap H^+) \setminus \{0_n\}}} r_{x,d}.$$

Since any $x \in F \cap H^+ \cap H = F \cap H$ belongs to $r_{x,d}$ for all $d \in 0^+(F \cap H^+) \setminus \{0_n\}$, we get

$$F \cap H^+ = \bigcup_{\substack{x \in F \cap H \\ d \in 0^+(F \cap H^+) \setminus \{0_n\}}} r_{x,d}.$$

□

In Example 13, given $a \in \mathbb{R}^2_{++}$ and $\alpha \geq \max\{a_1, a_2\}$,

$$F \cap H^+_{a,\alpha} = \operatorname{conv}\left\{\left(\frac{\alpha}{a_1}, 0\right), \left(0, \frac{\alpha}{a_2}\right)\right\}$$

is a truncation of F satisfying

$$\operatorname{extr}\left(F \cap H^+_{a,\alpha}\right) = \left\{\left(\frac{\alpha}{a_1}, 0\right), \left(0, \frac{\alpha}{a_2}\right)\right\} \subset H_{a,\alpha},$$

and so it is a union of halflines emanating from $H_{a,\alpha}$. In general, $\operatorname{extr}(F \cap H^+) \subset H$ does not imply that $F \cap H^+$ is the truncation of some translated closed convex cone (take as F a truncated cylinder).

Lemma 29 *Let $F \subset \mathbb{R}^n$ be an unbounded, closed, convex set without lines. Then F is Motzkin decomposable if and only if there exists a hyperplane H such that one of the truncations induced by H is compact and the other one is a union of closed halflines emanating from H.*

Proof: Assume first that F is M-decomposable. By Lemma 21 there exists a hyperplane H such that $F \cap H^+ = F \cap H + 0^+F$. Then $\operatorname{extr}(F \cap H^+) \subset F \cap H$ and hence, by Proposition 28, $F \cap H^+_{a,\alpha}$ is a union of closed halflines emanating from H.

For proving the converse, assume the existence of H as in the statement and let K be the compact set obtained by taking the intersection of F with H^+ one of the closed halfspaces determined by H. We will see that $F = K + 0^+F$; since K is convex, this will show that F is M-decomposable. We only have to prove the inclusion $\subset$, as the opposite one follows immediately from $K \subset F$. Let $x \in F$. If $x \in K$, then $x = x + 0_n \in K + 0^+F$. If, on the contrary, $x \notin K$ then, by the assumption, $x \in r_{h,d} \subset F \cap H^-$ for some $h \in H$ and $d \in \mathbb{R}^n \setminus \{0_n\}$, H^- being the other closed halfspace determined by H. Since $h \in r_{h,d} \subset F$ and $d \in 0^+F$, we have $h \in H \cap F \subset K$, and therefore from $x \in r_{h,d}$ we conclude that $x \in K + 0^+F$, which ends the proof. $\square$

In Example 13, the hyperplane $H := \{x \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ satisfies the conditions of Lemma 29.

Corollary 30 *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be a convex, lsc, proper function such that $\operatorname{dom} f$ contains no lines. Then f is Motzkin decomposable if it is inf-compact and there exists a sublevel set $L_\alpha(f) := f^{-1}(-\infty, \alpha]$, with $\alpha \in \mathbb{R}$, such that that $\operatorname{dom} f \setminus L_\alpha(f)$ is a union of halflines on each of which f is affine. In this case, the truncation of f by α, $\max\{f, \alpha\}$, is Motzkin decomposable.*

Proof: Let α be as in the statement. The hyperplane $H := \{(x, x_{n+1}) : x_{n+1} = \alpha\}$ induces in $\operatorname{epi} f$ two truncations:

$$\{(x, x_{n+1}) \in \operatorname{epi} f : x_{n+1} \leq \alpha\} = \{(x, x_{n+1}) : f(x) \leq x_{n+1} \leq \alpha\},$$

which is compact by the inf-compactness of f, and

$$\begin{aligned}\{(x, x_{n+1}) \in \operatorname{epi} f : x_{n+1} \geq \alpha\} &= \{(x, x_{n+1}) : \max\{f(x), \alpha\} \leq x_{n+1}\} \\ &= \operatorname{epi} \max\{f, \alpha\}.\end{aligned}$$

