**Title:** Funkcje prawie ortogonalnie addytywne

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# FUNKCJE PRAWIE ORTOGONALNIE ADDYTYWNE

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# Wstęp

Mając grupy $G, H$ oraz pewną *ortogonalność* $\perp \subset G^2$ będziemy mówić, że funkcja $f\colon G \to H$ jest ortogonalnie addytywna, jeżeli

$$f(x+y) = f(x) + f(y) \quad \text{dla takich } x, y \in G, \text{ że } x \perp y.$$

W czterech pracach stanowiących rozprawę, dwóch napisanych tylko przeze mnie oraz dwóch napisanych wspólnie z Tomaszem Kochankiem, zajmuję się funkcjami *prawie* ortogonalnie addytywnymi z rozumieniem słowa *prawie* na dwa różne sposoby. W pracach [16] oraz [11] badamy postać funkcji niekoniecznie ortogonalnie addytywnych, czyli takich, dla których różnica Cauchy'ego jest równa zero dla punktów prostopadłych, ale takich, że ta różnica dla punktów prostopadłych należy do pewnej podgrupy dyskretnej grupy wartości. Przy tym ortogonalnością, którą rozważamy, jest ortogonalność zdefiniowana w pracy [4], natomiast o funkcji zakładamy dodatkowo ciągłość w punkcie - w pracy [16] - lub mierzalność - w pracy [11]. Z kolei w pracy [12] rozważamy funkcje spełniające warunek addytywności dla punktów prostopadłych spoza pewnego zbioru *małego* rozumianego jako podzbioru zbioru $\perp \subset \mathbb{R}^{2n}$.

Praca [17] to przeniesienie pewnych wyników z prac [16] i [11] na przypadek pexiderowski, a więc zamiast różnicy Cauchy'ego rozważamy różnicę Pexidera.

W pracy [7] J. Brzdęk jako ortogonalność rozważa za J. Rätzem [14] taką relację $\perp \subset X^2$ na rzeczywistej przestrzeni liniowej $X$ wymiaru co najmniej 2, że spełnione są następujące warunki:

(01) $x \perp 0$ oraz $0 \perp x$ dla każdego $x \in X$.

(02) Jeżeli $x, y \in X \setminus \{0\}$ oraz $x \perp y$, to $x$ oraz $y$ są liniowo niezależne.

(03) Jeżeli $x, y \in X$ oraz $x \perp y$, to $ax \perp by$ dla dowolnych liczb rzeczywistych $a, b$.

(04') Jeżeli $P$ jest dwuwymiarową podprzestrzenią liniową przestrzeni $X$, $x \in P$ oraz $a$ jest rzeczywistą liczbą dodatnią, to istnieje takie $y \in P$, że $x \perp y$ oraz $x + y \perp ax - y$.

Dla tej ortogonalności J. Brzdęk pokazuje, że funkcja $f\colon X \to H$, określona na rzeczywistej przestrzeni liniowo-topologicznej o wartościach w przemiennej grupie topologicznej bez elementów rzędu 2, ciągła w zerze spełnia

$$f(x+y) - f(x) - f(y) \in K \quad \text{dla takich } x, y \in X, \text{ że } x \perp y,$$

gdzie $K$ jest dyskretną podgrupą grupy $H$, wtedy i tylko wtedy, gdy istnieją: ciągła funkcja addytywna $a\colon X \to H$ oraz taka ciągła w punkcie $(0,0)$ funkcja dwuaddytywna i symetryczna $b\colon X \times X \to H$, że

$$f(x) - a(x) - b(x,x) \in K \quad \text{dla } x \in X$$

oraz

$$b(x,y) = 0 \quad \text{dla takich } x, y \in X, \text{ że } x \perp y;$$

ponadto funkcje $a$ oraz $b$ są wyznaczone jednoznacznie.

Celem pracy [16] było przeniesienie powyższego wyniku na przypadek ortogonalności zdefiniowanej przez K. Barona i P. Volkmanna w [4] następująco: Niech $G$ będzie taką grupą, że odwzorowanie

$$x \mapsto 2x, x \in G,$$

jest bijekcją. Relację $\perp \subset G^2$ nazywamy ortogonalnością, jeśli spełnia ona poniższe dwa warunki:

(O) $0 \perp 0$, a jeżeli $x \perp y$, to $-x \perp -y$ oraz $\frac{x}{2} \perp \frac{y}{2}$.

(P) Jeżeli funkcja ortogonalnie addytywna określona na $G$ o wartościach w grupie przemiennej jest nieparzysta, to jest ona addytywna, zaś jeżeli jest parzysta, to jest ona kwadratowa.

Powyższa definicja ortogonalności obejmuje pojęcie przytoczonej wcześniej ortogonalności Rätza, a udowodnione w [16] twierdzenie jest uogólnieniem zacytowanego powyżej twierdzenia J. Brzdęka. W szczególności ciągłość w zerze rozważanej funkcji jest tam zastąpiona ciągłością w jakimś punkcie, a w tezie otrzymujemy ciągłość funkcji $b$ w każdym punkcie. Implikuje ono także następujący rezultat K. Barona oraz P. Volkmanna z pracy [4]: Załóżmy, że $G$ jest grupą przemienną z jednoznacznym dzieleniem przez 2, $H$ grupą przemienną, a $\perp \subset G^2$ relacją spełniającą warunki (O) i (P). Funkcja $f\colon G \to H$ jest ortogonalnie addytywna wtedy i tylko wtedy, gdy

$$f(x) = a(x) + b(x,x) \quad \text{dla } x \in G,$$

gdzie $a\colon G \to H$ jest funkcją addytywną, natomiast $b\colon G \times G \to H$ jest funkcją dwuaddytywną i symetryczną oraz

$$b(x,y) = 0 \quad \text{dla takich } x, y \in G, \text{ że } x \perp y;$$

ponadto, funkcje $a$ oraz $b$ są wyznaczone jednoznacznie.

W pracy [11] ciągłość w punkcie zastępujemy mierzalnością. Otrzymujemy podobne wyniki, ale pod pewnymi dodatkowymi założeniami, które można nieco osłabić jeśli nie żądamy ciągłości funkcji dwuaddytywnej z tezy, a tylko jej ciągłość względem każdej ze zmiennych. O rozważanym $\sigma$-ciele $\mathfrak{M}$ podzbiorów przemiennej grupy topologicznej $G$ zakładamy, że

$$x \pm 2A \in \mathfrak{M} \quad \text{dla } x \in G, A \in \mathfrak{M}$$

oraz istnienie właściwego $\sigma$-ideału $\mathfrak{J}$ podzbiorów grupy $G$, dla którego zachodziłoby twierdzenie Steinhausa:

$$0 \in \operatorname{Int}(A - A) \quad \text{dla } A \in \mathfrak{M} \setminus \mathfrak{J}.$$

Główne rezultaty [11] to twierdzenia 1 i 2, z których wyciągamy wnioski dla szczególnych przypadków: mierzalności w sensie Baire'a i Christensena. Rozwiązania mierzalne w sensie Baire'a oraz Christensena były rozważane wcześniej przez J. Brzdęka w pracy [6] dla ortogonalności wyznaczonej przez iloczyn skalarny oraz w pracy [8] dla ortogonalności Rätza w przestrzeni liniowo-topologicznej.

Celem pracy [17] było przeniesienie rezultatów z prac [16] (twierdzenie 1) oraz [11] (twierdzenie 1) na sytuację, gdy zamiast różnicy Cauchy'ego rozważamy różnicę Pexidera oraz zakładamy ciągłość w punkcie lub mierzalność choć jednej z występujących w niej trzech funkcji (ortogonalność pozostaje ta sama co w [16] i [11]). W celu wykazania głównego twierdzenia dowodzimy najpierw lemat pozwalający na przedstawienie dowolnej spośród trzech funkcji z założenia twierdzenia jako przesunięcia funkcji, dla której już różnica Cauchy'ego (a nie Pexidera) spełnia odpowiednie założenia i można zastosować udowodnione wcześniej twierdzenia: 1 z [16] oraz 1 z [11]. Jedna z części tego lematu została już wcześniej udowodniona przez K. Barona i PL. Kannappana w pracy [2], a dla podgrupy trywialnej, ale pod słabszymi pozostałymi założeniami, także w pracy [15] J. Sikorskiej.

Twierdzenie z pracy [17] jako bardzo szczególne przypadki zawiera też niektóre wyniki pracy [2] K. Barona i PL. Kannappana.

Niech $E$ będzie rzeczywistą przestrzenią unitarną wymiaru co najmniej 2, $H$ grupą przemienną, a $\perp$ zbiorem tych par wektorów przestrzeni $E$, dla których iloczyn skalarny się zeruje. R. Ger, Gy. Szabó, J. Rätz (por. wniosek 10 z pracy [14]) oraz K. Baron i J. Rätz [3] wykazali, że każda funkcja ortogonalnie addytywna $f\colon E \to H$ ma postać

$$f(x) = a(\|x\|^2) + b(x) \quad \text{dla } x \in E,$$

gdzie $a\colon \mathbb{R} \to H$ oraz $b\colon E \to H$ są funkcjami addytywnymi. N.G. de Bruijn w [5], W.B. Jurkat w [10] oraz R. Ger w [9] rozważali z kolei równanie Cauchy'ego spełnione prawie wszędzie, tj. poza pewnym zbiorem *małym* (dla funkcji określonej na grupie). W pracy [12] zajęliśmy się funkcjami ortogonalnie addytywnymi prawie wszędzie w $\perp$.

Zbiory *małe* są zwykle rozumiane jako elementy pewnego właściwego (liniowo-niezmienniczego) ideału, a każdy taki ideał podzbiorów pewnej przestrzeni $X$ generuje odpowiedni ideał podzbiorów przestrzeni $X^2$ poprzez twierdzenie Fubiniego (patrz [13], część 17.5). Chcemy jednak, aby zbiory te były małe w $\perp$, a nie tylko w $E^2$, zatem $\perp$ powinien być takim zbiorem, by te własności uwzględniać. Z tego powodu ograniczamy się do przestrzeni euklidesowej $\mathbb{R}^n$; wówczas bowiem $\perp$ jest $(2n-1)$-wymiarową rozmaitością w $\mathbb{R}^{2n}$.

Dla każdego $m \in \mathbb{N}$ niech $\mathfrak{I}_m$ oznacza taki właściwy $\sigma$-ideał podzbiorów przestrzeni $\mathbb{R}^m$, że spełnione są następujące cztery warunki:

($\mathrm{H}_0$) $\{0\} \in \mathfrak{I}_1$;

($\mathrm{H}_1$) jeżeli $\varphi$ jest $C^\infty$-dyfeomorfizmem określonym na zbiorze otwartym $U \subset \mathbb{R}^m$ oraz $A \in \mathfrak{I}_m$, to $\varphi(A \cap U) \in \mathfrak{I}_m$;

($\mathrm{H}_2$) jeżeli $m, n \in \mathbb{N}$ oraz $A \in \mathfrak{I}_{m+n}$, to $\{x \in \mathbb{R}^m : A[x] \notin \mathfrak{I}_n\} \in \mathfrak{I}_m$;

($\mathrm{H}_3$) jeżeli $m, n \in \mathbb{N}$ oraz $A \in \mathfrak{I}_n$, to $\mathbb{R}^m \times A \in \mathfrak{I}_{m+n}$.

Rodzina zbiorów miary Lebesgue'a zero oraz rodzina zbiorów pierwszej kategorii Baire'a spełniają powyższe założenia. Niepuste podzbiory otwarte przestrzeni $\mathbb{R}^m$ nie należą do $\mathfrak{I}_m$.

Dla $m$-rozmaitości $M \subset \mathbb{R}^n$ ($m \leqslant n$) wyposażonej w atlas $\mathcal{A}$, $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$, definiujemy właściwy $\sigma$-ideał $\mathfrak{I}_M \subset 2^M$ przyjmując

$$\mathfrak{I}_M = \{A \subset M : \varphi_i(A \cap U_i) \in \mathfrak{I}_m \text{ dla każdego } i \in I\}.$$

Definicja ta nie zależy od wyboru atlasu $\mathcal{A}$. Zbiory *małe* definiujemy następująco: jeżeli $n \geqslant 2$, a $\langle\cdot|\cdot\rangle$ jest (dowolnym) iloczynem skalarnym w $\mathbb{R}^n$, to mówimy, że zbiór $Z \subset \perp$ jest *mały* w $\perp$ wtedy i tylko wtedy, gdy $Z \in \mathfrak{I}_{\perp^*}$, gdzie $\perp^* := \perp \setminus \{0\}$ ($\perp^*$ jest $(2n-1)$-rozmaitością).

Głównym wynikiem pracy [12] jest twierdzenie mówiące, że jeżeli funkcja $f$ odwzorowuje $\mathbb{R}^n$ w grupę przemienną $H$ oraz

$$f(x+y) = f(x) + f(y) \quad \mathfrak{I}_\perp \text{ - p.w.},$$

to istnieje dokładnie jedna taka funkcja ortogonalnie addytywna $g\colon \mathbb{R}^n \to H$, że

$$f(x) = g(x) \quad \mathfrak{I}_n \text{ - p.w.}$$

Jednym z lematów dowodzonych w celu wykazania prawdziwości powyższego twierdzenia jest lemat mówiący, że jeżeli $A \in \mathfrak{I}_{S^{n-1}}$, gdzie $S^{n-1}$ jest sferą jednostkową w przestrzeni $\mathbb{R}^n$, to istnieje baza ortogonalna przestrzeni $\mathbb{R}^n$ złożona z elementów sfery $S^{n-1}$ nie należących do zbioru $A$.

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Aequationes Mathematicae

# Orthogonally additive functions modulo a discrete subgroup

Wirginia Wyrobek

**Summary.** Under appropriate conditions on the abelian groups $G$ and $H$ and the orthogonality $\perp \subset G^2$ we prove that a function $f : G \to H$ continuous at a point is orthogonally additive modulo a discrete subgroup $K$ if and only if there exist a unique continuous additive function $a : G \to H$ and a unique continuous biadditive and symmetric function $b : G \times G \to H$ such that $f(x) - b(x,x) - a(x) \in K$ for $x \in G$ and $b(x,y) = 0$ for $x, y \in G$ such that $x \perp y$.

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In this paper we work with the following orthogonality proposed by K. Baron and P. Volkmann in [4]:

Let $G$ be a group such that the mapping

$$x \mapsto 2x, \quad x \in G, \tag{1}$$

is a bijection onto the group $G$. A relation $\perp \subset G^2$ is called *orthogonality* if it satisfies the following two conditions:

(O) $0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

(P) If an orthogonally additive function from $G$ to an abelian group is odd, then it is additive; if it is even, then it is quadratic.

According to Theorems 5 and 6 from [7] the orthogonality considered by J. Rätz in [7] satisfies both (O) and (P).

Throughout this paper for a subset $U$ of a given group and for $n \in \mathbb{N}$ the symbol $nU$ denotes the set $\{nx : x \in U\}$.

Our main result reads as follows:

**Theorem 1.** *Assume that $G$ is an abelian topological group such that the mapping (1) is a homeomorphism and the following condition holds:*

(H) *every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that*

$$U \subset 2U \tag{2}$$

*and*

$$G = \bigcup\{2^n U : \ n \in \mathbb{N}\}. \tag{3}$$

*Assume $\perp \subset G^2$ is an orthogonality, $H$ is an abelian topological group and $K$ is a discrete subgroup of $H$. Then a function $f : G \to H$ continuous at a point satisfies*

$$f(x+y) - f(x) - f(y) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y \tag{4}$$

*if and only if there exist a continuous additive function $a : G \to H$ and a continuous biadditive and symmetric function $b : G \times G \to H$ such that*

$$f(x) - b(x,x) - a(x) \in K \quad \text{for } x \in G \tag{5}$$

*and*

$$b(x,y) = 0 \quad \text{for } x, y \in G \text{ such that } x \perp y. \tag{6}$$

*Moreover, the functions $a$ and $b$ are uniquely determined.*

Note that this theorem generalizes Theorem 2.9 from [6] and, in view of Theorem 9 from [7] and Theorem 4.2 from [3], also implies the result obtained in [1].

The proof of Theorem 1 will be presented after some lemmas. The first three lemmas and Lemma 4(i) are very similar to some results from [2], [6] and [5], but for the reader's convenience we formulate them explicitly; however, we omit their proofs. Note that Lemma 1(ii) [6, Lemma 2.3] is applied in the proof of Lemma 2 [6, Proposition 2.4], Lemma 1(i) [2, Lemma 1] and Lemma 2 in the proof of Lemma 3 [2, Theorem 3; 6, Theorem 2.6] and Lemma 3 in the proof of Lemma 4. Our Lemma 4(ii) can be proved in the same way as Lemma 4(i) [5, Lemma 4], so we also omit the proof.

**Lemma 1.** *Assume that $G$ is an abelian group such that (1) is a bijection onto $G$, $H$ is an abelian group and $U \subset G$ is a set with properties (2) and (3).*

(i) *If $f : U \to H$ satisfies*

$$f(x+y) = f(x) + f(y) \quad \text{for } x, y \in U \text{ with } x + y \in U,$$

*then it has a unique extension to an additive mapping of $G$ into $H$.*

(ii) *If $f : U \to H$ satisfies*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{for } x, y \in U \text{ with } x+y, x-y \in U$$

*and $f(0) = 0$, then it has a unique extension to a quadratic mapping of $G$ into $H$.*

**Lemma 2.** *Assume that $G$ is an abelian group such that* (1) *is a bijection onto $G$, $H$ is an abelian group, $K$ is a subgroup of $H$, $U \subset G$ is a set with properties* (2) *and* (3) *and $W$ is a subset of $H$ such that*

$$0 \in W, \quad W = -W \quad \text{and} \quad (W + W + W + W + W + W) \cap K = \{0\}.$$

*If $f : G \to H$ satisfies*

$$f(U) - f(0) \subset K + W$$

*and*

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) \in K \quad \text{for } x, y \in G, \tag{7}$$

*then $2f(0) \in K$ and there exists a quadratic function $q : G \to H$ such that*

$$f(x) - q(x) - f(0) \in K \quad \text{for } x \in G, \tag{8}$$

*$q(0) = 0$ and $q(U) \subset W$.*

**Lemma 3.** *Assume that $G$ is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds, $H$ is an abelian topological group and $K$ is a discrete subgroup of $H$.*

(i) *If $f : G \to H$ is continuous at zero and*

$$f(x + y) - f(x) - f(y) \in K \quad \text{for } x, y \in G,$$

*then there exists a continuous additive function $a : G \to H$ such that*

$$f(x) - a(x) \in K \quad \text{for } x \in G.$$

(ii) *If a function $f : G \to H$ continuous at zero satisfies* (7), *then there exists a unique quadratic function $q : G \to H$ continuous at zero such that $q(0) = 0$ and* (8) *holds.*

In the rest of this paper we consider for an abelian topological group $H$ and a subgroup $K$ of $H$, the quotient group $H/K$ with the quotient topology:

$$\{W \subset H/K : \ p^{-1}(W) \text{ is an open subset of } H\},$$

where $p : H \to H/K$ is the canonical mapping: $p(x) = x + K$.

