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General description of neutrino oscillations with non-standard interactions.

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Abstract:

We present a formalism describing neutrino oscillations in any application beyond the Standard Model theory. Instead of using the standard pure states approach, we apply the density matrix formalism. In general, in New Physics models, neutrino states are no longer as pure as they are in the Standard Model. We discuss the details of the appearance of a mixed state, following which possible New Physics effects are taken at the levels of both the production and detection processes. We present a number of examples of calculations with our formalism, using muons as a source of neutrinos and different detection process. We also show the connection between normal formulae, derived by assuming pure states, and proper results based on the density matrix approach. The difference occurs at the second order in parameters describing the departure from the Standard Model. Finally, as an application of our formalism, we also demonstrate that it is possible to distinguish Dirac and Majorana neutrinos in oscillations when New Physics scalar interactions are present.

Contents

List of Abbreviations

CM	Center of Mass frame
DM	Density Matrix
LAB	Laboratory frame
MNSP	Maki, Nakagawa, Sakata, Pontecorvo mixing matrix
NMPDF	Neutrino Momentum Probability Density Function
NP	New Physics
NSI	Non-Standard Interactions
QFT	Quantum Field Theory
QM	Quantum Mechanical
SM	Standard Model
SVD	Singular Value Decomposition

CHAPTER 1

Introduction

Neutrinos are considered the most mysterious particles of the Standard Model (SM). Their existence was first proposed by Pauli [1] in order to save conservation lows, but a major problem appeared in the electron spectrum observed in neutron beta decay. If it was two-body decay, then energy and momentum conservation lows required the electron to have one specific value of energy and not a continuous spectrum, as was observed. Following the introduction of the neutrino, the decay was now considered three-body decay, and the predicted electron spectrum started to be in agreement with experiments. The fact that neutrinos have spin $1/2$ follows from the angular momentum conservation.

When Pauli postulated the existence of neutrinos, it seemed that it would be impossible ever to detect them. Even today, they are very hard to detect because they are the only fermions with no electric charge; moreover, their tiny mass also distinguishes them from other particles whose masses are usually many orders of magnitude greater. That is why for many years neutrinos were assumed to be massless. In principle, SM is a consistent, well-defined theory with or without neutrino masses, but massive neutrinos have very a interesting phenomenology; for example, they can oscillate – as proposed by Pontecorvo [2] – or CP violation can occur in a lepton sector of the SM. Neutrino oscillations were considered for a long time not only the best place to confirm that neutrinos have non-zero masses (even today, oscillations are the only experimental results that indicate non-zero neutrino masses), but also as a way to test theoretical models. Of course, the data obtained in neutrino oscillation experiments are helpful for constraining neutrino mass models [3], although models of new interactions, astrophysical models and many others can also be tested using neutrino data. One of the most important of these examples is the Standard Solar Model, the predictions of which are in agreement with experimental data, but only after taking into account the neutrino oscillation effect in solar matter [4, 5]. Neutrino oscillations are the result of a misalignment of flavour and mass bases in which neutrino fields are written. Those two bases are related by a unitary rotation, which leads to the mixing matrix proposed by Z. Maki, M. Nakagawa, S. Sakata [6] and Pontecorvo [2] (MNSP). This mechanism is similar to well-known quark mixing.

Physicists are still searching for 'New Physics' (NP) effects that may show up in neutrino oscillation. Although SM is now in a perfect agreement with all the data[1] [10], there are a number of reasons for believing that SM is only a low energy effective theory of a more fundamental model [11]. What is usually considered the reason to look for an NP effect is a hierarchy problem, the smallness of neutrino masses, dark matter, dark energy, the unification of gravity with other forces, matter-antimatter asymmetry or a strong CP problem [12].

Two basic strategies are employed to search for deviations from the SM, i.e. NP effects can be searched for by building colliders that operate at higher energies or by increasing the precision involved in determining low energy observables. There is also another possibility connected with neutrinos, namely effects that accumulate across distance such as NP effects in neutrino matter oscillations. However, in order to understand the precise neutrino experiments, not only is a theory of matter oscillations required, but also the effects of NP in production and detection processes need to be taken into account [13, 14, 15]. Today, after more than 50 years since the discovery of neutrinos [16], as physicists we have now entered an era of precision neutrino experiments. It is therefore of great importance to understand all the possible effects connected with neutrino production and detection that may affect experimental results.

The aim of this work is to present a formalism which is valid for a large class of NP models. We assume only that the model is a Quantum Field Theory (QFT), which enables us to calculate an amplitude for the production process. Starting from the basic principles of Quantum Mechanics (QM), we try to construct a framework that enables us to calculate the effect of an NP model on production states. We use a density matrix (DM) formalism [17, 18], which is far more appropriate than usual approaches based on effective QM pure states [19]. The DM approach is valid in a large class of models, while using a pure state entails neglecting the entanglement of neutrinos and other particles that interact with it during the production process. Many different NP models lead to new types of neutrino interactions, which is commonly called 'neutrino Non Standard Interaction (NSI)'. These NSIs can lead in general to the appearance of mixed states in a QM sense, as was first demonstrated in [13]. NSIs are usually considered to be of vector type (see e.g. [20, 21, 22, 19]) because they make the biggest contribution to the case of neutrino matter potential [23]. In this work we also consider other types of interactions, in particular the scalar interaction. NSIs of different types can be generated in the most of the SM extension, for example the general version of two Higgs doublet models [24, 25] and multi Higgs doublet models [26] leads to a scalar-charged particle which modifies the SM $W^{\pm}$ amplitude. Another good example is left-right symmetric models [27] in which both scalar and vector NSIs can appear with both the left and right chiral couplings, while the vector NSI can also appear in other models with extended gauge

[1]There are three exceptions, one is the muon anomalous magnetic moment, see e.g. [7], the second is an interesting result in B physics, i.e. asymmetry in the production of $\mu^+\mu^+$ and $\mu^-\mu^-$ see [8] and the last one is forward-backward asymmetry in $t\bar{t}$, see [9]; however, in all these cases the difference between theory and experiment can still be merely a statistical fluctuation or systematic error connected to either an experiment or a theoretical prediction.

groups, such as the 331 model [28, 29, 30]. Furthermore, models such as Zee-Babu [31, 32], models with triplet Higgs fields [33] and other extensions of SM Higgs models with scalar fields different to the usual Higgs doublet representation of an SM gauge group can introduce neutrino NSIs. Their strength depends on the specific type of models – even couplings with Higgs fields may be not negligible, since they are usually proportional to Yukawa couplings, which even though neutrinos masses are small, can be large in see-saw types of models.

The construction of a DM is presented in Chapter 2, where criteria for the appearance of pure or mixed QM states are discussed. The usual pure states are shown to be in agreement with our DM approach in the lowest order approximation. In Chapter 3, we demonstrate how to correctly include the propagation of neutrinos in matter and in a vacuum in our formalism. We also discuss the Lorentz transformation properties of DM while changing the Centre of Mass (CM) frame to a Laboratory (LAB) frame and then determining how this transformation affects the number of neutrinos that reach the detector at the same distance away from the source of the neutrinos. Chapter 4 describes calculations of the detection cross-section, we discuss the problem of defining oscillation probability beyond the SM and we also calculate examples of oscillation probability and compare them with usual approximations. In Chapter 5, we show possible implications of NP in the neutrino sector on the possibility of distinguishing Dirac and Majorana neutrinos. This chapter also contains numerical results for the most promising scalar right-handed interactions in muon decay. Finally, we make our conclusions.

CHAPTER 2

Neutrino production state

Contents

In this chapter we present a DM formalism, which will enable us to describe initial neutrino states. The first part contains general considerations, and in the second part we present a few examples.

2.1 Production process

Let us assume that a neutrino is produced in a production process, such as muon decay

$$\mu^- \to e^- \ \nu_\mu \ \bar{\nu}_e, \tag{2.1}$$

or pion decay

$$\pi^+ \to \mu^+ \ \nu_\mu. \tag{2.2}$$

To make the discussion generic we will not assume any specific process, and we will write the production process symbolically as

$$i \to f + \nu(\lambda, k), \tag{2.3}$$

where $\nu(\lambda, k)$ is a neutrino with helicity λ and in k'th mass state and i is a particular initial state which belongs to a set of all possible initial states $i \in \mathcal{I}$ for a given type of reaction. For example, in the case of muon decay (2.1), the set $\mathcal{I}$ is given by

$$\mathcal{I} = \{|\mu(+)\rangle,\ |\mu(-)\rangle\}, \tag{2.4}$$

where $+$ $(-)$ denotes the right (left) helicity state. Analogously, $f \in \mathcal{F}$ where $\mathcal{F}$ is a set of final states from which we exclude the observed neutrino; in case of muon decay in SM (neglecting the neutrino masses in kinematic variables,) it is given by

$$\begin{aligned}\mathcal{F} = \{\ & |e(+)\bar{\nu_e}(+,1)\rangle,\ |e(+)\bar{\nu_e}(+,2)\rangle,\ |e(+)\bar{\nu_e}(+,3)\rangle, \\ & |e(-)\bar{\nu_e}(+,1)\rangle,\ |e(-)\bar{\nu_e}(+,2)\rangle,\ |e(-)\bar{\nu_e}(+,3)\rangle\}.\end{aligned} \tag{2.5}$$

Next, we assume a model (a local, causal QFT) which enables us to calculate an amplitude for the production process (2.3). In general, this amplitude will depend on neutrino energy E, momentum $\vec{p}$, mass eigenstate k and helicity λ, as well as on the quantum numbers of the initial state i[1], its kinematics p_i, final state f and its kinematics p_f. Therefore, we can write the amplitude as $A_k(E,\vec{p},\lambda;i,p_i,f,p_f)$. From this point we assume that the kinematics of the initial state is given (e.g. we chose a laboratory frame or a rest frame when the neutrino is produced in a decay process) and we omit argument p_i in amplitude; consequently, we write $A_k(E,\vec{p},\lambda;i,f,p_f)$.

2.2 Density matrix

Now we proceed to constructing a density matrix (statistical operator) that will describe neutrinos produced in a process (2.3). First of all we must look at the initial state. If it is a pure state, then the final state is also pure and is given by

$$|final\rangle = N\sum_{k,f,\lambda}\int_\Omega d\mu(\vec{p},p_f)A_k(E,\vec{p},\lambda;i,f,p_f)|\nu_k(\vec{p},\lambda),f\rangle, \tag{2.6}$$

where $\int_\Omega d\mu(\vec{p},p_f)$ is an integral over the reaction's product (2.3) phase space with measure $\mu(\vec{p},p_f)$, such that it takes into account all Dirac delta functions connected with four momentum conservations. and on shell relations for external particles, N is a normalisation factor. The initial state may be pure, for example in the case of pion decay, but in general it does not have to be pure. For example, in the case of a planned neutrino factory, muons are very unlikely to be perfectly polarised [34], although they will have some degree of polarisation $0<p<1$ so they can be described by density operator

$$\varrho_i = p\ \ |\mu(+)\rangle\langle\mu(+)|\ \ + (1-p)\ \ |\mu(-)\rangle\langle\mu(-)|\ . \tag{2.7}$$

Using $(\varrho_i)_{n,n'}$ we denote the matrix element of any statistical operator which describes an initial state. So, in general the final state of the production process (2.3) is described by a statistical operator

$$\begin{aligned}\varrho_f = N\textstyle\sum_{n,n'}(\varrho_i)_{n,n'}\sum_{k,k',\lambda,\lambda'}\sum_{f,f'}\int_\Omega d\mu(\vec{p},p_f)\int_\Omega d\mu(\vec{p}',p'_f)\\ A_k(E,\vec{p},\lambda;n,f,p_f)\ \ |\nu_k(\vec{p},\lambda),f\rangle\langle\nu_{k'},(\vec{p}',\lambda'),f'|\ \ A^*_{k'}(E',\vec{p}',\lambda';n',f',p'_f)\end{aligned} \tag{2.8}$$

where N is chosen such that $Tr(\varrho_f)=1$[2]. We can now prove the following theorem:

Theorem 2.1. *If an amplitude can be factorised such that* $A_k(E,\vec{p};i,f,p_f) = B_i \times C_k(E,\vec{p},\lambda;f,p_f)$*, then* ϱ_f *describes a pure state.*

[1]We denote the quantum numbers of an initial state in the same way as for an element of $\mathcal{I}$, but this does not lead to any confusion, since specifying any element of $\mathcal{I}$ is equivalent to writing all of its quantum numbers, the same remark applies to elements of $\mathcal{F}$.

