Title: On the structure of immediate extensions of valued fields

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Abstract

The connection between a valued field extension and the corresponding extensions of the value group and the residue field is meaningful for the theory of valued fields. When this connection is interrupted, the structure of valued field extensions is much more complicated. This causes one of the main hurdles to solve many important questions in valuation theory and related areas of mathematics. Crucial examples of this situation are defect extensions and immediate extensions of valued fields. A better understanding of both types of extensions turned out to be important for questions in algebraic geometry, like resolution of singularities, problems in real algebra and the model theory of valued fields.

In this thesis we study the structure and constructions of immediate as well as defect extensions of valued fields. In particular, we focus on the structure of maximal immediate extensions of valued fields.

In connection with local uniformization, a local version of resolution of singularities, we investigate the problems related to defect extensions. We describe properties of distances of elements in valued field extensions, which turned out to be a useful tool for the study of the structure of defect extensions of valued fields of positive characteristic. We also give an upper bound of the number of distinct distances of immediate elements of a bounded degree.

We further study the problem of existence of infinite towers of Galois defect extensions of prime degree. We give conditions for a valued field to admit such towers and present constructions of them. In connection with questions related to local uniformization we present constructions of infinite towers of Artin-Scheier defect extensions of rational function fields in two variables over fields of positive characteristic. We consider the classification of Artin-Schreier defect extensions into "dependent" and "independent" ones (according to whether they are connected with purely inseparable defect extensions, or not). To understand the meaning of the classification for the issue of local uniformization, we consider various valuations of the above mentioned rational function fields and investigate for which they admit an infinite tower of Artin-Schreier defect extensions of each type.

The existence of infinite towers of Galois defect extensions of prime degree turned out to be important for the structure of maximal immediate extensions of valued fields, which is the next problem treated in this thesis. We give conditions for a valued field to admit maximal immediate extensions of infinite transcendence degree. This problem is tightly connected with the description of the possible extensions of a valuation from a given field to an algebraic function field. We further consider algebraic extensions of maximal fields and study the structure of immediate extensions of such fields. We also investigate the problem of uniqueness of maximal immediate extensions. We prove that there is a class of valued fields which admit an algebraic maximal immediate extension as well as one of infinite transcendence degree, which can be seen as the worst possible case of non-uniqueness.

In our studies of maximal immediate extensions we consider also valued fields (K, v) with p-divisible value group and perfect residue field, where p is the characteristic exponent of the residue field Kv. Maximal immediate extensions of such fields are tame fields. Because of their good valuation theoretical and model theoretical properties, tame fields play an important role in the theory of valued fields and its applications. We discuss first the case of fields with maximal immediate extensions of finite transcendence degree and describe the structure of such extensions. We then relate the existence of defect extensions of the field (K, v) with the structure of the maximal immediate extensions of this field. We prove that if the field (K, v) admits a nontrivial separable-algebraic defect extension, then every maximal immediate extension of K is of infinite transcendence degree. We finally apply the results to the description of the structure of valued rational function fields. In particular, we give necessary and sufficient conditions on (K, v) to admit an extension of the valuation to a rational function field F over K such that vF/vK is a torsion group and the residue field extension $Fv|Kv$ is algebraic.

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1. Introduction

Special types of valuations were exploited in number theory and function theory already in the 19th century. However, the theory of valuations, as a separate branch of mathematics, was started in 1912 by J. Kürschák, who formulated in [34] the axioms for valued fields. The notion was introduced as a foundation for the theory of p-adic fields, presented by K. Hensel in [14]. From that time on one can observe a quick development of valuation theory. It turned out that valuation theoretical methods allow us to understand better important issues of algebraic number theory, algebraic geometry and the theory of ordered fields.

During the next decades, mathematicians like H. Hasse, F. K. Schmidt, A. Ostrowski, O. F. G. Schilling and S. MacLane were developing the theory introduced by Kürschák (cf. [40]). The first results and research in this area were connected mainly with discrete complete valuations. Complete valued fields are important for the study of analytic properties of fields, whereas for the investigation of algebraic properties the henselian fields are more adequate. After W. Krull had introduced the theory of general valuations of arbitrary rank, it turned out that complete valued fields of higher rank are not necessarily henselian (cf. [3], Chapter 6, §8). This problem can be solved by passing to a minimal henselian algebraic extension of a given valued field, called a henselization of the field (cf. Section 2.2.2). The completion of a given valued field as well as the henselization of the field have an interesting property: the corresponding value group and residue field extensions are trivial. Such valued field extensions are called **immediate** (cf. Section 2.2.3). They appear also in a natural way while studying power series fields with their canonical valuations. The notion of immediate extensions is due to Schmidt, but was first published by Krull in [20].

While the completion and the henselization of a valued field and their structure were known, other immediate extensions turned out to be more problematic. A good example of this is the Relative Uniqueness Theorem, proved by MacLane in [36] for discrete complete valued fields. In his subsequent paper MacLane claimed that the result can be proved also for non-discrete valuations. However, it was pointed out by his student I. Kaplansky that such a general fact is not true. The reason is the existence of nontrivial immediate extensions of complete fields in the non-discrete case. The research of Kaplansky on immediate extensions of valued fields ([16]), which was a continuation of the work of Ostrowski ([39]), laid the basis for the theory of immediate extensions, maximal immediate extensions, and their description. The theory was further developed by many other mathematicians (see, e.g., [9], [25], [32] and [42]).

The investigation of valued fields and related areas showed the importance of the connection between a valued field extension and the corresponding extensions of the value group and the residue field. It turned out that the valued field extensions for which the value group and the residue field extensions carry the maximal possible information are usually easier to deal with. Difficulties often appear when the connection between a valued field extension and the extensions of its invariants, i.e., the value groups and the residue fields, is interrupted. Crucial examples of this situation are immediate algebraic extensions of henselian fields. Therefore, a better understanding of the structure of immediate extensions of valued fields is meaningful for problems in various areas of research.

Since specific properties of an ordered field depend on specific properties of the valuations of the field, the structure of extensions of valued fields is important for the theory of ordered fields. As an example, the immediate extensions play a role in the problem of extending an ordering from a given field to a rational function field over this field (see, e.g., [22], [24] and [30]).

Immediate extensions are also meaningful for questions in the model theory of valued fields, like classification of valued fields up to elementary equivalence (cf. [27] and [28]), and in algebraic geometry, especially for the local uniformization problem (see, e.g., [19] and [27]). Local uniformization is essentially a local version of the problem of resolution of singularities, which can be described in the language of valued fields. For valued fields of characteristic 0, local uniformization was proved in 1940 by O. Zariski ([46]). Over twenty years later H. Hironaka proved resolution of singularities over fields of characteristic 0 ([15]). For fields of positive characteristic both problems are solved only in special cases (see, e.g., [1], [2], [6], [7], [18] and [19]), but the general case remains widely open.

One of the hurdles for the attempt to prove local uniformization in positive characteristic are so-called defect extensions, which do not appear when the residue field has characteristic 0. If $(L|K, v)$ is a finite extension of valued fields such that v admits a unique extension from K to L, then the **Lemma of Ostrowski** (see [47], Chapter VI, §12, Corollary to Theorem 25) says that

$$[L : K] \; = \; p^n (vL : vK)[Lv : Kv] \tag{1.1}$$

with $n \geq 0$ and p the **characteristic exponent** of Kv, that is, $p = \mathrm{char} Kv$ if it is positive and $p = 1$ otherwise (for the notions and basic facts see Chapter 2). The factor $d(L|K, v) := p^n$ is called the **defect** of the extension $(L|K, v)$. If it is nontrivial, that is, if $n > 0$, then $(L|K, v)$ is called **a defect extension**. If $d(L|K, v) = 1$, then $(L|K, v)$ is called a **defectless extension**. The nature of the defect was studied and described first by Ostrowski in [39]. However, there are still many open problems about defect extensions. Since such extensions play a role also in deep open problems of the model theory of valued fields as well as the theory of valued rational function fields (see, e.g., [22], [27] and [28]), we need to better understand defect extensions and their structure.

In this thesis we study the structure of maximal immediate extensions of valued fields and constructions of such extensions. In connection with the problem of local uniformization, we also study the issue of defect extensions of valued fields, especially of valued rational function fields.

We start with describing properties of a useful tool in the study of defect extensions of valued fields. Elements of a valued field extension induce cuts in the value group of the base field (in a sense that we will explain in Section 2.3). These cuts, called distances, turned out to be important for the study of the structure of defect extensions of valued fields of positive characteristic (cf. [25]). F.-V. Kuhlmann and O. Piltant in their joint work [31] relate defect extensions with higher ramification groups, in connection with the local uniformization problem. They use distances to describe the relation. In Chapter 3 we introduce a new definition of distance which carries more information about field extensions (cf. Section 3.1). We then apply the properties of distances that we have proved to the case of defectless extensions of prime degree, describing all possible distances of elements of such extensions and the properties of their distances (cf. Section3.2).

In the study of valuations of rational function fields, which continues the work of S. D. Cutcosky and O. Piltant ([8]), the problem of an upper bound of the number of distinct distances of elements in immediate extensions came up. The answer in the case of extensions of prime degree was given partially by Kuhlmann. In Section 3.3 we fill gaps in his sketch of proof and give an upper bound for the number of distinct distances in the case of extensions of higher degree (Theorem 3.23.).

We further study Galois defect extensions of prime degree. The importance of studying the structure of such extensions comes from the fact that towers of Galois extensions play a central role in the issue of defect extensions. This follows from the fact that every finite separable extension $(L|K, v)$, lifted up to the absolute ramification field of K, becomes a tower of Galois extensions of degree equal to the characteristic exponent of Kv. Furthermore, the defect of the lifted extension remains unchanged (for the details see Section 2.4). Since the existence of defect extensions shows a "bad behaviour" of the valuation, we are interested in the question whether the problem of defect extensions appears only in finite extensions, after which the defect vanishes. In Chapter 4 we show that the situation is not that simple by giving criteria for valued fields of positive residue characteristic p with p-divisible value group and perfect residue field to admit infinite towers of Galois defect extensions of degree p. We prove that under certain conditions the existence of at least one Galois defect extension of prime degree implies the existence of an infinite tower of such extensions. We give constructions of such towers.

The existence of infinite towers of Galois defect extensions of prime degree of a given field (K, v) turned out to be important for the description of maximal immediate extensions of (K, v). We apply the facts proven in Chapter 4 in our further investigation of this description (cf. Chapters 6 and 7).

The main results of the thesis are contained in Section 5.1 and in Chapters 6 and 7. Chapter 5 is devoted to defect extensions of rational function fields of positive characteristic in two variables. We focus on the issue of towers of Galois defect extensions of prime degree of such fields. If the characteristic of the valued field is positive, then Galois defect extensions of prime degree are Artin-Schreier defect extensions (cf. Section 2.4). The structure of defect extensions of function fields is especially interesting for the problems related to resolution of singularities, such as local uniformization. In particular, in the case of two dimensional algebraic function fields of positive characteristic, a strong relative form of local uniformization presented in Theorem 7.35 of [8] does not hold in case of nontrivial defect. This can be shown by an example which consists of a tower of two Artin-Schreier defect extensions of a rational function field in two variables (cf. Theorem 7.38 of [8]). In connection with these results and the importance of towers of Artin-Schreier extensions, we study the problem of constructing infinite towers of Artin-Schreier defect extensions of rational function fields in two variables. An example of such a construction was given by Kuhlmann ([22]). He showed the existence of a valuation of the rational function field in two variables over an algebraically closed field such that the valuation is trivial on the field of coefficients and the valued function field admits an infinite tower of Artin-Schreier defect extensions. We generalize this fact proving the following theorem.

Theorem 1.1. *Assume that K is a field of positive characteristic p and that it admits a perfect subfield of cardinality κ. Then there is a valuation v on the rational function field $K(x,y)|K$ whose restriction to K is trivial, such that $(K(x,y),v)$ admits κ many pairwise linearly disjoint infinite towers of Artin-Schreier defect extensions.*

The above theorem shows that even relatively simple fields can admit valuations such that the algebraic extensions of the valued fields have a very bad structure.

We further consider a classification of Artin-Schreier defect extensions (introduced by Kuhlmann in [25]) into "dependent" and "independent" ones, according to whether they are connected with purely inseparable defect extensions, or not. There are indications that considering this classification in connection with the problem of local uniformization is meaningful. M. Temkin's work (especially [41]) appears to show that the dependent Artin-Schreier defect extensions may be more harmful. This seems to be confirmed by the mentioned example of Cutkosky and Piltant (Theorem 7.38 of [8]). Work in progress of L. Ghezzi and S. ElHitti indicates that the tower of two Artin-Schreier defect extensions constructed in the example consists of dependent extensions. We therefore consider various valuations of the rational function fields and investigate for which they admit an infinite tower of dependent or independent Artin-Schreier defect extensions. In Section 5.1.1 we give examples of valuations for which the rational function fields admit towers of both kinds of extensions. We also prove that there are valuations for which the rational function fields admit no dependent, but infinite towers of independent Artin-Schreier defect extensions (cf. Section 5.1.2). As all of the proofs are constructive, this provides valuable examples for the further study of the local uniformization problem. The results of Chapter 5 are presented in [4].

In Chapter 6 we study the structure of maximal immediate extensions of valued fields. We focus mainly on the question of transcendence degree of the maximal immediate extensions of a given valued field. This turned out to be important for the description of the possible extensions of a valuation from a given field to an algebraic function field. Ramification theory enables us to deal with the algebraic extensions and reduces the problem to describing the possible extension of a valuation from a given field to a rational function field over this field. The case of rational function fields in one variable was studied already by Ostrowski ([39]) and investigated later by many authors (see the references in [22] for a selection from the literature on this problem). The case of higher transcendence degree was considered in [22]. The problem turned out to be tightly connected with the question whether the maximal immediate extensions of a given valued field have finite or infinite transcendence degree. The following theorem gives conditions for a valued field to admit maximal immediate extensions of the latter kind (see Chapter 2 for the notions).

Theorem 1.2. *Take a valued field extension $(L|K,v)$ of finite transcendence degree ≥ 0, with v nontrivial on L. Assume that one of the following four cases holds:*

valuation-transcendental case: vL/vK is not a torsion group, or $Lv|Kv$ is transcendental;

value-algebraic case: vL/vK contains elements of arbitrarily high order, or there is a subgroup $\Gamma \subseteq vL$ containing vK such that Γ/vK is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of Kv;

residue-algebraic case: Lv contains elements of arbitrarily high degree over Kv;

separable-algebraic case: $L|K$ contains a separable-algebraic subextension $L_0|K$ such that within some henselization of L, the corresponding extension $L_0^h|K^h$ is infinite.

Then each maximal immediate extension of (L,v) has infinite transcendence degree over L. If the cofinality of vL is countable (which for instance is the case if vL contains an element γ such that $\gamma > vK$), then already the completion of (L,v) has infinite transcendence degree over L.

This theorem is a far-reaching generalization of the constructive proof of the fact that the Laurent power series field $K((x))$ over a field K is of infinite transcendence degree over the rational function field $K(x)$, given by MacLane and Schilling in [37]. The proof of Theorem 1.2 in particular presents effective methods for the construction of infinitely many algebraically independent elements over various fields.

Because of the meaning of immediate extensions for various questions in valuation theory and related areas, fields which do not admit any proper immediate extensions are of particular interest. Such valued fields are called **maximal**. It is well known that a finite extension of a maximal field is again maximal (cf. Section 2.5 for details and references). In Section 6.2 we answer the question when an infinite algebraic extension (L,v) of a maximal field (K,v) can be again maximal and discuss the possible form of the maximal immediate extensions of (L,v) if it is not maximal.

Another important question connected with immediate extensions is the problem of uniqueness of maximal immediate extensions of valued fields. Kaplansky proved that under a certain condition, which he called "hypothesis A", a valued field admits maximal immediate extensions which are unique up to isomorphism (see Section 2.5 for details). He also gave an example showing that if hypothesis A is violated, then uniqueness may not hold. It is an open question whether the uniqueness of maximal immediate extensions of a given valued field always fails when the field does not satisfy hypothesis A and its completion is not maximal. Interesting is also the question how much the maximal immediate extensions of a given field can differ. In Theorem 6.10 we prove the existence of a class of valued fields which admit an algebraic maximal immediate extension as well as one of infinite transcendence degree. Results of Sections 6.1 and 6.2 are joint work with F.-V. Kuhlmann and are presented in [5].

Section 6.3 of Chapter 6 is devoted to the problem of the structure of valued rational function fields. We investigate the question when a valuation v of a given field K admits an extension to a rational function field $K(x_1,\dots,x_n)$ such that $vK(x_1,\dots,x_n)/vK$ is a torsion group and the residue field extension $K(x_1,\dots,x_n)v|Kv$ is algebraic. The results presented in this section are generalizations, comments and corrections of facts proved in [22].

In the last chapter we consider maximal immediate extensions of a valued field (K,v) with p-divisible value group and perfect residue field, where p is the characteristic exponent of the residue field Kv. Maximal immediate extensions of such fields are tame (cf. Section 2.2.3 for definition and properties). Since tame fields have good valuation theoretical and model theoretical properties, they play an important role in the theory of valued fields and its applications (cf., e.g., [21] and [22]).

We consider first the case when (K,v) admits a maximal immediate extension (M,v) algebraic over K. It was proven in [29] that if $M|K$ is finite, then the extension is trivial.

We extend the result by showing that if $M|K$ is an algebraic extension, then it is purely inseparable and equal to the completion of (K, v) (see Corollary 7.5). We consider further the case when $M|K$ is of finite transcendence degree, and prove in particular that the maximal immediate extension of (K, v) is then unique up to isomorphism (Theorem 7.7). This enables us to prove the following theorem, which relates the problem of the existence of nontrivial separable-algebraic defect extensions of (K, v) with the structure of the maximal immediate extensions of the field.

Theorem 1.3. *Take a valued field (K, v) of positive residue characteristic p, with p-divisible value group and perfect residue field. Assume that at least one of the following cases holds:*

1) (K, v) admits a finite separable-algebraic extension $(F|K, v)$ such that the valuation v extends in a unique way from K to F and $(F|K, v)$ has nontrivial defect;

2) $\operatorname{char} K = p$ and the perfect hull of K is not contained in the completion of K.

Then every maximal immediate extension of (K, v) is of infinite transcendence degree over K.

We then apply the results to the description of the extensions of a valuation v from the field K to a rational function field F over K. We give necessary and sufficient conditions on (K, v) to admit an extension of the valuation v to F such that vF/vK is a torsion group and the extension $Fv|Kv$ is algebraic (Theorem 7.16).

2. Preliminaries

This chapter briefly outlines main definitions and facts that will be used in this dissertation. We introduce also terminology and notions we will work with.

2.1 Linearly disjoint and algebraically disjoint extensions

In this section we recall a few properties of linearly and algebraically disjoint extensions. For the details see for instance [35], Chapter VII, and [29]. We assume that all considered fields are contained in a common extension field Ω. For an arbitrary field K, we will denote by $\tilde{K}$ the algebraic closure of K and by K^{sep} the separable-algebraic closure of K.

Take field extensions $L|K$ and $F|K$. We say that $L|K$ is **linearly disjoint** from $F|K$ if for every $n \in \mathbb{N}$, any $a_1, \ldots, a_n \in L$ linearly independent over K will also be linearly independent over F. This holds if and only if every finite subextension $E|K$ of $L|K$ is linearly disjoint from $F|K$, that is, if $[E : K] = [E.F : F]$ for every such subextension, where $E.F$ denotes the compositum of E and F inside Ω. This property is symmetrical with respect to L and F, hence we will also say that L and F are K-linearly disjoint.

A direct consequence of the above definition is the following transitivity property (cf. Proposition 3.1, Chapter 8 of [35]).

Lemma 2.1. *Let $L|K$ and $F \supseteq E \supseteq K$ be field extensions. Then $L|K$ is linearly disjoint from $F|K$ if and only if $L|K$ is linearly disjoint from $E|K$ and $L.E|E$ is linearly disjoint from $F|E$.*

The next lemma gives a useful criterion for linear disjointness if at least one of the extensions is Galois.

Lemma 2.2. *Suppose $L|K$ is a Galois and $F|K$ an arbitrary field extension. Then L and F are linearly disjoint over K if and only if $L \cap F = K$.*

For the proof, see [45], Chapter VII, Theorem 10.

Take an arbitrary field extension $K'|K$. If $L|K$ is a Galois extension, then also $L.K'|K'$ and $L|L \cap K'$ are Galois extensions. Moreover, the restriction of the automorphisms of $L.K'|K'$ to L induces a topological isomorphism $\mathrm{Gal}\,(L.K'|K') \cong \mathrm{Gal}\,(L|L \cap K')$ (cf. Proposition 6.5, Chapter 6 of [17]). Together with the previous lemma these facts proves the following property.

Corollary 2.3. *Take an arbitrary field extension $K'|K$ and a Galois extension $L|K$, linearly disjoint from $K'|K$. Then the extension $L.K'|K'$ is also Galois and the restriction of the automorphisms of $L.K'|K'$ to the field L is a topological isomorphism of $Gal(L.K'|K')$ and $Gal(L|K)$.*

A field extension $L|K$ will be called **separable** if it is linearly disjoint from $K^{1/p^\infty}|K$, or equivalently, from $K^{1/p}|K$. Note that in the case of algebraic extensions the definition coincides with the standard notion of separable extensions. Such extensions will be called separable-algebraic.

Lemma 2.4. *If $F|K$ is a field extension such that K is relatively algebraically closed in F, then $F|K$ is linearly disjoint from every separable-algebraic extension of K.*

Proof. Take a separable-algebraic extension L of K and a finite subextension $L'|K$. Since $L|K$ is separable-algebraic, $L' = K(a)$ for some $a \in L$, by the Theorem of Primitive Element. Then a and all its conjugates over K are algebraic over F. If h is the minimal polynomial of a over F, then each of its coefficients is a symmetric function in the conjugates of a over F, which are among the conjugates of a over K. Hence, the coefficients of h are algebraic over K. Since K is relatively algebraically closed in F, all of the coefficients lie in K and thus h is also the minimal polynomial of a over K. Consequently, $[K(a) : K] = [F(a) : F]$. Therefore, F and L' are K-linearly disjoint. It follows that also F and L are K-linearly disjoint. □

Take field extensions $L|K$ and $F|K$. The extension $L|K$ is called **algebraically disjoint** from $F|K$ if for every $n \in \mathbb{N}$, any $a_1, \ldots, a_n \in L$ algebraically independent over K will also be algebraically independent over F. Hence, $L|K$ is algebraically disjoint from $F|K$ if every finitely generated subextension $E|K$ of $L|K$ satisfies trdeg $E|K$ =trdeg $E.F|F$. As in the case of linear disjointness, the property of algebraic disjointness is symmetrical. Thus if $L|K$ is algebraically disjoint from $F|K$ we also say that L and F are K-algebraically disjoint. Directly from the definition of linear and algebraic disjointness it follows that if $L|K$ is linearly disjoint from $F|K$ then it is also algebraically disjoint from $F|K$. The converse holds only under additional assumptions. We are going to use it in the following form:

Lemma 2.5. *Let $L|K$ and $F|K$ be algebraically disjoint field extensions. If K is relatively algebraically closed in L and $F|K$ is separable then $L|K$ and $F|K$ are also linearly disjoint.*

The above fact is a special case of Theorem 4.12, Chapter VIII of [35].

2.2 Extensions of valued fields

In this section we recall basic notions and facts related to valued fields and their extensions. For the details we refer the reader to [10], [11], [29], [43] and [47] . For an abelian group Γ we will denote by $\tilde{\Gamma}$ the divisible hull of Γ.

2.2.1 Valued fields and their extensions

Take a field K and an ordered abelian group $(\Gamma, +, 0, <)$. Extend the operation and the ordering of Γ to $\Gamma \cup \{\infty\}$ by setting $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$ and $\gamma < \infty$ for every $\gamma \in \Gamma$. A mapping $v\colon K \to \Gamma \cup \{\infty\}$ is a valuation of K if it satisfies the following conditions:

(V0) $v(x) = \infty \Leftrightarrow x = 0$
(VH) $v(xy) = v(x) + v(y)$
(VU) $v(x+y) \geq \min\{v(x), v(y)\}$,

for all $x, y \in K$. In this case the pair (K, v) is called a **valued field**. We will write also K in place of (K, v) if the valuation v is fixed. For simplicity we will write vx in place of $v(x)$. From the definition we obtain that that $v1 = 0$ and thus $vx^{-1} = -vx$, for every $x \in K$. Moreover, if $x, y \in K$ are such that $vx \neq vy$, then $v(x+y) = \min\{vx, vy\}$. Hence in particular, $v(-x) = vx$ for every $x \in K$.

The set $vK := \{vx \mid x \in K^\times\}$ is a subgroup of the ordered abelian group Γ and is called a **value group** of (K, v). The valuation v is called **trivial** if $vK = \{0\}$. We further define the **rank of** (K, v) to be the rank of vK, that is, the order type of the chain of all proper convex subgroups of vK. Hence, v is of rank 1 if and only if vK is archimedean, that is, embeddable in $\mathbb{R}$ with its natural ordering.

From the definition of a valuation it follows that the set

$$\mathcal{O}_v := \{x \in K \mid vx \geq 0\}$$

is a subring of K. It is called the **valuation ring of** (K, v). It will be also denoted by $\mathcal{O}_K$. Further,

$$\mathcal{M}_v := \{x \in K \mid vx > 0\}$$

is an ideal of $\mathcal{O}_v$, called the **valuation ideal of** (K, v).

Since $\mathcal{O}_v \setminus \mathcal{M}_v = \{x \in K \mid vx = 0\}$ is the set of all invertible elements of $\mathcal{O}_v$, the valuation ring is a local ring and $\mathcal{M}_v$ is the unique maximal ideal of $\mathcal{O}_v$. The field $\mathcal{O}_v/\mathcal{M}_v$ is called the **residue field of** (K, v) and denoted by Kv. The element $a + \mathcal{M} \in Kv$ will be denoted by av. Similarly, if $f = a_n X^n + \cdots + a_1 X + a_0 \in \mathcal{O}_v[X]$, then by fv we will denote the reduction $(a_n v)X^n + \cdots + (a_1 v)X + a_0 v \in Kv[X]$ of f modulo v. The characteristic of Kv is called the **residue characteristic** of (K, v). Note that if $\operatorname{char} K = p > 0$, then also $\operatorname{char} Kv = p$. The converse is not true. The valued fields of positive residue characteristic will be for us of particular interest because of the phenomenon of defect, which can appear in this case (see Section 2.2.3).

By an **isomorphism of valued fields** (K, v) and (L, w) we will mean a field isomorphism $\sigma\colon K \to L$ preserving the valuation: $\sigma(\mathcal{O}_K) = \mathcal{O}_L$. Such an isomorphism preserves all valuation theoretical properties.

Given any subset S of K, we define

$$vS = \{va \mid 0 \neq a \in S\} \text{ and } Sv = \{av \mid a \in S, va \geq 0\}.$$

Take a valued field (K, v) and a field extension L of K. Then the valuation v admits an extension to a valuation of the field L (cf. Theorem 13.2 of [10]). By $(L|K, v)$ we denote an extension of valued fields, where v is a valuation of L and K is equipped with the restriction of this valuation. We will omit the valuation, writing $L|K$ for the valued field extension, if v is fixed. The natural embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ enable us to consider also the value group and residue field extensions. The (finite or infinite) cardinal $e(L|K, v) := (vL : vK)$ is called the **ramification index**, and $f(L|K, v) := [Lv : Kv]$ is called the **inertia degree** of $(L|K, v)$. The next lemmas state the relation between a valued field extension and the respective extensions of value group and residue field.

Lemma 2.6. *Let $(L|K,v)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values vx_i, $i \in I$, are rationally independent over vK, and the residues y_jv, $j \in J$, are algebraically independent over Kv. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.*

Moreover, if

$$f \;=\; \sum_{k\in S} c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J], \tag{2.1}$$

where S is a finite subset of $\mathbb{N}$ and for every $k \neq \ell$ there is some $i \in I$ s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some $j \in J$ s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf \;=\; \min_{k\in S} v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_{k\in S} \left(vc_k \,+\, \sum_{i\in I} \mu_{k,i} vx_i \right). \tag{2.2}$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i \in I, j \in J) \;&=\; vK \oplus \bigoplus_{i\in I} \mathbb{Z} vx_i \\ K(x_i, y_j \mid i \in I, j \in J)v \;&=\; Kv\,(y_jv \mid j \in J)\,. \end{aligned}$$

Moreover, the valuation v on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues y_jv.

For the proof, see [3], chapter VI, §10.3, Theorem 1 together with §10.1, Propositions 1 and 2.

The algebraic analogue to the transcendental case discussed in Lemma 2.6 is the following lemma:

Lemma 2.7. *Let $(L|K,v)$ be an extension of valued fields. Suppose that $\eta_1, \ldots, \eta_k \in L$ are such that $v\eta_1, \ldots, v\eta_k \in vL$ belong to distinct cosets modulo vK. Further, assume that $\vartheta_1, \ldots, \vartheta_\ell \in \mathcal{O}_L$ are such that $\vartheta_1 v, \ldots, \vartheta_\ell v$ are Kv-linearly independent. Then the elements $\eta_i\vartheta_j$, $1 \leq i \leq k$, $1 \leq j \leq \ell$, are K-linearly independent, and for every choice of elements $c_{ij} \in K$, we have that*

$$v \sum_{i\leq k, j\leq \ell} c_{ij}\eta_i\vartheta_j \;=\; \min_{i\leq k, j\leq \ell} vc_{ij}\eta_i\vartheta_j \;=\; \min_{i\leq k, j\leq \ell} (vc_{ij} \,+\, v\eta_i)\,. \tag{2.3}$$

If the elements $\eta_i\vartheta_j$ form a K-basis of L, then

$$vL \;=\; vK + \bigoplus_{1\leq i\leq k} \mathbb{Z} v\eta_i \qquad \textit{and} \qquad Lv \;=\; Kv(\vartheta_j v \mid 1 \leq j \leq \ell)\,.$$

The lemma follows from the proof of Theorem 30.14 of [43].

A direct consequence of the first assertion of the above lemma is the following

Corollary 2.8. *If $(L|K,v)$ is a finite extension of valued fields, then*

$$[L : K] \;\geq\; (vL : vK)[Lv : Kv]. \tag{2.4}$$

Under the conditions of the corollary one can say even more. If $v_1 = v, \dots, v_g$ are the distinct extensions of the valuation v of K to the field L, then $L|K$ satisfies the **fundamental inequality** (cf. Corollary 17.5 of [10]):

$$[L : K] \;\geq\; \sum_{i=1}^{g} (v_i L : v_i K)[L v_i : K v_i]. \tag{2.5}$$

Note that from the above corollary it follows that if $(L|K, v)$ is a finite extension of valued fields, then $Lv|Kv$ is an algebraic extension and the group vL/vK is torsion. Thus $vL \subseteq \widetilde{vK}$ and $Lv \subseteq \widetilde{Kv}$. Furthermore, we have (see Lemma 2.16 of [22]) :

Lemma 2.9. *Take a valued field (K, v) and assume that the valuation v is nontrivial. Take an extension of v to the algebraic closure $\tilde{K}$ of K. Then the value groups of $\tilde{K}$ and of K^{sep} are equal to the divisible hull of vK, and the residue fields of $\tilde{K}$ and of K^{sep} are equal to the algebraic closure of Kv.*

For any element x in a field extension of K and every nonnegative integer n, we set

$$K[x]_n \;:=\; K + Kx + \ldots + Kx^n \;.$$

Since $\dim_K K[x]_n \leq n+1$, we obtain the following corollary from Lemma 2.7:

Corollary 2.10. *Take a valued field extension $(K(x)|K, v)$. Then for every $n \geq 0$,*

a) the elements of $vK[x]_n$ lie in at most $n+1$ many distinct cosets modulo vK,

b) the Kv-vector space $K[x]_n v$ is of dimension at most $n+1$.

2.2.2 Henselian fields and henselizations

Although for every valued field (K, v) the valuation can be extended to any field extension L of K, such an extension does not need to be unique, even in the case of algebraic extensions. However, if K is of positive characteristic p and L is a purely inseparable extension of K, then any element of L is of the form $a^{p^{-n}}$ for some $a \in K$ and $n \geq 0$. From the definition of a valuation we obtain that $va^{p^{-n}} = p^{-n} va$, hence v admits a unique extension to L. In general, if $L|K$ is an algebraic extension, every two extensions v_1, v_2 of the valuation v from K to L are conjugate, that is, $v_2 = \tilde{v_1} \circ \iota$ for some extension of $\tilde{v_1}$ to $\widetilde{L}$ and an embedding ι of L in $\tilde{L}$ over K (see Chapter VI, §11 of [47]). Together with Corollary 2.3 this yields the following fact.

Lemma 2.11. *Take a valued field (L, v). If $(L(b), v_1)$ is an algebraic extension and $(L(x), v_2)$ a transcendental extension of the field (L, v), then there is an extension w of the valuation v to the field $L(x, b)$ such that the restrictions of w to the fields $L(b)$ and $L(x)$ coincide with v_1 and v_2, respectively.*

Proof. Take L_0 to be the separable algebraic closure of L in $L(b)$ and define F to be the normal hull of L_0 over K. Then $F|L$ is a Galois extension. Take an extension v_1' of the valuation v_1 to the field $F(b)$ and an extension v_2' of v_2 to the field $L(x).F = F(x)$. Since v_1' and v_2' coincide on L, the valuations $v_1'|_F$ and $v_2'|_F$ are conjugate. Take $\sigma \in$Gal$(F|L)$ such that $v_1'|_F = v_2' \circ \sigma$.

By Lemma 2.4, the extensions $F|L$ and $L(x)|L$ are linearly disjoint. Since $F|L$ is a Galois extension, it follows from Corollary 2.3 that σ can be extended to an authomorphism $\widetilde{\sigma}\in$Gal$(F(x)|L(x))$. Setting $w_0 := v_2' \circ \widetilde{\sigma}$, we obtain an extension of the valuations $v_1|_{L_0}$ and v_2 to the field $L_0(x)$.

As $L(b,x) = L_0(x).L(b)$ is a purely inseparable extension of $L_0(x)$, the valuation w_0 admits a unique extension w to $L(x,b)$. Since $L(b)|L_0$ is also purely inseparable, $w|_{L(b)}$ must coincide with v_1. □

Take a valued field (K,v). If the valuation v admits a unique extension to the algebraic closure $\tilde{K}$ of K, equivalently, admits a unique extension to any finite extension L of K, then (K,v) is called **henselian**. Note that every algebraic extension of a henselian field is henselian.

For a valued field (K,v) there is a henselian field extension which admits a unique embedding in every other henselian extension of (K,v). This extension is separable-algebraic and unique up to isomorphism over K (cf. Theorem 17.11 of [10]). It is called the **henselization** of (K,v) and denoted by $(K,v)^h$ or, if v is fixed, by K^h. Furthermore, $K^h|K$ is an immediate extension (cf. [10], Theorem 17.19). Fix the extension of v to $\tilde{K}$ and take any algebraic extension F of K. Then F^h must contain K^h and F, hence $K^h.F \subseteq F^h$. Conversely, since every algebraic extension of a henselian field is henselian, $K^h.F$ contains F^h . Therefore,

$$F^h = K^h.F.$$

Since $K^h|K$ is separable-algebraic, it is linearly disjoint from any purely inseparable extension of K. On the other hand, we know that the valuation v of K extends uniquely to any such extension. Generally, we have:

Lemma 2.12. *If L is a finite extension of a valued field (K,v), then the extension of v to L is unique if and only if $L|K$ is linearly disjoint from some (equivalently, every) henselization of (K,v).*

Proof. By Corollary 7.48 of [29], we have that

$$[L:K] = \sum_{i=1}^{g} [L^{h(v_i)} : K^{h(v_i)}] = [K^{h(v_i)}.L : K^{h(v_i)}], \tag{2.6}$$

where $v_1,\dots,v_g$ are the distinct extensions of v to L and $L^{h(v_i)}$, $K^{h(v_i)}$ are henselizations of L and K with respect to an extension of v_i to $\tilde{L} = \tilde{K}$. If $L|K$ was not linearly disjoint from $K^{h(v_i)}|K$ for some $i \le g$, then $[K^{h(v_i)}.L : K^{h(v_i)}] < [L:K]$. Thus $g \ge 2$ and the extension of v from K to L is not unique.

On the other hand, if $L|K$ is linearly disjoint from some henselization K^h of K, then $[L:K] = [K^h.L : K^h]$ and from equation (2.6) we deduce that v admits a unique extension from K to L. □

For the valued field (K,v) the property of being henselian is equivalent to various criteria. Two of them are presented in the next two theorems.

Theorem 2.13 (Hensel's Lemma)**.** *A valued field (K,v) is henselian if and only if it satisfies the following property: for every monic polynomial $f \in \mathcal{O}_v[X]$, if fv admits a simple root $\xi \in Kv$, then f admits a root $a \in \mathcal{O}_v$ such that $av = \xi$.*

For the proof, see for instance [10], Corollary 16.6.

A direct application of Hensel's Lemma is the following fact.

Lemma 2.14. *Assume (L, v) to be henselian and K to be relatively separable-algebraically closed in L. Then Kv is relatively separable-algebraically closed in Lv. If in addition $Lv|Kv$ is algebraic, then the torsion subgroup of vL/vK is a p-group, where p is the characteristic exponent of Kv.*

Proof. Take $\zeta \in Lv$ separable-algebraic over Kv. Choose a monic polynomial $g(X) \in K[X]$ whose reduction $gv(X) \in Kv[X]$ modulo v is the minimal polynomial of ζ over Kv. Then ζ is a simple root of gv. Hence by Hensel's Lemma, there is a root $a \in L$ of g whose residue is ζ. As all roots of gv are distinct, we can lift them all to distinct roots of g. Since g is monic, $\deg g = \deg gv$ and thus a is separable-algebraic over K. From the assumption of the lemma, it follows that $a \in K$, showing that $\zeta \in Kv$. This proves that Kv is relatively separable-algebraically closed in Lv.

Now assume in addition that $Lv|Kv$ is algebraic. Then Kv is relatively separable-algebraically closed in Lv, by what we have proved already. Take $\alpha \in vL$ and $n \in \mathbb{N}$ not divisible by p such that $n\alpha \in vK$. Choose $a \in L$ and $b \in K$ such that $va = \alpha$ and $vb = n\alpha$. Then $v(a^n/b) = 0$. Since $Lv|Kv$ is a purely inseparable extension, there exists $m \in \mathbb{N}$ such that $((a^n/b)v)^{p^m} \in Kv$. We choose $c \in K$ satisfying $vc = 0$ and $cv = ((a^n/b)v)^{p^m}$, to obtain that $(a^{np^m}/cb^{p^m})v = 1$. So the reduction of the polynomial $X^n - a^{np^m}/cb^{p^m}$ modulo v is $X^n - 1$. Since n is not divisible by p, 1 is a simple root of this polynomial. Hence by Hensel's Lemma, there is a simple root $d \in L$ of the polynomial $X^n - a^{np^m}/cb^{p^m}$ with $dv = 1$, whence $vd = 0$. Consequently, a^{p^m}/d is a simple root of the polynomial $X^n - cb^{p^m}$ and thus is separable algebraic over K. Since K was assumed to be relatively separable-algebraically closed in L, we find that $a^{p^m}/d \in K$. As n is not divisible by p, there are $k, l \in \mathbb{Z}$ such that $1 = kn + lp^m$. This yields:

$$\alpha \;=\; kn\alpha + lp^m\alpha \;=\; kn\alpha + l(p^m va - vd) \;=\; k(n\alpha) + lv\left(\frac{a^{p^m}}{d}\right) \in vK.$$

□

Let (K, v) be any valued field and $a \in \tilde{K} \setminus K$ not purely inseparable over K. Choose an extension of v from K to $\tilde{K}$. The **Krasner constant of** a **over** K is defined as

$$\mathrm{kras}(a, K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma, \tau \in \mathrm{Gal}\,(\tilde{K}|K) \;\text{ and }\; \tau a \neq \sigma a\} \;\in\; v\tilde{K}.$$

Since all extensions of v from K to $\tilde{K}$ are conjugate, this definition does not depend on the choice of the particular extension of v. For the same reason, over a henselian field (K, v) we have that

$$\mathrm{kras}(a, K) \;=\; \max\{v(a - \sigma a) \mid \sigma \in \mathrm{Gal}\,(\tilde{K}|K) \;\text{ and }\; a \neq \sigma a\}\;.$$

Theorem 2.15 (Krasner's Lemma). *A valued field (K, v) is henselian if and only if the following condition holds. Take an extension of v to K^{sep} and call it again v. If $a \notin K$ is separable-algebraic over K, then for any $b \in K^{sep} \setminus K$ such that*

$$v(a - b) > \mathrm{kras}(a, K)$$

the element a lies in $K(b)$.

For the proof, see e.g., [10], Lemma 16.8.

Lemma 2.16. *Take an extension $(K(a)|K,v)$ of henselian fields, where a is an element in the separable-algebraic closure of K with $va \geq 0$. Then*

$$va \;\leq\; \mathrm{kras}(a,K)\,, \tag{2.7}$$

and for every polynomial $f = d_m X^m + \ldots + d_0 \in K[X]$ of degree $m < [K(a):K]$,

$$vf(a) \;\leq\; vd_m \,+\, m\,\mathrm{kras}(a,K)\,. \tag{2.8}$$

Proof. Since (K,v) is henselian and every two extensions of v from K to $K(a)$ are conjugate, $v\sigma a = a$ and therefore, $v(a-\sigma a) \geq \min\{va, v\sigma a\} = va$ for all σ. This yields inequality (2.7).

Take any element b in the separable-algebraic closure of K with $[K(b):K] < [K(a):K]$. Then $v(a-b) \leq \mathrm{kras}(a,K)$ since otherwise, Krasner's Lemma would yield that $a \in K(b)$ and $[K(b):K] \geq [K(a):K]$.

If we write $f(X) = d_m \prod_{i=1}^{m}(X-b_i)$, then $[K(b_i):K] \leq \deg(f) < [K(a):K]$. Hence by our observation we obtain that

$$vf(a) \;=\; vd_m \,+\, \sum_{i=1}^{m} v(a-b_i) \;\leq\; vd_m \,+\, m\,\mathrm{kras}(a,K)\,.$$

This proves inequality (2.8). □

For a rational function field $K(x_1,\ldots,x_n)|K$ equipped with a valuation v, denote by $IC(K(x_1,\ldots,x_n)|K,v)$ the relative algebraic closure of K in $K(x_1,\ldots,x_n)^h$. We will call it the **implicit constant field of** $(K(x_1,\ldots,x_n)|K,v)$. The implicit constant field depends on the choice of an extension of v to the algebraic closure of $K(x_1,\ldots,x_n)$. However, since the henselization of $K(x_1,\ldots,x_n)$ is unique up to valuation preserving isomorphism over $K(x_1,\ldots,x_n)$, also $IC(K(x_1,\ldots,x_n)|K,v)$ is unique up to valuation preserving isomorphism over K. Since for a fixed extension of v to the algebraic closure of $K(x_1,\ldots,x_n)$ the henselization K^h of K is an algebraic subextension of $K(x_1,\ldots,x_n)^h|K$, the implicit constant field of $(K(x_1,\ldots,x_n)|K,v)$ contains K^h and is itself henselian. Further, since $K(x_1,\ldots,x_n)^h|K$ is separable, $IC(K(x_1,\ldots,x_n)|K,v)$ is a separable-algebraic extension of K. The following lemma was proved in [22] (cf. Lemma 3.13).

Lemma 2.17. *Assume that $K(a)|K$ is a separable-algbraic extension, $K(x)|K$ is a rational function field and v a valuation on $\widetilde{K(x)}$ such that*

$$v(x-a) > \mathrm{kras}(a,K).$$

Then $K(a) \subseteq IC(K(x)|K,v)$ and consequently, $vK(a) \subseteq vK(x)$ and $K(a)v \subseteq K(x)v$.

2.2.3 Tame, defect and immediate extensions

Take a valued field (K,v) and a finite extension L of K. Assume that the extension of the valuation v from K to L is unique, which holds in particular if (K,v) is henselian. Then equality (2.5) is of the form (2.4) and the nature of the "missing factor" on the right hand side of the inequality is determined by the Lemma of Ostrowski (cf. equation (1.1)).

An infinite algebraic extension $(F|K,v)$ such that the valuation v admits a unique extension from K to F is called **defectless** if every finite subextension $(E|K,v)$ of $(F|K,v)$ is defectless, that is, if $d(E|K,v)=1$ for every such subextension. The field (K,v) is called **defectless** if equality holds in the fundamental inequality (2.5) for every finite extension $L|K$. Note that a nontrivial defect can appear only in the positive residue characteristic case. Every field of residue characteristic 0 is defectless (see, e.g., Corollary 20.23 of [10]).

Take a valued field (K,v) and finite field extensions $K\subseteq L\subseteq M$. Assume that v extends in a unique way to a valuation of the field M and denote this extension again by v. Since degree of a field extension, ramification index and inertia degree are multiplicative, the defect is also multiplicative:

$$d(M|K,v) \;=\; d(M|L,v)\cdot d(L|K,v).$$

The defect of a valued field extension $(L|K,v)$ destroys the tight connection between this extension and the extensions of the invariants of (K,v) and (L,v), that is, their value groups and residue fields. If $(vF:vK)=1=[Fv:Kv]$ (i.e., the canonical embeddings of vK in vL and of Kv in Lv are surjective), which means that the extension $L|K$ is immediate, then the value group and residue field extensions carry minimal information about the extension $L|K$.

Note that by Lemma 2.12, for a finite extension $(L|K,v)$ of valued fields the defect of the extension is equal to its degree if and only if $(L|K,v)$ is an immediate extension and it is linearly disjoint from some henselization of (K,v). Directly from the definition of an immediate extension it follows that an infinite algebraic extension of valued fields is immediate if and only if every finite subextension of the extension is immediate.

Lemma 2.18. *Take a valued field extension $(\Omega|K,v)$. Assume that $(L|K,v)$ is a finite defectless and $(E|K,v)$ an immediate subextension of $(\Omega|K,v)$. If v admits a unique extension from K to L, then $L|K$ is linearly disjoint from $E|K$ and the extension $(E.L|L,v)$ is immediate.*

For the proof, see Lemma 2.5, [25].

Take a finite extension $(L|K,v)$ and assume that the valuation of K admits a unique extension to the field L. Fix an extension of this valuation to $\tilde{K}$ and denote it again by v. Then from Lemma 2.12 it follows that $[L:K]=[L.K^h:K^h]=[L^h:K^h]$. Since $(K^h|K,v)$ and $(L^h|L,v)$ are immediate extensions, $(vL:vK)[Lv:Kv]=(vL^h:vK^h)[L^hv:K^hv]$. Together with the definition of defect this yields that

$$d(L|K,v) \;=\; d(L^h|K^h,v). \tag{2.9}$$

There are also other extensions which do not change the defect. An algebraic extension $(L|K,v)$ of henselian fields is called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(T1) the ramification index $(vE:vK)$ is prime to the characteristic exponent of Kv,

(T2) the residue field extension $Ev|Kv$ is separable,

(T3) $(E|K,v)$ is a defectless extension.

Assume that (L,v) is a henselian field with $\mathrm{char}Lv=0$. Then the first two conditions of the above definition are trivially satisfied and the third one follows immediately from the Lemma of Ostrowski. Hence every algebraic extension of such a field is tame.

A henselian field (K,v) is said to be a **tame field** if $(\tilde{K}|K,v)$ is a tame extension. If p is the residue characteristic exponent of K, then directly from the definition of a tame extension it follows that (K,v) is tame if and only if

(TF1) the value group vK is p-divisible

(TF2) the residue field Kv is perfect,

(TF3) (K,v) is a defectless field.

Take a valued field (K,v), fix an extension of v to K^{sep} and call it again v. The fixed field of the closed subgroup

$$G^r := \{\sigma \in \mathrm{Gal}\,(K^{sep}|K) \mid v(\sigma a - a) > va \text{ for all } a \in \mathcal{O}_{K^{sep}} \setminus \{0\}\}$$

of $\mathrm{Gal}(K^{sep}|K)$ (cf. Corollary 20.6 of [10]) is called the **absolute ramification field of** (K,v) and is denoted by $(K,v)^r$ or K^r if v is fixed. If p is the characteristic exponent of the residue field of (K,v), then Lemma 2.7 of [25] states that $K^{sep}|K^r$ is a p**-extension**, that is, a Galois extension with a pro-p-group as its Galois group. Moreover, for any algebraic extension L of K we have that

$$L^r = K^r.L \tag{2.10}$$

(cf., Theorem 4.10 of [27]). From Section 4. of [27] (cf. equation (4.5) and the definition of henselization presented in that paper) it follows that $K^h \subseteq K^r$. In particular, together with the equation (2.10) it shows that

$$(K^h)^r = K^r. \tag{2.11}$$

If (K,v) is henselian, then K^r is a Galois extension of K, and it is also the unique maximal tame extension of (K,v) (see Theorem 20.10 of [10] and Proposition 4.1 of [32]). This yields that every tame extension of valued fields is separable-algebraic. Furthermore, we obtain that (K,v) is a tame field if and only if $K^r = \widetilde{K}$. An important property of the absolute ramification field is the following fact, which follows via Galois correspondence from the fact that G^r is a pro-p-group (For the proof, see Lemma 2.9 of [25]).

Lemma 2.19. *Let (K,v) be a valued field extension and take p to be the characteristic exponent of Kv. Then every finite extension of K^r is a tower of normal extensions of degree p. If $L|K$ is a finite extension, then there is already a finite tame extension N of K^h such that $L.N|N$ is such a tower.*

The above lemma together with the next proposition are of key importance for our further studies.

Proposition 2.20. *Take a henselian field (K,v) and a tame extension N of K. Then for any finite extension $L|K$,*

$$d(L|K,v) = d(L.N|N,v).$$

For the proof, see [25], Proposition 2.8.

2.3 Cuts and distances

We recall basic notions and facts connected with cuts of ordered abelian groups and distances of elements of valued field extensions, which will be among of our main tools in our further studies. For the details see Section 2.3 of [25] and Section 3 of [33]. In Section 3.1 we will introduce another definition of distance of elements in the case of algebraic extensions. The alternative definition carries more information about the valued field extension; however, the definition presented in this section allows us to determine distances of elements in any extensions of valued fields.