**Lemma 4.** *Assume that $G$ is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds, $H$ is an abelian topological group and $K$ is a discrete subgroup of $H$.*

(i) *If $A : G \to H/K$ is a continuous additive function, then there exists a continuous additive function $a : G \to H$ such that*

$$a(x) \in A(x) \quad \text{for } x \in G.$$

(ii) *If $Q : G \to H/K$ is a function which is continuous at zero and $Q(0) = K$, then there exists a continuous at zero quadratic function $q : G \to H$ such that $q(0) = 0$ and*

$$q(x) \in Q(x) \quad \text{for } x \in G.$$

The proof of the next lemma was kindly communicated to me by K. Baron.

**Lemma 5.** *Assume that $G$ is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds and $H$ is an abelian topological group. If a function $b : G \times G \to H$ is biadditive and continuous at $(0,0)$, then it is continuous.*

*Proof.* First we prove that $b(x,\cdot)$ is continuous at zero for every $x \in G$. Take $x_0 \in G$ and a neighbourhood $W \subset H$ of zero. It follows from the continuity at zero of $b$ and from (H) that there exists a neighbourhood $U \subset G$ of zero such that (3) and

$$b(U \times U) \subset W$$

hold. Consequently $x_0 = 2^n u_0$ with an $n \in \mathbb{N}$ and a $u_0 \in U$, and for $u \in U$ we have

$$b(x_0, 2^{-n}u) = b(2^n u_0, 2^{-n}u) = 2^n b(u_0, 2^{-n}u) = b(u_0, u) \in W.$$

Hence

$$b(x_0, 2^{-n}U) \subset W,$$

which shows that $b(x_0,\cdot)$ is continuous at zero. Clearly, the same concerns $b(\cdot, y_0)$ for every $y_0 \in G$. To finish the proof it is enough to observe now that

$$b(x,y) - b(x_0,y_0) = b(x - x_0, y_0) + b(x - x_0, y - y_0) + b(x_0, y - y_0)$$

holds for $x, y, x_0, y_0 \in G$. □

Our last lemma generalizes Theorem 4.3 from [3].

**Lemma 6.** *Assume that $G$ is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds, $\perp \subset G^2$ is an orthogonality and $H$ is an abelian topological group. If an orthogonally additive function $f : G \to H$ is continuous at some point, then it is continuous; more precisely, it is of the form*

$$f(x) = a(x) + b(x,x) \quad \text{for } x \in G, \tag{9}$$

*where $a : G \to H$ is a continuous additive function, $b : G \times G \to H$ is a continuous biadditive and symmetric function and* (6) *holds.*

*Proof.* According to Theorem 1 from [4] the function $f$ has form (9), where $a : G \to H$ is additive, $b : G \times G \to H$ is biadditive, symmetric and satisfies (6); moreover,

$$b(x,y) = 2\Big(f\Big(\frac{x+y}{4}\Big) + f\Big(\frac{-x-y}{4}\Big) - f\Big(\frac{x-y}{4}\Big) - f\Big(\frac{-x+y}{4}\Big)\Big) \quad \text{for } x, y \in G. \tag{10}$$

Let $x_0 \in G$ be a continuity point of $f$. It follows from (9) that

$$f(x + x_0) - f(x) - f(x_0) = 2b(x, x_0) \quad \text{for } x \in G,$$

whence continuity at zero of $f + 2b(\cdot, x_0)$ follows. Consequently also the function

$$x \mapsto f(-x) + 2b(-x, x_0), \quad x \in G,$$

is continuous at zero. Summing up those two functions we get continuity at zero of

$$x \mapsto f(x) + f(-x), \quad x \in G.$$

Since (1) is a homeomorphism, this jointly with (10) gives continuity at $(0,0)$ of $b$ and applying Lemma 5 we see that $b$ is continuous (at each point of $G \times G$). Hence and from (9) continuity of $a$ (at $x_0$ and, consequently, everywhere) follows. This ends the proof. □

*Proof of Theorem 1.* The proof of the "if" part is easy, so we omit it. The "only if" part is divided into Parts I and II.

Part I. Assume that $f$ satisfies (4) and define the function $\hat{f} : G \to H/K$ by the formula

$$\hat{f} = p \circ f.$$

Clearly $\hat{f}$ is continuous at a point, and (4) implies that $\hat{f}$ is orthogonally additive. According to Lemma 6 there exist a continuous additive function $\hat{a} : G \to H/K$ and a continuous quadratic function $\hat{q} : G \to H/K$ such that $\hat{q}(0) = K$ and

$$\hat{f}(x) = \hat{a}(x) + \hat{q}(x) \quad \text{for } x \in G.$$

By Lemma 4 we get a continuous additive function $a : G \to H$ and a quadratic function $q : G \to H$ continuous at zero such that $q(0) = 0$,

$$p \circ a = \hat{a} \quad \text{and} \quad p \circ q = \hat{q}.$$

Consequently, $f(x) - q(x) - a(x) + K = \hat{f}(x) - \hat{q}(x) - \hat{a}(x) = K$, i.e.,

$$f(x) - q(x) - a(x) \in K \quad \text{for } x \in G. \tag{11}$$

It follows from Lemma 2 from [4] that $q$ has the form

$$q(x) = b(x, x) \quad \text{for } x \in G, \tag{12}$$

where $b : G \times G \to H$ is biadditive, symmetric and continuous at $(0,0)$. Applying Lemma 5 we see that $b$ is continuous.

Part II. Now we prove that $q$ is orthogonally additive and that (6) holds.

Since $K$ is discrete, there exists a neighbourhood $W \subset H$ of zero such that

$$K \cap W = \{0\}.$$

Let $W_0 \subset H$ be a symmetric neighbourhood of zero with

$$W_0 + W_0 + W_0 \subset W$$

and $U \subset G$ be a neighbourhood of zero such that $q(U) \subset W_0$, (2) and (3) hold.

Take $x, y \in G$ with $x \perp y$ and, making use of (3) and (2), choose an $n \in \mathbb{N}$ such that

$$2^{-n}x,\ 2^{-n}y,\ 2^{-n}(x+y) \in U.$$

Then

$$q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y) \in W_0 - W_0 - W_0 \subset W.$$

On the other hand, by (11) and (4),

$$q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y) \in f(2^{-n}(x+y)) \\ -f(2^{-n}x) - f(2^{-n}y) + K = K.$$

Consequently,

$$q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y) = 0.$$

Moreover, by (12),

$$q(2^k z) = 2^{2k} q(z) \quad \text{for } z \in G \text{ and } k \in \mathbb{N}.$$

This yields

$$q(x+y) - q(x) - q(y) = 2^{2n}(q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y)) = 0$$

and, as $\frac{x}{2}$ and $\frac{y}{2}$ are also orthogonal,

$$b(x,y) = 4b\left(\frac{x}{2},\frac{y}{2}\right) = 2\left(q\left(\frac{x}{2}+\frac{y}{2}\right) - q\left(\frac{x}{2}\right) - q\left(\frac{y}{2}\right)\right) = 0.$$

Part III: Uniqueness. Suppose $a_1 : G \to H$ is additive and continuous, $b_1 : G \times G \to H$ is biadditive, symmetric and continuous, and

$$f(x) - b_1(x,x) - a_1(x) \in K \quad \text{for } x \in G. \tag{13}$$

Putting

$$a_0 = a - a_1, \quad b_0 = b - b_1,$$

we get in view of (5) and (13)

$$a_0(x) + b_0(x,x) \in K \quad \text{for } x \in G, \tag{14}$$

which jointly with additivity of $a_0$ and biadditivity of $b_0$ gives

$$a_0(2x) = (a_0(x) + b_0(x,x)) - (a_0(-x) + b_0(-x,-x)) \in K$$

for $x \in G$. Consequently, since (1) is a bijection, $a_0(G) \subset K$. Hence, taking into account that $K$ is discrete and $a_0$ is continuous and vanishes at zero, we infer that $a_0$ vanishes on a neighbourhood of zero and making use of (H) we see that $a_0$ vanishes everywhere. Thus $a_1 = a$ and (14) takes the form

$$b_0(x,x) \in K \quad \text{for } x \in G.$$

Reasoning as above we show that

$$b_0(x,x) = 0 \quad \text{for } x \in G,$$

whence

$$2b_0(x,y) = b_0(x+y,x+y) - b_0(x,x) - b_0(y,y) = 0$$

for $x, y \in G$ and, consequently,

$$b_0(x,y) = 4b_0\left(\frac{x}{2},\frac{y}{2}\right) = 0$$

for $x, y \in G$, which means that $b_1 = b$. □

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# MEASURABLE ORTHOGONALLY ADDITIVE FUNCTIONS MODULO A DISCRETE SUBGROUP*

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**Abstract.** Under appropriate conditions on Abelian topological groups $G$ and $H$, an orthogonality $\perp \subset G^2$ and a $\sigma$-algebra $\mathfrak{M}$ of subsets of $G$ we decompose an $\mathfrak{M}$-measurable function $f : G \to H$ which is orthogonally additive modulo a discrete subgroup $K$ of $H$ into its continuous additive and continuous quadratic part (modulo $K$).

## 1. Introduction

Throughout all the paper $G$ and $H$ are Abelian topological groups, $K$ is a discrete subgroup of $H$.

Following K. Baron and P. Volkmann [2], in the case when $G$ is uniquely 2-divisible, a relation $\perp \subset G^2$ is called *orthogonality* if it satisfies the following two conditions:

(O) $\quad 0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

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(P) $\begin{cases}\text{If an orthogonally additive function from } G \text{ to an Abelian group is} \\ \text{odd, then it is additive; if it is even, then it is quadratic.}\end{cases}$

For instance, the orthogonality considered by J. Rätz in [13] fulfils both (O) and (P), according to Theorems 5 and 6 therein. For further examples the reader is referred to [2].

All along we assume that $\mathfrak{M}$ is a $\sigma$-algebra and $\mathfrak{I}$ is a proper $\sigma$-ideal of subsets of $G$ which fulfil the condition:

(S) $$0 \in \operatorname{Int}(A - A), \text{ if } A \in \mathfrak{M} \setminus \mathfrak{I}.$$

We deal with the problem: under what assumptions an $\mathfrak{M}$-measurable mapping $f : G \to H$ which is orthogonally additive modulo $K$, i.e.

(1) $$f(x+y) - f(x) - f(y) \in K \quad \text{for} \quad x, y \in G \quad \text{such that} \quad x \perp y,$$

admits a factorization of the type

(2) $$f(x) - b(x,x) - a(x) \in K \quad \text{for} \quad x \in G$$

with a continuous additive $a : G \to H$ and a separately/jointly continuous biadditive $b : G \times G \to H$?

The main aim of this paper is to establish representation (2) with a *jointly* continuous biadditive function $b$. This is done in the next section under some reasonable assumptions (on $G$ or $\mathfrak{M}$). In the third section we obtain this decomposition with a separately continuous $b$ under somewhat weaker conditions.

## 2. Factorization with a jointly continuous biadditive term

The first lemma is a kind of folklore and has been established in special cases when $\mathfrak{M}$ is the $\sigma$-algebra of subsets having the Baire property or being Christensen measurable. In both cases the key property is condition (S), where $\mathfrak{I}$ is the family of meager or Christensen zero subsets of $G$, respectively (see [12, Theorem 9.9] and [8, Theorem 2] with [10]). For the proof of this lemma see e.g. [12, Theorem 9.10].

LEMMA 1. *Every $\mathfrak{M}$-measurable homomorphism from $G$ into a separable topological group is continuous.*

LEMMA 2. *Let $X$ be a topological space with a countable base. If the functions $f, g : G \to X$ are $\mathfrak{M}$-measurable, then so is the function $(f, g) : G \to X \times X$. Consequently, if $Y$ is a topological space and $\varphi : X \times X \to Y$ is a Borel function, then $\varphi(f, g)$ is $\mathfrak{M}$-measurable.*

PROOF. It is enough to observe that if $\mathcal{B}$ is a countable base of $X$, then $\{V \times W : V, W \in \mathcal{B}\}$ is a countable base of $X \times X$. $\square$

LEMMA 3. *Assume $H$ is separable metric and at least one of the conditions holds:*

(i) *$G$ is a first countable Baire group;*

(ii) *$G$ is separable metric;*

(iii) *$G$ is metric and $\mathfrak{M}$ contains all Borel subsets of $G$.*

*If a biadditive function $b : G \times G \to H$ has $\mathfrak{M}$-measurable sections $b(x, \cdot)$, $b(\cdot, y)$ for all $x, y \in G$, then $b$ is continuous.*

PROOF. If $G$ is a first countable Baire group, then [9, Proposition 2.3] implies that $(G, G, H)$ forms a Namioka–Troallic triple. Our assertion then follows from the fact that the sections of $b$ being $\mathfrak{M}$-measurable are, according to Lemma 1, continuous, and from the H. R. Ebrahimi-Vishki result [9, Theorem 3.2].

Let $d_G$, $d_H$ stand for invariant metrics for $G$, $H$, respectively (cf. [11, Theorem 8.3]), $B(r) = \{ z \in G : d_G(z, 0) \leqq r \}$ for positive $r \in \mathbb{R}$ and

$$F_{n,k} = \left\{ x \in G : d_H\big(b(x,u), b(x,v)\big) \leqq 2^{-n} \text{ for all } u, v \in B\big(2^{-k}\big) \right\}$$

for $n, k \in N$. By Lemma 1, the sections $b(\cdot, u)$ are continuous for $u \in G$, whence $F_{n,k}$ are closed for $n, k \in \mathbb{N}$. Consequently, in case (iii) we have

$$F_{n,k} \in \mathfrak{M} \quad \text{for } n, k \in \mathbb{N}. \tag{3}$$

To show that (3) holds also in case (ii) for every $k \in \mathbb{N}$ consider a countable and dense subset $D_k$ of $B(2^{-k})$. Then, due to continuity of $b(x, \cdot)$ for $x \in G$, we have

$$F_{n,k} = \bigcap_{(u,v) \in D_k} \left\{ x \in G : d_H\big(b(x,u), b(x,v)\big) \leqq 2^{-n} \right\} \quad \text{for } n, k \in \mathbb{N}.$$

Moreover, as follows from Lemma 2, the mapping $G \ni x \mapsto d_H\big(b(x,u), b(x,v)\big)$ is $\mathfrak{M}$-measurable for $u, v \in G$. Hence we have (3) also in case (ii).

Because of the continuity of $b(x, \cdot)$, we have

$$G = \bigcup_{k \in \mathbb{N}} F_{n,k} \quad \text{for } n \in \mathbb{N}.$$

Consequently, if $n \in \mathbb{N}$, then $F_{n,k(n)} \in \mathfrak{M} \setminus \mathfrak{J}$ for at least one $k(n) \in \mathbb{N}$. This fact, jointly with condition (S), yield

$$0 \in \operatorname{Int}\big(F_{n,k(n)} - F_{n,k(n)}\big). \tag{4}$$

On the other hand, if $k, n \in \mathbb{N}$, $n \geqq 2$, then for all $x, x' \in F_{n,k}$ and all $u, v \in B(2^{-k})$ we have

$$\begin{aligned} d_H\big(b(x-x',u), b(x-x',v)\big) &= d_H\big(b(x,u)-b(x',u), b(x,v)-b(x',v)\big) \\ &= d_H\big(b(x,u), b(x,v)+b(x',u-v)\big) \\ &\leqq d_H\big(b(x,u), b(x,v)\big) + d_H\big(b(x,v), b(x,v)+b(x',u-v)\big) \\ &= d_H\big(b(x,u), b(x,v)\big) + d_H\big(b(x',v), b(x',u)\big) \leqq 2^{-(n-1)}, \end{aligned}$$

which shows that $F_{n,k} - F_{n,k} \subset F_{n-1,k}$. Combining this with (4) we infer that for all $n \in \mathbb{N}$ there is $k(n) \in \mathbb{N}$ and $r(n) > 0$ such that

$$(5) \qquad d_H\big(b(x,u), b(x,v)\big) \leqq 2^{-n} \quad \text{for } x \in B\big(r(n)\big) \text{ and } u, v \in B\big(2^{-k(n)}\big).$$

Fix any $(x, u)$ and $(x', v)$ from $B\big(\frac{1}{2}r(n)\big) \times B\big(2^{-k(n)}\big)$. Then

$$x - x' \in B\big(0, r(n)\big)$$

and (5) yields

$$\begin{aligned} d_H\big(b(x,u), b(x',v)\big) &\leqq d_H\big(b(x,u), b(x,v)\big) + d_H\big(b(x,v), b(x',v)\big) \\ &\leqq 2^{-n} + d_H\big(b(x-x',v), 0\big) \\ &= 2^{-n} + d_H\big(b(x-x',v), b(x-x',0)\big) \leqq 2^{-(n-1)}. \end{aligned}$$

This proves the continuity of $b$ at $(0,0)$. Since

$$b(x,y) - b(x_0,y_0) = b(x-x_0,y_0) + b(x-x_0,y-y_0) + b(x_0,y-y_0)$$

for $x, y \in G$ and $b(\cdot, y_0)$, $b(x_0, \cdot)$ are continuous, $b$ is therefore continuous at every point $(x_0, y_0) \in G \times G$. $\square$

Note that in the special case when $\mathfrak{M}$ consists of all sets with the Baire property, the assumption that $G$ is Baire, or equivalently $G$ is non-meager (see e.g. [12, Proposition 9.8]), corresponds to our hypothesis $G \notin \mathfrak{I}$.

