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Riccati equation in studies of spin-boson systems

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Abstract

A Hamiltonian of a given composite qubit–environment system can be seen as a block operator matrix. We explore the relationship between such matrix and the operator Riccati equation to investigate qubit–boson systems, along with qubit's reduced dynamics. In particular, questions concerning the existence of symmetries in the Rabi model are addressed. It is recognised that, apart from the total energy conservation, there is a nonlocal $\mathbb{Z}_2$ and a somewhat hidden symmetry in this model. Conditions for the existence of this observable, its form, and its explicit construction in terms of the solution to the Riccati equation are presented.

Next, we introduce a special class of states called dephasing states. First, it is shown that the specific symmetry, present in dephasing–like models, between the interaction parties is not required to maintain energy lossless evolution of its constituents. Instead, a proper choice of an initial condition needs to be assured and we propose an adequate technique exactly for that purpose. Mathematically speaking, the initial (dephasing) state that guarantees the desired dynamics belongs to the graph of a linear solution to the related Riccati equation. This idea is generalized even further, beyond the energy, by showing how to prepare the qubit–environment system so that the information encoded in a preselected qubit observable cannot be erased by the environment.

Finally, we presents some results regarding time–dependent phenomena. We consider a qubit which maintains contact with a fermionic environment while a rotating (classical) magnetic field is applied upon it. We obtain the exact reduced dynamics for such system and analyse the adiabatic approximation.

Streszczenie

W pracy wskazujemy na związek pomiędzy Hamiltonianem danego układu złożonego typu qubitu–otoczenie a blokowymi macierzami operatorowymi. Następnie wykorzystujemy ich związek z równaniem Riccatiego do badania m.in układów typu spin–bozon oraz dynamiki zredukowanej qubitu. W szczególności rozważamy istnienie symetrii w słynnym, wciąż nie rozwiązanym całkowicie modelu Rabiego. Dowodzimy, iż prócz zachowania całkowitej energii istnieje nielokalna symetria $\mathbb{Z}_2$ (w pewnym sensie ukryta) zadana przez uogólniony operator parzystości. Prezentujemy warunki istnienia tej symetrii, jej postać oraz sposób jawnej konstrukcji z wykorzystaniem rozwiązania równania Riccatiego.

Dla otwartych układów dwustanowych wprowadzamy szczególną klasę stanów, zwanych stanami defazingowymi (ang. *dephasing states*). W pierwszej kolejności pokazujemy, iż specyficzna dla modeli defazingowych symetria pomiędzy oddziałującymi układami nie jest konieczna aby zapewnić bezstratną energetycznie ewolucję podukładów. Dowodzimy, że wystarczające są odpowiednio dobrane warunki początkowe–stany defazingowe oraz przedstawiamy adekwatną technikę ich konstrukcji. Następnie pokazujemy w jaki sposób przygotować stan początkowy układ qubit–otoczenie, tak aby informacja zakodowana w zadanej na przestrzeni qubitu obserwabli nie został utracona na rzecz otoczenia w wyniku ewolucji.

Na zakończenie rozważamy qubit w kontakcie z fermionowym otoczeniem oraz poddany działaniu zewnętrznego pola magnetycznego. Wyprowadzamy dokładną postać dynamiki zredukowanej oraz analizujemy przybliżenie adiabatyczne.

Contents

Chapter 1

Introduction

Comprehending the nature of interactions between quantum systems is still a challenge in contemporary physics. This problem is of great importance to various areas of fundamental and applied science. After all, the most interesting quantum systems *i.e.*, the real ones, are always exposed to the influence of an external environment. They no longer evolve unitarily and in that sense become *open* [1–3]. Ramification of which is the loss of quantum coherence due to various decoherence effects [4, 5] and as a result the emergence of classical behaviour [6].

Nowadays, the most important applications of the 'spooky interaction at a distance', known as the entanglement between quantum systems [7], is mainly focused on the area of controlled quantum technologies [8–10], including quantum computation, communication [11, 12] and information processing [13, 14].

Although significant progress in our understanding of such open systems has been made during the last decade, there are still many basic questions to which we do not have the answer. Take for example a single one that we ask within this thesis: Are there quantum states which allow conservation of certain quantities regardless of the existence of corresponding constants of motion?

Our motivation in writing this work has been to introduce some new physical ideas into the field. While paying attention to the concepts above all, we exemplify results using rather simple yet not trivial two–level open quantum systems. For instance, most often we assume that the environment influencing a quantum system consists with a single boson.

All the notions we present originate from a different mathematical technique in comparison with the ones usually invoked to approach open systems (for review see *e.g.* [15]). The idea is to think of the Hamiltonian of a given composite qubit–environment system as being a block operator matrix, a matrix with operator entries. One can then explore the relation of such matrix with its characteristic equation, which is known in this

context as the operator Riccati equation, to investigate the system's dynamic. Although the concept by itself is very simple, it is powerful enough to answer some recent questions regarding old problems and to offer a new insight into qubit's reduced dynamics.

The layout of this thesis can be summarised as follows. We begin in Chapter 2 with the famous and still not fully solved Rabi model. Next, we transform the Hamiltonian of this model into its arguably natural form in which it is represented by a matrix with operator entries, the Rabi matrix. A Rigorous analysis with regard to this idea is carried out. Then, we proceed by introducing our main tool–the Riccati operator equation.

In the chapter that follows, we address the question concerning the existence of symmetries in the Rabi model. It is recognised that, apart from the total energy conservation, there is a nonlocal $\mathbb{Z}_2$ and hidden in a sense symmetry in this model. Conditions for the existence of this observable, its form and explicit construction in terms of the solution to the Riccati equation are presented. It is worth a mention that these results are inconsistent with the ones recently anticipated in [16].

The word 'symmetry' almost always brings the questions regarding solvability. A possibility of finding the eigenvalues and eigenvector of the Rabi Hamiltonian is also investigated. Although we do not offer full resolution, compact and exact expressions allowing to some simplification of the problem are derived. This result allows generalisation of the so called parity chains [16] by showing how they originate from the block diagonalization. The latter can be performed provided a known solution to the Riccati equation. In particular, our findings in this regard may serve as a good starting point for developing new analytical approximations. It seems this idea has already been put into motion by others researcher [17].

In Chapter 4, we introduce a special class of states called dephasing states. First, it is shown that the specific symmetry, present in dephasing–like models, between the interaction parties (the qubit's Hamiltonian and the interaction operator commute with each other) is not required to maintain energy lossless evolution of its constituents. Instead, a proper choice of an initial condition needs to be assured and we propose an adequate technique exactly for that purpose. Mathematically speaking, the initial (dephasing) state that guarantees the desired dynamics belongs to the graph of a linear solution to the related Riccati equation. This idea is generalized even further, beyond the energy, by showing how to prepare a qubit–environment system so that the information encoded in a preselected qubit observable cannot be erased by the environment.

Chapter 5 presents some results regarding time–dependent phenomena. We consider a qubit which maintains contact with a fermionic environment while a rotating (classical) magnetic field is applied upon it. By means of the block diagonalization, an analytic formula for the qubit's reduced dynamics is obtained. As an application of this exact result,

we inspect the validity of the adiabatic approximation. Finally, Chapter 6 summarises the work.

Focusing only on the merit, we have kept all the chapters short yet long enough to cover all the relevant issues. To ensure a minimum degree of self–consistency an appendix has been included. As to the nomenclature and mathematical character of the work, all physical problems are stated within the usual quantum mechanical framework by using standard notation and terminology. Typical questions concerning *e.g.*, domains of operators, distinction between Hermitian (†) and self-adjoin (∗) conjugation, *etc.*, have not been taken into account. Exceptions are Chapters 2 and 3 where, due to the nature of the issue being analysed, a more rigorous approach needed to be applied.

One can argue such loss of generality is justified. Otherwise, this work would become much more technical and probably wouldn't interest people who are not experts in functional analysis. Not to mention technical difficulties resulting from 'unnecessary' formalization we would most likely encounter. On the other hand, by means of informal approach one cannot benefit from very powerful existential mathematical theorems (such as the famous Banach contraction mapping principle) simply because one cannot known whether the premises of these statements are met. It is needless to say that some sort of balance between a rigorous approach and the intuitive one, so to speak, is required. After all, it is pointless to invoke complicated mathematical theories to establish–let's say–the existence of a solution if it has already been found. Nonetheless, when the solution is unknown and the knowledge of its potential existence is crucial to 'make the case', it would be wise to formulate the problem rigorously from the very beginning. This is precisely the spirit we have tried to acquire in this work. It should be mentioned that the necessary background for this thesis requires familiarity only with quantum theory and functional analysis on a typical undergraduate level.

Chapter 2

The Rabi model

In what follows, we consider a quantum system consisting of a qubit (two–level system) interacting with a single mode bosonic field (representing *e.g.* electromagnetic radiation) with frequency ω. The Hamiltonian for that model is assumed to be of the form

$$\mathbf{H} = \beta\hat{\sigma}_z + \Delta\hat{\sigma}_x + \omega a^\dagger a + \hat{\sigma}_z \otimes \left(g^* a + g a^\dagger\right), \tag{2.1}$$

where a and $a^\dagger$ are the creation and annihilation operators of the field and they obey $[a, a^\dagger] = \mathbb{I}$. An interesting proposal for experimental realisation of such operators can be found [18]. Symbols $\hat{\sigma}_z$ and $\hat{\sigma}_x$ denote the Pauli spin operators. The term $\beta\sigma_z$ is an unperturbed energy of the qubit (with possible eigenenergies $\pm\beta$), while $\Delta\sigma_x$ describes a tunnelling of the two–level system in the absence of the bosonic field (spontaneous transition). Finally, the coupling constant g reflects the strength of the interaction between the two subsystems. We work with the units where $\hbar = 1$ and thus all the named constants have units of energy.

The above Hamiltonian constitutes the well known Rabi model [19]–probably the most influential model describing fully quantized interaction between matter and light. Although the model originates from quantum optics [20], its applications range from molecular physics [21], solid state (see Refs. in [22]) and the recent experiments involving cavity and circuit QED [23]. This model can be implemented by means of rich variety of different setups such as Josephson junctions [24], trapped ions [25], superconductors [26] or semiconductors [27], to name a few. The model (2.1) provides a highly simplified picture of a particular physical realisation being analysed, yet it often captures most of its essential properties.

Despite the simplicity of the model, its Hamiltonian cannot be diagonalised exactly when $\Delta \neq 0$. Although some progress has been reported recently [16], the exact analytical formulas for eigenvectors and eigenfunction, besides certain isolated (see *e.g.*, [28] and Ref.

therein) and quasi [29] solutions, are still missing. Nowadays, however, when powerful computers and accurate numerical methods are accessible, the lack of the resolution to this problem does not seem to be an obstacle. Also, there is a wide spectrum of available approximation techniques [30], including rotating wave approximation [31] (leading to the famous Jaynes–Cummings model [32]), which allow the problem to be approached from many different directions [33]. This should not be a surprise, since the model has been investigated for over half a century.

Let us consider the three body problem as an example [34]. It cannot be solved in an exact manner, yet most aspects of the system's dynamics is known to us. Nothing interesting which we haven't already thought of can happen. The same could be said with regard to the Rabi model. Nevertheless, one should always remember that there are questions which cannot be addressed by means of numerical methods. A typical example concerns the existence of symmetries. Roughly speaking, if we do not have a clue where and what to look for we probably will not find it, regardless of the technique's accuracy. We will return to this matter in Chapter 3.

2.1 Operator matrices and the Riccati equation

Suppose we write down a 2×2 matrix (a_{ij}). One can easily imagine that its entries, a_{ij}, instead of being complex numbers, are linear operators acting on some Hilbert space $\mathcal{H}$. It would be natural to refer to such a mathematical object as a (block) operator matrix [35].

As we will shortly see, the idea of having a matrix over non–commuting 'field' is much more powerful that this brief introduction may indicate. One can argue that such matrices provide a link between linear operators on $\mathcal{H}$ and those acting on $\mathcal{H} \oplus \mathcal{H}$ and therefore are naturally suited to investigate two-level quantum systems interacting with an external environment.

2.1.1 Rabi matrix

By virtue of the standard matrix representation of the Pauli spin operators (A1.1) σ_i, one can rewrite, for now only formally, the Rabi Hamiltonian (2.1) as

$$\mathbf{H} = \begin{bmatrix} \mathrm{H}_+ & \Delta \\ \Delta & \mathrm{H}_- \end{bmatrix}, \quad \text{where} \quad \mathrm{H}_\pm := \omega a^\dagger a \pm \left(g^* a + g a^\dagger\right) \pm \beta. \tag{2.2}$$

Customarily, the parameters Δ and β denote $\Delta \mathbb{I}_\mathrm{B}$ and $\beta \mathbb{I}_\mathrm{B}$, respectively. $\mathbb{I}_\mathrm{B}$ stands for the identity on the bosonic Hilbert space $\mathcal{H}_\mathrm{B}$. It should be mentioned that often the Pauli

operators $\hat{\sigma}_i$ are not expressed in the 'standard' basis. This results in the different form of (2.2). Of course, this has no effect on the general properties of the system.

The correspondence between (2.2) and (2.1) is established via the natural isomorphism $\mathbb{C}^2 \otimes \mathcal{H}_B \sim \mathcal{H}_B \oplus \mathcal{H}_B$. Interestingly, the idea of such identification is not new in physics and it is usually invoked when purely algebraic calculations need to be simplified [36]. It is not 'an old wine in a new bottle' we are selling. Instead, we will attack problems concerning the Rabi model (2.1) by using the concept of the Rabi matrix (2.2) along with its relation to the operator Riccati equation [37].

Since we are dealing with unbounded operators a and $a^\dagger$, a careful treatment of the above matrix is required. In particular, questions concerning domain $\mathcal{D}(\mathbf{H})$ need to be addressed, so that $\mathbf{H}$ may become a meaningful mathematical object. As a first step toward this objective, we will define domains $\mathcal{D}_\pm := \mathcal{D}(\mathrm{H}_\pm)$ on which both operators $\mathrm{H}_\pm$ are self–adjoint. Since the off–diagonal elements of $\mathbf{H}$ are bounded, it follows immediately that $\mathbf{H}^* = \mathbf{H}$ on $\mathcal{D}(\mathbf{H}) = \mathcal{D}_+ \oplus \mathcal{D}_-$.

It's worth a mention that the fundamental commutation relation between a and $a^\dagger$ is the core reason why these operators cannot even be considered bounded [38]. A commutator of two bounded operators always vanishes, yet we would like to have $\mathrm{Tr}([a, a^\dagger]) \neq 0$.

Let $\mathcal{D}_1$ be a dense set on which both a and $a^\dagger$ are adjoint of each other, that is $(a^\dagger)^* = a$ and $a^* = a^\dagger$. Assume in addition that $\mathcal{D}_2$ is a (dense) subspace of $\mathcal{H}_B$ on which we have $[a, a^\dagger] = \mathbb{I}$. One should not expect that $a\mathcal{D}_1 \subset \mathcal{D}_1$ meaning we cannot set $\mathcal{D}_2 = \mathcal{D}_1$. At this stage, the existence of $\mathcal{D}_1$ and $\mathcal{D}_2$ having the desired properties is not obvious. An interested reader can find the detailed construction of $\mathcal{D}_i$ *e.g.* in [39]. Here, we briefly summarise what was covered therein. We have

$$\mathcal{D}_k = \left\{ \sum_{n=0}^{\infty} \xi_n |n\rangle \in \mathcal{H}_B : \quad \sum_{n=0}^{\infty} n^k |\xi_n|^2 < \infty \right\}, \quad k = 1, 2, \tag{2.3}$$

where $\{|n\rangle\}_{n=0}^{\infty}$ is the canonical (orthonormal) basis in l^2 ($\sim \mathcal{H}_B$). Considering the fact that a, $a^\dagger$ and $a^\dagger a$ need to produce normalizable states at least, (2.3) should not come as a surprise to us. Having (2.3) in place, we define (see [40])

$$a|\psi\rangle := \sum_{n=1}^{\infty} \sqrt{n}\xi_n |n-1\rangle, \quad a^\dagger|\psi\rangle := \sum_{n=0}^{\infty} \sqrt{n+1}\xi_n |n+1\rangle, \quad |\psi\rangle \in \mathcal{D}_1. \tag{2.4}$$

It follows immediately from (2.4) that

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \text{and} \quad a|n\rangle = \sqrt{n}|n-1\rangle. \tag{2.5}$$

The letter relations capture the whole physical idea encoded in the creation and annihilation operators (2.4). Probably for that reason, (2.5) serves as the very definition of a and $a^\dagger$ in most textbooks on quantum mechanics. To be meticulous one should also define $|-1\rangle$ or exclude the $n = 0$ case from the last equation. It really does not matter which way one prefers since $a|0\rangle = 0$.