[2]The trace here is taken over discrete and continuous variables.

Proof. Since $A_k(E,\vec{p},\lambda;i,f,p_f) = B_i \times C_k(E,\vec{p},\lambda;f,p_f)$, we can introduce $N' = N\sum_{i,i'}(\varrho_i)_{i,i'}B_iB^*_{i'}$, in which case ϱ_f can be written as

$$\begin{aligned}\varrho_f &= N' \sum_{k,k',\lambda,\lambda'} \sum_{f,f'} \int_\Omega d\mu(\vec{p},p_f) \int_\Omega d\mu(\vec{p}',p'_f) \\ & C_k(E,\vec{p},\lambda;f,p_f) \ |\nu_k(\vec{p},\lambda),f\rangle\langle\nu_{k'},(\vec{p}',\lambda'),f'| \ C^*_{k'}(E',\vec{p}',\lambda';f',p'_f),\end{aligned} \tag{2.9}$$

which is obviously a projection on the state

$$|final\rangle = \sqrt{N'}\sum_{k,\lambda,f}\int_\Omega d\mu(\vec{p},p_f)C_k(E,\vec{p},\lambda;f,p_f)|\nu_k(\vec{p},\lambda),f\rangle. \tag{2.10}$$

□

Note that eq. (2.6) can be viewed as a special case of application of this theorem. Now that we have ϱ_f, we can define a neutrino state, which we achieve by taking a partial trace over all possible states in $\mathcal{F}$. The neutrino state is then given by

$$\begin{aligned}\varrho &= N\sum_{i,i'}(\varrho_i)_{i,i'} \sum_{k,k',\lambda,\lambda'} \sum_{f} \int_\Omega d\mu(\vec{p},p_f) \int_\Omega d\mu(\vec{p}',p'_f)\delta(p_f-p'_f) \\ & A_k(E,\vec{p},\lambda;i,f,p_f) \ |\nu_k(\vec{p},\lambda)\rangle\langle\nu_{k'},(\vec{p}',\lambda')| \ A^*_{k'}(E',\vec{p}',\lambda';i',f,p'_f).\end{aligned} \tag{2.11}$$

We can now formulate a theorem about the purity of the neutrino state.

Theorem 2.2. *In a process where only a left-handed neutrino is produced, for a given neutrino energy E and momentum $\vec{p}$, if an amplitude can be factorised such that $A_k(E,\vec{p},-1;i,f,p_f) = B_k \times C(E,\vec{p};i,f,p_f)$, then ϱ describes a pure state.*

Proof. The theorem can be proved in the same way as Theorem (2.1), Here the neutrino state is given by:

$$|\nu(\vec{p},-1)\rangle = \sum_k B_k|\nu_k(\vec{p},-1)\rangle. \tag{2.12}$$

□

As an immediate consequence of this theorem we note that relativistic neutrinos produced in any charged current process within SM are in a pure state. This is because only left-handed neutrinos interact and any charged current amplitude, when neglecting the dependence on neutrino mass in kinematics, can by written as $A_k(E,\vec{p},-1;i,f,p_f) = U^*_{\alpha\,k}C(E,\vec{p};i,f,p_f)$ with $U_{\alpha\,k}$ being an MNSP mixing matrix. We obtain the neutrino state within SM

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha\,k}|\nu_k\rangle, \tag{2.13}$$

which is the starting point of any standard derivation of oscillation probability. Let us also consider the integrals appearing in (2.11)

$$\int_\Omega d\mu(\vec{p},p_f)\int_\Omega d\mu(\vec{p}',p'_f)\delta(p_f-p'_f), \tag{2.14}$$

whereby the measure $\mu(\vec{p}, p_f)$ contains an overall energy momentum conservation delta function, i.e. $\mu(\vec{p}, p_f) = \delta(p_i - p_f - p) \times \mu'(\vec{p}, p_f)$, where p is a neutrino four momentum $(E, \vec{p})$. Therefore, using the convolution properties of delta functions, we obtain

$$\begin{aligned}
\int_\Omega d\mu(\vec{p}, p_f) \int_\Omega d\mu(\vec{p}', p'_f)\delta(p_f - p'_f) = \\
\int_\Omega d\mu'(\vec{p}, p_f) \int_\Omega d\mu'(\vec{p}', p'_f)\delta(p_f - p'_f)\delta(p_i - p_f - p)\delta(p_i - p'_f - p') = \\
\int_\Omega d\mu'(\vec{p}, p_f) \int_\Omega d\mu'(\vec{p}', p'_f)\delta(p_f - p'_f)\delta(p_i - p_f - p)\delta(p_i - p_f - p') = \\
\int_\Omega d\mu'(\vec{p}, p_f) \int_\Omega d\mu'(\vec{p}', p'_f)\delta(p_f - p'_f)\delta(p_i - p_f - p)\delta(p - p'),
\end{aligned} \tag{2.15}$$

from which we conclude that the density matrix is diagonal in momentum indices. As a result, we can write

$$\begin{aligned}
\varrho &= N\sum_{i,i'}(\varrho_i)_{i,i'} \sum_{k,k',\lambda,\lambda'} \sum_f \int_\Omega d\mu(\vec{p}, p_f) \\
&\quad A_k(E, \vec{p}, \lambda; i, f, p_f)\ |\nu_k(\vec{p}, \lambda)\rangle\langle\nu_{k'}(\vec{p}, \lambda')|\ A^*_{k'}(E', \vec{p}, \lambda'; i', f, p_f).
\end{aligned} \tag{2.16}$$

Unfortunately, this leads to serious problems, since from eq. (2.15) it follows that $m_i^2 = m_j^2$, even for $i \neq j$. The reason for that is because we wrote an amplitude for an unlocalised case, and this cannot be the case for an oscillation experiment. Nonetheless, we have two options for dealing with this problem. We can still use the plane wave in order to keep calculations as simple as possible, but in all kinematical variables we must set everywhere the mass of the neutrinos equal to zero. As such, equation (2.16) is still valid, and this approximation is good in practical terms because of the smallness of the neutrino mass in comparison to other energy scales appearing in the experiment, such as the energies and masses of particles accompanying neutrino production. From a theoretical point of view, wave packets would be a better option, but they would complicate the calculations significantly. Therefore, the best way to introduce the wave packets into our picture is to convolute an amplitude with function $f_\sigma(p_i - p_i^c, p_f - p_f^c)$ peaked around central point $(0,0)$ so that p_f^c and p_i^c are central values of the final and initial momentum, and σ parametrises the spread of the function. Now, equation (2.14) is replaced by

$$\begin{aligned}
\int_\Omega d\mu(\vec{p}, p_f) \int_\Omega d\mu(\vec{p}', p'_f) d\mu(p_i) d\mu(p'_i) f_\sigma(p_i - p_i^c, p_f - p_f^c) f^*_\sigma(p'_i - p_i^c, p'_f - p_f^c) \\
= \int_\Omega d\mu(\vec{p}, p_f) \int_\Omega d\mu(\vec{p}', p'_f)\Delta_\sigma(p_f, p'_f),
\end{aligned} \tag{2.17}$$

so instead of the Dirac delta function we have function $\Delta_\sigma(p_f, p'_f) = d\mu(p_i)d\mu(p'_i)f_\sigma(p_i - p_i^c, p_f - p_f^c)f^*_\sigma(p'_i - p_i^c, p'_f - p_f^c)$, which, because $f_\sigma(x, y)$ peaks around $(0,0)$, tends to the Dirac delta function as σ goes to zero $\Delta_\sigma(p_f, p'_f) \xrightarrow{\sigma \to 0} \delta(p_f - pf')\delta(p_f - p_f^c)^3$. Now we see that the neutrinos' energies and momentum need only to be equal approximately, up to spread σ, so, as a consequence, conditions $m_i^2 = m_j^2$ need not be fulfilled for reasonable values of momentum and energy

[3]Actually, $f_\sigma(x, y)$ must be normalised properly in order to produce this limiting behaviour, although we assume that it was chosen such that it was possible.

uncertainty values. Unfortunately, equation (2.16) can no longer be used and we are only able to conclude that the density matrix is nearly diagonal in momentum space.

Let us now go back to the plane wave approximation, i.e. in all kinematic variables we treat neutrinos as massless particles and the amplitude of neutrino production depends on neutrino mass only via the coupling constant. Finally, we can factorise the dependence of ϱ on discrete and continuous variables. Let us define a

$$\tilde{\varrho}(\vec{p}) = \frac{\varrho}{\mathbf{Tr}(\varrho)}, \tag{2.18}$$

where $\mathbf{Tr}()$ denotes taking a trace over all discrete variables only. Let us also introduce a quantity which represents the probability of finding a neutrino with momentum in interval $[\vec{p}, \vec{p} + d\vec{p}]$:

$$\frac{dj}{d\vec{p}} d\vec{p} = \mathbf{Tr}(\varrho) d\vec{p} \tag{2.19}$$

in which case we observe that due to the overall normalisation of statistical operator ϱ, the introduced neutrino momentum probability density function (NMPDF) is normalised such that $\int d\vec{p} \frac{dj}{d\vec{p}} = 1$. Now we can write the neutrino density matrix as

$$\varrho = \tilde{\varrho}(\vec{p}) \times \frac{dj}{d\vec{p}}. \tag{2.20}$$

Each of these terms is separately normalised and $\tilde{\varrho}(\vec{p})$ represents the neutrino density matrix for a given neutrino momentum, while $\frac{dj}{d\vec{p}}$ is a probability density function in a momentum space.

2.3 Muon decay

We now demonstrate the application of the presented formalism on the example of neutrino produced in muon decay within an effective model with different types of interactions. In order to achieve this, we analyse two cases, beginning with only left-handed neutrinos interacting in the effective model and later discussing the scenario whereby right-handed couplings also appear.

2.3.1 Left-handed neutrinos

We now present an explicit calculation for the neutrino density matrix in the muon decay example, using the effective Lagrangian interaction. We assume that only left-handed neutrinos are produced and that the interactions are of scalar or vector type, i.e.

$$\mathcal{L}_I = -2\sqrt{2} G_F \left[g^S_{ij} \left(\overline{\nu}_i P_R e\right) \left(\overline{\mu} P_L \nu_j\right) + g^V_{ij} \left(\overline{\nu}_i \gamma^\alpha P_L e\right) \left(\overline{\mu} \gamma_\alpha P_L \nu_j\right) \right] + h.c. \tag{2.21}$$

Left-handed vector interactions

Let us first consider only the vector part, i.e. we put $g^S_{ij} = 0$ into eq. (2.21) and as usual we neglect the neutrino mass in kinematics. The flux can be calculated immediately, if we assume that the initial muons are not polarised. Therefore, as the NMPDF in the muon rest frame is independent of neutrino direction, it is convenient to move to spherical coordinates and integrate over angles

$$\int \frac{dj}{d\vec{p}} d\Omega = \frac{1}{E^2}\frac{dj}{dE}, \tag{2.22}$$

where E is neutrino energy $E = \| \vec{p} \|$. It is also convenient to introduce a dimensionless quantity $x = \frac{2E}{M}$, where M is muon mass. As such, $\frac{dj}{dE}$ can be related to $\frac{dj}{dx} = \frac{M}{2}\frac{dj}{dE}$, which is given by [35]

$$\frac{dj}{dx} = 2x^2(3-2x). \tag{2.23}$$

In the above formula we have also neglected the electron mass. Furthermore, the neutrino spectrum in the laboratory frame is given by the same formula, although with $x = \frac{E}{E_\mu}$ where E_μ is a muon energy in the LAB. As we can see, the amplitude factorises for the part that depends on a neutrino and antineutrino mass index, as well as the part that depends on the spin indices of the electron and muon. The density matrix used to describe the neutrino is given by:

$$\tilde{\varrho}(\vec{p}) = \frac{(g^V)^\dagger g^V}{Tr[(g^V)^\dagger g^V]}. \tag{2.24}$$

In SM, the following relation $g^V_{ij} = U^*_{ei}U_{\mu j}$ holds, so by using the unitarity of the MNSP matrix U we obtain

$$(\tilde{\varrho}(\vec{p})_{SM})_{ij} = U^*_{\mu\, i}U_{\mu\, j}, \tag{2.25}$$

in agreement with general result (2.13). In general, the state given by Eq. (2.24) is not a pure QM state because state (2.24) is pure if and only if

$$(g^V)^\dagger g^V (g^V)^\dagger g^V = (g^V)^\dagger g^V Tr[(g^V)^\dagger g^V], \tag{2.26}$$

which is equivalent to g^V having only one non-zero singular value[4]. Condition (2.26) can be written also in term of traces

$$Tr[((g^V)^\dagger g^V)^2] = Tr[(g^V)^\dagger g^V]^2. \tag{2.27}$$

The reason for the appearance of the mixed state is the entanglement between both the neutrino and antineutrino mass states. Let us now formulate another theorem. Firstly, we introduce normalised coupling constants matrices g_N as follows $g_N = \frac{g^V}{||g^V||}$, where $||A|| = \sqrt{Tr]A^\dagger A]}$ is a Hilbert-Schmidt norm.