A key role in the above proof is played by condition (S). Even in the case when $G$ is a real separable normed space and $\mathfrak{M}$ is the $\sigma$-algebra of its Borel subsets, a suitable $\sigma$-ideal $\mathfrak{I}$ which satisfies (S) does not have to exist. Consider, for instance, the space of all real polynomials of one variable with the norm $\|f\| = \int_0^1 |f(t)|\,dt$ and the bilinear functional $B(f,g) = \int_0^1 f(t)g(t)\,dt$ which is separately but not jointly continuous. In view of our last lemma, such a space does not admit a $\sigma$-ideal $\mathfrak{I}$ which would fulfil condition (S). For the essentiality of the above assumptions cf. also Example 3.3 in [9].

LEMMA 4. *If $H$ is separable metric, then the quotient group $H/K$ is an Abelian separable metric group.*

PROOF. Since $K$ is closed in $H$, the group $H/K$ is Hausdorff (see [11, Theorem 5.21]). Because $H$ has a countable base, so has also $H/K$. In the light of the Birkhoff–Kakutani theorem [11, Theorem 8.3], $H/K$ is thus metrizable. Separability follows again from the existence of a countable base. □

Now we are prepared to proceed to our main result. The technical assumptions appearing below have been already considered (see [7], [3], [6] and [14]). In the last section we present a counterexample showing that condition (G2) is essential.

THEOREM 1. *Assume $H$ is separable metric,*

(G1) *the mapping $G \ni x \mapsto 2x$ is a homeomorphism,*

(G2) *every neighbourhood of zero in $G$ contains a zero neighbourhood $U$ such that*

$$U \subset 2U \quad \textit{and} \quad G = \bigcup \{2^n U : n \in \mathbb{N}\}, \tag{6}$$

(G3) *either $G$ is a first countable Baire group, or $G$ is metric separable, or $G$ is metric and $\mathfrak{M}$ contains all Borel subsets of $G$,*

(G4) *$x \pm 2A \in \mathfrak{M}$ for all $x \in G$ and $A \in \mathfrak{M}$.*

*Then an $\mathfrak{M}$-measurable function $f : G \to H$ satisfies (1) if and only if there exist a continuous additive function $a : G \to H$ and a continuous biadditive symmetric function $b : G \times G \to H$ such that the factorization (2) is valid, and*

$$b(x,y) = 0 \quad \textit{for} \quad x, y \in G \quad \textit{such that} \quad x \perp y; \tag{7}$$

*moreover, the functions $a$ and $b$ are uniquely determined.*

PROOF. Define $\hat{f} : G \to H/K$ as $\hat{f} = p \circ f$ where $p$ stands for the canonical projection. Condition (1) yields the orthogonal additivity of $\hat{f}$. By [2, Theorem 1], there exist an additive function $\hat{a} : G \to H/K$ and a quadratic function $\hat{q} : G \to H/K$ such that $\hat{f} = \hat{a} + \hat{q}$. Moreover the function $\hat{a}$ is defined by the formula

$$\hat{a}(x) = \hat{f}\left(\frac{x}{2}\right) - \hat{f}\left(-\frac{x}{2}\right)$$

and $\hat{q}(x) = \hat{b}(x,x)$, $x \in G$, with a biadditive and symmetric function $\hat{b} : G \times G \to H/K$ given by

$$\hat{b}(x,y) = 2\left[\hat{f}\left(\frac{x+y}{4}\right) + \hat{f}\left(\frac{-x-y}{4}\right) - \hat{f}\left(\frac{x-y}{4}\right) - \hat{f}\left(\frac{-x+y}{4}\right)\right].$$

The above equalities, jointly with $\mathfrak{M}$-measurability of $\hat{f}$, condition (G4) and Lemma 2, imply the $\mathfrak{M}$-measurability of $\hat{a}$ and the sections $\hat{b}(x,\cdot)$ for every $x \in G$. By Lemmas 4, 1 and 3, the functions $\hat{a}$ and $\hat{b}$ are continuous.

According to [14, Lemma 4] there exist a continuous additive function $a : G \to H$ and a continuous at zero quadratic function $q : G \to H$ such that $q(0) = 0$ and $p \circ a = \hat{a}$, $p \circ q = \hat{q}$. Hence $f(x) - q(x) - a(x) \in K$ for $x \in G$. As in the proof of [14, Theorem 1] we recall [2, Lemma 2] and [14, Lemma 5] to obtain $q(x) = b(x,x)$ with a continuous biadditive symmetric function $b : G \times G \to H$. To finish the proof of the "only if" part it remains to apply Lemma 5 given below.

The proof of the "if" part is a simple verification. $\square$

LEMMA 5. *Assume* (G1) *and* (G2). *Let the functions* $a_1, a_2 : G \to H$ *be continuous additive and let* $b_1, b_2 : G \times G \to H$ *be biadditive symmetric and continuous in each variable.*

(i) *If* $\big(a_1(x) + b_1(x,x)\big) - \big(a_2(x) + b_2(x,x)\big) \in K$ *for* $x \in G$, *then* $a_1 = a_2$ *and* $b_1 = b_2$.

(ii) *If* $b_1(x,y) \in K$ *for* $x, y \in G$ *such that* $x \perp y$, *then* $b_1(x,y) = 0$ *for* $x, y \in G$ *such that* $x \perp y$.

PROOF. (i) Let $a := a_1 - a_2$, $b := b_1 - b_2$. For $x \in G$ we have $a(x) + b(x,x) \in K$. Hence

$$a(2x) = \big(a(x) + b(x,x)\big) - \big(a(-x) + b(-x,-x)\big) \in K,$$

which implies $a(G) \subset K$. Now, condition (G2) guarantees that the function $a$, being continuous and additive, is constantly equal to zero.

We have just obtained that $b(x,x) \in K$ for $x \in G$, thus

$$b(x,2y) = 2b(x,y) = b(x+y,x+y) - b(x,x) - b(y,y) \in K \quad \text{for} \quad x, y \in G.$$

Arguing as above we infer that the section $b(\cdot, 2y)$ is constantly equal to zero for every $y \in G$, so $b = 0$.

(ii) Fix $x, y \in G$ such that $x \perp y$. Choose zero neighbourhoods $W \subset G$ such that $K \cap W = \{0\}$ and $U \subset G$ such that

$$b(U,y) \subset W \text{ and } G = \bigcup \{2^n U : n \in \mathbb{N}\}.$$

For some $n \in \mathbb{N}$ we have $x \in 2^n U$, whence $b\big(\frac{x}{2^n}, y\big) \in W$. Plainly, $2^{-n}x \perp 2^{-n}y$, which implies

$$b\left(\frac{x}{2^n}, y\right) = 2^n b\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \in K.$$

Consequently, $b(2^{-n}x, y) = 0$ and

$$b(x,y) = 2^n b\left(\frac{x}{2^n}, y\right) = 0,$$

as desired. $\square$

As a consequence of Theorem 1 we obtain the following result.

COROLLARY 1. *Assume $H$ is separable metric and* (G1), (G2) *hold. If either $G$ is a first countable Baire group and $f : G \to H$ is Baire measurable, or $G$ is a Polish group and $f : G \to H$ is Christensen measurable, then $f$ satisfies* (1) *if and only if there exist a continuous additive function $a : G \to H$ and a continuous biadditive symmetric function $b : G \times G \to H$ such that* (2) *and* (7) *hold; moreover, the functions $a$ and $b$ are uniquely determined.*

Baire and Christensen measurable solutions of (1) have been already examined by J. Brzdęk in [4] for the orthogonality given by an inner product and in [5] for a more abstract orthogonality in linear topological spaces.

## 3. Factorization with a separately continuous biadditive term

Under weaker assumptions we obtain the factorization (2) with a separately continuous biadditive term only (as it is in [5, Theorem 1]).

THEOREM 2. *Assume* (G1), (G2), (G4) *and let $H$ be separable metric. Then an $\mathfrak{M}$-measurable function $f : G \to H$ satisfies* (1) *if and only if there exist a continuous additive function $a : G \to H$ and a function $b : G \times G \to H$ biadditive symmetric and continuous in each variable such that the factorization* (2) *is valid and* (7) *holds; moreover, the functions $a$ and $b$ are uniquely determined.*

To get this result we argue as in the proof of Theorem 1 but without referring to Lemma 3 and applying the following Lemma 6 instead of [14, Lemma 4(ii)].

LEMMA 6. *Assume* (G1) *and* (G2). *If $\hat{b} : G \to H/K$ is biadditive, symmetric and continuous in each variable, then there exists a function $b : G \times G \to H$ biadditive, symmetric and continuous in each variable such that*

$$b(x,y) \in \hat{b}(x,y) \quad \textit{for } (x,y) \in G \times G. \tag{8}$$

PROOF. It follows from [14, Lemma 4(i)] that there exists a function $b : G \times G \to H$ such that for every $y \in G$ the function $b(\cdot,y)$ is additive, continuous and (8) holds. To show that $b$ is symmetric fix $x, y \in G$ and a neighbourhood $W$ of zero in $H$ with

$$(W + W - W) \cap K = \{0\}.$$

Since $b(\cdot,y)^{-1}(W) \cap b(\cdot,2y)^{-1}(W) \cap b(\cdot,x)^{-1}(W)$ is a neighbourhood of zero, it follows from (G2) that there exists a zero neighbourhood $U$ such that

$$U \subset b(\cdot,y)^{-1}(W) \cap b(\cdot,2y)^{-1}(W) \cap b(\cdot,x)^{-1}(W)$$

and (6) holds. In particular, $x = 2^n u_1$ and $y = 2^n u_2$ for some $n \in \mathbb{N}$ and $u_1, u_2 \in U$. Moreover,

$$2b(u_1, y) - b(u_1, 2y) \in (2W - W) \cap \big(2\hat{b}(u_1, y) - \hat{b}(u_1, 2y)\big)$$
$$= (2W - W) \cap K = \{0\},$$

whence $2b(u_1, y) = b(u_1, 2y)$ and, consequently,

$$2b(x, y) = 2b(2^n u_1, y) = 2^n \cdot 2b(u_1, y) = 2^n b(u_1, 2y) = b(x, 2y).$$

Now, having the equality $b(x, 2y) = 2b(x, y)$ for any $x, y \in G$ we see that

$$b(x, u_2) = b(2^n u_1, u_2) = b(u_1, 2^n u_2) = b(u_1, y) \in W,$$

whence

$$b(x, u_2) - b(u_2, x) \in (W - W) \cap \big(\hat{b}(x, u_2) - \hat{b}(u_2, x)\big) = (W - W) \cap K = \{0\}$$

and

$$b(x, y) = b(x, 2^n u_2) = 2^n b(x, u_2) = 2^n b(u_2, x) = b(2^n u_2, x) = b(y, x). \qquad \square$$

As a consequence we obtain a corollary asserting that if $G$ is Baire and we consider the Baire measurability, then we do not need to assume the first countability of $G$ in order to get the desired factorization with a separately continuous biadditive term only (cf. Corollary 1).

COROLLARY 2. *Assume $H$ is separable metric and* (G1), (G2) *hold. If $G$ is Baire and $f : G \to H$ is Baire measurable, then $f$ satisfies* (1) *if and only if there exist a continuous additive function $a : G \to H$ and a function $b : G \times G \to H$ biadditive symmetric and continuous in each variable such that* (2) *and* (7) *hold; moreover, the functions $a$ and $b$ are uniquely determined.*

If we take $\perp = G^2$, then Theorem 2 gives us Corollary 3 below. Of course, again it leads to another conclusions in the case when the measurability that we consider is Baire or Christensen.

COROLLARY 3. *Assume* (G1), (G2), (G4) *and let $H$ be separable metric. Then an $\mathfrak{M}$-measurable function $f : G \to H$ satisfies*

$$f(x + y) - f(x) - f(y) \in K \quad \textit{for} \quad x, y \in G$$

*if and only if there exists a (unique) continuous additive function $a : G \to H$ such that*

$$f(x) - a(x) \in K \quad \textit{for} \quad x \in G.$$

## 4. A counterexample

Hypothesis (G2) is supposed to be a substitute for the condition that every zero neighbourhood is absorbing – the condition which we dispose of in linear topological spaces. The following example shows that we cannot run too far away from this linear topological structure. Although for the simplest counterexample we may consider $(\mathbb{R},+)$ with the discrete topology, we present a more interesting one. Our aim is to demonstrate that the validity of all of the assumptions, just with the exception of (G2), does not guarantee the factorization (2) even if the domain is a "nice" structure with a non-discrete topology.

Let $\mathbb{R}^{\mathbb{N}}$ stand for the group of all real sequences (with the ordinary addition). In this group we introduce the so called *Krull topology,* the Tychonov (product) topology with the discrete topology in $\mathbb{R}$. Observe that we obtain in this manner an Abelian topological group metrizable by a complete metric. In particular, it is a Baire group. Note also that the family $\{V_I : I \in \mathcal{F}\}$, where

$$\mathcal{F} := \{I \subset \mathbb{N} : \operatorname{card} I < \aleph_0\}$$

and

$$V_I := \{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : x_i = 0 \text{ for } i \in I\} \quad \text{for} \quad I \in \mathcal{F}$$

is a zero neighbourhood basis.

Clearly, $\mathbb{R}^{\mathbb{N}}$ is uniquely 2-divisible (it is even a real linear space) and the orthogonality $\perp$ defined as $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ fulfils both (O) and (P). Obviously, the mapping $\mathbb{R}^{\mathbb{N}} \ni x \mapsto 2x$ is a homeomorphism. However, since $V_I$ is a subgroup of $\mathbb{R}^{\mathbb{N}}$, we have

$$\bigcup\{nV_I : n \in \mathbb{N}\} = V_I \subsetneqq \mathbb{R}^{\mathbb{N}} \quad \text{for} \quad I \in \mathcal{F},\ I \neq \emptyset.$$

Let $\mathfrak{B}$ be the $\sigma$-algebra of all Borel subsets of $\mathbb{R}^{\mathbb{N}}$ and let $\mathfrak{J}$ be the (proper) $\sigma$-ideal of all meager subsets of $\mathbb{R}^{\mathbb{N}}$. The classical theorem of Pettis [12, Theorem 9.9] asserts that $0 \in \operatorname{Int}(A - A)$, whenever $A \in \mathfrak{B} \setminus \mathfrak{J}$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any function fulfilling the congruence

$$\varphi(x+y) - \varphi(x) - \varphi(y) \in \mathbb{Z} \quad \text{for} \quad x, y \in \mathbb{R}$$

which is not a sum of an additive and a $\mathbb{Z}$-valued function (see [1, Remark 2] for a suitable example). Define $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ by the formula

$$f(x) = \varphi(x_1) \quad \text{for} \quad x = (x_n)_{n\in\mathbb{N}}.$$

Plainly, $f$ is a continuous (hence Borel) solution of the congruence

$$f(x+y) - f(x) - f(y) \in \mathbb{Z} \quad \text{for} \quad x, y \in \mathbb{R}^{\mathbb{N}}.$$

Now, suppose that $f(x) - b(x,x) - a(x) \in \mathbb{Z}$ for $x \in \mathbb{R}^{\mathbb{N}}$ with an additive function $a : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ and a function $b : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ which fulfils (7). Since our orthogonality is the trivial one, we have $b = 0$ and hence

$$f(x) - a(x) \in \mathbb{Z} \quad \text{for} \quad x \in \mathbb{R}^{\mathbb{N}}. \tag{9}$$

Defining $\alpha : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = a(x,0,0,\dots)$ we see that it is additive and (9) implies that

$$\varphi(x) - \alpha(x) = f(x,0,0,\dots) - a(x,0,0,\dots) \in \mathbb{Z} \quad \text{for} \quad x \in \mathbb{R},$$

contrary to the choice of $\varphi$.

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# OŚWIADCZENIE

o indywidualnym wkładzie współautora
w powstanie artykułu

*Measurable orthogonally additive functions modulo a discrete subgroup*,
Acta Mathematica Hungarica **123** (2009), 239–248.

Moim wkładem w powstanie wymienionej pracy było wyodrębnienie kilku grup założeń na temat grup topologicznych, które pozwalają wnosić o ciągłości funkcji dwuaddytywnych (lemat 3 z dowodem), a także podanie kontrprzykładu w sekcji 4.

Inne techniczne lematy i wnioski były wynikiem wspólnych rozważań prowadzonych z panią mgr W. Wyrobek-Kochanek. Poza tym miała ona indywidualny wkład w ogólne sformułowanie problemu i postawienie hipotezy o faktoryzacji mierzalnych funkcji ortogonalnie addytywnych modulo podgrupa dyskretna, która po doprecyzowaniu założen stała się docelowym wynikiem pracy. Podała też plan dowodu twierdzenia 1, którego realizacja, po udowodnieniu stosownych lematów, stała się natychmiastowa.

(-) Tomasz Kochanek

# ORTHOGONALLY PEXIDER FUNCTIONS MODULO A DISCRETE SUBGROUP

WIRGINIA WYROBEK-KOCHANEK

**Abstract**

Under appropriate conditions on abelian topological groups $G$ and $H$, an orthogonality $\perp \subset G^2$ and a $\sigma$-algebra $\mathfrak{M}$ of subsets of $G$ we prove that if at least one of the functions $f, g, h\colon G \to H$ satisfying

$$f(x+y) - g(x) - h(y) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y$$

is continuous at a point or $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and constants $a, b \in H$ such that

$$\begin{cases} f(x) - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b \in K \end{cases}$$

for $x \in G$ and

$$B(x,y) = 0 \text{ for } x, y \in G \text{ such that } x \perp y.$$

We would like to obtain some results similar to the main results from papers [5] and [3] but for the Pexider difference instead of the Cauchy difference. We start with the following result.