Although a definition like this is much more natural in comparison with (2.4), it has at least one serious drawback. Technically speaking, (2.5) introduces closeable operators that are not closed. This, on the other hand, leads to a variety of mathematical difficulties typical for such class of operators. Usually, one can avoid them by taking the closures of (2.5) as a definition instead. Interestingly, those happen to be given by (2.4).

A basic result from operator theory (see *e.g.*, Theorem 4.2.7 in [41]) states that if A is closed on $\mathcal{D}(A)$ then A^*A is positive, self-adjoint and its domain is a core of A (*i.e.* A is the closure of its restriction $A_{|\mathcal{D}(A^*A)}$). On $\mathcal{D}_2$, the operators $H_\pm$ can be written as

$$H_\pm = \omega\left(a \pm \frac{g}{\omega}\right)^*\left(a \pm \frac{g}{\omega}\right) \pm \beta - \frac{|g|^2}{\omega}, \quad \text{on} \quad \mathcal{D}_2. \tag{2.6}$$

As a result, they are both self–adjoint and their common domain $\mathcal{D}_2$ is a core of both a and $a^\dagger$. Therefore, the Rabi matrix (2.2) is a well–defined self–adjoint operator on $\mathcal{D}(\mathbf{H}) = \mathcal{D}_2 \oplus \mathcal{D}_2$. In fact, if we replace $\mathcal{D}_2$ with any common core of a and $a^\dagger$, this conclusion will stay true.

2.1.2 Riccati equation

Resolving all the technical nuances with respect to the Rabi matrix, we proceed by formally introducing a quadratic second order operator equation of the form

$$\Delta X^2 + XH_+ - H_-X - \Delta = 0. \tag{2.7}$$

From now on, we will refer to this equation as the Riccati equation associated with the Rabi matrix (2.2). As we will shortly see, most of the relevant problems regarding the Rabi model (2.1), including possibility of its exact diagonalization, can be reduced to the questions concerning solvability of this equation.

In general, when a self–adjoint block operator matrix reads

$$\mathbf{H} = \begin{bmatrix} H_+ & V \\ V^\dagger & H_- \end{bmatrix}, \tag{2.8}$$

by the corresponding Riccati equation it is meant the following operator equation

$$XVX + XH_{+} - H_{-}X - V^{\dagger} = 0. \tag{2.9}$$

It should be pointed out that neither the form nor even the existence of a solution to the above equation can be taken for granted in general. Nevertheless, some useful criteria for solvability applicable to the broad range of relevant physical systems can be found in literature [42]. Also, the very definition of a solution needs to be clarified which we will take care of in the next chapter.

We find it remarkable that very often in physics when one faces a difficult mathematical problem the possibility of its resolution happens to somehow be related to questions concerning solvability of an adequate Riccati type equation. Factorization of the second order differential operator, which one should emphasise is a generic problem in quantum mechanics, may serve as an illuminating example in this context. As if this equation reflects in a sense the difficulty of the problem being studied. It surely sheds a light on the complexity of the diagonalization of the Rabi model as it will be explained in the next chapter.

Chapter 3

New symmetry

A symmetry can be seen as an equivalence of different physical situations [43]. For quantum systems, such a similarity manifests itself in an invariance of a certain set of observables. It is often formalized in terms of commutation relations between these observables and the Hamiltonian of the system in question.

The existence of conserved quantities, including those having no classical counterpart, extends an amount of information accessible for a given quantum system. This is a kind of truism in studies of quantum systems. In extreme cases, one can meet analytically solvable models such as harmonic oscillator, Jaynes–Cummings model or hydrogen atom. The more symmetries recognised (together with related conserved quantities), the more different approaches to study the system's dynamics are at our disposal. Symmetries not only deepen our understanding of the system itself but also can be included to engineer its physical realization more effectively [44].

Provided that $\beta = 0$, the Hamiltonian (2.1) remains unchanged when $\sigma_z \to -\sigma_z$ and $a \to -a$ (hence $a^\dagger \to -a^\dagger$). The symmetry operator $\mathbf{J_0}$ that generates this transformation (*e.g.* fulfills $[\mathbf{H}, \mathbf{J_0}] = 0$) reads $\mathbf{J_0} = \sigma_x \otimes \mathrm{P}$, where $\mathrm{P} = \exp(\mathrm{i}\pi a^\dagger a)$ is the bosonic parity [45]. This is the well–known result: still being unsolvable, the Rabi model possesses a discrete symmetry if $\beta = 0$.

When $\beta \neq 0$, on the other hand, we can still leave $\mathbf{H}$ unaffected after changing $\sigma_z \to -\sigma_z$, $a \to -a$ if we change the sign of β as well (*i.e.* $\beta \to -\beta$). This instantly raises a question: What does the corresponding generator of such transformation, $\mathbf{J}$, look like? Unfortunately, this question has not been answered so far. Moreover, it was quite recently conjectured [16] that the Rabi model does not possess any symmetry at all, except the trivial one related to the total energy conservation, as long as $\beta \neq 0$. If that were true, the only self–adjoint operator $\mathbf{J}$ such that $[\mathbf{H}, \mathbf{J}] = 0$ would be the Hamiltonian $\mathbf{H}$ itself.

On the basis of the results reported here (see also [46]), we prove that this conjecture

is false. In particular, we show how one can find a self–adjoint involution $\mathbf{J}$, that is $\mathbf{J}^2 = \mathbb{I}_{\mathrm{B}}$, such that $\mathbf{HJ} = \mathbf{JH}$. Also, we discuss the possibility of the exact diagonalization of the Rabi Hamiltonian (2.1).

3.1 Riccati equation and symmetry

We begin by showing that if a solution to (2.7) exists then there is also an operator generating symmetry in the system (2.2). Next, we argue that under certain conditions imposed on the parameters α, β, and ω, (2.7) is solvable.

For equations with operator coefficients, there is often more than one notion of a solution. For instance, in a Hilbert space one can define a solution by involving the scalar product (weak solution). On the other hand, one may require for operators to be equal when they produce the same results while acting on the same states. Those kind of solutions, which are of great importance in quantum mechanics, are known as strong ones. Let us briefly clarify these two notions for the Riccati equation in question.

A bounded operator X_0 acting on $\mathcal{H}_{\mathrm{B}}$ is said to be a weak solution to the Riccati equation (2.7) if

$$\Delta\langle \mathrm{X}_0^2\phi, \psi\rangle + \langle \mathrm{X}_0\mathrm{H}_+\phi, \psi\rangle - \langle \mathrm{X}_0\phi, \mathrm{H}_-\psi\rangle - \Delta\langle \phi, \psi\rangle = 0, \quad \text{for} \quad |\psi\rangle, |\phi\rangle \in \mathcal{D}_2. \tag{3.1}$$

On the other hand, we call a bounded operator X_0 acting on $\mathcal{H}_{\mathrm{B}}$ a strong solution to (2.7) if $\mathrm{Ran}(\mathrm{X}_{0|\mathcal{D}_2}) \subset \mathcal{D}_2$ and

$$\Delta \mathrm{X}_0^2|\psi\rangle + \mathrm{X}_0\mathrm{H}_+|\psi\rangle - \mathrm{H}_-\mathrm{X}_0|\psi\rangle - \Delta|\psi\rangle = 0, \quad \text{for} \quad |\psi\rangle \in \mathcal{D}_2. \tag{3.2}$$

Of course, a strong solution is also a weak solution. In fact, these two notions are equivalent [47]. This result is of great importance since it is often easier to prove the existence of a weak rather than a strong solution.

Among all operators which solve the Riccati equation (2.7), we consider only linear and bounded ones. Nevertheless, different kinds of solution can be introduced in principle. For instance, for some classes of Riccati equation it is possible to define an unbounded linear solution [42]. One should also mention antilinear solutions (*e.g.*, time reversal operator) found for some Riccati equations [48]. In addition, an interesting example where all the coefficients are unbounded, yet the solution exists as a bounded operator, have been provided in [49].

3.2 Generator

Assume there exists a weak solution X_0 to the Riccati equation (2.7). Our objective is to show that there is also a self–adjoint involution $\mathbf{J}$ such that $\mathbf{JH} = \mathbf{HJ}$, where $\mathbf{H}$ is the Rabi Hamiltonian (2.2). Moreover, in terms of X_0 the generator of the symmetry $\mathbf{J}$ reads

$$\mathbf{J} = \begin{bmatrix} J_0 - \mathbb{I}_B & J_0 X_0^* \\ X_0 J_0 & X_0 J_0 X_0^* - \mathbb{I}_B \end{bmatrix}, \quad \text{where} \quad J_0 = 2(\mathbb{I}_B + X_0^* X_0)^{-1}. \tag{3.3}$$

In order to prove this statement, take $\mathcal{G}(X_0)$ to be the graph of X_0, that is

$$\mathcal{G}(X_0) = \left\{ \begin{bmatrix} |\psi\rangle \\ X_0|\psi\rangle \end{bmatrix} \in \mathcal{H}_B \oplus \mathcal{H}_B : |\psi\rangle \in \mathcal{H}_B \right\}. \tag{3.4}$$

Since X_0 is also a strong solution to (2.7), we have $X_0|\psi\rangle \in \mathcal{D}_2$ and $X_0(H_+ + \Delta X_0)|\psi\rangle = (H_- X_0 + \Delta)|\psi\rangle$ for $|\psi\rangle \in \mathcal{D}_2$. Therefore, we obtain

$$\begin{bmatrix} H_+ & \Delta \\ \Delta & H_- \end{bmatrix} \begin{bmatrix} |\psi\rangle \\ X_0|\psi\rangle \end{bmatrix} = \begin{bmatrix} (H_+ + \Delta X_0)|\psi\rangle \\ X_0 (H_+ + \Delta X_0)|\psi\rangle \end{bmatrix} \in \mathcal{G}(X_0), \tag{3.5}$$

that is, $\mathbf{H}(\mathcal{G}(X_0) \cap \mathcal{D}_2) \subset \mathcal{D}_2$. Making use of the same arguments, one can verify that $\mathcal{G}(X_0)^\perp$, which in this case reads

$$\mathcal{G}(X_0)^\perp = \left\{ \begin{bmatrix} -X_0^*|\psi\rangle \\ |\psi\rangle \end{bmatrix} \in \mathcal{H}_B \oplus \mathcal{H}_B : |\psi\rangle \in \mathcal{H}_B \right\}, \tag{3.6}$$

is $\mathbf{H}$–invariant as well. Since X_0 is bounded its graph forms a closed subspace of $\mathcal{H}_B \oplus \mathcal{H}_B$ and hence the decomposition $\mathcal{H}_B \oplus \mathcal{H}_B = \mathcal{G}(X_0) \oplus \mathcal{G}(X_0)^\perp$ holds. Therefore, each state $|\Psi\rangle \in \mathcal{D}(\mathbf{H})$ of the composite system can be uniquely decomposed as $|\Psi\rangle = |\Psi_1\rangle \oplus |\Psi_2\rangle$, where $|\Psi_1\rangle \in \mathcal{G}(X_0)$ and $\langle\Psi_2|\Psi_1\rangle = 0$.

Let $\mathbf{P}_+$ denotes a projection onto $\mathcal{G}(X_0)$, then

$$\mathbf{P}_+\mathbf{H}|\Psi_1\rangle = \mathbf{H}|\Psi_1\rangle \quad \text{and} \quad \mathbf{P}_+\mathbf{H}|\Psi_2\rangle = 0. \tag{3.7}$$

Assuming for a moment that $\mathbf{P}_+\mathcal{D}_2 \subset \mathcal{D}_2$, we obtain

$$\mathbf{H}\left(\mathbf{P}_+|\Psi_1\rangle \oplus \mathbf{P}_+|\Psi_2\rangle\right) = \mathbf{H}|\Psi_1\rangle, \quad \text{and} \quad \mathbf{P}_+\left(\mathbf{H}|\Psi_1\rangle \oplus \mathbf{H}|\Psi_2\rangle\right) = \mathbf{H}|\Psi_1\rangle. \tag{3.8}$$

Therefore, $\mathbf{H}\mathbf{P}_+|\Psi\rangle = \mathbf{P}_+\mathbf{H}|\Psi\rangle$ for all $|\Psi\rangle \in \mathcal{D}(\mathbf{H})$. The inverse $(\mathbb{I}_B + X_0^* X_0)^{-1}$ exists and

it is a bounded self–adjoint operator on $\mathcal{H}_B$. As a result, $\mathbf{P}_+$ can be expressed as

$$\mathbf{P}_+ = \frac{1}{2}\begin{bmatrix} J_0 & J_0X_0^* \\ X_0J_0 & X_0J_0X_0^* \end{bmatrix}. \tag{3.9}$$

Straightforward calculations show that (3.9) indeed projects onto $\mathcal{G}(X_0)$.

Due to the fact that $\mathbf{J} = 2\mathbf{P}_+ - \mathbf{1}$, the only question which we need to address to conclude the proof is whether $\mathbf{P}_+|\Psi\rangle$ is again in $\mathcal{D}(\mathbf{H})$ for $|\Psi\rangle \in \mathcal{D}(\mathbf{H})$. Because X_0 is a weak (and hence strong) solution of (2.7), we have $X_0\mathcal{D}_2 \subset \mathcal{D}_2$. Moreover, the function $f(\psi) := \langle H_+\psi, X_0^*\phi\rangle$ is continuous on $\mathcal{D}_2$ for every $|\phi\rangle \in \mathcal{D}_2$. Indeed, it follows from (3.1) that

$$|f(\psi)| \leq M_\phi\|\psi\|, \quad \text{where} \quad M_\phi = \alpha\|\phi\|\|X_0\|^2 + \|H_-\phi\|\|X_0\| + \alpha\|\phi\|. \tag{3.10}$$

As a result, $X_0^*|\phi\rangle \in \mathcal{D}(H_+^*) = \mathcal{D}_2$, *i.e.* $X_0^*\mathcal{D}_2 \subset \mathcal{D}_2$ and therefore $J_0^{-1}\mathcal{D}_2 \subset \mathcal{D}_2$. J_0^{-1} is invertible hence $J_0\mathcal{D}_2 = \mathcal{D}_2$. In summary, $\mathbf{P}_+\mathcal{D}(\mathbf{H}) \subset \mathcal{D}(\mathbf{H})$, which concludes the proof.

3.2.1 Hidden symmetry

(3.3) establishes a correspondence between symmetries of the Rabi Hamiltonian and solutions to the related characteristic equation. To prove that there is a symmetry it is sufficient to demonstrate that this equation is solvable. Unfortunately, we cannot do so by simply solving the equation since we don't know how. Nonetheless, there are criteria of solvability which we can use instead, one of which is included in the appendix (Theorem 1). In what follows, we check that the premises of this statement are met provided β, ω, $\Delta \neq 0$, and

$$\frac{2\beta}{\omega} \neq \mathbb{N} \quad \text{and} \quad \frac{\Delta}{\beta} > \frac{\pi}{2}. \tag{3.11}$$

Indeed, since $V_x = i(xa^\dagger - x^*a)$ is self–adjoint for $x \in \mathbb{C}$, the unitary Weyl operator $D(x) = \exp(iV_x)$ is well defined. Therefore, one can rederive the well–known result,

$$H_\pm = D(\pm\frac{g}{\omega})\left(\omega a^\dagger a \pm \beta - \frac{|g|^2}{\omega}\right)D(\pm\frac{g}{\omega})^\dagger. \tag{3.12}$$

By virtue of $a^\dagger a|n\rangle = n|n\rangle$, we thus obtain

$$\sigma(H_\pm) = \left\{\omega n \pm \beta - \frac{|g|^2}{\omega} : n \in \mathbb{N}\right\} = \omega\mathbb{N} \cup \{\pm\beta\} - \frac{|g|^2}{\omega}. \tag{3.13}$$

If 2β is not multiple of ω, as we have assumed in (3.11), then the distance

$$\mathrm{dist}(\sigma(\mathrm{H}_+),\sigma(\mathrm{H}_-)) = \inf\{|\omega(n-m)+2\beta| : n,m\in\mathbb{N}\} = 2\beta \neq 0. \tag{3.14}$$

Hence, the spectra $\sigma(\mathrm{H}_\pm)$ are disjoint, *i.e.* (A3.1) holds. In addition, both the smallness assumption (A3.2) and (A3.6) imposed on the off-diagonal elements are satisfied as long as $2\Delta > \pi\beta$. According to Theorem 1 there is exactly one solution X_0 to (2.7) such that $\|\mathrm{X}_0\| < 1$.