[4] Note that in order for the neutrino to be in a pure state, the condition for $\tilde{\varrho}(\vec{p})$ to have only one non-vanishing eigenvalue is trivial, but since $\tilde{\varrho}(\vec{p}) \sim (g^V)^\dagger g^V$, and from the definition of a singular value, it directly follows that g^V must have only one non-zero singular value in order for the neutrino be in a pure state.

Theorem 2.3. *$\tilde{\varrho}(\vec{p})$, in form (2.24) $\tilde{\varrho}(\vec{p}) = g_N^\dagger g_N$, describes the pure neutrino state if and only if g_N are rank-one matrices, i.e they can be written in the form $g_N = vu^\dagger$, where u and v are unit vectors.*

Proof. The neutrino density matrix has a form

$$\tilde{\varrho}(\vec{p}) = g_N^\dagger g_N, \tag{2.28}$$

so if we write $g_N = vu^\dagger$ with unit vectors u and v, then

$$\tilde{\varrho}(\vec{p}) = uu^\dagger, \tag{2.29}$$

which is obviously pure. Now we observe that g_N is the square root of $\tilde{\varrho}(\vec{p})$, so from theorem B.2 we know that any other roots can be written as $g'_N = Ug_N$ for unitary matrix U. As a consequence, $g'_N = Uvu^\dagger = v'u^\dagger$, which again is a rank-one matrix. □

We interpret this theorem by noting that the two particle neutrino-antineutrino states in our case are given by $|\nu_i\bar{\nu_j}\rangle = (g_N)_{ji}|\nu_i\rangle \otimes |\bar{\nu_j}\rangle$. Thus, the simple result of theorem 2.3 just means that the neutrino state is pure, but only if the neutrino-antineutrino state is not entangled, i.e. $|\nu_i\bar{\nu_j}\rangle = (g_N)_{ji}|\nu_i\rangle\otimes|\bar{\nu_j}\rangle = (u_i|\nu_i\rangle)\otimes(v_j|\bar{\nu_j}\rangle)$.

We can now return to theorem 2.2 to make it more general. In order to do so, we arrange the amplitude's indices such that we form matrices $A_{a,b} = A_{if,k\lambda} = A_k(E, \vec{p}, \lambda; i, f, p_f)$, which we then normalise as $A_N = \frac{A}{||A||}$. Comparing with our construction of a density matrix, we observe that A_N is the square root of the neutrino density matrix. Proceeding in a same way as in the proof of Theorem 2.3, we note that the neutrino will be in a pure state only when $A_N = bc^\dagger$ for unit vectors b and c. Theorem 2.2 is a special case demonstrating this fact.

We can also formulate theorem 2.3 in a slightly different way. In order for the neutrino to be in a pure state, we noted that g^V can have only one non-zero singular value, which we can refer to as $\sigma_1(g^V) \equiv ||g^V||$. We now introduce matrix $\Sigma(g^V)$, which is diagonal, and its non-zero elements are the singular values of g^V, i.e. $(\Sigma(g^V)_{ij} = \sigma_i(g^V)\delta_{ij}$. Then, using the SVD theorem (see B.4), we can write

$$g^V = V\Sigma(g^V)U^\dagger, \tag{2.30}$$

where U and V are unitary matrices. Equation (2.30) gives us another form of g^V, which leads to the $\tilde{\varrho}(\vec{p})$ representing a pure state in the case of $\Sigma(g^V)$ having only one non-zero element on a diagonal. It is of course equivalent to the result obtained in theorem 2.3, since $\Sigma(g_N)$ is a projector.

For completeness, let us also formulate a theorem that provides the criteria for a neutrino state to be maximally mixed.

Theorem 2.4. *Neutrino state $\tilde{\varrho}(\vec{p}) = \frac{(g^V)^\dagger g^V}{Tr[(g^V)^\dagger g^V]}$ is maximally mixed, i.e. $\tilde{\varrho}(\vec{p}) = \frac{1}{N}I$ if and only if matrix g^V is proportional to the unitary matrix.*

Proof. It therefore follows directly from the definition of the unitary matrix in a finite dimensional space. □

In practice, we do not expect to obtain a maximally mixed neutrino state, since in SM the neutrino state is pure and NP contributions are suppressed. Let us summarise our results in a table:

Neutrino state	Matrix g^V
Pure	proportional to partial isometry in the form $g^V \sim u^\dagger v$
Maximally mixed	proportional to isometry, i.e. unitary matrix

Let us now derive the neutrino state in a linear approximation. In SM we can write in a flavour base (we use a unitary MNSP matrix to change from the initial mass base) $(g^V)_{\alpha\beta} = \delta_{e\beta}\delta_{\mu\alpha}$. We assume that NP introduces a small correction $\varepsilon_{\alpha\beta}$. We work in a linear approximation with respect to NP parameters, and then up to the normalisation factor the density matrix is given by

$$\begin{aligned}(\tilde{\varrho}(\vec{p}))_{\alpha\alpha'} &= (\delta_{e\beta}\delta_{\mu\alpha} + \varepsilon_{\alpha\beta})(\delta_{e\beta}\delta_{\mu\alpha'} + \varepsilon^*_{\alpha'\beta}) \\ &= \delta_{\mu\alpha}\delta_{\mu\alpha'} + \delta_{\mu\alpha}\varepsilon^*_{\alpha' e} + \delta_{\mu\alpha'}\varepsilon_{\alpha e}. \end{aligned} \tag{2.31}$$

This can be represented as a pure state

$$|\nu\rangle = N' \sum_\alpha (\delta_{\alpha\mu} + \varepsilon_{\alpha e})|\nu_\alpha\rangle, \tag{2.32}$$

when neglecting higher order correction. The state should be normalised, and the normalisation constant in our approximation is given by

$$N' = 1 - 2Re(\varepsilon_{\mu e}). \tag{2.33}$$

Left-handed scalar interactions

The situation becomes more complicated if we allow for a non-zero value of g^S. Now, there is also entanglement between neutrino mass states and the spin states of the electron. Using amplitudes given in Appendix D, we calculate the NMPDF, which is given by

$$\frac{dj}{dx} = \frac{4x^2(2Tr[(g^V)^\dagger g^V](3-2x) + 3Tr[(g^S)^\dagger g^S](1-x))}{Tr[g^V)^\dagger g^V + 4(g^S)^\dagger g^S]}, \tag{2.34}$$

and the density matrix

$$\tilde{\varrho}(\vec{p}) = \frac{2(g^V)^\dagger g^V(3-2x) + 3(g^S)^\dagger g^S(1-x)}{2Tr[(g^V)^\dagger g^V](3-2x) + 3Tr[(g^S)^\dagger g^S](1-x)}, \tag{2.35}$$

where we observe that the state is mixed in general. Let us now expand the denominator of (2.35) for small values of scalar interaction coupling constants. Our expansion parameter is $Tr[(g^S)^\dagger g^S]$[5]

$$\frac{1}{2Tr[(g^V)^\dagger g^V](3-2x)+3Tr[(g^S)^\dagger g^S](1-x)} =$$
$$\frac{1}{2Tr[(g^V)^\dagger g^V](3-2x)} - \frac{3(1-x)}{(2Tr[(g^V)^\dagger g^V](3-2x))^2} Tr[(g^S)^\dagger g^S] + O(Tr[(g^S)^\dagger g^S]^2), \quad (2.36)$$

Therefore, density matrix (2.35) keeping only leading order correction is given by

$$\tilde{\varrho}(\vec{p}) = \frac{(g^V)^\dagger g^V}{Tr[(g^V)^\dagger g^V]} +$$
$$\frac{3(1-x)}{2Tr[(g^V)^\dagger g^V](3-2x)} \left((g^S)^\dagger g^S - (g^V)^\dagger g^V \frac{Tr[(g^S)^\dagger g^S]}{Tr[(g^V)^\dagger g^V]} \right) + O([(g^S)^\dagger g^S]^2). \quad (2.37)$$

Let us now assume that the vector part has the same flavour structure as found in SM, i.e. $g_{ij}^V = y^V U_{ei}^* U_{\mu j}$, with $|y^V|$ being the vector coupling strength close to one[6]. Next, the first term of (2.37) is just an SM term, while NP contribution is represented by just one function $f(x) = \frac{3(1-x)}{2(3-2x)}$ and a combination of coupling constants $\frac{1}{|y^V|^2}\left(((g^S)^\dagger g^S)_{ij} - U_{\mu i} U_{\mu j}^* Tr[(g^S)^\dagger g^S] \right)$. If we are interested in the antineutrino state, then formula (2.37) is still valid with the following changes $g^X \to (g^X)^*$ and function $f(x)$ has to be changed to function $h(x) = \frac{3-2x}{24(1-x)}$[7]. Note also the following relation between functions $f(x)$ and $h(x)$: $16h(x)f(x) = 1$. In Figure 2.1, we have plotted functions $f(x)$ and $h(x)$, from which we see that in the neutrino case the biggest effect caused by a new interaction is connected with low energy neutrinos, whereas in the antineutrino case the situation is the opposite. For $x = \frac{3}{4}$, both functions are equal, so this energy region may be interesting for experimental study because NP effects in neutrino and antineutrino initial states will be of comparable magnitude.

2.3.2 Right- and left-handed neutrinos

Let us now also allow for right-handed neutrino interaction. We consider the following Lagrangian with only vector left-handed neutrinos and right-handed both vector and scalar terms:

$$\mathcal{L}_I = -2\sqrt{2}G_F \Big[r_{ij}^S \left(\overline{\nu}_i P_L e\right)\left(\overline{\mu} P_R \nu_j\right) + r_{ij}^V \left(\overline{\nu}_i \gamma^\alpha P_R e\right)\left(\overline{\mu}\gamma_\alpha P_R \nu_j\right) +$$
$$g_{ij}^V \left(\overline{\nu}_i \gamma^\alpha P_L e\right)\left(\overline{\mu}\gamma_\alpha P_L \nu_j\right) \Big] + h.c. \quad (2.38)$$

[5] $Tr[(g^S)^\dagger g^S]$ is actually quadratic in scalar interaction strength, but since there is no linear dependence on g^S, we use it as an expansion parameter.

[6] $|y^V|$ can not be equal to one because normalisation of the overall decay rate require $||g^V||^2 + \frac{1}{4}||g^S||^2 = 1$.

[7] For the antineutrino case, the approximation is not good if x is very close to one, although this is not problematic for us because in practice x cannot be equal to one due to the non-vanishing electron mass which we neglected in the present derivations. Also, the antineutrino spectrum decreases very fast when x approaches unity. The point $x = 1$ is specific because in this pion the SM contribution is zero for antineutrinos.