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LEMMA. *Let $G$ be a groupoid with a neutral element, $H$ an abelian group, $K$ a subgroup of $H$. Let $\Delta \subset G \times G$ be a set with*

$$(0,x),(x,0) \in \Delta \quad \text{for all } x \in G. \tag{1}$$

*If functions $f,g,h\colon G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in K \quad \text{for } (x,y) \in \Delta, \tag{2}$$

*then the following are true:*

(a) *There are functions $k_1, l_1\colon G \to K$, $\varphi_1\colon G \to H$ and constants $a, b \in H$ such that*

$$\varphi_1(x+y) - \varphi_1(x) - \varphi_1(y) \in K \quad \text{for } (x,y) \in \Delta$$

*and*

$$\begin{cases} f(x) = \varphi_1(x) + a, \\ g(x) = \varphi_1(x) + k_1(x) + b, \\ h(x) = \varphi_1(x) - k_1(x) + l_1(x) + a - b \end{cases} \tag{3}$$

*for all $x \in G$.*

(b) *There are functions $k_2, l_2\colon G \to K$, $\varphi_2\colon G \to H$ and constants $a, b \in H$ such that*

$$\varphi_2(x+y) - \varphi_2(x) - \varphi_2(y) \in K \quad \text{for } (x,y) \in \Delta$$

*and*

$$\begin{cases} f(x) = \varphi_2(x) + k_2(x) + a, \\ g(x) = \varphi_2(x) + b, \\ h(x) = \varphi_2(x) + l_2(x) + a - b \end{cases}$$

*for all $x \in G$.*

(c) *There are functions $k_3, l_3\colon G \to K$, $\varphi_3\colon G \to H$ and constants $a, b \in H$ such that*

$$\varphi_3(x+y) - \varphi_3(x) - \varphi_3(y) \in K \quad \text{for } (x,y) \in \Delta$$

*and*

$$\begin{cases} f(x) = \varphi_3(x) + k_3(x) + a, \\ g(x) = \varphi_3(x) + l_3(x) + b, \\ h(x) = \varphi_3(x) + a - b \end{cases}$$

*for all $x \in G$.*

*Moreover, each of assertions* (a), (b), (c) *gives a complete description of solutions of* (2), *that is, every triple $(f,g,h)$, being of one of the forms described above, is a solution of* (2).

*Proof.* Setting $y = 0$ in (2), by (1) we get

$$\mu(x) := f(x) - g(x) - h(0) \in K \quad \text{for } x \in G \tag{4}$$

and setting $x = 0$ we have

$$\nu(y) := f(y) - g(0) - h(y) \in K \quad \text{for } y \in G. \tag{5}$$

In particular,

$$f(0) - g(0) - h(0) \in K. \tag{6}$$

Denote $a = f(0)$, $b = g(0)$ and define $\varphi_i, k_i, l_i \colon G \to H$ for $i = 1, 2, 3$ by

$$\begin{array}{lll} \varphi_1 = f - a, & k_1 = g - \varphi_1 - b, & l_1 = h + k_1 - \varphi_1 - a + b, \\ \varphi_2 = g - b, & k_2 = f - \varphi_2 - a, & l_2 = h - \varphi_2 - a + b, \\ \varphi_3 = h - a + b, & k_3 = f - \varphi_3 - a, & l_3 = g - \varphi_3 - b. \end{array}$$

Using (4), (5), (2) and (6) for every $(x, y) \in \Delta$ we get

$$\begin{aligned} \varphi_1(x+y) - \varphi_1(x) - \varphi_1(y) &= f(x+y) - a - f(x) + a - f(y) + a \\ &= f(x+y) - \mu(x) - g(x) - h(0) - \nu(y) - g(0) - h(y) + a \in K; \end{aligned}$$

$$\begin{aligned} \varphi_2(x+y) - \varphi_2(x) - \varphi_2(y) &= g(x+y) - b - g(x) + b - g(y) + b \\ &= f(x+y) - \mu(x+y) - h(0) - g(x) + \mu(y) - f(y) + h(0) + b \\ &= f(x+y) - \mu(x+y) - g(x) + \mu(y) - \nu(y) - g(0) - h(y) + b \in K; \end{aligned}$$

$$\begin{aligned} \varphi_3(x+y) - \varphi_3(x) - \varphi_3(y) &= h(x+y) - a + b - h(x) + a - b - h(y) + a - b \\ &= f(x+y) - g(0) - \nu(x+y) + \nu(x) - f(x) + g(0) - h(y) + a - b \\ &= f(x+y) - \nu(x+y) + \nu(x) - \mu(x) - g(x) - h(0) - h(y) + a - b \\ &\in K. \end{aligned}$$

We also have

$$k_1(x) = g(x) - f(x) + a - b = -\mu(x) - h(0) + a - b \in K,$$

$$k_2(x) = f(x) - g(x) + b - a = \mu(x) + h(0) + b - a \in K,$$

$$k_3(x) = f(x) - h(x) + a - b - a = \nu(x) + g(0) - b \in K,$$

$$l_1(x) = h(x) + k_1(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_1(x) + b \in K,$$

$$l_2(x) = h(x) + k_2(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_2(x) + b \in K,$$

$$l_3(x) = g(x) + k_3(x) - f(x) + a - b = -\mu(x) - h(0) + k_3(x) + a - b \in K$$

for $x \in G$. □

The part (b) of this lemma in the case when $\perp = G^2$ was also obtained by K. Baron and PL. Kannappan in [1], even under some weaker assumptions. Some variations of (2) for functions with values in groupoids were studied by J. Sikorska in [4].

We work with the following orthogonality proposed by K. Baron and P. Volkmann in [2]:

Let $G$ be a group such that the mapping

$$x \mapsto 2x, \quad x \in G, \tag{7}$$

is a bijection onto the group $G$. A relation $\perp \subset G^2$ is called *orthogonality* if it satisfies the following two conditions:

(O) $0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

(P) If an orthogonally additive function from $G$ to an abelian group is odd, then it is additive; if it is even, then it is quadratic.

For a subset $U$ of a given group and for $n \in \mathbb{N}$ the symbol $nU$ denotes the set $\{nx : x \in U\}$.

THEOREM. *Assume $G$ is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that*

$$U \subset 2U \quad \textit{and} \quad G = \bigcup \{2^n U : \ n \in \mathbb{N}\}. \tag{8}$$

*Let $\perp \subset G^2$ be an orthogonality, $H$ an abelian topological group, $K$ a discrete subgroup of $H$ and*

$$x \perp 0 \textit{ and } 0 \perp x \qquad \textit{for } x \in G. \tag{9}$$

*Assume that functions $f, g, h\colon G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in K \quad \textit{for } x, y \in G \textit{ such that } x \perp y. \tag{10}$$

(i) *If at least one of the functions $f, g, h$ is continuous at a point, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and constants $a, b \in H$ such that*

$$\begin{cases} f(x) - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b \in K \end{cases} \tag{11}$$

*for $x \in G$ and*

$$B(x,y) = 0 \quad \textit{for } x, y \in G \textit{ such that } x \perp y. \tag{12}$$

(ii) *Let $\mathfrak{M}$ be a $\sigma$-algebra of subsets of $G$ such that*

$$x \pm 2A \in \mathfrak{M} \quad \textit{for all } x \in G \textit{ and } A \in \mathfrak{M} \tag{13}$$

*and there is a proper $\sigma$-ideal $\mathfrak{I}$ of subsets of $G$ with*

$$0 \in \operatorname{Int}(A - A) \quad \textit{for } A \in \mathfrak{M} \setminus \mathfrak{I}. \tag{14}$$

*Assume moreover that $H$ is separable metric and the following condition* (G) *is fulfilled:*

(G) *either $G$ is a first countable Baire group, or $G$ is metric separable, or $G$ is metric and $\mathfrak{M}$ contains all Borel subsets of $G$.*

*If at least one of the functions $f, g, h$ is $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and constants $a, b \in H$ such that* (11) *and* (12) *hold.*

*Moreover, each of assertions* (i), (ii) *gives a complete description of solutions of* (10).

*Proof.* (i): Case 1. Assume that $f$ is continuous at a point. Let $k_1, l_1\colon G \to K$, $\varphi_1\colon G \to H$ be as in Lemma (a). Then the function $\varphi_1$ is continuous at a point. According to Theorem 1 from [5] we get a continuous additive function $A\colon G \to H$ and a continuous biadditive and symmetric function $B\colon G \times G \to H$ such that

$$\varphi_1(x) - B(x,x) - A(x) \in K \quad \text{for } x \in G$$

and (12) hold. Then, according to (3),

$$f(x) - B(x,x) - A(x) - a = \varphi_1(x) + a - B(x,x) - A(x) - a \in K,$$

$$g(x) - B(x,x) - A(x) - b = \varphi_1(x) + k_1(x) + b - B(x,x) - A(x) - b \in K,$$

$$\begin{aligned} h(x) - B(x,x) - A(x) - a + b = \; & \varphi_1(x) - k_1(x) + l_1(x) + a - b \\ & - B(x,x) - A(x) - a + b \in K \end{aligned}$$

for all $x \in G$.

Case 2. If the function $g$ is continuous at a point then instead of Lemma (a) we use Lemma (b).

Case 3. If the function $h$ is continuous at a point then we use Lemma (c).

(ii): If one of the functions $f, g, h$ is $\mathfrak{M}$-measurable then we use Theorem 1 from [3] instead of Theorem 1 from [5]. □

For $\perp = G^2$ some special cases were obtained in [1] (cf. Corollaries 6 and 7 there).

If in the Theorem $G$ is Baire and we consider the Baire measurability, then we do not need to assume the first countability of $G$ in order to get the factorization with a separately continuous biadditive term only (cf. Corollary 2 in [3]).

COROLLARY 1. *Assume $G$ is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that (8) holds. Let $\perp \subset G^2$ be an orthogonality satisfying (9), $H$ an abelian separable metric group, $K$ a discrete subgroup of $H$ and functions $f, g, h\colon G \to H$ satisfy (10). If $G$ is Baire and at least one of the functions $f, g, h$ is Baire measurable, then there exist: a continuous additive function $A\colon G \to H$, a function $B\colon G \times G \to H$ biadditive, symmetric and continuous in each variable, and constants $a, b \in H$ such that (11) and (12) hold.*

If we take $\perp = G^2$, then our Theorem gives us Corollary 2 below. It also leads to another conclusions in the case when we consider Baire or Christensen measurability.

COROLLARY 2. *Assume $G$ is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that (8) holds. Let $H$ be an abelian separable metric group, $K$ a discrete subgroup of $H$, $\mathfrak{M}$ a $\sigma$-algebra of subsets of $G$ satisfying (13) and such that there is a proper $\sigma$-ideal $\mathfrak{I}$ of subsets of $G$ with property (14). If functions $f, g, h\colon G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in K \quad \text{for } x, y \in G$$

*and at least one of them is $\mathfrak{M}$-measurable, then there exist a continuous additive function $A\colon G \to H$ and constants $a, b \in H$ such that*

$$\begin{cases} f(x) - A(x) - a \in K, \\ g(x) - A(x) - b \in K, \\ h(x) - A(x) - a + b \in K \end{cases}$$

*for $x \in G$.*

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# ALMOST ORTHOGONALLY ADDITIVE FUNCTIONS

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ABSTRACT. If a function $f$, acting on a Euclidean space $\mathbb{R}^n$, is "almost" orthogonally additive in the sense that $f(x+y)=f(x)+f(y)$ for all $(x,y)\in\perp\setminus Z$, where $Z$ is a "negligible" subset of the $(2n-1)$-dimensional manifold $\perp\subset\mathbb{R}^{2n}$, then $f$ coincides almost everywhere with some orthogonally additive mapping.

## 1. INTRODUCTION

Let $(E,\langle\cdot|\cdot\rangle)$ be a real inner product space, $\dim E\geq 2$, and let $(G,+)$ be an Abelian group. A function $f\colon E\to G$ is called *orthogonally additive* iff it satisfies the equation

$$f(x+y)=f(x)+f(y) \tag{1}$$

for all $(x,y)\in\perp:=\{(x,y)\in E^2:\langle x|y\rangle=0\}$. It was proved independently by R. Ger, Gy. Szabó and J. Rätz [13, Corollary 10] that such a function has the form

$$f(x)=a\left(\|x\|^2\right)+b(x) \tag{2}$$

with some additive mappings $a\colon\mathbb{R}\to G$, $b\colon E\to G$ provided that $G$ is uniquely 2-divisible. This divisibility assumption was dropped by K. Baron and J. Rätz [2, Theorem 1].

We are going to deal with the situation where equality (1) holds true for all orthogonal pairs $(x,y)$ outside from a "negligible" subset of $\perp$. Considerations of this type go back to a problem [7], posed by P. Erdős, concerning the unconditional version of Cauchy's functional equation (1). It was solved by N. G. de Bruijn [3] and, independently, by W. B. Jurkat [11], and also generalized by R. Ger [10]. Similar research concerning mappings which preserve inner product was made by J. Chmieliński and J. Rätz [5] and by J. Chmieliński and R. Ger [4].

While studying unconditional functional equations, "negligible" sets are usually understood as the members of some proper linearly invariant ideal. Moreover, any such ideal of subsets of an underlying space $X$ automatically generates another such ideal of subsets of $X^2$ via the Fubini theorem (see R. Ger [9] and M. Kuczma [12, §17.5]). However, we shall assume that equation (1) is valid for $(x,y)\in\perp\setminus Z$, where $Z$ is "negligible" in $\perp$ (not only in $E^2$), and therefore the structure of $\perp$ should be appropriate to work with "linear invariance" and Fubini-type theorems. This is the reason why we restrict our attention to Euclidean spaces $\mathbb{R}^n$ and regard $\perp$ as a smooth $(2n-1)$-dimensional manifold lying in $\mathbb{R}^{2n}$.

## 2. PRELIMINARY RESULTS

For completeness let us recall some definitions concerning the manifold theory (for further information see, e.g., R. Abraham, J. E. Marsden and T. Ratiu [1], and L. W. Tu [14]). Let $S$ be a topological space; by an *$m$-dimensional $\mathcal{C}^\infty$-atlas* we mean a family $\mathcal{A}=\{(U_i,\varphi_i)\}_{i\in I}$ such that $\{U_i\}_{i\in I}$ is an open covering of $S$, for each $i\in I$ the mapping $\varphi_i$ is a homeomorphism which maps $U_i$ onto an open subset of $\mathbb{R}^m$, and for each $i,j\in I$ the mapping $\varphi_i\circ\varphi_j^{-1}$ is a $\mathcal{C}^\infty$-diffeomorphism defined on $\varphi_j(U_i\cap U_j)$. Brouwer's theorem of dimension invariance implies that each two atlases on $S$ are of the same dimension.

We say that atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* iff $\mathcal{A}_1\cup\mathcal{A}_2$ is an atlas. A *$\mathcal{C}^\infty$-differentiable structure* $\mathcal{D}$ on $S$ is an equivalence class of atlases on $S$; the union $\bigcup\mathcal{D}$ forms a maximal atlas on $S$ and any of its element is called an *admissible chart*. By a *$\mathcal{C}^\infty$-differentiable manifold* (briefly: *manifold*) $M$ we

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mean a pair $(S,\mathcal{D})$ of a topological space $S$ and a $\mathcal{C}^\infty$-differentiable structure $\mathcal{D}$ on $S$; we shall then identify $M$ with the space $S$ for convenience. A manifold is called an *$m$-manifold* iff its every atlas is $m$-dimensional.

Having an $m_1$-manifold $M_1 = (S_1,\mathcal{D}_1)$ and an $m_2$-manifold $M_2 = (S_2,\mathcal{D}_2)$ we may define the *product manifold* $M_1 \times M_2 = (S_1 \times S_2, \mathcal{D}_1 \times \mathcal{D}_2)$, where the differentiable structure $\mathcal{D}_1 \times \mathcal{D}_2$ is generated by the atlas

$$\left\{(U_1 \times U_2, \varphi_1 \times \varphi_2) : (U_i,\varphi_i) \in \bigcup \mathcal{D}_i \text{ for } i = 1,2\right\}.$$

Then $M_1 \times M_2$ forms an $(m_1+m_2)$-manifold. For an arbitrary set $A \subset M_1 \times M_2$ and any point $x \in M_1$ we will be using the notation $A[x] = \{y \in M_2 : (x,y) \in A\}$.

In what follows, we will be considering only manifolds $M \subset \mathbb{R}^n$, for some $n \in \mathbb{N}$, equipped with the natural topology and a differentiable structure which is determined by the following condition: for every $x \in M$ there is a $\mathcal{C}^\infty$-diffeomorphism $\varphi$ defined on an open set $U \subset \mathbb{R}^n$ with $x \in U$ such that $\varphi(M \cap U) = \varphi(U) \cap (\mathbb{R}^m \times \{0\})$, where $m$ is the dimension of $M$. In particular, every open subset of $\mathbb{R}^n$ yields an $n$-manifold with the atlas consisting of a single identity map. Any set $M \subset \mathbb{R}^n$ satisfying the above condition forms a *submanifold* of $\mathbb{R}^n$ in the sense of [1, Definition 3.2.1], or a *regular submanifold* of $\mathbb{R}^n$ in the sense of [14, Definition 9.1]. Generally, if $M_1$ is an $m_1$-manifold and $M_2$ is an $m_2$-manifold, then $M_1$ is called a *(regular) submanifold* of $M_2$ iff $M_1 \subset M_2$ and for every $x \in M_1$ there is an admissible chart $(U,\varphi)$ of $M_2$ with $x \in U$ such that $\varphi(M_1 \cap U) = \varphi(U) \cap (\mathbb{R}^{m_1} \times \{0\})$.

If $M_1$ and $M_2$ are manifolds with atlases $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, then a mapping $\Phi\colon M_1 \to M_2$ is said to be *of the class* $\mathcal{C}^\infty$ iff it is continuous and for all $(U,\varphi) \in \mathcal{A}_1$, $(V,\psi) \in \mathcal{A}_2$ the composition $\psi \circ \Phi \circ \varphi^{-1}$ is of the class $\mathcal{C}^\infty$ (in the usual sense) in its domain. This condition is independent on the choice of particular atlases generating differentiable structures of $M_1$ and $M_2$; see [1, Proposition 3.2.6]. We say that $\Phi$ is a *$\mathcal{C}^\infty$-diffeomorphism* iff $\Phi$ is a bijection between $M_1$ and $M_2$, and both $\Phi$ and $\Phi^{-1}$ are of the class $\mathcal{C}^\infty$. According to the above explanation, such a definition is compatible with the usual notion of a $\mathcal{C}^\infty$-diffeomorphism. If any $\mathcal{C}^\infty$-diffeomorphism between $M_1$ and $M_2$ exists, then we write $M_1 \sim M_2$. Of course, in such a case the manifolds $M_1$ and $M_2$ are of the same dimension.

Finally, a mapping $\Phi\colon M_1 \to M_2$ between an $m_1$-manifold $M_1$ and an $m_2$-manifold $M_2$ is called a *$\mathcal{C}^\infty$-immersion* [*$\mathcal{C}^\infty$-submersion*] iff it is of the class $\mathcal{C}^\infty$ and for every $x \in M_1$ there exist admissible charts $(U,\varphi)$ and $(V,\psi)$ of $M_1$ and $M_2$, respectively, such that $x \in U$, $\Phi(x) \in V$, and the derivative of the function $\psi \circ \Phi \circ \varphi^{-1}$ at any point of $\varphi(U)$ is an injective [a surjective] linear mapping from $\mathbb{R}^{m_1}$ to $\mathbb{R}^{m_2}$ (see [14, Proposition 8.12] for another, equivalent definition). We will find useful the following lemma; for the proof see R. W. R. Darling [6, §5.5.1].