3.3 Why haven't we found it earlier?

First, let us take a closer look at the $\beta = 0$ case. The spectra (3.13) overlap in this situation and thus the separability condition (A3.1) is not satisfied. Therefore, one cannot invoke Theorem 1 to establish the existence of a solution to (2.7). However, in this particular case the spectra $\sigma(\mathrm{H}_\pm)$ are identical and $\mathrm{H}_\pm$ can be transformed one into another by the same bosonic parity operator that generates the symmetry $\mathbf{J_0}$. As we shall shortly see, this is not an accidental coincidence as P happens to be a solution to (2.7). At first, this fact might be surprising since P does not depend on any of the parameters appearing in the model (β, ω, g).

We begin by noticing that the action of P on a given state $|\psi\rangle$ reads

$$\mathrm{P}|\psi\rangle = \sum_{n=0}^{\infty}(-1)^n\xi_n|n\rangle, \quad \text{with} \quad \xi_n = \langle n|\psi\rangle. \tag{3.15}$$

This can be taken as the definition of P.

It immediately follows from (3.15) that P is bounded ($\|\mathrm{P}\| = 1$) and hence it can be defined for all for $|\psi\rangle \in \mathcal{H}_\mathrm{B}$. In addition to that $\mathrm{Ran}(\mathrm{P}_{|\mathcal{D}_2}) \subset \mathcal{D}_2$. This is due to the fact that if the sequence $n\xi_n$ is square–summable, $\sum_n |n\xi_n|^2 < \infty$, so is $(-1)^n n\xi_n$. (3.15) together with (2.4) give

$$\mathrm{P}a\mathrm{P}|\psi\rangle = \sum_{n=1}^{\infty}\sqrt{n}(-1)^{2n-1}\xi_n|n-1\rangle = -a|\psi\rangle, \tag{3.16}$$

for all $|\psi\rangle \in \mathcal{D}_1$. Since P is a self–adjoint involution we also have $\mathrm{P}a^\dagger\mathrm{P} = -a^\dagger$. Therefore, $\mathrm{PH}_\pm\mathrm{P} = \mathrm{H}_\mp$ on $\mathcal{D}_2$ and hence (3.15) solves the Riccati equation (2.7) as stated. This solution is not unique as *e.g.*, $-\mathrm{P}$ satisfies (2.7) as well. In fact, any solution to (2.7) is given by $\mathrm{X} = \mathrm{P} + \mathrm{Y}$ where

$$\alpha\mathrm{Y}^2 + \mathrm{Y}(\mathrm{H}_+ + \alpha\mathrm{P}) - (\mathrm{H}_- - \alpha\mathrm{P})\mathrm{Y} = 0. \tag{3.17}$$

One can easily confirm that $Y = -2P$ solves this equation.

If conditions (3.11) are met, in particular for $\beta \neq 0$, the spectra $H_\pm$ are separated and (2.7) possesses exactly one solution, which corresponds to a symmetry generator $\mathbf{J}$. The only problem is that this solution is unknown. However, one can attempt to simplify the problem by putting $X_0 = Y_\beta P$ where

$$\alpha Y_\beta P Y_\beta + [Y_\beta, H_+] + 2\beta Y_\beta - \alpha P = 0. \tag{3.18}$$

Above, we have redefined H_+ so that it reads (2.6) with $\beta = 0$. When $\beta = 0$, this equation becomes trivial and the solution can be retrieved to be $Y_0 = \mathbb{I}_B$. On the other hand, as long as $\beta = 0$, under (3.11), (3.18) satisfies premises of Theorem 1 and hence a unique Y_β such that $\|Y_\beta\| \leq 1$ exists. Note, if the inverse Y_β^{-1} exists as well then

$$\alpha Y_\beta^{-1} P Y_\beta^{-1} + \left[Y_\beta^{-1}, H_+\right] + 2(-\beta) Y_\beta^{-1} - \alpha P = 0, \tag{3.19}$$

and therefore $Y_{-\beta} = Y_\beta^{-1}$. Although we cannot solve this equation either, the latter equality indicates to what class Y_β belongs. One can also verify that Y_β is not anti–self–adjoint ($Y_\beta^* = -Y_\beta$). Indeed, assuming to the contrary that there is a solution to the above equation such that $Y_\beta = -Y_\beta^*$, one can separate this equation into a self–adjoint and an anti–self–adjoin part so that

$$\alpha Y_\beta P Y_\beta + [Y_\beta, H_+] - \alpha P = 0 \quad \text{and} \quad 2\beta Y_\beta = 0, \tag{3.20}$$

which contradicts (3.18) unless $\beta = 0$. Similar arguments show that the assumption of self–adjointness of the solution leads to

$$\alpha Y_\beta P Y_\beta + 2\beta Y_\beta - \alpha P = 0 \quad \text{and} \quad [Y_\beta, H_+] = 0. \tag{3.21}$$

Both these equations can be solved separately but the (obvious) solutions do not agree with each other as long as $\beta = 0$. To see this, suppose that

$$Y_\beta = \sum_n \xi_n |g_n\rangle\langle g_n|, \tag{3.22}$$

where $|g_n\rangle$ are the eigenstates of H_+. Y_β given in (3.22) can be seen as a function of H_+, $Y_\beta = f(H_+)$. This assures that the second equation from the pair (3.21) is automatically satisfied for every ξ_n. Now, substituting (3.22) into the first of the equations (3.21) we find

$$\sum_n \left(p_n\xi_n^2 + 2\beta\xi_n - p_n\right)|g_n\rangle\langle g_n| + \sum_{n\neq m} p_{nm}\left(\xi_n\xi_m - 1\right)|g_n\rangle\langle g_m| = 0, \tag{3.23}$$

where $p_{nm} = \alpha\langle g_n|\mathrm{P}|g_m\rangle$. Since operators $|g_n\rangle\langle g_m|$ form a basis, both the above terms must vanish independently. Therefore,

$$\xi_n = \frac{-\beta \pm \sqrt{\beta^2 + p_n^2}}{p_n} \quad \text{and} \quad \xi_n\xi_m = 1, \quad \text{for each} \quad n, m. \tag{3.24}$$

This is an obvious contradiction for every value of β other than zero.

One can also try to solve the equation which involves P first. Assuming the solution of the form

$$\mathrm{Y}_\beta = \sum_n \xi_n|n\rangle\langle n| \quad \text{with} \quad a^\dagger a|n\rangle = n|n\rangle, \tag{3.25}$$

we obtain similar expression as before for the coefficients ξ_n except $p_n = \alpha$. However, this solution does not fulfill $[\mathrm{Y}_\beta, \mathrm{H}_+] = 0$.

One way or another we encounter a contradiction when β is non zero. The interesting part is that we cannot rule out a possibility of the existence of a self–adjoint solution. The reason for that is an operator may exist which is neither the function of P nor H_+ and yet it is a common solution to (3.21). Take the following pair

$$\mathrm{X}a + a\mathrm{X} = 0 \quad \text{and} \quad a^\dagger\mathrm{X}^2 - a^\dagger = 0, \tag{3.26}$$

which are much simpler than the ones in the question, as an example. The solution reads $\mathrm{X} = \mathrm{P}$ and it is neither a function of a nor $a^\dagger$.

Also, it is interesting to notice that a parity operator cannot solve (2.7) if $\beta \neq 0$. Indeed, for every operator X such that $\mathrm{X}^2 = \mathbb{I}$ we have

$$\mathrm{X}(\mathrm{H}_+ + \beta) - (\mathrm{H}_- - \beta)\mathrm{X} = 0. \tag{3.27}$$

As we know, this equation has exactly one solution as long as $\beta \neq 0$. Obviously, this solution is $\mathrm{X} = 0$, which contradicts the idempotent condition.

It seems that our inability to solve the Riccati equation (2.7) when $\beta = 0$ is the core reason why the symmetry (3.3) hasn't been recognised earlier. An interesting point is that although the solution exists as we proved, it may not be expressible by standard (well–known) operators. In that case it is hard to expect to find it by using different methods regardless of what their nature might be. Despite the fact that the both symmetries $\mathbf{J}_0$ and $\mathbf{J}$ have the same origin, only the first one has been known all along. As we have seen,

the corresponding Riccati can be solved relatively easily in this case.

3.4 Diagonalization of the Rabi model

The unique solution to (2.7) not only gives rise to the symmetry in the Rabi model, but also can be explored to obtain its eigenfunctions and corresponding eigenvalues. Let us briefly discuss this concept.

Since both $\mathcal{G}(X_0)$ and $\mathcal{G}(X_0)^\perp$ are $\mathbf{H}$–invariant, thus if $|\Psi\rangle$ is an energy eigenstate it follows that either $|\Psi\rangle \in \mathcal{G}(X_0)$ or $|\Psi\rangle \in \mathcal{G}(X_0)^\perp$. We can actually say more than that. Namely, let us define (on $\mathcal{D}_2$) closed operators $K_+ = H_+ + \Delta X_0$ and $K_- = H_- - \Delta X_0^*$. Then we have form (3.5) that all eigenstates of $\mathbf{H}$ belonging to $\mathcal{G}(X_0)$ are of the following form

$$|\Psi_\lambda\rangle = \begin{bmatrix} |\psi_\lambda\rangle \\ X_0|\psi_\lambda\rangle \end{bmatrix}, \quad \text{where} \quad K_+|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle. \tag{3.28}$$

Similar arguments show that for all eigenvector from $\mathcal{G}(X_0)^\perp$ we have

$$|\Phi_\lambda\rangle = \begin{bmatrix} -X_0^*|\phi_\lambda\rangle \\ |\phi_\lambda\rangle \end{bmatrix}, \quad \text{where} \quad K_-|\phi_\lambda\rangle = \lambda|\phi_\lambda\rangle. \tag{3.29}$$

It can be proven that $K_\pm$ are self–adjoint on Hilbert spaces $(\mathcal{H}_B, \langle(1 + X_0^*X_0)\cdot,\cdot\rangle)$ and $(\mathcal{H}_B, \langle(1 + X_0X_0^*)\cdot,\cdot\rangle)$, respectively [35]. By means of different arguments, we will establish this fact in Sec. 4.2. As a result, we have $\sigma(\mathbf{H}) = \sigma(K_+) \cup \sigma(K_-)$.

To see how all of this connects with the notion of diagonalization we show

$$\mathbf{S}^{-1}\begin{bmatrix} H_+ & \Delta \\ \Delta & H_- \end{bmatrix}\mathbf{S} = \begin{bmatrix} K_+ & 0 \\ 0 & K_- \end{bmatrix}, \quad \text{where} \quad \mathbf{S} = \begin{bmatrix} 1 & -X_0^* \\ X_0 & 1 \end{bmatrix}. \tag{3.30}$$

Indeed, in terms of matrix elements the statement that X_0 solves the Riccati equation (2.7) formally means that $\mathbf{SH} = \mathbf{HS}$. Since X_0 is bounded so is the similarity matrix $\mathbf{S}$; moreover $\mathcal{D}(H_\pm) = \mathcal{D}(K_\pm)$ and thus $\mathcal{D}(\mathbf{SH}) = \mathcal{D}(\mathbf{HS})$. Furthermore, $\mathbf{S}$ can also be written as

$$\mathbf{S} = \mathbf{1} + \mathbf{X_0}, \quad \text{where} \quad \mathbf{X_0} = \begin{bmatrix} 0 & -X_0^* \\ X_0 & 0 \end{bmatrix}. \tag{3.31}$$

Because $\mathbf{X_0^*} = -\mathbf{X_0}$, it has purely imaginary spectrum. In particular, $-1 \notin \sigma(\mathbf{X_0})$ and therefore $0 \notin \sigma(\mathbf{S})$. This means that $\mathbf{S}$ has a bounded inverse and hence (3.30) holds true.

Formally, the similarity of the Rabi matrix $\mathbf{H}$ to a block diagonal matrix extends the notion of the parity chains introduced in [16] for the $\beta = 0$ case. For instance, when

$\beta = 0$ we have

$$\mathbf{H} \sim \begin{bmatrix} \mathrm{H}_+ + \Delta\mathrm{P} & 0 \\ 0 & \mathrm{H}_- - \Delta\mathrm{P} \end{bmatrix} \sim \begin{bmatrix} \mathrm{H}_+ + \Delta\mathrm{P} & 0 \\ 0 & \mathrm{H}_+ - \Delta\mathrm{P} \end{bmatrix}. \tag{3.32}$$

The first similarity is due to (3.30), the second one results from $\mathrm{PH\_P} = \mathrm{H}_+$ and it can be accomplished by $\mathrm{diag}[1, \mathrm{P}]$. Thus, to solve the Rabi model exactly when $\beta = 0$, the eigenproblems for tridiagonal infinite matrices $\mathrm{K}(\epsilon) = \mathrm{H}_+ \pm \epsilon\Delta\mathrm{P}$ ($\epsilon = \pm 1$) need to be resolved. This has not been done yet, despite the relatively simple form of $\mathrm{K}(\epsilon)$. This idea, however, has recently been pursued to obtain some useful analytical approximations [17].

Chapter 4

Dephasing and stationary states

An interaction between two quantum systems almost always modifies a set of quantum numbers suitable for the description of non–interacting components. Generically, only in the absence of interactions the set of good quantum numbers of a composite system consists of good quantum numbers of its subsystems. Let us consider energy as an example. It is well known that (Hamiltonian–type) interactions cause an energy flow from one system to the other and/or vice–verse. The total system remains conservative but the interacting parts become *open* which changes their character qualitatively [1].

One can named at least one case, commonly referred to as *dephasing* [50], where the energy exchange between two systems is absent during the entire evolution regardless of the initial conditions. In open quantum systems theory, where it is often assumed that the second subsystem is much larger than the first one, such phenomenon is also known as the pure decoherence [51]. It induces the loss of quantum coherence (and thus the emergence of classical behaviour [6]) without affecting the system's energy.

For dephasing to occur a rather specific type of interaction is required, yet nothing which might contradict our intuition is responsible for this phenomenon. The absence of the energy exchange is due to the existence of a conservation law (the total Hamiltonian commutes with the interaction operator). It is also the case if, instead of the energy, one considers a quantum number related to a local observable (*e.g.* acting nontrivially only on the subsystem). In general, one cannot expect such a number to be conserved unless the corresponding observable happens to be a symmetry in the subsystem.

One can ask: Is it possible to design such time evolution of a composite quantum system that makes a given local quantum number a good quantum number regardless of the existence of the related symmetry in the subsystem? This is a very intriguing question considering the fact that we are really asking whether the conservation of a quantum number in an open system is possible not despite the destructive influence of

its environment but rather because of it. At face value it may sound paradoxical, to say the least, yet similar behaviour has been shown to take place in *e.g.* quantum Markovian systems [52].

Application of such a paradigm might serve as a model of a different type of computation, dissipative quantum computation, in which it is the environment which does all the work. Instead of isolating a system (and then applying tractable unitary dynamics on it) as in the standard paradigm of quantum computation [53], we enginer its coupling with the environment to do the desired tasks [54,55] (also thermodynamic ones [56]). In a scenario like this the interaction is described by a Markovian master equation [57], the computation task is encoded in Lindblad operators which model the environment, whereas the unique outcome (result of the computation) is written into the steady state reached rapidly by the open system.

This chapter is devoted to show that the destructive nature of the environment can also be explored to induce conservation of information encoded in open quantum systems (see also [58]). We only consider two–level systems, the general case remains open. There are many potential applications of such a mechanism. Take for example quantum information science.

Quantum devices of the future, like *e.g.* quantum computers [59], will most likely be composed of components consisting of a large amount of qubits (quantum memory [60], quantum register [61], etc.). The challenge is not only to stabilise such devices (shield against decoherence) but also to program them. For instance, from various reasons (*e.g.* to establish a reference point or to cache the results) it may be desirable to freeze the spin of a single qubit (or a cluster of them) in a given, preselected, direction during computational cycles. In principle, this could be achieved by means of quantum control usually via suitable drivings. Instead of (either open loop or feedback) quantum control, which continuously affects the system being analysed, we propose a method in which it is sufficient to measure the system only once at the beginning of the evolution (initial preparation). No dynamical control is needed to maintain the desired dynamics. Such an alternative approach can be more beneficial as the goal can be achieved without introducing 'numerical' errors resulting from the measurement. Of course, to fully explore this idea in real environments much work, both experimental and theoretical, needs to be done. We hope that our results will serve as a simple theoretical starting point.