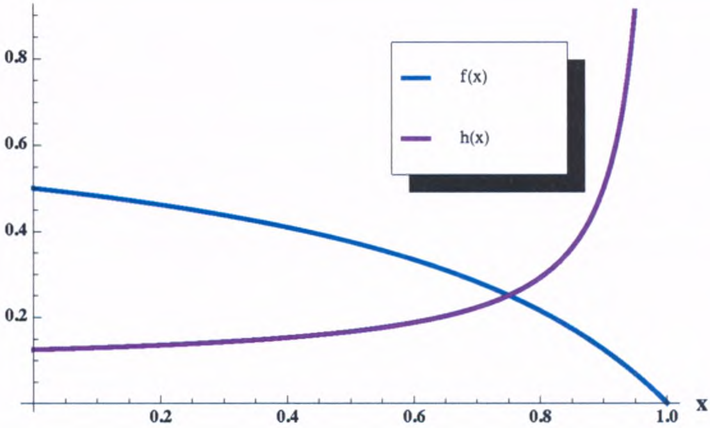

Figure 2.1: Functions $f(x)$ and $h(x)$ show how the NP contributions changes with neutrino energy.

Right-handed vector interactions

As before, we firstly analyse only the vector couplings, i.e we set $r^S_{ij} = 0$. In the unpolarised case, the density matrix has a simple form

$$\tilde{\varrho}(\vec{p}) = \frac{1}{Tr[(r^V)^\dagger r^V + (g^V)^\dagger g^V]}\left((r^V)^\dagger r^V P_{+1} + (g^V)^\dagger g^V P_{-1}\right), \tag{2.39}$$

where P_{+1} (P_{-1}) is the positive (negative) helicity projector operators $P_x = |\lambda = x\rangle\langle\lambda = x|$. This state is mixed because the entanglement between neutrino helicites and those of other particles has occurred. This is an obvious conclusion, but let us present some form of proof by using DM formalism. We can assume that both the left- and right-handed parts of neutrino states are separately pure, i.e. $Tr[((r^V)^\dagger r^V)^2] = (Tr[(r^V)^\dagger r^V])^2$, and the same in the left part $Tr[((g^V)^\dagger g^V)^2] = (Tr[(g^V)^\dagger g^V])^2$. Let us also introduce abbreviations $Tr[(g^V)^\dagger g^V] \equiv T_L$ and $Tr[(r^V)^\dagger r^V] \equiv T_R$, in which case the condition for the purity of the full neutrino state (2.39) is

$$T_R^2 + T_L^2 = (T_R + T_L)^2, \tag{2.40}$$

which means that the neutrino state will be pure in the presence of right-handed interactions, if both the left- and right-handed parts of the neutrino states are pure and $T_R = 0$ or $T_L = 0$. However, this is equivalent to $r^V = 0$ or $g^V = 0$[8] and the state is proven to be mixed, as expected. So, we observe that if right-handed neutrino interactions contribute to the production process, then the state is mixed. Let us also consider a case where the initial muon is polarised. Let $0 \leqslant P \leqslant 1$ be a polarisation degree (see eq. (2.7)) and θ denotes an angle between the polarisation

[8]See B.1 for proof.

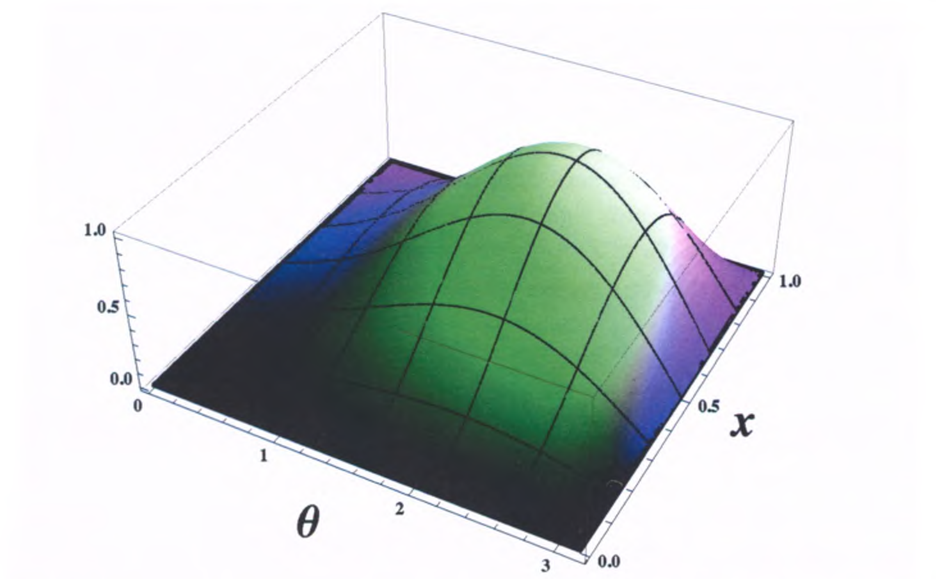

Figure 2.2: The antineutrino momentum probability distribution function for $P = -1$ and $T_L + T_R = 1$, $\sqrt{T_L} = 0.9$. We see that observing the neutrino for $\theta \sim 0$ would be very difficult.

vector and neutrino direction. Then, using amplitudes given in Appendix D, we obtain

$$\begin{aligned}\tilde{\varrho}(\vec{p}) &= \tfrac{(g^V)^\dagger g^V (P(2x-1)\cos(\theta)+2x-3)}{(2x-3)(T_L+T_R)+P(2x-1)(T_L-T_R)\cos(\theta)} P_{-1} + \\ &\quad \tfrac{(r^V)^\dagger r^V (-P(2x-1)\cos(\theta)+2x-3)}{(2x-3)(T_L+T_R)+P(2x-1)(T_L-T_R)\cos(\theta)} P_{+1}. \end{aligned} \tag{2.41}$$

We see that by adjusting the muon polarisation we can change the magnitude of the left and right parts of the neutrino density matrix. For completeness, let us also calculate the density matrix for antineutrino

$$\begin{aligned}\tilde{\varrho}(\vec{p}) &= \tfrac{(r^V)^T (r^V)^* (-P\cos(\theta)+1)}{T_L+P(T_L-T_R)\cos(\theta)+T_R} P_{-1} + \\ &\quad \tfrac{(g^V)^T (g^V)^* (P\cos(\theta)+1)}{T_L^2+P(T_L-T_R)\cos(\theta)+T_R} P_{+1}. \end{aligned} \tag{2.42}$$

This result is interesting because, in principle, for $\theta = \pi$ and $P = 1$ we can isolate the admixture of right-handed states. Unfortunately though, the NMPDF in this case is very small.

$$\frac{dj}{dx d\cos\theta} = \frac{6(1-x)x^2\,(T_L + P(T_L - T_R)\cos(\theta) + T_R)}{T_L + T_R}. \tag{2.43}$$

In Figure (2.2) we plotted the NMPDF for $P = 1$ in a direction opposite to the Z axis, and other parameters were chosen to satisfy $T_L + T_R = 1$[9], $\sqrt{T_L} = 0.9$[10]. The

[9]This condition follows from the normalisation of the total decay width.

[10]This value is actually below the lower experimental limits.

NMPDF is very small because it is proportional to the small ratio $\frac{T_R}{T_L+T_R}$, which is at most of the order 0.001 for current experimental limits [36].

Right-handed scalar interactions

Finally, we consider a case where right-handed neutrino interactions are of a scalar type and there are no vector right-handed currents, i.e. $r^V = 0$. Let us introduce the following notation $S_R \equiv Tr[(r^S)^\dagger r^S]$, and similarly as before $T_L \equiv Tr[(g^V)^\dagger g^V]$ contains an SM left-handed vector contribution. The antineutrino density matrix has a form

$$\tilde{\varrho}(\vec{p}) = \frac{(r^S)^T(r^S)^*(2x-3)}{S_R(2x-3)+24T_L(x-1)}P_{-1} + \frac{24(g^V)^T(g^V)^*(x-1)}{S_R(2x-3)+24T_L(x-1)}P_{+1}, \quad (2.44)$$

and for simplicity, as before, we assume that both the left- and right-handed parts of the neutrino states are separately pure. Calculating $1 - Tr[\tilde{\varrho}(\vec{p})^2]$ as a measure of purity of the state, we obtain

$$1 - Tr[\tilde{\varrho}(\vec{p})^2] = \frac{12(4-S_R)S_R(x-1)(2x-3)}{(S_R(4x-3)-24(x-1))^2}, \quad (2.45)$$

where we have used a normalisation condition $T_L + \frac{1}{4}S_R = 1$ which follows from the value of the total width and a Fermi constant definition. For small value of S_R the following approximation holds:

$$1 - Tr[\tilde{\varrho}(\vec{p})^2] = \frac{(2x-3)S_R}{12(x-1)} + \frac{x(2x-3)S_R^2}{144(x-1)^2} + O\left(S_R^3\right) \quad (2.46)$$

which shows that obviously the antineutrino state tends to a pure state, as S_R goes to zero. The effect is small because it is proportional to S_R. It is interesting to find a maximum of (2.45), which is located for

$$x_0 = \frac{3}{8}\left(\frac{1}{S_R-3}+3\right) \simeq 1 - \frac{S_R}{24} - \frac{S_R^2}{72} - \frac{S_R^3}{216} - \frac{S_R^4}{648} + O\left(S_R^5\right), \quad (2.47)$$

so it is very close to the maximum possible energy of an antineutrino. Figure 2.3 shows the $1 - Tr[\tilde{\varrho}(\vec{p})^2]$ as a function of energy for different values of scalar coupling. The maximum value of $1 - Tr[\tilde{\varrho}(\vec{p})^2]$ for $x = x_0$ is always exactly $\frac{1}{2}$. We can also derive an interesting relation between the case when new interactions are either of a left or right scalar type. In the unpolarised case, if we assume that in the antineutrino version of formula (2.35) matrices g^V and g^S are rank-one, then, up to a multiplicative constant ξ, the value of $1 - Tr[\tilde{\varrho}(\vec{p})^2]$ for an antineutrino density matrix with scalar left-handed interactions is given by the same formula as (2.45), with the following substitutions $S_R \to Tr[g^S(g^S)^\dagger]$. The constant ξ is given by $\xi = 1 - \frac{Tr[g^S(g^S)^\dagger g^V(g^V)^\dagger]}{Tr[g^S(g^S)^\dagger]Tr[g^V(g^V)^\dagger]}$ and can take any values from 0 to 1. Therefore, relation (2.47) applies also in that case, which is interesting because limits on scalar left-handed interactions are much weaker than in their right-handed counterparts.