**Lemma 1.** *Let $M_1$ be a submanifold of an open set $U \subset \mathbb{R}^{n_1}$ and $M_2$ be a submanifold of an open set $V \subset \mathbb{R}^{n_2}$. If $\Phi\colon U \to V$ is a $\mathcal{C}^\infty$-immersion [$\mathcal{C}^\infty$-submersion] with $\Phi(M_1) \subset M_2$, then the restriction $\Phi|_{M_1}\colon M_1 \to M_2$ is a $\mathcal{C}^\infty$-immersion [$\mathcal{C}^\infty$-submersion].*

Recall that given a non-empty set $X$ a family $\mathfrak{J} \subset 2^X$ is said to be a *proper $\sigma$-ideal* iff the following conditions hold:

(i) $X \notin \mathfrak{J}$;
(ii) if $A \in \mathfrak{J}$ and $B \subset A$, then $B \in \mathfrak{J}$;
(iii) if $A_k \in \mathfrak{J}$ for $k \in \mathbb{N}$, then $\bigcup_{k=1}^\infty A_k \in \mathfrak{J}$.

From now on we suppose that for each $m \in \mathbb{N}$ a family $\mathfrak{J}_m$ forms a proper $\sigma$-ideal of subsets of $\mathbb{R}^m$ satisfying the following conditions:

($\mathrm{H}_0$) $\{0\} \in \mathfrak{J}_1$;
($\mathrm{H}_1$) if $\varphi$ is a $\mathcal{C}^\infty$-diffeomorphism defined on an open set $U \subset \mathbb{R}^m$ and $A \in \mathfrak{J}_m$, then $\varphi(A \cap U) \in \mathfrak{J}_m$;
($\mathrm{H}_2$) if $m,n \in \mathbb{N}$ and $A \in \mathfrak{J}_{m+n}$, then $\{x \in \mathbb{R}^m : A[x] \notin \mathfrak{J}_n\} \in \mathfrak{J}_m$;
($\mathrm{H}_3$) if $m,n \in \mathbb{N}$ and $A \in \mathfrak{J}_n$, then $\mathbb{R}^m \times A \in \mathfrak{J}_{m+n}$.

Note that by condition ($\mathrm{H}_1$), non-empty open subsets of $\mathbb{R}^m$ do not belong to $\mathfrak{J}_m$. Note also that if $\mathfrak{J}_m$ consists of all Lebesgue measure zero subsets of $\mathbb{R}^m$ for $m \in \mathbb{N}$, or $\mathfrak{J}_m$ consists of all first category subsets of $\mathbb{R}^m$ for $m \in \mathbb{N}$, then conditions ($\mathrm{H}_0$)-($\mathrm{H}_3$) are satisfied.

For an arbitrary $m$-manifold $M \subset \mathbb{R}^n$ $(m \leq n)$ with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ we define a proper $\sigma$-ideal $\mathfrak{I}_M \subset 2^M$ by putting

$$\mathfrak{I}_M = \{A \subset M : \varphi_i(A \cap U_i) \in \mathfrak{I}_m \text{ for each } i \in I\}.$$

By condition $(\mathrm{H}_1)$, this definition does not depend on the particular choice of $\mathcal{A}$. Indeed, let $\{(V_j, \psi_j)\}_{j\in J}$ be another atlas of $M$, equivalent to $\mathcal{A}$. Fix any $A \in \mathfrak{I}_M$ and $j \in J$. With the aid of Lindelöf's theorem we choose a countable set $I_0 \subset I$ such that $V_j \subset \bigcup_{i\in I_0} U_i$. For each $i \in I_0$ the mapping $\chi_i := \psi_j \circ \varphi_i^{-1}$ is a $\mathcal{C}^\infty$-diffeomorphism on $\varphi_i(V_j \cap U_i)$ and since $B_i := \varphi_i(A \cap V_j \cap U_i) \in \mathfrak{I}_m$, we have $\psi_j(A \cap V_j \cap U_i) = \chi_i(B_i) \in \mathfrak{I}_m$. Consequently, $\psi_j(A \cap V_j) = \bigcup_{i\in I_0} \psi_j(A \cap V_j \cap U_i) \in \mathfrak{I}_m$. This shows that if $A \in \mathfrak{I}_M$, then $\psi_j(A \cap V_j) \in \mathfrak{I}_m$ for each $j \in J$. Analogously we obtain the reverse implication. Note that, by this definition, $\mathfrak{I}_{\mathbb{R}^m} = \mathfrak{I}_m$ for each $m \in \mathbb{N}$.

**Lemma 2.** *Let $M_1$ be an $m_1$-dimensional submanifold of an $m_2$-manifold $M_2 \subset \mathbb{R}^n$. Then*

(a) *$M_1 \in \mathfrak{I}_{M_2}$, provided that $m_1 < m_2$;*
(b) *$\mathfrak{I}_{M_1} \subset \mathfrak{I}_{M_2}$.*

*Proof.* (a) By the submanifold property, we may choose an atlas $\mathcal{A}$ of $M_2$ such that $\varphi(M_1 \cap U) = \varphi(U) \cap (\mathbb{R}^{m_1} \times \{0\})$ for each $(U, \varphi) \in \mathcal{A}$. Since $(\mathrm{H}_0)$ and $(\mathrm{H}_3)$ imply $\mathbb{R}^{m_1} \times \{0\} \in \mathfrak{I}_{m_2}$, we get $\varphi(M_1 \cap U) \in \mathfrak{I}_{m_2}$, as desired.

(b) The case $m_1 < m_2$ reduces to assertion (a). If $m_1 = m_2$, then for every admissible chart of $M_2$ we have $\varphi(A \cap U) \in \mathfrak{I}_{m_1} = \mathfrak{I}_{m_2}$. □

We can prove the following strengthening of condition $(\mathrm{H}_1)$.

**Lemma 3.** *If $\Phi\colon M_1 \to M_2$ is a $\mathcal{C}^\infty$-diffeomorphism between manifolds $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$, then for every $A \in \mathfrak{I}_{M_1}$ we have $\Phi(A) \in \mathfrak{I}_{M_2}$.*

*Proof.* Let $\mathcal{A}_1 = \{(U_i, \varphi_i)\}_{i\in I}$ and $\mathcal{A}_2 = \{(V_j, \psi_j)\}_{j\in J}$ be atlases generating the differentiable structures of $M_1$ and $M_2$, respectively. Let also $m$ be the dimension of $M_1$ and $M_2$. Fix $j \in J$; we are to prove that $\psi_j(\Phi(A) \cap V_j) \in \mathfrak{I}_m$. Choose a countable set $I_0 \subset I$ with $A \subset \bigcup_{i\in I_0} U_i$ and for each $i \in I_0$ define a $\mathcal{C}^\infty$-diffeomorphism $\chi_i = \psi_j \circ \Phi \circ \varphi_i^{-1}$. Then

$$\psi_j(\Phi(A) \cap V_j) \subset \bigcup_{i\in I_0} \chi_i(\varphi_i(A \cap U_i) \cap \mathrm{Dom}(\chi_i)), \tag{3}$$

where $\mathrm{Dom}(\chi_i)$ stands for the domain of $\chi_i$. Moreover, since $A \in \mathfrak{I}_{M_1}$, we have $\varphi_i(A \cap U_i) \in \mathfrak{I}_m$ thus $(\mathrm{H}_1)$ implies that the both sets in (3) belong to $\mathfrak{I}_m$. □

Conditions $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ imply a general version of Fubini's theorem.

**Lemma 4.** *Let $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$ be manifolds. If $A \in \mathfrak{I}_{M_1\times M_2}$, then*

$$\{x \in M_1 : A[x] \notin \mathfrak{I}_{M_2}\} \in \mathfrak{I}_{M_1}.$$

*Proof.* Let $\mathcal{A}_1 = \{(U_i, \varphi_i)\}_{i\in I}$ and $\mathcal{A}_2 = \{(V_j, \psi_j)\}_{j\in J}$ be arbitrary countable atlases generating the differentiable structures of $M_1$ and $M_2$, respectively. Since $A \in \mathfrak{I}_{M_1\times M_2}$, for each $i \in I$, $j \in J$ we have

$$B_{ij} := (\varphi_i \times \psi_j)(A \cap (U_i \times V_j)) \in \mathfrak{I}_{m_1+m_2}.$$

Moreover,

$$B_{ij} = \{(\varphi_i(x), \psi_j(y)) \in \mathbb{R}^{m_1+m_2} : x \in U_i \text{ and } y \in A[x] \cap V_j\}$$

for $i \in I$, $j \in J$. Suppose (in search of a contradiction)

$$Z := \{x \in M_1 : A[x] \notin \mathfrak{I}_{M_2}\} \notin \mathfrak{I}_{M_1}.$$

Then we may find $i_0 \in I$ with $Z \cap U_{i_0} \notin \mathfrak{I}_{M_1}$. If for every $j \in J$ the set

$$C_j := \{x \in Z \cap U_{i_0} : A[x] \cap V_j \notin \mathfrak{I}_{M_2}\}$$

belonged to $\mathfrak{I}_{M_1}$, then we would have

$$Z \cap U_{i_0} = \{x \in Z \cap U_{i_0} : A[x] \notin \mathfrak{I}_{M_2}\} = \bigcup_{j\in J} C_j \in \mathfrak{I}_{M_1},$$

which is not the case. Therefore, we may find $j_0 \in J$ with $C_{j_0} \notin \mathfrak{I}_{M_1}$. Define

$$B = \{(\varphi_{i_0}(x), \psi_{j_0}(y)) \in \mathbb{R}^{m_1+m_2} : x \in Z \cap U_{i_0} \text{ and } y \in A[x] \cap V_{j_0}\}$$

and note that $B \subset B_{i_0,j_0}$, whence $B \in \mathfrak{I}_{m_1+m_2}$. However, $\varphi_{i_0}(C_{j_0}) \notin \mathfrak{I}_{m_1}$ and for each $x \in C_{j_0}$ and $t = \varphi_{i_0}(x)$ we have

$$B[t] = \psi_{j_0}(A[x] \cap V_{j_0}) \notin \mathfrak{I}_{m_2}.$$

This yields a contradiction with ($\mathrm{H}_2$). □

**Lemma 5.** *If $\Phi\colon M_1 \to M_2$ is a $C^\infty$-submersion between manifolds $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$, then for every $A \subset M_1$, $A \notin \mathfrak{I}_{M_1}$ we have $\Phi(A) \notin \mathfrak{I}_{M_2}$.*

*Proof.* By Lindelöf's theorem, there is a point $x_0 \in M_1$ such that for every its neighbourhood $U \subset M_1$ we have $A \cap U \notin \mathfrak{I}_{M_1}$. By the assumption, we may find admissible charts $(U, \varphi)$ and $(V, \psi)$ of $M_1$ and $M_2$, respectively, such that $x_0 \in U$, $\Phi(x_0) \in V$, and the derivative of $\psi \circ \Phi \circ \varphi^{-1}$ at any point of $\varphi(U)$ is a surjection from $\mathbb{R}^{m_1}$ onto $\mathbb{R}^{m_2}$ ($m_1$, $m_2$ being the dimensions of $M_1$, $M_2$, respectively). Hence, obviously, $m_1 \geq m_2$ and there is a sequence $1 \leq i_1 < \ldots < i_{m_2} \leq m_1$ such that

$$\frac{\partial(\psi \circ \Phi \circ \varphi^{-1})}{\partial y_{i_1} \ldots \partial y_{i_{m_2}}}(\varphi(x_0)) \neq 0.$$

By decreasing the neighbourhood $U$, we may guarantee that the above condition holds true for every $x \in U$ in the place of $x_0$, and that the mapping $\psi \circ \Phi \circ \varphi^{-1}$ is defined on the whole $\varphi(U)$. Let $\psi \circ \Phi \circ \varphi^{-1} = (G_1, \ldots, G_{m_2})$ and define a function $F = (F_1, \ldots, F_{m_1})\colon \varphi(U) \to \mathbb{R}^{m_1}$ by the formula

$$F_k(y) = \begin{cases} G_j(y), & \text{if } k = i_j \text{ for some } j \in \{1, \ldots, m_2\}, \\ y_k, & \text{otherwise.} \end{cases}$$

Then for each $y \in \varphi(U)$ we have

$$\left|\frac{\partial F}{\partial y_1 \ldots \partial y_{m_1}}(y)\right| = \left|\frac{\partial(\psi \circ \Phi \circ \varphi^{-1})}{\partial y_{i_1} \ldots \partial y_{i_{m_2}}}(y)\right| \neq 0,$$

thus, decreasing $U$ as required, we may assume that $F$ is a $C^\infty$-diffeomorphism. Enumerating the coordinates we may also modify $F$ in such a way that it is still a $C^\infty$-diffeomorphism and

$$F(\varphi(A \cap U)) \subset (\psi \circ \Phi \circ \varphi^{-1})(\varphi(A \cap U)) \times \mathbb{R}^{m_1 - m_2}. \tag{4}$$

In view of $A \cap U \notin \mathfrak{I}_{M_1}$, condition ($\mathrm{H}_1$) yields $F(\varphi(A \cap U)) \notin \mathfrak{I}_{m_1}$, whence (4) and ($\mathrm{H}_3$) imply $\psi(\Phi(A \cap U)) \notin \mathfrak{I}_{m_2}$. Therefore, $\Phi(A \cap U) \notin \mathfrak{I}_{M_2}$, since $\psi$ is an admissible chart of $M_2$ defined on $\Phi(U)$. □

In a similar manner we obtain the next lemma.

**Lemma 6.** *If $\Phi\colon M_1 \to M_2$ is a $C^\infty$-immersion between manifolds $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$, then for every $A \in \mathfrak{I}_{M_1}$ we have $\Phi(A) \in \mathfrak{I}_{M_2}$.*

From now on, let $n \geq 2$ be a fixed natural number and $\langle\cdot|\cdot\rangle$ be an arbitrary inner product in $\mathbb{R}^n$ inducing a norm which we denote by $\|\cdot\|$. For any set $A$ we define $A^* = A \setminus \{0\}$, where the meaning of 0 is clear from the context. Let $\perp$ be the set of all pairs of orthogonal vectors from $\mathbb{R}^n$. Then $\perp^* = F^{-1}(0)$, where $F\colon (\mathbb{R}^n \times \mathbb{R}^n)^* \to \mathbb{R}$ is given by $F(x, y) = \langle x|y\rangle$. Since 0 is a regular value of $F$, it follows from [14, Theorem 9.11] that $\perp^*$ forms a $(2n-1)$-manifold (being also a regular submanifold of $(\mathbb{R}^n \times \mathbb{R}^n)^*$).

We may therefore precise what being "negligible" in $\perp$ means. Namely, we say that a set $Z \subset \perp$ has this property iff $Z \in \mathfrak{I}_{\perp^*}$ and we will then write simply $Z \in \mathfrak{I}_\perp$. We are now ready to formulate our main result which we shall prove in the last section. For notational convenience, if $M$ is a manifold and some property, depending on a variable $x$, holds true for all $x \in M \setminus A$ with $A \in \mathfrak{I}_M$, then we write that it holds $\mathfrak{I}_M$-(a.e.).

**Theorem.** *Let $(G, +)$ be an Abelian group. If a function $f\colon \mathbb{R}^n \to G$ satisfies $f(x+y) = f(x) + f(y)$ $\mathfrak{I}_\perp$-(a.e.), then there is a unique orthogonally additive function $g\colon \mathbb{R}^n \to G$ such that $f(x) = g(x)$ $\mathfrak{I}_n$-(a.e.).*

Let us note some preparatory observations. For any $x \in \mathbb{R}^n$ define $P_x = \{y \in \mathbb{R}^n : (x,y) \in \perp\}$, which obviously forms an $(n-1)$-manifold diffeomorphic to $\mathbb{R}^{n-1}$, provided $x \neq 0$. We will need to "smoothly" identify the hyperplanes $P_x$, for different $x$'s, with one "universal"space $\mathbb{R}^{n-1}$. By virtue of the Hairy Sphere Theorem, it is impossible to do for all $x \in (\mathbb{R}^n)^*$ in the case where $n$ is odd. Nevertheless, it is an easy task when considering only the set of vectors for which one fixed coordinate is non-zero, e.g. the set $X := \mathbb{R}^{n-1} \times \mathbb{R}^*$.

Namely, for an arbitrary $x \in X$ the vectors $x, e_1, \dots, e_{n-1}$ are linearly independent, where $e_i$ stands for the $i$th vector from the canonical basis of $\mathbb{R}^n$. Let $\mathcal{B}(x) = (y_i(x))_{i=0}^{n-1}$ be an orthonormal basis of $\mathbb{R}^n$ with $y_0(x) = x/\|x\|$, produced by the Gram-Schmidt process applied to the sequence $(x, e_1, \dots, e_{n-1})$. Define $\psi_x \colon \mathbb{R}^n \to \mathbb{R}^n$ to be the mapping which to every $z \in \mathbb{R}^n$ assigns its coordinates with respect to $\mathcal{B}(x)$, i.e. $\psi_x(z) = \boldsymbol{Y}(x)^{-1}z$, where

$$\boldsymbol{Y}(x) = \left[\frac{x}{\|x\|}, y_1(x), \dots, y_{n-1}(x)\right]$$

is the matrix formed from the column vectors. Define also $\Phi \colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ by $\Phi(x,z) = (x, \psi_x(z))$. Plainly, $\Phi$ is a $\mathcal{C}^\infty$-mapping and its inverse $\Phi^{-1}(x,y) = (x, Y(x)y)$ is $\mathcal{C}^\infty$ as well. Therefore, $\Phi$ is a $\mathcal{C}^\infty$-diffeomorphism. Moreover, by the definition of $\psi_x$, the restriction $\psi_x|_{P_x}$ maps $P_x$ onto $\{0\} \times \mathbb{R}^{n-1}$, hence we have

$$\Phi^{-1}\left(X \times (\{0\} \times \mathbb{R}^{n-1})\right) = \{(x,z) \in \perp^* : x \in X\} =: \perp'. \tag{5}$$

Making use of [14, Theorem 11.20] and an easy fact that the restriction of a $\mathcal{C}^\infty$ mapping to a submanifold of its domain is $\mathcal{C}^\infty$ again[1], we infer by (5) that $\Phi|_{\perp'}$ yields a $\mathcal{C}^\infty$-diffeomorphism between $\perp'$ and $X \times (\{0\} \times \mathbb{R}^{n-1})$.