4.1 Conservation of the energy

We begin by showing how a proper choice of an initial preparation of a composite qubit–boson system can assure no energy exchange between interacting subsystems. Strictly speaking, we identify dephasing–like behaviour in a non–dephasing model. We start by analysing a simple exactly solvable Jaynes–Cummings model [62],

$$\mathbf{H} = \beta\sigma_z + \omega a^\dagger a + (g^*\sigma_+ \otimes a + h.c.) \sim \begin{bmatrix} \omega a^\dagger a + \beta & g^* a \\ g a^\dagger & \omega a^\dagger a - \beta \end{bmatrix}. \tag{4.1}$$

In terms of the eigevectors $|\pm\rangle$ of σ_z, the ladder operators $\sigma_\pm$ read $\sigma_+ = |+\rangle\langle -|$ and $\sigma_- = \sigma_+^\dagger$, respectively. All the remaining symbols denote exactly as in the Rabi model (2.1). The symbol $\sim$ should be understood here as '*its corresponds to*' as it was explained in Chapter 2.

In fact, the Jaynes–Cummings Hamiltonian (4.1) can be considered as an approximation to the Rabi Hamiltonian [20] when the latter is expressed in a suitable basis. This approximation is valid near the resonance, $|\omega - \beta| \ll \omega + \beta$, where the rapidly oscillating term can be neglected. For standard argumentation in regard to this famous rotating wave approximation we refer readers to see *e.g.* [31].

The above equation constitutes probably the most recognisable Hamiltonian in quantum optics. This model has also been studied in a wide range of other branches of physics (see *e.g.* [63]) for almost half–century within the broad variety of contexts [64]. It serves as a good starting point for our considerations.

We are interested in finding the initial state, ϱ, of the composite qubit–boson system such that the qubit energy, $E_\mathrm{Q} = \beta \mathrm{Tr}\left(\sigma_z \otimes \mathbb{I}_\mathrm{B}\varrho(t)\right) \sim \langle\sigma_z(t)\rangle$, is conserved. In other words, we wish to have $E_\mathrm{Q} = \beta \mathrm{Tr}\left(\sigma_z\rho(t)\right) = cst.$, where $\rho(t) = \mathrm{Tr}_\mathrm{B}(e^{-i\mathbf{H}t}\varrho e^{i\mathbf{H}t})$ stands for the reduced qubit dynamics [57]. $\mathrm{Tr}_\mathrm{B}(\cdot)$ refers to the partial trace, the trace with respect to the bosonic degrees of freedom A2.

There are two well–known and in that sense trivial statements one can make with regard to the Jaynes–Cummings model in the present context. First, the total number of excitation, $\mathrm{N} = a^\dagger a + \sigma_z$, is a constant of motion. Of course, it does not follow from this that $\langle\sigma_z\rangle(t) = cst.$ In fact, starting from an arbitrary initial state we will get $\langle\sigma_z\rangle(t) \neq cst.$ Exceptions are certain combinations of $|+\rangle \otimes |n\rangle$ and $|-\rangle \otimes |n+1\rangle$ that form eigenstates of the total Hamiltonian (4.1). This brings us to the second observation: The reduced dynamics $\rho(t)$ do not change with time at all for all such initial states. Therefore, the result $\langle\sigma_z\rangle(t) = cst.$ follows immediately. There is nothing new and surprising here. However, one can ask the question: Are there initial states of the composite system such that the

reduced qubit dynamic is nontrivial, $\rho(t) \neq cst.$, yet $\langle\sigma_z\rangle(t) = cst$?

The free qubit Hamiltonian $\beta\sigma_z$ is diagonal in the Jaynes–Cummings model. As a result, the reduced qubit dynamics $\rho(t)$ should read

$$\rho(t) = \alpha|+\rangle\langle+| + (1-\alpha)|-\rangle\langle-| + (c(t)|-\rangle\langle+| + h.c.)\,, \tag{4.2}$$

in order to ensure $\langle\sigma_z(t)\rangle = cst.$ for all $t \geq 0$. Above, α is a real constant and $c(t)$ denotes a function of time such that $|c(t)| \leq 1$ ($\rho(t)$ is positive). If in addition one has $c(t) = cst.$ then the above state is stationary. So, the question is: Which initial states of the composite system guarantee reduced dynamics (4.2)? Does nature allow quantum systems to exist in states which are dephasing (in the sense explained above), yet not stationary?

In order to answer this question we will construct such states explicitly. As a first step towards this objective, we define $\varrho = |\Psi\rangle\langle\Psi|$, where

$$|\Psi\rangle = C_\psi\left(|+\rangle \otimes |\psi\rangle + |-\rangle \otimes \mathrm{X}|\psi\rangle\right) \sim C_\psi \begin{bmatrix} |\psi\rangle \\ \mathrm{X}|\psi\rangle \end{bmatrix}, \tag{4.3}$$

with $|\psi\rangle$ being an arbitrary state of the bosonic field. C_ψ is a normalization constant such that $\langle\Psi|\Psi\rangle = 1$. Henceforward, for the sake of simplicity, we absorb it into $|\psi\rangle$ so that $\|\psi\| = |C_\psi|$.

Taking into account what has been said with regard to operator matrices so far, it should not come as a surprise that the possibility of constructing dephasing states relies on the ability of finding a solution to the Riccati equation associated with $\mathbf{H}$,

$$g^*\mathrm{X}a\mathrm{X} + \mathrm{X}(\omega a^\dagger a + \beta) - (\omega a^\dagger a - \beta)\mathrm{X} - ga^\dagger = 0. \tag{4.4}$$

In the light of the arguments used in Sec. 3.1 to prove the existence of the symmetry $\mathbf{J}$, it should be obvious what we are trying to accomplish here. Namely, the state $|\Psi\rangle$ belongs to the graph of X which is $\mathbf{H}$–invariant if X happens to solve (4.4). As one may suspect, this restricts possible evolution which the states $|\Psi\rangle$ can undergo. It turns out, as we will shortly see that $|\Psi\rangle$ does not leave the space $\mathcal{G}(\mathrm{X})$ during the entire evolution, that is

$$|\Psi\rangle \to |\Psi_t\rangle = \left(|+\rangle \otimes |\psi_t\rangle + |-\rangle \otimes \mathrm{X}|\psi_t\rangle\right) \sim \begin{bmatrix} |\psi_t\rangle \\ \mathrm{X}|\psi_t\rangle \end{bmatrix}. \tag{4.5}$$

The state $|\psi_t\rangle$ is yet to be found. The above evolution results in the following reduced dynamics

$$\alpha(t) = \langle\psi_t|\psi_t\rangle \quad \text{and} \quad c(t) = \langle\psi_t|\mathrm{X}|\psi_t\rangle, \tag{4.6}$$

which resemble the dephasing dynamics (4.2) if the map $|\psi\rangle \to |\psi_t\rangle$ itself preserves the norm. This is the case when there exists a Kamiltonian K such that $|\psi_t\rangle = e^{-i\mathrm{K}t}|\psi\rangle$. In other words, $|\psi_t\rangle$ satisfies a Schrödinger equation, $i|\dot{\psi}_t\rangle = \mathrm{K}|\psi_t\rangle$. The latter indeed holds true for $\mathrm{K} = (\omega a^\dagger a + \beta) + g^* a\mathrm{X}$. At face value, this choice may seem bizarre and to some extent artificial. Nevertheless, such K is precisely the upper diagonal operator from the decomposition (3.30). As we will explain in the next section, this is not a coincidence. Till then we focus on implications of the choice rather than its origin.

According to both (4.1) and (4.3), we have

$$\begin{aligned}\mathbf{H}|\Psi\rangle &= |+\rangle \otimes \mathrm{K}|\psi\rangle + |-\rangle \otimes \left[\left(\omega a^\dagger a - \beta\right)\mathrm{X} + g a^\dagger\right]|\psi\rangle \\ &= |+\rangle \otimes \mathrm{K}|\psi\rangle + |-\rangle \otimes \mathrm{XK}|\psi\rangle,\end{aligned} \tag{4.7}$$

which holds for every vector $|\psi\rangle$. Thus, by replacing $|\psi\rangle$ with $\mathrm{K}|\psi\rangle$ one finds

$$\mathbf{H}^2|\Psi\rangle = |+\rangle \otimes \mathrm{K}^2|\psi\rangle + |-\rangle \otimes \mathrm{XK}^2|\psi\rangle \sim \begin{bmatrix} \mathrm{K}^2|\psi\rangle \\ \mathrm{XK}^2|\psi\rangle \end{bmatrix}. \tag{4.8}$$

Repeating this procedure n times the general formula is found:

$$\mathbf{H}^n|\Psi\rangle = |+\rangle \otimes \mathrm{K}^n|\psi\rangle + |-\rangle \otimes \mathrm{XK}^n|\psi\rangle \sim \begin{bmatrix} \mathrm{K}^n|\psi\rangle \\ \mathrm{XK}^n|\psi\rangle \end{bmatrix}. \tag{4.9}$$

Therefore,

$$e^{-i\mathbf{H}t}|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}\mathbf{H}^n|\Psi\rangle = |+\rangle \otimes e^{-i\mathrm{K}t}|\psi\rangle + |-\rangle \otimes \mathrm{X}e^{-i\mathrm{K}t}|\psi\rangle. \tag{4.10}$$

It remains 'only' to identify X explicitly and to show that K is Hermitian. There is no general method allowing us to find solutions to (2.7), yet judging from its structure it seems reasonable to try $\mathrm{X} = \sum_n \xi_n |n+1\rangle\langle n|$. By an explicit substitution we obtain

$$\sum_{n=0}^{\infty} \left(g^*\sqrt{n+1}\xi_n^2 + 2\delta\xi_n - g\sqrt{n+1}\right)|n+1\rangle\langle n| = 0, \tag{4.11}$$

where $\delta \equiv \beta - \omega/2$ is the detuning frequency. Therefore one infers that this is indeed a good guess as long as

$$\xi_n = \frac{-\delta + \epsilon\sqrt{\delta^2 + |g|^2(n+1)}}{g^*\sqrt{n+1}}, \quad n \geq 0, \quad \epsilon = \pm 1. \tag{4.12}$$

We have discarded the negative solution ($\epsilon = -1$). The one we are left with generalises

the Susskind–Glogower operator [65], in the sense that $\mathrm{X} \to |g|(aa^\dagger)^{-1/2}a^\dagger/g^*$ as $\delta \to 0$. As a result, the sought–after generator K reads

$$\begin{aligned} \mathrm{K} &= \omega a^\dagger a + \beta + \sum_{n=0}^{\infty} \frac{-\delta + \sqrt{\delta^2 + |g|^2(n+1)}}{\sqrt{n+1}} a|n+1\rangle\langle n| \\ &= \omega\left(a^\dagger a + \frac{1}{2}\right) + \sqrt{\delta^2 + |g|^2(a^\dagger a + \mathbb{I}_\mathrm{B})} \end{aligned} \tag{4.13}$$

and it clearly is Hermitian. It follows from (4.6) that $\alpha = |C_\psi|^2$ and in addition

$$c(t) = e^{-i\omega t}\sum_{n=0}^{\infty} \xi_n e^{i\Omega_n t}\langle\psi|n+1\rangle\langle n|\psi\rangle \quad (\neq cst.), \tag{4.14}$$

where $\Omega_n = \sqrt{\delta^2 + |g|^2(n+2)} - \sqrt{\delta^2 + |g|^2(n+1)}$.

We still have the freedom to choose the vector $|\psi\rangle$. In particular, putting $|\psi\rangle = |m\rangle$, where $|m\rangle$ is a state with defined number of bosons, we not only have $\dot{c}(t) = 0$ meaning $\rho(t)$ is a steady state but also $c(t) = 0$ (*i.e.* the state exhibits its classical nature). One should anticipate such a result since $\mathrm{X}|m\rangle = \xi_m|m+1\rangle$ and

$$|\Psi\rangle = |+\rangle \otimes |m\rangle + \xi_m|-\rangle \otimes |m+1\rangle \tag{4.15}$$

is an eigenstate of the Jaynes–Cummings Hamiltonian.

In general, states like the one from (4.3) are entangled. Hence, there is no one–to–one correspondence between qubit–boson density operators $|\Psi\rangle\langle\Psi|$ and reduced density matrices $\mathrm{Tr}_\mathrm{B}(|\Psi\rangle\langle\Psi|)$ of the qubit [66]. If, however, $\mathrm{X}|\psi\rangle = \lambda|\psi\rangle$ (*e.g.* $|\psi\rangle$ is an eigenvector of X) then $|\Psi\rangle = (|+\rangle + \lambda|-\rangle) \otimes |\psi\rangle$ is separable. In this case the initial density matrix that guarantees the dephasing dynamics reads

$$\rho(0) = |C_\psi|^2 \begin{pmatrix} 1 & \lambda^* \\ \lambda & |\lambda|^2 \end{pmatrix}. \tag{4.16}$$

4.2 General case

So far, we have shown how to determine initial preparations of the specific composite qubit–boson system that guarantee no energy flow between qubit and its environment (boson). In that case, the reduced qubit density matrix mimics a pure dephasing evolution. Also, we have pointed out that, in principle, from this broad class of states one can choose separable ones.

Now, we are interested in finding an answer to a more general question: How to

prepare an initial state of a composite system (and determine its separability) which result in no 'information flow' between its subsystems. We assume that the information is encoded in a qubit observable Λ. As we will see, such states have very much in common with dephasing states and for that reason we will keep this terminology.

Until now, our analysis has been carried out for a very specific exactly solvable model. Currently, we will show that neither solvability nor the form of the interaction in the model is a *sine qua non* condition for designing the 'dephasing dynamics' for general qubit–environment models.

Let Λ be a 2×2 Hermitian matrix–a given qubit observable. Our objective is to determine $\rho \equiv \rho(0)$ such that $\langle \Lambda(t) \rangle \equiv \mathrm{Tr}\,(\Lambda\rho(t))$ remains constant during the evolution. As before, by $\rho(t)$ we denote the qubit reduced dynamics, $\rho(t) = \mathrm{Tr}_{\mathrm{E}}\left[e^{-i\mathbf{H}t}|\Psi\rangle\langle\Psi|e^{i\mathbf{H}t}\right]$, where $|\Psi\rangle$ is the initial qubit–environment state and

$$\mathbf{H} = \mathrm{H}_{\mathrm{Q}} \otimes \mathbb{I}_{\mathrm{E}} + \mathbb{I}_{\mathrm{Q}} \otimes \mathrm{H}_{\mathrm{E}} + \mathbf{H}_{\mathrm{int}}, \tag{4.17}$$

with all the symbols having their usual meaning, stands for the total Hamiltonian.

We begin with a very simple observation that in each moment of time and for every complex 2×2 matrix Λ, the partial trace, $\mathrm{Tr}_{\mathrm{E}}(\cdot)$, satisfies

$$\mathrm{Tr}\left[\Lambda \mathrm{Tr}_{\mathrm{E}}(\varrho(t))\right] = \mathrm{Tr}\left[(\Lambda \otimes \mathbb{I}_{\mathrm{E}})|\Psi(t)\rangle\langle\Psi(t)|\right]. \tag{4.18}$$

By virtue of this relation, we obtain

$$\langle \Lambda(t) \rangle = \mathrm{Tr}\left[(\Lambda_d \otimes \mathbb{I}_{\mathrm{E}})\,|\Omega(t)\rangle\langle\Omega(t)|\right], \tag{4.19}$$

where $|\Omega(t)\rangle = e^{-i\mathbf{K}t}|\Omega\rangle$, $|\Omega\rangle = \mathrm{U} \otimes \mathbb{I}_{\mathrm{E}}|\Psi\rangle$ and

$$\mathbf{K} = (\mathrm{U} \otimes \mathbb{I}_{\mathrm{E}})\mathbf{H}(\mathrm{U} \otimes \mathbb{I}_{\mathrm{E}})^{\dagger}. \tag{4.20}$$

U denotes the unitary matrix such that $\mathrm{U}^{\dagger}\Lambda\mathrm{U} \equiv \Lambda_{\mathrm{d}} = \mathrm{diag}(\lambda_+, \lambda_-)$. The Kamiltonian $\mathbf{K}$ can always be written as

$$\mathbf{K} = |+\rangle\langle+| \otimes \mathrm{H}_+ + |-\rangle\langle-| \otimes \mathrm{H}_- + (|+\rangle\langle-| \otimes \mathrm{V} + h.c.), \tag{4.21}$$

which formally corresponds to the operator matrix (2.8). Of course, an explicit form of $\mathrm{H}_{\pm}$ and V can easily be recovered when the Hamiltonians H_{Q}, H_{E}, and $\mathbf{H}_{\mathrm{int}}$ are provided. *A priory* we neither impose any physical restriction of their specification nor assume the existence of symmetries in the total system (resulting *e.g.*, in its solvability).