Take a totally ordered set $(T, <)$. For a nonempty subset S of T and an element $a \in T$ we will write $S < a$ if $s < a$ for every $s \in S$. A set $\Lambda^L \subseteq T$ is called an **initial segment** of T if for each $\alpha \in \Lambda^L$ every $\beta < \alpha$ also lies in Λ^L. A pair (Λ^L, Λ^R) of subsets of T is called a **cut** in T if Λ^L is an initial segment of T and $\Lambda^R = T \setminus \Lambda^L$. To compare cuts in $(T, <)$ we will use the lower cut sets comparison. That is, for two cuts $\Lambda_1 = (\Lambda_1^L, \Lambda_1^R)$, $\Lambda_2 = (\Lambda_2^L, \Lambda_2^R)$ in T we will write $\Lambda_1 < \Lambda_2$ if $\Lambda_1^L \subsetneq \Lambda_2^L$, and $\Lambda_1 \leq \Lambda_2$ if $\Lambda_1^L \subseteq \Lambda_2^L$.

For any $s \in T$, we define the cuts

$$s^- := (\{t \in T \,|\, t < s\}, \{t \in T \,|\, t \geq s\}),$$

$$s^+ := (\{t \in T \,|\, t \leq s\}, \{t \in T \,|\, t > s\}).$$

We identify the element s with s^+. Therefore, for a cut $\Lambda = (\Lambda^L, \Lambda^R)$ in T and an element $s \in T$ the inequality $\Lambda < s$ means that for every element $\beta \in \Lambda^L$ we have $\beta < s$. Similarly, for any subset M of T we define M^+ to be the cut (Λ^L, Λ^R) in T such that Λ^L is the least initial segment containing M, that is,

$$M^+ = (\{t \in T \,|\, \exists m \in M \ t \leq m\}, \{t \in T \,|\, t > M\}).$$

We denote by M^- the cut (Λ^L, Λ^R) in T such that Λ^L is the largest initial segment disjoint with M, i.e.,

$$M^- = (\{t \in T \,|\, t < M\}, \{t \in T \,|\, \exists m \in M \ t \geq m\}).$$

For every extension $(L|K, v)$ of valued fields and $z \in L$ we define

$$v(z - K) := \{v(z - c) \,|\, c \in K\}.$$

Take an element $\alpha \in v(z-K) \cap vK$ and $\beta \in vK$, $\beta < \alpha$. If $c, d \in K$ are such that $\alpha = v(z-c)$ and $vd = \beta$, then $\beta = \min\{vd, v(z-c)\} = v(z - (c+d)) \in v(z-K) \cap vK$. Hence, the set $v(z-K) \cap vK$ is an initial segment of vK and thus the lower cut set of a cut in vK. However, it is more convenient to work with the cut

$$\operatorname{dist}(z, K) := (v(z - K) \cap vK)^+ \quad \text{in the divisible hull } \widetilde{vK} \text{ of } vK.$$

We call this cut the **distance of z from K**. The lower cut set of $\operatorname{dist}(z, K)$ is the smallest initial segment of $\widetilde{vK}$ containing $v(z-K) \cap vK$. If the lower cut set of $\operatorname{dist}(z, K)$ is equal to $\widetilde{vK}$, we will write $\operatorname{dist}(z, K) = \infty$. Since $\operatorname{dist}(z, K)$ is always a cut in $\widetilde{vK}$, we can compare distances of elements over any algebraic extensions of (K, v), regardless of the respective value group extensions. If $(F|K, v)$ is an algebraic subextension of $(L|K, v)$ then $\widetilde{vF} = \widetilde{vK}$.

Thus dist (z, K) and dist (z, F) are cuts in the same group and we can compare these cuts by set inclusion of the lower cut sets. Since $v(z-K) \subseteq v(z-F)$ we deduce that

$$\mathrm{dist}\,(z, K) \ \leq \ \mathrm{dist}\,(z, F).$$

If dist $(z, K) = (\Lambda^L, \Lambda^R)$, then for any natural number n we define

$$n \cdot \Lambda^L \ := \ \{n\gamma \mid \gamma \in \Lambda^L\}.$$

Since dist (z, K) is a cut in the divisible group $\widetilde{vK}$, the set $n \cdot \Lambda^L$ is again an initial segment of $\widetilde{vK}$. We denote by $n \cdot$ dist (z, K) the cut in $\widetilde{vK}$ with the lower cut set $n \cdot \Lambda^L$. We say that the distance dist (z, K) is **idempotent** if

$$n \cdot \mathrm{dist}\,(z, K) \ = \ \mathrm{dist}\,(z, K)$$

for some natural number $n \geq 2$. If it holds, then obviously also $n^i \cdot \mathrm{dist}\,(z, K) \ = \ \mathrm{dist}\,(z, K)$, for any $i \in \mathbb{N}$. Denote by Λ^L the lower cut set of dist (z, K) and take any natural number m. If $0 \in \Lambda^L$, then $\Lambda^L \subseteq 2 \cdot \Lambda^L \subseteq \ldots \subseteq m \cdot \Lambda^L$. Otherwise, $\Lambda^L \supseteq 2 \cdot \Lambda^L \supseteq \ldots \supseteq m \cdot \Lambda^L$. This yields that if dist (z, K) is idempotent, then $n \cdot \mathrm{dist}\,(z, K) \ = \ \mathrm{dist}\,(z, K)$ for every natural number $n \in \mathbb{N}$. From Lemma 2.14 of [25] it follows that this condition holds if and only if dist $(z, K) = H^+$ or dist $(z, K) = H^-$ for some convex subgroup H of $\widetilde{vK}$.

If y is another element of L then we define $z \sim_K y$ to mean that

$$v(z-y) \ > \ \mathrm{dist}\,(z, K).$$

If this holds, then from the definition of distance it follows that $v(z-c) = v(y-c)$ for all $c \in K$ such that $v(z-c) \in vK$ and thus, dist $(z, K) =$ dist (y, K). The next lemma shows that the converse holds under an additional assumption.

Lemma 2.21. *Take a valued field extension $(L|K, v)$ and elements $z, y \in L$. If $v(z-K) \cap vK$ has no maximal element, then $z \sim_K y$ if and only if $v(z-c) = v(y-c)$ for every $c \in K$ such that $v(z-c) \in vK$.*

Proof. Assume that $v(z-c) = v(y-c)$ for every $c \in K$ such that $v(z-c) \in vK$. Then

$$v(z-y) = v(z-c+c-y) \geq \min\{v(z-c), v(c-y)\} = v(z-c),$$

for every $c \in K$ such that $v(z-c) \in K$. Since $v(z-K) \cap vK$ has no maximal element, $v(z-y) > v(z-c)$ for every such c and thus $v(z-y) \ > \ \mathrm{dist}\,(z, K)$.

Assume now that $z \sim_K y$. Then $v(z-y) \ > \ \mathrm{dist}\,(z, K)$. Take $c \in K$ with $v(z-c) \in vK$. Then by the definition of dist (z, K) we have that $v(z-y) > v(z-c)$. Hence,

$$v(z-c) \ = \ v(z-c-(z-y)) \ = \ v(y-c).$$

□

The following theorem gives us important information about the distance of immediate extensions.

Theorem 2.22. *If $(L|K, v)$ is an immediate extension of valued fields, then for every element $z \in L \backslash K$ the set $v(z-K) \subseteq vK$ and has no maximal element. In particular, $vz <$ dist (z, K).*

Proof. Take $z \in L \setminus K$. Then $\infty \notin v(z-K)$. Since $(L|K,v)$ is immediate, we have that $v(z-K) \subseteq vL = vK$. Take any $c \in K$. As $vK = vL$, there is $d \in K$ such that $v(z-c) = vd$ and thus $vd^{-1}(z-c) = 0$. Since $Lv = Kv$, we obtain that $d^{-1}(z-c)v = bv$ for some $b \in Kv$. It follows that $(d^{-1}(z-c)-b)v = d^{-1}(z-c)v - bv = 0$ and consequently, $v(d^{-1}(z-c)-b) > 0$. Therefore, $v(z-(c+bd)) > vd = v(z-c)$, which shows that $v(z-K)$ has no maximal element.

Note that $vz \in v(z-K) = v(z-K) \cap vK$ lies in the initial segment of $\operatorname{dist}(z,K)$ and, by the first part of the proof, z cannot be a maximal element of this segment. This proves the last assertion of the theorem. □

Take an extension $(L|K,v)$ of valued fields and elements $y, z \in L$. If $(K(z)|K,v)$ is an immediate extension, then the previous theorem shows that the relation $z \sim_K y$ is equivalent to the inequality $v(z-y) > v(z-K)$.

The following fact was proven in [26] (Theorem 2).

Theorem 2.23. *Take a valued field (K,v) and the henselization K^h of K with respect to some extension of the valuation v to the algebraic closure of K. Assume that $a, z \in \widetilde{K}$ are such that*

$$a \sim_K z.$$

If a lies in K^h, then the extensions $K(z)$ and K^h are not linearly disjoint over K. In particular, the extension $K(z)|K$ is not purely inseparable.

For any $\alpha \in vK$ and each cut Λ in vK we set $\alpha + \Lambda := (\alpha + \Lambda^L, \alpha + \Lambda^R)$. An immediate consequence of the above definitions is the following lemma:

Lemma 2.24. *Take an extension $(L|K,v)$ of valued fields. Then for every element $c \in K$ and $y, z \in L$:*

a) $\operatorname{dist}(z+c,K) = \operatorname{dist}(z,K)$,

b) $\operatorname{dist}(cz,K) = vc + \operatorname{dist}(z,K)$,

c) if $z \sim_K y$, then $z + c \sim_K y + c$,

d) if $c \neq 0$ and $z \sim_K y$, then $cz \sim_K cy$.

Assume that $(L|K,v)$ is a defectless extension of henselian fields. Set $e = (vL : vK)$ and $f = [Lv : Kv]$. Choose elements $\eta_1, \ldots, \eta_e \in L$ such that $v\eta_1, \ldots, v\eta_e \in vL$ are representatives of the distinct cosets modulo vK. Further, choose $\vartheta_1, \ldots, \vartheta_f \in \mathcal{O}_L$ such that $\vartheta_1 v, \ldots, \vartheta_f v$ form a basis of $Lv|Kv$. Without loss of generality we can assume that $\eta_1 = \vartheta_1 = 1$. Since $L|K$ is defectless, $[L:K] = e \cdot f$. Thus by Lemma 2.7, the elements $\eta_i \vartheta_j$, $i \leq e$ and $j \leq f$ form a K-basis of L. Such a basis will be called a **standard valuation basis**. The next lemma allows us to determine the distance of defectless extensions of henselian fields with the use of a standard valuation basis of the extension.

Lemma 2.25. *With the above assumptions on $(L|K,v)$ and on the elements η_i, $i \leq e$ and ϑ_j, $j \leq f$, the set $v(a-K)$ has a maximum, for every $a \in L$. If*

$$a = \sum_{\substack{i \leq e \\ j \leq f}} c_{i,j} \eta_i \vartheta_j,$$

then $v(a - c_{1,1})$ is the maximal element of $v(a-K)$.

Proof. From Lemma 2.7 it follows that

$$v(a - c_{1,1}) = \min_{(i,j)\neq(1,1)} (vc_{i,j} + v\eta_i).$$

On the other hand, for any $c \in K$ we have that

$$v(a - c) = \min\{v(c_{1,1} - c), vc_{i,j} + v\eta_i \mid (i,j) \neq (1,1)\} \leq \min_{(i,j)\neq(1,1)} (vc_{i,j} + v\eta_i) = v(a - c_{1,1}).$$

Thus $v(a - c_{1,1})$ is the maximal element of $v(a - K)$. □

Corollary 2.26. *Assume that $(K(a)|K, v)$ is an extension of valued fields of degree $p = char(Kv) > 0$ such that the extension of v from K to $K(a)$ is unique.*

1) If $v(a - K)$ has no maximal element, then the extension $(K(a)|K, v)$ is immediate.

2) If $(K(b)|K, v)$ is an immediate extension such that $b \sim_K a$ in some common valued field extension of $K(a)$ and $K(b)$, then also the extension $(K(a)|K, v)$ is immediate.

Proof. Note first that by the Lemma of Ostrowski,

$$p = [L : K] = p^n (vL : vK)[Lv : Kv]$$

for $n \in \{0, 1\}$. If $(K(a)|K, v)$ were not immediate, then $n = 0$ and the extension would be defectless. Thus, by the previous lemma, $v(a - K)$ would have a maximum, a contradiction. This proves assertion 1).

To prove the remaining part of the corollary, assume that $(K(b)|K, v)$ is an immediate extension. Then by Theorem 2.22 the set $v(b - K)$ has no maximal element. Suppose moreover that $b \sim_K a$. Then Lemma 2.21 together with the fact that $v(b - K) \subseteq vK$ yields that $v(a - c) = v(b - c)$ for every $c \in K$. Hence $v(a - K) = v(b - K)$ has no maximal element. From assertion 1) it follows that $(K(a)|K, v)$ is immediate. □

2.4 Artin-Schreier defect extensions

We recall a few facts about Artin-Schreier defect extensions of valued fields and their classification presented in detail in [25]. Recall that an **Artin-Schreier extension** of a field K of positive characteristic p is an extension of degree p generated by a root ϑ of a polynomial $X^p - X - a$ with $a \in K$. In this case, ϑ is called an **Artin-Schreier generator** of the extension. Since the other roots of the polynomial $X^p - X - a$ are of the form $\vartheta + 1, \dots, \vartheta + p - 1$, such an extension is always normal and hence Galois. On the other hand, every Galois extension of K of degree p is an Artin-Schreier extension (see, e.g., [35], Chapter VI Galois Theory, 6. Cyclic Extensions).

The importance of studying the structure of such extensions comes from the fact that towers of Artin-Schreier defect extensions play a central role in the issue of defect extensions. Take a valued field (K, v) of positive residue characteristic p. Fix an extension of v to K^{sep}. Denote by K^h and K^r the henselization and the absolute ramification field of K with respect to this extension. Take any finite extension $(L|K, v)$ such that the extension of the valuation v from K to L is unique. Then equation (2.9) together with Proposition 2.20 and equation (2.11) give that

$$d(L|K, v) = d(L.K^h|K^h, v) = d(L.K^r|K^r, v).$$

On the other hand, $L.K^r|K^r$ is a tower of normal extensions of degree p, by Lemma 2.19. Thus, if $L|K$ is separable, $L.K^r|K^r$ is a tower of Galois extensions of degree p and if $(L|K,v)$ is a defect extension, then so are some of these extensions. If additionally $\mathrm{char}\, K = p$, then every Galois extension of degree p is an Artin-Schreier extension. Hence in this case, if the extension $(L|K,v)$ has nontrivial defect, then the tower $L.K^r|K^r$ contains Artin-Schreier defect extensions.

Throughout the remaining part of this section we assume that (K,v) is a valued field of positive characteristic p and $K(\vartheta)|K$ an Artin-Schreier extension with $\vartheta^p - \vartheta - a = 0$ for some $a \in K$.

We will frequently use the following observations (for the proofs, see Lemma 2.27 and Lemma 2.28 of [25], cf. also Lemma 5.15 and Lemma 5.16).

Lemma 2.27. *If $va \le 0$, then $v\vartheta = \frac{1}{p}va$. If $va > 0$, then exactly one of the conjugates $\vartheta, \vartheta+1, \ldots, \vartheta+p-1$ has value va and the others have value 0.*

Lemma 2.28. *Assume that $va > 0$ or that $va = 0$ and the polynomial $X^p - X - av$ has a root in Kv. Then the Artin-Schreier generator ϑ lies in the henselization of K with respect to every extension of v to $\tilde{K}$, and the valuation v of K has exactly p many distinct extensions to $K(\vartheta)$. Therefore, equality holds in the fundamental inequality. If $va = 0$ and $X^p - X - av$ has no root in Kv, then the residue field extension $K(\vartheta)v|Kv$ is a separable extension of degree p and $(K(\vartheta)|K,v)$ is defectless.*

If $(K(\vartheta)|K,v)$ is a defect extension, then $va < 0$.

Take a field L. A polynomial $g \in L[X]$ is called **additive** if $g(b+c) = g(b) + g(c)$ for all b,c in every extension field L' of L. Since $\mathrm{char} K = p$, the Artin-Schreier polynomial $X^p - X$ is additive. Hence, for any $c \in K$ the element $\vartheta - c$ is a root of the Artin-Schreier polynomial $X^p - X - a + c^p - c$. It follows that this polynomial induces the same extension $K(\vartheta)|K$ as the polynomial $X^p - X - a$. Suppose there is $c \in K$ such that $v(\vartheta - c) \ge 0$. As we have seen, $\vartheta - c$ is another Artin-Schreier generator of the extension $K(\vartheta)|K$. From the previous lemma it follows that $K(\vartheta)|K$ is not a defect extension. This proves:

Corollary 2.29. *If $(K(\vartheta)|K,v)$ has nontrivial defect, then $v(\vartheta - c) < 0$ for every $c \in K$ and consequently,* $\mathrm{dist}\,(\vartheta - K) \le 0^-$.

Assume that $(K(\vartheta)|K,v)$ is a defect extension. Then from the Lemma of Ostrowski it follows that the defect is equal to the degree of the extension. Consequently, the extension is immediate. Furthermore, the valuation v of K admits a unique extension to $K(\vartheta)$. By Theorem 2.22 it follows that $v(\vartheta - K) \cap vK = v(\vartheta - K)$, and that $v(\vartheta - K)$ has no maximal element.

Take $\vartheta' \in K(\vartheta)$ to be another Artin-Schreier generator of the extension $K(\vartheta)|K$. One can show that the element ϑ' is of the form $i\vartheta + c$ for some $i \in \mathbb{F}_p^\times$ and $c \in K$ (cf. Lemma 2.26 of [25]). Hence from Lemma 2.24 it follows that $\delta := \mathrm{dist}\,(\vartheta, K)$ does not depend on the choice of the Artin-Schreier generator ϑ. We call δ the **distance of the Artin-Schreier extension** $(K(\vartheta)|K,v)$. Corollary 2.29 implies that $\delta \le 0^-$.

We will distinguish two types of Artin-Schreier defect extensions considering their connection with purely inseparable extensions. Assume that $K(\vartheta)|K$ is an Artin-Schreier defect extension. If there is an immediate purely inseparable extension $K(\eta)|K$ of degree p such that

$$\eta \sim_K \vartheta,$$

then $K(\vartheta)|K$ is called a **dependent Artin-Schreier defect extension**. Otherwise it is called an **independent Artin-Schreier defect extension**. The next proposition gives us a useful characterization of independent Artin-Schreier defect extensions by idempotent cuts (cf. Proposition 4.2. of [25]).

Proposition 2.30. *Assume that the extension $(K(\vartheta)|K, v)$ has nontrivial defect. Then $K(\vartheta)|K$ is an independent Artin-Schreier defect extension if and only if its distance is idempotent.*

Assume that $(K(\vartheta)|K, v)$ is a defect extension. Suppose moreover that $\delta := \mathrm{dist}\,(\vartheta, K)$ is an idempotent cut. In the previous section we have mentioned that the cut δ is idempotent if and only if $\delta = H^+$ or $\delta = H^-$ for some convex subgroup H of $\widetilde{vK}$. Furthermore, Corollary 2.29 implies that $\delta \leq 0^-$. Hence $\delta = H^-$. Together with the previous proposition this yields that if $(K(\vartheta)|K, v)$ has a nontrivial defect, then the extension is independent if and only if $\mathrm{dist}\,(\vartheta, K) = H^-$ for some convex subgroup H of $\widetilde{vK}$. In particular, if the value group of (K, v) is archimedean, then the unique proper convex subgroup of $\widetilde{vK}$ is $\{0\}$. This yields the following fact.

Corollary 2.31. *Assume that $(K(\vartheta)|K, v)$ has nontrivial defect and the valuation v is of rank one. Then the Artin-Schreier defect extension $(K(\vartheta)|K, v)$ is independent if and only if* $\mathrm{dist}\,(\vartheta, K) = 0^-$.

2.5 Maximal immediate extensions and completions

This section is devoted to properties of maximal immediate extensions and maximal fields. We will introduce also the notions of complete valued fields and of completions.

Take a valued field (K, v). Assume that the field is maximal, that is, admits no proper immediate extensions. Since the henselization K^h of K is an immediate extension of K, we have that $K^h = K$. From this and Theorem 31.21 of [43] we obtain the following fact.

Theorem 2.32. *Every maximal field is henselian and defectless.*

We say that (K, v) is **algebraically maximal** (or **separable-algebraically maximal**) if it admits no proper immediate algebraic (or separable-algebraic, respectively) extensions.

Theorem 2.33. *If a valued field (K, v) is maximal, then every finite extension L of K with the unique extension of the valuation v is again a maximal field.*

For the proof see Theorem 31.22 of [43].

Directly from the definition of a maximal field it follows that a maximal immediate extension of a valued field is a maximal field. It was shown by W. Krull in [20] that every valued field (K, v) admits a maximal immediate extension (M, v) (later, K. A. H. Gravett in [12] gave a nice and simple proof of the fact). However, the maximal immediate extension M does not need to be unique up to isomorphism. This was shown by I. Kaplansky in [16].

He proved also that under a certain condition (which he called "hypothesis A"), uniqueness holds. A valued field (K, v) of residue characteristic p is called a **Kaplansky field** if it satisfies:

(K1) if $p > 0$ then the value group vK is p-divisible,

(K2) the residue field Kv is perfect,

(K3) the residue field Kv admits no finite separable extension of degree divisible by p.

Note that conditions (K2) and (K3) can be replaced by:

(K2') the residue field Kv admits no finite extensions of degree divisible by p.

A direct consequence of the above definition is that every valued field of residue characteristic 0 is a Kaplansky field.

Theorem 2.34. *If (K, v) is a Kaplansky field, then the maximal immediate extension of (K, v) is unique up to valuation preserving isomorphism over K.*

For the proof of Kaplansky, see Theorem 5 of [16]. See also Theorem 1 of [44], which shows the equivalence of conditions (K1) and (K2') with the original "hypothesis A" assumed by Kaplansky.

Immediate extensions and maximal fields can be characterized with the use of pseudo Cauchy sequences and approximation types. Both notions will be described in the remaining part of this section and will be the main tools in our further studies of immediate extensions of valued fields.

2.5.1 Pseudo Cauchy sequences

Pseudo Cauchy sequences were studied by Kaplansky in [16]. He called such sequences "pseudo convergent sets". We will also use the name of "pseudo limit" in place of "limit" of a pseudo Cauchy sequence. Take a valued field (K, v) and a sequence $(a_\nu)_{\nu<\lambda}$ of elements of K indexed by ordinals $\nu < \lambda$, where λ is a limit ordinal. Then $(a_\nu)_{\nu<\lambda}$ is called a **pseudo Cauchy sequence** if

$$v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma) \quad \text{whenever } \rho < \sigma < \tau < \lambda. \tag{2.12}$$

If this holds, then from the above definition it follows that $(v(a_{\nu+1} - a_\nu))_{\nu<\lambda}$ is a strictly increasing sequence and

$$v(a_\mu - a_\nu) \;=\; v(a_{\nu+1} - a_\nu) \;\text{ whenever } \nu < \mu < \lambda.$$

If moreover $a \in K$, then from the dichotomy of equation (4) and inequality (5) of [16] (page 306) it follows that either

$$v(a - a_\nu) \;<\; v(a - a_\mu) \text{ whenever } \;\nu < \mu < \lambda, \tag{2.13}$$

or there is $\nu_0 < \lambda$ such that

$$v(a - a_\nu) \;=\; v(a - a_{\nu_0}) \text{ whenever } \nu_0 < \nu < \lambda.$$

From the definition of a pseudo Cauchy sequence we obtain that condition (2.13) is equivalent to the following:

$$v(a - a_\nu) \;=\; v(a_{\nu+1} - a_\nu) \text{ for every } \;\nu < \lambda. \tag{2.14}$$

If a satisfies condition (2.13) (or equivalently, condition (2.14)), then it is called a **pseudo limit** of $(a_\nu)_{\nu<\lambda}$. Note that a pseudo Cauchy sequence can admit more than one limit. More precisely:

Lemma 2.35. *Take a valued field (K, v) and a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in this field. If $a \in K$ is a pseudo limit of $(a_\nu)_{\nu<\lambda}$, then $b \in K$ is also a pseudo limit of the sequence if and only if $v(a - b) > v(a_{\nu+1} - a_\nu)$ for every $\nu < \mu < \lambda$.*

For the proof see Lemma 3 of [16].

The next theorem, proved by Kaplansky, shows the relation between pseudo Cauchy sequences and immediate extensions. As the relation is the basis of our further investigation, for the convinience of the reader we present the proof of the theorem.

Theorem 2.36. *Assume that $(L|K, v)$ is an immediate extension of valued fields. Then every element $a \in L \setminus K$ is a pseudo limit of a pseudo Cauchy sequence in (K, v) without a limit in K.*

Proof. Take an element $a \in L \setminus K$. By Theorem 2.22, the initial segment $v(a - K)$ has no maximal element. Choose a strictly increasing sequence $(\alpha_\nu)_{\nu<\lambda}$ in L such that the set $\{\alpha_\nu \mid \nu < \lambda\}$ is cofinal in $v(a-K)$. For every $\nu < \lambda$ take a_ν to be an element of K such that

$$v(a - a_\nu) = \alpha_\nu.$$

Take any $\rho < \sigma < \tau < \lambda$. Then from the above equality and the fact that $\alpha_\rho < \alpha_\sigma < \alpha_\tau$ we obtain that

$$v(a_\tau - a_\sigma) \;=\; v(a - a_\sigma - (a - a_\tau)) \;=\; \alpha_\sigma \;>\; \alpha_\rho \;=\; v(a - a_\rho - (a - a_\sigma)) \;=\; v(a_\sigma - a_\rho).$$

Thus $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K, v). It also follows that for $\nu < \lambda$ we have that $v(a_{\nu+1} - a_\nu) = \alpha_\nu = v(a - a_\nu)$. Hence a is a pseudo limit of this sequence.

Suppose now that $(a_\nu)_{\nu<\lambda}$ admits a pseudo limit b in K. Then, from Lemma 2.35 it follows that $v(a - b) > v(a_{\nu+1} - a_\nu) = \alpha_\nu$ for every $\nu < \lambda$. this contradicts the fact that $\{\alpha_\nu \mid \nu < \lambda\}$ is cofinal in $v(a - K)$. $\square$

Kaplansky proved that for every pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in a valued field (K, v) there is a pseudo limit a of $(a_\nu)_{\nu<\lambda}$ in some valued field extension (L, v) of (K, v), such that the extension $(K(a)|K, v)$ is immediate (see Theorems 2.40 and 2.41). In general, not every pseudo limit of a pseudo Cauchy sequence generates an immediate extension (see Example 2.44). To make it more precise, we start with describing the relation between pseudo Cauchy sequences and polynomials.

Lemma 2.37. *Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in a valued field (K, v) and take a polynomial $f \in K[X]$. Then there is $\nu_0 < \lambda$ such that either*

$$vf(a_\nu) \;=\; vf(a_\mu) \;\; \textit{whenever } \nu_0 < \nu < \mu < \lambda. \tag{2.15}$$

or

$$vf(a_\nu) \;<\; vf(a_\mu) \;\; \textit{whenever } \nu_0 < \nu < \mu < \lambda. \tag{2.16}$$

For the proof, see [16], paragraph after Lemma 5.

This fact enables us to introduce an important classification of pseudo Cauchy sequences. Assume that (K, v) is a valued field. Take a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in K and a polynomial $f \in K[X]$. We say that $(a_\nu)_{\nu<\lambda}$ **fixes the value of** f if equation (2.15) holds for some $\nu_0 < \lambda$. If $(a_\nu)_{\nu<\lambda}$ fixes the value of every polynomial in $K[X]$, then it is said to be of **transcendental type**. Otherwise it is said to be of **algebraic type**. By the previous lemma, the second means that the sequence $(vf(a_\nu))_{\nu<\lambda}$ is ultimately strictly increasing for some polynomial $f \in K[X]$. Note that if $(a_\nu)_{\nu<\lambda}$ does not fix the value of f, then it does not fix the value of any polynomial of the form cf with $c \in K$. Hence if $(a_\nu)_{\nu<\lambda}$ is of algebraic type, then we can find a monic polynomial f such that the sequence $(vf(a_\nu))_{\nu<\lambda}$ is ultimately strictly increasing. If $f \in K[X]$ is such a polynomial of minimal degree, then it is called an **associated minimal polynomial for** $(a_\nu)_{\nu<\lambda}$. Such a polynomial is always irreducible over K. Indeed, suppose that $f = g \cdot h \in K[X]$ with $h, g \in K[X]$ of degree less than f. Then $(a_\nu)_{\nu<\lambda}$ fixes the value of g and h. Hence $(vg(a_\nu))_{\nu<\lambda}$ and $(vh(a_\nu))_{\nu<\lambda}$ are ultimately constant. It follows that also $(v(g \cdot h)(a_\nu))_{\nu<\lambda}$ is ultimately constant, a contradiction.

Lemma 2.38. *Take a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in a valued field (K, v). Then an element $a \in K$ is a pseudo limit of $(a_\nu)_{\nu<\lambda}$ if and only if $X - a$ is an associated minimal polynomial for the sequence. If $(a_\nu)_{\nu<\lambda}$ is an algebraic pseudo Cauchy sequence then it admits no limit in K if and only if the degree of an associated minimal polynomial for $(a_\nu)_{\nu<\lambda}$ is at least 2.*

Proof. The first assertion was shown in the proof of Theorem 3 of [16]. The second assertion is a direct consequence of the first one. □

Lemma 2.39. *Take an algebraic field extension $(K(a)|K, v)$, where a is a pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K, v) without a pseudo limit in K. Then $(a_\nu)_{\nu<\lambda}$ does not fix the value of the minimal polynomial of a over K.*

Proof. We denote the minimal polynomial of a over K by $f(X) = \prod_{i=1}^{n}(X - \sigma_i a)$ with $\sigma_i \in \mathrm{Gal}\,(\tilde{K}|K)$. Since a is a pseudo limit of $(a_\nu)_{\nu<\lambda}$, the values $v(a_\nu - a)$ are ultimately increasing. If $v(a - \sigma_i a) > v(a_\nu - a)$ for all $\nu < \lambda$, then also the values

$$v(a_\nu - \sigma_i a) = \min\{v(a_\nu - a), v(a - \sigma_i a)\} = v(a_\nu - a)$$

are ultimately increasing. If on the other hand, $v(a - \sigma_i a) \le v(a_{\nu_0} - a)$ for some $\nu_0 < \lambda$, then for $\nu_0 < \nu < \lambda$, the value

$$v(a_\nu - \sigma_i a) = \min\{v(a_\nu - a), v(a - \sigma_i a)\} = v(a - \sigma_i a)$$

is fixed. We conclude that the values $vf(a_\nu) = \sum_{i=1}^{n} v(a_\nu - \sigma_i a)$ are ultimately increasing. □

By the above lemma we have that if $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence admitting a pseudo limit algebraic over K, then $(a_\nu)_{\nu<\lambda}$ is of algebraic type. The converse does not need to be true. Suppose that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence of algebraic type with an algebraic pseudo limit a. Take an element y transcendental over $K(a)$ in some valued field extension (F, v) of $(K(a), v)$ such that $v(y) > v(a_{\nu+1} - a_\nu)$, for every $\nu < \lambda$. Then Lemma 2.35 shows that $y + a$ is also a pseudo limit of $(a_\nu)_{\nu<\lambda}$.

The following facts will be of particular meaning for our further studies of immediate extensions (cf. Theorem 2 and Theorem 3 of [16]):

Theorem 2.40. *Take a valued field (K,v) and an element x in some valued field extension (L,v) of (K,v). If x is a pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) of transcendental type, then $(K(x)|K,v)$ is an immediate transcendental extension. If $(K(y),w)$ is another valued field extension of (K,v) such that y is a limit of $(a_\nu)_{\nu<\lambda}$, then y is transcendental over K and the isomorphism between $K(x)$ and $K(y)$ sending x to y is valuation preserving.*

Theorem 2.41. *Take a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) of algebraic type and an associated minimal polynomial f for the sequence. If a is a root of f, then there is an extension of v to $K(a)$ such that $(K(a)|K,v)$ is an immediate extension and a is a pseudo limit of $(a_\nu)_{\nu<\lambda}$.*

The next theorem characterizes maximal fields with the use of pseudo Cauchy sequences (cf. Theorem 4 of [16]).

Theorem 2.42. *A valued field (K,v) is maximal if and only if every pseudo Cauchy sequence in K admits a limit in this field.*

The following is a special case of Lemma 3.7 of [22]:

Lemma 2.43. *Let $(K(x)|K,v)$ be an extension of valued fields and choose any extension of v to $\widetilde{K(x)}$. Take K^h and $K(x)^h$ to be the henselizations of K and $K(x)$, respectively, in $(\widetilde{K(x)},v)$. If the element x is a pseudo limit of a pseudo Cauchy sequence in (K,v) of transcendental type, then K^h is relatively algebraically closed in $K(x)^h$, that is,*

$$K^h = IC(K(x)|K,v).$$

Example 2.44. We consider now an important class of valued fields introduced by Hahn in [13]. For an ordered abelian group Γ and a field k take $k((x^\Gamma))$ to be the set of all maps f from Γ to k with well ordered support $\{\gamma\in\Gamma \mid f(\gamma)\neq 0\}$. If $f(\gamma)=c_\gamma$, $\gamma\in\Gamma$, then for simplicity we will write

$$f=\sum_{\gamma\in\Gamma} c_\gamma x^\gamma .$$

Endow this set with componentwise addition:

$$\Big(\sum_{\gamma\in\Gamma} c_\gamma x^\gamma\Big)+\Big(\sum_{\gamma\in\Gamma} d_\gamma x^\gamma\Big) \;=\; \sum_{\gamma\in\Gamma}(c_\gamma+d_\gamma)x^\gamma$$

and multiplication defined in the following way:

$$\Big(\sum_{\gamma\in\Gamma} c_\gamma x^\gamma\Big)\cdot\Big(\sum_{\gamma\in\Gamma} d_\gamma x^\gamma\Big) \;=\; \sum_{\gamma\in\Gamma}\Big(\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta\Big)x^\gamma .$$

Hahn proved that the set $k((x^\Gamma))$ with operations of addition and multiplication defined above is a field. Such a field is called a **(generalized) power series field**.

We introduce the valuation v of $k((x^\Gamma))$ by setting $vf=\min \operatorname{supp} f$, for every $f\in k((x^\Gamma))$, and $v0=\infty$. That is, for every nonzero power series we have that

$$v\left(\sum_{\gamma\in\Gamma} c_\gamma x^\gamma\right)=\min\{\gamma\in\Gamma \mid c_\gamma\neq 0\}.$$

This valuation is called the **canonical valuation** or **x-adic valuation** of $k((x^\Gamma))$. Directly from the definition of the valuation it follows that $vk((x^\Gamma)) = \Gamma$ and $k((x^\Gamma))v = k$. Krull in [20] proved that $(k((x^\Gamma)), v)$ is a maximal field.

Take Γ to be the p-divisible hull $\frac{1}{p^\infty}\mathbb{Z}$ of $\mathbb{Z}$ and set $k =: \mathbb{F}_p$. Set further

$$K := \mathbb{F}_p(x^{1/p^i} \mid i \in \mathbb{N}) \subseteq \mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big),$$

which is the prefect hull of the rational function field $\mathbb{F}_p(x)$. Take the restriction of the canonical valuation of $\mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big)$ to the field K and denote it again by v. Since

$$vK \;=\; \frac{1}{p^\infty}\mathbb{Z} \;=\; v\mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big) \;\text{ and }\; Kv \;=\; \mathbb{F}_p \;=\; \mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big)v,$$

the extension $\big(\mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big)|K, v\big)$ is immediate.

For every $n \in \mathbb{N}$ we define

$$a_n \;:=\; \sum_{i=1}^{n} x^{-1/p^i} \in K.$$

Since for $k < n < m$ we have that $v(a_n - a_k) = -\frac{1}{p^{k+1}} < -\frac{1}{p^{n+1}} = (a_m - a_n)$, we deduce that $(a_n)_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence. Similarly, one can check that equation (2.14) holds for the sequence $(a_n)_{n\in\mathbb{N}}$ and the element

$$a \;:=\; \sum_{i=1}^{\infty} x^{-1/p^i} \in \mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big).$$

Thus, a is a pseudo limit of $(a_n)_{n\in\mathbb{N}}$.

Since $v(a_n^p - a_n - \frac{1}{x}) = v(-x^{-1/p^n}) = -\frac{1}{p^n}$, the pseudo Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ does not fix the value of the polynomial $Y^p - Y - \frac{1}{x} \in K[Y]$ and thus, $(a_n)_{n\in\mathbb{N}}$ is of algebraic type. As moreover $(K(a)|K, v)$ is a subextension of the immediate extension $\big(\mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big)|K, v\big)$, also the extension $(K(a)|K, v)$ is immediate.

Consider now the power series field $\mathbb{F}_p\big(\big(x^{\mathbb{Q}}\big)\big) \supseteq \mathbb{F}_p\big(\big(x^{\frac{1}{p^\infty}\mathbb{Z}}\big)\big)$ with the extension of v to the canonical valuation of $\mathbb{F}_p\big(\big(x^{\mathbb{Q}}\big)\big)$. Denote this extension again by v. Take a prime number $q \neq p$. Since $v(x^{1/q}) = \frac{1}{q} > -\frac{1}{p^{n+1}} = v(a_{n+1} - a_n)$ for every $n \in \mathbb{N}$, from Lemma 2.35 it follows that $a + x^{1/q}$ is also a pseudo limit of the pseudo Cauchy sequence $(a_n)_{n\in\mathbb{N}}$. However, the extension $(K(a + x^{1/q})|K, v)$ is not immediate, since

$$v\left((a + x^{1/q})^p - (a + x^{1/q}) - \frac{1}{x}\right) \;=\; v\left(x^{p/q} - x^{1/q}\right) \;=\; \frac{1}{q} \notin \frac{1}{p^\infty}\mathbb{Z} \;=\; vK.$$

This shows that not every limit of a pseudo Cauchy sequence of algebraic type generates an immediate extension. $\diamond$

2.5.2 Cauchy sequences and completions

Assume that (K, v) is a valued field and take a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in K. if the set of the values $v(a_{\nu+1} - a_\nu)$, $\nu < \lambda$, is cofinal in vK, then $(a_\nu)_{\nu<\lambda}$ is called a **Cauchy**

sequence. Note that if a Cauchy sequence $(a_\nu)_{\nu<\lambda}$ admits a pseudo limit $a \in K$, then by equation (2.14) the sequence of the values $v(a - a_\nu)$, $\nu < \lambda$, is cofinal in vK. Furthermore, from Lemma 2.35 it follows that the pseudo limit a is unique in K. The element a will be also called the **limit** of the Cauchy sequence $(a_\nu)_{\nu<\lambda}$. Note that if (L, v) is a valued field extension of (K, v) such that vK is cofinal in vL, then every Cauchy sequence in (K, v) is also a Cauchy sequence in (L, v).

Remark 2.45. The usual definition of Cauchy sequences (as for instance presented in [11]) is slightly more general. However, every Cauchy sequence in the general sense is ultimately constant or contains a subsequence which satisfies the condition of the above definition of Cauchy sequence. In the construction of the completion and in the argument that leads up to Lemma 2.48, one should also view constant sequences as Cauchy sequences in order to simplify the arguments. But otherwise, they will play no role in our investigations.

The proof of the following equivalence is straightforward.

Lemma 2.46. *Take a valued field extension $(L|K, v)$ and an element $a \in L \setminus K$. Then the following conditions are equivalent:*
1) a is the limit of a Cauchy sequence in (K, v),
2) for every $\gamma \in vK$ there is $c \in K$ such that $v(a - c) \geq \gamma$,
3) $v(a - K) = vK$,
4) $\mathrm{dist}\,(a, K) = \infty$.

If (L, v) is a valued field and $A \subseteq L$ is such that for every $a \in L$ and $\alpha \in vL$ there is $c \in A$ such that $v(a - c) > \alpha$, then we say that A is **dense** in (L, v). In particular, if $(L|K, v)$ is an extension of valued fields and K is dense in (L, v), then the extension $(L|K, v)$ is immediate. Indeed, take an element $a \in L$. Then there is $c \in K$ such that $v(a - c) > va$. Thus $va = vc \in vK$. If $va = 0$, then $v(a - c) > 0$ and consequently $av - cv = (a - c)v = 0$. Thus $av = cv \in Kv$. The property "dense" is transitive: if (K, v) is dense in (F, v) and (F, v) is dense in (E, v), then (K, v) is dense in (E, v).

An immediate consequence Lemma 2.46 is the following:

Corollary 2.47. *Take a valued field extension $(L|K, v)$. If K is dense in L, then every element in L is a limit of a Cauchy sequence in (K, v).*

Take a valued field extension $(L|K, v)$ with vK cofinal in vL. Note that if $a, b \in L$ are both limits of Cauchy sequences in (K, v), then so are $a+b$, ab and $1/a$. This is a consequence of the continuity of addition, multiplication and inversion in valued fields. By an induction over the complexity of the representation of elements in $K(a)$ one proves:

Lemma 2.48. *Take a valued field extension $(K(a)|K, v)$ such that vK is cofinal in $vK(a)$. If a is the limit of a Cauchy sequence in (K, v), then K is dense in $K(a)$.*

A valued field (K, v) is called **complete** if every Cauchy sequence in (K, v) admits a limit in K. Every valued field (K, v) admits a unique (up to valuation preserving isomorphism) valued field extension which is complete and in which (K, v) is dense (see [11], Theorem 2.4.3). Such a field is called the **completion of** (K, v) and denoted by $(K, v)^c$ or K^c if v is fixed.

From Corollary 2.47 it follows that the completion of (K, v) is an extension that is maximal with respect to the property that K is dense in it.

Since a valued field (K, v) is dense in its completion (K^c, v), the extension $(K^c|K, v)$ is immediate. Thus in particular, every maximal field is complete. If moreover v is a valuation of rank one, then the completion of K is henselian, hence it contains a henselization of K (see Theorem 17.18 of [10]).

Proposition 2.49. *Assume that $(L|K, v)$ is a valued field extension such that vK is cofinal in vL. If (L, v) is complete, then it contains the completion of (K, v).*

Proof. Denote by K' the maximal subfield of L in which K is dense. We wish to show that (K', v) is complete. If it is not, then there is a Cauchy sequence in (K', v) without a limit in K'. Since $vK' = vK$ is cofinal in vL, this is also a Cauchy sequence in (L, v). As (L, v) is complete, the Cauchy sequence has a limit $a \in L$. Then by the foregoing lemma, K' and hence also K is dense in $K'(a)$, which contradicts the maximality. □

Corollary 2.50. *Take a valued field extension $(L|K, v)$ such that vK cofinal in vL. If $a \in L$ is a limit of a Cauchy sequence in (K, v), then a lies in the completion of (K, v).*

Proof. By passing to the completion of (L, v), we can assume that L is complete. Hence by the above proposition, it contains a completion K^c of K. Assume that a is a limit of a Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K, v). Since vK is cofinal in vL, $(a_\nu)_{\nu<\lambda}$ is also a Cauchy sequence in (L, v). Take b to be the limit of $(a_\nu)_{\nu<\lambda}$ in K^c. As $K^c \subseteq L$, we obtain that $a, b \in L$ are limits of the same Cauchy sequence in (L, v). From the uniqueness of the limit of a Cauchy sequence we obtain that $a = b \in K^c$. □

The next lemma shows in particular that a finite extension of a complete field is also complete. For the proof, see Lemma 6.25 of [29].

Lemma 2.51. *Take a finite extension $(L|K, v)$ of valued fields. Then there is a unique extension of v from K^c to $L.K^c$ which coincides with v on L. With this extension, $L^c = K^c.L$.*

2.5.3 Approximation types

Pseudo Cauchy sequences are a very handy tool in studies of immediate extensions of valued fields, but an element of an immediate extension $(L|K, v)$ of valued fields can be a pseudo limit of many different pseudo Cauchy sequences in (K, v). We can eliminate the problem of non-uniqueness by replacing pseudo Cauchy sequences by approximation types. Take an extension $(L|K, v)$ of valued fields and an element $x \in L$. For every $\alpha \in vK_\infty := vK \cup \{\infty\}$ set

$$\mathrm{appr}\,(x, K)_\alpha \;:=\; \{c \in K \mid v(x - c) \geq \alpha\}.$$

Note that for $\alpha \leq \beta$ we have that $\mathrm{appr}\,(x, K)_\beta \subseteq \mathrm{appr}\,(x, K)_\alpha$. Furthermore, $\mathrm{appr}\,(x, K)_\alpha \neq \emptyset$ if and only if there is $c \in K$ such that $v(x - c) \geq \alpha$. Hence if $x \notin K$, then

$$S := \{\alpha \in vK_\infty \mid \mathrm{appr}\,(x, K)_\alpha \neq \emptyset\} \;=\; v(x - K) \cap vK \;=\; v(x - K) \cap vK_\infty .$$

and the set S is an initial segment of vK (cf. Section 2.3). If $x \in K$, we obtain that $S = vK_\infty = v(x - K) \cap vK_\infty$. The set

$$\mathrm{appr}\,(x, K) \;:=\; \{\mathrm{appr}\,(x, K)_\alpha \mid \alpha \in v(x - K) \cap vK_\infty\}$$

will be called the **approximation type of x over** (K, v).

Lemma 2.52. *Take an extension $(L|K, v)$ of valued fields, and elements $x, x' \in L$. Then the following assertions hold.*
a) For every $\alpha \in v(x - K) \cap vK_\infty$,

$$\mathrm{appr}\,(x, K)_\alpha = \mathrm{appr}\,(x', K)_\alpha \textit{ if and only if } v(x - x') \geq \alpha.$$

b) If $\mathrm{appr}\,(x, K) = \mathrm{appr}\,(x', K)$*, then* $v(x - x') \geq \mathrm{dist}\,(x, K) = \mathrm{dist}\,(x', K)$*.*
c) If $v(x - x') \geq \max\{\mathrm{dist}\,(x, K), \mathrm{dist}\,(x', K)\}$*, then* $\mathrm{appr}\,(x, K) = \mathrm{appr}\,(x', K)$*.*

For the proof, see [33], Lemma 3.1.

Take a valued field extension $(L|K, v)$ and an element $x \in L$. Set $S := v(x - K) \cap vK_\infty$. The approximation type $\mathrm{appr}\,(x, K)$ is called **immediate** if

$$\bigcap_{\alpha \in S} \mathrm{appr}\,(x, K)_\alpha \;=\; \emptyset.$$

Note that if $x \in K$, then $S = vK_\infty$ and $\bigcap_{\alpha \in S} \mathrm{appr}\,(x, K)_\alpha = \mathrm{appr}\,(x, K)_\infty = \{x\}$. Hence, if the approximation type $\mathrm{appr}\,(x, K)$ is immediate, then the extension $K(x)|K$ is nontrivial.

The next lemma shows the relation between immediate approximation types and immediate extensions of valued fields (cf. Lemma 4.1 of [33]).

Lemma 2.53. *Assume that $(L|K, v)$ is an extension of valued fields. If $x \in L$, then the approximation type* $\mathrm{appr}\,(x, K)$ *is immediate if and only if the set $v(x - K)$ has no maximal element. If this holds, $v(x - K) \cap vK_\infty = v(x - K)$. Furthermore, the extension $(L|K, v)$ is immediate if and only if for every $x \in L \setminus K$ the approximation type* $\mathrm{appr}\,(x, K)$ *is immediate.*

Take a valued field (K, v) and an element x in some valued field extension (L, v) of (K, v). Similarly to the case of pseudo Cauchy sequences, we say that the approximation type $\mathrm{appr}\,(x, K)$ **fixes the value of** $f \in K[X]$ if there is $\alpha \in v(x - K) \cap vK_\infty$ such that $vf(c) = vf(d)$ for every $c, d \in \mathrm{appr}\,(x, K)_\alpha$. Since $\mathrm{appr}\,(x, K)_\beta \subseteq \mathrm{appr}\,(x, K)_\alpha$, for every $\beta \geq \alpha$, it follows that in this case $vf(c) = vf(d)$ for every $c, d \in \mathrm{appr}\,(x, K)_\beta$ and every $\beta \geq \alpha$. Assume that $\mathrm{appr}\,(x, K)$ is an immediate approximation type. If $\mathrm{appr}\,(x, K)$ fixes the value of every polynomial $f \in K[X]$, then we call it a **transcendental approximation type**. Otherwise, $\mathrm{appr}\,(x, K)$ is called an **algebraic approximation type**.

Assume that $\mathrm{appr}\,(x, K)$ is algebraic and take a polynomial f of minimal degree whose value is not fixed by $\mathrm{appr}\,(x, K)$. The same arguments an in the case of pseudo Cauchy sequences show that f is irreducible and can be chosen to be monic. Such a polynomial will be called an **associated minimal polynomial for** $\mathrm{appr}\,(x, K)$. Take a polynomial $g = X - a \in K[X]$. Since $\mathrm{appr}\,(x, K)$ is immediate, there is $\alpha \in v(x - K)$ such that $a \notin \mathrm{appr}\,(x, K)_\alpha$. Then for every $c, c' \in \mathrm{appr}\,(x, K)_\alpha$ we have that $v(x - c) \geq \alpha > v(x - a)$ and similarly $v(x - c') > v(x - a)$. Hence,

$$v(c - a) \;=\; v(c - x + x - a) \;=\; v(x - a) \;=\; v(x - c' + c' - a) \;=\; v(c' - a).$$

This shows that $\mathrm{appr}\,(x, K)$ fixes the value of g. Consequently, an immediate approximation type $\mathrm{appr}\,(x, K)$ fixes the value of every linear polynomial in $K[X]$.

Lemma 2.54. *Assume that* $\operatorname{appr}(x, K)$ *is an immediate approximation type over* (K, v), *where* x *is an element in some valued field extension of* K.

a) If x *is algebraic over* K, *then* $\operatorname{appr}(x, K)$ *does not fix the value of the minimal polynomial of* x *over* K.

b) If $\operatorname{appr}(x, K)$ *is an algebraic approximation type and* f *an associated minimal polynomial for* $\operatorname{appr}(x, K)$, *then for every* $g \in K[X]$ *such that* $\deg g < \deg f$, *the approximation type* $\operatorname{appr}(g(x), K)$ *is also immediate.*

For the proof, see Corollary 5.5 and Lemma 8.2 of [33]. Assertion a) of the above lemma yields that if x is algebraic over K then $\operatorname{appr}(x, K)$ is an algebraic approximation type.

The next theorem is an approximation type version of Theorem 2.41 (cf. Theorem 6.4 of [33]).