Consequently, if a function $h \colon \mathbb{R}^n \to G$ satisfies $h(x+y) = h(x) + h(y)$ $\mathfrak{I}_\perp$-(a.e.), then with the notation

$$Z(h) := \{(x,y) \in \perp^* : h(x+y) \neq h(x) + h(y)\}$$

it follows from Lemma 4 that

$$\left\{x \in X : \{\psi_x(z) : (x,z) \in Z(h)\} \notin \mathfrak{I}_{\{0\}\times\mathbb{R}^{n-1}}\right\} \in \mathfrak{I}_X.$$

Since $P_x \sim \{0\} \times \mathbb{R}^{n-1}$, by the mapping $\psi_x|_{P_x}$ for $x \in X$, we infer that the set

$$D(h) := \{x \in X : h(x+y) = h(x) + h(y) \ \mathfrak{I}_{P_x}\text{-(a.e.)}\}$$

fulfils $X \setminus D(h) \in \mathfrak{I}_X$. For any $x \in \mathbb{R}^n$ put

$$E_x(h) = \{y \in P_x : h(x+y) = h(x) + h(y)\};$$

then $P_x \setminus E_x(h) \in \mathfrak{I}_{P_x}$, provided $x \in D(h)$.

We end this section with a lemma, which will be useful in the "odd" part of the proof of our Theorem. Despite it will be applied only in the case $n = 2$, we present it in a full generality, since the lemma seems to be interesting independently on the problem considered. Let $S^{n-1}$ be the unit sphere of the normed space $(\mathbb{R}^n, \|\cdot\|)$. Since the function $F \colon \mathbb{R}^n \to \mathbb{R}$ given by $F(x) = \|x\|^2$ is $\mathcal{C}^\infty$ with the regular value 1 and $S^{n-1} = F^{-1}(1)$, we infer that $S^{n-1}$ is an $(n-1)$-manifold.

**Lemma 7.** *If $A \in \mathfrak{I}_{S^{n-1}}$, then there exists an orthogonal basis $(x_1, \dots, x_n)$ of $\mathbb{R}^n$ such that $x_i \in S^{n-1} \setminus A$ for each $i \in \{1, \dots, n\}$.*

*Proof.* It is enough to prove the assertion in the case where $\langle\cdot|\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$, since between any two inner product structures in $\mathbb{R}^n$ there is a linear isometry, which yields a $\mathcal{C}^\infty$-diffeomorphism between their unit spheres.

Consider the group $\mathrm{GL}(n)$ of $n \times n$ real matrices with non-zero determinant. It may be identified with an open subset of $\mathbb{R}^{n^2}$ and hence - it is an $n^2$-manifold. It is well-known that the orthogonal group

$$\mathrm{O}(n) = \{\boldsymbol{A} \in \mathrm{GL}(n) : \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}_n\}$$

forms a submanifold of $\mathrm{GL}(n)$ and its dimension equals $n(n-1)/2$ (see [1, §3.5.5C]). For any $i \in \{1, \dots, n\}$ let $\pi_i \colon \mathrm{O}(n) \to S^{n-1}$ be given by $\pi_i(\boldsymbol{A}) = \boldsymbol{A}e_i$ (which is nothing else but the $i$th column

[1] In the sequel, we will be using these two assertions without explicit mentioning.

vector of $\boldsymbol{A}$). Then $\pi_i$ is the restriction of the mapping $\overline{\pi}_i\colon \mathrm{GL}(n) \to \mathbb{R}^n$ defined by the formula analogous to the previous one. Since

$$\mathrm{D}\overline{\pi}_i(\boldsymbol{A})\boldsymbol{B} = \boldsymbol{B}e_i \quad \text{for } \boldsymbol{A} \in \mathrm{GL}(n), \boldsymbol{B} \in \mathbb{R}^{n^2},$$

the derivative $\mathrm{D}\overline{\pi}_i(\boldsymbol{A})$ is onto for any $\boldsymbol{A} \in \mathrm{GL}(n)$, thus $\overline{\pi}_i$ is a $\mathcal{C}^\infty$-submersion. By Lemma 1, $\pi_i$ is a $\mathcal{C}^\infty$-submersion as well.

Now, suppose on the contrary that each orthonormal basis of $\mathbb{R}^n$ has at least one entry belonging to $A$. In other words, for each $\boldsymbol{A} \in \mathrm{O}(n)$ there is $i \in \{1, \dots, n\}$ with $\pi_i(\boldsymbol{A}) \in A$, i.e.

$$\mathrm{O}(n) = \bigcup_{i=1}^{n} \pi_i^{-1}(A).$$

Therefore, for a certain $i \in \{1, \dots, n\}$ we would have $\pi_i^{-1}(A) \notin \mathfrak{I}_{\mathrm{O}(n)}$. However, $A = \pi_i(\pi_i^{-1}(A)) \in \mathfrak{I}_{S^{n-1}}$, which contradicts the assertion of Lemma 5, as $\pi_i$ is a $\mathcal{C}^\infty$-submersion. □

## 3. Proof of the Theorem

For the uniqueness part of our Theorem suppose that there are two orthogonally additive functions $g_1$ and $g_2$ equal to $f$ $\mathfrak{I}_n$-(a.e.). By the general form (2) of orthogonally additive mappings, we see that both $g_1$ and $g_2$ satisfy the Fréchet functional equation $\Delta_y^3 g(x) = 0$, thus arguing as in the proof of the uniqueness part of [8, Theorem 1], or making use of [12, Lemma 17.7.1], we get $g_1 = g_2$.

The proof of existence relies on some ideas from [2] and [13]. Assume $G$ and $f$ are as in the Theorem. We start with the following trivial observation.

**Lemma 8.** *The functions $f_1, f_2\colon \mathbb{R}^n \to G$ given by*

$$f_1(x) = f(x) - f(-x) \quad \textit{and} \quad f_2(x) = f(x) + f(-x)$$

*satisfy*

$$f_1(x+y) = f_1(x) + f_1(y) \quad \textit{and} \quad f_2(x+y) = f_2(x) + f_2(y) \quad \mathfrak{I}_\perp\text{-(a.e.)}.$$

In the sequel we will be using hypothesis ($\mathrm{H}_0$)-($\mathrm{H}_3$) and Lemmas 2-4 without explicit mentioning.

For $k, m \in \mathbb{N}$ with $2 \le k \le m$ we define $\mathrm{O}(k, m)$ as the set of all $k$-tuples of mutually orthogonal (with respect to the usual scalar product) vectors from $\mathbb{R}^m$ with at most one of them being zero. Put

$$\mathcal{R}_{k,m} = \{(x^{(1)}, \dots, x^{(k)}) \in (\mathbb{R}^m)^k : x^{(i)} = 0 \text{ for at most one } i = 1, \dots, k\}.$$

Then $\mathrm{O}(k.m) = F^{-1}(0)$, where $F\colon \mathcal{R}_{k,m} \to \mathbb{R}^{\frac{k(k-1)}{2}}$ is given by

$$\begin{aligned} F(x^{(1)}, \dots, x^{(k)}) = (&\langle x^{(1)}|x^{(2)}\rangle, \langle x^{(1)}|x^{(3)}\rangle, \dots, \langle x^{(1)}|x^{(k)}\rangle, \\ &\langle x^{(2)}|x^{(3)}\rangle, \dots, \langle x^{(2)}|x^{(k)}\rangle, \\ &\vdots \\ &\langle x^{(k-1)}|x^{(k)}\rangle). \end{aligned}$$

Since 0 is a regular value of $F$, [14, Theorem 9.11] implies that $\mathrm{O}(k, m)$ is a submanifold of $\mathbb{R}^{km}$ with dimension $km - \frac{1}{2}k(k-1)$. In particular, $\mathrm{O}(2, n) = \perp^*$.

**Lemma 9.** *Let $k \in \mathbb{N}$, $k \ge 2$ and let $A \subset \mathrm{O}(2, k)$ be a set such that*

$$\{(x^{(1)}, \dots, x^{(k)}) \in \mathrm{O}(k, k) : (x^{(1)}, x^{(2)}) \in A\} \in \mathfrak{I}_{\mathrm{O}(k,k)}.$$

*Then $A \in \mathfrak{I}_{\mathrm{O}(2,k)}$.*

*Proof.* Denote the above subset of $\mathrm{O}(k, k)$ by $B$. We may clearly assume that for each $(x^{(1)}, x^{(2)}) \in A$ we have $x^{(1)} \ne 0 \ne x^{(2)}$. For $i, j \in \{1, \dots, k\}$ define

$$D_{ij} = \left\{(x^{(1)}, x^{(2)}) \in \mathrm{O}(2, k) : \det\begin{bmatrix} x_i^{(1)} & x_j^{(1)} \\ x_i^{(2)} & x_j^{(2)} \end{bmatrix} \ne 0\right\},$$

$$B_{ij} = \{(x^{(1)}, \dots, x^{(k)}) \in B : (x^{(1)}, x^{(2)}) \in D_{ij}\},$$

and observe that

$$A = \bigcup_{\substack{i,j=1\\ i\neq j}}^{k} (A \cap D_{ij}) \quad \text{and} \quad B = \bigcup_{\substack{i,j=1\\ i\neq j}}^{k} B_{ij}. \tag{6}$$

For the former equality suppose that for some $(x^{(1)}, x^{(2)}) \in A$ and each pair of indices $1 \le i, j \le k$, $i \neq j$, we have

$$\det\begin{bmatrix} x_i^{(1)} & x_j^{(1)} \\ x_i^{(2)} & x_j^{(2)} \end{bmatrix} = 0. \tag{7}$$

Then for each $1 \le i \le k$ we have $x_i^{(1)} = 0$ if and only if $x_i^{(2)} = 0$. Indeed, choosing any $1 \le j \le k$ such that $x_j^{(1)} \neq 0$ we see from (7) that $x_i^{(1)} = 0$ implies $x_i^{(2)} = 0$; the reverse implication holds by symmetry. Now, let $1 \le i_1 < \ldots < i_\ell \le k$ be the indices of all non-zero coordinates of $x^{(1)}$ (and $x^{(2)}$). For each pair of $1 \le i, j \le k$ one of the rows of the determinant in (7) is a multiple of the other. Applying this observation consecutively for the pairs $(i_1, i_2), (i_2, i_3), \ldots, (i_{\ell-1}, i_\ell)$ we infer that $x^{(1)}$ and $x^{(2)}$ are parallel. Since they are also orthogonal, one of them should be zero which is the case we have excluded. The former equality in (6) is thus proved, and its easy consequence is the latter one.

We are now to show that $A \cap D_{ij} \in \mathfrak{I}_{\mathrm{O}(2,k)}$ for each pair of indices $i, j \in \{1, \ldots, k\}$ with $i \neq j$. So, fix any such pair and assume that $i < j$. Then for every $(x^{(1)}, x^{(2)}) \in D_{ij}$ the vectors

$$x^{(1)}, x^{(2)}, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{j-1}, e_{j+1}, \ldots, e_k$$

form a basis of $\mathbb{R}^k$. Let

$$\mathcal{B}(x^{(1)}, x^{(2)}) = \big(y_i(x^{(1)}, x^{(2)})\big)_{i=1}^k$$

be an orthonormal basis produced by the Gram-Schmidt process applied to that sequence of vectors. If $x^{(1)}$ and $x^{(2)}$ are orthogonal, then we may take

$$y_1(x^{(1)}, x^{(2)}) = \frac{x^{(1)}}{\|x^{(1)}\|} \quad \text{and} \quad y_2(x^{(1)}, x^{(2)}) = \frac{x^{(2)}}{\|x^{(2)}\|}.$$

For $(x^{(1)}, x^{(2)}) \in D_{ij}$ define $\vartheta_{x^{(1)},x^{(2)}} \colon \mathbb{R}^k \to \mathbb{R}^k$ as the mapping which to every $z \in \mathbb{R}^k$ assigns its coordinates with respect to $\mathcal{B}(x^{(1)}, x^{(2)})$, i.e.

$$\vartheta_{x^{(1)},x^{(2)}}(z) = \boldsymbol{Y}(x^{(1)}, x^{(2)})^{-1} z,$$

where

$$\boldsymbol{Y}(x^{(1)}, x^{(2)}) = \left[\frac{x^{(1)}}{\|x^{(1)}\|}, \frac{x^{(2)}}{\|x^{(2)}\|}, y_3(x^{(1)}, x^{(2)}), \ldots, y_k(x^{(1)}, x^{(2)})\right]$$

is formed from the column vectors. Obviously, every $z$ belonging to the orthogonal complement $V(x^{(1)}, x^{(2)})^\perp$ of the subspace spanned by $x^{(1)}$ and $x^{(2)}$ is mapped onto a certain vector of the form $(0, 0, t_3, \ldots, t_k)$ which may be naturally identified with an element of $\mathbb{R}^{k-2}$. Hence, we get a linear isomorphism $\gamma_{x^{(1)},x^{(2)}} \colon V(x^{(1)}, x^{(2)})^\perp \to \mathbb{R}^{k-2}$ and we may define a mapping

$$\Gamma\colon \{(x^{(1)}, \ldots, x^{(k)}) \in \mathrm{O}(k,k) \colon (x^{(1)}, x^{(2)}) \in D_{ij}\} \to \big(\mathrm{O}(2,k) \cap D_{ij}\big) \times \mathrm{O}(k-2, k-2)$$

by the formula

$$\Gamma(x^{(1)}, \ldots, x^{(k)}) = \big((x^{(1)}, x^{(2)}), (\gamma_{x^{(1)},x^{(2)}}(x^{(3)}), \ldots, \gamma_{x^{(1)},x^{(2)}}(x^{(k)}))\big).$$

The definition is well-posed, since $\vartheta_{x^{(1)},x^{(2)}}$, and hence also $\gamma_{x^{(1)},x^{(2)}}$, is an isometry for each orthogonal $(x^{(1)}, x^{(2)}) \in D_{ij}$. Moreover, it is easily seen that $\Gamma$ is a $\mathcal{C}^\infty$-diffeomorphism (the formulas of the Gram-Schmidt procedure are $\mathcal{C}^\infty$).

It easily follows from $B \in \mathfrak{I}_{\mathrm{O}(k,k)}$ that $B_{ij}$ belongs to the corresponding ideal of subsets of

$$\{(x^{(1)}, \ldots, x^{(k)}) \in \mathrm{O}(k,k) \colon (x^{(1)}, x^{(2)}) \in D_{ij}\},$$

thus $\Gamma(B_{ij})$ belongs to the ideal corresponding to $(\mathrm{O}(2,k) \cap D_{ij}) \times \mathrm{O}(k-2, k-2)$. Finally, observe that

$$\Gamma(B_{ij}) = (A \cap D_{ij}) \times \mathrm{O}(k-2, k-2),$$

which yields $A \cap D_{ij} \in \mathfrak{J}_{\mathrm{O}(2,k)\cap D_{ij}}$ and hence also $A \cap D_{ij} \in \mathfrak{J}_{\mathrm{O}(2,k)}$. □

**Lemma 10.** *If an odd function $h\colon \mathbb{R}^n \to G$ satisfies $h(x+y) = h(x) + h(y)$ $\mathfrak{J}_\perp$-(a.e.), then there is an additive function $b\colon \mathbb{R}^n \to G$ such that $h(x) = b(x)$ $\mathfrak{J}_n$-(a.e.).*

*Proof.* Due to some isometry formalities, we may suppose $\langle\cdot|\cdot\rangle$ to be the standard inner product in $\mathbb{R}^n$.
Define

$$W = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0 \text{ for some } i\}$$

and

$$S_+^{n-1} = \{x = (x_1, \dots, x_n) \in S^{n-1} : x_n > 0\}.$$

Since $S_+^{n-1}$ is an open subset of $S^{n-1}$, it is an $(n-1)$-manifold. For any $x \in S_+^{n-1}$ define

$$T_x = \{(\lambda, y) \in \mathbb{R}^* \times P_x^* : \lambda^2 = \|y\|^2\}$$

and $\Phi_x\colon (\mathbb{R}^* \times P_x^*) \setminus T_x \to \perp^*$ as

$$\Phi_x(\lambda, y) = \Big(\lambda x + y, \frac{\|y\|^2}{\lambda} x - y\Big), \tag{8}$$

and put $Q(x) = \Phi_x\big((\mathbb{R}^* \times P_x^*) \setminus T_x\big)$. We are going to show that for every $x \in P := S_+^{n-1} \setminus W$ the set $Q(x)$ forms a submanifold of $\perp^*$.

At the moment, let $x \in S_+^{n-1}$. For brevity, denote $\mu = \mu(\lambda, y) = \|y\|^2/\lambda$. It is easily seen that for each $(t, u) = (\lambda x + y, \mu x - y) \in Q(x)$ all four vectors: $t$, $u$, $x$, $y$ belong to the subspace $V(t, x)$ of $\mathbb{R}^n$ spanned by $t$ and $x$. Choose an arbitrary non-zero vector $z(t, x) \in V(t, x)$, orthogonal to $x$. Then $z(t, x)$ is collinear with $y$, hence the equality $t = \lambda x + y$ represents $t$ in terms of the basis $(x, z(t, x))$ of $V(t, x)$. Therefore, $\lambda$ and $y$ are uniquely determined by $t$, which proves that $\Phi_x$ is injective.

In order to show that $\Phi_x^{-1}$ is continuous fix an arbitrary $(t, u) \in Q(x)$. Now, put $z(t, x) = \langle t|x\rangle x - t$; then $(x, z(t, x))$ is an orthogonal basis of $V(t, x)$. Since $t = \lambda x + y$ for certain $\lambda \in \mathbb{R}^*$ and $y \in P_x^*$, we have $t = \lambda x + \alpha z(t, x)$ for some $\alpha \in \mathbb{R}$, whence we find that $\lambda = \langle t|x\rangle$ and $y = t - \langle t|x\rangle x$. We have thus shown that $\Phi_x$ is a homeomorphism.