Essentially, to some point we apply the same technique as before. Namely, the composite system is assumed to be in the state $\varrho = |\Psi\rangle\langle\Psi|$ initially, where

$$|\Psi\rangle = C_\psi \left(|\lambda_+\rangle \otimes |\psi\rangle + |\lambda_-\rangle \otimes \mathrm{X}|\psi\rangle\right) \sim \mathrm{U}^\dagger \begin{bmatrix} |\psi\rangle \\ \mathrm{X}|\psi\rangle \end{bmatrix} \tag{4.22}$$

with $|\lambda_\pm\rangle = \mathrm{U}^\dagger|\pm\rangle$. $|\psi\rangle$ is a freely chosen state of the environment. As before, one can redefine this state so that $|\psi\rangle \to C_\psi|\psi\rangle$. Also, we assume that X solves the Riccati equation associated with $\mathbf{K}$, *i.e.*, the equation (2.9).

By virtue of the very same arguments which lead us to (4.10), the evolution generated by $\mathbf{K}$ is easily obtained to be

$$|\Omega(t)\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}\mathbf{K}^n|\Omega\rangle \sim \begin{bmatrix} |\psi_t\rangle \\ \mathrm{X}|\psi_t\rangle \end{bmatrix} \quad \text{with} \quad |\psi_t\rangle = e^{-i\mathrm{K}_+ t}|\psi\rangle, \tag{4.23}$$

where $\mathrm{K}_+ = \mathrm{H}_+ + \mathrm{VX}$.

Having (4.23) in place, we can begin to investigate conditions under which the expectation value (4.19) does not depend on time. Note, the density matrix $\mathrm{Tr}_\mathrm{E}(|\Omega(t)\rangle\langle\Omega(t)|)$ has a structure similar to the dephasing matrix (4.2), yet with time–dependent α:

$$\alpha(t) = \langle\psi|e^{i\mathrm{K}_+^\dagger t}e^{-i\mathrm{K}_+ t}|\psi\rangle \quad \text{and} \quad c(t) = \langle\psi|e^{i\mathrm{K}_+^\dagger t}\mathrm{X}e^{-i\mathrm{K}_+ t}|\psi\rangle; \tag{4.24}$$

therefore, it follows from (4.19) that

$$\langle\Lambda(t)\rangle = \alpha(t)\lambda_+ + (1-\alpha(t))\,\lambda_-. \tag{4.25}$$

Clearly, if $\alpha(t)$ is a constant so is $\langle\Lambda(t)\rangle$. Thus, the question is: Upon which conditions can we have $\alpha(t) = cst.$? Obviously, the answer depends on properties of K_+. So, what are they? First of all, if VX is Hermitian, as in our opening example, so is K_+ and hence the evolution $|\psi\rangle \to |\psi_t\rangle$ is unitary. Thus, we have $\alpha(t) = \alpha(0)$ and yet $c(t) \neq c(0)$ which result in not trivial dephasing dynamics regardless of the initial vector $|\psi\rangle$.

So, the really interesting question here is: What about all the cases when K_+ is not Hermitian? Are those even possible? It is hard to address these questions since we have no knowledge regarding properties of the solution X. Nevertheless, with a fair amount of intuition combined with justifiable assumptions concerning the operator K_+ we can overcome this problem.

It seams reasonable to assume that the Kamiltonian K_+ has a complete set of eigenstates. For the sake of argument, let's also assume that its spectrum is discrete and

not degenerated so that $\mathrm{K}_+|\psi_n\rangle = E_n|\psi_n\rangle$. As a result, we obtain

$$\mathrm{K}_+ = \sum_n E_n|\psi_n\rangle\langle\psi_n| \quad \text{and} \quad \mathrm{K}_+^\dagger = \sum_n E_n^*|\psi_n\rangle\langle\psi_n|. \tag{4.26}$$

Now, the interesting part follows: An eigenvalue E_n of K_+ is also an eigenvalue of $\mathbf{K}$ as noted in Sec. 3.4. Thus, $E_n^* = E_n$ proving that K_+ is in fact Hermitian.

If one cannot find a basis that consists with eigenstates of the Kamiltonian K_+, we still can design the dephasing dynamics as long as K_+ is diagonalizable. This condition is much weaker than Hermicity and it seems to reflect an absolute minimal physical requirement one can impose on K_+ in this context [67].

As before, we only investigate discrete and not degenerated cases. Saying that K_+ can be diagonalized means that there is a basis $\{|n\rangle\}$, linear invertible transformation S and complex numbers E_n (which we known are in fact real) such that

$$\mathrm{S}^{-1}\mathrm{K}_+\mathrm{S} = \sum_n E_n|n\rangle\langle n|. \tag{4.27}$$

Now, we can introduce two sets of vectors $|\psi_n\rangle = \mathrm{S}|n\rangle$ and $|\phi_n\rangle = (\mathrm{S}^{-1})^\dagger|n\rangle$ which form a complete set of biorthonormal eigenvectors [68] meaning

$$\begin{aligned} \mathrm{K}_+|\psi_n\rangle = E_n|\psi_n\rangle, \quad \mathrm{K}_+^\dagger|\phi_n\rangle = E_n|\phi_n\rangle, \\ \mathbb{I}_\mathrm{E} = \sum_n |\psi_n\rangle\langle\phi_n|, \quad \langle\psi_n|\phi_m\rangle = \delta_{nm}. \end{aligned} \tag{4.28}$$

At this point, it should be stressed that $\langle\psi_n|\psi_m\rangle \neq \delta_{nm}$ and $\langle\phi_n|\phi_m\rangle \neq \delta_{nm}$ in general. In view of these properties, we have

$$\mathrm{K}_+ = \sum_n E_n|\psi_n\rangle\langle\phi_n| \quad \text{and} \quad \mathrm{K}_+^\dagger = \sum_n E_n|\phi_n\rangle\langle\psi_n|. \tag{4.29}$$

Note, if the similarity operator S is unitary so is the evolution $|\psi\rangle \to |\psi_t\rangle$ since the two bases $\{|\psi_n\rangle\}$,$\{|\psi_m\rangle\}$ are identical and thus K_+ is Hermitian. Henceforward, we assume this is not the case because we have already examined it. It follows from both (4.24) and (4.28) that

$$\alpha(t) = \sum_n |\langle\psi|\phi_n\rangle|^2\|\psi_n\|^2 + \sum_{n\neq m} e^{i(E_n-E_m)t}\langle\psi|\phi_n\rangle\langle\psi|\phi_m\rangle^*\langle\psi_n|\psi_m\rangle \tag{4.30}$$

and in addition

$$c(t) = \sum_n |\langle\psi|\phi_n\rangle|^2 \|\mathrm{X}\psi_n\|^2 + \sum_{n\neq m} e^{i(E_n - E_m)t}\langle\psi|\phi_n\rangle\langle\psi|\phi_m\rangle^*\langle\psi_n|\mathrm{X}|\psi_m\rangle. \tag{4.31}$$

Both these expressions are time–dependent as they should be (in a Hilbert space one cannot have a linear and non unitary evolution which preserves the norm). We can, however, make $\alpha(t)$ constant at least for certain initial states $|\psi\rangle$. In particular, if $|\psi\rangle = |\psi_n\rangle$ then both $\alpha(t)$ and $c(t)$ do not depend on time. The latter observation is in perfect agreement with the results regarding stationary states reported in [49].

4.3 Remarks and observations

It is interesting and it should not be surprising that the dephasing dynamics can also be designed by starting from any state orthogonal to (4.22). Indeed, such states are found to be of the following form

$$|\Phi\rangle = C_\phi\left(|\lambda_-\rangle \otimes |\phi\rangle - |\lambda_+\rangle \otimes \mathrm{X}^\dagger|\phi\rangle\right) \sim \mathrm{U}^\dagger \begin{bmatrix} -\mathrm{X}^\dagger|\psi\rangle \\ |\psi\rangle \end{bmatrix}, \tag{4.32}$$

with C_ϕ being a normalization constant. This can be verified by noticing that $\langle\Psi|\Phi\rangle = 0$ for every $|\psi\rangle$, $|\phi\rangle$.

Repeating exactly the same arguments as before, we conclude that the qubit's reduced dynamics in this case takes the form

$$\alpha(t) = 1 - \langle\phi_t|\phi_t\rangle, \quad c(t) = -\langle\phi_t|\mathrm{X}^\dagger|\phi_t\rangle, \quad |\phi_t\rangle = e^{-i\mathrm{K}_- t}|\phi\rangle. \tag{4.33}$$

As before, the dephasing evolution is possible provided $\mathrm{K}_- = \mathrm{H}_- - \mathrm{V}^\dagger\mathrm{X}^\dagger$ is at least diagonalizable. For instance, in the Jaynes–Cummings model which we have discussed previously one finds that (compare with (4.13))

$$\mathrm{K}_- = \omega\left(a^\dagger a - \frac{1}{2}\right) - \sqrt{\delta^2 + |g|^2 a^\dagger a}. \tag{4.34}$$

4.3.1 Is the formalism valid?

A closer look at the structure of both the diagonal entries $\langle\psi_t|\psi_t\rangle$ and $\langle\psi_t|\mathrm{X}^\dagger\mathrm{X}|\psi_t\rangle$ makes one wonder whether the formalism we have introduced is valid. If one of these elements is time–independent so must be the other one since the reduced dynamics, is trace preserving, meaning that

$$\langle\psi|\psi\rangle + \langle\psi|\mathrm{X}^\dagger\mathrm{X}|\psi\rangle = \langle\psi_t|\psi_t\rangle + \langle\psi_t|\mathrm{X}^\dagger\mathrm{X}|\psi_t\rangle. \tag{4.35}$$

However, it is not obvious at all why $\langle\psi_t|\mathrm{X}^\dagger\mathrm{X}|\psi_t\rangle$ should be constants regardless of the choice of initial states even when $\langle\psi_t|\psi_t\rangle$ is. So, it seems either we have made a mistake performing calculations or our arguments on which we build on are logically invalid. The same issue arises if one considers the pair $\langle\phi_t|\phi_t\rangle$ and $\langle\phi_t|\mathrm{X}\mathrm{X}^\dagger|\phi_t\rangle$ instead.

Let us resolve this puzzle. As we will show, not only are there no contradictions here, but there is also good reason for diagonal elements to have aforementioned properties. Namely, $\mathrm{K}_\pm$ are pseudo Hermitian *i.e.*, there exist some invertible η and ξ such that $\mathrm{K}_+^\dagger = \eta\mathrm{K}_+\eta^{-1}$ and $\mathrm{K}_-^\dagger = \xi\mathrm{K}_-\xi^{-1}$ [69].

Indeed, if we define $\mathbf{U}$ to be the unitary matrix from the polar decomposition of $\mathbf{S}$ from (3.30), that is $\mathbf{S} = \mathbf{U}\sqrt{\mathbf{S}^\dagger\mathbf{S}}$, then it follows

$$\mathbf{U}^\dagger \begin{bmatrix} \mathrm{H}_+ & \Delta \\ \Delta & \mathrm{H}_- \end{bmatrix} \mathbf{U} = (\mathbf{S}^\dagger\mathbf{S})^{1/2} \begin{bmatrix} \mathrm{K}_+ & 0 \\ 0 & \mathrm{K}_- \end{bmatrix} (\mathbf{S}^\dagger\mathbf{S})^{-1/2}. \tag{4.36}$$

Since unitary transformations preserve the Hermiticity, the latter equality proves the pseudo–Hermicity conditions given above for $\eta = \mathbb{I}_\mathrm{E} + \mathrm{X}^\dagger\mathrm{X}$ and $\xi = \mathbb{I}_\mathrm{E} + \mathrm{X}\mathrm{X}^\dagger$, respectively.

Now, we can understand why (4.35) holds on a more fundamental level. The equality reflects the pseudo–unitary evolution which the vector $|\psi_t\rangle$ undergoes. The right side of the equation reads

$$\langle\psi|e^{i\mathrm{K}_+^\dagger}\eta e^{-i\mathrm{K}_+}|\psi\rangle = \langle\psi|\eta e^{i\mathrm{K}_+}\eta^{-1}\eta e^{-i\mathrm{K}_+}|\psi\rangle = \langle\psi|\eta|\psi\rangle. \tag{4.37}$$

4.3.2 Structure of the solution

Let us note that η and ξ induce positive–defined inner products $\langle\eta\cdot,\cdot\rangle$ and $\langle\xi\cdot,\cdot\rangle$ (because of that they are called metric operators) with respect to which K_+ and K_- are Hermitian, respectively. Furthermore, a pseudo Hermitian operator is Hermitian (with respect to the intrinsic inner product) if it commutes with the metric operator. In particular, $\mathrm{K}_+ = \mathrm{K}_+^\dagger$ provided $[\mathrm{K}_+, \mathrm{X}^\dagger\mathrm{X}] = 0$. On the other hand, if K_+ is only diagonalizable then $\langle\psi|\mathrm{X}^\dagger\mathrm{X}|\psi\rangle$ is diagonal with respect to the basis that consists with eigenvectors of K_+. We can confirm this result by noticing that

$$\langle\psi_t|\mathrm{X}^\dagger\mathrm{X}|\psi_t\rangle = \sum_{n,m} e^{i(E_n - E_m)t}\langle\psi|\phi_n\rangle\langle\psi_n|\mathrm{X}^\dagger\mathrm{X}|\psi_m\rangle\langle\phi_m|\psi\rangle, \tag{4.38}$$

which cannot be time–independent for every vector $|\psi\rangle$ unless $\langle\psi_n|\mathrm{X}^\dagger\mathrm{X}|\psi_m\rangle = |x_n|^2\delta_{nm}$ for some x_n. The latter observation suggests expressing X and $\mathrm{X}^\dagger$ in terms of the basis $\{|\psi_n\rangle\}$ as

$$\mathrm{X} = \sum_n x_n|\chi_n\rangle\langle\psi_n| \quad \text{and} \quad \mathrm{X}^\dagger = \sum_n x_n^*|\psi_n\rangle\langle\chi_n|, \tag{4.39}$$

where $\{|\chi_n\rangle\}$ is a set of orthonormal vectors.

At this point some remarks are in order. First of all, we know neither $\{|\chi_n\rangle\}$, x_n nor $\{|\psi_n\rangle\}$. But still, the above parametrization might be useful to us since it indicates what structure the solution may or may not have. Second, the vectors $|\chi_n\rangle$ we have used to express the solution do not necessarily form a basis. In particular, the set in question may not be completed as it is in the Jaynes–Cummings model where $|\chi_n\rangle = |n+1\rangle$. The vacuum state $|0\rangle$ is missing from the set in this case. Finally, even if we are able to derive the solution by means of purely algebraic method it does not mean it will have already the structure (4.39). Take the parity operator (3.15) as an example. It is diagonal in the $\{|n\rangle\}$ basis and yet $|\psi_n\rangle \neq |n\rangle$.