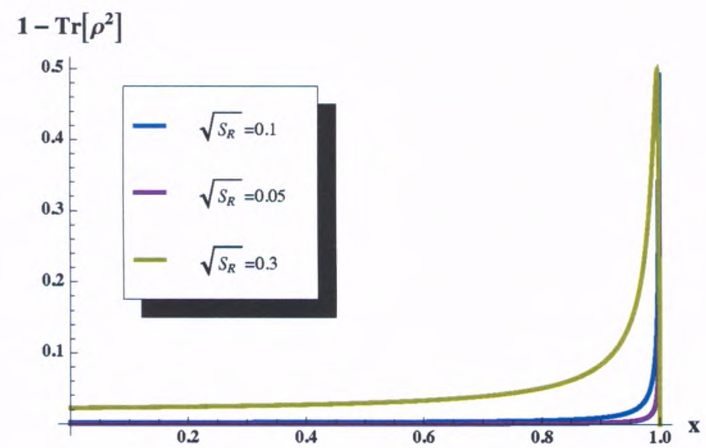

Figure 2.3: $1 - Tr[\tilde{\varrho}(\vec{p})^2]$ as a function of the neutrino energy for different values of couplings. The maximum is at $x_0 \simeq 1 - \frac{S_R}{24}$

CHAPTER 3

Oscillation process

Contents

3.1 Transformation form CM to LAB

In the previous chapter, because of convenience, the density matrix and NMPDF were calculated in the rest frame of the decaying particle. In general, we perform explicit calculations in a frame in which we expect the formulas to have the simplest form, and then we need to transform it to another frame of reference. In our case, we need to perform a boost from a CM to a LAB frame in order to calculate the neutrino oscillation process. Hopefully, the spin-mass structure of the density matrix will be unaffected in practice (it is caused by the smallness of neutrino masses in comparison with the energy in a typical neutrino experiment, see Appendix A for details of calculations) and the transformations will follow

$$\varrho \longmapsto \varrho' = \varrho|_{p\to p'}, \tag{3.1}$$

in which case all we need to take care of is the neutrino momentum transformation. All the formulae appearing here are well known, see e.g. [37] for a review. We assume that the neutrino in the CM frame has a momentum vector lying in x-z plane $\vec{p} = E(\sin\theta, 0, \cos\theta)$ and energy E. We then perform a boost in z direction with velocity β and obtain

$$\begin{aligned}
E' &= \gamma E(1+\beta\cos\theta) \\
p'_x &= E'\sin\theta' = E\sin\theta \\
p'_y &= p_y = 0 \\
p'_z &= E'\cos\theta' = \gamma E(\beta+\cos\theta),
\end{aligned} \tag{3.2}$$

from which we can solve

$$\begin{aligned}
\sin\theta' &= \frac{\sin\theta}{(1+\beta\cos\theta)\gamma}, \\
\cos\theta' &= \frac{\cos\theta+\beta}{1+\beta\cos\theta}.
\end{aligned} \tag{3.3}$$

Both relations can now be merged into

$$\tan\frac{\theta'}{2} = \sqrt{\frac{1-\beta}{1+\beta}}\tan\frac{\theta}{2}. \tag{3.4}$$

This formula enables us to calculate the angular distribution in the LAB, provided that we know this distribution in the CM. We therefore can calculate which of a produced neutrinos will reach the detector. It is important that θ is a monotonic function of θ'. So, assuming the baseline in the oscillation experiment to be L and r is the radius of the detector, then from e.q. 3.4 we obtain a maximum value of θ in CM for which the neutrino will reach the detector

$$\theta_{max} = 2Arctan(\sqrt{\frac{1+\beta}{1-\beta}}\frac{r}{L}). \tag{3.5}$$

If we assume that we know the energy-angular distribution of neutrinos in CM $\frac{dj}{dEd\cos\theta}$, in order to calculate this distribution in the LAB frame $\frac{dj}{dE'd\cos\theta'}$ we also need to know the Jacobian of the transformations, which is as follows:

$$J = \gamma(1+\beta\cos\theta) = \frac{1}{\gamma(1-\beta\cos\theta')}, \tag{3.6}$$

such that we obtain the relation

$$\frac{dj}{dE'd\cos\theta'} = \gamma(1+\beta\cos\theta)\frac{dj}{dEd\cos\theta}. \tag{3.7}$$

3.2 Oscillation

The density matrix obeys a relativistic equation (compare [38, 39])

$$i\partial_\mu\varrho = [P_\mu, \varrho], \tag{3.8}$$

which can be formally solved

$$\varrho(x) = e^{-iP_\mu x^\mu}\varrho(0)e^{iP_\mu x^\mu}, \tag{3.9}$$

so we can interpret the oscillation process as a translation of a neutrino DM in space and time.

3.2.1 Oscillations in a vacuum

We now present the simple derivation of an oscillation length in our formalism. Let us first analyse the oscillation of a neutrino in a vacuum, i.e. energy and momentum are related in a standard way $E^2 = p^2 + m^2$. We first analyse the phase factor proportional to $P_\mu x^\mu$, and we choose a four-vector x^μ in order to have the time component equal to T, the spatial component equal to L and the direction parallel to the neutrino's momentum. Consequently, $P_\mu x^\mu = ET - PL$. Due to the propagation process, the elements of the density matrix acquire the phase factor, which can by written as $(E_i - E_j)T$ because of the equal momenta assumption. Defining average neutrino velocity $v = \frac{L}{T} = \frac{2P}{E_i+E_j}$, we have $(E_i - E_j)T = \frac{(E_i^2-E_j^2)}{2P}L = \frac{(E_i^2-P^2-E_j^2+P^2)}{2P}L = \frac{(m_i^2-m_j^2)}{2P}L$. As such, the non-diagonal terms oscillate with the standard oscillation phase in the plane wave approximation.

For completeness, let us also mention the other derivations of the oscillation phase based on various assumptions and approximations; a review of the subject can be found for example in [40]. We still use the equal momentum approximation, but now we expand the energy such that $E_i = P + \frac{m_i^2}{2P}$ and we also approximate $L = T$ such that we obtain the oscillation phase difference equal to

$$\frac{(m_i^2 - m_j^2)}{2P}L = \frac{\Delta m_{ij}^2}{2P}L, \tag{3.10}$$

which agrees with our previous results.

The derivation of the oscillation phase still is a subject of some controversy and appears in the literature from time to time (see e.g. [41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51]). Nevertheless, we do not discuss this issue here and assume that the oscillation length in a vacuum is given by phase difference (3.10) such that

$$L_{osc} = 2\pi\frac{2P}{\Delta m_{ij}^2}. \tag{3.11}$$

3.2.2 Oscillation in matter

We derive effective potential by describing neutrino oscillations in matter with general forms of interactions. We assume that both charged current (after appropriate Fierz rearrangement [52]) and neutral current interactions are described by the following effective Lagrangian

$$-\mathcal{L}_{eff} = \frac{G_F}{\sqrt{2}}\sum_a(\bar{\nu}\,\Gamma^a\,\nu)\ \left[\bar{\psi}_f\,\Gamma_a\,(g_a + g_a'\gamma^5)\,\psi_f\right]\ , \tag{3.12}$$

where a numerates different Lorentz structures $a = S, V, T, P, A$, and f stands for any fermions present in the matter. In order to calculate the effect of the medium, we introduce an effective potential V_a^f

$$-\,\mathcal{L}_{\rm int} = \sum_{a,f}(\bar{\nu}\,\Gamma^a\,\nu)\,V_a^f\,, \tag{3.13}$$

such that

$$V_a^f = \frac{G_F}{\sqrt{2}}\sum_\lambda\int\frac{d^3p}{(2\pi)^3}\rho_f(p,\lambda)\mathcal{J}_a^f\,, \tag{3.14}$$

with a current $\mathcal{J}_a$ given by

$$\mathcal{J}_a^f = \langle f,p,\lambda|\bar{\psi}_f\,\Gamma_a\,(g_a + g_a'\gamma^5)\,\psi_f|f,p,\lambda\rangle,$$

and a fermion distribution function $p_f(p,\lambda)$ normalised to the number of fermions n_f such that $n_f = \sum_\lambda \int p_f(p,\lambda) d^3p$. The effective potentials calculated by Bergmann, Grossman and Nardi [53], as well as the neutrino currents $\bar{u}_i \Gamma_r u_j$, are given in Appendix C. These relations indicate that, for realistic unpolarised media, only vector and axial vector terms make a significant contribution in NP models. Also, tensor interactions may be important, but we neglect them because, for phenomenological reasons, we do not expect to obtain a tree-level contribution of tensor interaction. In general, dependence on medium polarisation and on average momentum may appear in astrophysical environments. In these cases, the dissipative term also needs to be taken into account while considering neutrino evolution equation [39]. We do not, however, discuss those cases here. Taking into account the relations given in Appendix C, we can write the approximate Hamiltonian for a neutrino with mass m

$$H_{eff} = E + \frac{m^2}{2E} + V^{LL} P_L + V^{RR} P_R, \tag{3.15}$$

where $V^{LL} = V_0^V - V_0^A$ and $V^{RR} = V_0^V + V_0^A$ with V_μ^V and V_μ^A are given by (C.4) and (C.5), respectively. P_L and P_R are projector operators on helicity eigenstates. If we restrict ourselves only to the model given by (D.1), i.e. a Lagrangian which describes muon decay and inverse muon decay, of course no NP contribution to the matter's effective potential will appear, since it is impossible to describe a coherent scattering within this Lagrangian only. More general NP Lagrangians, however, will produce a contribution to neutrino oscillations in matter, which can be easily calculated using the formalism presented in this chapter.

The evolution operator for a Hamiltonian (3.15) is given by

$$U(T) = \exp(iH_{eff}T) = e^{i(E+\frac{m^2}{2E})}(e^{iV_{LL}}P_L + e^{iV_{RR}}P_R), \tag{3.16}$$

and since the overall phase is irrelevant[1] we can use the following operator

$$U(T) = e^{i\frac{m^2}{2E}}(e^{iV_{LL}}P_L + e^{iV_{RR}}P_R). \tag{3.17}$$

As a solution to the evolution equation (3.8), with relativistic approximation and low environment density we obtain a density matrix in the detection place (at distance $L = T$ from a production site in some specific direction)

$$\varrho(L) = U(L)\varrho U(L)^\dagger. \tag{3.18}$$

As our probability distribution in momentum space (2.19) does not depend on neutrino mass or neutrino spin, we can write

$$\varrho(L) = \frac{dj}{d\vec{p}} U(L)\tilde{\varrho}(\vec{p})U(L)^\dagger \equiv \frac{dj}{d\vec{p}}\tilde{\varrho}(\vec{p},L). \tag{3.19}$$

[1] This is equivalent to redefining the Hamiltonian by subtracting a constant energy E.

CHAPTER 4

Detection process

Contents

In this chapter we analyse the detection of neutrinos at distance L from a production site.

4.1 General formulae for the detection process

We learned in previous chapters that neutrinos arrive at the detector in a quantum state, described by the density matrix $\tilde{\varrho}(\vec{p}, L)$ (3.19). This matrix contains all the information about the neutrinos' state. In addition, we have learned about the neutrino momentum distribution given by the function $\frac{dj}{dp}$ (2.19). It now follows that we should choose a specific process that will serve us as a reaction to neutrino detection. In order to generalise our discussion, though, we do not choose any particular reaction; instead, we assume that the detection process is known and we are able to calculate an amplitude B to detect a neutrino with mass m_i, helicity λ and momentum $\vec{p}$: $B_i(\lambda, \vec{p}, x)$ where x denotes all other variables not directly related to the neutrino, such as the momentum and helicity of other particles participating in the detection process. The cross-section for the detection process is given by

$$\sigma(\vec{p}, L) = \int \frac{1}{F} \sum_{i,i',\lambda,\lambda'} B_i(\lambda, \vec{p}, x)[\tilde{\varrho}(\vec{p}, L)]_{\lambda i;\lambda' i'} B^*_{i'}(\lambda', \vec{p}, x) dLips(x), \tag{4.1}$$

where F is a flux in the case of two particle collisions $1+2 \to ...$, given by $F = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$[1]. The integral in (4.1) is calculated over all available phase space $dLips(x)$ which depends on a specific type of reaction, also the sum over spin states of final particles and the average over spins of initial state particles are included in $dLips(x)$.