Theorem 2.55. *Assume that* $\operatorname{appr}(x, K)$ *is an immediate algebraic approximation type over* (K, v), *where* x *is an element in some valued field extension over* K. *If* f *is an associated minimal polynomial for* $\operatorname{appr}(x, K)$ *and* a *is a root of* f, *then there is an extension of the valuation* v *to* $K(a)$ *such that* $(K(a)|K, v)$ *is an immediate extension and* $\operatorname{appr}(x, K) = \operatorname{appr}(a, K)$.

The next proposition is a consequence of the above theorem and the Lemma of Ostrowski (for the details of the proof, see [33], Proposition 6.5).

Proposition 2.56. *Assume that* $\operatorname{appr}(x, K)$ *is an algebraic approximation type over a henselian field* (K, v), *where* x *is an element of some valued field extension over* K. *Then the degree of an associated minimal polynomial for* $\operatorname{appr}(x, K)$ *is greater than* 1 *and is a power of the residue characteristic of* (K, v). *Hence in particular,* $\operatorname{char} Kv > 0$.

3. Distances of elements in valued field extensions

In this chapter we present an alternative definition of the distance of elements in valued field extensions. We consider further the case of valued field extensions of prime degree to show that the new notion of distance carries more information about the field extension than the previous one, given in Section 2.3. However, with the use of the new definition we can define only distances of elements in algebraic extensions of a given field. We then describe possible distances of elements in defectless extensions of prime degree of a henselian field. Finally, we give an upper bound for the number of distinct distances of immediate elements in the extensions of a given degree.

3.1 Distances of elements in algebraic extensions

In Section 2.3 we have introduced a notion of the distance of an element in a valued field extension. The notion enables us to define distances of elements in any valued field extension (L, v) of a given field (K, v). Nevertheless, since for an element $z \in L$ the lower cut set of $\mathrm{dist}\,(z, K)$ depends only on $v(z-K) \cap vK$, in the case of a nontrivial value group extension we can lose some information about $v(z-K)$. The next easy observation shows that $v(b-K)$ and $v(b-K) \cap vK$ differ by at most one element.

Lemma 3.1. *Assume that $(K(b)|K, v)$ is an algebraic extension of valued fields.*

1) If $v(b-K)$ has no maximal element, then $v(b-K) \subseteq vK$.

2) If $v(b-K)$ admits a maximal element α, then $v(b-K) \setminus \{\alpha\} \subseteq vK$ and

$$v(b-K) \setminus \{\alpha\} \;=\; \{\beta \in vK : \beta < \alpha\}. \tag{3.1}$$

If moreover $\alpha \in vK$, then $v(b-K) \subseteq vK$ and for any $c, d \in K$ such that $v(b-c) = \alpha = vd$ we have that $d^{-1}(b-c)v \notin Kv$.

Proof. Assume that $v(b-K)$ has no maximal element. Take any $c \in K$. Then there is $d \in K$ such that $(b-c) < v(b-d)$. Hence

$$v(b-c) \;=\; v(b-c-(b-d)) \;=\; v(d-c) \in vK.$$

Suppose that $v(b-K)$ admits a maximal element α. Take $c \in K$ such that $v(b-c) = \alpha$. Then for any $c' \in K$ with $v(b-c') < \alpha$ we obtain that

$$v(b-c') \;=\; v(b-c'-(b-c)) \;=\; v(c-c') \in vK.$$

This proves that $v(b-K)\setminus\{\alpha\}\subseteq vK$.

Since α is the maximl element of $v(b-K)$, the inclusion $v(b-K)\setminus\{\alpha\}\subseteq\{\beta\in vK:\beta<\alpha\}$ is obvious. For the proof of the converse, take any $\beta\in vK$, $\beta<\alpha$. If $\beta=vd$ and $\alpha=v(b-c)$, for some $c,d\in K$, then

$$\beta\;=\;vd\;=\;v(b-c+d)\in v(b-K).$$

Assume now that $\alpha\in vK$. Then from the first part of the proof we obtain that $v(b-K)\subseteq vK$. Take $c,d\in K$ such that $v(b-c)=\alpha=vd$. Suppose that $d^{-1}(b-c)v\in Kv$. Take $d'\in K$ with $d'v=d^{-1}(b-c)v$. Then $(d^{-1}(b-c)-d')v=0$ and thus $v(d^{-1}(b-c)-d')>0$. It follows that

$$\alpha\;=\;vd\;<\;v(b-c-dd')\in v(b-K),$$

a contradiction. $\square$

Assume that (K,v) is a henselian field with vK densely ordered, and (L,v) is a finite extension of (K,v) such that $(vL:vK)=[L:K]$. Then $(L|K,v)$ is defectless and from Lemma 2.25 if follows that for every $a\in L\setminus K$ the set $v(a-K)$ admits a maximal element $\alpha_a\notin vK$. By the previous lemma,

$$v(a-K)\cap vK=\{\beta\in vK:\beta<\alpha_a\}.$$

Since vK is densely ordered and $\alpha_a\in\widetilde{vK}$, from the above equality it follows that the set $v(a-K)\cap vK$ has no maximal element.

Take now an immediate algebraic extension (E,v) of (K,v). From Theorem 2.22 we infer that also in this case, for every $a\in E\setminus K$ the set $v(a-K)\cap vK=v(a-K)$ has no maximal element. On the other hand, if for every $a\in E\setminus K$ the set $v(a-K)$ has no maximal element, then by Lemma 2.53, the extension is immediate. The above paragraph shows that the last assertion may not hold if we replace the sets $v(a-K)$ by $v(a-K)\cap vK$.

Note that for every $b\in\widetilde{K}$, the distance of b from K depends only on $v(b-K)\cap vK$. Hence the distances of elements of a given valued field extension may not carry any information if the considered extension is immediate or not. We introduce now a different notion of distance which gives us more complete information about algebraic extensions of valued fields.

Take a valued field (K,v). Fix an extension of v to the algebraic closure $\widetilde{K}$ of K and denote it again by v. For an element $b\in\widetilde{K}$ define the **distance of b from K over $\widetilde{K}$** to be the cut in $\widetilde{vK}$ of the form

$$\mathrm{dist}_{\widetilde{K}}(b,K)\;:=\;v(b-K)^+$$

(cf. the definition of the distance of an element of a valued field from a subset of this field as presented in [33]). Hence, $\mathrm{dist}_{\widetilde{K}}(b,K)$ is the cut in $\widetilde{vK}$ having as the lower cut set the smallest initial segment in $\widetilde{vK}$ containing $v(b-K)$. Note that as the initial segment of $\mathrm{dist}_{\widetilde{K}}(b,K)$ contains the whole set $v(b-K)$, this notion of distance carries more information about the set $v(b-K)$ than $\mathrm{dist}\,(b,K)$.

We have introduced the new notion of distance only for elements algebraic over (K,v), since for every $a,b\in\widetilde{K}$, the sets $v(a-K)$ and $v(b-K)$ are contained in the divisible hull of vK. Hence, we can consider the cuts $v(a-K)^+$ and $v(b-K)^+$ in the same group $\widetilde{vK}$. This enables us to compare the distances $\mathrm{dist}_{\widetilde{K}}(a,K)$ and $\mathrm{dist}_{\widetilde{K}}(b,K)$.

As a direct consequence of the above definition we obtain the following properties of the distance (cf. Lemma 2.24).

Lemma 3.2. *Take an element b algebraic over K. Then for any $c \in K$ we have that*
1) $\operatorname{dist}_{\widetilde{K}}(b+c, K) = \operatorname{dist}_{\widetilde{K}}(b, K)$,
2) $\operatorname{dist}_{\widetilde{K}}(cb, K) = vc + \operatorname{dist}_{\widetilde{K}}(b, K)$.

Take an element $b \in \widetilde{K}$. As $v(b-K) \cap vK \subseteq v(b-K)$, from the definitions of the distances it follows that

$$\operatorname{dist}(b, K) \leq \operatorname{dist}_{\widetilde{K}}(b, K). \tag{3.2}$$

Furthermore, if $v(b-K) \subseteq vK$, then $\operatorname{dist}(b, K) = \operatorname{dist}_{\widetilde{K}}(b, K)$. Hence both definitions of distance coincide in particular in the case of algebraic extensions with trivial value group extensions, thus also for immediate extensions.

For elements $a, b \in \widetilde{K}$ define $a \approx_K b$ to mean that

$$v(a-b) \;\geq\; \max\{\operatorname{dist}_{\widetilde{K}}(a, K), \operatorname{dist}_{\widetilde{K}}(b, K)\}.$$

By inequality (3.2) we obtain that in this case also $v(a-b) \geq \max\{\operatorname{dist}(a, K), \operatorname{dist}(b, K)\}$. Together with Lemma 2.52, this yields:

Lemma 3.3. *If $a, b \in \widetilde{K}$ are such that $a \approx_K b$, then* $\operatorname{appr}(a, K) = \operatorname{appr}(b, K)$.

The next lemma gives an important characterization of the relation $\approx_K$.

Lemma 3.4. *Take elements a and b algebraic over K. Then $a \approx_K b$ if and only if $v(a-c) = v(b-c)$ for every $c \in K$.*

Proof. Assume that $a \approx_K b$. Take an element $c \in K$. Then $v(a-c) \leq v(a-b)$, by definition of $\approx_K$. If $v(a-c) < v(a-b)$, then $v(a-c) = v(a-c-(a-b)) = v(b-c)$. Assume that $v(a-c) = v(a-b)$. Then

$$v(b-c) \;=\; v(a-c-(a-b)) \;\geq\; \min\{v(a-c), v(a-b)\} \;=\; v(a-b).$$

On the other hand, as $a \approx_K b$, we have that $\operatorname{dist}_{\widetilde{K}}(b, K) \leq v(a-b)$. Thus in particular, $v(b-c) \leq v(a-b)$. Therefore, $v(b-c) = v(a-b)$ and consequently, $v(b-c) = v(a-c)$.

Suppose now that $v(a-c) = v(b-c)$ for every $c \in K$. Take any element c in K. Then

$$v(a-b) = v(a-c-(b-c)) \;\geq\; \min\{v(a-c), v(b-c)\} \;=\; v(a-c) \;=\; v(b-c).$$

By definition of the distance of an element from K over $\widetilde{K}$, we obtain that

$$v(a-b) \geq \operatorname{dist}_{\widetilde{K}}(a, K) = \operatorname{dist}_{\widetilde{K}}(b, K)$$

and thus $a \approx_K b$. □

As direct consequences of the lemma we obtain the following properties, which correspond to the ones that hold for $\operatorname{dist}(a, K)$ and the relation $\sim_K$, defined in Section 2.3.

Corollary 3.5. *If $a, b \in \widetilde{K}$ are such that $a \approx_K b$, then* $\operatorname{dist}_{\widetilde{K}}(a, K) = \operatorname{dist}_{\widetilde{K}}(b, K)$.

Corollary 3.6. *Take any elements a and b algebraic over K. Then for every $c \in K$ we have:*

1) if $a \approx_K b$, then $a + c \approx_K b + c$;

2) if $a \approx_K b$ and $c \neq 0$, then $ca \approx_K cb$.

We know already that in the case of an immediate algebraic extension $(K(a)|K, v)$, the two notions of distances $\operatorname{dist}_{\widetilde{K}}(a, K)$ and $\operatorname{dist}(a, K)$ coincide. The next lemma shows in particular that if $(K(b)|K, v)$ is another immediate extension, also the relation $a \approx_K b$ can be equivalently replaced by $a \sim_K b$.

Lemma 3.7. *Take elements a, b algebraic over K. If the sets $v(a - K)$ and $v(b - K)$ have no maximal elements, then*

$$a \approx_K b \text{ if and only if } a \sim_K b.$$

Proof. Assume that $a \approx_K b$. Then by Corollary 3.5 we have that

$$v(a - b) \geq \operatorname{dist}_{\widetilde{K}}(a, K) = \operatorname{dist}_{\widetilde{K}}(b, K).$$

Since $v(a - K)$ has no maximal element, $v(a - b) > \operatorname{dist}_{\widetilde{K}}(a, K)$. Furthermore, Lemma 3.1 yields that the sets $v(a - K)$ and $v(b - K)$ are contained in vK. It follows that $\operatorname{dist}_{\widetilde{K}}(a, K) = \operatorname{dist}(a, K)$ and $\operatorname{dist}_{\widetilde{K}}(b, K) = \operatorname{dist}(b, K)$. Hence

$$v(a - b) > \operatorname{dist}(a, K) = \operatorname{dist}(b, K),$$

which gives $a \sim_K b$.

Assume now that $a \sim_K b$. Then

$$v(a - b) > \operatorname{dist}(a, K) = \operatorname{dist}(b, K).$$

As we have already noticed, $\operatorname{dist}(a, K) = \operatorname{dist}_{\widetilde{K}}(a, K)$ and $\operatorname{dist}(b, K) = \operatorname{dist}_{\widetilde{K}}(b, K)$. Together with the previous inequality this gives

$$v(a - b) > \operatorname{dist}_{\widetilde{K}}(a, K) = \operatorname{dist}_{\widetilde{K}}(b, K).$$

Therefore, $a \approx_K b$. □

Take an algebraic extension L of K with the extension of the valuation of K to $\widetilde{K} = \widetilde{L}$ that we have previously fixed. Take an element b which is algebraic over K and hence also over L. Since $\widetilde{vK} = v\widetilde{K} = v\widetilde{L} = \widetilde{vL}$, we have that $\operatorname{dist}_{\widetilde{L}}(b, L)$ and $\operatorname{dist}_{\widetilde{K}}(b, K)$ are cuts in the same group. Hence we can compare the distances.

Lemma 3.8. *Take an algebraic extension $L|K$ and an element b algebraic over K. Then* $\operatorname{dist}_{\widetilde{K}}(b, K) \leq \operatorname{dist}_{\widetilde{L}}(b, L)$. *Furthermore, if*

$$\operatorname{dist}_{\widetilde{K}}(b, K) < \operatorname{dist}_{\widetilde{L}}(b, L),$$

then there is $a \in L$ such that $a \approx_K b$. Then in particular,

$$\operatorname{dist}_{\widetilde{K}}(b, K) = \operatorname{dist}_{\widetilde{K}}(a, K).$$

Proof. The first inequality follows immediately from the definition of the distance.

Assume that $\operatorname{dist}_{\widetilde{K}}(b,K) \;<\; \operatorname{dist}_{\widetilde{L}}(b,L)$. The there is an element $a \in L$ such that

$$v(b-a) \;>\; v(b-K).$$

Then for any $c \in K$ we have that $v(b-c) = v(b-c-(b-a)) = v(a-c)$, which by Lemma 3.4 yields that $a \approx_K b$. Thus in particular, $\operatorname{dist}_{\widetilde{K}}(b,K) \;=\; \operatorname{dist}_{\widetilde{K}}(a,K)$, by Corollary 3.5. $\square$

Corollary 3.9. *Take K^h to be the henselization of K with respect to the valuation v of $\widetilde{K}$. Assume that $b \in \widetilde{K}$ is such that $b \approx_K a$ for some $a \in K^h$. Then K^h and $K(b)$ are not linearly disjoint over K.*

Proof. Suppose that $b \approx_K a$. Then in particular, $v(b-a) \geq \operatorname{dist}_{\widetilde{K}}(a,K) \geq \operatorname{dist}(a,K)$. Since $a \in K^h$, the extension $(K(a)|K,v)$ is immediate. Thus $v(a-K)$ has no maximal element, by Theorem 2.22. Therefore, $v(b-a) > \operatorname{dist}(a,K)$ and consequently, $a \sim_K b$. From Theorem 2.23 it follows that K^h and $K(b)$ are not linearly disjoint over K. $\square$

Proposition 3.10. *Take an algebraic extension $L|K$ such that the valuation of K admits a unique extension to L. Then for every $b \in L \setminus K$,*

$$\operatorname{dist}_{\widetilde{K}}(b,K^h) \;=\; \operatorname{dist}_{\widetilde{K}}(b,K).$$

Proof. Take K^h to be the henselization of K with respect to the fixed valuation v of $\widetilde{K}$. As v extends uniquely to L, the extension $L|K$ is linearly disjoint from $K^h|K$, by Lemma 2.12. Suppose that $\operatorname{dist}_{\widetilde{K}}(b,K^h) > \operatorname{dist}_{\widetilde{K}}(b,K)$. Then by Lemma 3.8 there would be an element $d \in K^h$ such that $d \approx_K b$. From the previous corollary it would follow that $K(a)$ and K^h are not linearly disjoint over K, a contradiction. $\square$

We consider now the following property of an algebraic extension $(K(a)|K,v)$ of degree n:
(EF) for every polynomial $g \in K[X]$ of degree less than n there is $\alpha \in v(a-K)$ such that for every $c \in K$ with $v(a-c) \geq \alpha$ the value $vg(c)$ is fixed.
Note that the property (EF) states that the approximation type $\operatorname{appr}(a,K)$ fixes the value of every polynomial of degree less than n. If in addition $\operatorname{appr}(a,K)$ is immediate, then the above property together with Lemma 2.54, part a) yields that the minimal polynomial of a over K is an associated minimal polynomial for $\operatorname{appr}(a,K)$. Hence, every associated minimal polynomial for $\operatorname{appr}(a,K)$ is of degree n. On the other hand, if g is an associated minimal polynomial for $\operatorname{appr}(a,K)$ of degree n, then by the definition, $\operatorname{appr}(a,K)$ fixes the value of every polynomial of degree less that n. We thus obtain the following property.

Lemma 3.11. *Assume that $(K(a)|K,v)$ is an algebraic extension of degree n. If the approximation type appr(a,K) is immediate, then $K(a)|K$ has property (EF) if and only if every associated minimal polynomial for* $\operatorname{appr}(a,K)$ *is of degree n.*

Lemma 3.12. *Take an algebraic extension $(K(a)|K,v)$ with the property (EF).*

1) If v(a-K) has no maximal element, then $(K(a)|K,v)$ is an immediate extension.

2) Assume that $(K(a)|K,v)$ is an immediate extension. If $b \in \widetilde{K}$ is such that $a \approx_K b$ and $[K(a):K] = [K(b):K]$, then the extension $(K(b)|K,v)$ is also immediate.

Proof. Assume that $v(a-K)$ has no maximal element. Then by Lemma 2.53, the approximation type $\operatorname{appr}(a,K)$ is immediate. Lemma 2.54 yields that $\operatorname{appr}(a,K)$ is an algebraic approximation type. Take any element $b \in K(a)$. Then $b = h(a)$ for some polynomial $h \in K[X]$ of degree less than $n := [K(a):K]$. From the previous lemma and part b) of Lemma 2.54 it follows that $\operatorname{appr}(b,K) = \operatorname{appr}(h(a),K)$ is an immediate approximation type. Hence by Lemma 2.53, the extension $(K(a)|K,v)$ is immediate.

For the proof of part 2), assume that $b \in \widetilde{K}$ is such that $a \approx_K b$ and that $[K(a):K] = [K(b):K]$. By Lemma 3.3 we obtain that $\operatorname{appr}(a,K) = \operatorname{appr}(b,K)$. As property (EF) says that the approximation type $\operatorname{appr}(a,K)$ fixes the value of every polynomial of degree less than n, the same holds for $\operatorname{appr}(b,K)$. Thus also $(K(b)|K,v)$ has property (EF). Furthermore, by Lemma 3.4 we have that $v(b-K) = v(a-K)$. If $(K(a)|K,v)$ is an immediate extension, then from Theorem 2.22 we deduce that $v(a-K)$ has no maximal element. Therefore, $(K(b)|K,v)$ satisfies the assumptions of part 1) of the lemma and consequently is an immediate extension. □

Lemma 3.13. *Take an algebraic extension $K(a)|K$ with property (EF). If $b \in \widetilde{K}$ is such that $a \approx_K b$, then for every polynomial $f \in K[X]$ of degree less than $[K(a):K]$,*

$$f(a) \approx_K f(b).$$

For the proof we will need the following property (cf. Lemma 5.2 of [33]).

Lemma 3.14. *Take an algebraic approximation type* $\operatorname{appr}(a,K)$ *and a polynomial $f \in K[X]$ of degree less than or equal to the degree of an associated minimal polynomial for* $\operatorname{appr}(a,K)$. *Assume that* $\operatorname{appr}(a,K)$ *fixes the value of g and take an element $\alpha \in v(a-K) \cap vK_\infty$ such that $vg(c)$ is fixed for every $c \in K$ with $v(a-c) \geq \alpha$. Then $vg(a) = vg(c)$.*

Proof of Lemma 3.13 As $a \approx_K b$, by Lemma 3.3 we obtain that $\operatorname{appr}(a,K) = \operatorname{appr}(b,K)$. By the assumption on $K(a)|K$, the approximation type $\operatorname{appr}(a,K)$ fixes the value of every polynomial of degree less than $n := [K(a):K]$. Take such a polynomial f and an element $c \in K$. Then also $\deg(f-c) < n$. Hence there is $\alpha \in v(a-K)$ such that for all $d \in K$ with $v(a-d) > \alpha$ the value $v((f-c)(d)) = v(f(d)-c)$ is fixed. Lemma 3.4 yields that $\alpha \in v(b-K)$ and for every $d \in K$ with $v(b-d) > \alpha$ the value $v(f(d)-c)$ is fixed. Applying the above lemma to $g = f-c$ and elements a and b respectively, we obtain that

$$v(f(a)-c) \;=\; v(f(d)-c) \;=\; v(f(b)-c).$$

Since the equality $v(f(a)-c) = v(f(b)-c)$ holds for every $c \in K$, by Lemma 3.4 we obtain the relation $f(a) \approx_K f(b)$. □

3.2 Distances of elements in extensions of prime degree

We apply now the notions of the distance and the relation $\approx_K$ considered in the previous section to the case of algebraic extensions of prime degree. We show that for such extensions the relation $a \approx_K b$ implies a strong connection between the value group and between the residue field extensions of the extensions $(K(a)|K,v)$ and $(K(b)|K,v)$. We also consider possible distances of elements in defectless extensions of prime degree, which are linearly disjoint from the henselization.

Throughout this section we assume that (K, v) is a valued field, fix an extension of v to the algebraic closure $\widetilde{K}$ of K and denote it again by v. Furthermore, we take p to be a prime number.

Note that if $K(a)|K$ is an extension of degree p such that the valuation v admits a unique extension from K to $K(a)$, then from the Lemma of Ostrowski it follows that either $(K(a)|K, v)$ is an immediate extension (which is possible only in the case of $p = \operatorname{char} Kv$), $(vK(a) : vK) = p$ and the residue field extension is trivial, or $[K(a)v : Kv] = p$ and the value group extension is trivial.

Assume that $(vK(a) : vK) = p$ or $[K(b)v : Kv] = p$. Then the extension $K(a)|K$ is defectless. As from Proposition 3.10 it follows that $\operatorname{dist}_{\widetilde{K}}(a, K) = \operatorname{dist}_{\widetilde{K}}(a, K^h)$, we deduce that $v(a - K)$ has a maximal element, equal to the maximal element of $v(a - K^h)$.

If the value group extension $vK(a)|vK$ is nontrivial, then by Lemma 2.25 the maximal element α of $v(a - K^h)$, hence also of $v(a - K)$, lies in $vK(a) \setminus vK$. Thus α has order p modulo vK and $vK(a) = vK + \alpha\mathbb{Z}$. Obviously, if $v(a - K)$ has a maximal element $\alpha \notin vK$, then the value group extension is nontrivial and as before we deduce that $vK(a) = vK + \alpha\mathbb{Z}$.

A similar argumentation together with Lemma 3.1 shows that the residue field extension $K(a)v|Kv$ is nontrivial if and only if $v(a - K)$ admits a maximal element which lies in vK. Then $K(a)v = Kv(d^{-1}(a - c)v)$ for $c, d \in K$ such that $v(a - c) = \alpha = vd$.

By Theorem 2.22 we obtain that if $K(a)|K$ is an immediate extension, then the set $v(a - K)$ has no maximal element. Note that also the converse holds. It follows from the fact that if the extension $K(a)|K$ is not immediate, then the above arguments show that $K(a)|K$ is defectless and $v(a - K)$ admits a maximal element.

We thus can read off the information about the value group and the residue field extensions from the distance.

Lemma 3.15. *Take an algebraic extension $K(a)|K$ of degree p and assume that the valuation v of K extends in a unique way to $K(a)$.*

1) The set $v(a - K)$ has no maximal element if and only if the extension $(K(a)|K, v)$ is immediate.

2) The set $v(a - K)$ admits a maximal element $\alpha \notin vK$ if and only if the value group extension $vK(a)|vK$ is nontrivial. If this holds, $vK(a) = vK + \alpha\mathbb{Z}$.

3) The set $v(a - K)$ admits a maximal element $\alpha \in vK$ if and only if the residue field extension $K(a)v|Kv$ is nontrivial. If this holds, for every $c, d \in K$ such that $v(a-c) = \alpha = vd$ we have that $K(a)v = Kv(d^{-1}(a - c)v)$.

Proposition 3.16. *Take algebraic extensions $K(a)|K$ and $K(b)|K$ of degree p and assume that the valuation v of K extends in a unique way to the fields $K(a)$ and $K(b)$. Assume moreover that $a \approx_K b$.*

1) If $(K(a)|K, v)$ is an immediate extension, then also the extension $(K(b)|K, v)$ is immediate.

2) If the value group extension $vK(a)|vK$ is nontrivial, then $vK(a) = vK(b)$.

3) If the residue field extension $K(a)v|Kv$ is nontrivial, then also the extension $K(b)v|Kv$ is nontrivial. If moreover $v(b - a) > \operatorname{dist}_{\widetilde{K}}(a, K)$, then $K(a)v = K(b)v$.

Proof. If $v(a-K)$ has no maximal element, then by Corollary 3.5 also the set $v(b-K)$ has no maximal element. Now part 1) of the proposition follows from Lemma 2.26.

Assume that the group $vK(a)/vK$ is nontrivial. Then by the previous lemma $v(a-K)$ has a maximal element $\alpha \in vK(a)\setminus vK$ and $vK(a) = vK + \alpha\mathbb{Z}$. Since $a \approx_K b$, by Lemma 3.4 we have that $v(a-K) = v(b-K)$. Therefore $\alpha \in v(b-K) \subseteq vK(b)$ is the maximal element of $v(b-K)$. Together with the previous lemma this yields that $vK(b) = vK + \alpha\mathbb{Z} = vK(a)$.

Suppose now that the extension $K(a)v|Kv$ is nontrivial. Then $[K(a)v : Kv] = p$ and $vK(a) = vK$. This yields that also $K(b)v|Kv$ is of degree p, since otherwise $K(b)|K$ would be immediate or the value group extension $vK(b)|vK$ would be nontrivial. By parts 1) and 2) of the lemma, that would imply that also $K(a)|K$ is immediate or $vK(a)|vK$ is nontrivial, respectively, a contradiction.

Assume additionally that $v(a-b) > \mathrm{dist}_{\tilde{K}}(a, K)$. By the above lemma, the set $v(b-K)$ admits a maximal element $\alpha \in vK$ and for elements $c, d \in K$ such that $v(b-c) = \alpha = vd$ we have that $\xi := d^{-1}(b-c)v$ generates the extension $K(b)v|Kv$. By the assumption on $\mathrm{dist}_{\tilde{K}}(a, K) = \mathrm{dist}_{\tilde{K}}(b, K)$ we have that $v(b-a) > \alpha = vd$. Hence

$$v(d^{-1}(b-c) - d^{-1}(a-c)) \;=\; vd^{-1}(a-b) \;>\; 0.$$

Thus $d^{-1}(b-c)v = d^{-1}(a-c)v$ and $\xi \in K(a)v$. From the fact that $K(a)v|Kv$ is of prime degree, we deduce that $K(a)v = Kv(\xi) = K(b)v$. □

Another easy consequence of Lemma 2.25 is the following fact, which describes the possible distances of elements of a defectless extension $(K(a)|K, v)$ of degree p with unique extension of the valuation v from K to $K(a)$.

Corollary 3.17. *Take a defectless extension $(L|K, v)$ of degree p and assume that the valuation v of K extends in a unique way to L.*

1) If $vL = vK$, then the distance of every element $b \in L \setminus K$ from K over $\tilde{K}$ is of the form α^+ for some $\alpha \in vK$. Conversely, for every $\alpha \in vK$ there is $b \in L \setminus K$ such that $\mathrm{dist}_{\tilde{K}}(b, K) = \alpha^+$.

2) If the value group extension $vL|vK$ is nontrivial, then the distance of every element $b \in L \setminus K$ from K over $\tilde{K}$ is of the form α^+ for some $\alpha \in vL \setminus vK$. Furthermore, for every $\alpha \in vL \setminus vK$ there is $b \in L \setminus K$ such that $\mathrm{dist}_{\tilde{K}}(b, K) = \alpha^+$.

Proof. Assume first that $vL = vK$. As $(L|K, v)$ is defectless, the Lemma of Ostrowski yields that $[Lv : Kv] = p$. Take an element $a \in L$ such that $av \notin Kv$. Then the elements $1, av, \ldots, av^{p-1}$ are Kv-linearly independent and thus $1, a, \ldots, a^{p-1}$ form a standard valuation basis of $(L|K, v)$. Take an element $b \in L$. Then $b = \sum_{i=0}^{p-1} c_i a^i$ for some $c_i \in K$ and Lemma 2.25 together with Lemma 2.7 yield that the set $v(b-K)$ admits a maximal element

$$\alpha \;=\; v(b-c_0) \;=\; \min\{vc_i \mid 1 \le i \le p-1\} \in vK.$$

Thus $\mathrm{dist}_{\tilde{K}}(b, K) = \alpha^+$ with $\alpha \in vK$. Take now an element β in vK. Then from the first part of the proof we have that $\mathrm{dist}_{\tilde{K}}(a, K) = 0^+$. Hence, Lemma 3.2 yields that for $b = ca$ with $c \in K$ of value β we have that

$$\mathrm{dist}_{\tilde{K}}(b, K) = vc + \mathrm{dist}_{\tilde{K}}(a, K) = \beta^+.$$

Assume that $vL|vK$ is nontrivial. Then $vL = vK + \gamma\mathbb{Z}$ for some element $\gamma \in \widetilde{vK}$ of order p modulo vK. Similarly to the previous case, we deduce that for an element $a \in L$ of value γ the elements $1, a, \ldots, a^{p-1}$ form a standard valuation basis of $(L|K, v)$. Furthermore, for an element $b = \sum_{i=0}^{p-1} c_i a^i \in L \setminus K$, with $c_i \in K$, the set $v(b - K)$ admits a maximal element

$$\alpha \;=\; v(b - c_0) \;=\; \min\{vc_i + i\gamma \mid 1 \leq i \leq p-1\} \in vL \setminus vK.$$

Hence $\mathrm{dist}_{\widetilde{K}}(b, K) = \alpha^+$ with $\alpha \in vL \setminus vK$. On the other hand, if β is an element of $vL \setminus vK$, then it is of the form $vc + i\gamma$ for some $c \in K$ and $1 \leq i \leq p-1$, since the order of $\gamma + vK$ in vL/vK is equal to p. As we have seen, $\mathrm{dist}_{\widetilde{K}}(a^i, K) = (i\gamma)^+$. Then for an element $b = ca^i$, Lemma 3.2 yields that

$$\mathrm{dist}_{\widetilde{K}}(b, K) = vc + \mathrm{dist}_{\widetilde{K}}(a^i, K) = \beta^+.$$

□

Assume that $(L|K, v)$ is a defectless extension of degree p such that the valuation v admits a unique extension from K to L. The previous corollary shows that then the number of distinct distances $\mathrm{dist}_{\widetilde{K}}(b, K)$ modulo vK of elements $b \in L \setminus K$ is at most p. In the next section we consider the number of distinct distances of elements in immediate extensions in the case of fields of positive characteristic.

3.3 The number of distinct distances in immediate elements in valued field extensions

Throughout this section we shall work under the following assumptions. We take (K, v) to be a valued field of positive characteristic p and finite p-degree n, that is, $[K^{1/p} : K] = p^n$. We assume that v is a valuation of finite rank r such that $(\frac{1}{p}vK : vK) = p^k$ and that $d(K^{1/p}|K, v) = p^m$ for some nonnegative integers k, m. Then $m + k \leq n$. We fix an extension of the valuation v to $\widetilde{K}$ and denote it again by v. We further denote by K^h the henselization of K with respect to this valuation.

Take an element b algebraic over K. Assume that v extends in a unique way from K to $K(b)$. We say that the element b is **immediate over** K if the set $v(b - K)$ has no maximal element. By Lemma 2.53, this means that $\mathrm{appr}\,(b, K)$ is an immediate approximation type. Note that if b is immediate over K, then $\infty \notin v(b - K)$, hence the extension $K(b)|K$ is nontrivial.

If the extension $(K(b)|K, v)$ is immediate, then Theorem 2.22 yields that the set $v(b - K)$ has no maximal element. The converse does not hold. As an example consider the extension $(K(a + x^{1/q})|K, v)$ constructed in Example 2.44. Since the extension $(K(a)|K, v)$ is immediate, the set $v(a - K)$ has no maximal element. More precisely, in Example 3.12 of [27] it is shown that

$$v(a - K) = \frac{1}{p^\infty}\mathbb{Z}^{<0},$$

where $\frac{1}{p^\infty}\mathbb{Z}^{<0}$ is the set of all negative elements of $\frac{1}{p^\infty}\mathbb{Z}$. Since for every $c \in K$ we have that $v(x^{1/q}) = \frac{1}{q} > 0 > v(a - c)$, we deduce that $v(a - c) = v(a + x^{1/q} - c)$. Consequently,

$v(a+x^{1/q}-K)=v(a-K)$ has no maximal element, but we have seen in Example 2.44 that the extension $(K(a+x^{1/q})|K,v)$ is not immediate.

Take an algebraic extension $(L|K,v)$ such that the valuation v admits a unique extension from K to L. We define $\mathrm{ndd}\,(L|K,v)$ to be the number of essentially distinct distances of elements of L immediate over K modulo vK, by which we mean the minimal nonnegative integer j such that there are elements $a_1,\dots,a_j\in L$ immediate over K so that for every $b\in L$ immediate over K we have that

$$\mathrm{dist}\,(b,K)=\alpha+\mathrm{dist}\,(a_i,K)$$

for some $i\in\{1,\dots,j\}$ and $\alpha\in vK$. If for every element $b\in L$ the set $v(b-K)$ admits a maximal element, then we set $\mathrm{ndd}\,(L|K,v)=0$. Note that if an element $b\in L$ is immediate over K, then by Lemma 3.1 we obtain that $\mathrm{dist}\,(b,K)=\mathrm{dist}_{\widetilde{K}}(b,K)$. Hence the definition of $\mathrm{ndd}\,(L|K,v)$ does not depend on the choice of the definition of distance and in our further investigation we will use the notion of $\mathrm{dist}_{\widetilde{K}}(b,K)$.

If $\mathrm{dist}\,(a,K)=\alpha+\mathrm{dist}\,(b,K)$ for some a,b algebraic over K and $\alpha\in vK$, then we will also write that $\mathrm{dist}\,(a,K)\equiv\mathrm{dist}\,(b,K)\pmod{vK}$.

Lemma 3.18. *Assume that $(E(a)|E,v)$ is an immediate extension of degree p^j with property (EF) and such that the valuation v admits a unique extension from E to $E(a)$. Then for every nonconstant polynomial $f\in E[X]$ of degree less than $[E(a):E]$ there is $\alpha\in vE$ and $i\in\{0,\dots,j-1\}$ such that*

$$\mathrm{dist}_{\widetilde{E}}(f(a),E)=\alpha+p^i\mathrm{dist}_{\widetilde{E}}(a,E).$$

Proof. Since the extension $(E(a)|E,v)$ is immediate, Lemma 2.53 implies that $\mathrm{appr}\,(a,E)$ is an immediate approximation type. Now the assertion of the lemma follows from Lemma 8.2 of [33]. □

Corollary 3.19. *If $E(a)|E$ is an immediate extension of degree p such that the valuation v admits a unique extension from E to $E(a)$, then for every $b\in E(a)\setminus E$, there is $\alpha\in vE$ such that*

$$\mathrm{dist}_{\widetilde{E}}(b,E)=\alpha+\mathrm{dist}_{\widetilde{E}}(a,E).$$

Proof. Take an extension of v from E to $\widetilde{E}$ and denote it again by v. Take E^h to be the henselization of E with respect to this extension. From Proposition 3.10 we deduce that $\mathrm{dist}_{\widetilde{E}}(a,E)=\mathrm{dist}_{\widetilde{E}}(a,E^h)$ and $\mathrm{dist}\,(b,E)=\mathrm{dist}\,(b,E^h)$. Since $E(a)|E$ is an immediate algebraic extension,

$$vE^h(a)=vE(a)^h=vE(a)=vE=vE^h\quad\text{and}\quad E^h(a)v=E(a)^hv=E(a)v=Ev=E^hv.$$

Thus also the extension $(E^h(a)|E^h,v)$ is immediate. Furthermore, Lemma 2.12 yields that the extension is of degree p. Hence without loss of generality we can assume that the field (E,v) is henselian.

Since $(E(a)|E,v)$ is an immediate extension, Lemma 2.53 implies that the approximation type $\mathrm{appr}\,(a,E)$ is immediate. By Proposition 2.56 this implies that the extension $(E(a)|E,v)$ has property (EF). The assertion of the corollary follows now from the previous lemma. □

The above observation allows us to prove the following fact.

Proposition 3.20. *Under the assumptions on (K,v) in this section, we have that*

$$\mathrm{ndd}\,(K^{1/p}|K,v) \leq m.$$

Furthermore, if $j := \mathrm{ndd}\,(K^{1/p}|K,v) > 0$, then there are elements $a_1,\dots,a_j \in K^{1/p}$ such that $(K(a_1,\dots,a_j)|K,v)$ is a nontrivial immediate subextension of $K^{1/p}|K$ and for every $b \in K^{1/p}$ immediate over K,

$$\mathrm{dist}\,(b,K) \equiv \mathrm{dist}\,(a_i,K) \pmod{vK}.$$

Proof. Suppose first that $(K^{1/p}|K,v)$ is a defectless extension, that is, $m = 0$. Then Lemma 2.25 yields that $v(b-K)$ admits a maximal element for every $b \in K^{1/p}$ and thus $\mathrm{ndd}\,(K^{1/p}|K,v) = 0$.

Assume now that $\mathrm{ndd}\,(K^{1/p}|K,v) > 0$. This means that $K^{1/p}$ contains elements that are immediate over K. Take a_1 to be such an element. Since v admits a unique extension from K to the purely inseparable extension $K^{1/p}$, it follows from Lemma 3.15 that $(K(a_1)|K,v)$ is an immediate extension. Take any $b \in K(a_1)\setminus K$. By Corollary 3.19 we have that $\mathrm{dist}_{\widetilde{K}}(b,K) \equiv \mathrm{dist}_{\widetilde{K}}(a_1,K) \pmod{vK}$.

Assume that we have chosen $a_1,\dots,a_i \in K^{1/p}$ in such a way that $K(a_1,\dots,a_i)|K$ is an immediate extension of degree p^i and $\mathrm{dist}_{\widetilde{K}}(a_{i_1},K) \not\equiv \mathrm{dist}_{\widetilde{K}}(a_{i_2},K) \pmod{vK}$ for any distinct $i_1,i_2 \in \{1,\dots,i\}$. Assume also that $i \leq m$ and for every $b \in K(a_1,\dots,a_i)\setminus K$ there is $t \in \{1,\dots,i\}$ such that $\mathrm{dist}_{\widetilde{K}}(b,K) \equiv \mathrm{dist}_{\widetilde{K}}(a_t,K) \pmod{vK}$.

If for every element $b \in K^{1/p}$ immediate over K there is a natural number $t \leq i$ such that

$$\mathrm{dist}_{\widetilde{K}}(b,K) \equiv \mathrm{dist}_{\widetilde{K}}(a_i,K) \pmod{vK},$$

then the proposition holds with $j = i$. Otherwise, there is an element $a_{i+1} \in K^{1/p}$ immediate over K and such that

$$\mathrm{dist}_{\widetilde{K}}(a_{i+1},K) \not\equiv \mathrm{dist}_{\widetilde{K}}(a_t,K) \pmod{vK}$$

for every $t \in \{1,\dots,i\}$. Then, by the assumption on $K(a_1,\dots,a_i)$, we have that

$$\mathrm{dist}_{\widetilde{K}}(a_{i+1},K) \not\equiv \mathrm{dist}_{\widetilde{K}}(b,K) \pmod{vK}$$

for every $b \in K(a_1,\dots,a_i)$. Hence $a_{i+1} \notin K(a_1,\dots,a_i)$ and by Lemma 3.8,

$$\mathrm{dist}_{\widetilde{K}}(a_{i+1},K) \;=\; \mathrm{dist}_{\widetilde{K}}(a_{i+1},K(a_1,\dots,a_i)). \tag{3.3}$$

Since the set $v(a_{i+1}-K)$ has no maximal element, from the above equality it follows that also the set $v(a_{i+1}-K(a_1,\dots,a_i))$ has no maximal element. We thus obtain that $K(a_1,\dots,a_{i+1})|K(a_1,\dots,a_i)$ is an extension of degree p such that the element a_{i+1} is immediate over $K(a_1,\dots,a_i)$. Therefore, it follows from Lemma 3.15 that the extension $(K(a_1,\dots,a_{i+1})|K(a_1,\dots,a_i),v)$ is immediate. Consequently, also $(K(a_1,\dots,a_{i+1})|K,v)$ is an immediate extension. Then in particular, $i+1 \leq m$, since otherwise we would have an immediate purely inseparable subextension of $K^{1/p}|K$ of degree p^{m+1}. That would imply that $d(K^{1/p}|K,v) \geq p^{m+1}$, a contradiction.

Take an element $b \in K(a_1,\dots,a_{i+1})$. Suppose that $\mathrm{dist}_{\widetilde{K}}(b,K) = \mathrm{dist}_{\widetilde{K}}(d,K)$ for some $d \in K(a_1,\dots,a_i)$. Then, by the assumption on $K(a_1,\dots,a_i)$,

$$\mathrm{dist}_{\widetilde{K}}(b,K) \equiv \mathrm{dist}_{\widetilde{K}}(a_t,K) \pmod{vK}$$

for some $t \in \{1, \ldots, i\}$. Otherwise, $b \in K(a_1, \ldots, a_{i+1}) \setminus K(a_1, \ldots, a_i)$ and applying again Lemma 3.8 we obtain that

$$\mathrm{dist}_{\widetilde{K}}(b, K) \equiv \mathrm{dist}_{\widetilde{K}}(b, K(a_1, \ldots, a_i)) \pmod{vK}.$$

This together with Corollary 3.19 and equation (3.3) yields that

$$\mathrm{dist}_{\widetilde{K}}(b, K) \equiv \mathrm{dist}_{\widetilde{K}}(a_{i+1}, K(a_1, \ldots, a_i)) \equiv \mathrm{dist}_{\widetilde{K}}(a_{i+1}, K) \pmod{vK}.$$

Since we have seen that the number of elements a_t cannot be greater than m, the construction finishes after j steps with $j \leq m$. Then $K(a_1, \ldots, a_j)$ is an immediate extension of K. From the above construction if follows that for every $b \in K^{1/p}$ immediate over K we have that

$$\mathrm{dist}_{\widetilde{K}}(b, K) \equiv \mathrm{dist}_{\widetilde{K}}(a_t, K) \pmod{vK}$$

for some $t \leq j$.

Hence in particular, $\mathrm{ndd}\,(K^{1/p}|K, v) = j \leq m$. □

Take an Artin-Schreier defect extension $(K(\vartheta)|K, v)$. In Section 2.4 we mentioned that $(K(\vartheta)|K, v)$ is an independent Artin-Schreier defect extension if and only if $\mathrm{dist}_{\widetilde{K}}(\vartheta, K) = H^-$ for some proper convex subgroup H of vK. By Corollary 3.19 the distance of every element of $K(\vartheta) \setminus K$ from K is equal to H^- modulo vK. Therefore, the number of the possible distances of elements of independent Artin-Schreier defect extensions modulo vK is bounded by r.

If $(K(\vartheta)|K, v)$ is a dependent Artin-Schreier defect extension, then $\vartheta \sim_K \eta$ for some $\eta \in K^{1/p}$ which generates an immediate extension of K. Then $\mathrm{dist}_{\widetilde{K}}(\vartheta, K) = \mathrm{dist}_{\widetilde{K}}(\eta, K)$. Again, from Corollary 3.19 we deduce that the distance of every element of $K(\vartheta) \setminus K$ from K is equal to $\mathrm{dist}_{\widetilde{K}}(\eta, K)$ modulo vK. By the above proposition, we have at most m possible distances $\mathrm{dist}_{\widetilde{K}}(\eta, K)$ modulo vK. This proves the following fact.

Proposition 3.21. *The number of distinct distances of elements in Artin-Schreier defect extensions of (K, v) modulo vK is bounded by $m + r$.*

For every natural number i denote by $\mathrm{ndd}_i(K, v)$ the number of distinct distances modulo vK of elements $b \in \widetilde{K} \setminus K$ satisfying the following conditions:

$$\left.\begin{array}{l} [K(b) : K] \leq p^i, \\ v \text{ extends in a unique way from K to } K(b), \\ b \text{ is immediate over } K. \end{array}\right\} \qquad (3.4)$$

We will show now that for every $i \in \mathbb{N}$ the number $\mathrm{ndd}_i(K, v)$ is finite. To prove it, we need the following fact.

Lemma 3.22. *Take an algebraic extension $(L|E, v)$ and a defectless algebraic extension $(F|E, v)$ such that v extends in a unique way from E to F. Then every $b \in L$ immediate over E is also immediate over F, with*

$$\mathrm{dist}_{\widetilde{F}}(b, F) \;=\; \mathrm{dist}_{\widetilde{E}}(b, E).$$

Proof. Suppose that there is an element $b \in L$ immediate over E, but not immediate over F. Since $v(b-E) \subseteq v(b-F)$, the inclusion is proper and consequently $\operatorname{dist}_{\widetilde{F}}(b,F) > \operatorname{dist}_{\widetilde{E}}(b,E)$. By Lemma 3.8 there is an element $a \in F$ such that $\operatorname{dist}_{\widetilde{E}}(b,E) = \operatorname{dist}_{\widetilde{E}}(a,E)$. It follows that $v(a-E)$ has no maximal element, as the set $v(b-E)$ has no maximal element. On the other hand, since $(E(a)|E,v)$ is defectless, as a subextension of a defectless extension, Lemma 2.25 yields that the set $v(a-E)$ has a maximal element, a contradiction. Hence $v(b-F)$ has no maximal element and $\operatorname{dist}_{\widetilde{F}}(b,F) = \operatorname{dist}_{\widetilde{E}}(b,E)$. □

Theorem 3.23. *Under the assumptions on (K,v) of this section, for every natural number i we have that*

$$\operatorname{ndd}_i(K,v) \le (r+m)\sum_{s=0}^{i-1} p^{ks}.$$

Proof. Take an element $b \in \widetilde{K}$ satisfying the assumptions (3.4). From Proposition 3.10 we deduce that $\operatorname{dist}_{\widetilde{K}}(b,K) = \operatorname{dist}_{\widetilde{K}}(b,K^h)$. This implies in particular that $v(b-K^h)$ has no maximal element, that is, b is immediate over K^h. Furthermore, the assumptions (3.4) together with Lemma 2.12 yield that $[K^h(b):K^h] = [K(b):K]$. Hence, for every natural number i we have that $\operatorname{ndd}_i(K,v) \le \operatorname{ndd}_i(K^h,v)$ and we can assume that (K,v) is henselian.

Take K^r to be the absolute ramification field of K. Since the extension $K^r|K$ is tame, it is defectless and Lemma 3.22 yields that every element $b \in \widetilde{K}$ immediate over K is immediate also over K^r, with $\operatorname{dist}_{\widetilde{K^r}}(b,K^r) = \operatorname{dist}_{\widetilde{K}}(b,K)$. Moreover, $[K^r(b):K^r] \le [K(b):K]$. Therefore, $\operatorname{ndd}_i(K,v) \le \operatorname{ndd}_i(K^r,v)$ for any $i \in \mathbb{N}$, and we can assume that $K^r = K$. Note that by Lemma 2.19 this means that for every element b algebraic over K the extension $K(b)|K$ is a tower of normal extensions of degree p. In particular, it is of degree p^t for some $t \ge 0$.

Take an element b of degree p over K and assume that b is immediate over K. By Lemma 3.15, the extension $(K(b)|K,v)$ is immediate. If $K(b)|K$ is purely inseparable, it follows from Proposition 3.20 that we have at most m possible distances $\operatorname{dist}_{\widetilde{K}}(b,K)$ modulo vK. Assume now that $K(b)|K$ is not purely inseparable. Since it is of degree p, this means that the extension is separable. Since furthermore $K(b)|K$ is a normal extension, it is Galois. As (K,v) is a henselian field of characteristic p and the extension $K(b)|K$ is immediate, it is an Artin-Schreier defect extension. If the Artin-Schreier extension is dependent, then we have already seen that the distance $\operatorname{dist}_{\widetilde{K}}(b,K)$ is equal modulo vK to a distance of an element $\eta \in K^{1/p} \setminus K$, immediate over K. If $K(b)|K$ is an independent Artin-Schreier defect extension, then we have proved that the number of distinct distances $\operatorname{dist}_{\widetilde{K}}(b,K)$ modulo vK is bounded by r. Consequently, we obtain that there are at most $m+r$ distinct distances of immediate elements of degree p modulo vK. This proves the theorem in the case of $i=1$.

Take $i \ge 2$ and assume that

$$\operatorname{ndd}_{i-1}(K,v) \le (r+m)\sum_{s=0}^{i-2} p^{ks}.$$

To give an upper bound for $\operatorname{ndd}_i(K,v)$, it is enough to consider elements of degree p^i over K, immediate over the field. This follows from the fact that the distance of any element b of degree at most p^{i-1}, immediate over K, is already counted in the upper bound of

$\mathrm{ndd}_{i-1}(K,v)$. Take an element b immediate over K and assume that $[K(b):K]=p^i$. Since $v(b-K)$ has no maximal element, the approximation type $\mathrm{appr}\,(b,K)$ is immediate, by Lemma 2.53. Take g to be an associated minimal polynomial for $\mathrm{appr}\,(b,K)$. If the extension $K(b)|K$ does not have the property (EF), then $\deg g<p^i$, by Lemma 3.11. Take $d\in\widetilde{K}$ to be a root of g. Since (K,v) is henselian, Theorem 2.55 yields that $(K(d)|K,v)$ is an immediate extension and

$$\mathrm{appr}\,(b,K)=\mathrm{appr}\,(d,K).$$

Hence, by part b) of Lemma 2.52,

$$\mathrm{dist}_{\widetilde{K}}(b,K)=\mathrm{dist}_{\widetilde{K}}(d,K).$$

Since $[K(d):K]<p^i$, it follows that $[K(d):K]\le p^{i-1}$. Therefore, the distance $\mathrm{dist}_{\widetilde{K}}(b,K)$ appears already as a distance of some immediate element of degree at most p^{i-1}.