Now, fix $x \in P$. We shall prove that $\Phi_x$ is a $C^\infty$-immersion. To this end put

$$V_x = \Big\{(\lambda, y) \in \mathbb{R}^* \times (\mathbb{R}^n)^* : \lambda = \langle x|y\rangle \pm \sqrt{\langle x|y\rangle^2 + \|y\|^2}\Big\}$$

and define a mapping $\overline{\Phi}_x\colon (\mathbb{R}^* \times (\mathbb{R}^n)^*) \setminus V_x \to \mathbb{R}^n \times \mathbb{R}^n$ by the formula analogous to (8). Then $(\mathbb{R}^* \times P_x^*) \setminus T_x$ is a submanifold of $(\mathbb{R}^* \times (\mathbb{R}^n)^*) \setminus V_x$. Let $(\lambda, y) \in \mathbb{R}^* \times (\mathbb{R}^n)^*$. If we show that the derivative $\mathrm{D}\overline{\Phi}_x(\lambda, y)$ is injective, then, in view of Lemma 1, we will be done. Since

$$\mathrm{D}\overline{\Phi}_x(\lambda, y) = \left[\begin{array}{c|cccc} x_1 & 1 & 0 & & 0 \\ x_2 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x_n & 0 & 0 & \dots & 1 \\ \hline \dfrac{\partial(\mu x - y)}{\partial\lambda} & \multicolumn{4}{c}{\dfrac{\partial(\mu x - y)}{\partial y}} \end{array}\right],$$

we immediately get that $\operatorname{rank}\mathrm{D}\overline{\Phi}_x(\lambda, y) \geq n$, where the equality occurs only if the first column vector is a linear combination of the remaining $n$ column vectors with coefficients $x_1, \dots, x_n$. However, this would imply that for each $i \in \{1, \dots, n\}$ we have

$$\frac{\partial(\mu x_i - y_i)}{\partial\lambda} = \sum_{j=1}^n x_j \frac{\partial(\mu x_i - y_i)}{\partial y_j},$$

i.e.

$$\lambda^2 - 2\langle x|y\rangle\lambda - \|y\|^2 = 0,$$

which is not the case, since $(\lambda, y) \notin V_x$. As a result, we obtain $\operatorname{rank}\mathrm{D}\overline{\Phi}_x(\lambda, y) = n + 1$, thus $\mathrm{D}\overline{\Phi}_x(\lambda, y)$ is injective.

We have shown that $\Phi_x$ is an embedding (i.e. homeomorphic $C^\infty$-immersion) of $(\mathbb{R}^* \times P_x^*) \setminus T_x$ into $\perp^*$. By virtue of [14, Theorem 11.17], its image $Q(x)$ is a submanifold of $\perp^*$.

Observe that the manifolds $Q(x)$, for $x \in P$, are $C^\infty$-diffeomorphic each to others. Indeed, by the remarks following the statement of our Theorem, for each $x \in X$ there is a $C^\infty$-diffeomorphism $\Psi_x\colon \mathbb{R}^* \times P_x^* \to \mathbb{R}^* \times (\mathbb{R}^{n-1})^*$ defined by the formula

$$\Psi_x(\lambda, y) = (\lambda, \widetilde{\psi}_x(y)), \tag{9}$$

where $\psi_x(y) = \boldsymbol{Y}(x)^{-1}y$ is defined as earlier and the tilde operator deletes the first coordinate (which equals 0 for $y \in P_x$). Moreover, $\Psi_x$ maps $(\mathbb{R}^* \times P_x^*) \setminus T_x$ onto the set

$$U := \{(\lambda, y) \in \mathbb{R}^* \times (\mathbb{R}^{n-1})^* : \lambda^2 \neq \|y\|^2\},$$

which follows from the fact that $\widetilde{\psi}_x$ is an isometry. Therefore, for each $x, y \in P$, the mapping $\Phi_y \circ \Psi_y^{-1} \circ \Psi_x \circ \Phi_x^{-1}$ yields a $C^\infty$-diffeomorphism between $Q(x)$ and $Q(y)$. So, we pick any $x_0 \in P$ and we regard the set $Q := Q(x_0)$ as a "model" manifold for all $Q(x)$'s.

Define

$$\perp^{(1)} = \{(t, u) \in \perp^* : t_n + u_n \neq 0 \text{ and } \|t\| \neq \|u\|\}$$

(which is an open subset, and hence - a submanifold, of $\perp^*$) and observe that

$$\perp^{(1)} = \bigcup_{x \in S_+^{n-1}} Q(x). \tag{10}$$

In fact, for any $(t.u) \in \perp^{(1)}$ put

$$x = \pm \frac{t + u}{\|t + u\|}, \tag{11}$$

where the sign is the same as the sign of $t_n + u_n$. Then $x \in S_+^{n-1}$ and $(t, u) \in Q(x)$. Indeed, if we choose any $y_0 \in P_x^* \cap V(t, u)$ with $\|y_0\| = 1$ (which is unique up to a sign), then $t$ and $u$ are represented in terms of the basis $(x, y_0)$ of $V(t, u)$ as follows:

$$t = \langle t|x\rangle x + \langle t|y_0\rangle y_0 \quad \text{and} \quad u = \langle u|x\rangle x + \langle u|y_0\rangle y_0,$$

and we have

$$\langle t|y_0\rangle = \langle t + u|y_0\rangle - \langle u|y_0\rangle = \pm\|t + u\|\langle x|y_0\rangle - \langle u|y_0\rangle = -\langle u|y_0\rangle.$$

Hence, after substitution $\lambda = \langle t|x\rangle$ and $y = \langle t|y_0\rangle y_0$, we obtain $t = \lambda x + y$ and $u = \langle u|x\rangle x - y$. The coefficient $\langle u|x\rangle$ equals $\|y\|^2/\lambda$, since $\langle t|u\rangle = \langle x|y\rangle = 0$. Moreover, $\lambda \neq 0$, $y_0 \neq 0$, and it follows from $\|t\| \neq \|u\|$ that $\lambda^2 \neq \langle u|x\rangle^2 = \|y\|^4/\lambda^2$, which gives $\lambda^2 \neq \|y\|^2$. Consequently, $(t, u) \in Q(x)$ and thus we have proved the inclusion "$\subseteq$". The reverse inclusion is a straightforward calculation.

We shall now prove that the mapping $\Lambda\colon S_+^{n-1} \times U \to \perp^{(1)}$ defined by

$$\Lambda(x, \lambda, y) = \Phi_x \circ \Psi_x^{-1}(\lambda, y)$$

is a $C^\infty$-diffeomorphism.

First, in view of (10), it is easily seen that the image of $\Lambda$ is $\perp^{(1)}$. According to the definition, $\Lambda$ is $C^\infty$. Moreover, for each $(t, u) = \Phi_x(\lambda, \widetilde{\psi}_x^{-1}(y)) \in Q(x)$ we have

$$\left(\lambda + \frac{\|\widetilde{\psi}_x^{-1}(y)\|}{\lambda}\right)x = t + u, \tag{12}$$

which, jointly with the fact that $x \in S_+^{n-1}$, uniquely determines $x$. By the injectivity of $\Phi_x$, we infer that $\lambda$ and $y$ are then uniquely determined by $t$ and $u$ as well. Therefore, $\Lambda$ is injective.

In order to get a formula for $\Lambda^{-1}$, observe that for each $(t, u) = \Phi_x(\lambda, \widetilde{\psi}_x^{-1}(y)) \in \perp^{(1)}$ equality (12) yields (11). This means that $x$ is expressed as a function of $t$ and $u$, which is $C^\infty$ on both components of the set $\perp^{(1)}$. By the formula for $\Phi_x^{-1}$, we get

$$\lambda = \pm\frac{\langle t|t + u\rangle}{\|t + u\|} \quad \text{and} \quad y = \widetilde{\psi}_x\left(t - \frac{\langle t|t + u\rangle}{\|t + u\|^2}(t + u)\right),$$

and since the value of $\widetilde{\psi}_x$ at a given point is a $C^\infty$ function of $x$, we infer that $\Lambda^{-1}$ is $C^\infty$. Consequently, $\Lambda$ is a $C^\infty$-diffeomorphism.

Let $\chi\colon \perp^{(1)} \to S_+^{n-1} \times Q$ be given by

$$\chi = (\mathrm{id}_{S_+^{n-1}} \times \Phi_{x_0}) \circ (\mathrm{id}_{S_+^{n-1}} \times \Psi_{x_0}^{-1}) \circ \Lambda^{-1};$$

then $\chi$ is a $\mathcal{C}^\infty$-diffeomorphism. Since $Z(h) \in \mathfrak{I}_\perp$ and $\perp^{(1)}$ is an open subset of $\perp^*$, we have $Z(h)\cap\perp^{(1)} \in \mathfrak{I}_{\perp^{(1)}}$. Therefore,

$$\{x \in S_+^{n-1} : \chi(Z(h) \cap \perp^{(1)})[x] \notin \mathfrak{I}_Q\} \in \mathfrak{I}_{S_+^{n-1}}. \tag{13}$$

By the definition of $\chi$, for each $x \in S_+^{n-1}$ we have

$$\chi(Z(h) \cap \perp^{(1)})[x] = \{q \in Q : \Lambda(x, (\Psi_{x_0} \circ \Phi_{x_0}^{-1})(q)) \in Z(h)\}.$$

If additionally $x \in P$, then

$$\begin{aligned}
&\chi(Z(h) \cap \perp^{(1)})[x] \notin \mathfrak{I}_Q \\
&\quad \Leftrightarrow \{(\lambda, y) \in \mathbb{R}^* \times (\mathbb{R}^{n-1})^* : \Lambda(x, \lambda, y) \in Z(h)\} \notin \mathfrak{I}_n \\
&\quad \Leftrightarrow \{(\lambda, y) \in \mathbb{R}^* \times (\mathbb{R}^{n-1})^* : \Phi_x(\lambda, \widetilde{\psi}_x^{-1}(y)) \in Z(h)\} \notin \mathfrak{I}_n \\
&\quad \Leftrightarrow \{(\lambda, y) \in \mathbb{R}^* \times P_x^* : \Phi_x(\lambda, y) \in Z(h)\} \notin \mathfrak{I}_{\mathbb{R}^* \times P_x^*} \\
&\quad \Leftrightarrow Z(h) \cap Q(x) \notin \mathfrak{I}_{Q(x)}.
\end{aligned}$$

Thus (13) gives

$$\{x \in P : Z(h) \cap Q(x) \notin \mathfrak{I}_{Q(x)}\} \in \mathfrak{I}_{S_+^{n-1}}.$$

Since $S_+^{n-1} \setminus P \in \mathfrak{I}_{S_+^{n-1}}$, we have also

$$Z(h) \cap Q(x) \in \mathfrak{I}_{Q(x)} \quad \mathfrak{I}_{S_+^{n-1}}\text{-(a.e.)}. \tag{14}$$

For any $x \in S_+^{n-1}$ define $\Gamma_x\colon \mathbb{R}^* \times P_x^* \to \perp^*$ and $\Theta_x\colon \mathbb{R}^* \times P_x^* \to \perp^*$ as

$$\Gamma_x(\lambda, y) = \Big(\frac{\|y\|^2}{\lambda} x, -y\Big) \quad \text{and} \quad \Theta_x(\lambda, y) = (\lambda x, y),$$

and put $R(x) = \Gamma_x(\mathbb{R}^* \times P_x^*)$, $S(x) = \Theta_x(\mathbb{R}^* \times P_x^*)$. An argument similar to the one above shows that $R(x)$, for $x \in S_+^{n-1}$, are submanifolds of $\perp^*$, $\mathcal{C}^\infty$-diffeomorphic each to others, and the same is true for $S(x)$'s. Moreover, the set

$$\perp^{(2)} := \{(t, u) \in \perp^* : t_n \neq 0 \text{ and } u \neq 0\} = \bigcup_{x \in S_+^{n-1}} R(x) = \bigcup_{x \in S_+^{n-1}} S(x)$$

is $\mathcal{C}^\infty$-diffeomorphic to $S_+^{n-1} \times R$ and $S_+^{n-1} \times S$, where $R$ and $S$ are "model" manifolds for all $R(x)$'s and for all $S(x)$'s, respectively. Arguing further, analogously as above, we also infer that

$$Z(h) \cap R(x) \in \mathfrak{I}_{R(x)} \quad \text{and} \quad Z(h) \cap S(x) \in \mathfrak{I}_{S(x)} \quad \mathfrak{I}_{S_+^{n-1}}\text{-(a.e.)}. \tag{15}$$

According to (14) and (15) there is a set $S_0 \in \mathfrak{I}_{S_+^{n-1}}$ with

$$\begin{cases} Z(h) \cap Q(x) \in \mathfrak{I}_{Q(x)}, \\ Z(h) \cap R(x) \in \mathfrak{I}_{R(x)}, \\ Z(h) \cap S(x) \in \mathfrak{I}_{S(x)} \end{cases} \tag{16}$$

for $x \in S_+^{n-1} \setminus S_0$.

At the moment, assume that $n = 2$. Applying Lemma 7 to the set

$$A := S_0 \cup (-S_0) \cup \{(-1, 0), (1, 0)\} \in \mathfrak{I}_{S^1},$$

and changing signs of vectors of the obtained basis as required, we get an orthogonal basis $(x^{(1)}, x^{(2)})$ of $\mathbb{R}^2$ whose each element $x$ satisfies conditions (16).

Now, we shall prove that for each $i \in \{1, 2\}$ the function $h_i\colon \mathbb{R} \to G$ given by $h_i(\lambda) = h(\lambda x^{(i)})$ satisfies

$$h_i(\lambda + \mu) = h_i(\lambda) + h_i(\mu) \quad \Omega(\mathfrak{I}_{(0,\infty)})\text{-(a.e.)}, \tag{17}$$

where $\Omega(\mathfrak{I}_{(0,\infty)}) = \{A \subset (0,\infty)^2 : A[x] \in \mathfrak{I}_{(0,\infty)}\ \mathfrak{I}_{(0,\infty)}\text{-(a.e.)}\}$ is the so called conjugate ideal. Plainly, condition (17) would imply that the same is true with $(0,\infty)$ replaced by $(-\infty,0)$, due to the oddness of the function $h$.

Fix $i \in \{1,2\}$. In view of (16), with $x$ replaced by $x^{(i)}$, there is a set $C_i \in \mathfrak{I}_{\mathbb{R}^* \times P^*_{x^{(i)}}}$ such that

$$(18)\qquad \begin{cases} \left(\lambda x^{(i)} + y, \dfrac{\|y\|^2}{\lambda} x^{(i)} - y\right) \in \perp^* \setminus Z(h), \\ \left(\dfrac{\|y\|^2}{\lambda} x^{(i)}, -y\right) \in \perp^* \setminus Z(h), \\ \left(\lambda x^{(i)}, y\right) \in \perp^* \setminus Z(h) \end{cases}$$

for $(\lambda, y) \in (\mathbb{R}^* \times P^*_{x^{(i)}}) \setminus C_i$ (note that $T_{x^{(i)}} \in \mathfrak{I}_{\mathbb{R}^* \times P^*_{x^{(i)}}}$, so we may include the set $T_{x^{(i)}}$ into $C_i$ and we see that the difference between the domain of $\Phi_{x^{(i)}}$ and the domains of $\Gamma_{x^{(i)}}$, $\Theta_{x^{(i)}}$ causes no trouble at all). Therefore, for all $\lambda \in \mathbb{R}$ except a set $\Lambda_i \in \mathfrak{I}_1$ the conjunction (18) holds true for all $y \in P_{x^{(i)}} \setminus Y_i(\lambda)$ with $Y_i(\lambda) \in \mathfrak{I}_{P_{x^{(i)}}}$. Let

$$B_i(\lambda) = \left\{ \frac{\|y\|^2}{\lambda} : y \in P_{x^{(i)}} \setminus Y_i(\lambda) \right\}.$$

Then, obviously, $\mathbb{R} \setminus B_i(\lambda) \in \mathfrak{I}_{(0,\infty)}$ for each positive $\lambda \notin \Lambda_i$, whereas $\mathbb{R} \setminus B_i(\lambda) \in \mathfrak{I}_{(-\infty,0)}$ for each negative $\lambda \notin \Lambda_i$. For every pair $(\lambda, \mu)$ with $\lambda \notin \Lambda_i$ and $\mu \in B_i(\lambda)$, $\mu = \frac{\|y\|^2}{\lambda}$, we have

$$\begin{aligned} h_i(\lambda + \mu) &= h\left(\lambda x^{(i)} + y + \frac{\|y\|^2}{\lambda} x^{(i)} - y\right) = h(\lambda x^{(i)} + y) + h\left(\frac{\|y\|^2}{\lambda} x^{(i)} - y\right) \\ &= h(\lambda x^{(i)}) + h(y) + h\left(\frac{\|y\|^2}{\lambda} x^{(i)}\right) + h(-y) = h_i(\lambda) + h_i(\mu), \end{aligned}$$

which proves (17). Applying the theorem of de Bruijn [3] separately to the functions $h_i|_{(0,\infty)}$ and $h_i|_{(-\infty,0)}$ we get two additive mappings $b_i' \colon (0,\infty) \to G$ and $b_i'' \colon (-\infty, 0) \to G$ which coincide with these two restrictions of $h_i$ almost everywhere in $(0,\infty)$ and $(-\infty,0)$, respectively. However, since $h$ is odd, the extensions of both $b_i'$ and $b_i''$ to the whole real line have to be the same. As a result, there is an additive function $b_i \colon \mathbb{R} \to G$ such that $h_i(\lambda) = b_i(\lambda)$ for $\lambda \in \mathbb{R} \setminus Z_i$ with a certain $Z_i \in \mathfrak{I}_1$.

Define a function $b \colon \mathbb{R}^2 \to G$ by $b(x) = b_1(\lambda_1) + b_2(\lambda_2)$, where $\lambda_i$ is the $i$th coordinate of $x$ with respect to the basis $(x^{(1)}, x^{(2)})$. Plainly, $b$ is an additive function. It remains to show that $h(x) = b(x)$ $\mathfrak{I}_2$-(a.e.).