4.3.3 Symmetry

In Chapter 3, we have shown how a solution to the Riccati equation associated with the Rabi matrix gives rise to a symmetry in the Rabi model. Now, we are in a position to argue that this connection can be established for general systems. Indeed, let

$$\mathbf{J} = \begin{bmatrix} 2\eta^{-1} - \mathbb{I}_\mathrm{E} & 2\eta^{-1}\mathrm{X} \\ 2\mathrm{X}\eta^{-1} & 2\mathrm{X}\eta^{-1}\mathrm{X}^\dagger - 1 \end{bmatrix}, \tag{4.40}$$

then, in the light of pseudo Hermiticity of K_+, we have

$$[\mathbf{J}, \mathbf{H}] = 2\begin{bmatrix} \eta^{-1}\mathrm{K}_+^\dagger - \mathrm{K}_+\eta^{-1} & \left(\eta^{-1}\mathrm{K}_+^\dagger - \mathrm{K}_+\eta^{-1}\right)\mathrm{X}^\dagger \\ \mathrm{X}\left(\eta^{-1}\mathrm{K}_+^\dagger - \mathrm{K}_+\eta^{-1}\right) & \mathrm{X}\left(\eta^{-1}\mathrm{K}_+^\dagger - \mathrm{K}_+\eta^{-1}\right)\mathrm{X}^\dagger \end{bmatrix} = 0. \tag{4.41}$$

In general, we can write

$$\eta = \sum_n |\phi_n\rangle\langle\phi_n| \quad \text{and} \quad \eta^{-1} = \sum_n |\psi_n\rangle\langle\psi_n|. \tag{4.42}$$

The above result means that every composed qubit–environment system has a $\mathbb{Z}_2$ symmetry, provided the Riccati equation related to the Hamiltonian of that system is solvable. Once again, we can invoke the Jaynes–Cummings model as an example. Simple calculation shows that

$$\mathbf{J} = \sum_{n=0}^{\infty} \frac{1}{1+|\xi_n|^2} \begin{pmatrix} (1-|\xi_n|^2)|n\rangle\langle n| & 2\xi_n|n\rangle\langle n+1| \\ 2\xi_n^*|n+1\rangle\langle n| & (|\xi_n|^2-1)|n+1\rangle\langle n+1| \end{pmatrix}. \tag{4.43}$$

It is well known that a continuous symmetry in the Jaynes–Cummings model is generated by the total number of excitations, N. To be more precise, we have $\mathrm{G}(\alpha) = \exp(i\alpha\mathrm{N})$ where $\alpha \in \mathbb{R}$. In contrast to $\mathrm{G}(\pi)$, which resembles the local parity of the total system $\sigma_z \otimes \mathrm{P}$, (4.43) introduces a nonlocal (hidden in a sense) parity of that system.

When the counter–rotating terms are included in the Jaynes–Cummings Hamiltonian and we end up with the Rabi model, the continuous symmetry $\mathrm{G}(\alpha)$ breaks down to a discrete parity symmetry. This symmetry, depending on the value of β is either $\mathbf{J}_0 = \sigma_x \otimes \mathrm{P}$ when $\beta = 0$ or $\mathbf{J}$ given by (3.3) otherwise.

4.3.4 Separability of initial dephasing states

Let us return to the problem of separability of initial dephasing states. Since X is not Hermitian it may possess no eigenvectors at all (like *e.g.* the creation operator $a^\dagger$). In such cases, one cannot disentangle initial dephasing states (4.22). However, they may exist eigenvectors of $\mathrm{X}^\dagger$. Once again, $a^\dagger$ serves as a good example. In this case we have $a|z\rangle = z|z\rangle$, where $z \in \mathbb{C}$ and $|z\rangle$ is a coherent state [70]. Therefore, separate initial states guaranteeing dephasing dynamics can be found among the states (4.33).

The nonexistence of eigenstates of either X or $\mathrm{X}^\dagger$ is not the only problem we can encounter trying to disentangle initial states. The second kind of issue, so to speak, arises when eigenstates do exist but they cannot be normalised. Let us consider the operator X found in Sec. (4.1) as an example. In the limited case, $\delta \to 0$ provided g is real, X takes the form

$$\mathrm{X}_0 = \sum_n |n+1\rangle\langle n|. \tag{4.44}$$

For every $\xi \in \mathbb{C}$, the operator $\mathrm{X}^\dagger$ possesses eigenstates $|\xi\rangle = \sum_n e^{in\xi}|n\rangle$ with corresponding eigenvalues $e^{i\xi}$. However, $\langle\xi|\xi'\rangle \neq \delta(\xi-\xi')$. Even if the state $|\xi\rangle$ could be normalized to the Dirac delta it would not solve all problems. The inner product $\langle\xi|\xi\rangle$ would become infinite in this case and could not represent the probability of finding the qubit in its excited state as it is supposed to,

4.4 Further examples

In the examples provided below, we focus mostly on explaining how one can find a solution of a given Riccati equation since it is the hardest part of the entire method. Everything else, *e.g.* construction of the initial states, generators $\mathrm{K}_{\pm}$, symmetry and finally dynamics itself follows immediately.

4.4.1 Multi–photon Rabi model

We consider the k-photon Rabi model [71] for which $\mathrm{H_Q} = \beta\sigma_z$, $\mathrm{H_E} = \omega a^\dagger a$ read exactly as in the Jaynes–Cummings model (4.1) but the interaction is given by

$$\mathbf{H}_{\mathrm{int}} = \sigma_x \otimes (g^* a^k + g(a^\dagger)^k). \tag{4.45}$$

In this model, we have not only included counter–rotating–wave terms [72] which make its analytical treatment much more complicated in comparison with (4.1) but also have generalized the single mode case incorporating k photons.

Let us suppose that this time we want to find initial state(s) of the composite system such that the x–component of the qubit spin operator remains constant during the evolution *i.e.*, $S_x = \frac{1}{2}\mathrm{Tr}\left(\sigma_x \rho(t)\right) = cst$. First, one needs to determine U which transforms σ_x to its diagonal form. It is an easy task to do and the answer is

$$\mathrm{U} = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.46}$$

In this case, we have $\mathrm{U}\sigma_x\mathrm{U}^\dagger = \sigma_z$, that is $\lambda_\pm = \pm 1$. Next, we transform $\mathbf{H}$ into $\mathbf{K}$ according to (4.20) and then recover $\mathrm{H}_\pm$, V from the decomposition (4.21). Straightforward calculation shows

$$\mathrm{H}_\pm = \omega a^\dagger a \pm \left(g^* a^k + g(a^\dagger)^k\right), \quad \mathrm{V} = \beta. \tag{4.47}$$

The corresponding Riccati equation (2.9) reads as follows

$$\beta \mathrm{X}^2 + \mathrm{X}\mathrm{H}_+ - \mathrm{H\_X} - \beta = 0. \tag{4.48}$$

We can solve this equation by introducing the generalise parity operator [73]:

$$\mathrm{X}_k = \sum_{l=1}^{k}\sum_{n=0}^{\infty}(-1)^n |n,l\rangle\langle n,l|, \tag{4.49}$$

where $|n,l\rangle := |kn+l-1\rangle$ and $l \leq k$. This operator is both Hermitian and unitary. Moreover, it simplifies to the bosonic parity operator $\mathrm{P} = \exp(i\pi a^\dagger a)$ introduced in Sec. 3.3 when $k = 1$. Also, it shares with P similar properties, namely

$$\mathrm{X}_k a^k \mathrm{X}_k = -a^k \quad \text{from which} \quad \mathrm{X}_k \mathrm{H}_+ \mathrm{X}_k = \mathrm{H}_-. \tag{4.50}$$

As a result, X_k solves (4.48).

The dephasing dynamics is generated by $\mathrm{K}_\pm = \mathrm{H}_\pm \pm \beta \mathrm{X}_k$, respectively. Introducing projections $\mathrm{P}_\pm = \frac{1}{2}(\mathbb{I}_\mathrm{E} \pm \mathrm{X}_k)$ onto subspaces $\mathcal{H}_\pm$ consist of states with defined parity (with respect to the generalized parity (4.49)) and taking into account both (4.3) and (4.32) we have

$$|\Psi_\epsilon\rangle = \frac{1}{2}\left(|+\rangle \otimes \mathrm{P}_\epsilon|\psi\rangle + |-\rangle \otimes \mathrm{P}_{-\epsilon}|\psi\rangle\right), \quad \epsilon = \pm 1, \tag{4.51}$$

which are separable if $|\psi\rangle \in \mathcal{H}_\pm$.

To assure the energy lossless evolution, in this case the following Riccati equation

$$\mathrm{X}\left(g^* a^k + g(a^\dagger)^k\right)\mathrm{X} + \omega\left[\mathrm{X}, a^\dagger a\right] + 2\mathrm{X}\beta - \left(g^* a^k + g(a^\dagger)^k\right) = 0, \tag{4.52}$$

needs to be solved. Unfortunately, the solution of this equation is unknown when $\beta \neq 0$ (even for $k = 1$). Therefore, the problem of determining states which guarantee no energy exchange between qubit and bosons in the multi–photon Rabi model remains open.

4.4.2 Almost Jaynes–Cummings model

In our second example, we will find generators of the dephasing dynamics for a system that consists with a qubit, $\mathrm{H}_\mathrm{Q} = \beta\sigma_z$ as before, interacting with a general quantum system having integer spin j_0 for which we take

$$\mathrm{H}_\mathrm{E} = \omega \mathrm{J}_z, \quad \mathbf{H}_\mathrm{int} = g^* \sigma_+ \otimes \mathrm{J}_- + g\sigma_- \otimes \mathrm{J}_+. \tag{4.53}$$

Above, J_z and $\mathrm{J}_\pm$ refer to the standard angular momentum operators, well known from the general theory of angular momentum,

$$\begin{aligned} \mathrm{J}_z|j,m\rangle &= m|j,m\rangle, \quad \langle j,m|i,n\rangle = \delta_{ji}\delta_{mn}, \\ \mathrm{J}_\pm|j,m\rangle &= \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle, \end{aligned} \tag{4.54}$$

with $-j \leq m \leq j$ and $j \leq j_0$. Let $\Lambda = \sigma_z$ then

$$\mathrm{H}_\pm = \omega \mathrm{J}_z \pm \beta, \quad \mathrm{V} = g^* \mathrm{J}_- \tag{4.55}$$

and thus the Riccati equation takes the form

$$g^*\mathrm{X}\mathrm{J}_-\mathrm{X} + 2\beta\mathrm{X} - \omega\,[\mathrm{J}_z, \mathrm{X}] - g\mathrm{J}_+ = 0. \tag{4.56}$$

We try to solve this equation with

$$\mathrm{X} = \sum_{j=0}^{j_0} \sum_{m=-j}^{j} c_{jm} |j, m+1\rangle\langle j, m|. \tag{4.57}$$

For such X we have $[\mathrm{J}_z, \mathrm{X}] = \mathrm{X}$ and therefore (4.56) simplifies considerably,

$$g^*\mathrm{X}\mathrm{J}_-\mathrm{X} + 2\delta\mathrm{X} - g\mathrm{J}_+ = 0, \quad \text{where} \quad \delta = \beta - \frac{1}{2}\omega. \tag{4.58}$$

By substituting (4.57) into (4.58), one finds

$$c_{jm} = \frac{-\delta \pm \sqrt{\delta^2 + |g|^2(j-m)(j+m+1)}}{g^*\sqrt{(j-m)(j+m+1)}}, \quad \epsilon = \pm 1, \tag{4.59}$$

as long as $j \neq m$. Otherwise we set $c_{jj} = 0$. Generators of the dephasing dynamics read

$$\mathrm{K}_\pm^\epsilon = \omega\left(\mathrm{J}_z \mp \tfrac{1}{2}\right) \pm \epsilon\sqrt{\delta^2 + |g|^2 \mathrm{J}_\mp \mathrm{J}_\pm}. \tag{4.60}$$

Interestingly, these operators are similar to the ones obtained in the example with Jaynes–Cummings. However, the two models are not isomorphic, so to speak, since the commutation relation between $\mathrm{J}_\pm$, J_z and the one between a, $a^\dagger$ and $a^\dagger a$ are quite different.

4.4.3 Two interacting particles with spin half

In the following example, we consider a system consisting of two interacting qubits. The Hamiltonian of the model is assumed to be of the form:

$$\mathbf{H} = \omega_1\sigma_z^{(1)} + \omega_1\sigma_z^{(2)} + g\sigma_x \otimes \sigma_x \sim \begin{bmatrix} \omega_2\sigma_z + \omega_1 & g\sigma_x \\ g\sigma_x & \omega_2\sigma_z - \omega_1 \end{bmatrix}, \tag{4.61}$$

where ω_1, ω_2 and g are real parameters denoting energies of the qubits and the interaction between them, respectively. From an experimental point of view, one may be interested how to prepare an initial state of this system so that an expectation value of the spin operator $\mathrm{S}(\theta, \varphi)$ in a preselected direction $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ remains constant during the evolution. Let us briefly sketch the necessary steps which allow to do so by using the concept of dephasing states.

A basic result of quantum mechanics shows that

$$\mathrm{S}(\theta,\varphi)=\frac{1}{2}\begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix}, \tag{4.62}$$

is the spin operator in the direction $\mathbf{n}$. Its eigenvectors,

$$|\lambda_+(\theta,\varphi)\rangle=\begin{pmatrix}e^{-i\varphi/2}\cos(\theta/2)\\ e^{i\varphi/2}\sin(\theta/2)\end{pmatrix},\quad |\lambda_-(\theta,\varphi)\rangle=\begin{pmatrix}-e^{-i\varphi/2}\sin(\theta/2)\\ e^{i\varphi/2}\cos(\theta/2)\end{pmatrix}, \tag{4.63}$$

represent the 'spin up' and 'spin down' states along the $\mathbf{n}$–direction with the corresponding eigenvalues $\lambda_\pm=\pm 1/2$. Simple calculation shows that the operators $\mathrm{H}_\pm$ and V from (4.21) read

$$\mathrm{H}_\pm=\begin{pmatrix}\pm\omega_1\cos\theta\pm\omega_2 & \mp g\sin\theta\\ \mp g\sin\theta & \pm\omega_1\cos\theta\mp\omega_2\end{pmatrix},\mathrm{V}=e^{-i\varphi}\begin{pmatrix}\omega_1\sin\theta & g\cos\theta\\ g\cos\theta & \omega_1\sin\theta\end{pmatrix}. \tag{4.64}$$

The similarity matrix $\mathbf{U}$ which transforms $\mathbf{H}$ into $\mathbf{K}$ is simply given by

$$\mathbf{U}=\begin{pmatrix}e^{-i\varphi/2}\cos(\theta/2) & -e^{-i\varphi/2}\sin(\theta/2)\\ e^{i\varphi/2}\sin(\theta/2) & e^{i\varphi/2}\cos(\theta/2)\end{pmatrix}. \tag{4.65}$$

One could continue this general analysis further. In the next step one could assume $\mathrm{X}=(x_{ij})_{2\times 2}$ and insert this form into (2.9) which would result in four coupled quadratic equations for the coefficients x_{ij}. However, such approach would involve long and complicated formulas which are of no use to our purposes. It is more beneficial to consider particular cases one by one. We investigate two common ones, $\mathrm{S}(0,\varphi)$, $\mathrm{S}(\frac{\pi}{2},0)$, only.

For $\theta=0$ and $\varphi\in\mathbb{R}$ the aforementioned system of quadratic equation reads

$$\begin{aligned}x_{11}e^{-i\varphi}[(x_{12}+x_{21})g+2e^{i\varphi}\omega_1] &= 0\\ (x_{12}^2+x_{11}x_{22})e^{-i\varphi}g-e^{i\varphi}g+2x_{12}\omega_1-2x_{12}\omega_2 &= 0\\ (x_{21}^2+x_{11}x_{22})e^{-i\varphi}g-e^{i\varphi}g+2x_{21}\omega_1+2x_{21}\omega_2 &= 0\\ x_{22}e^{-i\varphi}[(x_{21}+x_{12})g+2e^{i\varphi}\omega_1] &= 0,\end{aligned} \tag{4.66}$$

which has the following solutions

$$\begin{aligned} x_{11} &= 0, \quad x_{22} = 0 \\ x_{12} &= e^{i\varphi} \frac{\epsilon_1(\omega_1 - \omega_2) - \epsilon_2\sqrt{(\omega_1 - \omega_2)^2 + g^2}}{g}, \\ x_{21} &= e^{i\varphi} \frac{\epsilon_1(\omega_1 + \omega_2) - \epsilon_2\sqrt{(\omega_1 + \omega_2)^2 + g^2}}{g} \quad (\text{note} \quad x_{12}^* \neq x_{21}), \end{aligned} \tag{4.67}$$

where $\epsilon_{1,2} = \pm 1$. If $\theta = \pi/2$ and $\varphi = 0$ then $\mathrm{X} = \pm\sigma_z$. In both these cases the Kamiltonians $\mathrm{K}_\pm$ are Hermitian.

4.4.4 Infinite environment

In the following example we consider to the Rabi model (2.2) again. This time, however, we assume that $\beta = 0$ and

$$\mathrm{H}_\pm = \sum_{k=1}^{N} \omega_k a_k^\dagger a_k \pm \sum_{k=1}^{N} \left(g_k^* a_k + g_k a_k^\dagger \right). \tag{4.68}$$

This choice reflects a situation in which a qubit is immersed within an environment consisting of N bosons, including $N \to \infty$. The solution to the Riccati equation formally reads (2.7) but with $\mathrm{H}_\pm$ given above reads

$$\mathrm{X} = \exp\left(i\pi \sum_{k=1}^{N} a_k^\dagger a_k \right) = \prod_{k=1}^{N} e^{i\pi a_k^\dagger a_k}. \tag{4.69}$$

The operators $\mathrm{K}_\pm$ which generate dephasing evolution guaranteeing conservation of the spin σ_x read exactly as in the Rabi model *i.e.*, $\mathrm{K}_\pm = \mathrm{H}_\pm + \Delta\mathrm{X}$.