Hence, it is enough to consider the case when b is immediate over K and the extension $K(b)|K$ is of degree p^i with property (EF). By Lemma 3.12 this yields that the extension is immediate. Assume first that the extension $K(b)|K$ is purely inseparable. Then from Lemma 3.8 we deduce that $\mathrm{dist}_{\widetilde{K}}(b,K)=\mathrm{dist}_{\widetilde{K}}(d,K)$ for some $d\in K^{1/p^{i-1}}$ or $\mathrm{dist}_{\widetilde{K}}(b,K)=\mathrm{dist}_{\widetilde{K}}(b,K^{1/p^{i-1}})$. In the former case, $\mathrm{dist}_{\widetilde{K}}(b,K)$ appears already as a distance of some immediate element of degree p^{i-1}. Assume that the second case holds. Then $K^{1/p^{i-1}}(b)|K^{1/p^{i-1}}$ is a purely inseparable extension of degree p and the element b is immediate over $K^{1/p^{i-1}}$. Since $d(K^{1/p^i}|K^{1/p^{i-1}},v)=d(K^{1/p}|K,v)=p^m$, Proposition 3.20 yields that there are at most m distinct distances of elements of K^{1/p^i} immediate over $K^{1/p^{i-1}}$, modulo $vK^{1/p^{i-1}}=\frac{1}{p^{i-1}}vK$. Let $\delta_1,\dots,\delta_j$ with $j\le m$ be these distances. Then

$$\mathrm{dist}_{\widetilde{K}}(b,K)\;=\;\mathrm{dist}_{\widetilde{K}}(b,K^{1/p^{i-1}})\;=\;\delta_t+\alpha$$

for some $t\le j$ and $\alpha\in\frac{1}{p^{i-1}}vK$. As $\left(\frac{1}{p}vK:vK\right)=p^k$, we have that $\left(\frac{1}{p^{i-1}}vK:vK\right)=p^{k(i-1)}$. This gives $jp^{k(i-1)}$ possible distances $\mathrm{dist}_{\widetilde{K}}(b,K)$ modulo vK. As $j\le m$, we have at most $mp^{k(i-1)}$ such distances.

Suppose now that $K(b)|K$ is not purely inseparable. Then $E:=(K(b)|K)^{sep}$ is a nontrivial separable subextension of $K(b)|K$. Furthermore, $E|K$ is a tower of Galois extensions of degree p, as $K(b)|K$ is a tower of normal extensions of degree p. This yields that K admits an Artin-Schreier extension $K(\vartheta)\subseteq K(b)$, where ϑ is an Artin-Schreier generator. Since $K(b)|K$ is an immediate extension of henselian fields, the same holds for $K(\vartheta)|K$ and thus $K(\vartheta)|K$ is an Artin-Schreier defect extension. Take a polynomial $f\in K[X]$ such that $\vartheta=f(b)$ and $\deg f<p^i$. Then, by Lemma 3.18,

$$\mathrm{dist}_{\widetilde{K}}(\vartheta,K)\;=\;\mathrm{dist}_{\widetilde{K}}(f(b),K)\;=\;\alpha+p^s\mathrm{dist}_{\widetilde{K}}(b,K)\tag{3.5}$$

for some $\alpha\in vK$ and $s<i$. Take $c\in K$ such that $vc=\alpha$.

Assume that the Artin-Schreier defect extension $K(\vartheta)|K$ is dependent. Then $\vartheta\sim_K\eta$, for some $\eta\in K^{1/p}$, such that the extension $K(\eta)|K$ is immediate. Hence,

$$\mathrm{dist}_{\widetilde{K}}(\eta,K)\;=\;\mathrm{dist}_{\widetilde{K}}(\vartheta,K)\;=\;vc+p^s\mathrm{dist}_{\widetilde{K}}(b,K).$$

By Lemma 3.2, this yields that

$$p^s\mathrm{dist}_{\widetilde{K}}(b,K)\;=\;\mathrm{dist}_{\widetilde{K}}\big(\frac{\eta}{c},K\big).$$

Therefore, we obtain that

$$\operatorname{dist}_{\widetilde{K}}(b,K) \;=\; \operatorname{dist}_{\widetilde{K}}\Big(\Big(\frac{\eta}{c}\Big)^{1/p^s}, K^{1/p^s}\Big), \tag{3.6}$$

since $\frac{1}{p^s}v(\frac{\eta}{c} - K) = v\big((\frac{\eta}{c})^{1/p^s} - K^{1/p^s}\big)$. As $v(b-K)$ has no maximal element, it follows from equation (3.6) that also $v\big((\frac{\eta}{c})^{1/p^s} - K^{1/p^s}\big)$ has no maximal element, and thus $(\frac{\eta}{c})^{1/p^s}$ is immediate over K^{1/p^s}. Moreover, $K^{1/p^s}((\frac{\eta}{c})^{1/p^s})|K^{1/p^s}$ is a purely inseparable extension of degree p. Hence, the argument used above for the case of $K(b)|K$ purely inseparable shows that $\operatorname{dist}_{\widetilde{K}}\big((\frac{\eta}{c})^{1/p^s}, K^{1/p^s}\big)$ is already considered as a possible distance of an immediate purely inseparable element over K of degree $p^{s+1} \leq p^i$ over the field.

Suppose that $K(\vartheta)|K$ is an independent Artin-Schreier defect extension. Then Proposition 2.30 yields that

$$\operatorname{dist}_{\widetilde{K}}(\vartheta,K) = p^s \operatorname{dist}_{\widetilde{K}}(\vartheta,K).$$

Hence, from equation 3.5 we obtain that

$$p^s \operatorname{dist}_{\widetilde{K}}(\vartheta,K) = vc + p^s \operatorname{dist}_{\widetilde{K}}(b,K)$$

and consequently,

$$\operatorname{dist}_{\widetilde{K}}(b,K) = -\frac{1}{p^s}vc + \operatorname{dist}_{\widetilde{K}}(\vartheta,K). \tag{3.7}$$

Since the valuation v is of rank r, we have already noticed that there are r possible distances $\operatorname{dist}_{\widetilde{K}}(\vartheta,K)$ modulo vK. Moreover, $(\frac{1}{p^s}vK : vK) \leq (\frac{1}{p^{i-1}}vK : vK) = p^{k(i-1)}$. Hence in the case under consideration we have at most $rp^{k(i-1)}$ possible distances $\operatorname{dist}_{\widetilde{K}}(b,K)$ modulo vK.

Consequently, we obtain that

$$\operatorname{ndd}_i(K,v) \;\leq\; \operatorname{ndd}_{i-1}(K,v) + rp^{k(i-1)} + mp^{k(i-1)}.$$

By the induction hypothesis, it follows that

$$\operatorname{ndd}_i(K,v) \;\leq\; (r+m)\sum_{s=0}^{i-1} p^{ks}.$$

□

4. Infinite towers of Galois defect extensions of prime degree

In this chapter we develope constructions of towers of Galois defect extensions of prime degree. We give criteria for valued fields of positive residue characteristic p with p-divisible value group and perfect residue field to admit an infinite tower of such extensions. As we have noticed in Section 2.4, towers of Galois defect extensions of prime degree play a central role in the problem of defect extensions. While in that case we consider mainly finite towers of Galois defect extensions of prime degree, we will see in Chapter 7 that for some problems the existence of infinite towers of such extensions is crucial. We prove first a useful criterion for a valued field of positive characteristic p with p-divisible value group and perfect residue field to admit an infinite tower of dependent Artin-Schreier defect extensions.

4.1 Towers of dependent Artin-Schreier defect extensions of valued fields

Throughout this section we assume that (K, v) is a valued field of positive characteristic p with perfect residue field and p-divisible value group. Due to the importance of the dependent Artin-Schreier defect extensions for the problems related to local uniformization which we have mentioned in the introduction, an interesting question is under which conditions the field (K, v) admits dependent Artin Schreier defect extensions or towers of such extensions, and how to construct them.

Take a valued field (L, v) and suppose that the field L admits an immediate purely inseparable extension $L(\eta)|L$ of degree p such that $\eta \in L^{1/p} \setminus L^c$. For any element $b \in L$ consider the polynomial

$$f_b \;=\; Y^p - b^{p-1}Y - \eta^p.$$

With each of the polynomials f_b we can associate through the transformation $Y = bX$ the Artin-Schreier polynomial

$$g_b(X) \;=\; X^p - X - \left(\frac{\eta}{b}\right)^p.$$

Note that ϑ_b is a root of the polynomial g_p if and only if $b\vartheta_b$ is a root of f_b. The next theorem shows when such deformation of a polynomial $Y^p - \eta^p$ generating a purely inseparable extension into an Artin-Schreier polynomial leads to a dependent Artin-Schreier defect extension.

Theorem 4.1. *Suppose that the polynomial $X^p - \eta^p \in L[X]$ induces an immediate extension of (L, v) which does not lie in the completion of L. Then for each $b \in L^\times$ such that*

$$(p-1)vb \;>\; p\,\mathrm{dist}\,(\eta, L) - v\eta \tag{4.1}$$

the polynomial $g_b = X^p - X - (\frac{\eta}{b})^p$ induces a dependent Artin-Schreier defect extension. Moreover, every root ϑ_b of g_b satisfies

$$\vartheta_b \sim_L \frac{\eta}{b}.$$

Proof. Take ϑ to be a root of the polynomial g_b. If condition (4.1) holds, then

$$(p-1)vb + v\eta \;>\; p\,\mathrm{dist}\,(\eta, L) \;>\; pv\eta, \tag{4.2}$$

by Theorem 2.22. Thus $vb > v\eta$ and consequently $v(\frac{\eta}{b})^p < 0$. By Lemma 2.27 it follows that

$$v\vartheta = v\frac{\eta}{b} = v\eta - vb. \tag{4.3}$$

By definition of ϑ,

$$\eta^p + b^p\vartheta = b^p\left(\left(\frac{\eta}{b}\right)^p + \vartheta\right) = (b\vartheta)^p. \tag{4.4}$$

Take an element $c \in L$. By equalities (4.3), (4.2) and from the definition of $\mathrm{dist}\,(\eta, L)$ we have that

$$vb^p\vartheta \;=\; pvb + v\eta - vb \;=\; (p-1)vb + v\eta \;>\; p\mathrm{dist}\,(\eta, L) > pv(\eta - c) \;=\; v(\eta^p - c^p).$$

Together with equation (4.4) this implies that

$$v(\eta - c) \;=\; \frac{1}{p}v(\eta^p - c^p) = \frac{1}{p}v(\eta^p - c^p + b^p\vartheta) \;=\; \frac{1}{p}v((b\vartheta)^p - c^p) = v(b\vartheta - c). \tag{4.5}$$

Hence $v(\eta - L) = v(b\vartheta - L)$. Since $L(\eta)|L$ is immediate, from Theorem 2.22 follows that $v(\eta - L) = v(\eta - L) \cap vL$ has no maximal element. Now Lemma 2.21 together with (4.5) imply that $b\vartheta \sim_L \eta$. Hence part d) of Lemma 2.24 yields that

$$\vartheta \sim_L \frac{\eta}{b}.$$

Furthermore, $v(b\vartheta - L)$ has no maximal element, thus from Lemma 2.26 we deduce that the extension $(L(\vartheta)|L, v)$ is immediate.

Since $\vartheta \sim_L \frac{\eta}{b}$ and $\frac{\eta}{b}$ is purely inseparable over L, from Theorem 2.23 it follows that $\vartheta \notin L^h$. As $L(\vartheta)|L$ is of prime degree, this implies that the extension is linearly disjoint from $L^h|L$. By Lemma 2.12 this means that the valuation v admits a unique extension from L to $L(\vartheta)$. Since moreover $(L(\vartheta)|L, v)$ is immediate, equation (1.1) shows that the extension has nontrivial defect. From the relation $\vartheta \sim_L \frac{\eta}{b}$ it follows that $(L(\vartheta)|L, v)$ is a dependent Artin-Schreier defect extension. □

Theorem 4.1 was proved in [25]. The above argument fills a gap in the proof of the fact that the valuation v admits a unique extension from L to $L(\vartheta)$.

We use the deformation of purely inseparable extensions to prove the following fact.

Theorem 4.2. *If there is a purely inseparable extension of (K, v) which does not lie in the completion of the field, then K admits an infinite tower of dependent Artin-Schreier defect extensions. If every purely inseparable extension of degree p lies in the completion of K, then the field admits no dependent Artin-Schreier defect extensions.*

Proof. Suppose there is an element $a \in K$ such that a^{1/p^n} does not lie in the completion K^c of K. Take

$$k := \min\{i \in \mathbb{N} \mid a^{1/p^i} \notin K^c\}.$$

As $a^{1/p^k} \notin K^c$, by Corollary 2.50 and Lemma 2.46 there is $\gamma \in vK$ such that $v(a^{1/p^k} - K) < \gamma$. By definition of k, the element $a^{1/p^{k-1}}$ lies in the completion of K. Hence, $v(a^{1/p^{k-1}} - d) > p\gamma$ for some $d \in K$. Thus

$$v(a^{1/p^k} - K) < \gamma < \frac{1}{p} v(a^{1/p^{k-1}} - d) = v(a^{1/p^k} - d^{1/p}).$$

It follows that also $v(d^{1/p} - K) < \gamma$ and consequently $d^{1/p} \notin K^c$.

Since the value group of the field K is p-divisible and its residue field is perfect, $d^{1/p}$ generates an immediate purely inseparable extension of K which does not lie in the completion of K. By Theorem 4.1, we can choose an element $b_1 \in K^\times$ of large enough value, such that a root ϑ_1 of the polynomial

$$f_1 = Y^p - Y - \frac{d}{b_1^p}$$

generates a dependent Artin-Schreier defect extension $K_1 = K_0(\vartheta_1)$ of $K_0 := K$.

Take a natural number n and assume that we have chosen $K_1, \ldots, K_n$ to be algebraic extensions of K such that each $K_i = K_{i-1}(\vartheta_i)$ is a dependent Artin-Schreier defect extension of K_{i-1}, where ϑ_i is a root of the polynomial

$$f_i = Y^p - Y - \frac{1}{b_i^p} \vartheta_{i-1}$$

for some $b_i \in K^\times$. Assume in addition that $\vartheta_i^{1/p} \notin K_i^c$ for every $i \leq n - 1$. Note that then also $\vartheta_n^{1/p}$ does not lie in the completion of K_n. Indeed, suppose that $\vartheta_n^{1/p} \in K_n^c$. By Lemma 2.51 we have that $K_n^c = K_{n-1}^c(\vartheta_n)$, hence $\vartheta_n^{1/p} \in K_{n-1}^c(\vartheta_n)$. Since $\vartheta_n^p - \vartheta_n = \frac{1}{b_n^p} \vartheta_{n-1}$, we have that $\vartheta_n - \vartheta_n^{1/p} = \frac{1}{b_n} \vartheta_{n-1}^{1/p}$. Therefore,

$$\vartheta_{n-1}^{1/p} = b_n(\vartheta_n - \vartheta_n^{1/p}) \in K_{n-1}^c(\vartheta_n).$$

By assumption $\vartheta_{n-1}^{1/p} \notin K_{n-1}^c$, thus we would obtain that $K_{n-1}^c(\vartheta_{n-1}^{1/p}) = K_{n-1}^c(\vartheta_n)$, but $K_{n-1}^c(\vartheta_{n-1}^{1/p})$ is a nontrivial purely inseparable and $K_{n-1}^c(\vartheta_n)$ a separable extension of K_{n-1}^c, a contradiction.

Then, using the same argument as before, we can choose an element $b_{n+1} \in K^\times$ such that the polynomial

$$f_{n+1} = Y^p - Y - \frac{1}{b_{n+1}^p} \vartheta_n$$

induces a dependent Artin-Schreier defect extension $K_n(\vartheta_{n+1})|K_n$, where ϑ_{n+1} is a root of the polynomial f_{n+1}. By induction we obtain an infinite chain $K_n|K_{n-1}$ of dependent Artin-Schreier defect extensions.

Assume now that every purely inseparable extension of (K, v) lies in the completion of K. Suppose there were a dependent Artin-Schreier defect extension $(K(\vartheta)|K, v)$. Then there would be $\eta \in K^{1/p}$ such that $\eta \sim_K \vartheta$, that is,

$$v(\vartheta - \eta) \;>\; v(\eta - K).$$

Since $\eta \in K^{1/p} \subseteq K^c$ we would obtain $\eta = \vartheta$, a contradiction. $\square$

The above theorem shows that (K, v) either admits an infinite tower or admits no Artin-Schreier defect extensions at all. In particular, this yields the following property:

Corollary 4.3. *If (K, v) admits at least one dependent Artin-Schreier defect extension, then it admits an infinite tower of such extensions.*

We modify now the above construction to obtain infinitely many parallel dependent Artin-Schreier defect extensions of the field K. Set $L_0 := K$. As before, we choose an element $d \in K$ whose p-th root does not lie in the completion of K and take $c_1 \in K^\times$ such that a root η_1 of the polynomial

$$h_1 \;=\; Y^p - Y - \frac{d}{c_1^p}$$

generates a dependent Artin-Schreier defect extension $L_1 = L_0(\eta_1)$ of $L_0 := K$. Take a natural number n and assume that we have chosen $L_1, \ldots, L_n$ to be algebraic extensions of K such that $L_i = L_{i-1}(\eta_i)$ is a dependent Artin-Schreier defect extension of L_{i-1} generated by a root η_i of the polynomial

$$h_i \;=\; Y^p - Y - \frac{d}{c_i^p}$$

for some $c_i \in K^\times$. Assume in addition that $d^{1/p} \notin L_i^c$ for every $i \leq n-1$. Suppose that $d^{1/p} \in L_n^c = L_{n-1}^c(\eta_n)$. Since $d^{1/p}$ does not lie in the completion of L_{n-1}, we have that $[L_{n-1}^c(d^{1/p}) : L_{n-1}^c] = p$. Therefore we would obtain $L_{n-1}^c(d^{1/p}) = L_{n-1}^c(\eta_n)$, but $L_{n-1}^c(d^{1/p})$ is a nontrivial purely inseparable extension of L_{n-1}^c and the extension $L_{n-1}^c(\eta_n)|L_{n-1}^c$ is separable, a contradiction. Consequently $d^{1/p} \notin L_n^c$. By Theorem 4.1, we can choose an element $c_{n+1} \in K^\times$ of large enough value, such that a root η_{n+1} of the polynomial

$$h_{n+1} \;=\; Y^p - Y - \frac{d}{c_{n+1}^p}$$

generates a dependent Artin-Schreier defect extension $L_{n+1} = L_n(\eta_{n+1})$ of L_n. Hence we obtain an infinite chain of dependent Artin-Schreier defect extensions $L_n|L_{n-1}$.

Take a natural number n. Since every polynomial h_n has coefficients in K, the field $K(\eta_n)$ is an Artin-Schreier extension of K. By what we have shown, the valuation v of K has a unique extension to the field L_n and the extension $L_n|K$ is immediate. Since $K(\eta_n)|K$ is a subextension of $L_n|K$, we deduce that v has also a unique extension to $K(\eta_n)$ and $K(\eta_n)|K$ is immediate. Hence $K(\eta_n)|K$ has nontrivial defect. From Theorem 4.1 it follows that $\eta_n \sim_{L_{n-1}} \frac{1}{c_n} d^{1/p}$. This means that

$$v\left(\eta_n - \frac{d^{1/p}}{c_n}\right) \;>\; \mathrm{dist}\left(\frac{d^{1/p}}{c_n}, L_{n-1}\right) \;\geq\; \mathrm{dist}\left(\frac{d^{1/p}}{c_n}, K\right).$$

It follows that $\eta_n \sim_K \frac{1}{c_n} d^{1/p}$ and $K(\eta_n)|K$ is a dependent Artin-Schreier defect extension. Since for every $n \in \mathbb{N}$ the extension $L_n|L_{n-1}$ is nontrivial, we deduce that K admits infinitely many dependent Artin-Schreier extensions.

Take $n \in \mathbb{N}$, any distinct natural numbers $i_1, \ldots, i_n$ and consider the compositum $K(\eta_{i_1}, \ldots, \eta_{i_n})$ of the fields $K(\eta_{i_1}), \ldots, K(\eta_{i_n})$. Since $K(\eta_{i_1}, \ldots, \eta_{i_n})|K$ is a subextension of some $L_m|K$, we deduce that the valuation v of K has a unique extension to $K(\eta_{i_1}, \ldots, \eta_{i_n})$ and the extension $K(\eta_{i_1}, \ldots, \eta_{i_n})|K$ is immediate. Consequently, the defect of the extension is equal to its degree. By what we have proved, $L_m|K$ is of degree p^m. From the definition of L_m it follows that $K(\eta_{i_1}, \ldots, \eta_{i_n})|K$ must be of degree p^n. Furthermore the extension is Galois, as a compositum of Galois extensions of the field K. We have thus proved:

Proposition 4.4. *If there is a purely inseparable extension of (K, v) which does not lie in the completion of the field, then K admits infinitely many dependent Artin-Schreier defect extensions such that the compositum of any n of the extensions is a Galois extension of K of degree and defect p^n.*

Corollary 4.5. *If (K, v) admits at least one dependent Artin-Schreier defect extension, then it admits infinitely many dependent Artin-Schreier defect extensions such that the compositum of any n of the extensions is a Galois extension of K of degree and defect p^n.*

From the above corollary it follows immediately that if the field (K, v) admits only finitely many Artin-Schreier defect extensions, then all of the extensions are independent.

4.2 Towers of Galois defect extensions of prime degree of Kaplansky fields

Assume that (K, v) is a valued field of positive characteristic p with p-divisible value group and perfect residue field. In the previous section we proved that if the field admits at least one dependent Artin-Schreier defect extension, then it admits an infinite tower of Artin-Schreier defect extensions, that is, an infinite tower of Galois defect extensions of prime degree. We consider now the case of Kaplansky fields by assuming additionally that Kv admits no finite separable extensions of degree divisible by p. Under this assumption we can relax the condition that the field admits a dependent Artin-Schreier defect extension. We prove that in this case it is enough to assume that (K, v) admits any Artin-Schreier extension, linearly disjoint from the henselization of K. We also give conditions for Kaplansky fields of characteristic 0 and positive residue characteristic to admit an infinite tower of Galois defect extensions of prime degree.

We start with general remarks about the existence of infinite towers of Galois extensions of degree p for any field of positive characteristic p. As for such a field an extension of degree p is Galois if and only if it is an Artin-Schreier extension, we consider the existence of infinite towers of Artin-Schreier extensions. The basis for the constructions of such towers will be the following lemma, proved in Chapter 8 of [17] (cf. Lemma 1.10).

Lemma 4.6. *Take a field L of positive characteristic p and an element $a \in L$. Assume that the polynomial $f = X^p - X - a$ is irreducible over L. If ϑ is a root of f, then the polynomial $X^p - X - a\vartheta^{p-1}$ is irreducible over $L(\vartheta)$.*

Lemma 4.7. *Assume that K is a field of positive characteristic. If it admits an Artin-Schreier extension, then it admits already an infinite tower of such extensions.*

Proof. Assume that $K_0 := K$ admits an Artin-Schreier extension K_1 of degree $p = \operatorname{char} K$. Then $K_1 = K_0(\vartheta_1)$, where ϑ_1 is a root of an irreducible polynomial $f_1 = X^p - X - a_1 \in K[X]$. Consider the following construction. For every $n > 1$:

$$\left.\begin{array}{l} \text{if } K_n = K_{n-1}(\vartheta_n) \text{ with } \vartheta_n^p - \vartheta_n = a_n \in K_{n-1}, \\ \text{take } K_{n+1} = K_n(\vartheta_{n+1}) \text{ with } \vartheta_{n+1} \text{ a root of } f_{n+1} := X^p - X - a_n \vartheta_n^{p-1}. \end{array}\right\} \tag{4.6}$$

Take a natural number n and suppose that $K_n|K_{n-1}$ is an Artin-Schreier extension, that is, the polynomial f_n is irreducible over K_{n-1}. Then Lemma 4.6 yields that the polynomial f_{n+1} is irreducible over K_n and $K_{n+1}|K_n$ is an Artin-Schreier extension.

By induction on n we obtain an infinite tower of Artin-Schreier extensions $K_n|K_{n-1}$. $\square$

We now apply the above result to the valued field extensions of Kaplansky fields.

Theorem 4.8. *Take a Kaplansky field (K, v) of positive characteristic. Assume that K admits an Artin-Schreier extension $E|K$ such that v extends in a unique way to a valuation of E. Then $(E|K, v)$ has nontrivial defect and (K, v) admits an infinite tower of Artin-Schreier defect extensions.*

Proof. Set $p := \operatorname{char} K$. Since $K_1 := E$ is an Artin-Schreier extension of K, it is of the form $K(\vartheta_1)$, where ϑ_1 is a root of a polynomial $f_1 = X^p - X - a_1 \in K[X]$. From Lemma 4.7 we deduce that $K_0 := K$ admits an infinite tower of Artin-Schreier extensions $K_n|K_{n-1}$, which can be obtained by construction (4.6).

By our assumption, the valuation v admits a unique extension from K_0 to K_1. Take $n \in \mathbb{N}$ and assume that we have already shown that v admits a unique extension from K_{n-1} to K_n. From Lemma 2.12 it follows that $K_n|K_{n-1}$ is linearly disjoint from $K_{n-1}^h|K_{n-1}$. Hence $[K_{n-1}^h(\vartheta_n) : K_{n-1}^h] = p$ and thus the polynomial f_n is irreducible over K_{n-1}^h. By Lemma 4.6 the polynomial f_{n+1} is irreducible over $K_{n-1}^h(\vartheta_n) = K_n^h$. Hence $[K_n^h(\vartheta_{n+1}) : K_n^h] = p$ and $K_{n+1}|K_n$ is linearly disjoint from $K_n^h|K_n$. Applying again Lemma 2.12 we deduce that v admits a unique extension from K_n to K_{n+1}. By induction on n, this holds for every extension in the tower.

Take a natural number n. By the Lemma of Ostrowski,

$$p \;=\; [K_n : K_{n-1}] \;=\; d(K_n|K_{n-1}, v)(vK_n : vK_{n-1})[K_n v : K_{n-1} v].$$

Since (K, v) is a Kaplansky field, vK is p-divisible and Kv admits no finite extension of degree divisible by p. This yields that the extension $(K_n|K_{n-1}, v)$ is immediate and $d(K_n|K_{n-1}, v) = p$. As $E = K_1$, we have that also $(E|K, v)$ is a defect extension. $\square$

Note that if (K, v) is a Kaplansky field of positive characteristic p, then vK is p-divisibe and Kv is perfect. Thus (K, v) satisfies the assumptions of the previous section. From Theorem 4.2 we know that (K, v) admits an infinite tower of Galois extensions of degree and defect p if the perfect hull of K is not contained in the completion of the field.

For fields of characteristic 0 we will consider separately the case of odd primes and $p = 2$. A field extension $E|K$ is called **irreducible radical** if it is generated by a root of an irreducible polynomial $X^n - a \in K[X]$. If K is a field of characteristic 0 containing a primitive n-th root of unity, then a field extension $E|K$ of degree n is cyclic if and only if it is irreducible radical (cf. Lemma 1.1, Chapter 8 of [17]). We will consider the existence of infinite towers of Galois extensions of prime degree p of fields admitting primitive p-th root of unity. Hence we will investigate when such fields admit infinite towers of irreducible radical extensions of degree p. We treat first the case of an odd prime p. The next lemma is a special case of Theorem 1.6, Chapter 8 of [17].

Lemma 4.9. *Let L be an arbitrary field. Take an element $a \in L$ and an odd prime p. Then for any natural number n, the polynomial $X^{p^n} - a$ is irreducible over L if and only if $a \notin L^p$.*

Lemma 4.10. *Assume that K is a field of characteristic 0. Take an odd prime p and $\varepsilon_p \in \widetilde{K}$ to be a primitive p-th root of unity. If K admits a Galois extension of degree p, then $K(\varepsilon_p)$ admits an infinite tower of Galois extensions of degree p.*

Proof. Assume that $L|K$ is a Galois extension of degree p. Set $K_0 := K(\varepsilon_p)$. Then $K_1 := L.K_0 = L(\varepsilon_p)$ is a Galois extension of K_0 of degree p. Hence $K_1 = K_0(\eta_1)$, where η_1 is a root of an irreducible polynomial $f_1 := X^p - a \in K_0[X]$. By the previous lemma, $a \notin K_0^p$ and consequently, for every natural number n the polynomial

$$f_n := X^{p^n} - a \tag{4.7}$$

is irreducible over K_0. Denote by K_n, $n > 1$ the extension of K_0 generated by a root η_n of the polynomial f_n. Then $[K_n : K_o] = p^n$, $n \in \mathbb{N}$. Hence each of the extensions $K_n|K_{n-1}$ is of degree p. Assume additionally that we have chosen the roots η_n in a way that $\eta_{n+1}^p = \eta_n$, $n \in \mathbb{N}$. Then for every natural number n, the extension $K_n|K_{n-1}$ is generated by a root η_n of the polynomial $X^p - \eta_{n-1} \in K_{n-1}[X]$. As $[K_n : K_{n-1}] = p$, the extension $K_n|K_{n-1}$ is irreducible radical, hence Galois. □

It remains to consider the case of $\operatorname{char} K = 0$ and $p = 2$. We will need the following lemma (cf. Lemma 1.5, Chapter 8 of [17]).

Lemma 4.11. *Assume that L is a field of characteristic distinct from 2. Take an element $a \in L$ and a natural number n. Then the polynomial $X^{2^n} - a$ is irreducible over L if and only if $a \notin F^2$ and $a \notin -4F^4$.*

Lemma 4.12. *Take a field K of characteristic 0 and assume that at least one of the following conditions holds:*

1) K contains a square root i of -1 and admits a Galois extension of degree 2,

2) K admits a tower of two Galois extensions of degree 2.

Then the field admits an infinite tower of Galois extensions of degree 2.

Proof. Assume that the first case holds and take $K_1|K_0$ with $K_0 := K$ to be a Galois extension of degree 2. As $-1 \in K$, the extension is generated by a root η_1 of an irreducible polynomial $X^2 - a \in K[X]$. By the previous lemma, a is not a square in K and $a \notin -4K^4$. Therefore, the same lemma yields that for every $n \in \mathbb{N}$ the polynomial

$$f_n := X^{2^n} - a \tag{4.8}$$

is irreducible over K_0. As in the proof of Lemma 4.10 we construct an infinite tower of Galois extensions of degree 2.

Assume now that case 2) holds and take $E|K$ to be a tower of two Galois extensions of degree 2. Then $E(i)|K(i)$ is either again a tower of two Galois extension of degree 2 or a Galois extension of degree 2. In both cases $K(i)$ satisfies the assumptions of the case 1). Thus, from the first part of the proof we deduce that $K(i)$ admits an infinite tower of Galois extensions of degree 2. Since $K(i)|K$ is either trivial or a Galois of degree 2, the assertion holds already for the field K. □

Assume that K is a field of characteristic 0. Note that if the field admits a Galois extension E of degree 4, then $\mathrm{Gal}(E|K)$ is abelian and thus $E|K$ is a tower of two Galois extensions of degree 2. Also, if a square root i of -1 does not lie in K and the field admits a Galois extension E of degree 2 distinct from $K(i)$, then $K \subseteq K(i) \subseteq E(i)$ form a tower of two Galois extensions of degree 2. Hence, in both cases the field K satisfies the assumptions of part 2) of the above lemma.

We are now able to prove the counterpart of Theorem 4.8 in the case of Kaplansky fields of characteristic 0 and positive residue characteristic.

Theorem 4.13. *Take a Kaplansky field (K, v) of characteristic* 0 *and* $\mathrm{char}\, Kv = p > 0$. *Assume that at least one of the following cases holds:*

1) $p \neq 2$, the field K contains a primitive p-th root ε_p of unity and admits a Galois extension E of degree p such that v extends in a unique way to a valuation of E,

2) $p = 2$, the field K contains a square root i of -1 and admits a Galois extension E of degree 2 *such that v extends in a unique way to a valuation of E,*

3) $p = 2$, the field K admits an extension E which is a tower of two Galois extensions of degree 2 *and v extends in a unique way to a valuation of $E(i)$.*

Then $(E|K, v)$ has defect equal to the degree of the extension and (K, v) admits an infinite tower of Galois extensions of degree and defect p.

Proof. Fix an extension of the valuation v to $\tilde{K}$ and call it again v. If $\mathrm{char}\, Kv \neq 2$ or $i \in K$, then set $K_0 := K$. Otherwise define $K_0 := K(i)$. In the latter case, from point 3) it follows that $E(i)|K(i)$ is a Galois extension of degree 2 or a tower of two such extensions. Furthermore, as the valuation v extends in a unique way from K to $E(i)$, we have that v admits also a unique extension from $K(i)$ to $E(i)$. Hence, in each of the cases K_0 admits a Galois extension E' of degree p, and the valuation v extends in a unique way from K_0 to E'. As $\varepsilon_p \in K_0$, the extension is generated by a root η_1 of a polynomial $f_1 = X^p - a \in K_0[X]$. From Lemmas 4.10 and 4.12 it follows that K_0 admits an infinite tower of Galois extensions $K_n|K_{n-1}$ of degree p. Furthermore, we can choose $K_1 = E'$ and $K_n = K_0(\eta_n)$, where η_n is a root of a polynomial (4.7) or (4.8), depending on $\mathrm{char}\, Kv$, and $\eta_n^p = \eta_{n-1}$, for every $n \geq 2$.

Since by assumption the valuation v extends uniquely to $K_1 = K_0(\eta_1)$, by Lemma 2.12 we have that K_1 is linearly disjoint from K_0^h over K_0, and we obtain that $[K_0^h(\eta_1) : K_0^h] = p$ and f_1 is irreducible over K_0^h. By Lemmas 4.9 and 4.11, this yields that also the polynomials f_n, $n \geq 2$, are irreducible over K_0^h. Hence, $[K_0^h(\eta_n) : K_0^h] = p^n$ and the extensions $K_n|K_0$ and $K_0^h|K_0$ are linearly disjoint. It follows that v has a unique extension from K_{n-1} to K_n for every n.

As in the proof of Theorem 4.8 we deduce that each of the extensions $K_n|K_{n-1}$ is immediate and thus has defect equal to the degree of the extension. Additionally, in the case of $p = 2$ the valuation v admits a unique extension from K to $K(i)$, thus if $i \notin K$, then also the Galois extension $K_0|K$ has defect equal p. This yields that in each of the cases (K, v) admits an infinite tower of Galois extensions of degree and defect p.

Repeating the above arguments, we obtain that also $E|K$ has defect equal to the degree of the extension. $\square$

5. Towers of Artin-Schreier defect extensions of rational function fields

In this section we study the problem of constructing infinite towers of Artin-Schreier defect extensions of rational function fields. We consider various types of valuations of the rational function field and investigate for which it admits an infinite tower of dependent or independent Artin-Schreier defect extensions.

5.1 Constructions of towers of Artin-Schreier defect extensions of rational function fields

Throughout this section we shall work under the following assumptions. We assume that K is a field of positive characteristic p. We take $(K(x)|K, v)$ to be the rational function field equipped with the x-adic valuation v, that is, v is trivial on K and $vx = 1$. The field can be considered as a subfield of the power series field $(K((x^\Gamma)), v_x)$ with the canonical valuation v_x and a group $\Gamma \subseteq \mathbb{Q}$.

In the following constructions we choose the element $y \in K((x^\Gamma))$ to be a pseudo limit of a pseudo Cauchy sequence of transcendental type in some subfield of $K((x^\Gamma))$ containing $K(x)$. The field $K(x, y)$ is equipped with the restriction v of the x-adic valuation v_x of $K((x^\Gamma))$. By Theorem 2.40, such element y is transcendental over $K(x)$. Hence $(K(x, y), v)$ can be viewed as a rational function field with valuation v described by its restriction to $K(x)$ and the choice of y.

Take y to be a power series

$$y = \sum_{i=1}^{\infty} x^{n_i p^{-e_i}} \in K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right), \tag{5.1}$$

where $(e_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers such that

$$e_{i+1} - e_i \geq i$$

for every $i \in \mathbb{N}$, and $(n_i)_{i\in\mathbb{N}}$ is a sequence of integers coprime with p and such that $(n_i p^{-e_i})_{i\in\mathbb{N}}$ is strictly increasing.

Then $(K(x,y),v)$ is a subfield of $\left(K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right), v_x\right)$ defined by the following conditions:

$$\left.\begin{array}{l} K \text{ is a field of characteristic } p>0, \\ K(x)|K \text{ is the rational function field,} \\ y \in K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right) \text{ is of the form (5.1),} \\ v \text{ is the restriction of the valuation } v_x \text{ to the field } K(x,y). \end{array}\right\} \tag{5.2}$$

Consequently we obtain that

$$K(x,y)v \subseteq K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)v = K \text{ and } vK(x,y) \subseteq vK\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right) = \frac{1}{p^\infty}\mathbb{Z}.$$

Moreover, since $K = K(x)v \subseteq K(x,y)v$, it follows that $K(x,y)v = K$. We show that equality holds also for the value groups of $K(x,y)$ and the power series field. For any natural number j we have that

$$z_j := \sum_{i=j+1}^{\infty} x^{n_i p^{e_j - e_i}} = y^{p^{e_j}} - \sum_{i=1}^{j} x^{n_i p^{e_j - e_i}} \in K(x,y),$$

by the assumption on $(e_i)_{i\in\mathbb{N}}$. Thus $vz_j = n_{j+1}p^{e_j - e_{j+1}}$. Since $e_j - e_{j+1} \leq -j$, the element $n_{j+1}p^{-j}$ lies in $vK(x,y)$. As n_{j+1} is coprime with p, also p^{-j} lies in $vK(x,y)$. This yields that $\frac{1}{p^\infty}\mathbb{Z} \subseteq vK(x,y)$ and consequently

$$vK(x,y) = \frac{1}{p^\infty}\mathbb{Z} = v_x K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right).$$

Therefore in particular, $\left(K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right) | K(x,y), v_x\right)$ is an immediate extension.

Consider the subfield $L := K(x^{p^{-i}} \mid i \in \mathbb{N})$ of the power series field. For every natural number n, set

$$a_n := \sum_{i=1}^{n} x^{n_i p^{-e_i}} \in L.$$

Then $(a_n)_{n\in\mathbb{N}}$ satisfies condition (2.12), hence is a pseudo Cauchy sequence in L. Furthermore, y is a pseudo limit of the pseudo Cauchy sequence $(a_n)_{n\in\mathbb{N}}$, since it satisfies inequality (2.13). We show that the sequence is of transcendental type. Suppose the sequence is of algebraic type. Then by Theorem 2.41, there exists an algebraic extension $(L(b)|L,v)$ such the element b is a pseudo limit of the sequence. Thus also the extension $K(x,b)|K(x)$ is algebraic and consequently finite. On the other hand, $\frac{1}{p^\infty}\mathbb{Z} \subseteq vK(x,b)$. Indeed, for every $j \in \mathbb{N}$ consider the value of the element

$$b^{p^{e_j}} - a_j^{p^{e_j}} = b^{p^{e_j}} - \sum_{i=1}^{j} x^{n_i p^{e_j - e_i}} \in K(x,b).$$

Since b is a pseudo limit of $(a_n)_{n\in\mathbb{N}}$, we have

$$v(b^{p^{e_j}} - a_j^{p^{e_j}}) = p^{e_j} v(b - a_j) = p^{e_j} v(a_{j+1} - a_j) = n_{j+1}p^{e_j - e_{j+1}}.$$

As before we deduce that also $p^{-j} \in vK(x,b)$. Therefore, we have that $\frac{1}{p^\infty}\mathbb{Z} \subseteq vK(x,b)$ and $(vK(x,b) : vK(x)) = \infty$, a contradiction to the fundamental inequality. Hence the pseudo Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ is of transcendental type and from Theorem 2.40 it follows that y is transcendental over $K(x)$.

Lemma 5.1. *The ring* $K\left[x, \frac{1}{x}, y\right]$ *is dense in the field* $(K(x,y), v)$.

Proof. Take any element $u \in K(x,y) \backslash K\left[x, \frac{1}{x}, y\right]$. Choose $f, g \in K\left[x, \frac{1}{x}, y\right]$ such that $u = \frac{f}{g}$. Without loss of generality we may assume that $vg = 0$. This can be seen as follows: suppose that $vg = \frac{a}{p^k}$, where a, k are integers and $k \geq 0$. Then

$$u = \frac{x^{-a} f g^{p^k - 1}}{x^{-a} g^{p^k}},$$

with $vx^{-a}g^{p^k} = 0$. Hence we can replace f, g by $x^{-a} f g^{p^k-1}, x^{-a} g^{p^k} \in K[x, \frac{1}{x}, y]$ if necessary to obtain that $vg = 0$.

Therefore, g is of the form

$$\sum_{q \in \frac{1}{p^\infty}\mathbb{Z},\, q \geq 0} a_q x^q$$

with $a_q \in K$ and $a_0 \neq 0$. Set $\tilde{f} := a_0^{-1} f$ and $h = -a_0^{-1}(g - a_0)$. Then $\tilde{f}$ and h are elements of $K\left[x, \frac{1}{x}, y\right]$ such that $vh > 0$ and

$$u = \frac{f}{g} = \frac{\tilde{f}}{1-h}.$$

Since $u \notin K\left[x, \frac{1}{x}, y\right]$, the element h is nonzero.

Take any $\alpha \in vK(x,y)$. As $vh > 0$, there is a natural number N such that $v\tilde{f} + Nvh > \alpha$. Hence for

$$u_N := \tilde{f} \sum_{j=0}^{N-1} h^j \in K\left[x, \frac{1}{x}, y\right]$$

we obtain that

$$v(u - u_N) = v\left(\frac{\tilde{f}}{1-h} - \tilde{f}\sum_{j=0}^{N-1} h^j\right) = v\tilde{f} + v\left(\frac{h^N}{1-h}\right) = v\tilde{f} + Nvh > \alpha.$$

This shows that $K\left[x, \frac{1}{x}, y\right]$ is dense in $K(x,y)$. □

5.1.1 Towers of independent and dependent Artin-Schreier defect extensions of rational function fields

Throughout this section we will assume that $(K(x,y), v)$ is a valued rational function field satisfying the assumptions (5.2), unless stated otherwise. This section is devoted to the proof of Theorem 1.1. We show also that all infinite towers of Artin-Schreier defect extensions constructed in the proof of Theorem 1.1 consist of independent extensions. (cf. Theorem 5.6). Furthermore, we give an example of a valuation of the rational function field $K(x,y)$ under which it admits infinite towers of dependent Artin-Schreier defect extensions (Theorem 5.7).

To prove Theorem 1.1, we will need the following lemmas:

Lemma 5.2. *Assume that K is a field of positive characteristic p and take a rational function field $K(x,y)|K$. For any $a \in K$ take $L_a|K(y+\frac{a}{x})$ to be a separable algebraic extension such that K is relatively algebraically closed in L_a. Then for any two distinct elements $a, b \in K$ the extensions $L_a(x)$ and $L_b(x)$ are linearly disjoint over $K(x,y)$.*

Proof. Take two distinct elements a and b of K. Since $K(y+\frac{a}{x}, y+\frac{b}{x}) = K(x,y)$, the elements $y+\frac{a}{x}$ and $y+\frac{b}{x}$ are algebraically independent over K. Thus the extensions $K(y+\frac{a}{x})|K$ and $K(y+\frac{b}{x})|K$ are algebraically disjoint. Furthermore, $L_a|K(y+\frac{a}{x})$ and $L_b|K(y+\frac{b}{x})$ are algebraic extensions, hence also $L_a|K$ and $L_b|K$ are algebraically disjoint. Since K is relatively algebraically closed in L_a and $L_b|K$ is a separable extension, Lemma 2.5 implies that L_a and L_b are K-linearly disjoint.

Applying Lemma 2.1 to the tower $K \subsetneq K(y+\frac{a}{x}) \subsetneq L_a$ and the extension $L_b|K$ we deduce that L_a and $L_b.K(y+\frac{a}{x}) = L_b(x)$ are linearly disjoint over $K(y+\frac{a}{x})$. Again, since $K(y+\frac{a}{x}) \subseteq K(x,y) \subseteq L_b(x)$, from the same lemma it follows that the extensions $L_b(x)$ and $L_a.K(x,y) = L_a(x)$ are linearly disjoint over $K(x,y)$. □

Lemma 5.3. *Take an element*

$$u = \sum_{i=m}^{\infty} a_i x^{-p^{-i}} \in K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right),$$

where m is an integer and the coefficients a_i lie in some perfect subfield E of K. Then for every natural number n a root η of the polynomial $Y^{p^n} - Y - u$ can be chosen to be of the form

$$\eta = \sum_{i=m+n}^{\infty} c_i x^{-p^{-i}}$$

with $c_i \in E$.

Proof. One can easily check that the element

$$\vartheta = \sum_{i=m+n}^{\infty} c_i x^{-p^{-i}},$$

where $c_i \in E$ are of the form

$$\begin{array}{ll} c_i = a_{i-n}^{p^{-n}} & \text{for } i = m+n, \ldots, m+2n-1, \\ c_i = (a_{i-n} + c_{i-n})^{p^{-n}} & \text{for } i \geq m+2n, \end{array}$$

satisfies $\vartheta^{p^n} - \vartheta = u$. □

Now we are able to give the

Proof of Theorem 1.1: Assume that $(K(x,y), v)$ satisfies the assumptions (5.2) with $n_i = -1$ for every $i \in \mathbb{N}$ in (5.1). Then the element y is of the form

$$y = \sum_{i=1}^{\infty} x^{-p^{-e_i}}.$$

Suppose that E is a perfect subfield of K of cardinality κ. Take any $a \in E$ and consider the field $(K(y+\frac{a}{x}), v)$. Since $v(y+\frac{a}{x}) = -1$ and the valuation v is trivial on K, for any $f = c_n(y+\frac{a}{x})^n + \ldots + c_1(y+\frac{a}{x}) + c_0 \in K[y+\frac{a}{x}]$ with $c_n \neq 0$ we have that $v(f) = -n$. This yields that $vK(y+\frac{a}{x}) = \mathbb{Z}$. We show now that the element y is a pseudo limit of a pseudo Cauchy sequence in the perfect hull $F_a := K(y+\frac{a}{x})^{1/p^\infty}$ of $K(y+\frac{a}{x})$. More precisely, we construct a sequence of elements $b_k \in F_a$ of the form

$$b_k \;=\; y - \sum_{i=m_k}^{\infty} a_i^{(k)} x^{-p^{-i}}, \tag{5.3}$$

where $(m_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers and $a_i^{(k)} \in K$, $a_{m_k}^{(k)} \neq 0$. Then for every k and l such that $k < l$ we obtain $v(b_k - b_l) = -p^{-m_k}$. Hence, $(b_k)_{k\in\mathbb{N}}$ satisfies condition (2.12) and thus is a pseudo Cauchy sequence. Since

$$v(y - b_k) \;=\; -p^{-m_k} \;=\; v(b_{k+1} - b_k)$$

for $k \in \mathbb{N}$, the element y is a pseudo limit of the pseudo Cauchy sequence $(b_k)_{k\in\mathbb{N}}$.

We start the construction with

$$b_1 \;:=\; a^{-p^{-e_1}} \left(y + \frac{a}{x}\right)^{p^{-e_1}} \;=\; x^{-p^{-e_1}} + a^{-p^{-e_1}} y^{p^{-e_1}}.$$

Then, using the fact that $(e_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers we obtain that

$$\begin{aligned} y - b_1 \;&=\; \sum_{i=1}^{\infty} x^{-p^{-e_i}} - x^{-p^{-e_1}} - a^{-p^{-e_1}} \sum_{i=1}^{\infty} x^{-p^{-e_i - e_1}} \\ &=\; \sum_{i=2}^{\infty} x^{-p^{-e_i}} - a^{-p^{-e_1}} \sum_{i=1}^{\infty} x^{-p^{-e_i - e_1}} \;=\; \sum_{i=m_2}^{\infty} a_i^{(2)} x^{-p^{-i}}, \end{aligned}$$

where $m_2 \geq 2$ and $a_{m_2}^{(2)} \neq 0$. Assume that we have constructed $b_1, \ldots, b_j \in F_a$ of the form (5.3) for every $k \leq j$. Set

$$b_{j+1} \;:=\; b_j + a^{-p^{-m_j}} a_{m_j}^{(j)} \left(y + \frac{a}{x}\right)^{p^{-m_j}} \;=\; b_j + a^{-p^{-m_j}} a_{m_j}^{(j)} y^{p^{-m_j}} + a_{m_j}^{(j)} x^{-p^{-m_j}} \in F_a.$$

Then we have

$$\begin{aligned} y - b_{j+1} \;&=\; y - b_j - a^{-p^{-m_j}} a_{m_j}^{(j)} y^{p^{-m_j}} - a_{m_j}^{(j)} x^{p^{-m_j}} \\ &=\; \sum_{i=m_j+1}^{\infty} a_i^{(j)} x^{-p^{-i}} - a^{-p^{-m_j}} a_{m_j}^{(j)} \sum_{i=1}^{\infty} x^{-p^{-e_i - m_j}} \;=\; \sum_{i=m_{j+1}}^{\infty} a_i^{(j+1)} x^{-p^{-i}}, \end{aligned}$$

for some $a_i^{(j+1)} \in K$ and a natural number m_{j+1} such that $a_{m_{j+1}} \neq 0$. Since $e_1 + m_j \geq m_j + 1$ we have that $m_{j+1} > m_j$.