The cross-section defined by e.q. (4.1) already contains an oscillation probability. In general, it cannot be written as the product of two factors $\sigma(\vec{p}, L) = \sigma(\vec{p})P(L, \vec{p})$

[1]This expression can be further simplified in LAB frame we obtain $F = m_2|\vec{p_1}|$ or in CM $F = |\vec{p_1}|\sqrt{s}$.

as we see in an SM with $P(L,\vec{p})$ being the universal oscillation probability. However, with some general assumptions we can say something more about the cross-section and oscillation probability. First of all, if the density matrix is diagonal in the neutrino helicity indices, i.e. $\tilde{\varrho}(\vec{p},L) = \varrho_{++}P_R + \varrho_{--}P_L$, the cross-section is the sum of two independent terms

$$\sigma(\vec{p},L) = \sigma(\vec{p},L)_+ + \sigma(\vec{p},L)_-, \tag{4.2}$$

with

$$\begin{aligned}
\sigma(\vec{p},L)_+ &= \int \frac{1}{F}\sum_{i,i'} B_i(+1,\vec{p},x)[\varrho_{++}]_{ii'}B^*_{i'}(+1,\vec{p},x)dLips(x), \\
\sigma(\vec{p},L)_- &= \int \frac{1}{F}\sum_{i,i'} B_i(-1,\vec{p},x)[\varrho_{--}]_{ii'}B^*_{i'}(-1,\vec{p},x)dLips(x).
\end{aligned} \tag{4.3}$$

In this case we can independently treat left and right neutrinos. Let us suppose that only one of the contributions (4.2) is non-zero (for definiteness, let it be the negative helicity part, as in the SM). In this case, if the detection amplitude factorises such that the $B_i(-1,\vec{p},x) = V_i\tilde{B}(\vec{p},x)$, then we can write

$$\sigma(\vec{p},L)_- = \sigma_0 P(\vec{p},L), \tag{4.4}$$

with probability $P(\vec{p},L) = \sum_{i,i'} V_i V^*_{i'}[\varrho_{--}]_{ii'}$ and cross-section $\sigma_0 = \int \frac{1}{F}|\tilde{B}(\vec{p},x)|^2 dLips(x)$. This immediately enables us to find an SM limit which agrees with the standard oscillation formula, since in SM, for neutrinos with flavour α, we have

$$[\varrho_{--}]_{ii'} = U^*_{\alpha i}U_{\alpha i'}e^{i\frac{m_i^2-m_{i'}^2}{2E}}, \tag{4.5}$$

and σ_0 is a standard neutrino detection cross-section, while matrix $V = U$ is a standard MNSP mixing matrix.

In general, if factorisation $B_i(-1,\vec{p},x) = V_i\tilde{B}(\vec{p},x)$ holds, the V_i factors can be normalised such that $\sum_i |V_i|^2 = 1$, in which case we have as usual $P(\vec{p},L) \leq 1$, so this quantity can indeed be interpreted as an oscillation probability.

For further convenience we can introduce an operator, which in the mass-helicity base is given by

$$[D]_{i,\lambda;i',\lambda'} = \int \frac{1}{F}B^*_i(\lambda,\vec{p},x)B_{i'}(\lambda',\vec{p},x)dLips. \tag{4.6}$$

As the density matrix does not depend on the detection reaction's phase space, we can write a formula for detecting the cross-section in the form

$$\sigma(\vec{p},L) = Tr[\tilde{\varrho}(\vec{p},L)D^\dagger]. \tag{4.7}$$

Similarly, as in the case of production, we can factorise the D operator such that

$$D = \tilde{D}\sigma_0, \tag{4.8}$$

with $\tilde{D} = \frac{D}{Tr[D]}$ and $\sigma_0 = Tr[D|$. As a result, the quantity

$$P(L, E) = Tr[\tilde{D}\tilde{\varrho}(\vec{p}, L)] \tag{4.9}$$

can be interpreted as a probability[2] that is not as universal as an SM probability but instead depends on a chosen detection process. In the case of an SM, it reduces to a well-known probability formula and is equal to the probability defined in (4.4) in theories where only left-handed neutrinos are present. The quantity

$$\sigma_0 = \int \sum_{i,\lambda} \frac{1}{F} |B_i(\lambda, \vec{p}, x)|^2 dLips(x) \tag{4.10}$$

is the detection cross-section when no oscillation is present (at a close detector, i.e. for $L \ll L_{osc}$ and we simply approximate $L = 0$ in all equations). It is now possible to write $\sigma(\vec{p}, L) = \sigma_0 P(L, E)$, which is connected to the fact that in the detection process no entanglement between neutrinos and other accompanying particles appears. Unfortunately, in general, the probability (4.9) is process-dependent; however, in theories like SM, where amplitudes for different processes have the same mass-dependent factor V_i (4.4), the oscillation probability is universal.

4.2 Specific processes

We now apply our general theory to specific cases of detection reactions.

4.2.1 Inverse muon decay

We analyse an inverse muon decay as a detection process, which can be described with the same Lagrangian

$$\mathcal{L}_I = -2\sqrt{2} G_F \times$$
$$\left[(g^S_{\alpha\beta})_{ij} (\overline{\nu}_i P_{-\alpha} e) (\overline{\mu} P_\beta \nu_j) + (g^V_{\alpha\beta})_{ij} \left(\overline{\nu}_i \gamma^\delta P_\alpha e\right) \left(\overline{\mu} \gamma_\delta P_\beta \nu_j\right)\right] + h.c. \tag{4.11}$$

as muon decay. More details can be found in Appendix D. We now calculate the D operator (4.6). Let us denote an SM cross-section for the process $\nu_\mu + e \to \mu + \nu_e$ as

$$\sigma_{SM} = \frac{G_F^2 \left(s - M^2\right)^2}{\pi s}, \tag{4.12}$$

where, as always, M is a muon mass and G_F a Fermi constant. The matrix elements of D (4.6) for negative and positive neutrino helicity for inverse muon

[2]We can call this quantity a probability because it is a trace of two non-negative, hermitian operators with a unit trace each, so $0 \leq P(L, E) \leq 1$.

decay are given by

$$\begin{aligned}
D_{--} &= \frac{(M^2+2s)((g^S_{LL})^\dagger g^S_{LL} + (g^S_{RL})^\dagger g^S_{RL} + 4(g^V_{RL})^\dagger g^V_{RL})}{24s}\sigma_{SM} \\
&+ (g^V_{LL})^\dagger g^V_{LL}\sigma_{SM}, \\
D_{++} &= \frac{(M^2+2s)((g^S_{RR})^\dagger g^S_{RR} + (g^S_{LR})^\dagger g^S_{LR} + 4(g^V_{LR})^\dagger g^V_{LR})}{24s}\sigma_{SM} \\
&+ (g^V_{RR})^\dagger g^V_{RR}\sigma_{SM}. \qquad (4.13)
\end{aligned}$$

As we can see, there are no interference terms between different types of interactions. Assuming that only g^S_{LL} and g^V_{LL} are non-zero, then the σ_0 (4.10) is given by

$$\sigma_0 = \left(Tr[(g^S_{LL})^\dagger g^S_{LL}]\frac{(M^2+2s)}{24s} + Tr[(g^V_{LL})^\dagger g^V_{LL}]\right)\sigma_{SM}. \qquad (4.14)$$

Furthermore, quantity $\tilde{D}$ is equal to

$$\tilde{D} = \frac{(g^S_{LL})^\dagger g^S_{LL}(M^2+2s) + (g^V_{LL})^\dagger g^V_{LL}24s}{Tr[(g^S_{LL})^\dagger g^S_{LL}](M^2+2s) + Tr[(g^V_{LL})^\dagger g^V_{LL}]24s}. \qquad (4.15)$$

As evidenced above, the effects of different interactions are included in eq. (4.13). In order to estimate possible effects, we now calculate the oscillation probability (4.9) in a couple of simple examples. We assume that in our theory only vector NSI contributes to the production and detection processes, which means that D_{++} is zero and $\tilde{D} = \frac{(g^V_{LL})^\dagger g^V_{LL}}{Tr[(g^V_{LL})^\dagger g^V_{LL}]}$.

In the first example we assume that only an ordinary SM $W^\pm$ boson exchange contributes to the detection process with the usual flavour structure of coupling constants, i.e the detection operator (4.6) is given by $|\tilde{D}|_{ij} = \delta_{i\alpha}\delta_{j\alpha}$. For further convenience we also introduce an operator $[X^\mu]_{\beta,\beta'} = U^*_{\mu,\beta}U_{\mu\beta'}$ ($U = Exp(-iP^\mu x_\mu)$ (3.16)), which represents the propagation process. In the linear approximation (2.31) we obtain

$$P(L,E) = |N'|^2(P^{SM}(L,E) + 2Re([\varepsilon^T X^\mu]_{e\mu})), \qquad (4.16)$$

where ε, N', are defined in the same way as in eq. (2.31, 2.32, 2.33) and $P^{SM}(L,E)$ is an ordinary SM oscillation probability. If, instead of (2.31), we use a pure state (2.32) $|\nu\rangle = \sum_\alpha(\delta_{\alpha\mu} + \varepsilon_{\alpha e})|\nu_\alpha\rangle$, we obtain a different result. The difference is of the second order, i.e the term $[\varepsilon^* X^\mu \varepsilon^T]_{ee}$ also appears. The full calculation, without linear approximation, leads to the formula

$$P(L,E) = |N''|^2(P^{SM}(L,E) + 2Re([\varepsilon^T X^\mu]_{e\mu}) + Tr[\varepsilon^* X^\mu \varepsilon^T]), \qquad (4.17)$$

where now $N'' = N' - ||\varepsilon||^2$.

The differences between these two approaches, based on DM formalism and using pure QM states as usual, are small and can be neglected in present experiments because the bound on NSI [54] constrains $\varepsilon_{\alpha\beta} \lesssim 0.03^3$. This difference enables us to estimate the difference as being no greater than $||\varepsilon||^2$, which is of the order one per mille.

Since we consider inverse muon decay as a detection process, it is natural to assume that NSI contributes also to a detection operator. Assuming that the NP contribution to the detection and production process are given by the same Lagrangian, then in linear approximation we obtain

$$P(L,E) = |N'|^4(P^{SM}(L,E) + 2Re([\varepsilon^T X^\mu]_{e\mu}) + 2Re([\varepsilon^T \tilde{X}^\mu]_{e\mu})), \quad (4.18)$$

where we have introduced a $[\tilde{X}^\mu]_{\beta\beta'} = U^*_{\beta\mu}U_{\beta'\mu}$. A calculation up to a second order in NSI gives the result

$$\begin{aligned} P(L,E) = |N''|^4(P^{SM}(L,E) + 2Re([\varepsilon^T X^\mu]_{e\mu}) + 2Re([\varepsilon^T \tilde{X}^\mu]_{e\mu}) + Tr[\varepsilon^* X^\mu \varepsilon^T] + \\ 2Re(U_{\mu\nu}[\varepsilon^T U\varepsilon]_{ee} + [U^*\varepsilon]_{\mu e}[\varepsilon^T U]_{e\mu}) + Tr[\varepsilon \tilde{X}^\mu \varepsilon^\dagger]) \end{aligned} \quad (4.19)$$

Moreover, higher order terms appear, but we can safely neglect them.

4.2.2 Deep inelastic scattering

In this case we assume that there are no scalar interactions between quarks and neutrinos. This assumption is justified by strong bounds obtained in pion leptonic decay [55]. Let us assume the following form of leptonic tensor:

$$\begin{aligned} L^{\mu\nu} &= K_L 8[k^\mu k'^\nu + k'^\mu k^\nu + g^{\mu\nu}q^2/2 + i\varepsilon^{\mu\alpha\nu\beta}k'_\alpha k_\beta] \\ &+ K_R 8[k^\mu k'^\nu + k'^\mu k^\nu + g^{\mu\nu}q^2/2 - i\varepsilon^{\mu\alpha\nu\beta}k'_\alpha k_\beta], \end{aligned} \quad (4.20)$$

where K_L and K_R are coupling constant matrices, k and k' are neutrinos and lepton four momentum, while $q = k - k'$. To calculate the detection cross-section it is useful to note that the leptonic tensor for antineutrinos in SM is exactly equal to the leptonic tensor calculated with a right-handed current. Therefore, following the standard derivation of the DIS scattering formula, and assuming that the density matrix is diagonal in the helicity base, we obtain for (4.3)

$$\sigma(\vec{p},L)_+ = Tr[\varrho_{++}K_R^\dagger]\sigma_{\bar{\nu}}^{SM}, \quad (4.21)$$

$$\sigma(\vec{p},L)_- = Tr[\varrho_{--}K_L^\dagger]\sigma_{\nu}^{SM}, \quad (4.22)$$

where σ_ν^{SM} ($\sigma_{\bar{\nu}}^{SM}$) is the cross-section for the DIS of neutrinos (antineutrinos) measured under the assumption that SM is valid. Let us assume that there are no non-standard neutrino interactions of vector type with left-handed neutrinos, in which case part $Tr[\varrho_{--}K_L^\dagger]$ is of a purely SM origin and is therefore equal to the SM oscillation probability. Using present bounds [36, 54, 56] on New Physics parameters, we can estimate that quantity $Tr[\varrho_{++}K_R^\dagger]$ can take any value from 0 to 5×10^{-3}. This

[3] This bound is obtained assuming that only one element of the matrix ε is non-zero. We therefore must use the same assumption while estimating the possible effect of NSI

follows on from the fact that elements of matrix K_R do not exceed value 0.017[4]. In a similar way we can estimate ϱ_{++}, since it is proportional to the NSI parameters that produce right-handed neutrinos. If NSIs are of vector type, then elements of ϱ_{++} do not exceed 0.001, which is the square of a parameter $g^V_{RR} < 0.034$ [36, 56]. Conversely, if right-handed neutrinos can be produced by scalar interactions, then the bounds are much worse, $g^S_{RR} < 0.55$, so the elements of ϱ_{++} do not exceed 0.3. However, this value is much lower if we assume that scalar right-handed interactions are the only ones to contribute, in which case $g^S_{RR} < 0.058$ [56]. A detailed study of possible effects would require choosing some specific models instead of working with effective operators and is beyond the scope of this work.