Chapter 5

Further applications

The most established and useful time-dependent two-level quantum system is, perhaps, the one that describes a qubit in a rotating magnetic field [74]. The Hamiltonian of this system, in its basic variant, is usually written as

$$\mathrm{H_Q}(t) = -\frac{\gamma_e}{2}\boldsymbol{\sigma}\cdot\mathbf{B}(t), \tag{5.1}$$

where γ_e is the gyromagnetic ratio and $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$. Let's assume in addition that the magnetic field $\mathbf{B}(t)$ has the form

$$\mathbf{B}(t) = \left(B_1 \cos(\omega t), B_1 \sin(\omega t), B_0\right), \tag{5.2}$$

where B_0 and B_1 denote its amplitudes and ω is the frequency of the rotation. Introducing the familiar abbreviations, $\Delta \equiv -\frac{1}{2}\gamma_e B_1$, $\beta \equiv -\frac{1}{2}\gamma_e B_0$, one can rewrite the system's Hamiltonian as

$$\mathrm{H_Q}(t,\beta) = \beta\sigma_z + \Delta\left(\sigma_y \sin(\omega t) + \sigma_x \cos(\omega t)\right). \tag{5.3}$$

In the case of no coupling with the external environment, the exact form of the evolution operator U_t and hence the density matrix $\rho(t) = \mathrm{U}_t \rho \mathrm{U}_t^\dagger$ for the model (5.3) can be derived in an elegant and simple manner [75]. In fact, this problem is so common that it can be found in almost every modern textbook on quantum mechanics. However, if the aforementioned coupling is present the *exact* reduced dynamics has not yet been derived in general case.

Due to the dissipation, the evolution of an open system is not unitary yet it can always be described by trace preserving and completely positive map–dynamical map [76], $\Phi_t \colon \rho \to \rho(t)$. Unfortunately, finding its exact form is almost impossible in most cases, especially for the systems governed by the time–dependent Hamiltonian.

In this chapter, we focus on the case where the system (5.1) is immersed within the fermionic environment. We show how the methods we have been discussing so far can be used to obtain an exact form of the qubit's reduced dynamics. As an application of the formula, we investigate the adiabatic approximation for the system in question. It is demonstrated that in the weak coupling regime the well–known conditions ensuring the adiabatic evolution of the system (5.1) also lead to the adiabatic behaviour when the fermionic environment is present.

5.1 Model

In what follows, we assume that

$$\mathbf{H}(t,\beta) = \mathrm{H_Q}(t,\beta)\otimes\mathbb{I}_\mathrm{E} + \mathbb{I}_\mathrm{Q}\otimes\mathrm{H_E} + \mathbf{H}_\mathrm{int}, \tag{5.4}$$

with $\mathrm{H_Q}(t,\beta)$ given by (5.3), is the qubit–environment Hamiltonian. In addition, we take the environment to be composed of N independent and non-interacting qubits. The free evolution of which is governed by

$$\mathrm{H_E} = \sum_{n=1}^{N}\omega_n\sigma_n^z, \tag{5.5}$$

where ω_n are the energies of the qubits, $\sigma_n^z = \mathrm{I}_2\otimes\ldots\otimes\sigma^z\otimes\ldots\otimes\mathrm{I}_2$ stands for their spin operators (I_2 is the 2×2 identity matrix). The Hilbert space $\mathcal{H}_\mathrm{E}$ on which the Hamiltonian (5.5) acts is given by N-fold tensor product of $\mathbb{C}^2$ spaces, $\mathcal{H}_\mathrm{E} = \bigotimes_{n=1}^{N}\mathbb{C}^2$.

The coupling between the qubit and its environment is of the Ising-typ:

$$\mathbf{H}_\mathrm{int} = \sigma_z\otimes\sum_{n=1}^{N} g_n\sigma_n^z, \tag{5.6}$$

where g_n are the coupling constants.

The model described by (5.4)–(5.6) for $\Delta = 0$ (*i.e.*, dephasing and static magnetic field case) was investigated both in the context of the approximation methods in the open quantum systems and the capacities of the quantum channels [77]. For the detailed discussion and possible applications we refer the reader therein.

5.1.1 Total evolution

To derive the qubit's reduced dynamics, we simplify the problem to a time-independent one. First, we notice that the Hamiltonian (5.4) is of the special unitary time–dependence

type such that

$$\mathbf{H}(t,\beta) = e^{i\mathbf{K}t}\mathbf{H}(\beta)e^{-i\mathbf{K}t}, \quad \text{with} \quad \mathbf{H}(\beta) \equiv \mathbf{H}(0,\beta), \quad \mathbf{K} = -\frac{\omega}{2}\sigma_z \otimes \mathbb{I}_{\mathrm{E}}. \tag{5.7}$$

This observation allows a simple form of the evolution operator to be derived, namely

$$\mathbf{U}_t = e^{i\mathbf{K}t}e^{-i\mathbf{H}(\gamma)t}, \quad \text{where} \quad \gamma := \beta - \frac{\omega}{2}, \tag{5.8}$$

is an effective parameter. Indeed, the above operator satisfies Schrödinger type evolution equation,

$$\begin{aligned} i\dot{\mathbf{U}}_t &= -\mathbf{K}\mathbf{U}_t + e^{i\mathbf{K}t}\mathbf{H}(\gamma)e^{-i\mathbf{H}(\gamma)t} \\ &= -\mathbf{K}\mathbf{U}_t + e^{i\mathbf{K}t}\mathbf{H}(\beta)e^{-i\mathbf{K}t}\mathbf{U}_t + \mathbf{K}\mathbf{U}_t = \mathbf{H}(t,\beta)\mathbf{U}_t, \end{aligned} \tag{5.9}$$

where the equality $\mathbf{H}(\gamma) = \mathbf{H}(\beta) + \mathbf{K}$ has been used.

All the relevant information regarding the dynamics are encoded in $\mathbf{H}(\gamma)$. In order to extract these information we rewrite $\mathbf{H}(\gamma)$ as

$$\mathbf{H}(\gamma) = \begin{bmatrix} \mathrm{H}_-(\gamma) & \Delta \\ \Delta & \mathrm{H}_-(\gamma) \end{bmatrix}, \quad \text{with} \quad \mathrm{H}_\pm(\gamma) := \sum_{n=1}^{N} \left(\omega_n^{\pm}\sigma_n^z \pm \frac{\gamma}{N}\right), \tag{5.10}$$

where $\omega_n^{\pm} := \omega_n \pm g_n$. Henceforward, the explicit dependence of γ will be omitted. The evolution operator (5.8) can also be written as

$$\mathbf{U}_t = e^{i\mathbf{K}t}\mathbf{S}e^{-i\mathbf{H}_{\mathrm{d}}}\mathbf{S}^{-1}, \tag{5.11}$$

where $\mathbf{S}$ and $\mathbf{H}_{\mathrm{d}} = \mathrm{diag}[\mathrm{K}_+, \mathrm{K}_-]$ were defined in (3.30). The Riccati equation, a solution on which both these operators depend, takes the familiar form

$$\Delta \mathrm{X}^2 + \mathrm{X}\mathrm{H}_+ - \mathrm{H}_-\mathrm{X} - \Delta = 0. \tag{5.12}$$

If $\Delta = 0$, the operator (possibly not the only one) that satisfies this equation is simply $\mathrm{X}_0 = 0$. This reflects the fact that $\mathbf{H}$ is already diagonal in this case.

In order to obtain the solution for $\Delta \neq 0$ more subtle investigation is needed. Due to the fact that $[\mathrm{H}_-, \mathrm{H}_+] = 0$, it seems reasonable try to solve this equation by putting

$$\mathrm{X} = \sum_{\lambda} \xi_\lambda |\lambda\rangle\langle\lambda|, \tag{5.13}$$

where $|\lambda\rangle$ are the common eigenvectors of $\mathrm{H}_{\pm}$. Let's find them.

Clearly, if $i = i_1 i_2 ... i_N$ ($i_n = 0, 1$ and $n \leq N$) represents a binary expansion of an integer number $i \in [0, 2^N - 1]$ then for $|i\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle$, by virtue of $\sigma^z |i_n\rangle = (-1)^{i_n} |i_n\rangle$, we have

$$\mathrm{H}_{\pm} |i\rangle = \sum_{n=1}^{N} \left(\omega_n^{\pm} (-1)^{i_n} \pm \bar{\beta}\right) |i\rangle \equiv E_i^{\pm} |i\rangle, \quad 0 \leq i \leq 2^N - 1. \tag{5.14}$$

Substituting (5.13) into (5.12) and taking into account (5.14) one finds

$$\sum_i \left(\Delta \xi_i^2 + 2 E_i \xi_i - \Delta\right) |i\rangle\langle i| = 0, \quad \text{where} \quad E_i = \frac{E_i^+ - E_i^-}{2}, \tag{5.15}$$

and therefore

$$\xi_i = \frac{\epsilon \sqrt{E_i^2 + \Delta^2} - E_i}{\Delta}, \quad \epsilon = \pm 1. \tag{5.16}$$

At this point some remarks should be made. The choice between the negative ($\epsilon = -1$) and positive ($\epsilon = 1$) solution is entirely arbitrary in principle. However, it may have some influence on the analysis in certain cases. For instance, if we would decide to go about the negative solution then one may meet serious difficulties considering the limit when $\Delta \to 0$ since $\xi_i \to -\infty$ in this case. On the other hand, the positive defined solution guarantees $\mathrm{X} \to \mathrm{X}_0$ as Δ goes to 0. Indeed, as long as $E_i \neq 0$, we have

$$\xi_i = \frac{1}{\sqrt{(\frac{E_i}{\Delta})^2 + 1} + \frac{E_i}{\Delta}} \to 0 \quad \text{as} \quad \Delta \to 0. \tag{5.17}$$

From a physical point of view, this result means that we can control energy exchange between the systems in a continuous way. If $\Delta = 0$ then the energy transfer is absent. This scenario can be accomplished by taking the limit $\Delta \to 0$ of the final solution we are about to derive.

Combining all the results together, we obtain the similarity matrix,

$$\mathrm{S} = \sum_{i=0}^{2^N - 1} \mathrm{U}_i \otimes |i\rangle\langle i|, \quad \text{where} \quad \mathrm{U}_i = \begin{pmatrix} 1 & -\xi_i \\ \xi_i & 1 \end{pmatrix}, \tag{5.18}$$

the operators on the diagonal,

$$\mathrm{K}_{\pm} = \sum_i k_i^{\pm} |i\rangle\langle i| \pm \gamma, \quad \text{with} \quad k_i^{\pm} = E_i^{\pm} \pm \Delta \xi_i, \tag{5.19}$$

and finally the evolution operator of the composite system,

$$\mathrm{U}_t = \sum_i \begin{pmatrix} e^{-ik_i^+ t} + \xi_i^2 e^{-ik_i^- t} & e^{-i\omega t}\xi_i(e^{-ik_i^+ t} - e^{-ik_i^- t}) \\ e^{i\omega t}\xi_i(e^{-ik_i^+ t} - e^{-ik_i^- t}) & \xi_i^2 e^{-ik_i^+ t} + e^{-ik_i^+ t} \end{pmatrix} \otimes |i\rangle\langle i|. \tag{5.20}$$

5.1.2 Reduced dynamics

Let ρ_{Q} and ρ_{E} be arbitrary density operators of the qubit and its environment, respectively. Assuming the initial state ρ_{QE} of the composite system is factorable, $\rho_{\mathrm{QE}} = \rho_{\mathrm{Q}} \otimes \rho_{\mathrm{E}}$, we obtain

$$\rho_{\mathrm{QE}}(t) = \sum_{i,j} p_{ij} \mathrm{U}_i(t) \rho_{\mathrm{Q}} \mathrm{U}_j(t)^\dagger \otimes |i\rangle\langle j|, \quad \text{with} \quad p_{ij} = \langle i|\rho_E|j\rangle, \tag{5.21}$$

where $\mathrm{U}_i(t)$ is the 2×2 matrix from (5.20). Tracing out this relation over the environment degrees of freedom and introducing matrices $\mathrm{K}_{ij}(t) := \delta_{ij}\sqrt{p_i}\mathrm{U}_i(t)$, we find the Kraus representation of the reduced dynamics,

$$\Phi_t(\rho_{\mathrm{Q}}) = \sum_{ij} \mathrm{K}_{ij}(t) \rho_{\mathrm{Q}} \mathrm{K}_{ij}(t)^\dagger \quad \text{and} \quad \sum_{ij} \mathrm{K}_{ij}(t) \mathrm{K}_{ij}(t)^\dagger = \mathbb{I}_{\mathrm{Q}}. \tag{5.22}$$

This result is general in the sense that it holds for the arbitrary states ρ_{Q} and ρ_{E}. Although the separability of the initial state has been taken into account, the evolution operator (5.20) can be applied to *any* given initial state. Of course, if initial correlations are present then (5.21) and thus (5.22) no longer holds, but the reduced dynamics $\rho_{\mathrm{Q}}(t)$ can still easily be obtained.

It is worth mentioning that the model with similar properties was studied in [78], where the Authors have obtained the operator sum representation. However, the Kraus operators K_{ij} provided therein involve the time chronological operator. Needles to say, such expressions are much less manageable than the ones derived herein.

Anyone with an interest in looking into some common cases that may arise during the examination of the model (5.4) should not hesitate to read [79].

5.1.3 Dephasing and dephasing states

If Δ equals to zero the Hamiltonian $\mathbf{H}(t, \beta)$ becomes time-independent. In addition to that, $[\mathrm{H}_{\mathrm{Q}} \otimes \mathbb{I}_{\mathrm{E}}, \mathbf{H}_{\mathrm{int}}] = 0$. Therefore, the qubit does not lose its energy. It poses no problems to check that the reduced dynamics $\rho_{\mathrm{Q}}(t) = (\rho_{ij}(t))$ reads

$$\rho_{11}(t) = \rho_{11}, \quad \rho_{12}(t) = \left(\sum_i \rho_i e^{-i2E_i t}\right) \rho_{12}, \tag{5.23}$$

in this case. Naturally, $\rho_{22}(t) = 1 - \rho_{11}(t)$ and $\rho_{21}(t) = \rho_{21}(t)^*$. Note that the coherence $C(t) := |\rho_{12}(t)|$ does not decay exponentially (or anyhow for finite N) in the long time regime, but manifests oscillating behaviour.

One the other hand, when $\Delta \neq 0$, the energy transfer between the qubit and its environment is present. The question is: Can we design the dephasing dynamics in this case? The technique discussed in Chapter 4 does not cover time dependent cases, yet due to the special character of the evolution (5.8) the answer is: Yes, we can. Recall, the operators (5.19) are precisely the ones which generate the dephasing dynamics. Interestingly, $[\mathrm{K}_\pm, \mathrm{X}] = 0$ in this case. If the system would be time–independent, the latter equality would imply that dephasing states are stationary. However, due to the time dependence, according to (5.8) and (4.24), we have

$$\rho_{11}(t) = \rho_{11}, \quad \rho_{12}(t) = e^{-i\omega t} \sum_i \xi_i |\langle\psi|i\rangle|^2, \tag{5.24}$$

instead. $|\psi\rangle$ is a given state of the environment. It encodes an initial configuration of the external spins. The states given by (5.24) are stationary when the magnetic field is static ($\omega = 0$).

5.2 Adiabatic approximation

For *closed* quantum systems, the adiabatic theorem [80], in its basic variant, states that if the Hamiltonian of a given system, $\mathrm{H}(t)$, varies slowly [81] and if the system is initially in one of the eigenstates of $\mathrm{H}(0)$, say $|\psi_n(0)\rangle$, then the probability of finding it in the eigenstate $|\psi_n(t)\rangle$ is near to 1, at any given moment of time t.