We now use a similar argument as before to show that the pseudo Cauchy sequence $(b_k)_{k\in\mathbb{N}}$ is of transcendental type. Suppose the sequence were of algebraic type. Then, by Theorem 2.41 there would exist an algebraic extension $(F_a(b)|F_a, v)$ with b a pseudo limit of the sequence. Then the element b would be also algebraic over $K(y+\frac{a}{x})$. Thus the extension $K(y+\frac{a}{x},b)|K(y+\frac{a}{x})$ would be finite. On the other hand, $vK(y+\frac{a}{x},b)=\frac{1}{p^\infty}\mathbb{Z}$. Indeed, for any $j\in\mathbb{N}$ consider the value of the element

$$\begin{aligned} u_j &:= b^{p^{e_j}} - \sum_{i=1}^{j}\left(a^{-1}\left(y+\frac{a}{x}-b\right)\right)^{p^{e_j-e_i}} \\ &= b^{p^{e_j}} - \sum_{i=1}^{j} x^{-p^{e_j-e_i}} - \sum_{i=1}^{j}\left(a^{-1}(y-b)\right)^{p^{e_j-e_i}}. \end{aligned}$$

Using Lemma 2.11 for the field $L=K(y+\frac{a}{x})$ we can extend the valuations of $K(y+\frac{a}{x},b)$ and $K(x,y)$ to a valuation of $K(x,y,b)$. Denote this extension again by v. Then

$$vu_j = v\left((b-y)^{p^{e_j}} + y^{p^{e_j}} - \sum_{i=1}^{j} x^{-p^{e_j-e_i}} - \sum_{i=1}^{j}\left(a^{-1}(y-b)\right)^{p^{e_j-e_i}}\right).$$

Since $v(b-b_k)=v(y-b_k)=v(b_{k+1}-b_k)=-p^{-m_k}$, we have that

$$v(b-y) = v(b-b_k+b_k-y) \geq \min\{v(b-b_k),(b_k-y)\} - p^{-m_k} \geq -p^{-k}$$

for every natural number k. Hence $v(b-y)\geq 0$. Moreover,

$$v\left(y^{p^{e_j}} - \sum_{i=1}^{j} x^{-p^{e_j-e_i}}\right) = v\left(\sum_{i=j+1}^{\infty} x^{-p^{e_j-e_i}}\right) = -p^{e_j-e_{j+1}}$$

and thus $vu_j=-p^{e_j-e_{j+1}}$, where $e_j-e_{j+1}\leq -j$. It follows that $p^{-j}\in vK(y+\frac{a}{x},b)$ and the value group is p-divisible. This contradicts the fundamental inequality, since the extension $K(y+\frac{a}{x},b)|K(y+\frac{a}{x})$ was finite and $vK(y+\frac{a}{x})=\mathbb{Z}$. Therefore, the pseudo Cauchy sequence $(b_k)_{k\in\mathbb{N}}$ must be of transcendental type.

Using Lemma 2.43, we conclude that the field F_a^h is relatively algebraically closed in $F_a(y)^h$. Since $K(x,y)^h|K(y+\frac{a}{x})$ is separable and therefore linearly disjoint from $F_a|K(y+\frac{a}{x})$, Lemma 2.1 shows that $K(x,y)^h$ and $K(y+\frac{a}{x})^h.F_a=\left(K(y+\frac{a}{x})^h\right)^{1/p^\infty}$ are linearly disjoint over $K(y+\frac{a}{x})^h$. Hence the extension $K(x,y)^h|K(y+\frac{a}{x})^h$ is separable. We show that from these facts follows that $K(y+\frac{a}{x})^h$ is relatively algebraically closed in $K(x,y)^h$. Assume towards a contradiction that there is an element $z\in K(x,y)^h\setminus K(y+\frac{a}{x})^h$ algebraic over $K(y+\frac{a}{x})^h$. Then z is separable over $K(y+\frac{a}{x})^h$. Since $K(x,y)^h\subseteq F_a(y)^h$ and $F_a^h=K(y+\frac{a}{x})^h.F_a$ is a purely inseparable extension, we obtain that $F_a^h(z)|F_a^h$ is a nontrivial separable-algebraic subextension of $F_a(y)^h|F_a^h$. This contradicts the fact that F_a^h is relatively algebraically closed in $F_a(y)^h$.

Set $\eta_{a,0}:=y+\frac{a}{x}$. By induction on $i\in\mathbb{N}$ choose $\eta_{a,i}$ to be a root of the polynomial

$$Y^p-Y-\eta_{a,i-1}.$$

Since $v(y+\frac{a}{x})=-1$ we obtain $v(\eta_{a,i})=-\frac{1}{p^i}$ for every natural number i. Furthermore, $vK(y+\frac{a}{x})^h=vK(y+\frac{a}{x})=\mathbb{Z}$, hence the extension $K(y+\frac{a}{x})^h(\eta_{a,i})|K(y+\frac{a}{x})^h$ has ramification index at least p^i. On the other hand, the degree of this extension is at most p^i. Thus the fundamental inequality shows that it has degree and ramification index p^i. The same arguments hold for the extension $K(y+\frac{a}{x})(\eta_{a,i})|K(y+\frac{a}{x})$. Therefore,

$$\left[K\left(y+\frac{a}{x},\eta_{a,i}\right):K\left(y+\frac{a}{x}\right)\right] \;=\; \left[K\left(y+\frac{a}{x}\right)^h(\eta_{a,i}):K\left(y+\frac{a}{x}\right)^h\right]$$

and the chain of the extensions $K(y+\frac{a}{x},\eta_{a,i})$ is linearly disjoint from $K(y+\frac{a}{x})^h$ over $K(y+\frac{a}{x})$. Moreover, $K(y+\frac{a}{x})^h(\eta_{a,i}\,|\,i\in\mathbb{N})$ is a separable-algebraic extension of $K(y+\frac{a}{x})^h$. Since $K(y+\frac{a}{x})^h$ is relatively algebraically closed in $K(x,y)^h$, from Lemma 2.4 we deduce that $K(y+\frac{a}{x})^h(\eta_{a,i}\,|\,i\in\mathbb{N})$ and $K(x,y)^h$ are linearly disjoint over $K(y+\frac{a}{x})^h$. Hence, by Lemma 2.1 the extensions $K(y+\frac{a}{x})(\eta_{a,i}\,|\,i\in\mathbb{N})$ and $K(x,y)^h$ are linearly disjoint over $K(y+\frac{a}{x})$. Using again Lemma 2.1 we deduce finally that $K(x,y)(\eta_{a,i}\,|\,i\in\mathbb{N})$ is linearly disjoint from $K(x,y)^h$ over $K(x,y)$. By Lemma 2.12 it follows that the valuation v of $K(x,v)$ admits a unique extension to the field $K(x,y)(\eta_{a,i}\,|\,i\in\mathbb{N})$. Since y is transcendental over $K(y+\frac{a}{x})$ and $K(y+\frac{a}{x},y)=K(x,y)$, the extensions $K(x,y,\eta_{a,j})|K(x,y,\eta_{a,j-1})$ remain nontrivial. We therefore obtain an infinite tower of Arin-Schreier extensions $K(x,y,\eta_{a,j})|K(x,y,\eta_{a,j-1})$ such that for every j the valuation v of $K(x,y,\eta_{a,j-1})$ has unique extension to $K(x,y,\eta_{a,j})$.

Since

$$\eta_{a,0} \;=\; ax^{-1}+\sum_{i=1}^{\infty}x^{-p^{-e_i}},$$

from Lemma 5.3 by induction on i, if follows that each of the Artin-Schreier generators $\eta_{a,j}$ can be chosen to be of the form

$$\eta_{a,j} \;=\; \sum_{i=j}^{\infty}c_{a,j}^{(i)}x^{-p^{-i}}\in E\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right) \tag{5.4}$$

with $c_{a,j}^{(i)}\in E\subseteq K$. Therefore $(K(x,y,\eta_{a,j})|K(x,y),v)$ is a subextension of the immediate extension $\left(K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)|K(x,y),v_x\right)$, hence it is also immediate. Thus $K(x,y,\eta_{a,j})|K(x,y,\eta_{a,j-1})$ is an Artin-Schreier defect extension for every $j\in\mathbb{N}$.

Taking $L_a:=K(y+\frac{a}{x})(\eta_{a,i}\,|\,i\in\mathbb{N})$ in Lemma 5.2 we obtain that for every two distinct $a,b\in E$ the infinite towers of Artin-Schreier defect extensions $L_a(x)=K(x,y)(\eta_{a,i}\,|\,i\in\mathbb{N})$ and $L_b(x)=K(x,y)(\eta_{b,i}\,|\,i\in\mathbb{N})$ are linearly disjoint extensions of $K(x,y)$. □

Remark 5.4. Note that if the field K admits no finite extensions of degree divisible by p, we can prove Theorem 1.1 using the construction from the proof of Theorem 4.8. Indeed, by the additional assumption, K is perfect, and from the above proof we know that for any $a\in K$ a root $\eta_{a,1}$ of the polynomial $Y^p-Y-(y+\frac{a}{x})$ induces an Artin-Schreier extension such that v admits a unique extension from $K(x,y)$ to a valuation of $K(x,y,\eta_{a,1})$. As $vK(x,y)$ is p-divisible and $K(x,y)v=K$ admits no finite extensions of degree divisible by p, $(K(x,y),v)$ is a Kaplansky field. Thus it satisfies the assumptions of Theorem 4.8 with $E:=K(x,y,\eta_{a,1})$. Therefore, we obtain that $K(x,y)$ admits an infinite tower of Artin-Schreier defect extensions, defined by construction (4.6) with $a_1=y+\frac{a}{x}$, $\vartheta_1=\eta_{a,1}$ and $K_0=K(x,y)$.

From the construction it follows that roots of the polynomials f_n also generate separable-algebraic extension L_a of $K(y+\frac{a}{x})$. Thus, the assumptions of Lemma 5.2 are satisfied and the same arguments as in the proof of Theorem 1.1 show that we obtain $|K|$ many infinite towers of Artin-Schreier defect extensions, pairwise linearly disjoint.

The next lemma enables us to approximate the Artin-Schreier roots $\eta_{a,i}$ by elements from the field $K(x,y)$.

Lemma 5.5. *Assume that $n_i = -1$ for every $i \in \mathbb{N}$ in (5.1). For every power series*

$$\eta \;=\; \sum_{i=1}^{\infty} a_i x^{-p^{-i}} \in K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right) \tag{5.5}$$

there is a sequence $(\zeta_n)_{n\in\mathbb{N}}$ of elements of $K(x,y)$ such that $v(\eta - \zeta_n) \ge -\frac{1}{p^{n+1}}$ for every natural number n.

Proof. Assume that η is of the form (5.5). We construct a sequence $(\zeta_n)_{n\in\mathbb{N}}$ of elements of $K(x,y)$ such that

$$\eta - \zeta_n = \sum_{i=n+1}^{\infty} b_i^{(n)} x^{-p^{-i}}, \tag{5.6}$$

where $b_i^{(n)} \in K$ for $i \ge n+1$. Then, in particular, $v(\eta - \zeta_n) \ge -\frac{1}{p^{n+1}}$ for every natural number n. Set

$$\zeta_1 \;:=\; a_1 y^{p^{e_1-1}} \;=\; a_1 \sum_{i=1}^{\infty} x^{-p^{-e_i+e_1-1}} = a_1 x^{-p^{-1}} + \sum_{i=2}^{\infty} a_1 x^{-p^{-e_i+e_1-1}}.$$

By the assumption on $(e_i)_{i\in\mathbb{N}}$ we have $-e_2 + e_1 - 1 \le -2$. Hence,

$$\eta - \zeta_1 \;=\; \sum_{i=2}^{\infty} a_i x^{-p^{-i}} - \sum_{i=2}^{\infty} a_1 x^{-p^{-e_i+e_1-1}} \;=\; \sum_{i=2}^{\infty} b_i^{(1)} x^{-p^{-i}}$$

for some $b_i^{(1)} \in K$. Assume now that ζ_n is an element of $K(x,y)$ such that equation (5.6) holds for some $b_i^{(n)} \in K$. Take $j_{n+1} := e_{n+2} - e_{n+1} - (n+1)$. It is a nonnegative integer, since $e_{n+2} - e_{n+1} \ge (n+1)$. Putting

$$\tilde{\zeta}_{n+1} \;:=\; \left(y^{p^{e_{n+1}}} - \sum_{i=1}^{n+1} x^{-p^{e_{n+1}-e_i}} \right)^{p^{j_{n+1}}} \;\in K(x,y)$$

we obtain that

$$\tilde{\zeta}_{n+1} \;=\; \left(\sum_{i=n+2}^{\infty} x^{-p^{e_{n+1}-e_i}} \right)^{p^{j_{n+1}}} \;=\; x^{-p^{-(n+1)}} + \sum_{i=n+3}^{\infty} x^{-p^{e_{n+1}-e_i+j_{n+1}}}.$$

Set $\zeta_{n+1} := \zeta_n + b_{n+1}^{(n)} \tilde{\zeta}_{n+1}$. Then

$$\begin{aligned} \eta - \zeta_{n+1} \;&=\; \sum_{i=n+1}^{\infty} b_i^{(n)} x^{-p^{-i}} - b_{n+1}^{(n)} x^{-p^{-(n+1)}} - \sum_{i=n+3}^{\infty} b_{n+1}^{(n)} x^{-p^{e_{n+1}-e_i+j_{n+1}}} \\ &=\; \sum_{i=n+2}^{\infty} b_i^{(n)} x^{-p^{-i}} - \sum_{i=n+3}^{\infty} b_{n+1}^{(n)} x^{-p^{e_{n+1}-e_i+j_{n+1}}}. \end{aligned}$$

Since $e_{n+1} - e_{n+2} + j_{n+1} = -(n+1)$ and the sequence $(e_i)_{i\in\mathbb{N}}$ is strictly increasing, we have that $e_{n+1} - e_i + j_{n+1} \le -(n+2)$ for $i \ge n+3$. Therefore,

$$\eta - \zeta_{n+1} \;=\; \sum_{i=n+2}^{\infty} b_i^{(n+1)} x^{-p^{-i}},$$

where $b_i^{(n+1)} \in K$ for $i \ge n+2$.

$\square$

By this simple observation one can easily show that each of the towers of Artin-Schreier defect extensions constructed in the proof of Theorem 1.1 consist of independent extensions. Therefore we obtain the following theorem:

Theorem 5.6. *Take a field K of positive characteristic and assume that it admits a perfect subfield of cardinality κ. Then there is a valuation v of the rational function field $K(x,y)|K$, trivial on K, such that $(K(x,y),v)$ admits κ many pairwise linearly disjoint infinite towers of independent Artin-Schreier defect extensions.*

Proof. Take the valued rational function field $(K(x,y),v)$, the subfield E of K and elements $\eta_{a,j} \in \widetilde{K(x,y)}$ as in the proof of Theorem 1.1. Since for every $a \in E$ and $j \in \mathbb{N}$ the Artin-Schreier extension $K(x,y,\eta_{a,j})|K(x,y,\eta_{a,j-1})$ has nontrivial defect, from Corollary 2.29 it follows that $\operatorname{dist}(\eta_{a,j}, K(x,y,\eta_{a,j-1})) \le 0^-$.

On the other hand, since $\eta_{a,j}$ can be chosen to be of the form (5.4), from the above lemma we deduce that the set $(\frac{1}{p^\infty}\mathbb{Z})^{<0}$ of all negative elements of $vK(x,y)$ is contained in the initial segment $v(\eta_{a,j} - K(x,y,\eta_{a,j-1}))$. Consequently,

$$\operatorname{dist}(\eta_{a,j}, K(x,y,\eta_{a,j-1})) \;=\; 0^-.$$

By Corollary 2.31, this implies that $K(x,y,\eta_{a,j})|K(x,y,\eta_{a,j-1})$ is an independent Artin-Schreier defect extension. $\square$

Due to the importance of the classification of Artin-Schreier defect extensions for the problems related to local uniformization, an interesting question is whether such constructions are also possible with dependent in the place of independent Artin-Schreier defect extensions. The following theorem gives an answer:

Theorem 5.7. *If K is a perfect field of positive characteristic, then there is a valuation v of the rational function field $K(x,y)|K$, trivial on K, such that $(K(x,y),v)$ admits $\max\{|K|,\aleph_0\}$ many pairwise linearly disjoint infinite towers of dependent Artin-Schreier defect extensions.*

For the construction of towers of dependent extensions we use the idea of the transformation of purely inseparable polynomials into Artin-Schreier polynomials (cf. Theorem 4.2).

Lemma 5.8. *Assume that the field K is perfect and the sequence $(n_i p^{-e_i})_{i\in\mathbb{N}}$ of exponents of y is bounded from above. Then for every $a \in K$ and every nonnegative integer r the element $(\frac{y}{x^r} + \frac{a}{x})^{1/p}$ does not lie in the completion of $(K(x,y),v)$.*

Proof. Set

$$\gamma \ := \ \sup\{n_i p^{-e_i} \mid i \in \mathbb{N}\} \in \mathbb{R}.$$

Take $a \in K$ and a nonnegative integer r. We show first that

$$v\left(\left(\frac{y}{x^r} + \frac{a}{x}\right)^{1/p} - f\right) < \frac{1}{p}\gamma$$

for every $f \in K\left[x, \frac{1}{x}, y\right]$. Every such f can be written in a form

$$f \ = \sum_{\substack{-n \leq i \leq m \\ 0 \leq j \leq l}} a_{ij} x^i y^j,$$

for some $a_{ij} \in K$ and $m, n, l \in \mathbb{N}_0$. For every $i \in \{-n, \ldots, m\}$ and $j \in \{0, \ldots, l\}$, set

$$h_{ij} \ := \ a_{ij} x^i y^j \ = \sum_{(m_1, \ldots, m_j) \in \mathbb{N}^j} a_{ij} x^{i + n_{m_1} p^{-e_{m_1}} + \ldots + n_{m_j} p^{-e_{m_j}}}.$$

Take any $j \in \{0, \ldots, l\}$. We claim that there is $N_j \in \mathbb{N}$ such that for every $i \in \{-n \ldots, m\}$, $(m_1, \ldots, m_j) \in \mathbb{N}^j$ and $N \geq N_j$ we have

$$i + n_{m_1} p^{-e_{m_1}} + \ldots + n_{m_j} p^{-e_{m_j}} \ \neq \ n_N p^{-e_N - 1} - p^{-1} r$$

and consequently $n_N p^{-e_N - 1} - p^{-1} r \notin \operatorname{supp} h_{ij}$. Since

$$\left(\frac{y}{x^r} + \frac{a}{x}\right)^{1/p} \ = \ a^{p^{-1}} x^{-p^{-1}} + \sum_{i=1}^{\infty} x^{n_i p^{-e_i - 1} - p^{-1} r},$$

the condition means that $\operatorname{supp}\left(\frac{y}{x^r} + \frac{a}{x}\right)^{1/p}$ and $\operatorname{supp} h_{ij}$ have at most finitely many common elements.

If $j = 0$ then we can choose $N_j = 1$. Let $0 < j \leq l$ and suppose that

$$i + n_{m_1} p^{-e_{m_1}} + \ldots + n_{m_j} p^{-e_{m_j}} = n_N p^{-e_N - 1} - r p^{-1} \tag{5.7}$$

for some natural numbers $m_1, \ldots, m_j$ and N. Without loss of generality we may assume that $m_1 \leq \ldots \leq m_j$. Since n_i is coprime with p for every $i \in \mathbb{N}$, comparing denominators of both sides of the above inequality, we obtain that $e_{m_j} \geq e_N + 1$ and consequently, $e_{m_j} \geq e_{N+1}$.

Set $d := \min\{i \mid 1 \leq i < j \text{ and } m_i = m_j\}$ and $k := j - d + 1$. Multiplying both sides of equation (5.7) by $p^{e_{m_j}}$ we obtain

$$k n_{m_j} \ = \ n_N p^{e_{m_j} - e_N - 1} - r p^{e_{m_j} - 1} - i p^{e_{m_j}} - \sum_{t=1}^{d-1} n_{m_t} p^{e_{m_j} - e_{m_t}}.$$

Moreover, since $e_{N+1} - e_N \geq N$, we have $e_{m_j} - e_N - 1 \geq N - 1$ and

$$e_{m_j} - e_{m_s} \geq e_{m_j} - e_{m_j - 1} \ \geq \ e_{N+1} - e_N \geq N$$

for every $1 \leq s < d$. By assumption, n_{m_j} is coprime with p. Therefore, p^{N-1} divides k. Choose N_j such that $j < p^{N_j - 1}$ and take $N \geq N_j$. Then, since $k \leq j$, we have that k is not divisible by p^{N-1} and consequently, equality (5.7) does not hold for any $m_1, \ldots, m_j \in \mathbb{N}$.

Therefore, setting N_f to be the maximum of the N_j for $0 \leq j \leq l$, we have that $n_N p^{-e_N - 1} - rp^{-1}$ is not an element of supp f for any $N \geq N_f$. Hence,

$$v\left(\left(\frac{y}{x^r} + \frac{a}{x}\right)^{\frac{1}{p}} - f\right) \;\leq\; n_{N_f} p^{-e_{N_f} - 1} - rp^{-1} \;<\; \frac{1}{p}\gamma.$$

Lemma 5.1 yields that for every element u of $K(x,y)$ there is $f \in K\left[x, \frac{1}{x}, y\right]$ such that $v(f-u) > \frac{1}{p}\gamma$. Then

$$v\left(\left(\frac{y}{x^r} + \frac{a}{x}\right)^{\frac{1}{p}} - u\right) \;=\; \min\left\{v\left(\left(\frac{y}{x^r} + \frac{a}{x}\right)^{\frac{1}{p}} - f\right), v(f-u)\right\} < \frac{1}{p}\gamma.$$

It follows that $\left(\frac{y}{x^r} + \frac{a}{x}\right)^{\frac{1}{p}} \notin K(x,y)^c$. $\square$

We will also need the following observation:

Lemma 5.9. *Take a field K of characteristic $p > 0$ and a rational function field $K(x,y)|K$. For any nonnegative integer r take $L_r|K(\frac{y}{x^r})$ to be a (possibly infinite) tower of Artin-Schreier extensions such that K is relatively algebraically closed in L_r. Then for every two distinct nonnegative integers r, s the extensions $L_r(x)|K(x,y)$ and $L_s(x)|K(x,y)$ are linearly disjoint.*

Proof. Take r and s to be two distinct nonnegative integers. Without loss of generality we may assume that $t := r - s > 0$. Elements $\frac{y}{x^s}$ and $\frac{y}{x^r}$ are algebraically independent over K, thus the extensions $K(\frac{y}{x^s})|K$, $K(\frac{y}{x^r})|K$ are algebraically disjoint. Moreover, $L_r|K(\frac{y}{x^r})$ and $L_s|K(\frac{y}{x^s})$ are algebraic extensions, K is relatively algebraically closed in L_r and $L_s|K$ is separable, hence we can deduce as in the proof of Lemma 5.2 that $L_r|K$ and $L_s|K$ are linearly disjoint.

Applying Lemma 2.1 to the tower $K \subseteq K(\frac{y}{x^s}) \subseteq L_s$ and the extension $L_r|K$ we obtain that L_s and $L_r.K(\frac{y}{x^s}) = L_r(x^t)$ are linearly disjoint over $K(\frac{y}{x^s})$. Using again Lemma 2.1 for the tower $K(\frac{y}{x^s}) \subseteq K(\frac{y}{x^r}, \frac{y}{x^s}) \subseteq L_r(x^t)$ and the extension $L_s|K(\frac{y}{x^s})$ we deduce that $L_r(x^t)$ and $L_s.K(\frac{y}{x^r}, \frac{y}{x^s}) = L_s(x^t)$ are linearly disjoint over $K(\frac{y}{x^r}, \frac{y}{x^s}) = K(\frac{y}{x^r}, x^t)$.

We now show that the extensions $L_s(x^t)$ and $K(x,y)$ are linearly disjoint over $K(\frac{y}{x^s}, x^t)$. By assumption, $L_s = \bigcup_{i \in I} L_{s,i}$, where $L_{s,i}|L_{s,i-1}$ is a nontrivial Artin-Schreier extension for every $i \in I$, $L_{s,0} := K(\frac{y}{x^s})$ and $I = \{0, \ldots, n\}$ for some natural number n or $I = \mathbb{N}$. We prove, by induction on i, that each of the the extensions $L'_{s,i} := L_{s,i}(x^t)$ is linearly disjoint from $K(x,y)$ over $K(\frac{y}{x^s}, x^t)$. Write $t = p^k l$, where k is a nonnegative integer and $l \in \mathbb{N}$ is coprime with p. Since x^t is transcendental over L_s, also $L_{s,i}(x^t)|L_{s,i-1}(x^t)$ is a nontrivial Artin-Schreier extension for very $i \in I$. In the case of $i = 1$, $L'_{s,1}|K(\frac{y}{x^s}, x^t)$ is an Arin-Schreier extension, hence Galois extension of degree p. Suppose that $L'_{s,1}$ and $K(x,y)$ were not linearly disjoint over $K(\frac{y}{x^s}, x^t)$. Then by Lemma 2.2, there would exist $a \in L'_{s,1} \cap K(x,y)$ such that $a \notin K(\frac{y}{x^s}, x^t)$. Since $a \in L'_{s,1} \setminus K(\frac{y}{x^s}, x^t)$, we would have $K(\frac{y}{x^s}, x^t, a) = L'_{s,1}$, as $L'_{s,1}|K(\frac{y}{x^s}, x^t)$ is of prime degree. On the other hand $a \in K(x,y)$, hence $K(x,y)|K(\frac{y}{x^s}, x^t)$ would contain a separable subextension of degree p. But $K(x,y) = K(\frac{y}{x^s}, x^t)(x)$ is an irreducible radical extension of degree t of the field $K(\frac{y}{x^s}, x^t)$ and the separable degree of the extension is equal to l, which is not divisible by p, a contradiction.

Take $i \in I$, $i \geq 1$ and assume that $L'_{s,i}$ and $K(x,y)$ are linearly disjoint over $K(\frac{y}{x^s}, x^t)$. Hence, in particular, $L'_{s,i}.K(x,y) = L'_{s,i}(x)$ is an extension of $L'_{s,i}$ of degree t. Suppose that $L'_{s,i+1}$ and $K(x,y)$ were not linearly disjoint over $K(\frac{y}{x^s},x^t)$. Then by Lemma 2.1, also the extensions $L'_{s,i+1}$ and $L'_{s,i}.K(x,y)$ would not be linearly disjoint over $L'_{s,i}$. However, $L'_{s,i}.K(x,y)$ is an irreducible radical extension of $L'_{s,i}$ of degree t and $L'_{s,i+1}|L'_{s,i}$ is a nontrivial Artin-Schreier extension. The same argument as in the case of $i=1$ leads to a contradiction.

Therefore, $L_s(x^t)$ is linearly disjoint from $K(x,y)$ over $K(\frac{y}{x^s},x^t)$. This yields that $L_s(x)|L_s(x^t)$ is an irreducible radical extension of degree t. Since $L_r(x^t)|K(\frac{y}{x^s},x^t)$ and $L_s(x^t)|K(\frac{y}{x^s},x^t)$ are linearly disjoint, $L_s(x^t).L_r(x^t) = L_s(x^t).L_r$ is a separable-algebraic extension of $L_s(x^t)$, being a tower of Artin-Schreier extensions. Repeating the above reasoning we deduce that the extensions $L_s(x)$ and $L_s(x^t).L_r(x^t)$ are linearly disjoint over $L_s(x^t)$. As we have shown, the extensions $L_r(x^t)$ and $L_s(x^t)$ are linearly disjoint over $K(\frac{y}{x^r},x^t)$. Thus from Lemma 2.1 it follows that also $L_r(x^t)$ and $L_s(x)$ are linearly disjoint over $K(\frac{y}{x^r},x^t)$.

Finally, applying Lemma 2.1 to the tower $K(\frac{y}{x^s},x^t)\subseteq K(x,y)\subseteq L_s(x)$ and the extension $L_r(x^t)|K(\frac{y}{x^s},x^t)$, we obtain that $L_s(x)$ and $L_r(x^t).K(x,y)=L_r(x)$ are linearly disjoint over $K(x,y)$. □

Proof of Theorem 5.7: With the general assumptions (5.2) on $(K(x,y),v)$, take K to be a perfect field and suppose that the sequence $(n_ip^{-e_i})_{i\in\mathbb{N}}$ of exponents of y is bounded from above.

Take any $a\in K$. Define $K_{a,0}:=K(x,y)$ and $\vartheta_{a,0}:=y+\frac{a}{x}$. By Lemma 5.8, we have that $\vartheta_{a,0}^{1/p}$ does not lie in the completion $(K^c_{a,0},v)$ of $(K_{a,0},v)$. Since the value group of $K_{a,0}$ is p-divisible and the residue field $K_{a,0}v=K$ is perfect, the polynomial $Y^p-\vartheta_{a,0}$ induces an immediate extension which does not lie in the completion of $K_{a,0}$. Thus, from Theorem 4.2 we obtain that $K_{a,0}$ admits an infinite tower of dependent Artin-Schreier defect extensions $K_{a,n}|K_{a,n-1}$, $n\in\mathbb{N}$.

From the proof of Theorem 4.2 it follows that the tower can be constructed in the following way. By induction on n we choose $\vartheta_{a,n}\in\widetilde{K_{a,0}}$ to be a root of the polynomial

$$f_{a,n} \;=\; Y^p-Y-\frac{1}{b^p_{a,n}}\vartheta_{a,n-1}$$

with $b_{a,n}\in K(y+\frac{a}{x})^\times$ of large enough value. We set $K_{a,n}:=K_{a,n-1}(\vartheta_{a,n})$. Then, for every natural number n we obtain a dependent Artin-Schreier defect extension $K_{a,n}|K_{a,n-1}$.

We thus have an immediate algebraic extension $F_a:=\bigcup_{n\in\mathbb{N}}K_{a,n}$ of $K(x,y)$, which is an infinite tower of dependent Artin-Schreier defect extensions. By the choice of $b_{a,n}$, the field $L_a=K(y+\frac{a}{x})(\vartheta_{a,n}\,|\,n\in\mathbb{N})$ is an algebraic extension of $K(y+\frac{a}{x})$. From Lemma 5.2 we deduce that for two distinct elements $a,b\in K$ the extensions $F_a|K(x,y)$ and $F_b|K(x,y)$ are linearly disjoint. Hence $(K(x,y),v)$ admits $|K|$ many pairwise linearly disjoint infinite towers of dependent Artin-Schreier defect extensions. This proves the theorem in the case of an infinite field K.

If K is finite, then repeating the above construction for $\vartheta_{r,0}=\frac{y}{x^r}$ with $r\in\mathbb{N}\cup\{0\}$, we obtain an immediate extension

$$F_r \;:=\; K(x,y)(\vartheta_{r,i}\,|\,i\in\mathbb{N})$$

of $K(x,y)$ being an infinite tower of dependent Artin-Schreier defect extensions $K(x,y,\vartheta_{r,1},\dots,\vartheta_{r,i})|K(x,y,\vartheta_{r,1},\dots,\vartheta_{r,i-1})$, where $\vartheta_{r,i}$ is a root of the polynomial

$$Y^p-Y-\frac{1}{b^p_{r,i}}\vartheta_{r,i-1}$$

with $b_{r,i} \in K(\frac{y}{x^r})^\times$. From Lemma 5.9 it follows that for any two distinct $r, s \in \mathbb{N} \cup \{0\}$ the extensions $F_r|K(x,y)$ and $F_s|K(x,y)$ are linearly disjoint. Hence $(K(x,y),v)$ admits infinitely many pairwise linearly disjoint infinite towers of dependent Artin-Schreier defect extensions. $\square$

Putting, as in the proofs of the previous theorems, $n_i = -1$ for $i \in \mathbb{N}$ we obtain the series

$$y \;=\; \sum_{i=1}^{\infty} x^{-p^{-e_i}}$$

with the sequence of exponents bounded from above by 0. Theorems 5.6 and 5.7 imply that if K is perfect, then the field $(K(x,y),v)$ admits infinite towers of both types of Artin-Schreier defect extensions. More precisely, from the theorems we obtain:

Corollary 5.10. *Assume that K is a perfect field and*

$$y \;=\; \sum_{i=1}^{\infty} x^{-p^{-e_i}}.$$

Then the valued field $(K(x,y),v)$ admits $|K|$ many pairwise linearly disjoint infinite towers of independent and $\max\{|K|,\aleph_0\}$ *many pairwise linearly disjoint infinite towers of dependent Artin-Schreier defect extensions.*

5.1.2 Valued rational function fields admitting no dependent Artin-Schreier defect extensions.

As in the previous section we take $(K(x,y),v)$ to be a field satisfying the assumptions (5.2). In the foregoing constructions of Artin-Schreier defect extensions we chose y to be a series with a bounded sequence of exponents $n_i p^{-e_i}$. We show that in the case of a perfect field K this assumption is necessary for the existence of dependent Artin-Schreier defect extensions. **Throughout this section, we assume that the sequence $(n_i p^{-e_i})_{i\in\mathbb{N}}$ is unbounded.**

Under this additional condition we obtain:

Lemma 5.11. *For every natural number n the elements $x^{p^{-n}}$ and $y^{1/p}$ lie in the completion of $K(x,y)$.*

Proof. Since the field $\left(K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right), v_x\right)$ is maximal and thus complete, by Proposition 2.49 it contains the completion $(K(x,y)^c, v)$ of $(K(x,y),v)$. We show first that $x^{p^{-N}} \in K(x,y)^c$ for every natural number N. Take $N, j \in \mathbb{N}$ with $j \geq N$. Set $s_j := e_{j+1} - e_j - N$. By assumption, $e_{j+1} - e_j \geq j$, thus s_j is a nonnegative integer. Set

$$\begin{aligned}\widetilde{\xi}_j \;&:=\; \left(y^{p^{e_j}} - \sum_{i=1}^{j} x^{n_i p^{e_j - e_i}}\right)^{p^{s_j}} = \left(\sum_{i=j+1}^{\infty} x^{n_i p^{e_j-e_i}}\right)^{p^{s_j}} \\ &=\; x^{n_{j+1}p^{-N}} + \sum_{i=j+2}^{\infty} x^{n_i p^{s_j+e_j-e_i}}.\end{aligned}$$

Since p and n_{j+1} are coprime, there are integers l and $k > 0$ such that

$$kn_{j+1} + lp^N \;=\; 1.$$

Hence, putting $\xi_j := x^l(\widetilde{\xi_j})^k \in K(x,y)$ we obtain

$$\xi_j \;=\; x^l \left(x^{n_{j+1}p^{-N}} + \sum_{i=j+2}^{\infty} x^{n_i p^{s_j+e_j-e_i}} \right)^k \;=\; x^{l+kn_{j+1}p^{-N}} + \ldots \;=\; x^{p^{-N}} + \ldots,$$

where

$$\begin{aligned} v\left(\xi_j - x^{p^{-N}}\right) \;&=\; l + (k-1)n_{j+1}p^{-N} + n_{j+2}p^{e_j+s_j-e_{j+2}} \\ &=\; p^{-N} - n_{j+1}p^{-N} + n_{j+2}p^{-N-e_{j+2}+e_{j+1}} \\ &=\; p^{-N} + p^{-N+e_{j+1}}\left(-n_{j+1}p^{-e_{j+1}} + n_{j+2}p^{-e_{j+2}}\right). \end{aligned}$$

Note that $-n_{j+1}p^{-e_{j+1}} + n_{j+2}p^{-e_{j+2}} > p^{-j}$ for infinitely many $j \geq N$. Indeed, suppose that $-n_{j+1}p^{-e_{j+1}} + n_{j+2}p^{-e_{j+2}} \leq p^{-j}$ for all but finitely many j. Then the fact that $-n_{j+1}p^{-e_{j+1}} + n_{j+2}p^{-e_{j+2}} > 0$ and the series $\sum\limits_{j=1}^{\infty} p^{-j}$ is convergent contradicts the assumption that the sequence $(n_jp^{-e_j})_{j\in\mathbb{N}}$ is unbounded. Therefore,

$$v\left(\xi_j - x^{p^{-N}}\right) \;\geq\; p^{-N} + p^{-N}p^{e_{j+1}-j} \geq p^{-N} + p^{-N}p^{e_j}$$

for infinitely many $j \geq N$. By assumption $(e_j)_{j\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, hence for arbitrary large elements $\gamma \in \frac{1}{p^\infty}\mathbb{Z}$ we can choose $j \geq N$ such that $v\left(x^{p^{-N}} - \xi_j\right) > \gamma$. Thus by Lemma 2.46 and Corollary 2.50 we obtain that $x^{p^{-N}} \in K(x,y)^c$.

Consider now the element $y^{1/p}$. Since $(n_jp^{-e_j})_{j\in\mathbb{N}}$ is a strictly increasing unbounded sequence and

$$v\left(y^{1/p} - \sum_{i=1}^{k} x^{n_ip^{-e_i-1}}\right) = n_{k+1}p^{-e_{k+1}-1} = p^{-1}(n_{k+1}p^{-e_{k+1}})$$

for every $k \in \mathbb{N}$, the values are cofinal in $\frac{1}{p^\infty}\mathbb{Z}$. By what we have shown,

$$\sum_{i=1}^{k} x^{n_ip^{-e_i-1}} \in K(x,y)^c.$$

Therefore also $y^{1/p}$ lies in the completion of $K(x,y)$.

$\square$

Proposition 5.12. *If the field K is perfect then $(K(x,y),v)$ admits no dependent Artin-Schreier defect extensions.*

Proof. Since K is perfect, by the above lemma $K(x,y)^{1/p} = K(x^{1/p}, y^{1/p}) \subseteq K(x,y)^c$. Then, from Theorem 4.2 if follows that $K(x,y)$ admits no dependent Artin-Schreier defect extensions. $\square$

Nevertheless, the next proposition shows that the field $(K(x,y),v)$ can still admit independent Artin-Schreier defect extensions.

Proposition 5.13. *Assume that the element y is of positive value. If K admits a perfect subfield of cardinality κ then $(K(x,y),v)$ admits κ many pairwise linearly disjoint infinite towers of independent Artin-Schreier defect extensions.*

Proof. Take an extension of the valuation v to the algebraic closure of $K(x,y)$ and denote it again by v. Since y is a pseudo limit of a pseudo Cauchy sequence of transcendental type in $L := K(x^{p^{-i}} \mid i \in \mathbb{N})$, then by Lemma 2.43 the field L^h is relatively algebraically closed in $L(y)^h$. Furthermore, $L^h = K(x)^h.L$ is a purely inseparable extension of $K(x)^h$ and $K(x,y)^h|K(x)^h$ is separable. Hence, using the fact that $K(x,y)^h \subseteq L(y)^h$ we deduce that $K(x)^h$ is relatively algebraically closed in $K(x,y)^h$. Indeed, if there were an element $z \in K(x,y)^h \setminus K(x)^h$ algebraic over $K(x)^h$, then z would be separable over $K(x)^h$. Thus $L^h(z)|L^h$ would be a nontrivial separable-algebraic subextension of $L(y)^h|L^h$, a contradiction.

Assume that E is a perfect subfield of cardinality κ and take $a \in E$. Set $K_{a,0} := K(x,y)$, $\xi_{a,0} := \frac{a}{x}$ and by induction on n choose $\xi_{a,n}$ to be a root of the polynomial

$$Y^p - Y - \xi_{a,n-1}.$$

Set $K_{a,n} := K_{a,n-1}(\xi_{a,n}) = K(x,y,\xi_{a,n})$. Since $v\left(\frac{a}{x}\right) = -1$, for every natural number n we have $v(\xi_{a,n}) = -\frac{1}{p^n}$. Therefore, from the fact that $vK(x)^h = vK(x) = \mathbb{Z}$ we obtain

$$(vK(x)^h(\xi_{a,n}) : vK(x)^h) \geq p^i.$$

On the other hand, the degree of the extension $K(x)^h(\xi_{a,n})|K(x)^h$ is at most p^i. Hence, the fundamental inequality shows that it has degree and ramification index p^i. This implies in particular that the extension $K(x,\xi_{a,n})|K(x)$ is also of degree p^i. Consequently, chain of the extensions $K(x,\xi_{a,i})$ is linearly disjoint from $K(x)^h$ over $K(x)$. Moreover, $K(x)^h(\xi_{a,i} \mid i \in \mathbb{N})$ is a separable-algebraic extension of $K(x)^h$. As we have shown, $K(x)^h$ is relatively algebraically closed in $K(x,y)^h$. Thus from Lemma 2.4 we deduce that the extensions $K(x)^h(\xi_{a,i} \mid i \in \mathbb{N})$ and $K(x,y)^h$ are linearly disjoint over $K(x)^h$. Hence, by Lemma 2.1 the extensions $K(x)(\xi_{a,i} \mid i \in \mathbb{N})|K(x)$ and $K(x,y)^h|K(x)$ are linearly disjoint. Using again Lemma 2.1 we deduce finally that $K(x,y)(\xi_{a,i} \mid i \in \mathbb{N})$ is linearly disjoint from $K(x,y)^h$ over $K(x,y)$. Since y is transcendental over $K(x)$ and $[K(x,\xi_{a,n}) : K(x)] = p^n$, we obtain that also each of the extensions $K_{a,n}|K(x,y)$ has degree p^n and, as we have shown, is linearly disjoint from $K(x,y)^h|K(x,y)$.

As $\xi_{a,0} = ax^{-1}$, from Lemma 5.3 by induction on n it follows that each of the Artin-Schreier generators $\xi_{a,n}$ can be chosen to be of the form

$$\xi_{a,n} \ = \ \sum_{i=n}^{\infty} d_{a,n}^{(i)} x^{-p^{-i}}.$$

Hence $K(x,y) \subseteq K_{a,n} \subseteq K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)$. The fact that $\left(K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)|K(x,y),v_x\right)$ is an immediate extension implies that also $(K_{a,n}|K(x,y),v)$ is immediate.

Set $\xi_0 := y$. Since $vy > 0$, from Lemma 2.27 if follows that there is a root ξ_1 of the polynomial $Y^p - Y - \xi_0$ of positive value. Take a natural number m. Suppose that for

$n \leq m$ we have chosen ξ_n to be a root of the polynomial $Y^p - Y - \xi_{n-1}$ such that $v\xi_n > 0$. Again, by Lemma 2.27 we can choose a root ξ_{m+1} of the polynomial $Y^p - Y - \xi_m$ of positive value.

For every $n \geq 0$ set

$$\eta_{a,n} := \xi_n + \xi_{a,n}.$$

By the additivity of the Artin-Schreier polynomial $Y^p - Y$, for every natural number n the element $\eta_{a,n}$ is a root of the polynomial

$$Y^p - Y - \eta_{a,n-1}.$$

Since $v\eta_{a,0} = -1$, we deduce that $vK(\eta_{a,0}) = \mathbb{Z}$. Furthermore, $v\eta_{a,n} = -\frac{1}{p^n}$ and as in the proof of Theorem 1.1 one can show that $(K(y + \frac{a}{x}, \eta_{a,n})|K(y + \frac{a}{x}, \eta_{a,n-1}), v)$ is an Artin-Schreier extension of ramification index p. Since the element y is transcendental over $K(y+\frac{a}{x})$ and $K(y + \frac{a}{x}, y) = K(x, y)$, the extension $K(x, y, \eta_{a,n})|K(x, y, \eta_{a,n-1})$ remains nontrivial.

We use now the properties of the extensions $K_{a,n}|K_{a,n-1}$ we have constructed to show that the extensions $K(x, y, \eta_{a,n})|K(x, y, \eta_{a,n-1})$ form an infinite tower of independent Artin-Schreier defect extensions. Consider the henselizations of $K(x, y, \xi_n)$ with respect to the fixed extension of the valuation v of $K(x, y)$ to the algebraic closure of $K(x, y)$. Then $K(x, y, \xi_n)^h = K(x, y)^h$ for every natural number n. Indeed, since $v\xi_0 > 0$, by Lemma 2.28 the Artin-Schreier generator ξ_1 lies in the henselization of $K(x, y)$. Thus $K(x, y, \xi_1)^h = K(x, y)^h$. Take any $n \in \mathbb{N}$ and assume that $K(x, y, \xi_n)^h = K(x, y)^h$. By our choice, $v\xi_n > 0$ hence using again Lemma 2.28 we deduce that

$$K(x, y, \xi_{n+1})^h = K(x, y, \xi_n)^h(\xi_{n+1}) = K(x, y, \xi_n)^h = K(x, y)^h.$$

Set $L_{a,0} := K(x, y)$ and $L_{a,n} := L_{a,n-1}(\eta_{a,n})$ for every $n \in \mathbb{N}$. We claim that for every natural number n the extension $L_{a,n}|L_{a,n-1}$ is linearly disjoint from $L^h_{a,n-1}|L_{a,n-1}$. Since $\xi_1 \in K(x, y)^h$ we obtain that

$$L^h_{a,1} = K(x, y)^h(\xi_1 + \xi_{a,1}) = K(x, y)^h(\xi_{a,1}) = K^h_{a,1}.$$

Take $n \in \mathbb{N}$ and assume that $L^h_{a,n} = K^h_{a,n}$. As we have shown $\xi_{n+1} \in K(x, y)^h \subseteq L^h_{a,n}$, hence

$$L^h_{a,n+1} = L^h_{a,n}(\xi_{n+1} + \xi_{a,n+1}) = L^h_{a,n}(\xi_{a,n+1}) = K^h_{a,n}(\xi_{a,n+1}) = K^h_{a,n+1}.$$

Therefore, by induction we obtain the equality $L^h_{a,n} = K^h_{a,n}$ for every $n \in \mathbb{N}$.

Suppose that the Artin-Schreier extension $L_{a,n}|L_{a,n-1}$ were not linearly disjoint from $L^h_{a,n-1}|L_{a,n-1}$ for some natural number n. Then $\eta_{a,n} \in L^h_{a,n-1}$. Since $\eta_{a,n} = \xi_n + \xi_{a,n}$ and $\xi_n \in K(x, y)^h \subseteq L^h_{a,n-1}$, we would have that $\xi_{a,n} \in L^h_{a,n-1} = K^h_{a,n-1}$. On the other hand, we have proved that the valuation v of $K_{a,n-1}$ has a unique extension to the field $K_{a,n} = K_{a,n-1}(\xi_{a,n})$, a contradiction with Lemma 2.12. Therefore, again by Lemma 2.12, the valuation v of $L_{a,n-1}$ has a unique extension to the field $L_{a,n}$ for every $n \in \mathbb{N}$.

Since the value group of $K(x, y)$ is p-divisible, each of the extensions $L_{a,n}|L_{a,n-1}$ has ramification index equal to 1. Take a natural number n. Using the fact that the henselization is an immediate field extension, we obtain that

$$L_{a,n}v = L^h_{a,n}v = K^h_{a,n}v = K_{a,n}v = K.$$

Therefore $L_{a,n}v = L_{a,n-1}v$ and the extension $L_{a,n}|L_{a,n-1}$ is immediate. Consequently, each of the Artin-Schreier extensions $L_{a,n}|L_{a,n-1}$ has nontrivial defect.

Take a natural number n. From Corollary 2.29 it follows that $\operatorname{dist}(\eta_{a,n}, L_{a,n-1}) \leq 0^-$. Note that

$$\eta_{a,n} = \sum_{i=n}^{\infty} d_{a,n}^{(i)} x^{-p^{-i}} + \xi_n$$

with $v\xi_n > 0$. By Lemma 5.11, for every $j \geq n$ we have that

$$\sum_{i=n}^{j} d_{a,n}^{(i)} x^{-p^{-i}} \in K(x,y)^c.$$

Thus there is $u_{n,j} \in K(x,y)$ such that $v\left(\sum\limits_{i=n}^{j} d_{a,n}^{(i)} x^{-p^{-i}} - u_{n,j}\right) > 0$. Then

$$\begin{aligned} v(\eta_{a,n} - u_{n,j}) &= v\left(\left(\sum_{i=n}^{j} d_{a,n}^{(i)} x^{-p^{-i}} - u_{n,j} + \xi_n\right) + \sum_{i=j+1}^{\infty} d_{a,n}^{(i)} x^{-p^{-i}}\right) \\ &= v\left(\sum_{i=j+1}^{\infty} d_{a,n}^{(i)} x^{-p^{-i}}\right) = -p^{-(j+1)}. \end{aligned}$$

Thus the set of values $v(\eta_{a,n}-u_{n,j})$ is cofinal in $\left(\frac{1}{p^\infty}\mathbb{Z}\right)^{<0}$. Hence $\operatorname{dist}(\eta_{a,n}, L_{a,n-1}) = 0^-$. From Corollary 2.31 it follows that $L_{a,n}|L_{a,n-1}$ is an independent Artin-Schreier defect extension.

Using Lemma 5.2 we obtain that for any two distinct elements $a, b \in E$ the extensions $\bigcup\limits_{n\in\mathbb{N}} L_{a,n}|K(x,y)$ and $\bigcup\limits_{n\in\mathbb{N}} L_{b,n}|K(x,y)$ are linearly disjoint. □

Assume that the field K is perfect and $vy > 0$. Then from Proposition 5.12 it follows that $(K(x,y),v)$ admits no dependent Artin-Schreier defect extensions. However, by Proposition 5.13 the field admits $|K|$ many pairwise linearly disjoint infinite towers of independent Artin-Schreier defect extensions. In Chapter 7 we will give an example of a rank 1 valuation of a rational function field $L(x,y)$ of positive characteristic p, with p-divisible value group and perfect residue field, admitting neither dependent not independent Artin-Schreier defect extensions. However, the constructed valuation will not be trivial on the field L. An open question is how to construct a rank 1 valuation w on $L(x,y)$, trivial on L, such that $(L(x,y),w)$ admits no Artin-Schreier defect extensions, or more generally, is a defectless field.

Note also that the above construction of towers $\bigcup_{n\in\mathbb{N}} L_{a,n}|K(x,y)$ of Artin-Schreier defect extensions does not depend on the fact that the sequence $(n_i p^{-e_i})_{i\in\mathbb{N}}$ of exponents of y is unbounded. We use the assumption only to show that all of the Artin-Schreier defect extensions in the towers are independent. Hence, we obtain the following corollary to the proof of Proposition 5.13:

Corollary 5.14. *Take a valued rational function field $(K(x,y),v)$ satisfying conditions (5.2). Assume that the element y is of positive value, but not necessarily that its exponents are unbounded. If K admits a perfect subfield of cardinality κ, then $(K(x,y),v)$ admits κ many pairwise linearly disjoint infinite towers of Artin-Schreier defect extensions.*

Therefore, the construction of towers of Artin-Schreier defect extensions as in the proof of Proposition 5.13 gives us another proof of Theorem 1.1.

5.2 p-elementary extensions of rational function fields

Take a field L of characteristic $p > 0$. If a polynomial $f \in L[X]$ is of the form $f = \mathcal{A} + c$, where $\mathcal{A} \in L[X]$ is an additive polynomial and c is a constant in L, then f is called a **p-polynomial**. An important example of p-polynomials are Artin-Schreier polynomials $X^p - X - c$ with $c \in L$. We have already mentioned, that if an Artin-Schreier polynomial is irreducible, any of its roots generates a Galois extension of degree p.

We consider now a more general class of Galois extensions of degree a power of p. An algebraic extension $L'|L$ is called a **p-elementary extension** if it is a finite Galois extension and its Galois group $\mathrm{Gal}(L'|L)$ is an elementary-abelian p-group, that is, $\mathrm{Gal}(L'|L)$ is an abelian p-group such that every nonzero element of the group has order p; if $[L' : L] = p^n$, then the group is a direct sum of n cyclic subgroups of order p. Hence $L'|L$ is a compositum of n many parallel Galois extensions of degree p, thus a tower of Artin-Schreier extensions. Every Artin-Schreier extension is generated by a root of a p-polynomial of degree p over L. More generally, one can show that every p-elementary extension is generated by a root of some irreducible p-polynomial (cf. Theorem 34 of [23]).