We will prove that the latter set is a union of closed halflines emanating from H. Let $(x, x_{n+1}) \in \operatorname{epi} \max\{f, \alpha\}$. Then $x_{n+1} \geq \alpha$ and $f(x) \leq x_{n+1}$. If $f(x) \leq \alpha$ then (x, x_{n+1}) belongs to the vertical line emanating from $(x, \alpha) \in H$. Suppose now that $f(x) > \alpha$. Then $x \in \operatorname{dom} f \setminus L_\alpha(f)$, and hence $x \in r_{y,x-y}$ for some $y \in f^{-1}(\alpha)$ such that $y - x \in 0^+ \operatorname{dom} f$ and f is affine on $r_{y,x-y}$. We will next show that $r_{(y,\alpha),(x-y,x_{n+1}-\alpha)} \subset \operatorname{epi} \max\{f, \alpha\}$, which, as $(x, x_{n+1}) \in r_{(y,\alpha),(x-y,x_{n+1}-\alpha)}$ and $(y, \alpha) \in H$, will finish the proof. Consider a point $(y, \alpha) + \mu(x - y, x_{n+1} - \alpha) = (y + \mu(x - y), \alpha + \mu(x_{n+1} - \alpha))$, with $\mu \geq 0$. Since f is affine on $r_{y,x-y}$, we have $f(y + \mu(x - y)) = f(y) + \mu(f(x) - f(y)) = \alpha + \mu(f(x) - \alpha) \leq \alpha + \mu(x_{n+1} - \alpha)$; on the other hand, from $x_{n+1} \geq \alpha$ and $\mu \geq 0$ it follows that $\alpha \leq \alpha + \mu(x_{n+1} - \alpha)$. This proves that $(y, \alpha) + \mu(x - y, x_{n+1} - \alpha) \in \operatorname{epi} \max\{f, \alpha\}$. From 29 we conclude that f is M-decomposable. Since, as we already observed at the beginning of this section, truncations of an M-decomposable set are themselves M-decomposable, we conclude that $\max\{f, \alpha\}$ is M-decomposable too. $\square$

The function f in Example 17 is M-decomposable but not inf-compact, so that the converse of Corollary 30 does not hold.

Theorem 31 *Let $F \subset \mathbb{R}^n$ be an unbounded, closed, convex set. Then F is Motzkin decomposable if and only if there exists a hyperplane H parallel to $\operatorname{lin} F$ such that H induces truncations of $F \cap (\operatorname{lin} F)^{\perp}$ and F which are compact and union of closed halflines emanating from H, respectively.*

Proof: Denote $L = \operatorname{lin} F$. If $L = \{0_n\}$, the statement reduces to that of Lemma 29; we will thus assume w.l.o.g. that $L \neq \{0_n\}$.

It is known that F is M-decomposable iff $F \cap L^{\perp}$ is M-decomposable iff (by Lemma 29) there exists a hyperplane $\widetilde{H}$ in $L^{\perp}$ such that the intersection K of $F \cap L^{\perp}$ with one of the closed halfspaces $\widetilde{H^+}$ in $L^{\perp}$ determined by $\widetilde{H}$ is compact and its intersection with the other closed halfspace $\widetilde{H^-}$ determined by $\widetilde{H}$ is a union of closed halflines emanating from $\widetilde{H}$.

If F is M-decomposable, take $\widetilde{H}$ as above and denote $H = \widetilde{H} + L$. Clearly, H is a hyperplane parallel to L, $H^+ = \widetilde{H^+} + L$ is one of the closed halfspaces determined by H, and the intersection of H^+ with $F \cap L^{\perp}$ is compact: $H^+ \cap F \cap L^{\perp} = \widetilde{H^+} \cap F \cap L^{\perp} = K$. Let H^- be the other closed halfspace determined by H; one clearly has $H^- = \widetilde{H^-} + L$. Let $x \in F \cap H^-$ and consider the projection $\widetilde{x}$ of x on $L^{\perp}$. We have $\widetilde{x} \in F \cap L^{\perp} \cap H^- = F \cap L^{\perp} \cap \widetilde{H^-}$ and therefore there exists $r_{h,d} \subset F \cap L^{\perp} \cap \widetilde{H^-} \subset F \cap H^-$, with $h \in F \cap L^{\perp} \cap \widetilde{H} \subset H$ and $d \in 0^+ \left(F \cap L^{\perp} \cap H^- \right)$, such that $\widetilde{x} \in r_{h,d}$. Having this in mind and the fact that H is parallel to L, we get that $x \in r_{h,x-h} \subset F \cap H^-$ in the case when $x \neq h$. If, on the contrary, $x = h$, taking any l in the nonempty set $L \setminus \{0_n\}$ we have $x \in r_{h,l} \subset F \cap H^-$.