One can introduce a measure of the adiabatic approximation,

$$F(t) = \mathrm{Tr}(\rho(t)\rho_\psi(t)), \tag{5.25}$$

where $\rho_\psi(t) = |\psi_t\rangle\langle\psi_t|$ and $\rho(t) = \mathrm{U}_t|\psi_0\rangle\langle\psi_0|\mathrm{U}_t^\dagger$. Here, $|\psi_t\rangle$ is the eigenvector of $\mathrm{H}(t)$ and U_t is the (unitary) evolution generated by $\mathrm{H}(t)$. If the system evolves adiabatically then $\mathrm{U}_t|\psi_0\rangle \simeq |\psi_t\rangle$ and thus $F(t) \simeq 1$. One could say that $F(t)$ is a measure (although it is not a metric) of the 'closeness' of the two states $\mathrm{U}_t|\psi_0\rangle$ and $|\psi_t\rangle$. In fact, $F(t)$ coincides with the standard definition of the quantum fidelity $F(\rho(t), \rho_\psi(t))$ when one of the states $\rho(t)$, $\rho_\psi(t)$ is *pure* [82].

This notion can easily be extended to open systems if one replaces the unitary evolving state $\rho(t)$ with $\rho(t) = \Phi_t(|\psi_0\rangle\langle\psi_0|)$, where Φ_t is a quantum map which reflects the dissipative character of the evolution. In what follows, will use this observation to compare the behaviour of the two models (5.1) and (5.4) in the adiabatic regime.

5.2.1 Without the environment

For the sake of simplicity, we put $\beta = \beta_0 \cos\theta$ and $\Delta = \beta_0 \sin\theta$ for certain θ and β_0. This assumption leads to the following form of the Hamiltonian (5.3)

$$\mathrm{H_Q}(t) = \beta_0 \begin{pmatrix} \cos\theta & e^{-i\omega t}\sin\theta \\ e^{i\omega t}\sin\theta & -\cos\theta \end{pmatrix}, \tag{5.26}$$

which formally has the same structure as the spin operator (4.62) examined in Sec. 4.4.3. Thus, the eigenvalues of this matrix are $E_\pm = \pm\beta_0$ and the corresponding eigenvectors $|\psi_t^\pm\rangle$ read

$$|\psi_t^+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\omega t}\sin(\theta/2) \end{pmatrix}, \quad |\psi_t^-\rangle = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\omega t}\cos(\theta/2) \end{pmatrix}. \tag{5.27}$$

Let's assume

$$\rho_\psi(t) = |\psi_t^+\rangle\langle\psi_t^+| = \begin{pmatrix} \cos^2(\theta/2) & \frac{1}{2}e^{-i\omega t}\sin\theta \\ \frac{1}{2}e^{i\omega t}\sin\theta & \sin^2(\theta/2) \end{pmatrix}. \tag{5.28}$$

For the model (5.26), the condition that guarantees the adiabatic evolution is known to be $\beta_0 \gg \omega$. This is very intuitive, since it means that the magnetic field rotates slowly in comparison with the phase of the state vector [83]. We can also confirm this by means of the adiabatic measure (5.25) which, in terms of the *adiabatic parameter* $x := \omega/2\beta_0$, reads

$$F(t) = 1 - \frac{x^2}{1+x^2}\sin^2(\Omega(x)t), \quad \text{where} \quad \Omega(x) = \beta_0\sqrt{1+x^2}. \tag{5.29}$$

Without loss of generality, we have set $\theta = \pi/2$ to obtain the above equation. The parameter x measures how slowly the magnetic field rotates in β_0^{-1} unit. In the adiabatic limit x is close to zero and the second term in (5.29) can be neglected. As a result, $F(t) \simeq 1$.

5.2.2 With the environment

For the sake of simplicity and without essential loss of generality, we assume that $\omega_n = \Omega$ and $g_n = g$ for $n \leq N$. In this case, $E_i = g(N-2k) + \beta$ where k is the Hamming weight of

the integer number i (a number of nonzero element in a binary expansion of i). Since there are $\binom{N}{k}$ integer number $i \in [0, 2^N - 1]$ with the same Hamming weight k the dynamical map (5.22) takes the form ($\mathrm{K}_i \equiv \mathrm{K}_{ii}$)

$$\Phi_t(\rho) = \sum_{i=0}^{N} \binom{N}{i} \mathrm{K}_i(t) \rho \mathrm{K}_i(t)^\dagger. \tag{5.30}$$

By the analogy of the previous case, one can write

$$F(t) == \sum_{k=0}^{N} \binom{N}{k} p_k F_k(t), \quad \text{with} \quad F_k(t) = 1 - \frac{x_k^2}{1 + x_k^2} \sin^2(\Omega(x_k)t), \tag{5.31}$$

where $x_k = G(N - 2k)/\beta_0 - x$. We have put $\theta = \pi/2$, as before. It is customary to write $F(t)$ as

$$F(t) = 1 - R(t), \quad \text{with} \quad R(t) = \sum_{k=0}^{N} \binom{N}{k} \frac{x_k^2}{1 + x_k^2} \rho_k \sin^2(\Omega(x_k)t). \tag{5.32}$$

In the adiabatic domain ($x \to 0$), we have $R(t) \neq 0$ and thus $F(t) < 1$. Therefore, the standard condition that leads to the adiabatic behaviour of the closed system (5.1) does *not* guarantee adiabatic evolution of the system (5.4). However, if one additionally assumes that coupling with the environment is weak, in comparison with the energy split between the states $|0\rangle$ and $|1\rangle$ ($G/\beta_0 \ll 1$), then $x_k \ll 1$ (for finite N) and $R(t) \simeq 0$, thus $F(t) \simeq 1$.

Chapter 6

Summary

In this work, we have presented some new ideas on how to approach problems concerning two–level open quantum systems. The technique is based on the concept of operator matrices along with their relation to the operator Riccati equation. It was argued that essentially most of the relevant questions regarding such systems can be answered when the solution to this equation is known. We have also demonstrated how the well–known mathematical tools closely related to the solution (*e.g.* its graph) can give rise to new physical concepts (*e.g.* dephasing states). Moreover, we have also pointed out that for a given model there is a correspondence between solutions to the Riccati equation associated with the Hamiltonian and symmetries in the system.

In particular, we have introduced a novel symmetry in the Rabi model and constructed its generator ($\mathbf{J}$) by using a solution to the Riccati equation associated with the Rabi matrix (block operator form of the Rabi Hamiltonian). Although this symmetry is nonlocal, unlike *e.g.* $\mathbf{J}_0 = \sigma_z \otimes e^{i\pi a^\dagger a}$, it is a self–adjoint involution and therefore can be considered as a generalised parity of the Rabi model. Invoking physical nomenclature, the Rabi model is invariant with respect to this parity or it has an *unbroken* $\mathbb{Z}_2$ symmetry. One should mention that the latter terminology is often used in a different (local) context when it is stated that $\beta \neq 0$ corresponds to the broken $\mathbb{Z}_2$ symmetry as in this case $[\mathbf{H}, \mathbf{J_0}] \neq \mathbf{0}$. Our aim was to generalise the notion of local parity constructed for total system (combined by parity operators of the individual subsystems, σ_x and $e^{i\pi a^\dagger a}$) to the nonlocal one for $\beta \neq 0$. In essence, this result, generalised in Chapter 4, proves that the presence of the extra term $\beta\sigma_z$ does not break $\mathbb{Z}_2$ symmetry but rather hides it.

At this point one should mention that usually the existence of a discrete symmetry in a quantum system is not enough by itself to fully understand its dynamics. Also, there is no obvious and direct guideline suggesting usefulness of symmetries given by discrete operators, especially nonlocal ones, in construction of solutions to the equations of motion

of quantum systems. However, discrete symmetries, local or not, allow the decomposition of the system Hilbert space into two subspaces with states having certain properties. One can then seek for the solution to the equation of motion in the individual subspaces, and then try to combine the results to obtain a full solution. For the Rabi model, this idea can be realised in terms of the parity chains, as we have seen in Chapter 3.

Moreover, nonlocal discrete symmetries can help in classification and grouping of known solutions. They can also be used in constructing new solutions from the ones which are already known such as Juddian solutions or quasi–exact solutions. Symmetries of the type presented here can also serve as a tool helping to verify certain conjectures concerning solutions of the Rabi model such as the celebrated Reik conjecture.

We have proposed a method of finding time evolution of two–level quantum systems guaranteeing conservation of certain quantities in the absence of related conservation laws. This is done by a proper choice of initial states which we suggest calling dephasing states. In a sense, such states lead to the qubit's reduced density matrix which is intermediate between stationary evolution and the ones with full time–dependence.

Although preparation of initial dephasing states would require highly sophisticated quantum engineering, there is at least one advantage of this procedure. It is performed only once at the beginning of the evolution–no dynamical control is needed to maintain the desired dynamics. One can hope that, together with continuous development of quantum state engineering techniques, the construction proposed in this thesis can become useful for applications.

Besides obvious applications, one can attempt to use the proposed method to design quantum evolution having the desired property (conserved quantity) instead of using quantum control (usually making use of external drivings). The other possibility is to utilise our method either as a test for precision of quantum engineering of initial preparations of qubit–enviroment systems or for examining if the choice of Hamiltonian describing the system is correct.

The time–dependent quantum system discussed herein may pose a useful prototype for describing a spin 'resonance' phenomenon, like for instance Nuclear Magnetic Resonance [84, 85]. In such a picture, the spin 1/2 particle is the open system the time evolution of which we wish to describe. The spin-bath models the influence of the other spins on the system in question. Finally, the rotating magnetic field is used to induce the resonance and to control the system.

We have derived the exact reduced dynamics for such a system and studied the adiabatic approximation in it. It was shown that the standard condition that guarantees the adiabatic evolution in the the case of the closed systems is not valid for the open system generalisation. This is not an unexpected result. It is interesting, however, that

the aforementioned condition ensures the adiabatic behaviour of the open system under consideration in the weak coupling limit.

6.1 Plans for future

In the near future, we would like to apply our results to investigate two–level 'truly open' quantum systems, *i.e.* the ones in contact with an infinite thermodynamic heat–bath at a given temperature. One can then analyse the condition under which the system may or may not preserve information encoded in its initial state. Although we cannot offer a nontrivial example with this regard at this stage, the results like the one given in Sec. (4.4.4) seem to indicate we are headed in the right direction.

Besides the problem of the exact diagonalization, there are still open questions concerning the Rabi model which need to be answered. For instance, can the generator $\mathbf{J}$ from (3.3) exist under conditions other than the one provided in (3.11)? Does this generator actually reflect the symmetry with respect to the changes $\sigma_z \to -\sigma_z$, $a \to -a$, $\beta \to -\beta$? And of course, what is the solution of (3.18) which determines the symmetry generator $\mathbf{J}$ explicitly?

The time–dependent model we have investigated may serve as a simple prototype allowing us to trace the time evolution of a single qubit in a quantum device. The magnetic field may be applied to program the device. Those are just a few potential applications of the simple theoretical time-dependent spin-spins model (5.4). To fully control such quantum systems both dynamic (suitable external drivings) and static (a proper initial preparation) need to be combined together. This is our agenda for future research.

Appendix

A1 Pauli spin matrices and their algebra

The standard matrix representation of the Pauli spin operators $\hat{\sigma}_i$ $(i = x, y, z)$ reads

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A1.1}$$

By 'standard' we mean represented in the basis that consists of the eigenvectors $|\pm\rangle$ of $\hat{\sigma}_z$. The above matrices are traceless, Hermitian and unitary. In addition to that, they obey $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}$. Depending on the context, we sometimes write σ^i instead of σ_i.

A2 Partial trace and reduced dynamics

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and A_i given observables acting on $\mathcal{H}_i$. The partial trace with respect to $\mathcal{H}_2$, denotes as $\mathrm{Tr}_{\mathcal{H}}(\cdot)$, is a linear map from $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ into $\mathcal{L}(\mathcal{H}_1)$ such that

$$\mathrm{Tr}_{\mathcal{H}_1}(\mathrm{A}_1 \otimes \mathrm{A}_2) = \mathrm{A}_1 \mathrm{Tr}(\mathrm{A}_2), \tag{A2.1}$$

where $\mathrm{Tr}(\cdot)$ stands for the trace on $\mathcal{H}_2$. The above equation provides a recipe of computing the partial trace only for local observables (*i.e.*, the ones of the form $\mathrm{A}_i \otimes \mathrm{A}_j$). However, due to the linearity of the trace and by the fact that every operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be expressed as a certain combination of $\mathrm{A}_i \otimes \mathrm{A}_j$, we have

$$\mathrm{Tr}_{\mathcal{H}_1}(\mathrm{A}) = \sum_{i,j} c_{ij} \mathrm{A}_i \mathrm{Tr}(\mathrm{A}_j). \tag{A2.2}$$

$\mathrm{Tr}_{\mathcal{H}_1}(\cdot)$ can be defined in a similar fashion.

If $\mathcal{H}_1 = \mathbb{C}^2$ and $\mathcal{H}_2 = \mathcal{H}$ ($\mathcal{H}$ being a given Hilbert space), one can define the partial

trace with respect to $\mathcal{H}$ in a more intuitive way. Namely, as a map from $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ into $M_2(\mathbb{C}^2)$,

$$\mathrm{Tr}_{\mathcal{H}}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{pmatrix} \mathrm{Tr}(A_{11}) & \mathrm{Tr}(A_{12}) \\ \mathrm{Tr}(A_{21}) & \mathrm{Tr}(A_{22}) \end{pmatrix}, \tag{A2.3}$$

from which it is clear that $\mathrm{Tr}_{\mathcal{H}}(\cdot)$ transforms operator matrices into the 'ordinary' ones. Technically speaking, all the entries A_{ij} must at least be trace class operators [86] for this definition to be true. Note, if $A_{ij} = |\psi_i\rangle\langle\phi_j|$ then after tracing the first subsystem with respect to the second one we are left with $a_{ij} = \langle\psi_j|\phi_j\rangle$.

Suppose we are interested in finding the state of a qubit $\rho_{\mathrm{Q}}(t)$ while it is interacting with its enviromnent. If the Hamiltonian of the compsed qubit–environment system reads $\mathbf{H}$, the state of the total system is known to be $\rho_{\mathrm{QE}}(t) = e^{-i\mathbf{H}t}\rho_{\mathrm{QE}}e^{i\mathbf{H}t}$. To obtain the reduced state of the qubit, the enviroment degrees of freedom need to be traced out. As a result,

$$\rho_{\mathrm{Q}}(t) = \mathrm{Tr}_{\mathrm{E}}\left(e^{-i\mathbf{H}t}\rho_{\mathrm{QE}}e^{i\mathbf{H}t}\right). \tag{A2.4}$$

In particular, when $\rho_{\mathrm{QE}} = |\Psi\rangle\langle\Psi|$ where $|\Psi\rangle = |+\rangle \otimes |\psi\rangle + |-\rangle \otimes |\phi\rangle$ we obtain a very useful formula:

$$\rho_{\mathrm{Q}}(t) = \begin{pmatrix} \langle\psi_t|\psi_t\rangle & \langle\psi_t|\phi_t\rangle \\ \langle\phi_t|\psi_t\rangle & \langle\phi_t|\phi_t\rangle \end{pmatrix}. \tag{A2.5}$$

A3 Existence and uniqueness of the solution to the operator Riccati equation

Since there is no general method allowing us to find either a weak or a strong solution to Riccati equations, the following theorem is of great importance to us. It offers a criteria of solvability.

Theorem 1. *Let $H_{\pm}$ be possibly unbounded self–adjoint operators on domains $\mathcal{D}(H_{\pm})$ in separable Hilbert space $\mathcal{H}$. Also assume $V_1 \neq 0$ and V_2 are bounded operators on $\mathcal{H}$. If the spectra $\sigma(H_{\pm})$ are disjoint, i.e.,*

$$d := dist(\sigma(H_+), \sigma(H_-)) > 0, \tag{A3.1}$$

and if one imposes the smallness assumption on operators V_1, V_2 of the form

$$\sqrt{\|V_1\|\|V_2\|} < \frac{d}{\pi}, \tag{A3.2}$$

then the Riccati equation

$$XV_1X + XH_+ - H_-X - V_2 = 0, \tag{A3.3}$$

has a unique weak solution X_0 *in the ball*

$$\left\{ X \in \mathcal{B}(\mathcal{H}) : \|X\| < \frac{d}{\pi\|V_1\|} \right\}. \tag{A3.4}$$

The weak solution satisfies the estimate

$$\|X_0\| \leq \frac{1}{\|V_2\|} \left(\frac{d}{\pi} - \sqrt{\frac{d^2}{\pi^2} - \|V_1\|\|V_2\|} \right). \tag{A3.5}$$

In particular, if

$$\|V_1\| + \|V_2\| < \frac{2d}{\pi}, \tag{A3.6}$$

then X_0 *is a strict contraction, that is,* $\|X_0\| < 1$.

An elegant and compact proof of this statement, based on the Banach fixed point theorem, can be found in [87].

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