Take any natural number n and assume that $\mathbb{F}_{p^n} \subseteq L$. Consider the polynomial

$$f \;=\; X^{p^n} - X - a \in L[X].$$

Note that for $n = 1$ we obtain an Artin-Schreier polynomial. Assume that f is irreducible over L and consider the extension $L(\vartheta)|L$ generated by a root ϑ of f. Since the elements $\vartheta + c$ with $c \in \mathbb{F}_{p^n}$ form the set of all roots of f, the extension $L(\vartheta)|L$ is normal and separable, hence Galois. Furthermore,

$$\mathrm{Gal}(L(\vartheta)|L) = \{\sigma_c \,|\, c \in \mathbb{F}_{p^n}\},$$

where $\sigma_c(\vartheta) = \vartheta + c$. Thus the Galois group of $L(\vartheta)|L$ is an elementary-abelian p-group. Consequently, $L(\vartheta)|L$ is a p-elementary extension of degree p^n. As in the case of Artin-Schreier extensions, for the extensions of valued fields generated by roots of polynomials $X^{p^n} - X - a$ we obtain the following facts.

Lemma 5.15. *Assume that (L, v) is a valued field and ϑ a root of the polynomial $f = X^{p^n} - X - a \in L[X]$. If $va \le 0$, then $v\vartheta = \frac{1}{p^n} va$. If $va > 0$, then exactly one of the conjugates of ϑ has value va and the other roots of f have value* 0.

Proof. If $v\vartheta \neq 0$ then $v\vartheta^{p^n} \neq v\vartheta$. Therefore from equality $\vartheta^{p^n} - \vartheta = a$ it follows that $va = \min\{v\vartheta, p^n v\vartheta\}$. Thus, if $va = 0$, we have that $v\vartheta = 0$. Assume that $va < 0$. Then also $v\vartheta < 0$ and consequently $va = p^n v\vartheta$. This yields that $v\vartheta = \frac{1}{p^n} va$. Note that

$$a \;=\; \vartheta^{p^n} - \vartheta \;=\; \prod_{c \in \mathbb{F}_{p^n}} (\vartheta + c).$$

Thus, if $va > 0$, there must be a conjugative $\vartheta' = \vartheta + c$ of ϑ of positive value. Since $v(\vartheta' + d) = 0$ for every $d \in \mathbb{F}_{p^n}^*$, the other roots of f have value 0. □

Lemma 5.16. *Assume that (L, v) is a valued field of positive characteristic p and $\mathbb{F}_{p^n} \subseteq Lv$ for some $n \in \mathbb{N}$. Take a polynomial $f = X^{p^n} - X - a \in L[X]$. If $va > 0$ or $va = 0$ and $X^{p^n} - X - av$ has a root in Lv then every root of f lies in the henselization of L (with respect to every extension of v to $\tilde{L}$).*

Proof. If $va > 0$, then the polynomial $X^{p^n} - X$ is the reduction of f modulo v. Since $\mathbb{F}_{p^n} \subseteq Lv$, the polynomial $X^{p^n} - X$ splits into linear factors in Lv. Assume that $va = 0$ and $X^{p^n} - X - av$ has a root ϑ in Lv. Since all other roots of the polynomial are of the form $\vartheta + c$ with $c \in \mathbb{F}_{p^n}$, also in this case the reduction of f modulo v splits into linear factors in Lv. Therefore, in both cases it follows from Hensel's Lemma that $X^{p^n} - X - a$ splits into linear factors in every henselization of (L, v). □

The similarities between the Artin-Schreier extensions and the more general class of p-elementary extensions generated by roots of polynomials $X^{p^n} - X - a$ give rise to the question if we can use the techniques from Theorems 1.1 and 5.7 to construct towers of p-elementary extensions of degree and defect p^n. The next theorem shows that the constructions from the proof of Theorem 1.1 can indeed be generalized in this way.

Theorem 5.17. *Take a field K of positive characteristic p and a natural number n such that $\mathbb{F}_{p^n} \subseteq K$. Assume that K contains a perfect subfield of cardinality κ. Then there is a valuation v of the rational function field $K(x, y)|K$, trivial on K, such that $(K(x, y), v)$ admits κ many pairwise linearly disjoint infinite towers of p-elementary extensions of degree and defect p^n.*

Proof. Take $(K(x, y), v)$ to be the valued rational function field defined as in the proof of Theorem 1.1. Namely, we assume that $(K(x, y), v)$ satisfies assumptions (5.2) with $n_i = -1$ for every $i \in \mathbb{N}$ in (5.1). Then y is of the form

$$y = \sum_{i=1}^{\infty} x^{-p^{-e_i}}.$$

Take a natural number n such that $\mathbb{F}_{p^n} \subseteq K$. Suppose that E is a perfect subfield of K of cardinality κ and choose $a \in E$. From the proof of Theorem 1.1 we know that y is a pseudo limit of a pseudo Cauchy sequence of transcendental type in the perfect hull $F_a = K(y + \frac{a}{x})^{1/p^\infty}$ of $K(y + \frac{a}{x})$ and consequently, $K(y + \frac{a}{x})^h$ is relatively algebraically closed in $K(x, y)^h$.

Set $\eta_{a,0}^{(n)} := y + \frac{a}{x}$. By induction on $i \in \mathbb{N}$ choose $\eta_{a,i}^{(n)}$ to be a root of the polynomial

$$Y^{p^n} - Y - \eta_{a,i-1}^{(n)}.$$

Since $v(y + \frac{a}{x}) = -1$ we obtain that $vK(y + \frac{a}{x})^h = vK(y + \frac{a}{x}) = \mathbb{Z}$ and $v\eta_{a,i}^{(n)} = -\frac{1}{p^{in}}$ for every $i \in \mathbb{N}$. Therefore, using arguments similar to those in the proof of Theorem 1.1 we deduce that

$$\left[K\left(y + \frac{a}{x}\right)^h \left(\eta_{a,i}^{(n)}\right) : K\left(y + \frac{a}{x}\right)^h\right] = p^{in}$$

and $K(x, y, \eta_{a,i}^{(n)})|K(x, y, \eta_{a,i-1}^{(n)})$ is an extension of degree p^n such that the valuation v of $K(x, y, \eta_{a,i-1}^{(n)})$ has a unique extension to the field $K(x, y, \eta_{a,i}^{(n)})$ for every $i \in \mathbb{N}$. Since $[K(x, y, \eta_{a,i}^{(n)}) : K(x, y, \eta_{a,i-1}^{(n)})] = p^n$, the polynomial $Y^{p^n} - Y - \eta_{a,i-1}^{(n)}$ is irreducible. Thus, $K(x, y, \eta_{a,i}^{(n)})|K(x, y, \eta_{a,i-1}^{(n)})$ is a p-elementary extension for every $i \in \mathbb{N}$.

Note that the element $\eta_{a,0}^{(n)}$ is of the form

$$\eta_{a,0}^{(n)} = ax^{-1} + \sum_{i=1}^{\infty} x^{-p^{-e_i}} \in E\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right),$$

thus from Lemma 5.3 by induction on i, it follows that each of the generators $\eta_{a,j}^{(n)}$ can be chosen to be of the form

$$\eta_{a,j}^{(n)} = \sum_{i=nj}^{\infty} c_{a,j}^{(n)}(i) x^{-p^{-i}}$$

with $c_{a,j}^{(n)}(i) \in E \subseteq K$. Therefore $(K(x,y,\eta_{a,j}^{(n)})|K(x,y),v)$, as a subextension of the immediate extension $(K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)|K(x,y),v_x)$, is also immediate. Hence we obtain an infinite tower of p-elementary extensions $K(x,y,\eta_{a,j}^{(n)})|K(x,y,\eta_{a,j-1}^{(n)})$ of degree and defect p^n.

Finally, from Lemma 5.2 it follows that for every two distinct $a,b \in E$ the extensions $K(x,y)(\eta_{a,i}^{(n)} \mid i \in \mathbb{N})|K(x,y)$ and $K(x,y)(\eta_{b,i}^{(n)} \mid i \in \mathbb{N})|K(x,y)$ are linearly disjoint. □

Note that since every p-elementary extension is a tower of Artin-Schreier extensions, $K(x,y)(\eta_{a,i}^{(n)} \mid i \in \mathbb{N})|K(x,y)$ is in particular an infinite tower of Artin-Schreier defect extensions.

Consider now the methods used in the proof of Theorem 5.7, or more generally in the proof of Theorem 4.2, to show the existence of infinite towers of dependent Artin-Schreier defect extensions. The constructions of dependent extensions are based on the deformation of purely inseparable polynomials into Artin-Schreier polynomials. Take a valued field (L,v) of characteristic $p>0$. From Theorem 4.1 we know that a suitable deformation of a purely inseparable polynomial $X^p - \eta^p \in L[X]$ with $\eta \notin L^c$ into a polynomial $X^p - X - (\frac{\eta}{b})^p$ yields a dependent Artin-Schreier defect extension generated by a root ϑ of the Artin-Schreier polynomial. The proof of Theorem 4.1 shows that the fact that $(L(\vartheta)|L,v)$ has nontrivial defect follows from the relation $\vartheta \sim_L \frac{\eta}{b}$ between the generators of the purely inseparable and the Artin-Schreier extension.

The following lemma shows that also a suitable deformation of purely inseparable polynomials of higher degrees into separable p-polynomials induces a relation between roots of the two polynomials. We obtain it by replacing in the first part of the proof of Theorem 4.1 the prime p by its power p^n.

Lemma 5.18. *Assume that (L,v) is a valued field of positive characteristic p. Take an immediate purely inseparable extension $L(\eta)|L$ of degree at most p^n such that η does not lie in the completion (L^c,v) of (L,v). Then for every $b \in L^\times$ such that*

$$(p^n-1)vb + v\eta > p^n dist(\eta, L), \tag{5.8}$$

a root ϑ of the polynomial

$$g_b = X^{p^n} - X - \left(\frac{\eta}{b}\right)^{p^n}$$

satisfies the condition

$$\vartheta \sim_L \frac{\eta}{b}. \tag{5.9}$$

Nevertheless, the next example shows that in the case of extensions of degree higher than the characteristic of the field, the relation (5.9) does not suffice to prove that $L(\vartheta)|L$ is disjoint from the henselization. Therefore, the direct generalization of Theorem 4.1 to the purely inseparable extensions of higher degrees is not possible.

Example 5.19. Take a perfect field K of characteristic $p > 0$ and consider the valued rational function field $(K(x,y),v)$ defined by the conditions (5.2). Assume that $n_i = -1$ for every $i \in \mathbb{N}$ in (5.1). Then

$$y \;=\; \sum_{i=1}^{\infty} x^{-p^{-e_i}}.$$

Take any extension of v to $\widetilde{K(x,y)}$ and denote by $K(x,y)^h$ the henselization of $K(x,y)$ with respect to v.

We construct a p-elementary extension $K(x,y,\vartheta)|K(x,y)$ of degree p^2 generated by a root ϑ of some polynomial $X^{p^2} - X - z$ such that $K(x,y,\vartheta)|K(x,y)$ is not linearly disjoint from $K(x,y)^h|K(x,y)$, even though it is derived from an immediate purely inseparable extension of degree p^2 not contained in the completion $(K(x,y)^c, v)$ of $(K(x,y),v)$.

Set $d := y^{-p^{e_1}} \in K(y)$ and consider the Artin-Schreier polynomial

$$h_1 \;:=\; Y^p - Y - \frac{1}{d^p}y.$$

Since the value group of $K(x,y)$ is p-divisible and the residue field $K(x,y)v = K$ is perfect, by Lemma 5.8 the polynomial $Y^p - y$ induces an immediate purely inseparable extension, which does not lie in the completion $K(x,y)^c$ of $K(x,y)$. Moreover, from the proof of Lemma 5.8 it follows that $\operatorname{dist}(y^{1/p}, K(x,y)) \le 0^-$. Since $vy = -\frac{1}{p^{e_1}}$ and $vd = 1$, we have that

$$(p-1)vd + vy \;>\; 0 \;\ge\; p \operatorname{dist}(y^{1/p}, K(x,y)).$$

Thus the element d satisfies the condition (4.1) of Theorem 4.1 and consequently a root η of the polynomial h_1 generates a dependent Artin-Schreier defect extension $K(x,y,\eta)|K(x,y)$. Note that η is a root of the polynomial

$$\tilde{h}_1 \;:=\; Y^{p^2} - Y - \frac{1}{d^p}y - \frac{1}{d^{p^2}}y^p.$$

Set $n = p^{e_1+2} + 1$ and take a p-polynomial

$$h_2 \;:=\; Y^{p^2} - Y - \frac{1}{d^{p^2}}y^{-n}.$$

Since $v\frac{1}{d^{p^2}}y^{-n} = -p^2 + \frac{n}{p^{e_1}} = \frac{1}{p^{e_1}} > 0$ and $\mathbb{F}_{p^2} \subseteq K = K(x,y)v$, from Lemma 5.16 it follows that a root ϑ' of the polynomial h_2 generates an extension $K(x,y,\vartheta')$ of $K(x,y)$ contained in the henselization $K(x,y)^h$ of $K(x,y)$.

Define $\vartheta = \eta + \vartheta' \in \widetilde{K(x,y)}$. By the additivity of the polynomial $Y^{p^2} - Y$, the element ϑ is a root of the polynomial

$$f \;:=\; Y^{p^2} - Y - \frac{1}{d^p}y - \frac{1}{d^{p^2}}y^p - \frac{1}{d^{p^2}}y^{-n}.$$

We show that the extension $K(x,y,\vartheta)|K(x,y)$ is of degree p^2. Consider the rational function field $K(y)|K$ with the y-adic valuation w. Then

$$w\frac{1}{d^{p^2}}y^{-n} \;=\; p^{e_1+2} - n \;=\; -1,$$

and $w\frac{1}{d^p}y$, $w\frac{1}{d^{p^2}}y^p > 0$. Hence

$$w\left(\frac{1}{d^p}y + \frac{1}{d^{p^2}}y^p + \frac{1}{d^{p^2}}y^{-n}\right) \;=\; -1 < 0$$

and consequently $w\vartheta = -\frac{1}{p^2}$, by Lemma 5.15. Therefore $(wK(y,\vartheta) : wK(y)) \geq p^2$. On the other hand $[K(y,\vartheta) : K(y)] \leq p^2$, hence by the fundamental inequality we obtain that $K(y,\vartheta)|K(y)$ is of degree p^2. The element x is transcendental over $K(y)$, thus we have also that

$$[K(x,y,\vartheta) : K(x,y)] = p^2.$$

Since $K(x,y,\eta)|K(x,y)$ is an Artin-Schreier defect extension, the valuation v of $K(x,y)$ admits a unique extension to $K(x,y,\eta)$. From Lemma 2.12 it follows that $K(x,y,\eta)|K(x,y)$ is linearly disjoint from $K(x,y)^h|K(x,y)$. As $\vartheta' \in K(x,y)^h$, from the equality $\vartheta = \eta + \vartheta'$ we deduce that

$$[K(x,y)^h(\vartheta) : K(x,y)^h] \;=\; [K(x,y)^h(\eta) : K(x,y)^h] \;= p < p^2 =\; [K(x,y,\vartheta) : K(x,y)].$$

Therefore $K(x,y,\vartheta)|K(x,y)$ is not linearly disjoint from $K(x,y)^h|K(x,y)$.

On the other hand, the polynomial f can be derived by a deformation of a purely inseparable polynomial inducing an immediate extension which does not lie in the completion of $K(x,y)$ in the following way. Define

$$\xi \;:=\; d^{1-p^{-1}}y^{p^{-2}} - y^{p^{-1}} - y^{-np^{-2}}.$$

The value group of $K(x,y)$ is p-divisible and the residue field of $K(x,y)$ is perfect, therefore $K(x,y,\xi)|K(x,y)$ is an immediate purely inseparable extension of degree p^2. Since $\operatorname{dist}(y^{1/p}, K(x,y)) \leq 0^-$, we have that $v(y^{1/p} - K(x,y)) < 0$. The values $vd^{1-p^{-1}}y^{p^{-2}}$ and $vy^{-np^{-2}}$ are positive, thus also $v(\xi - K(x,y)) \;<\; 0$. It follows that

$$\operatorname{dist}(\xi, K(x,y)) \leq 0^-$$

and consequently ξ does not lie in the completion of $K(x,y)$. Note that

$$v\xi = vy^{1/p} = -\frac{1}{p^{e_1+1}}$$

and thus

$$(p^2-1)vd + v\xi \;=\; p^2 - 1 - \frac{1}{p^{e_1+1}} \;>\; 0 \;\geq\; p^2\operatorname{dist}(\xi, K(x,y)).$$

Therefore, from Lemma 5.18 it follows that every root of the polynomial

$$Y^{p^2} - Y - \left(\frac{\xi}{d}\right)^{p^2} \;=\; Y^{p^2} - Y - \frac{1}{d^p}y - \frac{1}{d^{p^2}}y^p - \frac{1}{d^{p^2}}y^{-n} \;=\; f$$

is in relation $\sim_{K(x,y)}$ with $\frac{\xi}{d}$. Thus $\vartheta \sim_{K(x,y)} \frac{\xi}{d}$.

Hence by the deformation of the purely inseparable polynomial $Y^{p^2} - \xi^{p^2}$ we obtain the p-polynomial $f = Y^{p^2} - Y - (\frac{\xi}{d})^{p^2}$ generating the extension $K(x,y,\vartheta)|K(x,y)$ which is not linearly disjoint from the henselization of $K(x,y)$. This is because while only the element $\frac{1}{d^{p^2}}y^p$ matters for the approximation of ξ and hence of ϑ, the elements $\frac{1}{d^{p^2}}y^{-n}$ and $\frac{1}{d^p}y$ are responsible for the extensions $K(x,y,\xi)|K(x,y)$ and $K(x,y,\vartheta)|K(x,y)$ having degree p^2. $\diamond$

As we have seen, the deformation of a polynomial inducing an immediate purely inseparable extension of prime degree not contained in the completion of the field leads to an Artin-Schreier defect extension, because of the relation between the generators of the purely inseparable and the Artin-Schreier extensions. From that relation, by Theorem 2.23, it follows in particular that the Artin-Schreier extension is disjoint from the henselization (see proof of Theorem 4.1). Example 5.19 shows that for extensions of higher degrees this implication does not hold. Note that this example also shows that we cannot generalize Theorem 2.23 by replacing the condition "a lies in K^h" by "$K(a)|K$ is not linearly disjoint from $K^h|K$". Indeed, as we have shown, $\vartheta \sim_{K(x,y)} \frac{\xi}{d}$ and $K(x,y,\vartheta)|K(x,y)$ is not linearly disjoint from $K(x,y)^h|K(x,y)$. On the other hand, the element $\frac{\xi}{d}$ is purely inseparable over $K(x,y)$. It follows that $K(x,y,\frac{\xi}{d})|K(x,y)$ is linearly disjoint from $K(x,y)^h|K(x,y)$, as the henselization is a separable extension of $K(x,y)$.

6. Algebraic independence of elements in immediate extensions of valued fields

This chapter is devoted to the proof of Theorem 1.2 and its applications. The theorem gives several criteria for a valued field that guarantee that all of its maximal immediate extensions have infinite transcendence degree. We apply the criteria to the question which algebraic extensions of a maximal valued field are again maximal. We give further examples of applications of Theorem 1.2 to the problems related to valued rational function fields. In the case of valued fields of infinite p-degree, Theorem 1.2 enables us also to prove the existence of valued fields which admit an algebraic maximal immediate extension as well as one of infinite transcendence degree.

6.1 Valued fields admitting immediate extensions of infinite transcendence degree

In this section we give the proof of Theorem 1.2. Our first goal is a basic independence lemma. For the proof we will need a Taylor expansion that works in all characteristics.

Take a polynomial $f \in K[X]$ of degree n. We define the **i-th formal derivative of f** as

$$f_i(X) := \sum_{j=i}^{n} \binom{j}{i} c_j X^{j-i} = \sum_{j=0}^{n-i} \binom{j+i}{i} c_{j+i} X^j . \qquad (6.1)$$

Then regardless of the characteristic of K, we have the **Taylor expansion** of f at c in the following form:

$$f(X) = \sum_{i=0}^{n} f_i(c)(X-c)^i . \qquad (6.2)$$

Take $i \in \mathbb{N}$, any field K and a polynomial $f \in K[X_1, \ldots, X_i]$. With respect to the lexicographic order on $\mathbb{Z}^i$, let $(\mu_1, \ldots, \mu_i)$ be maximal with the property that the coefficient of $X_1^{\mu_1} \cdots X_i^{\mu_i}$ in f is nonzero. Then define c_f to be this coefficient and call $(\mu_1, \ldots, \mu_i)$ the **crucial exponent** of f.

For our basic independence lemma, we consider the following situation. We choose a function

$$\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

such that

$$\varphi(k,\ell) > \max\{k,\ell\} \quad \text{and} \quad \varphi(k+1,\ell) > \varphi(k,\ell) \quad \text{for all } k,\ell \in \mathbb{N},$$

and for each $i \in \mathbb{N}$ a strictly increasing sequence $(E_i(k))_{k\in\mathbb{N}}$ of integers ≥ 2 such that for all $k \geq 1$ and $i \geq 2$,

$$\left.\begin{array}{rcl} E_1(k+1) & \geq & \varphi(k, E_1(k)) + 1\,, \\ E_i(k+1) & \geq & E_{i-1}(\varphi(k, E_i(k)) + 1)\,. \end{array}\right\} \tag{6.3}$$

From the above assumptions it follows that for every $i, k \in \mathbb{N}$,

$$E_i(k) \;>\; k \quad \text{and} \quad E_i(k+1) \;\geq\; \varphi(k, E_i(k)) + 1 \;>\; E_i(k) + 1\,. \tag{6.4}$$

Further, we take an extension $(L|K, v)$ of valued fields, choose elements

$$a_j \in L \quad \text{and} \quad \alpha_j \in vL \quad \text{for all } j \in \mathbb{N},$$

and K-subspaces

$$S_j \;\subseteq\; L\,, \quad j \in \mathbb{N}\,.$$

We assume that for all $i, k, \ell \in \mathbb{N}$, the following conditions are satisfied:

(A1) $0 \leq va_k \leq \alpha_k < va_{k+1}$ and $k\alpha_{E_i(k)} \leq \alpha_{\varphi(k,E_i(k))}$,

(A2) $a_1, \ldots, a_k \in S_k$ and $S_k \subseteq S_{k+1}$,

(A3) if $d_0, \ldots, d_k \in S_k$ and $u \in S_\ell$, then

$$d_0 + d_1 u + \ldots + d_k u^k \;\in\; S_{\varphi(k,\ell)}\;,$$

(A4) if $m \leq k$ and $d_0, \ldots, d_m \in S_k$, then

$$v(d_0 + d_1 a_{k+1} + \ldots + d_m a_{k+1}^m) \;\leq\; vd_m + m\alpha_{k+1}\,.$$

For every natural number i consider the sequence

$$\mathbf{A}_i := \left(\sum_{j=1}^{k} a_{E_i(j)} \right)_{k\in\mathbb{N}} \,. \tag{6.5}$$

From the choice of the numbers $E_i(j)$ and elements a_k it follows that $(E_i(j))_{j\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers and $va_k < va_{k+1}$ for every $k \in \mathbb{N}$. Thus the sequence $\mathbf{A}_i$ satisfies condition (2.12) and is a pseudo Cauchy sequence in (L, v). Now we choose any maximal immediate extension (M, v) of (L, v). By Theorem 2.42, the pseudo Cauchy sequence $\mathbf{A}_i$ admits a pseudo limit in M. Take $y_i \in M$ to be an arbitrary pseudo limit of $\mathbf{A}_i$. In this situation, we can prove the following basic independence lemma:

Lemma 6.1. *Suppose that $k \geq 2$ is an integer and $f \in L[X_1, \ldots, X_i]$ is a polynomial with coefficients in $S_{k-1} \cap \mathcal{O}_L$ such that $\alpha_{E_i(k)} \geq vc_f$ and that f has degree less than k in each variable. Then*

$$vf(y_1, \ldots, y_i) \;<\; va_{E_i(k+1)}\,. \tag{6.6}$$

Proof. We shall prove the lemma by induction on i. We start with $i = 1$. Set $m = \deg f$ and define

$$u := \sum_{j=1}^{k} a_{E_1(j)} \qquad \text{and} \qquad z := y_1 - u \,.$$

Then $u \in S_{E_1(k)}$ because of (A2) and the fact that the S_k are vector spaces. Since y_1 is a pseudo limit of $\mathbf{A}_1$, assertion (2.14) shows that $vz = va_{E_1(k+1)}$. Hence by (A1), the definition of E_1 and our assumption that $\alpha_{E_1(k)} \geq vc_f$,

$$vz \;\geq\; va_{\varphi(k,E_1(k))+1} \;>\; \alpha_{\varphi(k,E_1(k))} \;\geq\; k\alpha_{E_1(k)} \;\geq\; vc_f + (k-1)\alpha_{E_1(k)} \,. \tag{6.7}$$

We use the Taylor expansion

$$f(y_1) \;=\; f(u+z) \;=\; f(u) \,+\, zf_1(u) \,+\, z^2 f_2(u) \,+\, \ldots \,+\, z^m f_m(u) \tag{6.8}$$

where $f_j(X) \in \mathcal{O}_L[X]$ is the j-th formal derivative of f as defined in (6.1). As $va_k > 0$ for every k, we have that $z, f_j(u) \in \mathcal{O}_L$ for all j. Thus $f_1(u) \,+\, zf_2(u) \,+\, \ldots \,+\, z^{m-1} f_m(u) \in \mathcal{O}_L$ and

$$v(zf_1(u) \,+\, z^2 f_2(u) \,+\, \ldots \,+\, z^m f_m(u)) \;\geq\; vz \,. \tag{6.9}$$

We wish to prove that $vf(u) < vz$. We set

$$u' := \sum_{j=1}^{k-1} a_{E_1(j)} \;\in\; S_{E_1(k-1)}$$

so that $u = u' + a_{E_1(k)}$. We use the Taylor expansion

$$f(u) \;=\; f(u' + a_{E_1(k)}) \;=\; f(u') \,+\, f_1(u') a_{E_1(k)} \,+ \ldots +\, f_m(u') a_{E_1(k)}^m$$

where $m = \deg f < k$. By definition, c_f is the leading coefficient of f, which in turn is equal to the constant $f_m(u') = f_m(X) \in L$. Since f has coefficients in the vector space S_{k-1} , we know from (6.1) that also all f_j have coefficients in S_{k-1} . Thus, (A3) and (A2) together with (6.4) show that

$$f(u'),\; f_j(u') \;\in\; S_{\varphi(k-1,E_1(k-1))} \;\subseteq\; S_{E_1(k)-1}$$

for each j. Further, $m \leq k-1 \leq E_1(k) - 1$. Hence by (A4) and (6.7),

$$vf(u) \;\leq\; vf_m(u') + m\alpha_{E_1(k)} \;\leq\; vc_f \,+\, (k-1)\alpha_{E_1(k)} \;<\; vz \,.$$

From this together with (6.8) and (6.9), we deduce that

$$vf(y_1) \;=\; vf(u) \;<\; vz \;=\; va_{E_1(k+1)} \,,$$

which gives the assertion of our lemma for the case of $i = 1$.

In the case of $i > 1$ we assume that the assertion of our lemma has been proven for $i - 1$ in place of i, and we set

$$u := \sum_{j=1}^{k} a_{E_i(j)} \;\in\; S_{E_i(k)} \,, \quad u' := \sum_{j=1}^{k-1} a_{E_i(j)} \;\in\; S_{E_i(k-1)} \,, \quad \text{and} \quad z := y_i - u \,.$$

As in the case of $i = 1$ we deduce that $vz = va_{E_i(k+1)}$. Then by (6.4), (A1) and our assumption that $\alpha_{E_i(k)} \geq vc_f$,

$$vz \;\geq\; va_{\varphi(k,E_i(k))+1} \;>\; \alpha_{\varphi(k,E_i(k))} \;\geq\; k\alpha_{E_i(k)} \;\geq\; vc_f + (k-1)\alpha_{E_i(k)} \,.$$

We use the Taylor expansion

$$\begin{aligned} f(y_1,\ldots,y_{i-1},u+z) \;&=\; f(y_1,\ldots,y_{i-1},u) \,+\, zf_1(y_1,\ldots,y_{i-1},u) \\ &+\; z^2 f_2(y_1,\ldots,y_{i-1},u) \,+\, \ldots \,+\, z^m f_m(y_1,\ldots,y_{i-1},u) \end{aligned}$$

where $f_j \in \mathcal{O}_L[X_1,\ldots,X_i]$ is the j-th formal derivative of f with respect to X_i and $m = \deg_{X_i} f$. We obtain the analogue of inequality (6.9); hence it will suffice to prove that

$$vf(y_1,\ldots,y_{i-1},u) \;<\; vz \,. \tag{6.10}$$

We set

$$g(X_1,\ldots,X_{i-1}) \;:=\; f\,(X_1,\ldots,X_{i-1},u)$$

so that $g(y_1,\ldots,y_{i-1}) = f(y_1,\ldots,y_{i-1},u)$. Viewing f as a polynomial in the variables $X_1,\ldots,X_{i-1}$ with coefficients in $L[X_i]$, we denote by $h(X_i)$ the coefficient of $X_1^{\mu_1}\cdots X_{i-1}^{\mu_{i-1}}$ in f. Note that h has coefficients in S_{k-1}, its leading coefficient is c_f and its degree is $\mu_i < k$. Again, since h has coefficients in S_{k-1} , definition (6.1) shows that the same is true for the j-th formal derivative h_j of h, for all j. Thus, (A3), (6.4) and (A2) imply that

$$h(u'),\, h_j(u') \;\in\; S_{\varphi(k-1,E_i(k-1))} \;\subseteq\; S_{E_i(k)-1}$$

for each j. As in the first part of our proof we find that

$$vh(u) \;\leq\; vh_{\mu_i}(u') + \mu_i\alpha_{E_i(k)} \;=\; vc_f \;+\; \mu_i\alpha_{E_i(k)} \tag{6.11}$$

since $h_{\mu_i}(u') = c_f$. In particular, this shows that $h(u) \neq 0$. Hence if $(\mu_1,\ldots,\mu_i)$ is the crucial exponent of f, then $(\mu_1,\ldots,\mu_{i-1})$ is the crucial exponent of g, and

$$c_g \;=\; h(u) \,.$$

We set

$$k' \;:=\; \varphi(k,E_i(k)) \;>\; \max\{k,E_i(k)\} \,.$$

Since $\mu_i \leq k-1$ and $vc_f \leq \alpha_{E_i(k)}$ by assumption, and by virtue of (6.11), (A1) and (6.4), it follows that

$$vc_g \;=\; vh(u) \;\leq\; vc_f + (k-1)\alpha_{E_i(k)} \;\leq\; k\alpha_{E_i(k)} \;\leq\; \alpha_{k'} \;<\; \alpha_{E_{i-1}(k')} \,.$$

Since every coefficient of g is of the form $h(u)$ with h a polynomial of degree less than k and coefficients in S_{k-1} , we know from (A3), our conditions on φ and (A2) that the coefficients of g lie in

$$S_{\varphi(k-1,E_i(k))} \;\subseteq\; S_{\varphi(k,E_i(k))-1} \;=\; S_{k'-1} \,.$$

Also, its degree in each variable is less than k, hence less than k'. Therefore, we can apply the induction hypothesis to the case of $i-1$, with k' in place of k. We obtain, by (A1) and our choice of the numbers $E_i(k)$:

$$vf(y_1,\ldots,y_{i-1},u) \;<\; va_{E_{i-1}(\varphi(k,E_i(k))+1)} \;\leq\; va_{E_i(k+1)} \;=\; vz \,.$$

This establishes our lemma. □

By (A2),

$$S_\infty := \bigcup_{k\in\mathbb{N}} S_k$$

contains a_k for all k. We set

$$K_\infty := K(S_\infty) .$$

Further, we note that condition (A1) implies that

$$\Gamma := \{\alpha \in vK_\infty \mid -va_k \leq \alpha \leq va_k \text{ for some } k\}$$

is a convex subgroup of vK_∞.

Corollary 6.2. *Assume that every element of K_∞ with value in Γ can be written as a quotient r/s with $r, s \in S_\infty$ such that $0 \leq vs \in \Gamma$. Then the elements y_i, $i \in \mathbb{N}$, are algebraically independent over K_∞.*

Proof. We have to check that $g(y_1, \dots, y_i) \neq 0$ for all $i \in \mathbb{N}$ and all nonzero polynomials $g(X_1, \dots, X_i) \in K_\infty[X_1, \dots, X_i]$. After division by some coefficient of g with minimal value we may assume that g has coefficients in $K_\infty \cap \mathcal{O}_L$ and at least one of them has value 0. We write all its coefficients which have value in Γ in the form as given in our assumption. We take $\tilde{s}$ to be the product of all appearing denominators. Then $v\tilde{s} \in \Gamma$ and $v\tilde{s} \geq 0$. After multiplication with $\tilde{s}$, all coefficients of g with value in Γ are elements of S_∞. There is at least one coefficient with value $v\tilde{s} \in \Gamma$, so at least one coefficient is an element of S_∞. Furthermore, all coefficients of g have nonnegative value. Now we write $g(X_1, \dots, X_i) = f(X_1, \dots, X_i) + h(X_1, \dots, X_i)$ where every coefficient of f is in S_∞ and has value less than va_k for some k, and every coefficient of h has value bigger than va_k for all k (we allow h to be the zero polynomial). Since g has coefficients of value $v\tilde{s}$, the polynomial f is nonzero. Since y_i is a pseudo limit of the pseudo Cauchy sequence $\mathbf{A}_i$, from equation (2.14) it follows that $v(y_i - a_{E_i(1)}) = va_{E_i(2)} > va_{E_i(1)}$. Thus, $vy_i = va_{E_i(1)} \geq 0$ for all i, we have that $vh(y_1, \dots, y_i) > va_k$ for all k.

We choose k such that the assumptions of Lemma 6.1 hold; note that k exists since by our definition of f, the coefficient c_f has value less than va_j for some j and f has coefficients in $S_\infty \cap \mathcal{O}_L$. We obtain that

$$vf(y_1, \dots, y_i) < va_{E_i(k+1)} < vh(y_1, \dots, y_i) .$$

This gives that

$$vg(y_1, \dots, y_i) = v(f(y_1, \dots, y_i) + h(y_1, \dots, y_i)) = vf(y_1, \dots, y_i) < va_{E_i(k+1)} < \infty ,$$

that is, $g(y_1, \dots, y_i) \neq 0$. □

Now we are able to give the

Proof of Theorem 1.2:

In all cases of the proof, we will choose functions φ that have the previously required properties. We will choose a suitable sequence $(b_k)_{k\in\mathbb{N}}$ of elements in L and a sequence $(c_k)_{k\in\mathbb{N}}$ in K. Then we will set $a_k := c_k b_k$ and choose some values $\alpha_k \geq va_k$.

First, let us consider the valuation-transcendental case. We set

$$\varphi(k,\ell) \;:=\; k+k\ell\;,$$

and note that equations (6.3) now read as follows:

$$\left.\begin{array}{rcl} E_1(k+1) & \geq & k+kE_1(k)+1, \\ E_i(k+1) & \geq & E_{i-1}(k+kE_i(k)+1)\;. \end{array}\right\} \tag{6.12}$$

Further, we will work with a suitable element $t\in\mathcal{O}_L$ transcendental over K and set, after a suitable choice of the sequence $(c_k)_{k\in\mathbb{N}}$,

$$\begin{array}{rcl} a_k & := & c_k t^k\;, \\ \alpha_k & := & va_k\;, \\ S_k & := & K[t]_k = K+Kt+\ldots+Kt^k\;. \end{array}$$

Conditions (A2) and (A3) are immediate consequences of our choice of S_k as the set of all polynomials in $K[t]$ of degree at most k.

Suppose that vL/vK is not a torsion group. Then we pick $t\in\mathcal{O}_L$ such that vt is rationally independent over vK (that is, $nvt\notin vK$ for all integers $n>0$). Further, for all k we set $b_k=t^k$ and $c_k=1$ so that $a_k=t^k$. Then condition (A1) is satisfied since we have that

$$0\leq va_k=\alpha_k=vt^k=kvt<(k+1)vt=vt^{k+1}=va_{k+1}$$

and

$$k\alpha_{E_i(k)}=kva_{E_i(k)}=kvt^{E_i(k)}=kE_i(k)vt<(k+kE_i(k))vt=\alpha_{\varphi(k,E_i(k))}\;.$$

Suppose now that vL/vK is a torsion group. In this case, $Kv|Lv$ is transcendental by assumption, and we note that since v is assumed nontrivial on L, it must be nontrivial on K. We pick $t\in\mathcal{O}_L$ such that $vt=0$ and tv is transcendental over Kv. Further, we choose a sequence $(c_k)_{k\in\mathbb{N}}$ in $\mathcal{O}_K$ such that

$$vc_{k+1}\;\geq\;kvc_k$$

for all k. Since $va_k=vc_k+kvt=vc_k$, we obtain that $va_k=\alpha_k<va_{k+1}$ and

$$k\alpha_k\;=\;kva_k\;\leq\;va_{k+1}\;. \tag{6.13}$$

Then by (6.4),

$$k\alpha_{E_i(k)}\;<\;E_i(k)\alpha_{E_i(k)}\;\leq\;va_{E_i(k)+1}\;\leq\;va_{\varphi(k,E_i(k))}\;\leq\;\alpha_{\varphi(k,E_i(k))}\;. \tag{6.14}$$

Hence again, condition (A1) is satisfied.

Now we have to verify (A4), simultaneously for all of the above choices for a_k . Take $d_0,\ldots,d_m\in S_k$, $m\leq k$, and write $d_j=\sum_{\nu=0}^k d_{j\nu}t^\nu$ with $d_{j\nu}\in K$. Then

$$d_0+d_1a_{k+1}+\ldots+d_m a_{k+1}^m\;=\;\sum_{j=0}^m\sum_{\nu=0}^k c_{k+1}^j d_{j\nu}t^{j(k+1)+\nu}\;.$$

In this sum, each power of t appears only once. So we have, by Lemma 2.6,

$$v(d_0+d_1a_{k+1}+\ldots+d_m a_{k+1}^m)\;=\;\min_{j,\nu} vc_{k+1}^j d_{j\nu}t^{j(k+1)+\nu}\;:=\;\beta\;.$$

If this minimum is obtained at $j = j_0$ and $\nu = \nu_0$, then

$$\begin{aligned}\beta &= vc_{k+1}^{j_0} d_{j_0\nu_0} t^{j_0(k+1)+\nu_0} = \min_\nu vc_{k+1}^{j_0} d_{j_0\nu} t^{j_0(k+1)+\nu} \\ &= (\min_\nu vd_{j_0\nu} t^\nu) + va_{k+1}^{j_0} = vd_{j_0} a_{k+1}^{j_0} ,\end{aligned}$$

where the last equality again holds by Lemma 2.6. For all j,

$$\beta \le \min_\nu vc_{k+1}^j d_{j\nu} t^{j(k+1)+\nu} = (\min_\nu vd_{j\nu} t^\nu) + va_{k+1}^j = vd_j a_{k+1}^j .$$

This gives that

$$v(d_0 + d_1 a_{k+1} + \ldots + d_m a_{k+1}^m) = \beta = \min_j vd_j a_{k+1}^j \le vd_m + mva_{k+1} = vd_m + m\alpha_{k+1} ,$$

as required. Finally, we have to verify the assumption of Corollary 6.2. Each element in $K_\infty = K(t)$ can be written as a quotient r/s of polynomials in t with coefficients in K, that is, of elements of S_∞. After multiplying both r and s with a suitable element from K we may assume that s has coefficients in $\mathcal{O}_K$ and one of them is 1. If this is the coefficient of t^i, say, then it follows by Lemma 2.6 that $0 \le vs \le vt^i \le va_i$ and thus, $vs \in \Gamma$.

Now we take any maximal immediate extension (M, v) of (L, v) and y_i as defined preceeding to Lemma 6.1. Then we can infer from Corollary 6.2 that the elements y_i are algebraically independent over K_∞; that is, the transcendence degree of M over K_∞ is infinite. Since the transcendence degree of L over K and thus also that of L over K_∞ is finite, we can conclude that the transcendence degree of M over L is infinite.

Next, we consider the value-algebraic case and the residue-algebraic case. We will assume for now that there is an algebraic subextension $L_0|K$ of $L|K$ such that vL_0/vK contains elements of arbitrarily high order, or L_0v contains elements of arbitrarily high degree over Kv. The remaining cases will be treated at the end of the proof of our theorem.

For the present case as well as the separable-algebraic case, we work with any function φ that satisfies the conditions outlined in the beginning of this section, and with

$$S_k := K(a_1, \ldots, a_k) .$$

Then S_∞ is a field and the assumption of Corollary 6.2 are trivially satisfied (taking $s = 1$). Further, condition (A2) is trivially satisfied. To prove that condition (A3) holds, take any $u \in S_\ell = K(a_1, \ldots, a_\ell)$. If $n = \max\{k, \ell\}$, then $d_0, \ldots, d_k, u \in K(a_1, \ldots, a_n) = S_n$ and therefore,

$$d_0 + d_1 u + \ldots + d_k u^k \in S_n \subseteq S_{\varphi(k,\ell)} .$$

This shows that (A3) holds.

By induction, we define $a_k \in L_0$ as follows, and we always take $\alpha_k = va_k$. We start with $a_1 = 1$ and $\alpha_1 = 0$. Suppose that $a_1, \ldots, a_k$ are already defined. Since $K(a_1, \ldots, a_k)|K$ is a finite extension, also $vK(a_1, \ldots, a_k)/vK$ and $K(a_1, \ldots, a_k)v|Kv$ are finite. Hence by our assumption in the algebraic case, there is some $b_{k+1} \in L_0$ such that

$$0, vb_{k+1}, 2vb_{k+1}, \ldots, kvb_{k+1} \text{ lie in distinct cosets modulo } vK(a_1, \ldots, a_k), \text{ or} \tag{6.15}$$

$$1, b_{k+1}v, (b_{k+1}v)^2, \ldots, (b_{k+1}v)^k \text{ are } K(a_1, \ldots, a_k)v\text{-linearly independent.} \tag{6.16}$$

If L_0v contains elements of arbitrarily high degree over Kv, we always choose b_{k+1} such that (6.16) holds; in this case, $vb_{k+1}=0$ and we choose the elements c_k as in the residue-transcendental case above. Otherwise, vL_0/vK contains elements of arbitrarily high order, and we always choose b_{k+1} such that (6.15) holds. In this case, we choose c_{k+1} such that for $a_{k+1}:=c_{k+1}b_{k+1}$ we obtain $k\alpha_k=kva_k\leq va_{k+1}$; this is possible since the values of b_k and hence of all a_k lie in the convex hull of vK in vL. As in the residue-transcendental case above, we obtain (6.13) and (6.14), showing that condition (A1) is satisfied.

To prove that (A4) holds, take any $k\geq 1$ and $d_0,\ldots,d_k\in S_k=K(a_1,\ldots,a_k)$. By Lemma 2.7 applied to b_{k+1} ,

$$\begin{aligned} v(d_0+d_1a_{k+1}+\ldots+d_ka_{k+1}^k) &= v(d_0+d_1c_{k+1}b_{k+1}+\ldots+d_kc_{k+1}^kb_{k+1}^k)\\ &= \min_i vd_ic_{k+1}^ib_{k+1}^i \;=\; \min_i vd_ia_{k+1}^i\;. \end{aligned}$$

This shows that (A4) holds.

As in the valuation-transcendental case, we can now deduce our assertion about the transcendence degree of the maximal immediate extensions of (L,v).

Next, we consider the separable-algebraic case. We can assume that v is nontrivial on K, since otherwise we are in the valuation-transcendental case. Take (L^h,v) to be a henselization of (L,v) such that for the henselizations L_0^h and K^h of L_0 and K contained in L^h, the extension $L_0^h|K^h$ is infinite. Assume that $(L^{h(v')},v')$ is another henselization of (L,v). Then there is a valuation preserving isomorphism $\sigma\in\mathrm{Gal}\,(L)$ such that $\sigma(L^h)=L^{h(v')}$. Take K^h and $K^{h(v')}$ to be the henselization of (K,v) inside of L^h and $L^{h(v')}$ respectively. Since σ is also a valuation preserving K-isomorphism, $\sigma(K^h)=K^{h(v')}$. Furthermore, $\sigma(L_0)=L_0$, as $L_0\subseteq L$. Therefore,

$$\sigma(L_0^h)=\sigma(K^h.L_0)=\sigma(K^h).\sigma(L_0)=K^{h(v')}.L_0=L_0^{h(v')},$$

where $L_0^{h(v')}$ is the henselization of L_0 contained in $L^{h(v')}$. Since $L_0^h|K^h$ is infinite, so is $L_0^{h(v')}|K^{h(v')}$. It follows that $L_0^h|K^h$ is infinite inside of every henselization L^h of (L,v).

Note that we can without loss of generality assume that (K,v) is henselian. Indeed, each maximal immediate extension of (L,v) contains a henselization L^h of (L,v) and hence also a henselization K^h of (K,v), and our assumption on L_0 implies that the subfield $L_0^h=L_0.K^h$ of L^h is an infinite separable-algebraic extension of K^h.

We take S_k and $\varphi(k,\ell)$ as in the previous case, so that again, (A2), (A3) and the assumption of Corollary 6.2 hold. Then we take $a_1=b_1$ to be any element in $\mathcal{O}_{L_0}\setminus K$ and choose some $\alpha_1\in vK$ such that $\alpha_1\geq\mathrm{kras}(a_1,K)\in v\widetilde{K}$; this is possible since vK is cofinal in its divisible hull, which is equal to $v\widetilde{K}$. Inequality (2.7) of Lemma 2.16 shows that $\mathrm{kras}(a_1,K)\geq va_1$, so that $\alpha_1\geq va_1$. Suppose we have chosen $a_1,\ldots,a_k\in\mathcal{O}_{L_0}$. Since $L_0|K$ is infinite and separable-algebraic, the same is true for $L_0|K(a_1,\ldots,a_k)$. By the Theorem of the Primitive Element, we can therefore find an element $b_{k+1}\in L_0$ such that

$$[K(a_1,\ldots,a_k,b_{k+1}):K(a_1,\ldots,a_k)]\;\geq\;k+1\;.$$

We choose $c_{k+1}\in K$ such that for $a_{k+1}:=c_{k+1}b_{k+1}$ we have that $k\alpha_k\leq va_{k+1}$. Finally, we choose $\alpha_{k+1}\in vK$ such that

$$\alpha_{k+1}\;\geq\;\mathrm{kras}(a_{k+1},K)\;\geq va_{k+1}\;.$$

Again, we obtain that (6.14) and (A1) hold.

It only remains to show that (A4) holds. But this follows readily from inequality (2.8) of Lemma 2.16, where we take $K(a_1,\dots,a_k)$ in place of K and $a = a_{k+1}$, together with the fact that $\mathrm{kras}(a_{k+1}, K(a_1 \dots, a_k)) \le \mathrm{kras}(a_{k+1}, K)$.

As before, we now obtain our assertion about the transcendence degree of the maximal immediate extensions of (L, v).

It remains to prove the value-algebraic case and the residue-algebraic case for transcendental valued field extensions $(L|K, v)$ of finite transcendence degree. We assume that vL/vK is a torsion group containing elements of arbitrarily high order or the extension $Lv|Kv$ is algebraic and such that Lv contains elements of arbitrarily high degree over Kv.

Take any subextension $E|K$ of $L|K$. Then $(L|K, v)$ satisfies the above assumption if and only if at least one of the extensions $(L|E, v)$ and $(E|K, v)$ satisfies the assumption. Choose a transcendence basis $(x_1,\dots,x_n)$ of $L|K$ and set

$$F \;:=\; K(x_1,\dots,x_n)\,.$$

Then $L|F$ is algebraic. By what we have already proved, if vL/vF contains elements of arbitrarily high order or Lv contains elements of arbitrarily high degree over Fv, then any maximal immediate extension of (L, v) has infinite transcendence degree over L.

Suppose now that $(F|K, v)$ satisfies the assumption on the value group or the residue field extension. Take $s \in \mathbb{N}$ minimal such that $vK(x_1,\dots,x_s)/vK$ contains elements of arbitrarily high order or $K(x_1,\dots,x_s)v$ contains elements of arbitrarily high degree over Kv. Then the assertion holds also for the value group or the residue field extension of $(K(x_1,\dots,x_s)|K(x_1,\dots,x_{s-1}), v)$. We can replace K by $K(x_1,\dots,x_{s-1})$ and we will write x in place of x_s so that now we have a subextension $(K(x)|K, v)$ that satisfies the assertion for its value group or its residue field extension.

In both the value-algebraic and the residue-algebraic case we define $b_k \in K[x]$ by induction on k and set

$$S_k \;:=\; K[x]_{N_k} = K + Kx + \ldots + Kx^{N_k} \quad \text{with } N_k := \deg b_k.$$

Assume that $vK(x)$ contains elements of arbitrarily high order modulo vK. Then such elements can already be chosen from $vK[x]$, as for any $\frac{f}{g}$ with $f, g \in K[x]$ we have that $v(\frac{f}{g}) = vf - vg$. Set $b_1 = 1$. Suppose that $b_1,\dots,b_k$ are already chosen with $\deg b_{i-1} < \deg b_i$ for $1 < i \le k$. From Corollary 2.10 we know that vS_k contains only finitely many values that represent distinct cosets modulo vK. Since all of these values are torsion modulo vK, the subgroup $\langle vS_k\rangle$ of $vK(x)$ generated by vS_k satisfies $(\langle vS_k\rangle : vK) < \infty$. By assumption, there is $b_{k+1} \in K[x]$ for which the order of vb_{k+1} modulo vK is at least $(k+1)(\langle vS_k\rangle : vK)$; this forces $0, vb_{k+1}, 2vb_{k+1},\dots,kvb_{k+1}$ to lie in distinct cosets modulo $\langle vS_k\rangle$. Since $b_{k+1} \notin K[x]_{N_k}$, we have that $N_{k+1} = \deg b_{k+1} > N_k$.

Assume now that $K(x)v$ contains elements of arbitrarily high degree over Kv. Without loss of generality we can assume that $vK(x)/vK$ is then a torsion group with a finite exponent N. Otherwise, vL/vK is not a torsion group and we are in the valuation-transcendental case or $vK(x)/vK$ contains elements of arbitrarily high order and we are in the value-algebraic case.