If there exists a hyperplane H as in the statement, define $\widetilde{H} = H \cap L^{\perp}$. Since H is parallel to L, $\widetilde{H}$ is a hyperplane in $L^{\perp}$. Let H^+ be the closed halfspace determined by H such that $H^+ \cap F \cap L^{\perp}$ is compact, H^- be the opposite closed halfspace and denote $\widetilde{H_i} = H_i \cap L^{\perp}$ $(i = 1, 2)$. Then $\widetilde{H^+}$ and $\widetilde{H^-}$ are the closed halfspaces in $L^{\perp}$ determined by $\widetilde{H}$. We have that $\widetilde{H^+} \cap F \cap L^{\perp} = H^+ \cap F \cap L^{\perp}$ is compact and $\widetilde{H^-} \cap F \cap L^{\perp} = H^- \cap F \cap L^{\perp}$ is a union of closed halflines emanating from $H \cap L^{\perp} = \widetilde{H}$. The proof is complete. □

From now on we will deal only with closed and convex sets without lines, or equivalently, possessing extreme points.

Corollary 32 *Let $F \subset \mathbb{R}^n$ be a Motzkin decomposable set. Then F admits a Motzkin decomposition of type T if and only if it contains no lines.*

Proof: The "only if" statement is an immediate consequence of Corollary 15. For proving the converse, let H be a hyperplane as in Theorem 31, i.e.

such that $F \cap H^+$ is compact and $F \cap H^-$ is a union of halflines emanating from $F \cap H$, where H^+, H^- are the halfspaces determined by H. Clearly $F \cap H^+ + 0^+F \subset F$. For the opposite inclusion, use Proposition 28 to deduce that $H \supset \operatorname{extr}(F \cap H^-) \supset (\operatorname{extr} F) \cap H^-$, from where it follows that $\operatorname{extr} F \subset F \cap H^+$, and hence $Q(F) + 0^+F \subset F \cap H^+ + 0^+F$. Since $F = Q(F) + 0^+F$ by statement (i) in Theorem 1, we have established that $F = F \cap H^+ + 0^+F$. □

5 Minimal Motzkin decompositions of type T

The main result in this section shows that the MT-minimal component of a closed and convex set without lines F is $Q(F)$.

Theorem 33 *Let F be an unbounded, closed, convex set without lines. The MT-minimal component of F, when it exists, coincides with the M-minimal component of F.*

Proof: We must prove that, if H is a hyperplane which induces the minimal Motzkin decomposition of type T of F, then $F \cap H^- = Q(F)$, where H^- is one of the closed halfspaces determined by H.

Since $F = F \cap H^- + 0^+F$ and $F \cap H^-$ is compact, we conclude from Theorem 1 that $Q(F) \subset F \cap H^-$. Thus, it suffices to prove that $F \cap H^- \subset Q(F)$. We sketch next the proof of this fact. We will assume for the sake of contradiction that this inclusion does not hold, i.e. that there exists a point $z \in (F \cap H^-) \backslash Q(F)$, from which we construct a point $u \in (F \cap H) \backslash Q(F)$. We consider a hyperplane which separates u from $Q(F)$, and a positive combination of the normal vectors to this hyperplane and to H turns out to be normal to a hyperplane H_1 which also induces a Motzkin decomposition of F of type T, but such that $u \notin F \cap H_1^-$, contradicting the minimality of the Motzkin decomposition of type T induced by H. We proceed now to formalize this proofline.