[4]Elements of K_R are proportional to the square of NSI parameters governing neutrino quarks' interactions, which are bounded in pion decay and NOMAD experiments. The biggest parameter does not exceed 0.13. These details have been considered many times in the literature – see [36, 54].

CHAPTER 5
Majorana neutrinos

In this chapter we investigate the possible differences between Dirac [57] and Majorana [58] neutrinos in muon decay, by using the DM formalism presented in previous chapters. This is an interesting case because if a neutrino is a Majorana particle, additional interference between different amplitudes will appear, thus contributing to muon decay.

It is not easy to determine if a neutrino is a Dirac or Majorana particle in the SM. It has been shown that it is not possible to determine the neutrino's nature in the oscillation process [59], and a practical Dirac-Majorana confusion theorem [60] has been formulated. A reason for this theorem being true is that helicity of a high energetic neutrino can be seen as a quantity which, with very good accuracy, acts as a conserved charge. Therefore, even if the neutrino is its own antiparticle, both neutrino states can be distinguished by helicity, if the interactions are only of the left-handed type. This forces us to consider NP models with both left- and right-handed neutrino interactions, particularly as we want to distinguish between Dirac and Majorana particles[1]. It has also been shown [61] that when neutrinos are not observed in muon decay, it is not possible to distinguish between Dirac and Majorana particles, even if any type of NP interaction is present, so we will calculate our observables which are based on neutrinos only.

Therefore, let us investigate a case with both left- and right-handed neutrinos produced in muon decay. It is best to choose only g^S_{RR} and g^V_{LL} in $D.1$ as being non-zero because g^S_{RR} can produce the largest effects. From our considerations in chapter 2, it follows that in the Dirac neutrino case the density matrix has the form

$$\varrho = \frac{4x^2(2(g^V_{LL})^\dagger g^V_{LL}(2x-3)P_- + 3(g^S_{RR})^\dagger g^S_{RR}(1-x)P_+)}{Tr[(g^S_{RR})^\dagger g^S_{RR}] + 4Tr[(g^V_{LL})^\dagger g^V_{LL}]}, \tag{5.1}$$

while in the case of the Dirac antineutrino we have

$$\varrho = \frac{2x^2(24(g^V_{LL})^T(g^V_{LL})^*(1-x)P_+ + (g^S_{RR})^T(g^S_{RR})^*(2x-3)P_-)}{Tr[(g^S_{RR})^\dagger g^S_{RR}] + 4Tr[(g^V_{LL})^\dagger g^V_{LL}]}. \tag{5.2}$$

We can easily obtain amplitudes for Majorana neutrinos by antisymmetrising the amplitudes calculated in the Dirac case in Appendix D with respect to both neutrinos and we obtain

$$\varrho = \frac{K^\dagger K(2x-3)x^2P_- + 6K^*K^T(1-x)x^2P_+}{Tr[K^\dagger K]} \tag{5.3}$$

[1] Another possibility is to look for neutrino-less double beta decay, but we do not consider this here.

where the K matrix is given by $K = 2g^V_{LL} + (g^S_{RR})^T$. Now we must look carefully at the normalisation of our matrices. The coupling must be normalised such that it reproduces the measured decay rate, which means that for Dirac neutrinos we must have [56]

$$Tr[(g^S_{RR})^\dagger g^S_{RR}] + 4Tr[(g^V_{LL})^\dagger g^V_{LL}] = 4; \tag{5.4}$$

however, the in Majorana case, the condition is different [61] due to the interference between the scalar and vector amplitudes

$$Tr[K^\dagger K] = 4. \tag{5.5}$$

When comparing Dirac and Majorana neutrinos, mass-flavour structure is very important. If we assume that $L = 0$, i.e. we are interested in close detector results, we can calculate the probability of neutrino conversion. Even though the density matrices for Dirac (5.1 5.2) and Majorana (5.3) have different forms, the probability is the same for both the Dirac and Majorana neutrinos. We assume that the detection process is described only by the SM contribution, with $W^\pm$ exchange and the usual mass flavour relation. This assumption simplifies considerations significantly and can be justified. For example, if neutrinos are detected in the deep inelastic scattering reaction, new scalar interactions with quarks can be classed as negligible. As such, a simple formula for the conversion probability can be found[2]

$$P(0)_{\mu\to\mu} = 1 - \frac{||g^S_{RR}||^2}{4}(1-\xi) \tag{5.6}$$

$$P(0)_{\mu\to e,\tau} = \frac{||g^S_{RR}||^2}{4}(1-\xi) \tag{5.7}$$

where ξ is a number between zero and one ($\xi \in [0,1]$), which depends on the flavour structure of the scalar interactions, and $||\cdot||$ represents, as usual, the Hilbert-Schmidt norm of a matrix. This formula is true for both the Dirac and Majorana neutrinos; however, whereas for Dirac neutrinos a bound from inverse muon decay measurement $||g^S_{RR}|| < 0.55$ [36] exists, for the Majorana neutrinos this condition does not hold.

Therefore, the possibility exists that we can distinguish between Dirac and Majorana neutrinos in a nearby detector, although in reality this is very unlikely. In order to do that we have to measure inverse muon decay very precisely, so that we can constrain the bound for Dirac neutrinos such that we have $||g^S_{RR}|| < g_{max}$ for some measured value of g_{max}, and then if we observe $P(0)_{\mu\to e,\tau} > \frac{g^2_{max}}{4}$ or $P(0)_{\mu\to\mu} < 1 - \frac{g^2_{max}}{4}$, we could conclude that neutrinos are Majorana particles. Unfortunately, this is model-dependent and we would also need to establish a charged scalar contribution to muon decay to produce a definite conclusion in relation to this point. Present bounds are not robust enough to provide sufficient information

[2] In order to prove that we need to assume some general form of matrix g^S_{RR} and then perform explicit calculations, we can factorise the norm of matrix g^S_{RR} and observe that the remaining part is bounded between 0 and 1.

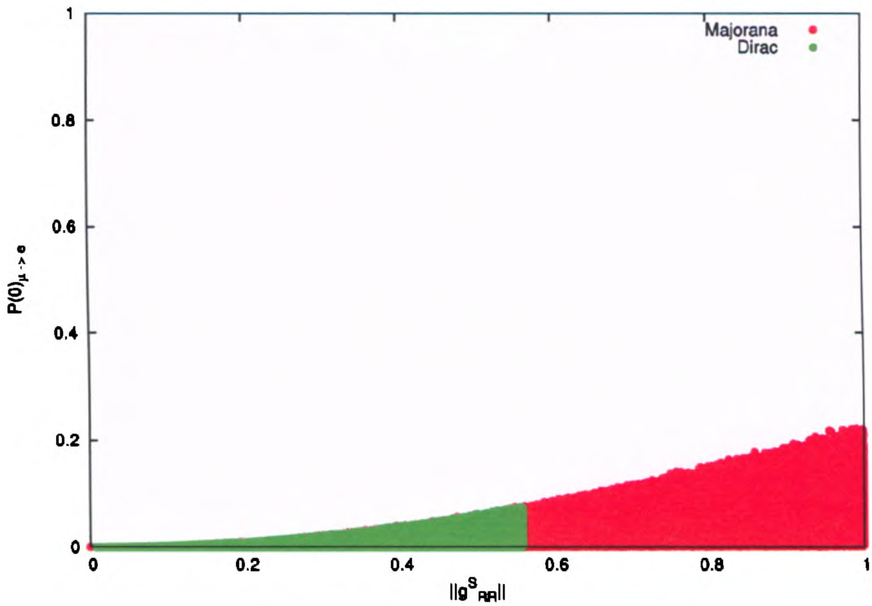

Figure 5.1: The $P(0)_{\mu\to e}$ for the Dirac and Majorana neutrinos. Matrices g^S_{RR} were generated randomly, while g^V_{LL} were assumed to have the SM flavour structure and $||g^V_{LL}|| > 0.96$, and only constraints coming from overall normalisation to G_F were applied.

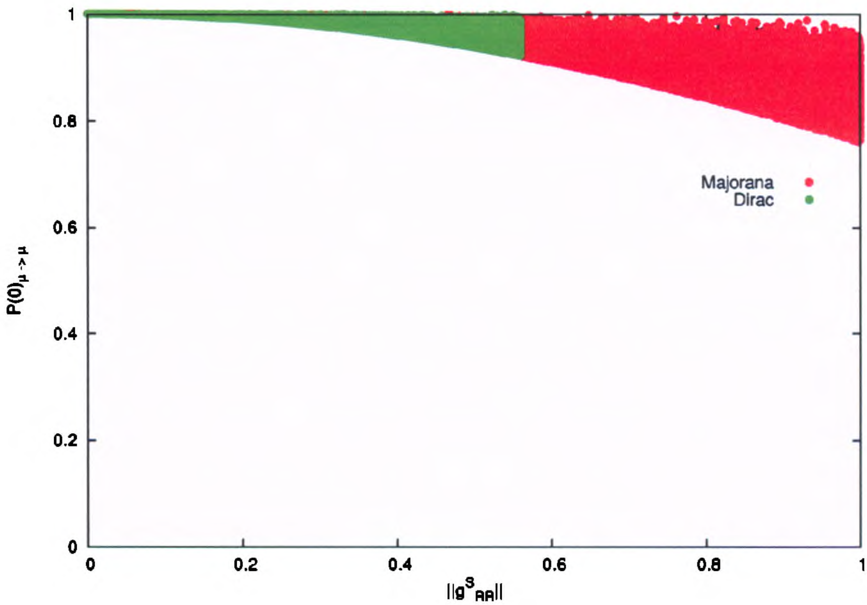

Figure 5.2: The $P(0)_{\mu\to\mu}$ for the Dirac and Majorana neutrinos. Matrices g^S_{RR} were generated randomly while g^V_{LL} were assumed to have the SM flavour structure and $||g^V_{LL}|| > 0.96$, and only constraints coming from overall normalisation to G_F were applied.

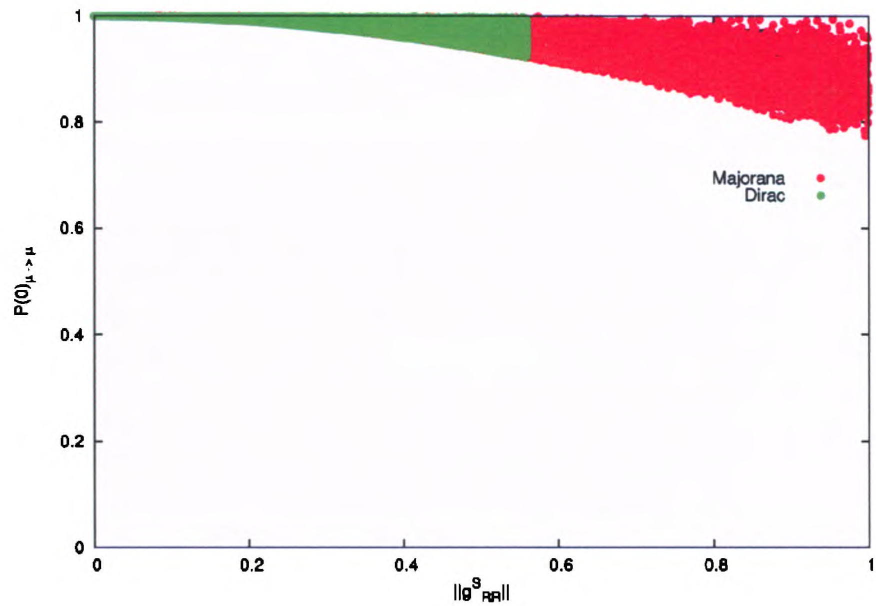

Figure 5.3: The $P(0)_{\mu\to\mu}$ for the Dirac and Majorana neutrinos. Matrices g^S_{RR} were generated randomly while g^V_{LL} were assumed to have the SM flavour structure and $||g^V_{LL}|| > 0.96$, and both constraints coming from overall normalisation to G_F and cuts from $P(0)_{\mu\to e}$ were applied.