The elements of arbitrarily high degree over Kv can be chosen from $K[x]v$. Indeed, suppose there is $m \in \mathbb{N}$ such that $[Kv(fv) : Kv] \leq m$ for every polynomial f of nonnegative value. Take any $r = \frac{h}{g}$, where $g, h \in K[x]$ and $vr = 0$. By the assumption on $vK(x)/vK$ we have that $nvh = vd$ for some natural number $n \leq N$ and $d \in K$. Then

$$r = \frac{d^{-1}h^n}{d^{-1}h^{n-1}g}$$

and $vd^{-1}h^{n-1}g = vd^{-1}h^n = 0$, since $vh = vg$. Therefore we may assume that $vh = vg = 0$. Hence,

$$[Kv(rv) : Kv] \leq [Kv(rv, gv) : Kv] = [Kv(hv, gv) : Kv] \leq m^2$$

for every $r \in K(x)$ with $vr = 0$, a contradiction to our assumption.

As in the value-algebraic case, we set $b_1 = 1$. Suppose that $b_1, \ldots, b_k$ are already chosen with $\deg b_{i-1} < \deg b_i$ for $1 < i \leq k$. By Corollary 2.10, there are at most $NN_k + 1$ many Kv-linearly independent elements in $K[x]_{NN_k}v$, and as all of them are algebraic over Kv, it follows that the extension $Kv\,(K[x]_{NN_k}v)\,|Kv$ is finite. By assumption, there is $b_{k+1} \in K[x]$ such that $vb_{k+1} = 0$ and the degree of $b_{k+1}v$ over Kv is at least $(k+1)[Kv\,(K[x]_{NN_k}v) : Kv]$, which forces the elements $1, b_{k+1}v, (b_{k+1}v)^2, \ldots, (b_{k+1}v)^k$ to be $Kv\,(K[x]_{NN_k}v)$-linearly independent. Since $b_{k+1} \notin K[x]_{NN_k}$, we have that $N_{k+1} = \deg b_{k+1} > NN_k \geq N_k$.

For the value-algebraic as well as for the residue-algebraic case we set

$$\varphi(k, l) := N_k + N_k N_l.$$

Since in both cases $(N_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, φ has the required properties. As in the first part of the proof of the value-algebraic and the residue-algebraic case, one can show that the elements $c_k \in K$ can be chosen in such a way that condition (A1) holds for $a_k := c_k b_k$ and $\alpha_k := va_k$. Since $(N_k)_{k\in\mathbb{N}}$ is strictly increasing, condition (A2) is trivially satisfied. Moreover, $N_k \geq k$ for every $k \in \mathbb{N}$. Hence for any $d_0, \ldots, d_k \in S_k$ and $u \in S_l$,

$$\deg(d_0 + d_1 u + \cdots + d_k u^k) \leq N_k + kN_l \leq \varphi(k, l) \leq N_{\varphi(k,l)}\,.$$

Thus, $d_0 + d_1 u + \cdots + d_k u^k \in S_{\varphi(k,l)}$. This shows that (A3) holds.

To verify (A4), we take any $k, m \in \mathbb{N}$ with $m \leq k$, and $d_0, \ldots, d_m \in S_k$. We wish to estimate the value of the element $d_0 + d_1 a_{k+1} + \cdots d_m a_{k+1}^m$. We discuss first the value-algebraic case. Note that the values $v(d_i a_{k+1}^i)$, $0 \leq i \leq m$, lie in distinct cosets modulo vK. Indeed, $vd_i a_{k+1}^i = vd_i c_{k+1}^i + ivb_{k+1}$, where $d_i c_{k+1}^i \in S_k$. Therefore, if

$$vd_i a_{k+1}^i + vK = vd_j a_{k+1}^j + vK$$

for some $0 \leq i \leq j \leq m$, then also

$$ivb_{k+1} + \langle vS_k \rangle = jvb_{k+1} + \langle vS_k \rangle\,,$$

which by our choice of b_{k+1} yields that $i = j$. Hence, from Lemma 2.7 it follows that

$$v(d_0 + d_1 a_{k+1} + \cdots d_m a_{k+1}^m) = \min_i vd_i a_{k+1}^i = \min_i (vd_i + ia_{k+1}) \leq vd_m + mva_{k+1}\,.$$

We obtain the same assertion also in the residue-algebraic case. If $d_i = 0$ for all i, then it is trivially satisfied. If not, take i_0 so that

$$vd_{i_0}c_{k+1}^{i_0} \;=\; \min_i vd_ic_{k+1}^i \;=\; \min_i vd_ia_{k+1}^i\,.$$

We have that $vd_{i_0}^N = vc$ for some $c \in K$. Setting $d := c^{-1}c_{k+1}^{-i_0}d_{i_0}^{N-1}$, we obtain that

$$v(d_0 + d_1a_{k+1} + \cdots d_ma_{k+1}^m) \;=\; -vd + v\xi$$

with $\xi := dd_0 + dd_1c_{k+1}b_{k+1} + \cdots + dd_mc_{k+1}^mb_{k+1}^m$. Note that $dd_i \in K[x]_{NN_k}$ for $0 \leq i \leq m$, and that

$$vdd_ic_{k+1}^i \;\geq\; vdd_{i_0}c_{k+1}^{i_0} \;=\; vc^{-1}d_{i_0}^N \;=\; 0.$$

In particular, $v\xi \geq 0$, and

$$\xi v = (dd_0)v + (dd_1c_{k+1})vb_{k+1}v + \cdots + (dd_mc_{k+1}^m)v(b_{k+1}v)^m$$

is a linear combination of $1, b_{k+1}v, (b_{k+1}v)^2, \ldots, (b_{k+1}v)^m$ with coefficients from Kv ($K[x]_{N\cdot N_k}v$). Since at least one of them, the element $dd_{i_0}c_{k+1}^{i_0}v$, is nonzero, also the linear combination is nontrivial by our choice of b_{k+1}. Hence $v\xi = 0$ and

$$v(d_0 + d_1a_{k+1} + \cdots d_ma_{k+1}^m) \;=\; -vd \;=\; vd_{i_0}c_{k+1}^{i_0} \;\leq\; vd_m + mva_{k+1}\,.$$

Therefore, condition (A4) is satisfied in both cases.

It suffices now to verify the assumptions of Corollary 6.2. Take any element $\frac{h}{g}$ of $K_\infty = K(x)$, where $g, h \in S_\infty = K[x]$. In both the value-algebraic and the residue-algebraic case we assumed that $vK(x)/vK$ is a torsion group. Therefore, as in the residue-algebraic case above one can multiply h and g by a suitable polynomial to obtain that $vg = 0 \in \Gamma$. Hence the assumptions of the corollary are satisfied.

Since the transcendence degree of the extension $L|K(x)$ is finite, we can now deduce the assertion about about the transcendence degree of the maximal immediate extensions of (L, v) as in the previous cases.

In the value-algebraic case, we still have to deal with the case where there is a subgroup $\Gamma \subseteq vL$ containing vK such that Γ/vK is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of Kv. We may assume that $Lv|Kv$ is algebraic and vL/vK is a torsion group since otherwise, the assertion of our theorem follows from the valuation-transcendental case. Since every maximal immediate extension of (L, v) contains a henselization of (L, v), we may assume that both (L, v) and (K, v) are henselian. We take L' to be the relative separable-algebraic closure of K in L. Then by Lemma 2.14, vL/vL' is a p-group, which yields that $\Gamma \subseteq vL'$. In view of the fundamental inequality, we find that $L'|K$ must be an infinite extension. Now the assertion of our theorem follows from the separable-algebraic case.

Finally, we have to deal with our additional assertion about the completion. Since the transcendence degree of $L|K$ is finite, we know that vL/vK has finite rational rank. Therefore, vK is cofinal in vL or there exists some $\alpha \in vL$ such that the sequence $(i\alpha)_{i\in\mathbb{N}}$ is cofinal in vL. In the latter case (which always holds if vL contains an element γ such that $\gamma > vK$), we are in the value-transcendental case and we choose the element t such that $vt = \alpha$. In the former case, provided that the cofinality of vL is countable, we choose the elements c_i such that the sequence $(vc_ib_i)_{i\in\mathbb{N}}$ is cofinal in vL. In all of these cases, the sequence $(va_i)_{i\in\mathbb{N}}$ will be cofinal in vL. Take any maximal immediate extension (M,v) of (L,v) and y_i as defined preceeding to Lemma 6.1. Since (M,v) is a maximal field, it is complete. Hence by Proposition 2.49 it contains a completion (L^c,v) of (L,v). Fix $i\in\mathbb{N}$. Then

$$v\left(\sum_{j=1}^{k+1} a_{E_i(j)} - \sum_{j=1}^{k} a_{E_i(j)}\right) \;=\; va_{E_i(k+1)},$$

hence this sequence of values is cofinal in vL. Thus, $\mathbf{A}_i$ defined by (6.5), is a Cauchy sequence in (L,v). By Corollary 2.50 this yields that the element y_i lies in L^c. □

Note that the condition in the residue-algebraic case of Theorem 1.2 always holds when $Lv|Kv$ contains an infinite separable-algebraic subextension; this is a consequence of the Theorem of the Primitive Element. There is no analogue of this theorem in abelian groups; therefore, the first condition in the value-algebraic case does not follow from the second. As an example, take q to be a prime different from $\operatorname{char} Kv$ and consider the case where vL/vK is an infinite product of $\mathbb{Z}/q\mathbb{Z}$. Under the second condition, however, the result can easily be deduced from the separable-algebraic case, as we have seen in the proof.

The key assumption in the separable-algebraic case is that the separable-algebraic subextension remains infinite when passing to the respective henselizations. We show that this condition is crucial. To prove this, we need the following lemma.

Lemma 6.3. *Take a nontrivially valued field $(k(\mathcal{T}),v)$, where $\mathcal{T}$ is a nonempty set of elements algebraically independent over k. Then the henselization of $(k(\mathcal{T}),v)$ inside of any henselian valued extension field is an infinite extension of $k(\mathcal{T})$.*

Proof. Set $F := k(\mathcal{T})$ and take a henselization F^h of F inside of some henselian valued extension field. Pick an arbitrary $t\in\mathcal{T}$. Without loss of generality we can assume that $vt>0$. Then the reduction of the polynomial X^2-X-t is X^2-X. Since 0 is a simple root of the polynomial X^2-X, by Hensel's Lemma, F^h contains a root ϑ_1 of the polynomial X^2-X-t such that $\vartheta_1 v=0$. Thus $v\vartheta_1>0$. We proceed by induction. Once we have constructed ϑ_i with $v\vartheta_i>0$ for some $i\in\mathbb{N}$, we again use Hensel's Lemma to obtain a root $\vartheta_{i+1}\in F^h$ of the polynomial $X^2-X-\vartheta_i$ with $v\vartheta_{i+1}>0$.

It now suffices to show that the extension $F(\vartheta_i\,|\,i\in\mathbb{N})|F$ is infinite. To this end, we consider the t^{-1}-adic valuation w on $F=k(\mathcal{T}\setminus\{t\})(t^{-1})$ which is trivial on $k(\mathcal{T}\setminus\{t\})$. We note that $wF=\mathbb{Z}$. Since $wt<0$, we obtain that $w\vartheta_1=\frac{1}{2}wt$ and by induction, $w\vartheta_i=\frac{1}{2^i}wt$. Therefore, the 2-divisible hull of $\mathbb{Z}$ is contained in $wF(\vartheta_i\,|\,i\in\mathbb{N})$. In view of the fundamental inequality (2.5), this shows that $F(\vartheta_i\,|\,i\in\mathbb{N})|F$ cannot be a finite extension. □

Take a valued field (k,v) which has a transcendental maximal immediate extension (M,v). We know that (M,v) as a maximal field is henselian. Take a transcendence basis $\mathcal{T}$ of $M|k$ and set $K:=k(\mathcal{T})$. Then from Lemma 6.3 it follows that the henselization K^h of K inside of (M,v) is an infinite separable-algebraic subextension of $(M|K,v)$. But M is a maximal immediate extension of $L:=K^h$ and $M|L$ is algebraic. Hence the assertion of Theorem 1.2 does not necessarily hold without the condition that $L_0^h|K^h$ is infinite.

Example 6.4. Take any field K and consider the rational function field $K(t)$ with the t-adic valuation v. Then the power series field $\big(K((t)), v\big)$, where v is the t-adic valuation of the field, is an immediate extension of the field $(K(t), v)$. On the other hand, the power series field is maximal. Hence, $\big(K((t)), v\big)$ is a maximal immediate extension of $(K(t), v)$. As v is trivial on K, the extension $(K(t)|K, v)$ satisfies the valuation-transcendental case of Theorem 1.2. Hence, we obtain the well known fact that the transcendence degree of the extension $K((t))|K(t)$ is infinite. From the proof of the theorem we know that the power series y_i, $i \in \mathbb{N}$, algebraically independent over $K(t)$, can be chosen to be of the form

$$y_i = \sum_{k=1}^{\infty} c_{E_i(k)} t^{E_i(k)},$$

where $c_{E_i(k)} \in K^\times$ and the natural numbers $E_i(k)$ satisfy conditions (6.12). A special case of the above construction is the proof of the fact that the extension $K((t))|K(t)$ is of infinite transcendence degree, presented by MacLane and Schilling (cf. [37], Lemma 1). $\diamond$

6.2 Extensions of maximal fields

An interesting problem is given when (K, v) is itself a maximal field and we ask about the form of maximal immediate extension of a given extension (L, v) of (K, v). In this case, it is well known that if $(L|K, v)$ is a finite extension, then (L, v) is again a maximal field (cf. Theorem 2.33). So we would like to know what happens if $(L|K, v)$ is infinite algebraic, or transcendental of finite transcendence degree. Under which conditions could (L, v) be again a maximal field? We consider these problems in this section.

We start with the following theorem which gives a partial answer to the above question in the case of algebraic extensions.

Theorem 6.5. *Take a nontrivially valued maximal field (K, v) and an infinite algebraic extension $(L|K, v)$. Assume that $L|K$ contains an infinite separable subextension or that*

$$(vK : pvK)[Kv : Kv^p] \,<\, \infty\,, \tag{6.17}$$

where p is the characteristic exponent of Kv. Then every maximal immediate extension of (L, v) has infinite transcendence degree over L.

Proof. Take a maximal field (K, v) which satisfies (6.17), and denote by p the characteristic exponent of Kv. Further, take an infinite algebraic extension $(L|K, v)$. Denote the relative separable-algebraic closure of K in L by L'. Assume that $L'|K$ is infinite. Since K as a maximal field is henselian, $K^h = K$ and $L'^h = L'$. Thus the separable-algebraic case of Theorem 1.2 shows that any maximal immediate extension of (L, v) has infinite transcendence degree over L.

Assume now that $L'|K$ is a finite extension. Then from Theorem 2.33, the field (L', v) is maximal. Furthermore $(vL' : pvL')[L'v : L'v^p] = (vK : pvK)[Kv : Kv^p] < \infty$, and $L|L'$ is an infinite purely inseparable extension. Therefore at least one of the extensions $vL|vL'$ or $Lv|L'v$ is infinite. Indeed, suppose that $(vL : vL')$ and $[Lv : L'v]$ were finite. Take any finite subextension $E|L'$ of $L|L'$ such that $[E : L'] > (vL : vL')[Lv : L'v]$. Then

$$[E : L'] \,>\, (vL : vL')[Lv : L'v] \,\geq\, (vE : vL')[Ev : L'v]\,,$$

which contradicts the fact that L' as a maximal field is defectless. If vL/vL' contains elements of arbitrarily high order or Lv contains elements of arbitrarily high degree over $L'v$, then from the value-algebraic or residue-algebraic case of Theorem 1.2 we deduce that any maximal immediate extension of L is of infinite transcendence degree over L. Otherwise, vL/vL' is a p-group of finite exponent, $Lv|L'v$ is a purely inseparable extension with $(Lv)^{p^n} \subseteq L'v$ for some natural number n, and vL/vL' or $Lv|L'v$ is infinite. But this is not possible if $[L'v : L'v^p](vL' : pvL') < \infty$. □

It remains to discuss the case where $L|K$ is an infinite extension, its maximal separable subextension $L'|K$ is finite, and condition (6.17) fails. Since then also (L', v) is maximal by Theorem 2.33, we can replace K by L' and simply concentrate on the case where $L|K$ is purely inseparable. We will start with a following easy observation.

Lemma 6.6. *If (K, v) is a maximal field of characteristic $p > 0$, then also $K^{1/p}$ with the unique extension of the valuation v is a maximal field.*

Proof. If (a_ν) is a pseudo Cauchy sequence in $K^{1/p}$, then from the definition of a pseudo Cauchy sequence it follows that (a_ν^p) is a pseudo Cauchy sequence in K. Since (K, v) is maximal, by Theorem 2.42 it has a pseudo limit $b \in K$. But then, with the use of condition (2.14), we obtain that $a = b^{1/p} \in L$ is a pseudo limit of (a_ν). □

Note that if the maximal field (K, v) is of characteristic p, then condition (6.17) implies that the p-degree of K is finite, as it is equal to $(vK : pvK)[Kv : Kv^p]$. If condition (6.17) does not hold, then since $vK^{1/p} = \frac{1}{p}vK$ and $K^{1/p}v = (Kv)^{1/p}$, we have that $vK^{1/p}/vK$ has exponent p, every element in $K^{1/p}v \setminus Kv$ has degree p over Kv, and at least one of the two extensions is infinite. By the fundamental inequality, also the purely inseparable extension $K^{1/p}|K$ is infinite. On the other hand, Lemma 6.6 shows that the field $(K^{1/p}, v)$ is again maximal. This case shows that the assertion of Theorem 1.2 may fail even when vL/vK is an infinite torsion group or $Lv|Kv$ is an infinite algebraic extension. In fact, all possible cases can appear for infinite p-degree:

Theorem 6.7. *Take a maximal field (K, v) of characteristic $p > 0$ for which condition (6.17) fails (which is equivalent to K having infinite p-degree). Assume that the valuation v is nontrivial and take κ to be the maximum of $(vK : pvK)$ and $[Kv : Kv^p]$, considered as cardinals. Then:*

1) The valued field $(K^{1/p}, v)$ is again maximal, although $vK^{1/p}/vK$ is an infinite torsion group or $K^{1/p}v|Kv$ is an infinite algebraic extension.

2) For every $n \in \mathbb{N}$ and every infinite cardinal $\lambda \leq \kappa$, there are subextensions $(L_n|K, v)$ and $(L_\lambda|K, v)$ of $(K^{1/p}|K, v)$ such that $(K^{1/p}|L_\lambda, v)$ is an immediate algebraic extension of degree λ and $(K^{1/p}|L_n, v)$ is an immediate algebraic extension of degree p^n.

3) There is a purely inseparable extension $(L|K, v)$ with

- *$vL = \frac{1}{p}vK$ and $Lv = Kv$ if $(vK : pvK) = \infty$,*
- *$vL = vK$ and $Lv = (Kv)^{1/p}$ if $[Kv : Kv^p] = \infty$,*

such that every maximal immediate extension of (L, v) has transcendence degree at least κ. In both cases, L can also be taken to simultaneously satisfy $vL = \frac{1}{p}vK$ and $Lv = (Kv)^{1/p}$.

If the cofinality of vK is countable, then in 2), $K^{1/p}$ can be replaced by the completion of L_λ or L_n, respectively, and in 3), "maximal immediate extension" can be replaced by "completion".

We prove first the following observation.

Lemma 6.8. *Take a henselian field (K,v) of positive characteristic p and a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) without a pseudo limit in K. If $(K(a)|K,v)$ is a valued field extension of degree p such that a is a pseudo limit of $(a_\nu)_{\nu<\lambda}$, then $(K(a)|K,v)$ is immediate.*

Proof. By Lemma 2.39, the sequence $(a_\nu)_{\nu<\lambda}$ does not fix the value of the minimal polynomial f of a over K. On the other hand, we will show that $(a_\nu)_{\nu<\lambda}$ fixes the value of every polynomial of degree less than $\deg f = p$. We take $g \in K[X]$ to be a polynomial of smallest degree such that $(a_\nu)_{\nu<\lambda}$ does not fix the value of g. Since $(a_\nu)_{\nu<\lambda}$ admits no pseudo limit in (K,v), from Lemma 2.38 it follows that the polynomial g is of degree at least 2. Take a root b of g. By Theorem 2.41, there is an extension of the valuation v from K to $K(b)$ such that $(K(b)|K,v)$ is immediate. Since $[K(b):K] \geq 2$ and (K,v) is henselian, the Lemma of Ostrowski implies that $[K(b):K] \geq p$. This shows that f is a polynomial of smallest degree whose value is not fixed by $(a_\nu)_{\nu<\lambda}$. Hence again by Theorem 2.41, there is an extension of the valuation v from K to $K(a)$ such that $(K(a)|K,v)$ is immediate. Since (K,v) is henselian, this extension coincides with the given valuation on $K(a)$ and we have thus proved that the extension $(K(a)|K,v)$ is immediate. □

Proof of Theorem 6.7: Part 1) follows immediately from Lemma 6.6.

To prove assertions 2) and 3) we consider the following subsets of K. We take A to be a set of elements of K such that the cosets $\frac{1}{p}va + vK$, $a \in A$, form a basis of the $\mathbb{Z}/p\mathbb{Z}$-vector space $\frac{1}{p}vK/vK$. Similarly, we take B to be a set of elements of the valuation ring of (K,v) such that the residues $(bv)^{1/p}$, $b \in B$, form a basis of $(Kv)^{1/p}|Kv$. Then

$$\frac{1}{p}vK \;=\; vK + \sum_{a\in A}\frac{1}{p}va\mathbb{Z} \quad\text{and}\quad (Kv)^{1/p} \;=\; Kv((bv)^{1/p} \mid b \in B)\,.$$

In order to prove assertion 2) of our theorem, we set

$$L_\infty \;:=\; K(a^{1/p}, b^{1/p} \mid a \in A,\, b \in B) \;\subseteq\; K^{1/p} \tag{6.18}$$

and obtain that $vL_\infty = \frac{1}{p}vK$ and $L_\infty v = (Kv)^{1/p}$. So the extension $(K^{1/p}|L_\infty, v)$ is immediate. Lemma 6.6 shows that $(K^{1/p}, v)$ is a maximal immediate extension of (L_∞, v). Our goal is now to show that under the assumptions of the theorem, this extension is of degree at least κ. Once this is proved, we can take $X \subseteq K^{1/p}$ to be a minimal set of generators of the extension $K^{1/p}|L_\infty$. Then the elements of X are p-independent over L_∞. Take any natural number n. As X is infinite, we can choose $x_1,\ldots,x_n \in X$ and set $L_n := L_\infty(X\setminus\{x_1,\ldots,x_n\})$. Then $K^{1/p}|L_n$ is an immediate extension of degree p^n. Similarly, for any infinite cardinal $\lambda \leq \kappa$, take $Y \subseteq X$ of cardinality λ and set $L_\lambda := L_\infty(X\setminus Y)$. Then $K^{1/p}|L_\lambda$ is an immediate algebraic extension of degree λ.

We assume first that $\kappa = (vK : pvK)$, so the set A is infinite. Then we take a partition of A into κ many countably infinite sets A_τ, $\tau < \kappa$. We choose enumerations

$$A_\tau \;=\; \{a_{\tau,i} \mid i \in \mathbb{N}\}\,.$$

For every $\mu < \kappa$ we set $\mathcal{A}_\mu := \bigcup_{\tau<\mu} A_\tau$ and

$$K_\mu \;:=\; K(a^{1/p} \mid a \in \mathcal{A}_\mu)\,.$$

Note that $\mathcal{A}_0 = \emptyset$ and $K_0 = K$. We claim that

$$vK_\mu \;=\; vK + \sum_{a\in\mathcal{A}_\mu} \frac{1}{p} va\mathbb{Z} \quad\text{and}\quad K_\mu v \;=\; Kv\,. \tag{6.19}$$

The inclusions "$\supseteq$" are clear. For the converses, we observe that value group and residue field of K_μ are the unions of the value groups and residue fields of all finite subextensions of $K_\mu|K$. Such subextensions can be written in the form $F = K(a_1^{1/p},\ldots,a_k^{1/p})$ with distinct $a_1,\ldots,a_k \in \mathcal{A}_\mu$. We have that $\frac{1}{p}va_1,\ldots,\frac{1}{p}va_k \in vF$, hence

$$p^k \;\geq\; [F:K] \;\geq\; (vF:vK)[Fv:Kv] \;\geq\; p^k\cdot 1\,,$$

so equality holds everywhere. Consequently, $vF = vK + \sum_{i=1}^k va_i\mathbb{Z}$ and $Fv = Kv$. This proves our claim.

For every $\tau < \kappa$ we choose a sequence $(c_{\tau,i})_{i\in\mathbb{N}}$ of elements in K such that the sequence of values

$$(vc_{\tau,i}a_{\tau,i}^{1/p})_{i\in\mathbb{N}} \tag{6.20}$$

is strictly increasing. For every $n \in \mathbb{N}$, we set

$$\xi_{\tau,n} \;:=\; \sum_{i=1}^{n} c_{\tau,i}a_{\tau,i}^{1/p} \;\in\; K_{\tau+1}\,. \tag{6.21}$$

Then $(\xi_{\tau,n})_{n\in\mathbb{N}}$ satisfies condition (2.12), hence is a pseudo Cauchy sequence. By Theorem 2.42, the sequence $(\xi_{\tau,n})_{n\in\mathbb{N}}$ admits a pseudo limit ξ_τ in the maximal field $(K^{1/p},v)$. In order to show that the degree of $K^{1/p}|L_\infty$ is at least κ, we prove by induction that for every $\mu < \kappa$ and each K' such that $K_{\mu+1} \subseteq K' \subseteq L_\infty$, the pseudo Cauchy sequence $(\xi_{\mu,n})_{n\in\mathbb{N}}$ admits no pseudo limit in $K'(\xi_\tau \mid \tau < \mu)$ and the extension

$$(K'(\xi_\tau \mid \tau \leq \mu)|K',v) \tag{6.22}$$

is immediate.

Take $\mu < \kappa$ and assume that our assertions have already been shown for all $\mu' < \mu$. If $\mu = \mu'+1$ is a successor ordinal, then from (6.22) we readily get that the extension

$$(K'(\xi_\tau \mid \tau < \mu)|K',v) \tag{6.23}$$

is immediate for every K' such that $K_\mu \subseteq K' \subseteq L_\infty$. If μ is a limit ordinal, then (6.23) follows from the induction hypothesis since $K_{\mu'} \subseteq K_\mu \subseteq K'$ for each $\mu' < \mu$ and since the union over the increasing chain of immediate extensions $K'(\xi_\tau \mid \tau \leq \mu')$, $\mu' < \mu$, of (K',v) is again an immediate extension of (K',v).

In order to prove the induction step, suppose towards a contradiction that $(\xi_{\tau,n})_{n\in\mathbb{N}}$ admits a pseudo limit η_μ in $K'(\xi_\tau \mid \tau < \mu)$ for some K' such that $K_{\mu+1} \subseteq K' \subseteq L_\infty$. Then η_μ lies already in a finite extension

$$E \;:=\; K_\mu(\xi_\tau \mid \tau < \mu)(a_1^{1/p},\ldots,a_k^{1/p},b_1^{1/p},\ldots,b_\ell^{1/p}) \tag{6.24}$$

of $K_\mu(\xi_\tau \mid \tau < \mu)$ in $L_\infty(\xi_\tau \mid \tau < \mu)$, with distinct elements $a_1, \ldots, a_k \in A \setminus \mathcal{A}_\mu$ and $b_1, \ldots, b_\ell \in B$. We claim that

$$vE \;=\; vK_\mu + \sum_{i=1}^{k} \frac{1}{p} va_i\mathbb{Z} \quad \text{and} \quad Ev \;=\; K_\mu v((b_1 v)^{1/p}, \ldots, (b_\ell v)^{1/p})\,. \tag{6.25}$$

As the extension (6.23) is immediate for K_μ in place of K', the inclusions "$\supseteq$" are clear. Conversely, from these inclusions together with the equations in (6.19) and our assumption on the a_i, it follows that $(vE : vK) \geq p^k$ as well as $[Ev : Kv] \geq p^\ell$. Therefore, we have that $p^k \cdot p^\ell \geq [E : K] \geq (vE : vK)[Ev : Kv] \geq p^k \cdot p^\ell$, so equality holds everywhere. Consequently, $(vE : vK) = p^k$ and $[Ev : Kv] = p^\ell$, which proves that the inclusions are equalities.

Now we take n to be the minimum of all $i \in \mathbb{N}$ such that $a_{\mu,i}$ is not among the $a_1, \ldots, a_k$. We set $\xi_E := 0$ if $n = 1$, and $\xi_E := \xi_{\mu,n-1}$ otherwise. Then $\eta_\mu - \xi_E \in E$. In contrast, the fact that η_μ is a pseudo limit, together with the first equation of (6.25), yields that

$$v(\eta_\mu - \xi_E) \;=\; v(\xi_{\mu,n} - \xi_E) \;=\; vc_{\mu,n} + \frac{1}{p} va_{\mu,n} \;\notin\; vK_\mu + \sum_{i=1}^{k} \frac{1}{p} va_i\mathbb{Z} \;=\; vE\,.$$

This contradiction proves that $(\xi_{\mu,n})_{n\in\mathbb{N}}$ admits no pseudo limit in $K'(\xi_\tau \mid \tau < \mu)$. Thus in particular, $\xi_\mu \notin K'(\xi_\tau \mid \tau < \mu)$. As $K_{\mu+1} \subseteq K'$, we have that $(\xi_{\mu,n})_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence in $(K'(\xi_\tau \mid \tau < \mu), v)$. Since $[K'(\xi_\tau \mid \tau < \mu)(\xi_\mu) : K'(\xi_\tau \mid \tau < \mu)] = p$ is a prime and $K'(\xi_\tau \mid \tau < \mu)$, as an algebraic extension of the henselian field K, is also henselian, Lemma 6.8 shows that the extension $(K'(\xi_\tau \mid \tau \leq \mu)|K'(\xi_\tau \mid \tau < \mu), v)$ is immediate. As also the extension (6.23) is immediate, we find that the extension $(K'(\xi_\tau \mid \tau \leq \mu)|K', v)$ is immediate. This completes our induction step. Since every extension $L_\infty(\xi_\tau \mid \tau \leq \mu)|L_\infty(\xi_\tau \mid \tau < \mu)$ is nontrivial, it follows that the degree of $K^{1/p}|L_\infty$ is at least κ.

It remains to prove the assertion about field completions. Since $(K^{1/p}, v)$ as a maximal field is complete, Proposition 2.49 implies that $K^{1/p}$ contains a completion L_∞^c of L_∞. If the cofinality of vK is countable, then the elements $c_{\tau,i}$ can be chosen in such a way that the sequence $(vc_{\tau,i}a_{\tau,i}^{1/p})_{i\in\mathbb{N}}$ is cofinal in $\frac{1}{p}vK$. Then the sequence

$$v(\xi_{\tau,n+1} - \xi_{\tau,n}) \;=\; vc_{\tau,n+1}a_{\tau,n+1}^{1/p},$$

$n \in \mathbb{N}$, is cofinal in $\frac{1}{p}vK = vL_\infty$ and thus $(\xi_{\tau,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in L_∞. By Corollary 2.50, this proves that the element ξ_τ lies in L_∞^c. Consequently, $L_\infty^c|L_\infty$ is of degree at least κ. Now as in the case of maximal immediate extensions we choose subextensions L_λ and L_n of $L_\infty^c|L_\infty$ such that $L_\lambda^c|L_\lambda$ and $L_n^c|L_\lambda$ are of the required degrees.

A simple modification of the above arguments allows us to show the assertion of part 2) of the theorem in the case of $\kappa = [Kv : (Kv)^p]$, in which the set B is infinite. Let us describe these modifications.

We take a partition of B into κ many countably infinite sets B_τ, $\tau < \kappa$, and choose enumerations

$$B_\tau \;=\; \{b_{\tau,i} \mid i \in \mathbb{N}\}\,.$$

For every $\mu < \kappa$ we set $\mathcal{B}_\mu := \bigcup_{\tau<\mu} B_\tau$ and

$$K_\mu \;:=\; K(b^{1/p} \mid b \in \mathcal{B}_\mu)\,.$$

Similarly as before, it is shown that

$$vK_\mu \;=\; vK \quad\text{and}\quad K_\mu v \;=\; Kv((bv)^{1/p}) \mid b\in\mathcal{B}_\mu)\,. \tag{6.26}$$

We choose a sequence $(c_i)_{i\in\mathbb{N}}$ of elements in K with strictly increasing values. For every $\tau<\kappa$ and $n\in\mathbb{N}$, we set

$$\xi_{\tau,n} \;:=\; \sum_{i=1}^{n} c_i b_{\tau,i}^{1/p} \;\in\; K_{\tau+1}\,. \tag{6.27}$$

Now the only further part of the proof that needs to be modified is the one that shows that $\eta_\mu\in E$, where η_μ is a pseudo limit of $(\xi_{\mu,n})_{n\in\mathbb{N}}$, leads to a contradiction. In the present case, we take n to be the minimum of all $i\in\mathbb{N}$ such that $b_{\mu,i}$ is not among the $b_1,\ldots,b_\ell$. As before, we set $\xi_E:=0$ if $n=1$, and $\xi_E:=\xi_{\mu,n-1}$ otherwise. Then $\eta_\mu-\xi_E\in E$. In contrast, the fact that η_μ is a pseudo limit, together with the second equation of (6.25), yields that

$$\begin{array}{rcl} c_n^{-1}(\eta_\mu-\xi_E)v &=& c_n^{-1}(\xi_{\mu,n}-\xi_E)v \;=\; (b_{\mu,n}^{1/p})v \;=\; (b_{\mu,n}v)^{1/p}\\ &\notin& K_\mu v((b_1v)^{1/p},\ldots,(b_\ell v)^{1/p}) \;=\; Ev\,, \end{array}$$

a contradiction. This completes our modification and thereby the proof that the extension $(K^{1/p}|L_\infty,v)$ is of degree at least κ.

Again, if the cofinality of vK is countable, then the elements c_i can be chosen in such a way that the sequence of their values is cofinal in vK. Repeating the argument from the first part of the proof, we deduce the assertion about field completion.

We now turn to part 3) of the theorem. Again, we consider separately the cases of $\kappa=(vK:pvK)$ and of $\kappa=[Kv:(Kv)^p]$.

We assume first that $\kappa=(vK:pvK)$ and take a partition of A as in the proof of part 2). Further, we set $s(1)=0$ and $s(m)=1+2+\cdots+(m-1)$ for $m>1$. For every $\tau<\mu$ and every $m\in\mathbb{N}$, we set

$$z_{\tau,m} \;:=\; \sum_{i=1}^{m} d_{\tau,s(m)+i}\,a_{\tau,s(m)+i}^{p^{-i}} \in K^{1/p^\infty}\,,$$

where $d_{\tau,j}$ are elements from K such that for every $m\in\mathbb{N}$,

1) the sequence $(vd_{\tau,s(m)+i}a_{\tau,s(m)+i}^{p^{-i}})_{1\le i\le m}$ is strictly increasing,

2) $vd_{\tau,s(m)+m}a_{\tau,s(m)+m}^{p^{-m}} < vd_{\tau,s(m+1)+1}a_{\tau,s(m+1)+1}^{p^{-1}}$.

If the cofinality of vK is countable, then the elements $d_{\tau,i}$ can be chosen in such a way that the sequence that results from 1) and 2) is cofinal in $\frac{1}{p}vK$. Then also the sequence of the values $(vz_{\tau,m})_{m\in\mathbb{N}}$ is cofinal in $\frac{1}{p}vK$.

We note that $z_{\tau,m}^{p^m}\in K$ with

$$[K(z_{\tau,m}):K] \;=\; p^m \quad\text{and}\quad \frac{1}{p}va_{\tau,s(m)+1},\ldots,\frac{1}{p}va_{\tau,s(m)+m}\in vK(z_{\tau,m})\,. \tag{6.28}$$

We set

$$L_\mu \;:=\; K(z_{\tau,m}\mid \tau<\mu,\, m\in\mathbb{N}) \;\text{ for } \mu\le\kappa\,,\; \text{and} \quad L \;:=\; L_\kappa\,.$$

Further, we fix a maximal immediate extension (M,v) of (L,v). We claim that

$$vL_\mu \;=\; vK + \sum_{a\in\mathcal{A}_\mu} \frac{1}{p}va \quad\text{and}\quad L_\mu v \;=\; Kv\,. \tag{6.29}$$

In particular, this shows that

$$vL = \frac{1}{p}vK \quad\text{and}\quad Lv \;=\; Kv\,. \tag{6.30}$$

To prove our claim, we observe that the first inclusion "$\supseteq$" in (6.29) follows from (6.28). We choose any $\mu<\kappa$, $k\in\mathbb{N}$, $\tau_1,\ldots,\tau_k<\mu$ and $m_1,\ldots,m_k\in\mathbb{N}$ such that the pairs (τ_i,m_i), $1\le i\le k$, are distinct. Then we compute, using (6.28):

$$\begin{aligned}
p^{m_1}\cdot\ldots\cdot p^{m_k} \;&\geq\; [K(z_{\tau_1,m_1},\ldots,z_{\tau_k,m_k}):K]\\
&\geq\; (vK(z_{\tau_1,m_1},\ldots,z_{\tau_k,m_k}):vK)[K(z_{\tau_1,m_1},\ldots,z_{\tau_k,m_k})v:Kv]\\
&\geq\; (vK(z_{\tau_1,m_1},\ldots,z_{\tau_k,m_k}):vK)\\
&\geq\; \Big(vK+\sum_{j=1}^{k}\sum_{i=1}^{m_j}\frac{1}{p}va_{\tau_j,s(m_j)+i}\mathbb{Z} \,:\, vK\Big) \;\geq\; p^{m_1}\cdot\ldots\cdot p^{m_k}\,,
\end{aligned}$$

showing that equality holds everywhere. Therefore,

$$vK(z_{\tau_1,m_1},\ldots,z_{\tau_k,m_k}) \;=\; vK+\sum_{j=1}^{k}\sum_{i=1}^{m_j}\frac{1}{p}va_{\tau_j,s(m_j)+i}\mathbb{Z} \;\subseteq\; vK+\sum_{a\in\mathcal{A}_\mu}\frac{1}{p}va$$

and

$$K(z_{\tau_1,m_1},\ldots,z_{\tau_k,m_k})v \;=\; Kv\,.$$

Since the value group and residue field of L_μ are the unions of the value groups and residue fields of all subfields of the above form, this proves our claim.

For every $\tau<\kappa$ and $n\in\mathbb{N}$, we set

$$\zeta_{\tau,n} := \sum_{m=1}^{n} z_{\tau,m}\in L.$$

Then $(\zeta_{\tau,n})_{n\in\mathbb{N}}$ satisfies condition (2.12), thus is a pseudo Cauchy sequence in (L,v). Hence the sequence admits a pseudo limit ζ_τ in the maximal field (M,v). In order to show that the transcendence degree of $M|L$ is at least κ, we prove by induction that for every $\mu<\kappa$ and every field L' such that $L_{\mu+1}\subseteq L'\subseteq L$, the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$ is of transcendental type over $L'(\zeta_\tau\mid\tau<\mu)$, so that the extension $(L'(\zeta_\tau\mid\tau\le\mu)|L'(\zeta_\tau\mid\tau<\mu),v)$ is immediate and transcendental and then also the extension

$$(L'(\zeta_\tau\mid\tau\le\mu)|L',v) \tag{6.31}$$

is immediate.

Take $\mu<\kappa$ and assume that our assertions have already been shown for all $\mu'<\mu$. If $\mu=\mu'+1$ is a successor ordinal, then from (6.31) we readily get that the extension

$$(L'(\zeta_\tau\mid\tau<\mu)|L',v) \tag{6.32}$$

is immediate for every L' such that $L_\mu \subseteq L' \subseteq L$. If μ is a limit ordinal, then (6.32) follows from the induction hypothesis since $L_{\mu'} \subseteq L_\mu \subseteq L'$ for each $\mu' < \mu$ and since the union over an increasing chain of immediate extensions of (L', v) is again an immediate extension of (L', v).

In order to prove the induction step, take any L' such that $L_{\mu+1} \subseteq L' \subseteq L$. Suppose towards a contradiction that the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$ in $L_{\mu+1}$ is of algebraic type over $(L'(\zeta_\tau \mid \tau < \mu), v)$ (which includes the case where it has a pseudo limit in $L'(\zeta_\tau \mid \tau < \mu)$). Then by Theorem 2.41 there exists an immediate algebraic extension $(L'(\zeta_\tau \mid \tau < \mu)(d)|L'(\zeta_\tau \mid \tau < \mu), v)$ with d a pseudo limit of the sequence. The element d is also algebraic over $L_\mu(\zeta_\tau \mid \tau < \mu)$. On the other hand, we will now show that from the fact that d is a pseudo limit of $(\zeta_{\mu,n})_{n\in\mathbb{N}}$ it follows that the value group $vL_\mu(\zeta_\tau \mid \tau < \mu)(d)$ is an infinite extension of $vL_\mu(\zeta_\tau \mid \tau < \mu)$. Take $n \in \mathbb{N}$ and define

$$\eta_{\mu,n} \;:=\; \zeta_{\mu,n}^{p^{n-1}} - d_{\mu,s(n)+n}^{p^{n-1}} a_{\mu,s(n)+n}^{1/p} \;=\; \zeta_{\mu,n-1}^{p^{n-1}} + \sum_{i=1}^{n-1} d_{\mu,s(n)+i}^{p^{n-1}} a_{\mu,s(n)+i}^{p^{n-1-i}} \in K\;.$$

Since d is a pseudo limit of the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$, we deduce that

$$v(d-\zeta_{\mu,n}) = v(\zeta_{\mu,n+1}-\zeta_{\mu,n}) = vz_{\mu,n+1} = vd_{\mu,s(n+1)+1} a_{\mu,s(n+1)+1}^{1/p} > vd_{\mu,s(n)+n} a_{\mu,s(n)+n}^{p^{-n}}. \tag{6.33}$$

Therefore,

$$\begin{aligned}
v\left(d^{p^{n-1}} - \eta_{\mu,n}\right) &= v\left(d^{p^{n-1}} - \zeta_{\mu,n}^{p^{n-1}} + d_{\mu,s(n)+n}^{p^{n-1}} a_{\mu,s(n)+n}^{1/p}\right)\\
&= p^{n-1} v\left(d - \zeta_{\mu,n} + d_{\mu,s(n)+n} a_{\mu,s(n)+n}^{p^{-n}}\right)\\
&= p^{n-1} \min\left\{v(d-\zeta_{\mu,n})\,,\; v\left(d_{\mu,s(n)+n} a_{\mu,s(n)+n}^{p^{-n}}\right)\right\}\\
&= p^{n-1} v\left(d_{\mu,s(n)+n} a_{\mu,s(n)+n}^{p^{-n}}\right)\\
&= p^{n-1} vd_{\mu,s(n)+n} + \frac{1}{p} va_{\mu,s(n)+n}\;,
\end{aligned}$$

which shows that

$$\frac{1}{p} va_{\mu,s(n)+n} \;\in\; vL_\mu(\zeta_\mu \mid \mu < \tau)(d)$$

for all $n \in \mathbb{N}$. In view of (6.29), these values are not in vL_μ. Since the extension (6.32) is immediate for L_μ in place of L', they are also not in $vL_\mu(\zeta_\tau \mid \tau < \mu)$. It follows that the index $(vL_\mu(\zeta_\tau \mid \tau < \mu)(d) : vL_\mu(\zeta_\tau \mid \tau < \mu))$ is infinite. Together with the fundamental inequality this contradicts the fact that the extension $L_\mu(\zeta_\tau \mid \tau < \mu)(d)|L_\mu(\zeta_\tau \mid \tau < \mu)$ is finite. This contradiction proves that the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$ is of transcendental type over $L'(\zeta_\tau \mid \tau < \mu)$. From Theorem 2.40 it follows that $(L'(\zeta_\tau \mid \tau \leq \mu)|L'(\zeta_\tau \mid \tau < \mu), v)$ is an immediate transcendental extension. Since the extension (6.32) is immediate, we obtain that also $(L'(\zeta_\tau \mid \tau \leq \mu)|L', v)$ is immediate.

This completes our induction step. By induction on μ we have therefore shown that $(L(\zeta_\tau \mid \tau < \mu), v)$ is an immediate extension of (L, v) for each $\mu < \kappa$, which yields that also the union $(L(\zeta_\tau \mid \tau < \kappa), v)$ of these fields is an immediate extension of (L, v). As every extension $L(\zeta_\tau \mid \tau \leq \mu)|L(\zeta_\tau \mid \tau < \mu)$ is transcendental, the transcendence degree of $L(\zeta_\tau \mid \tau < \kappa)$ over L is at least κ.

If the cofinality of vK is countable, then as we have already noticed, the elements $d_{\tau,i}$ can be chosen in such a way that the sequence $(vz_{\tau,n})_{n\in\mathbb{N}}$ is cofinal in vL. Then also the sequence of the values

$$v(\zeta_{\tau,n+1}-\zeta_{\tau,n}) \;=\; vz_{\tau,n+1},$$

$n\in\mathbb{N}$, is cofinal in vL. Hence $(\zeta_{\tau,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in (L,v). Similarly to the proof of part b) we deduce that (M,v) contains a completion L^c of L and every ζ_τ lies in L^c.

Note that if we replace L_μ by the field

$$K(z_{\tau,m},\, b^{1/p}\mid \tau<\mu,\, m\in\mathbb{N}, b\in B) \;=\; L_\mu(b^{1/p}\mid b\in B),$$

for $\mu\leq\kappa$, and L by $L_\kappa(b^{1/p}\mid b\in B)$, then the above arguments remain true. However, $vL_\kappa(b^{1/p}\mid b\in B)=\frac{1}{p}vK$ and $L_\kappa(b^{1/p}\mid b\in B)v=(Kv)^{1/p}$ simultaneously. This proves the additional assertion of the case 3).

A simple modification of the above arguments allows us to show the assertion of part 3) of the theorem in the case of $\kappa=[Kv:(Kv)^p]$. We take the partition of B as in the proof of part 2). We now list the modifications.

Since the $vb=0$ for all $b\in B$, the only requirement for the elements $d_{\tau,i}$ that we need is that $vd_{\tau,i}<vd_{\tau,j}$ for $i<j$. If the cofinality of vK is countable, then the elements $d_{\tau,i}$ can be chosen in such a way that the sequence of their values is cofinal in $\frac{1}{p}vK=vL$. We set

$$z_{\tau,m} \;:=\; \sum_{i=1}^{m} d_{\tau,s(m)+i}b_{\tau,s(m)+i}^{p^{-i}}\in K^{1/p^\infty}\;,$$

Equation (6.28) is replaced by

$$[K(z_{\tau,m}):K] \;=\; p^m \quad\text{and}\quad (b_{\tau,s(m)+1}v)^{1/p},\ldots,(b_{\tau,s(m)+m}v)^{1/p}\in K(z_{\tau,m})v\;. \tag{6.34}$$

One proves in a similar way as before that

$$vL_\mu \;=\; vK \quad\text{and}\quad L_\mu v \;=\; Kv((bv)^{1/p}\mid b\in\mathcal{B}_\mu)\;. \tag{6.35}$$

In particular, this shows that

$$vL=vK \quad\text{and}\quad Lv \;=\; (Kv)^{1/p}\;. \tag{6.36}$$

Now the only further part of the proof that needs to be modified is the one that shows that the extension $L_\mu(\zeta_\tau\mid\tau<\mu)(d)|L_\mu(\zeta_\tau\mid\tau<\mu)$ cannot be finite. We define $\eta_{\mu,n}$ as before, with "b" in place of "a". Also (6.33) holds with "b" in place of "a", whence

$$v\,d_{\mu,s(n)+n}^{-p^{n-1}}\left(d^{p^{n-1}}-\zeta_{\mu,n}^{p^{n-1}}\right) \;=\; p^{n-1}vd_{\mu,s(n)+n}^{-1}\,(d-\zeta_{\mu,n}) \;>\; 0\,.$$

This leads to

$$\begin{aligned}
d_{\mu,s(n)+n}^{-p^{n-1}}\left(d^{p^{n-1}}-\eta_{\mu,n}\right)v \;&=\; d_{\mu,s(n)+n}^{-p^{n-1}}\left(d^{p^{n-1}}-\zeta_{\mu,n}^{p^{n-1}}+d_{\mu,s(n)+n}^{p^{n-1}}b_{\mu,s(n)+n}^{1/p}\right)v\\
&=\; \left(d_{\mu,s(n)+n}^{-p^{n-1}}(d^{p^{n-1}}-\zeta_{\mu,n}^{p^{n-1}})+b_{\mu,s(n)+n}^{1/p}\right)v\\
&=\; (b_{\mu,s(n)+n}^{1/p})v \;=\; (b_{\mu,s(n)+n}v)^{1/p}\;,
\end{aligned}$$

which shows that

$$(b_{\mu,s(n)+n}v)^{1/p} \in L_\mu(\zeta_\tau \mid \tau < \mu)(d)v$$

for all $n \in \mathbb{N}$. In view of (6.35), these residues are not in $L_\mu v$. As before, this is shown to contradict d being algebraic over $L_\mu(\zeta_\tau \mid \tau < \mu)$. This completes our modification and thereby the proof that $(M|L,v)$ is of transcendence degree at least κ.

To prove the additional assertion of case 3), we replace L_μ by the field

$$K(z_{\tau,m},\, a^{1/p} \mid \tau < \mu,\, m \in \mathbb{N}, a \in A) \;=\; L_\mu(a^{1/p} \mid a \in A),$$

$\mu \leq \kappa$ and L by $L_\kappa(a^{1/p} \mid a \in A)$. □

We note that in contrast to part 1) and 2), the field L of part 3) is *not* contained in $K^{1/p}$. Case 2) of this theorem is a generalization of Nagata's example ([38, Appendix, Example (E3.1), pp. 206-207]). Similar to that example, the valued fields in 2) are nonmaximal fields admitting an algebraic maximal immediate extension.

Example 6.9. As an example of a field satisfying the conditions of Theorem 6.7 we can take a power series field $k\big((x^\Gamma)\big)$ with the x-adic valuation v, where $\mathrm{char} k = p > 0$ and the quotient group $\Gamma/p\Gamma$ is infinite or k is of infinite p-degree. The first case holds for instance for $K_1 := \mathbb{F}_p\big((x^G)\big)$, where G is an ordered subgroup of the reals of the form $\bigoplus_{i\in\mathbb{N}} r_i\mathbb{Z}$. In the second case we can choose $K_2 := k\big((x^{\mathbb{Z}})\big)$ with $k := \mathbb{F}_p(t_i \mid i \in \mathbb{N})$, where t_i , $i \in \mathbb{N}$, are algebraically independent over $\mathbb{F}_p$.