Take $a \in \mathbb{R}^n \backslash \{0_n\}$ and $\alpha \in \mathbb{R}$ such that $H = H_{a,\alpha}$ and $H^- = H_{a,\alpha}^-$. As $F \cap H_{a,\alpha}^-$ is compact by assumption, $a \in \operatorname{int}(0^+F)^\circ$ (recall Theorem 12). Let $\varepsilon > 0$ be such that $a + \varepsilon v \in (0^+F)^\circ$ for all $v \in S^{n-1}$. Given $y \in (0^+F) \backslash \{0_n\}$, $-\frac{y}{\|y\|} \in S^{n-1}$, so that $a'y \geq \varepsilon \|y\| > 0$. Therefore

$$a'y > 0 \ \ \forall y \in \left(0^+F\right) \backslash \{0_n\}. \tag{10}$$

Now we assume that the inclusion $F \cap H_{a,\alpha}^- \subset Q(F)$ fails, and hence there exists a point $z \in \left(F \cap H_{a,\alpha}^-\right) \backslash Q(F)$. Since $F = Q(F) + 0^+F$ by Theorem 1, $z = w + d$ for some $w \in Q(F), d \in 0^+F$. Clearly, $d \neq 0_n$ (otherwise, z belongs to $Q(F)$). We claim now that

$$a'w < \alpha. \tag{11}$$

Otherwise, since $w \in Q(F) \subset F \cap H^-_{a,\alpha}$, we have $a'w = \alpha$, and hence, using (10),

$$a'z = a'w + a'd = \alpha + a'd > \alpha,$$

contradicting the fact that $z \in H^-_{a,\alpha}$. Hence $z \in \operatorname{int} H^-_{a,\alpha}$. Observe now that the halfline $r_{w,d}$ must cut H, because otherwise the whole halfline would be contained in $H^-_{a,\alpha}$, and since it is contained in F because $w \in Q(F), d \in 0^+F$, we would be contradicting the compactness of $F \cap H^-_{a,\alpha}$. Since w, the vertex of $r_{w,d}$, belongs to $\operatorname{int} H^-_{a,\alpha}$ by (11), $r_{w,d}$ cuts $H_{a,\alpha}$ at one point, say $u = w + td$. Note that $t \geq 1$, because $w = w + 0d \in \operatorname{int} H^-_{a,\alpha}$, $w + d = z \in H^-_{a,\alpha}$, so that points of the form $w + sd \in \operatorname{int} H^+_{a,\alpha}$ for all $s > t$. Thus, z is in the segment between w and u. Taking into account that $w \in Q(F)$, $z \notin Q(F)$, we conclude from the convexity of $Q(F)$ that $u \notin Q(F)$. We invoke now the Convex Separation Theorem to find $b \in \mathbb{R}^n$, with $\|b\| = 1$, and $\beta \in \mathbb{R}$ such that

$$b'u > \beta, \tag{12}$$

$$b'x < \beta \ \ \forall x \in Q(F). \tag{13}$$

Define

$$\delta = \min\left\{a'y : \ \ y \in S^{n-1} \cap 0^+F\right\}. \tag{14}$$

Note that $\delta > 0$ by (10) and the compactness of $S^{n-1} \cap 0^+F$. Take $\bar{\delta} \in]0, \delta[$ such that $c := a + \bar{\delta}b \neq 0_n$ and define $\gamma := \alpha + \bar{\delta}\beta$. We claim that $H_{c,\gamma}$ induces a Motzkin decomposition of F of type T, and in view of Corollary 2, the claim will be established if we prove that:

i) $Q(F) \subset F \cap H^-_{c,\gamma}$.

ii) $F \cap H^-_{c,\gamma}$ is compact.

For checking (i), take any $x \in Q(F)$, and note that

$$c'x = a'x + \bar{\delta}b'x < \alpha + \bar{\delta}\beta = \gamma,$$

using the fact that $x \in Q(F) \subset H^-_{a,\alpha}$ and (13) in the inequality.