Recall that for every $x \in X = \mathbb{R} \times \mathbb{R}^*$ the mapping $\Psi_x$ defined by (9) yields a $C^\infty$-diffeomorphism between $\mathbb{R}^* \times P_x^*$ and $\mathbb{R}^* \times \mathbb{R}^*$. In particular, we have $C := \Psi_{x^{(1)}}(C_1) \in \mathfrak{I}_2$ and

$$(19)\qquad (\lambda x^{(1)}, \widetilde{\psi}^{-1}_{x^{(1)}}(y)) \in \perp^* \setminus Z(h) \quad \text{for } (\lambda, y) \in \mathbb{R}^2 \setminus C.$$

Define $\Delta \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Delta(\lambda_1, \lambda_2) = \left(\lambda_1, \widetilde{\psi}_{x^{(1)}}(\lambda_2 x^{(2)})\right).$$

Plainly, $\Delta$ is a $C^\infty$-diffeomorphism, so $\Delta^{-1}(C) \in \mathfrak{I}_2$. Therefore,

$$\Delta^{-1}(C) \cup (Z_1 \times \mathbb{R}) \cup (\mathbb{R} \times Z_2) \in \mathfrak{I}_2$$

and for each pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ outside this set condition (19) implies $(\lambda_1 x^{(1)}, \lambda_2 x^{(2)}) \in \perp^* \setminus Z(h)$, thus

$$\begin{aligned} h(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) &= h(\lambda_1 x^{(1)}) + h(\lambda_1 x^{(1)}) = h_1(\lambda_1) + h_2(\lambda_2) \\ &= b_1(\lambda_1) + b_2(\lambda_2) = b(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}). \end{aligned}$$

By the isomorphism, which to every $x \in \mathbb{R}^2$ assigns its coordinates in the basis $(x^{(1)}, x^{(2)})$, we have $h(x) = b(x)$ $\mathfrak{I}_2$-(a.e.) and our assertion for $n = 2$ follows.

In the sequel, assume that $n \geq 3$ and the assertion holds true for $n-1$ in the place of $n$.

Define $\mathrm{O}(n-1,n)'$ to be the set of all $(n-1)$-tuples from $\mathrm{O}(n-1,n)$ generating a subspace of $\mathbb{R}^n$ whose orthogonal complement is spanned by a vector $(x_1, \ldots, x_n)$ with $x_n \neq 0$. In other words,

$$\mathrm{O}(n-1,n)' = \left\{ (x^{(1)}, \ldots, x^{(n-1)}) \in \mathrm{O}(n-1,n) : \pm \frac{x^{(1)} \wedge \ldots \wedge x^{(n-1)}}{\|x^{(1)} \wedge \ldots \wedge x^{(n-1)}\|} \in S_+^{n-1} \right\}.$$

This set, being an open subset of $\mathrm{O}(n-1,n)$, is its submanifold having the same dimension. Consider the mapping $\Omega\colon S^{n-1}_+ \times \mathrm{O}(n-1,n-1) \to \mathrm{O}(n-1,n)'$ defined by

$$\Omega(x,x^{(1)},\dots,x^{(n-1)}) = \big(\widetilde{\psi}_x^{-1}(x^{(1)}),\dots,\widetilde{\psi}_x^{-1}(x^{(n-1)})\big).$$

The values of $\Omega$ indeed belong to $\mathrm{O}(n-1,n)'$, since for each $x \in X$ the function $\psi_x$ is an isometry, being a linear map determined by the orthogonal matrix $\boldsymbol{Y}(x)^{-1}$. Furthermore, $\Omega$ is bijective with the inverse $\Omega^{-1}$ given by

$$\Omega^{-1}(y^{(1)},\dots,y^{(n-1)}) = \big(x,\widetilde{\psi}_x(y^{(1)}),\dots,\widetilde{\psi}_x(y^{(n-1)})\big),$$

where

$$x = \pm\frac{y^{(1)}\wedge\ldots\wedge y^{(n-1)}}{\|y^{(1)}\wedge\ldots\wedge y^{(n-1)}\|}$$

and the sign depends on which of the two components of $\mathrm{O}(n-1,n)'$ contains $(y^{(1)},\dots,y^{(n-1)})$. By the above formulas, $\Omega$ is a $\mathcal{C}^\infty$-diffeomorphism.

Put

$$Z = \{(y^{(1)},\dots,y^{(n-1)}) \in \mathrm{O}(n-1,n)' : (y^{(1)},y^{(2)}) \in Z(h)\}.$$

Then Lemma 5 implies $Z \in \mathfrak{I}_{\mathrm{O}(n-1,n)'}$, since $Z(h) \in \mathfrak{I}_\perp$ (i.e. $Z(h) \in \mathfrak{I}_{\mathrm{O}(2,n)}$) is the image of $Z$ through the $\mathcal{C}^\infty$-submersion $(y^{(1)},\dots,y^{(n-1)}) \mapsto (y^{(1)},y^{(2)})$. Therefore, we have $\Omega^{-1}(Z) \in \mathfrak{I}_{S^{n-1}_+\times \mathrm{O}(n-1,n-1)}$, hence $\Omega^{-1}(Z)[x] \in \mathfrak{I}_{\mathrm{O}(n-1,n-1)}$ is valid $\mathfrak{I}_{S^{n-1}_+}$-(a.e.), which translates into the fact that the set

$$A(x) := \big\{(x^{(1)},\dots,x^{(n-1)}) \in \mathrm{O}(n-1,n-1) : \big(\widetilde{\psi}_x^{-1}(x^{(1)}),\widetilde{\psi}_x^{-1}(x^{(1)})\big) \in Z(h)\big\}$$

belongs to $\mathfrak{I}_{\mathrm{O}(n-1,n-1)}$ for every $x \in S^{n-1}_+$ except a set from $\mathfrak{I}_{S^{n-1}_+}$. By virtue of Lemma 9, for each such $x$ we must have

$$\big\{(x^{(1)},x^{(2)}) \in \mathrm{O}(2,n-1) : \big(\widetilde{\psi}_x^{-1}(x^{(1)}),\widetilde{\psi}_x^{-1}(x^{(2)})\big) \in Z(h)\big\} \in \mathfrak{I}_{\mathrm{O}(2,n-1)}. \tag{20}$$

Hence, putting $\perp_x = \{(t,u) \in P_x \times P_x : (t,u) \in \perp\}$ we infer that the condition

$$h(t+u) = h(t) + h(u) \quad \mathfrak{I}_{\perp_x^*}\text{-(a.e.)} \tag{21}$$

is valid $\mathfrak{I}_{S^{n-1}_+}$-(a.e.). Consequently, we may pick a particular $x \in S^{n-1}_+$ satisfying both (16) and (21). By virtue of our inductive hypothesis and some isometry formalities (identifying $P_x$ with $\mathbb{R}^{n-1}$), condition (21) yields the existence of an additive function $b_x\colon P_x \to G$ such that $h(t) = b_x(t)$ for $t \in P_x \setminus Y$ with a certain $Y \in \mathfrak{I}_{P_x}$. Moreover, by an earlier argument, there is also an additive function $b_1\colon \mathbb{R} \to G$ such that $h(\lambda x) = b_1(\lambda)$ for $\lambda \in \mathbb{R}\setminus Z_1$ with a certain $Z_1 \in \mathfrak{I}_1$. Finally, there is a set $C_1 \in \mathfrak{I}_{\mathbb{R}\times P_x}$ with $(\lambda x, y) \in \perp^* \setminus Z(h)$ whenever $(\lambda,y) \in (\mathbb{R}\times P_x)\setminus C_1$.

Define a function $b\colon \mathbb{R}^n \to G$ by the formula $b(\lambda x + y) = b_1(\lambda) + b_x(y)$ for $\lambda \in \mathbb{R}$ and $y \in P_x$. Then $b$ is additive and for each pair $(\lambda,y) \in \mathbb{R}\times P_x$ outside the set

$$C_1 \cup (Z_1 \times P_x) \cup (\mathbb{R}\times Y) \in \mathfrak{I}_{\mathbb{R}\times P_x}$$

we have

$$h(\lambda x + y) = h(\lambda x) + h(y) = b_1(\lambda) + b_x(y) = b(\lambda x + y),$$

which completes the proof. □

**Lemma 11.** *If a function $h\colon \mathbb{R}^n \to G$ satisfies $h(x) = h(-x)$ $\mathfrak{I}_n$-(a.e.) and $h(x+y) = h(x) + h(y)$ $\mathfrak{I}_\perp$-(a.e.), then there is an additive function $a\colon \mathbb{R}\to G$ such that $h(x) = a\big(\|x\|^2\big)$ $\mathfrak{I}_n$-(a.e.).*

*Proof.* For any $r \geq 0$ let $S^{n-1}(r) = \{x \in \mathbb{R}^n : \|x\| = r\}$. By the natural identification, we have $(\mathbb{R}^n)^* \sim (0,\infty)\times S^{n-1}$. Therefore, for every $A \in \mathfrak{I}_n$ there is a set $R(A) \in \mathfrak{I}_{(0,\infty)}$ such that $A \cap S^{n-1}(r) \in \mathfrak{I}_{S^{n-1}(r)}$ for $r \in (0,\infty)\setminus R(A)$. In the first part of the proof we will show the following claim: there exists a set $A \in \mathfrak{I}_n$ such that for each $r \in (0,\infty)\setminus R(A)$ the function $h$ is constant $\mathfrak{I}_{S^{n-1}(r)}$-(a.e.) on $S^{n-1}(r)$, more precisely – that $h|_{S^{n-1}(r)}$ is constant outside the set $A \cap S^{n-1}(r)$.

We start with the following observation: there is $T \in \mathfrak{I}_\perp$ such that $h(t+u) = h(u-t)$ whenever $(t,u) \in \perp^* \setminus T$. Let $E = \{x \in \mathbb{R}^n : h(x) = h(-x)\}$ and $H = (-D(h)) \cap D(h) \cap E$; then $\mathbb{R}^n \setminus H \in \mathfrak{I}_n$. Define

$$T = \{(t,u) \in \perp^* : t \notin H\} \cup \{(t,u) \in \perp^* : t \in H \text{ and } u \notin E_t(h) \cap E_{-t}(h)\}. \tag{22}$$

Then for every $(t,u) \in \perp^* \setminus T$ we have $h(t+u) = h(t) + h(u)$ and $h(u-t) = h(u) + h(-t)$. Moreover, we have also $h(t) = h(-t)$, hence $h(t+u) = h(u-t)$, as desired. In order to show that $T \in \mathfrak{I}_\perp$ note that it is equivalent to $T \cap \perp' \in \mathfrak{I}_{\perp'}$, where $\perp'$ may be identified with $X \times \mathbb{R}^{n-1}$. The first summand in (22), after intersecting with $\perp'$, is then identified with $(X \setminus H) \times \mathbb{R}^{n-1} \in \mathfrak{I}_{2n-1}$, whereas for each pair $(t,u)$ from the second summand we have either $(t,u) \in Z(h)$, or $(-t,u) \in Z(h)$, which shows that it belongs to $\mathfrak{I}_\perp$. Consequently, $T \in \mathfrak{I}_\perp$.

Define $\Phi\colon \perp^* \to \mathbb{R}^n \times \mathbb{R}^n$ by putting $\Phi(t,u) = (t+u, u-t)$. It is evident that $\Phi$ is a $C^\infty$-immersion and yields a homeomorphism between $\perp^*$ and

$$M := \Phi(\perp^*) = \bigcup_{r \in (0,\infty)} (S^{n-1}(r) \times S^{n-1}(r)).$$

Therefore, [14, Theorem 11.17] implies that $M$ is a manifold. Moreover, $\Phi\colon \perp^* \to M$ is a $C^\infty$-diffeomorphism, thus $\Phi(T) \in \mathfrak{I}_M$. Since the mapping $(x,y) \mapsto (x, y/\|x\|)$ yields $M \sim (\mathbb{R}^n)^* \times S^{n-1}$, there exists a set $A \in \mathfrak{I}_n$ such that for every $x \in \mathbb{R}^n \setminus A$ we have

$$(x,y) \notin \Phi(T) \quad \mathfrak{I}_{S^{n-1}(\|x\|)}\text{-(a.e.)}.$$

By the property of the set $T$, $(x,y) \notin \Phi(T)$ implies $h(x) = h(y)$. Now, for any $r \in (0,\infty) \setminus R(A)$ and for arbitrary $x, y \in \mathbb{R}^n \setminus A$ with $\|x\| = \|y\| = r$, we have

$$(x,z), (y,z) \notin \Phi(T) \quad \mathfrak{I}_{S^{n-1}(r)}\text{-(a.e.)},$$

hence $h(x) = h(z) = h(y)$, which completes the proof of our claim.

There is a function $g\colon \mathbb{R}^n \to G$ which is constant on every sphere $S^{n-1}(r)$ and such that $h(x) = g(x)$ for $x \in \mathbb{R}^n \setminus A$. Therefore, there is also a function $\varphi\colon [0,\infty) \to G$ satisfying $g(x) = \varphi\left(\|x\|^2\right)$ for every $x \in \mathbb{R}^n$. We are going to show that

$$\varphi(\lambda + \mu) = \varphi(\lambda) + \varphi(\mu) \quad \Omega(\mathfrak{I}_{(0,\infty)})\text{-(a.e.)}. \tag{23}$$

Put

$$B = \{(x,y) \in \perp^* : \text{ either } x \in A, \text{ or } y \in A, \text{ or } x+y \in A\}$$

and observe that $B \in \mathfrak{I}_\perp$, whence also $Z := Z(h) \cup B \in \mathfrak{I}_\perp$. Let

$$D = \{x \in (\mathbb{R}^n)^* : (x,y) \notin Z \quad \mathfrak{I}_{P_x}\text{-(a.e.)}\}.$$

By an argument similar to the one applied to $D(h)$, we infer that $X \setminus D \in \mathfrak{I}_X$, hence $\mathbb{R}^n \setminus D \in \mathfrak{I}_n$. For each $x \in \mathbb{R}^n$ put $E_x = \{y \in P_x : (x,y) \notin Z\}$; then $P_x \setminus E_x \in \mathfrak{I}_{P_x}$ provided $x \in D$. Let also $D' = \{\|x\|^2 : x \in D\}$; then $(0,\infty) \setminus D' \in \mathfrak{I}_{(0,\infty)}$.

Fix arbitrarily $\lambda \in D'$ and choose any $x \in D$ satisfying $\sqrt{\lambda} = \|x\|$. Put $E(\lambda) = \{\|y\|^2 : y \in E_x\}$ (then $(0,\infty) \setminus E(\lambda) \in \mathfrak{I}_{(0,\infty)}$) and pick any $\mu \in E(\lambda)$. Then $\sqrt{\mu} = \|y\|$ for some $y \in E_x$, which implies $(x,y) \notin Z$. Applying the facts that $x + y \notin A$, $(x,y) \notin Z(h)$, $x \notin A$ and $y \notin A$, consecutively, we obtain

$$\begin{aligned}\varphi(\lambda+\mu) &= g(x+y) = h(x+y) \\ &= h(x) + h(y) = g(x) + g(y) = \varphi(\lambda) + \varphi(\mu),\end{aligned}$$

which proves (23).

By the theorem of de Bruijn, there is an additive function $a\colon \mathbb{R} \to G$ such that $\varphi(\lambda) = a(\lambda)$ for $\lambda \in [0,\infty) \setminus Y$ with $Y \in \mathfrak{I}_{[0,\infty)}$. Then the equality $h(x) = a\left(\|x\|^2\right)$ holds true for $x \in \mathbb{R}^n \setminus (A \cup C)$, where $C = \{x \in \mathbb{R}^n : \|x\|^2 \in Y\} \in \mathfrak{I}_n$. Thus, the proof has been completed. □

To finish the proof of our Theorem we shall combine Lemmas 8, 10 and 11 to get additive functions $a\colon \mathbb{R} \to G$ and $b\colon \mathbb{R}^n \to G$ such that

$$2\left(f(x) - a(\|x\|^2) - b(x)\right) = 0 \quad \mathfrak{I}_n\text{-(a.e.)}.$$

The only thing left to be proved is the following fact in the spirit of [2, Lemma 2].

**Lemma 12.** *If a function* $h\colon \mathbb{R}^n \to G$ *satisfies* $2h(x) = 0$ $\mathfrak{I}_n$-(a.e.) *and* $h(x+y) = h(x)+h(y)$ $\mathfrak{I}_\perp$-(a.e.), *then* $h(x) = 0$ $\mathfrak{I}_n$-(a.e.).

*Proof.* For every $x \in \mathbb{R}^n$ put $g(x) = h(x) - h(-x)$. Applying Lemmas 8 and 10 we get an additive function $b\colon \mathbb{R}^n \to G$ such that $g(x) = b(x)$ $\mathfrak{I}_n$-(a.e.). Therefore

$$g(x) = 2b\left(\frac{x}{2}\right) = 2h\left(\frac{x}{2}\right) - 2h\left(-\frac{x}{2}\right) = 0 \quad \mathfrak{I}_n\text{-(a.e.)},$$

i.e. $h(x) = h(-x)$ $\mathfrak{I}_n$-(a.e.). Now, by virtue of Lemma 11, there is an additive function $a\colon \mathbb{R} \to G$ satisfying $h(x) = a(\|x\|^2)$ $\mathfrak{I}_n$-(a.e.). Consequently,

$$h(x) = a\left(2\left\|\frac{1}{\sqrt{2}}x\right\|^2\right) = 2a\left(\left\|\frac{1}{\sqrt{2}}x\right\|^2\right) = 2h\left(\frac{1}{\sqrt{2}}x\right) = 0 \quad \mathfrak{I}_n\text{-(a.e.)}.$$

□

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# OŚWIADCZENIE

o indywidualnym wkładzie współautora
w powstanie artykułu

*Almost orthogonally additive functions*,
wysłanego do recenzji.

Po sformułowaniu ogólnego pytania badawczego o to, czy funkcja „prawie wszędzie" (wtedy jeszcze w niesprecyzowanym sensie) ortogonalnie addytywna musi być równa „prawie wszędzie" funkcji ortogonalnie addytywnej, pani mgr W. Wyrobek-Kochanek przeprowadziła szereg rozważań, niektóre natury heurystycznej, które stały się dla mnie cenną wskazówką do wprowadzenia definicji ideału na rozmaitości różniczkowej i formalnego sprecyzowania stosownej hipotezy.

Lematy 2–6, będące przygotowaniem do głównego twierdzenia artykułu, są wynikiem naszej wspólnej pracy z panią mgr W. Wyrobek–Kochanek.

Mojego autorstwa są dowody lematów 7 i 9. Lematy 10 i 11 są wynikiem wspólnych prac. Istotną rolę w ich dowodach odegrało kilka pomysłów mgr W. Wyrobek-Kochanek, np. zaproponowała ona próbę przeniesienia pewnego fragmentu rozumowania Jurga Rätza z pracy *On orthogonally additive mappings*, Aequationes Math. 28 (1985), 35–49. Widoczne jest to (po głębszej analizie) w warunku $Z(h) \cap Q(x) \in \mathfrak{J}_{Q(x)}$ (patrz: dowód lematu 10). Jest to jeden z wielu technicznych szczegółów, ale istotny.

(-) Tomasz Kochanek