Consider now the field (K_2, v). Set

$$L_\infty := K_2\big(x^{1/p}, t_i^{1/p} \mid i \in \mathbb{N}\big).$$

Then L_∞ is a subfield of $K_2^{1/p}$. It fits the definition in (6.18) with K_2 in place of K, $A = \{x\}$ and $B = \{t_i \mid i \in \mathbb{N}\}$. From the proof of Theorem 6.7 it follows that $(K_2^{1/p}, v)$ is a maximal immediate extension of (L_∞, v). We show that in this case we have even more: $K_2^{1/p}$ is the completion of L_∞. As $vL_\infty = vK_2^{1/p}$, Proposition 2.49 yields that $K_2^{1/p}$ contains a completion L^c_∞ of L_∞. For the proof of the converse inclusion, note first that $K_2^{1/p} = k'\big((x^{\frac{1}{p}\mathbb{Z}})\big)$, with $k' = \mathbb{F}_p(t_i^{1/p} \mid i \in \mathbb{N})$. Take any $\xi \in K_2^{1/p} \setminus L_\infty$. Then the element is of the form

$$\xi \;=\; \sum_{i=1}^{\infty} c_i x^{\frac{n_i}{p}}$$

with $c_i \in k'$ and a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$. Since $\xi \notin L_\infty$, infinitely many of the elements c_i are nonzero. Thus we can omit the summands with coefficients equal to zero, and we can assume that all the coefficients c_i are nonzero.

For every natural number N set

$$\xi_N \;:=\; \sum_{i=1}^{N} c_i x^{\frac{n_i}{p}} \in L_\infty.$$

Then $(\xi_N)_{N\in\mathbb{N}}$ satisfies condition (2.12), hence is a pseudo Cauchy sequence in L_∞. Furthermore, from the equation (2.14) it follows that ξ is a pseudo limit of the sequence. Since the sequence of the values $v(\xi_{N+1} - \xi_N) = \frac{n_{N+1}}{p}$, $N \in \mathbb{N}$, is cofinal in $\frac{1}{p}\mathbb{Z} = vL_\infty$, we obtain that

$(\xi_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in L_∞. By Corollary 2.50 this yields that $\xi \in L_\infty^c$. Therefore, $K_2^{1/p} = L_\infty^c$. Hence the maximal immediate extension of L_∞ is unique up to isomorphism.

The next theorem will show that if we replace K_2 by K_1, the assertion about uniqueness of maximal immediate extension is very far from being true. $\diamond$

Theorem 6.10. *Assume that (K,v) is a maximal field of characteristic $p>0$ satisfying one of the following conditions:*

1) vK/pvK is infinite and vK admits a set of representatives of the cosets modulo pvK which contains an infinite bounded subset, or

2) the residue field extension $Kv|(Kv)^p$ is infinite and the value group vK is not discrete.

Then there is an infinite purely inseparable extension (L,v) of (K,v) which admits $(K^{1/p},v)$ as an algebraic maximal immediate extension, but also admits a maximal immediate extension of infinite transcendence degree.

Proof. Note that a field (K,v) which satisfies the assumptions of Theorem 6.10 also satisfies the assumptions of Theorem 6.7. We choose the sets $A,B\subseteq K^{1/p}$ and define $L:=L_\infty$ as in the proof of part 2) of Theorem 6.7. Then, as we have already seen, $(K^{1/p},v)$ is a maximal immediate extension of (L,v).

To show the existence of an immediate extension of L of infinite transcendence degree over L, we consider separately the cases 1) and 2) of the theorem. We assume first that the conditions of case 1) hold. Then the set A can be chosen so as to contain an infinite countable subset A' such that the set of values $S=\{va \mid a\in A'\}$ is bounded. It must contain a bounded infinite strictly increasing or a bounded infinite strictly decreasing sequence. If it does not contain the former, we replace A' by $\{a^{-1}\mid a\in A'\}$, thereby passing from S to $-S$. Now we can choose a sequence $(a_j)_{j\in\mathbb{N}}$ of elements in A' such that the sequence $(va_j)_{j\in\mathbb{N}}$ is strictly increasing and bounded by some $\gamma\in vK$. We partition the sequence $(a_j)_{j\in\mathbb{N}}$ into countably many subsequences

$$(a_{N,i})_{i\in\mathbb{N}} \qquad (N\in\mathbb{N})\,.$$

As in the proof of Theorem 6.7, we define $K_N := K(a_{n,i}^{1/p} \mid n<N\,,\, i\in\mathbb{N})\subseteq K^{1/p}$.

For every $N\in\mathbb{N}$ we consider the pseudo Cauchy sequence $(\xi_{N,m})_{m\in\mathbb{N}}$ defined by

$$\xi_{N,m} \;:=\; \sum_{i=1}^{m} a_{N,i}^{1/p} \;\in\; K_{N+1}$$

and the pseudo limit ξ_N of the sequence in the maximal immediate extension $(K^{1/p},v)$ of (L,v). We show that for every N the pseudo limit ξ_N does not lie in the completion L^c of (L,v). Fix $N\in\mathbb{N}$ and take any $d\in L$. Then d lies already in some finite extension

$$E \;:=\; K(a_1^{1/p},\dots,a_k^{1/p},b_1^{1/p},\dots,b_l^{1/p})$$

of K in L. Choosing ξ_E as in the proof of Theorem 6.7, we obtain that $\xi_E - d\in E$. But from equalities (6.25) with $\mu=0$ it follows that $v(\xi_N-\xi_E)=\frac{1}{p}va_{N,n}\notin vE$. Thus,

$$v(\xi_N-d) \;=\; \min\{v(\xi_N-\xi_E),v(\xi_E-d)\}\le \frac{1}{p}\,va_{N,n}<\frac{1}{p}\,\gamma\,.$$

Hence the values $v(\xi_N - d)$, $d \in L$, are bounded by $\frac{1}{p}\gamma$ and consequently, $\xi_N \notin L^c$.

Again from the proof of Theorem 6.7 it follows that for every field K' such that $K_1 \subseteq K' \subseteq L$ the extension $(K'(\xi_1)|K', v)$ is immediate and purely inseparable of degree p. Since $\xi_1 \notin L^c$, from Theorem 4.1 we deduce that for an element $d_1 \in K^\times$ satisfying inequality (4.1) with $\eta = \xi_1$, a root ϑ_1 of the polynomial

$$f_1 \;:=\; X^p - X - \left(\frac{\xi_1}{d_1}\right)^p$$

generates an immediate Galois extension $(L(\vartheta_1)|L, v)$ of degree p. Take any field K' such that $K_1 \subseteq K' \subseteq L$. Then, Proposition 2.49 yields that $K'^c \subseteq L^c$. Thus $\xi_1 \notin K'^c$. Moreover, $\operatorname{dist}(\xi, K') \leq \operatorname{dist}(\xi, L)$ and thus the element d_1 satisfies inequality (4.1) with K' in place of L. Therefore also $(K'(\vartheta_1)|K', v)$ is an immediate extension of degree p.

Take any $m > 1$. Suppose that we have shown that for every $l < m$ there is $d_l \in K^\times$ such that a root ϑ_l of the polynomial

$$f_l \;:=\; X^p - X - \left(\frac{\xi_l}{d_l}\right)^p$$

generates, for any field K' with $K_{l+1} \subseteq K' \subseteq L$, an immediate Galois extension $(K'(\vartheta_1, \ldots, \vartheta_l)|K'(\vartheta_1, \ldots, \vartheta_{l-1}), v)$ of degree p. It follows in particular, that the extension $(K_{m+1}(\vartheta_1, \ldots, \vartheta_{m-1})|K_{m+1}, v)$ is immediate. Take any field K' such that $K_{m+1} \subseteq K' \subseteq L$. Replacing in the argumentation of the proof of part 2) of Theorem 6.7 the field $K'(\xi_l \mid l < m)$ by $K'(\vartheta_l \mid l < m)$, we deduce that $(K'(\vartheta_1, \ldots, \vartheta_{m-1})(\xi_m)|K'(\vartheta_1, \ldots, \vartheta_{m-1}), v)$ is an immediate purely inseparable extension of degree p. Since $\xi_m \notin L^c$ and $L(\vartheta_1, \ldots, \vartheta_{m-1})^c = L^c(\vartheta_1, \ldots, \vartheta_{m-1})$ is a separable extension of L, linearly disjoint from the purely inseparable extension $L^c(\xi_m)|L^c$, we obtain that

$$[L^c(\vartheta_1, \ldots, \vartheta_{m-1})(\xi_m) : L^c(\vartheta_1, \ldots, \vartheta_{m-1})] \;=\; p.$$

Therefore, ξ_m does not lie in $L(\vartheta_1, \ldots, \vartheta_{m-1})^c$. Thus, from Theorem 4.1 it follows that for an element $d_m \in K^\times$ satisfying inequality (4.1) with $\eta = \xi_m$ and $L(\vartheta_1, \ldots, \vartheta_{m-1})$ in place of L, a root ϑ_m of the polynomial $f_m := X^p - X - (\xi_m/d_m)^p$ generates an immediate Galois extension $(L(\vartheta_1, \ldots, \vartheta_m)|L(\vartheta_1, \ldots, \vartheta_{m-1}), v)$ of degree p. As in the case of $m = 1$ we deduce that also the extension $(K'(\vartheta_1, \ldots, \vartheta_m)|K'(\vartheta_1, \ldots, \vartheta_{m-1}), v)$ is immediate.

By induction, we obtain an infinite immediate separable-algebraic extension $F := L(\vartheta_m \mid m \in \mathbb{N})$ of L. Since K as a maximal field is henselian and $L|K$ is an algebraic extension, L is also a henselian field. Hence from the separable-algebraic case of Theorem 1.2 it follows that each maximal immediate extension (M, v) of (F, v) has infinite transcendence degree over F. Since $(F|L, v)$ is immediate, M is also a maximal immediate extension of L.

Similar arguments allow us to prove the assertion in the case of an infinite residue field extension $Kv|(Kv)^p$ when the value group vK is not discrete. Let us describe the modifications.

Take an infinite countable subset B' of B and an infinite partition of B' into infinite sets

$$B_N \;=\; \{b_{N,i} \mid i \in \mathbb{N}\} \qquad (N \in \mathbb{N})\;.$$

Since vK is not discrete, we can choose elements $c_i \in K$ such that the sequence $(vc_i)_{i\in\mathbb{N}}$ of their values is strictly increasing and bounded by some element $\gamma \in vK$. For every N we consider the pseudo Cauchy sequence $(\xi_{N,m})_{m\in\mathbb{N}}$ defined by (6.27).

The only further part of the proof that needs to be modified is the one that shows that $\xi_N \notin L^c$. More precisely, we need to show that for any element $d \in E$ we have that $v(\xi_N - d) < \gamma$. Take ξ_E as in the second case of the proof of part 2) of Theorem 6.7. From the equalities (6.25) with $\mu = 0$ we deduce that $c_n^{-1}(\xi_N - \xi_E)v = (b_{N,n}v)^{1/p} \notin Ev$. Suppose that $v(\xi_N - d) > v(\xi_N - \xi_E)$. Then

$$v\left(c_n^{-1}(\xi_N - \xi_E) - c_n^{-1}(\xi_E - d)\right) \;>\; vc_n^{-1}(\xi_N - \xi_E) \;=\; 0.$$

It follows that $c_n^{-1}(\xi_N - \xi_E)v = c_n^{-1}(\xi_E - d)v \in Ev$, a contradiction. Consequently,

$$v(\xi_N - d) \;\leq\; v(\xi_N - \xi_E) \;=\; vc_n \;<\; \gamma\,.$$

This completes our modification and thereby the proof that (L, v) admits a maximal immediate extension of infinite transcendence degree over L. □

Finally, let us discuss the case of transcendental extensions (L, v) of a maximal field (K, v). In view of the valuation-transcendental case of Theorem 1.2, it remains to consider the **valuation-algebraic case** where vL/vK is a torsion group and $Lv|Kv$ is algebraic.

Theorem 6.11. *Take a maximal field (K, v) and a transcendental extension (L, v) of (K, v) of finite transcendence degree. Assume that $Lv|Kv$ is separable-algebraic and vL/vK is a torsion group such that the characteristic of Kv does not divide the orders of its elements. Then $Lv|Kv$ or vL/vK is infinite and every maximal immediate extension of (L, v) has infinite transcendence degree over L.*

Proof. Take an extension $(L|K, v)$ as in the assumptions of the theorem. In view of the value-algebraic and residue-algebraic cases of Theorem 1.2, it suffices to show that at least one of the extensions $vL|vK$ or $Lv|Kv$ is infinite.

Take K' to be the relative algebraic closure of K in L^h. By the assumptions on the residue field and value group extensions of $(L|K, v)$, it follows from Lemma 2.14 that $vK' = vL^h = vL$ and $K'v = L^h v = Lv$. Therefore, $(L^h|K', v)$ is an immediate transcendental extension.

Suppose that the value group extension and the residue field extension of $(L|K, v)$ and hence of $(K'|K, v)$ were finite. Since K is henselian and a defectless field by Theorem 2.32, the degree $[K' : K]$ is equal to $(vK' : vK)[K'v : Kv]$ and hence would be finite, so (K', v) would again be a maximal field, which contradicts the fact that $(L^h|K', v)$ is a nontrivial immediate extension. □

6.3 Valued rational function fields

We will apply now Theorem 1.2 to the problem of describing the possible extensions of a valuation from a given valued field (K, v) to a rational function field over K. The problem was considered in [22]. The following theorem is a generalization of one of the cases treated in Theorem 1.6 of that paper.

Theorem 6.12. *Take a nontrivially valued field (K,v), an ordered abelian group extension Γ of vK such that Γ/vK is finite, and a finite field extension $k|Kv$. Assume that there is a separable-algebraic extension (L,v) of (K,v) such that $vL\subseteq\Gamma$, $Lv\subseteq k$, and that with respect to some extension of v to the algebraic closure of L, the corresponding extension $L^h|K^h$ is infinite. Then for any natural number n there is an extension w of the valuation v of K to the rational function field $K(x_1,\ldots,x_n)|K$ such that*

$$wK(x_1,\ldots,x_n) \;=\; \Gamma \quad \textit{and} \quad K(x_1,\ldots,x_n)w \;=\; k.$$

For the proof we will need the following facts (cf. Theorem 2.14 and Lemma 3.13 of [22]).

Lemma 6.13. *Let (K,v) be a non-trivially valued field, Γ an ordered abelian group extension of vK such that Γ/vK is finite and k a finite extension of Kv. Then there is a simple separable-algebraic extension (L,v) of (K,v) such that $vL=\Gamma$ and $Lv=k$.*

Lemma 6.14. *Assume that $K(a)|K$ is a separable-algebraic extension. Assume further that $K(x)$ is a rational function field over K and v is a valuation of $\widetilde{K(x)}$ such that*

$$v(x-a) \;>\; \mathrm{kras}(a,K).$$

Then $vK(a)\subseteq vK(x)$ and $K(a)v\subseteq K(x)v$.

Proof of Theorem 6.12: Assume that $(L^h|K^h,v)$ is an infinite extension. Since $(L^h|L,v)$ is an immediate extension, $vK\subseteq vL^h\subseteq\Gamma$ and $Kv\subseteq L^hv\subseteq k$. Thus Γ/vL^h is a finite group and $k|L^hv$ is a finite extension. As the valuation v is nontrivial on K, hence also on L^h, from Lemma 6.13 it follows that there is a separable-algebraic extension $(L^h(a),v)$ of (L^h,v) such that $vL^h(a)=\Gamma$ and $L^h(a)v=k$. Then $(L(a),v)$ is a separable-algebraic extension of (K,v) and $L^h(a)=L(a)^h$ is an infinite extension of K^h. Therefore, without loss of generality we can assume that $vL=\Gamma$ and $Lv=k$.

Since $L^h|K^h$ is an infinite separable-algebraic extension, from the separable-algebraic case of Theorem 1.2 it follows that (L,v) admits an immediate extension (M,v) of infinite transcendence degree. Take elements $x_1,\ldots,x_{n-1},y\in M$ algebraically independent over L and set

$$E \;:=\; K(x_1,\ldots,x_{n-1})\subseteq M.$$

As $(L(x_1,\ldots,x_{n-1})|L,v)$ is an immediate extension, we obtain that

$$vE \;\subseteq\; vL(x_1,\ldots,x_{n-1}) \;=\; \Gamma \quad\text{and}\quad Ev \;\subseteq\; L(x_1,\ldots,x_{n-1})v \;=\; k.$$

Since $L|K$ is a separable-algebraic extension, vL/vK is a finite group and the extension $Lv|Kv$ is finite, there is a finite subextension $L'|K$ of $L|K$ such that $vL'=vL$ and $L'v=Lv$. Moreover, by the Theorem of Primitive Element, we can choose $L'|K$ to be a simple extension $K(b)|K$ for some $b\in L$. Then $E(b)\subseteq L(x_1,\ldots,x_{n-1})$, hence $vE(b)=vL=\Gamma$ and $E(b)v=Lv=k$.

Multiplying y by an element in $K^\times$ of large enough value if necessary, we can assume that

$$vy \;>\; \mathrm{kras}(b,E)\in\widetilde{vE}=\widetilde{vK}.$$

Since $E(b)\subseteq E(y,b)\subseteq L(x_1,\ldots,x_{n-1},y)\subseteq M$, the extension $(L(x_1,\ldots,x_{n-1},y)|E(b),v)$ and hence also the extension $(E(y,b)|E(b),v)$ is immediate. Take an element x_n in some field extension of E, transcendental over E and define by $y \mapsto x_n - b$ an isomorphism of $E(y, b)$ onto $E(x_n, b)$. This isomorphism induces a valuation w on $E(x_n, b)$, which is an extension of the valuation v of $E(b)$ with $w(x_n - b) = vy$. Hence, $w(x_n - b) >$kras(b, E) and from Lemma 6.14 we deduce that

$$\begin{aligned} vL &= vE(b) \subseteq wE(x_n) = vE(y+b) \subseteq vL(x_1, \ldots, x_{n-1}, y) = vL, \\ vL &= E(b)v \subseteq E(x_n)w = E(y+b)v \subseteq L(x_1, \ldots, x_{n-1}, y)v = Lv, \end{aligned}$$

since $(L(x_1, \ldots, x_{n-1}, y)|L, v)$ is an immediate extension. Thus equality holds everywhere and w is an extension of v from K to $K(x_1, \ldots, x_n)$ such that

$$wK(x_1, \ldots, x_n) = wE(x_n) = vL = \Gamma \quad \text{and} \quad K(x_1, \ldots, x_n)w = E(x_n)w = Lw = k.$$

□

Theorem 6.12 was proved in [22] in the case of an immediate extension $L|K$. However, the assumption that the extension $L^h|K^h$ remains infinite is there omitted. The next example shows that without this assumption Theorem 6.12 may not hold.

Example 6.15. Assume that (F, v) is a valued field admitting a unique maximal immediate extension (M, v), transcendental over F. Take a transcendence basis $\mathcal{T}$ of $M|F$ and set $K := F(\mathcal{T})$. Then (M, v) is the unique maximal immediate extension of (K, v), as every immediate extension of K is also an immediate extension of F. Furthermore, $M|K$ is algebraic. From Lemma 6.3 it follows that $L = K^h$ is an infinite separable-algebraic extension of K.

Set $\Gamma := vK$, $k := Kv$ and take any natural number n. Suppose there was an extension w of the valuation v of K to the rational function field $K(x_1, \ldots, x_n)$ such that $wK(x_1, \ldots, x_n) = \Gamma = vK$ and $K(x_1, \ldots, x_n)w = k = Kv$. Then (K, v) would admit a transcendental immediate extension $(K(x_1, \ldots, x_n), w)$. This contradicts the fact, that the unique maximal immediate extension of (K, v) is algebraic over the field. ◇

Assume that (K, v) is a valued field. Take $F = K(x_1, \ldots, x_n)$ to be a rational function field over K. Assume that there is an extension v of the valuation of K to F such that vF/vK is a torsion group and the residue field extension $Fv|Kv$ is algebraic. Then for any extension of v from K to $\widetilde{K}$, the field $(\widetilde{K}, v)$ admits an immediate extension of transcendence degree n (cf. Theorem 1.7 of [22]). In certain cases, the above assertion holds already for possibly smaller algebraic extensions of K.

Lemma 6.16. *Take a field K and a natural number n. Assume that v is a valuation on the rational function field $F = K(x_1, \ldots, x_n)$ such that vF/vK is a torsion group and $Fv|Kv$ is an algebraic extension. Fix an extension of v to $\widetilde{F}$ and set $L_0 := IC(F|K, v)$.*

1) If the order of each element of vF/vK is prime to the characteristic exponent of Kv and $Fv|Kv$ is separable, then $vL_0 = vF$ and $L_0v = Fv$, and the extension $(L_0(x_1, \ldots, x_n)|L_0, v)$ is immediate.

2) If $p = \operatorname{char} K > 0$, then in general for $L := L_0^{1/p^\infty}$, the extension $(L(x_1, \ldots, x_n)|L, v)$ is immediate.

Proof. Fix an extension of the valuation v to $\widetilde{F}$ and denote it again by v. Denote by F^h the henselization of F with respect to v. Take p to be the characteristic exponent of Kv. Then from Lemma 2.14 it follows that vF^h/vL_0 is a p-group and the extension $F^hv|L_0v$ is purely inseparable. As $(F^h|F,v)$ is an immediate extension, the same assertions hold for vF/vL_0 and $Fv|L_0v$.

Assume first that every element of vF/vK is of degree coprime with p and $Fv|Kv$ is a separable-algebraic extension. Then $vF = vL_0$ and $Fv = L_0v$. Since $L_0(x_1,\dots,x_n) = F.L_0 \subseteq F^h$, we obtain that

$$\begin{aligned} vF &= vL_0 \subseteq vL_0(x_1,\dots,x_n) \subseteq vF^h = vF, \\ Fv &= L_0v \subseteq L_0(x_1,\dots,x_n)v \subseteq F^hv = Fv. \end{aligned}$$

Hence, equality holds everywhere and we obtain that the extension $(L_0(x_1,\dots,x_n)|L_0,v)$ is immediate.

Suppose now that $\operatorname{char} K = p > 0$. Then for $L := L_0^{1/p^\infty}$ we obtain that $vL = \frac{1}{p^\infty}vL_0$ and $Lv = (L_0v)^{1/p^\infty}$. As moreover $E := L(x_1,\dots,x_n) = F.L$ and $L_0 \subseteq F^h$, we have that

$$E^h = F^h.L_0^{1/p^\infty} \subseteq (F^h)^{1/p^\infty}.$$

Together with the fact that vF/vL_0 is a p-group and the extension $Fv|L_0v$ is purely inseparable, this yields that

$$\begin{aligned} vE &= vE^h \subseteq \frac{1}{p^\infty}vF^h = \frac{1}{p^\infty}vF = \frac{1}{p^\infty}vL_0 = vL \subseteq vE, \\ Ev &= E^hv \subseteq (F^hv)^{1/p^\infty} = (Fv)^{1/p^\infty} = (L_0v)^{1/p^\infty} = Lv \subseteq Ev. \end{aligned}$$

Hence, equality holds everywhere and the extension $(L(x_1,\dots,x_n)|L,v)$ is immediate. $\square$

The next example shows that in general, even if the assumptions of the lemma hold, (K,v) itself may not admit a transcendental immediate extension.

Example 6.17. Take a nontrivially valued maximal field (K,v) and denote the unique extension of the valuation from K to $\widetilde{K}$ again by v. Assume that the value group vK is not divisible or the residue field Kv is neither real closed nor algebraically closed. If the former holds, $v\widetilde{K}/vK$ contains elements of arbitrarily high order. Otherwise, $\widetilde{K}v$ contains elements of arbitrarily high degree over Kv. In both cases, by Theorem 1.2, the field $\widetilde{K}$ admits a maximal immediate extension (M,v) of infinite transcendence degree.

Take a natural number n and choose elements $x_1,\dots,x_n \in M$ algebraically independent over K. Set $F := K(x_1,\dots,x_n)$ and take the restriction of the valuation v of M to the field F. Then (F,v) is the rational function field over K in n variables with a valuation v such that $(\widetilde{K}(x_1,\dots,x_n)|\widetilde{K},v)$ is immediate. On the other hand, (K,v) is maximal, hence admits no proper immediate extensions. $\diamond$

Take a valued field (K,v) and an extension of the valuation of K to $\widetilde{K}$. Denote this extension again by v. It is stated in [22] that if $(\widetilde{K},v)$ admits a transcendental immediate extension, then also (K,v) admits such an extension, provided that (K,v) is a Kaplansky field. Since we can assume additionally in the above example that the considered field (K,v) is a Kaplansky field, it shows that the assertion is not true. However, in Chapter 7 we prove that the statement holds for a tame field K and finite extensions $vF|vK$, $Fv|Kv$ (cf. Theorem 7.16).

7. Immediate extensions of fields with p-divisible value group and perfect residue field

In Chapter 4 we gave conditions for a valued field of positive characteristic p with p-divisible value group and perfect residue field, or for a henselian Kaplansky field, to admit an infinite tower of Galois defect extensions of prime degree, hence an infinite immediate separable-algebraic extension. In this chapter we consider a more general question: what is the form of maximal immediate extensions of such fields? We will give also an example of valued fields, other than Kaplansky fields, for which the uniqueness of maximal immediate extensions holds. We will start with remarks about fields admitting maximal immediate extensions of finite transcendence degree.

Lemma 7.1. *Assume that (K, v) is a henselian field admitting a maximal immediate extension (M, v) of finite transcendence degree over K. Then the relative separable-algebraic closure of K in M is a finite extension of K.*

Proof. Take L to be the relative separable-algebraic closure of K in M. Suppose that $L|K$ is an infinite extension. Since K is henselian, the separable-algebraic case of Theorem 1.2 implies that every maximal immediate extension of (L, v) is of infinite transcendence degree over L. On the other hand, (M, v) is a maximal immediate extension of (L, v) of finite transcendence degree over L, a contradiction. □

In particular, if $\mathrm{char}\, K = 0$ and the maximal immediate extension M is algebraic over K, we obtain that $M|K$ is finite. A valued field (K, v) is called **maximal-by-finite** if it is not maximal, but a finite extension (L, v) of (K, v) is a maximal field. If this holds and (K, v) is a henselian field, then $(L|K, v)$ is a defect extension. Indeed, if the extension were defectless, then Lemma 2.18 would imply that for every nontrivial immediate extension $(F|K, v)$ the extension $(F.L|L, v)$ would be also nontrivial and immediate. This is not possible, since (L, v) is a maximal field. The fact that a finite extension of a tame field is tame, together with the next theorem, proved in [29] (Theorem 14.51), shows that in the above situation neither K nor L can be tame.

Theorem 7.2. *Assume that $(L|K, v)$ is a finite extension of henselian fields. If (L, v) is a tame field, then also (K, v) is a tame field and the extension $(L|K, v)$ is defectless.*

This shows in particular that a henselian field (K, v) of positive residue characteristic p, with p-divisible value group and perfect residue field cannot be maximal-by-finite. This follows from the fact that a maximal immediate extension (M,v) of (K,v) is a tame field, as $vM = vK$ is p-divisible, $Mv = Kv$ is perfect and a maximal field (M,v) is defectless, by Theorem 2.32. Hence, if $M|K$ were finite and nontrivial, it would be defectless by the above theorem. This means that if (K,v) is not maximal, then every maximal immediate extension of K is an infinite extension of the field. We extend this result by showing that a maximal immediate extension of K can be algebraic only in very particular cases. We will need the following fact (cf. Lemma 4.15 of [28]).

Lemma 7.3. *Assume that (L,v) is a tame field and K is a relatively algebraically closed subfield of L. If in addition $Lv|Kv$ is an algebraic extension, then K is also a tame field.*

Throughout the remaining part of the chapter we will assume that (K,v) is a valued field of positive residue characteristic p, with p-divisible value group and perfect residue field, unless stated otherwise.

Proposition 7.4. *Assume additionally that (K,v) is henselian and admits a maximal immediate extension (M,v) of finite transcendence degree over K. If L denotes the relative algebraic closure of K in M, then the following assertions hold:*

1) if $\operatorname{char} K = 0$*, then* $L = K$*;*

2) if $\operatorname{char} K = p$*, then K is relatively separable-algebraically closed in M and* $L = K^{1/p^\infty}$*.*

Proof. As we have already shown, (M,v) is a tame field. Since L is relatively algebraically closed in M, from Lemma 7.3 we deduce that (L,v) is also a tame field.

If $\operatorname{char} K = 0$ then set $F = K$, otherwise define F to be the prefect hull K^{1/p^∞} of K. In the latter case $F \subseteq M^{1/p^\infty}$. As $vM = vK$ is p-divisible and $Mv = Kv$ is perfect, $(M^{1/p^\infty}|M,v)$ is an immediate extension. Since M is maximal, it follows that $M^{1/p^\infty} = M$ and thus $F \subseteq L$. In both cases $L|F$ is a separable-algebraic extension. If $L|F$ were infinite, then as F is henselian, by the separable-algebraic case of Theorem 1.2 we would obtain that every maximal immediate extension of L is of infinite transcendence degree over L. On the other hand, M is a maximal immediate extension of L of finite transcendence degree, a contradiction. Thus $L|F$ is a finite extension. Since (L,v) is tame, by Theorem 7.2 we obtain that $(L|F,v)$ is defectless. As it is an immediate extension of henselian fields, it follows that $L = F$. This proves that K is relatively separable-algebraically closed in M, and in the case of positive characteristic, that $L = K^{1/p^\infty}$. □

Corollary 7.5. *Assume that (K,v) is henselian and admits a maximal immediate extension (M,v) algebraic over K.*

1) If $\operatorname{char} K = 0$*, then $M = K$, that is, (K,v) is a maximal field.*

2) If $\operatorname{char} K = p$*, then K is relatively separable-algebraically closed in M and $M = K^{1/p^\infty} = K^c$.*

Proof. Assertions 1) and 2), except for the last equality, follow directly from the above proposition. Assume that $\operatorname{char} K = p$. Suppose that there is an element $\zeta \in K^{1/p^\infty}$, which does not lie in the completion of K. As vK is p-divisible and Kv is perfect, by Theorem 4.2 we obtain that the henselian field (K,v) admits an infinite immediate separable-algebraic extension (E,v). Since $M = K^{1/p^\infty}$, the extension $M.E|E$ is purely inseparable. Furthermore, $vE = vK$ is p-divisible and $Ev = Kv$ is perfect, hence the extension $(M.E|E,v)$, with the unique extension of the valuation v to $M.E$, is immediate. Consequently, $(M.E|K,v)$ is an immediate extension. Thus also $(M.E|M,v)$ is immediate. As $M|K$ is a purely inseparable extension, $M.E|M$ is a nontrivial separable-algebraic extension. This contradicts the fact that (M,v) is maximal and shows that $M \subseteq K^c$. Since $K^c|K$ is an immediate extension, we deduce finally that $M = K^c$. □

A valued field (L,v) is called **almost maximal** if the completion of L is a maximal immediate extension of K. Since the completion of a valued field is unique up to valuation preserving isomorphism, the maximal immediate extensions of almost maximal fields are unique up to isomorphism. Assume that the valuation v of K is of rank one and (K,v) admits a maximal immediate extension (M,v) algebraic over K. As M is henselian by Theorem 2.32, it contains the henselization K^h of K. Since v is a rank one valuation, the henselization K^h is contained in the completion of K. Moreover, (M,v) is a maximal immediate extension of (K^h,v), which is algebraic over K^h. Thus, Corollary 7.5 yields that $M = (K^h)^c = K^c$. This proves:

Corollary 7.6. *Assume that the valuation v of K is of rank one. If (K,v) admits a maximal immediate extension algebraic over the field, then (K,v) it is almost maximal.*

The next theorem shows that the uniqueness of maximal immediate extensions of (K,v) holds not only in the cases described in the last two corollaries.

Theorem 7.7. *Assume that (K,v) admits a maximal immediate extension of finite transcendence degree. Then the maximal immediate extension of K is unique up to valuation preserving isomorphism.*

For the proof we will need the following lemma.

Lemma 7.8. *Assume that (K,v) is algebraically maximal. If $(K(x),v)$ is an immediate transcendental extension of (K,v), then $(K(x),v)$ can be K-isomorphically embedded into every maximal immediate extension of K.*

Proof. Since $(K(x),v)$ is an immediate extension of (K,v), Theorem 2.36 yields that x is a pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) without a pseudo limit in the field. As by assumption K admits no algebraic immediate extensions, it follows from Theorem 2.41 that the pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ is of transcendental type. Take a maximal immediate extension (M,w) of (K,v). Then $(a_\nu)_{\nu<\lambda}$ admits a pseudo limit y in M, by Theorem 2.42. From Theorem 2.40 we obtain that there is a valuation preserving K-isomorphism between $(K(x),v)$ and $(K(y),w)$, which gives a K-isomorphic embedding of $(K(x),v)$ into (M,w). □

Proof of Theorem 7.7: Assume that (M,v) is a maximal immediate extension of (K,v) and $\mathrm{trdeg} M|K$ is finite. Take (N,w) to be another maximal immediate extension of (K,v). It suffices to show that (M,v) can be K-isomorphically embedded in N, as a valued field. Indeed, if ϕ is the embedding, then $(\phi(M),w)$ is a maximal immediate extension of (K,v). Since the extension $(N|K,w)$ is immediate, also $(N|\phi(M),w)$ is immediate and hence trivial; so we obtain that $\phi(M) = N$. Thus the fields (M,v) and (N,w) are isomorphic.

Consider the family of all subextensions E of $M|K$ admitting a valuation preserving K-embedding $\tau\colon E \to N$. By Zorn's Lemma, there is a maximal such embedding $\sigma\colon L \to N$, where $(L|K,v)$ is a subextension of $(M|K,v)$. We wish to show that $L = M$.

Since M, N as maximal fields are henselian, M contains the henselization L^h of L and the embedding extends uniquely to an embedding $\sigma_1 \colon L^h \to N$. By the maximality of σ we have that $L^h = L$, that is, L is henselian.

Take F to be the relative algebraic closure of L in M. Proposition 7.4 yields that $F = L$ if $\operatorname{char} K = 0$ and $F = L^{1/p^\infty}$ otherwise. Assume that $\operatorname{char} K = p$. We know that in this case N, as a maximal immediate extension of a field with p-divisible value group and perfect residue field, is perfect. Hence, σ can be extended in a unique way to an embedding $\sigma_2 \colon F \to N$. From the maximality of σ it follows that $L = F$. Thus, regardless of the characteristic of K, we obtain that L is relatively algebraically closed in M. Furthermore, (M, v) is tame, as a maximal immediate extension of (K, v). Therefore, Lemma 7.3 implies that also (L, v) is a tame field.

Suppose that $L \subsetneq M$. We identify (L, v) with its isomorphic image $(\sigma(L), w)$ in (N, w). Take an element $x \in M \setminus L$. Then $(L(x)|L, v)$ is an immediate extension, and it is transcendental since L is relatively algebraically closed in M. Since every tame field is defectless, (L, v) admits no proper immediate algebraic extensions. Hence by Lemma 7.8, the field $L(x)$ can be L-isomorphically embedded into (N, w), a contradiction to the maximality of σ and L. Thus we obtain the required equality $M = L$. $\square$

An immediate consequence of the above theorem is the following fact.

Corollary 7.9. *If (K, v) admits an immediate extension of infinite transcendence degree over K, then every maximal immediate extension of (K, v) is of infinite transcendence degree over this field.*

Assume that (K, v) is a henselian field and admits a maximal immediate extension (M, v) of finite transcendence degree over K. Suppose that (K, v) admits an immediate separable-algebraic extension (E, v). Then (E, v) is contained in some maximal immediate extension (N, w) of (K, v). Theorem 7.7 implies that $N|K$ is of finite transcendence degree, hence by Proposition 7.4 the field K is relatively separable-algebraically closed in N. Thus the extension $E|K$ is trivial and consequently, (K, v) admits no proper immediate separable-algebraic extensions.

Assume additionally that $\operatorname{char} K = p$ and denote by L the relative algebraic closure of K in M. From Proposition 7.4 we know that $L = K^{1/p^\infty}$. If $M|K$ is algebraic, then moreover every element purely inseparable over K lies in the completion of the field, by Corollary 7.5. The last paragraph allows us to deduce the same if only $\operatorname{trdeg} M|K$ is finite. Indeed, suppose there is an element $\zeta \in K^{1/p^\infty}$, which does not lie in the completion of K, then by Theorem 4.2 we obtain that the henselian field (K, v) admits an infinite immediate separable-algebraic extension (E, v). But as we have seen, (K, v) admits no proper immediate separable-algebraic extensions, a contradiction. We have thus proved:

Corollary 7.10. *Assume that (K, v) is a henselian field. If the field admits a maximal immediate extension of finite transcendence degree, then (K, v) is separable-algebraically maximal. If additionally K is of positive characteristic, then the perfect hull of K is contained in the completion of the field.*

We will use these facts to give the proof of Theorem 1.3. We will also need the following lemma.

Lemma 7.11. *Take a normal extension F of a henselian field (K,v) and set $L = F \cap K^r$. Then vF/vL is a p-group, $Fv|Lv$ is purely inseparable and $(L|K,v)$ is a tame extension.*

Proof. Since $L|K$ is a subextension of the tame extension $K^r|K$, it is also tame. The assertions on value group and residue field follow from Theorem 7.16 and Lemma 7.20 of [29]. $\square$

We are now able to give the

Proof of Theorem 1.3: Assume that the first of the two cases holds. Since $d(F|K,v) = d(F^h|K^h,v)$ by equation (2.9), without loss of generality we can assume that K is a henselian field. As the normal hull of F over K is again a separable-algebraic extension with nontrivial defect, we can also assume that $F|K$ is normal. Set $L = F \cap K^r$. Lemma 7.11 yields that $(L|K,v)$ is a finite tame extension. Furthermore, vF/vL is a p-group and the residue field extension $Fv|Lv$ is purely inseparable. Since vL is p-divisible and Lv is perfect, as this holds already for the value group and the residue field of (K,v), the group vF/vL and the extension $Fv|Lv$ are trivial. Thus $(F|L,v)$ is an immediate extension. From this and Proposition 2.20 it follows that

$$[F:L] \;=\; d(F|L,v) \;=\; d(F|K,v) \;>\; 1.$$

This shows that the immediate separable-algebraic extension $(F|L,v)$ is nontrivial. Applying Corollary 7.10 to the field (L,v) in place of (K,v), we obtain that every maximal immediate extension of (L,v) is of infinite transcendence degree.

Take (E,w) to be a maximal immediate extension of (K,v). As E is henselian, w admits a unique extension to a valuation of $E.L$. Denote this extension again by w. Since we are assuming K to be henselian, the restriction of w to L coincides with v. As $L|K$ is tame, by Lemma 2.18 the extension $(E.L|L,w)$ is immediate. Since by Theorem 2.33 a finite extension of a maximal field is again maximal, we deduce that $(E.L,w)$ is a maximal immediate extension of (L,v). From the first part of the proof it follows that $\mathrm{trdeg}\,E.L|L$ is infinite. Therefore also the extension $E|K$ is of infinite transcendence degree.

Assume now that the second case holds, i.e., $\mathrm{char}\,K$ is positive and there is an element purely inseparable over K which does not lie in the completion of the field. Then from Corollary 7.10 it follows that every maximal immediate extension of (K,v) is of infinite transcendence degree over K. $\square$

Remark 7.12. Note that if $(L|K,v)$ is a finite separable extension, then the perfect hull of K is contained in the completion of K if and only if the same holds for L. Indeed, take (L^c,v) to be the completion of (L,v). By Proposition 2.49 we obtain that L^c contains a completion K^c of (K,v). Since the perfect hull K^{1/p^∞} of K is a perfect field, also its algebraic extension $K^{1/p^\infty}.L$ is perfect. Thus $L^{1/p^\infty} = K^{1/p^\infty}.L$. As $K^{1/p^\infty} \subseteq K^c \subseteq L^c$ and $L \subseteq L^c$, we obtain that $L^{1/p^\infty} = K^{1/p^\infty}.L \subseteq L^c$. Conversely, suppose there is an element a purely inseparable over K, which does not lie in K^c. Then a is purely inseparable also over K^c. Since $L|K$ is a finite extension, by Lemma 2.51 we obtain that $L^c = K^c.L$. Consequently the extension $L^c|K^c$ is finite and separable, as also $L|K$ is a finite separable extension. Therefore a does not lie in L^c. It follows that L^{1/p^∞} is not contained in L^c.

This shows that we can replace condition 2) of Theorem 1.3 by the equivalent condition: *2') $\mathrm{char}\,K = p$ and for some finite separable extension L of K, the perfect hull of L is not contained in the completion of L.*

Remark 7.13. Note that for a Kaplansky field (K,v) of positive residue characteristic p, where $p \neq 2$ or $\operatorname{char} K = p$, the assertion of Theorem 1.3 in case 1) follows also from Theorems 4.8 and 4.13. Indeed, suppose that $(L|K,v)$ is a finite separable defect extension such that the valuation v extends in a unique way from K to L. By equation (2.9) we have that $d(L^h|K^h,v) = d(L|K,v) > 1$. Therefore, and because $L^h = K^h.L$ is a finite separable extension of K^h, we can assume without loss of generality that (K,v) is henselian. By Lemma 2.19 there is a finite tame extension N of K such that $L.N|N$ is a tower of Galois extensions of degree p. From Proposition 2.20 it follows that

$$[L.N:N] \;\geq\; d(L.N|N,v) \;=\; d(L|K,v) > 1.$$

Thus the extension $L.N|N$ is nontrivial. If $\operatorname{char} K = 0$ and ε_p is a p-th primitive root of unity, then $N(\varepsilon_p)|N$ is also a tame extension. As the extension is trivial or of degree a divisor of $p-1$, it is linearly disjoint from $L.N|N$. Hence $L.N(\varepsilon_p)|N(\varepsilon_p)$ remains a nontrivial tower of Galois extensions of degree p and we can replace N by $N(\varepsilon_p)$.

Regardless of the characteristic of K, the field N admits a Galois extension of degree p. Furthermore, as an algebraic extension of a Kaplansky field, (N,v) is a Kaplansky field. Now Theorems 4.8 and 4.13 imply that N admits an infinite tower of Galois extensions of defect and degree p. Thus N admits an infinite immediate separable-algebraic extension F. By the separable-algebraic case of Theorem 1.2 we have that every maximal immediate extension of (N,v) is of infinite transcendence degree over N. As in the proof of Theorem 1.3, we deduce that also every maximal immediate extension of (K,v) is of infinite transcendence degree over K.

An easy consequence of the special properties of maximal immediate extensions of (K,v) is the following fact.

Corollary 7.14. *Assume that (K,v) is henselian and take a maximal immediate extension (M,v) of (K,v). Assume additionally that there is an element $a \in M \setminus K$ such that*

$$\operatorname{dist}(a,K) < \infty.$$

Then $M|K$ is a transcendental extension. If moreover a is algebraic over K, then $M|K$ is of infinite transcendence degree.

Proof. If a is transcendental over K, then the first assertion is trivial. Assume that the element a is algebraic over K. If K is not relatively separable-algebraically closed in M, then the second assertion follows from Proposition 7.4. Otherwise, $\operatorname{char} K = p$ and a is a purely inseparable element over K, which does not lie in the completion of the field. Thus Theorem 1.3 yields that every maximal immediate extension of K is of infinite transcendence degree. □

Note that the condition $\operatorname{dist}(a,K) < \infty$ is equivalent to the fact that a does not lie in the completion of K.

Example 7.15. Take $(K(x,y),v)$ to be a valued rational function field defined by conditions (5.2). Assume additionally that the field K is perfect. As we have noticed in Section 5.1, the power series field $K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)$ with the canonical valuation v_x is an immediate extension of $(K(x,y),v)$. Since the power series field is maximal (cf. Example 2.44), it is the maximal immediate extension of $(K(x,y),v)$. Furthermore, it is well known that the extension $K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)|K(x,y)$ is of infinite transcendence degree. This can be shown also with the use of Theorem 1.2. Since $vK(x)=\mathbb{Z}$ and $vK(x,y)=\frac{1}{p^\infty}\mathbb{Z}$, the group $vK(x,y)/vK(x)$ contains elements of arbitrarily high order. Thus the value algebraic case of Theorem 1.2 implies that every maximal immediate extension of $(K(x,y),v)$ is of infinite transcendence degree.

Take a transcendence basis T of the extension $K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)|K(x,y)$ and a natural number n. Choose any $t_1,\dots,t_n\in T$. Set $F:=K(x,y)(T\setminus\{t_1,\dots,t_n\})$ and take v to be the restriction of the x-adic valuation of the power series field to F. Since $(F|K(x,y),v)$ is a subextension of the immediate extension $\left(K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)|K(x,y),v_x\right)$, it is also immediate and thus $vF=\frac{1}{p^\infty}\mathbb{Z}$ is p-divisible and $Fv=K$ is perfect. As $K\left(\left(x^{\frac{1}{p^\infty}\mathbb{Z}}\right)\right)$ is a maximal immediate extension of F of finite transcendence degree, Theorem 7.7 yields that the power series field is the unique (up to isomorphism) maximal immediate extension of (F,v).

Define $L:=K(T\setminus\{t_1,\dots,t_n\})$. Then $F=L(x,y)$ and the elements x,y are algebraically independent over L. From Corollary 7.10 we deduce that every henselization of $(L(x,y),v)$ is separable-algebraically maximal. It follows that $(L(x,y),v)$ admits no Artin-Schreier defect extensions. Indeed, every Artin-Schreier defect extension is in particular an immediate separable-algebraic extension. Take a maximal separable-algebraic immediate extension of $L(x,y)$. Then it contains a henselization $L(x,y)^h$ of $L(x,y)$ and, by what we have proved, it is equal to the henselization. This shows that every immediate Artin-Schreier extension of $L(x,y)$ is contained in some henselization of $L(x,y)$. On the other hand, an Artin-Schreier defect extension is by Lemma 2.12 linearly disjoint from every henselization of $L(x,y)$. $\diamond$

We now go back to the question considered in Section 6.3. Theorem 1.6 of [22] and Theorem 6.12 of this thesis give conditions for a valued field (L,v) to admit an extension of v to a rational function field $L(x_1,\dots,x_n)$ such that $vL(x_1,\dots,x_n)/vL$ is torsion and $L(x_1,\dots,x_n)v|Lv$ is algebraic. The question if the conditions given in the theorems are necessary for the valuation v to admit the required extension is open in the general case. However, the next theorem answers the question positively for the class of fields considered in this chapter.

Theorem 7.16. *Take an ordered abelian group extension Γ of vK such that Γ/vK is a torsion group, an algebraic extension k of Kv and a natural number n. Then there is an extension of v from K to the rational function field $F:=K(x_1,\dots,x_n)$ with $vF=\Gamma$ and $Fv=k$ if and only if at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite or (K,v) admits an immediate extension of transcendence degree n. In this case, $L:=IC(F|K,v)$ is a separable-algebraic extension of (K,v) with $vL=\Gamma$ and $Lv=k$, and it is infinite over the henselization of K if $\Gamma|vK$ or $k|Kv$ is infinite.*

Proof. Assume that at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite or (K,v) admits an immediate extension of transcendence degree n. Then parts A2) and B4) of Theorem 1.6 of [22] state that in both cases the valuation v admits an extension to F such that $vF=\Gamma$ and $Fv=k$.

Assume now that there is an extension of v to F with $vF = \Gamma$ and $Fv = k$. Fix an extension of this valuation to $\widetilde{F}$ and denote it again by v. Take K^h and F^h to be the henselizations of K and F with respect to this extension. Set $L := IC(F|K, v)$. Then $L|K^h$ is a separable-algebraic extension. Since $K^h|K$ is also separable-algebraic, we obtain that L is a separable-algebraic extension of K. As vK is p-divisible and Kv is perfect, the order of each element of Γ/vK is prime to p and $k|Kv$ is a separable-algebraic extension. Hence, Lemma 6.16 yields that $(L(x_1, \ldots, x_n)|L, v)$ is an immediate extension with $vL = \Gamma$ and $Lv = k$. Moreover, if $\Gamma/vK = vL/vK^h$ is an infinite group or the extension $k|Kv = Lv|K^hv$ is infinite, then by the fundamental inequality, also the extension $L|K^h$ is infinite.

Suppose that the extensions $\Gamma|vK$ and $k|Kv$ are finite. Assume first that $L|K^h$ is an infinite extension. Take a finite subextension $E|K^h$ of degree bigger than $(\Gamma : vK)[k : Kv]$. Then

$$[E : K^h] \; > \; (\Gamma : vK)[k : Kv] \; = \; (vL : vK^h)[Lv : K^hv] \; \geq \; (vE : vK^h)[Ev : K^hv],$$

and thus the extension $(E|K^h, v)$ has a nontrivial defect. In this case, or if $L|K^h$ is itself a finite defect extension, Theorem 1.3 yields that every maximal immediate extension of (K^h, v) is of infinite transcendence degree. Thus the same holds for (K, v) and in particular, (K, v) admits an immediate extension of transcendence degree n.

It remains to consider the case of $(L|K^h, v)$ defectless. Since

$$(vL : vK^h)[Lv : K^hv] = (\Gamma : vK)[k : Kv] < \infty,$$

the extension $L|K^h$ is finite. Moreover, the order of each element of $\Gamma/vK = vL/vK^h$ is prime to p and the extension $k|Kv = Lv|K^hv$ is separable. Therefore, the extension $(L|K^h, v)$ is tame. As $(L(x_1, \ldots, x_n)|L, v)$ is an immediate extension, Theorem 7.7 yields that every maximal immediate extension of L is of transcendence degree at least n.

Take a maximal immediate extension (M, w) of (K^h, v). Take the unique extension of the valuation w of M to $M.L$ and call it again w. Since K^h is henselian, the restriction of w to L coincides with v. As $L|K^h$ is a finite defectless extension of henselian fields, by Lemma 2.18 it is linearly disjoint from $M|K^h$ and the extension $(M.L|L, w)$ is immediate. Since a finite extension of a maximal field is again maximal by Theorem 2.33, the field $(M.L, w)$ is a maximal immediate extension of (L, v). As we have already shown, $\mathrm{trdeg} M.L|L \geq n$. Hence also trdeg $M|K^h \geq n$. Since (M, v) is also a maximal immediate extension of (K, v), we deduce that K admits an immediate extension of transcendence degree n. $\square$

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