Now we look at (ii). Let $\bar{\delta} \in]0, \delta[$ be such that $c := a + \bar{\delta}b \in \operatorname{int}(0^+F)^\circ$. Then $c \neq 0_n$ (because F contains no lines) and $F \cap H^-_{c,\gamma}$ is compact by Theorem 12. This proves that $H_{c,\gamma}$ induces a Motzkin decomposition of type T.

Now, the minimality of the decomposition induced by $H_{a,\alpha}$ among Motzkin decompositions of type T implies that $F \cap H^-_{a,\alpha} \subset F \cap H^-_{c,\gamma}$. Since u belongs to $F \cap H_{a,\alpha} \subset F \cap H^-_{a,\alpha}$, we get that $u \in F \cap H^-_{c,\gamma}$, i.e., that

$$c'u \leq \gamma. \tag{15}$$

On the other hand

$$c'u = a'u + \bar{\delta}b'u = \alpha + \bar{\delta}b'u > \alpha + \bar{\delta}\beta = \gamma, \tag{16}$$

using the definition of c in the first equality, the fact that $u \in H_{a,\alpha}$ in the second one, and (12) in the inequality. The contradiction between (15) and (16) entails that $F \cap H^-_{a,\alpha} = Q(F)$, completing the proof. □

Corollary 34 *A closed and convex set F, without lines, has an MT-minimal component if and only if $Q(F)$ is a truncation of F.*

Proof: The "only if" part follows directly from Theorem 33; the "if" part is a consequence of Corollary 2 and Theorem 1. □

Corollary 34 will help us in the construction of sets in more than two dimensions with and without MT-minimal component (the MT-minimal component of any unbounded M-decomposable set in $\mathbb{R}^2$ containing no lines is one of the truncations induced by a line containing the vertices of its unbounded edges, see Example 13). Any application of Corollary 34 relies on the identification of $Q(F)$.

Example 35 *Define $F \subset \mathbb{R}^3$ as $F = C+D$, where $D = \left\{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} \leq x_3\right\}$ and C is the unit bidimensional disk $C = \{(x_1, 0, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq 1\}$. Clearly, $\operatorname{extr} F \subset \operatorname{extr} C = \{(x_1, 0, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 = 1\}$. In order to precisely determine $\operatorname{extr} F$, we must exclude from $\operatorname{extr} C$ those points which belong to a halfline with direction in D starting at another point in $\operatorname{extr} C$. After some elementary algebra, it can be seen that the points to be excluded are those with $x_3 > -\frac{\sqrt{2}}{2}$, so that*

$$\operatorname{extr} F = \left\{(x_1, 0, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 = 1, x_3 \leq -\tfrac{\sqrt{2}}{2}\right\},$$

and hence

$$Q(F) = \left\{(x_1, 0, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq 1, x_3 \leq -\tfrac{\sqrt{2}}{2}\right\},$$

which is not a face of F although $\dim Q(F) < \dim F$. Indeed, for $c = (0, 0, -1) \in C$, $d_1 = \left(0, 1 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right) \in D$ and $d_2 = \left(0, \frac{\sqrt{2}}{2} - 1, 1 - \frac{\sqrt{2}}{2}\right) \in D$ one has $c + d_1, c + d_2 \in F \setminus Q(F)$ and $\frac{1}{2}(c + d_1 + c + d_2) \in Q(F)$. Thus, $Q(F)$ is not a truncation of F (see the discussion at the beginning of Section 3). In view of Corollary 34, we conclude that F has no MT-minimal component.

Example 36 *We take now $F = \operatorname{cl} B_3 + D$, with D as in Example 35, i.e. the vertical "ice-cream cone" in $\mathbb{R}^3$. A computation similar to that of Example 35 shows that*

$$Q(F) = \left\{x \in \mathbb{R}^3 : \|x\| \leq 1, x_3 \leq -\frac{\sqrt{2}}{2}\right\}.$$

Observing that $Q(F) = F \cap H^{-}_{a,\alpha}$ *for* $a = (0, 0, 1)$ *and* $\alpha = -\frac{\sqrt{2}}{2}$*, by Corollary 34 we conclude that* $Q(F)$ *is the MT-minimal component of* F*.*

Observe that, in general, the intersection of all the compact components of F which are truncations does not coincide with its M-minimal component (recall Example 20), so that it is not necessarily the MT-minimal component either.

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