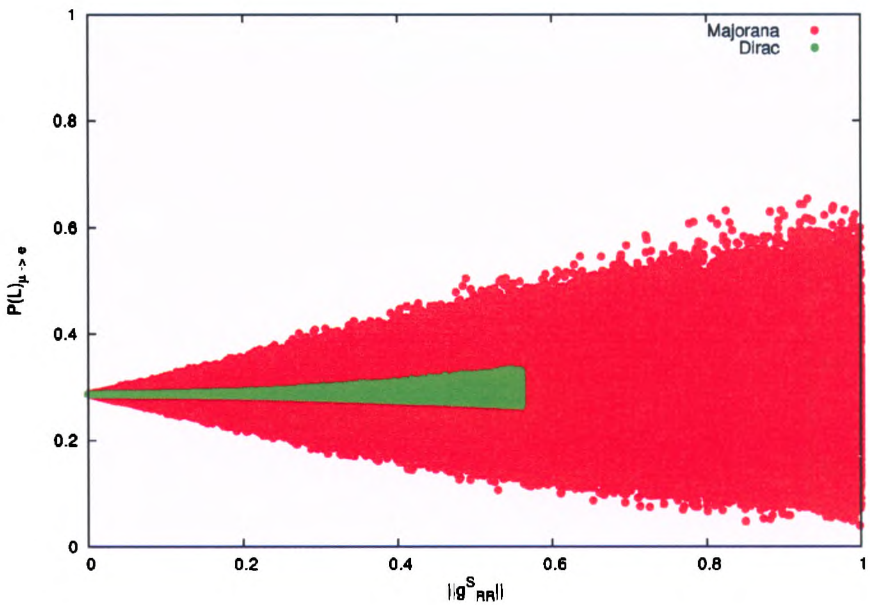

Figure 5.4: The $P(L)_{\mu\to e}$ for the Dirac and Majorana neutrinos. Matrices g^S_{RR} were generated randomly while g^V_{LL} were assumed to have the SM flavour structure and $||g^V_{LL}|| > 0.96$, and only constraints coming from overall normalisation to G_F were applied. We chosen $\frac{L}{E} = 15000$ and we assumed that the detector had perfect energy resolution.

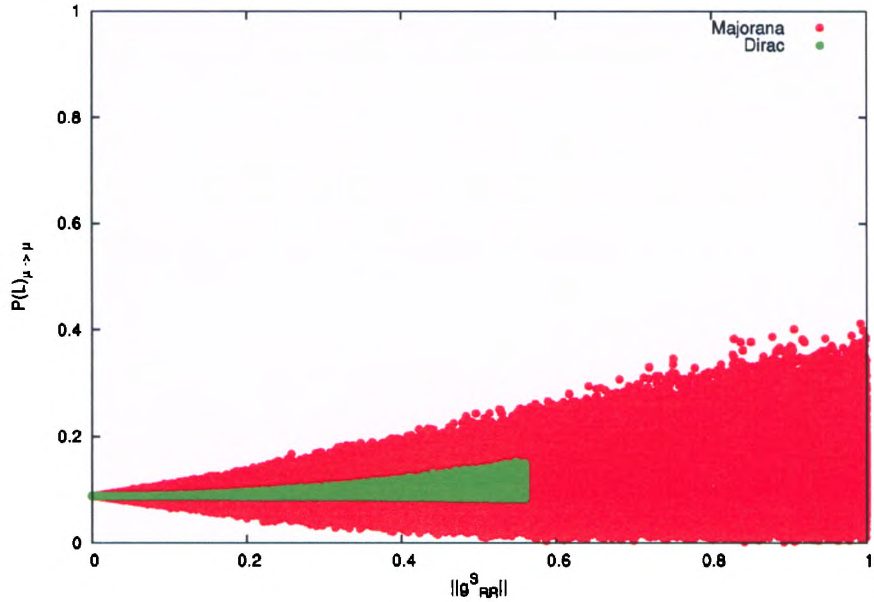

Figure 5.5: The $P(L)_{\mu\to\mu}$ for the Dirac and Majorana neutrinos. Matrices g^S_{RR} were generated randomly, while g^V_{LL} were assumed to have the SM flavour structure and $||g^V_{LL}|| > 0.96$, and only constraints coming from overall normalisation to G_F were applied. We chosen $\frac{L}{E} = 15000$ and we assumed that the detector had perfect energy resolution.

about neutrino nature. In fact, we have checked this numerically by generating 10^7 random complex matrices g^S_{RR}, after which we applied bounds from experiments measuring $P(0)_{\mu\to e}$. We plotted all possible values of $P(0)_{\mu\to e}$ in Figure 5.1 and $P(0)_{\mu\to\mu}$ in Figure 5.2 for the Dirac and Majorana neutrinos. We then applied the cuts from experimental results for $P(0)_{\mu\to e}$[3] (see [54] and references therein), but the result did not change in any significant way, as can be seen in Figure 5.3 (comparing Figure 5.2, where only overall normalisation constrains were applied, we see that the pattern does not change). We therefore conclude that it would be extremely difficult to distinguish Dirac and Majorana neutrinos in a nearby detector because it would require very high values of $||g^S_{RR}||$ and the specific flavour structure of g^S_{RR}, such that we would observe a large deficit of muon neutrinos. The situations improve dramatically if we look for neutrinos in a far detector. Assuming perfect energy resolution of the detector for $\frac{L}{E} = 15000$, we plotted possible values of oscillation probability. In Figure 5.4, we plotted $P(L)_{\mu\to e}$, while in Figure 5.5 , $P(L)_{\mu\to\mu}$ was depicted.

As we can see in this case, possible values of oscillation probability for Majorana

[3]Due to the different flavour structures of scalar interaction, the flavour conversion of neutrinos can appear at zero distance. KARMEN and NOMAD experiments [62, 63] have not detected any flavour conversion at short distance, so we can use this result to constrain $P(0)_{\mu\to e}$.

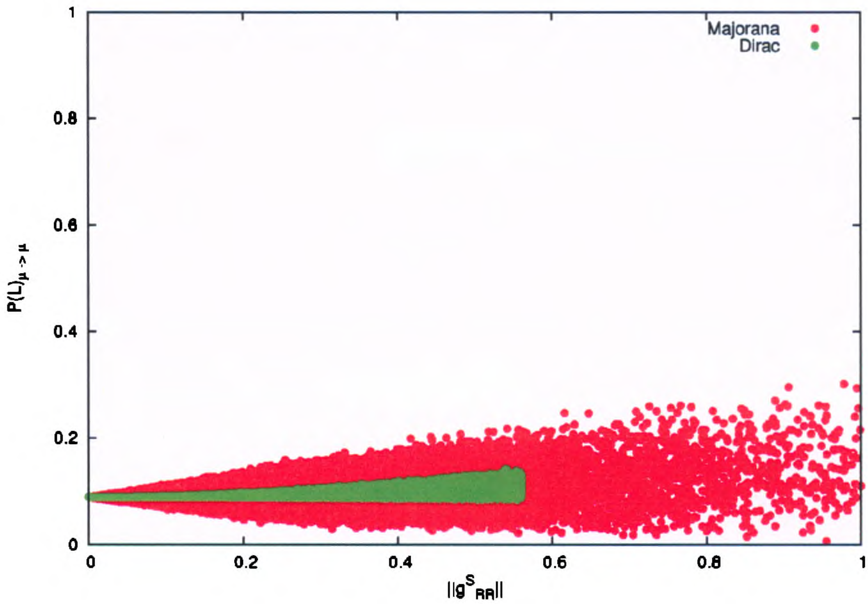

Figure 5.6: The $P(L)_{\mu\to\mu}$ for the Dirac and Majorana neutrinos. Matrices g^S_{RR} were generated randomly, while g^V_{LL} were assumed to have the SM flavour structure and $||g^V_{LL}|| > 0.96$, and only constraints coming from overall normalisation to G_F were applied. We chosen $\frac{L}{E} = 15000$ and we assumed that the detector had perfect energy resolution.

neutrinos exceed the region allowed for Dirac neutrinos, even for small values of $||g^S_{RR}||$. In this case, if we establish a scalar right-handed contribution to muon decay (which is very common in many models of NP), then observing the probability beyond the region allowed for Dirac neutrinos, we can conclude that neutrinos are Majorana particles. Similarly, as in the case of the near detector, the application of cuts from $P(0)_{\mu \to e}$ [54] does not change the possible patterns for the muon neutrino disappearance experiment, as can be seen in Figure 5.6.

CHAPTER 6

Summary

In this thesis we presented a formalism based on QM mixed states, which is important in the phenomenological analysis of planned neutrino oscillation experiments. Using the basic principles of QM, we provided the proper definition of neutrino states. Consequently, our formalism can be applied to any QFT that describes neutrino interactions. We only need to calculate the neutrino's production amplitudes, which later defines the neutrino state using the construction presented in Chapter 2 of this thesis. We also showed that when production amplitude satisfies certain conditions, the neutrino state can be pure. This situation occurs for usual production process in SM; however, in general, the neutrino states are mixed in the QM sense.

Furthermore, the Lorentz transformation properties of DM matrix were presented. We demonstrated that, due to small neutrino masses, they do not change the helicity structure of the neutrino DM, which subsequently simplifies calculations in a very significant way. Also, a condition was provided that enabled us to calculate which neutrinos will reach the detector, depending on its distance and size. We also devised a calculation for neutrino propagation in matter with general types of interaction, based on already known effective potential methods. This method is easily accommodated into the DM approach. However, we did not present a theory of neutrino propagation in a dense environment, although it could be taken into account in further research.

The calculations for the detection cross-section were presented and we discussed the problem of properly defining oscillation probability in the general case of NSI. Usually, oscillation probability depends on the detection process, so in practice it is best to use only a general formula for the detection cross-section, which includes oscillations. However, it was shown that it is possible to define a quantity that may be called oscillation probability and is helpful in the analysis of oscillation experiments. In addition, we presented example calculations using muon decay as a production process.

The example of neutrinos produced in muon decay showed that the usual effects connected with NSI in production and detection process are small and will require the next generation of more precise neutrino experiments to produce telling results. The most promising NP contributions in muon decay are scalar right-handed interactions that, as we calculated, can give significant effects which in general may be different for Dirac and Majorana neutrinos. Although the determination of neutrinos' nature in nearby detectors is extremely unlikely, if scalar right-handed interactions are present, oscillation experiments may confirm that neutrinos are indeed

Majorana particles.

Lorentz Transformations

In this appendix we will derive the Lorentz transformations of a density matrix by using the properties of the helicity state and following the original paper by Jacob and Wick [64] as well as a good review by Leader [65]. In addition, we will use a Jackson convention [66]. We will denote the helicity states by $|\vec{p}, \lambda\rangle$, while $|p_0, s, s_z = \lambda/2\rangle$ is a state of particle at rest with spin s and a third component of spin s_z. The p_0 will be a standard four-vector, i.e. $p_0 = (m, 0, 0, 0)$. By $r(\alpha, \beta, \gamma) = r_z(\alpha) r_y(\beta) r_z(\gamma)$ we mean a rotation through the Euler angles

$$r_z(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.1}$$

$$r_y(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\beta) & 0 & \sin(\beta) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \tag{A.2}$$

and by $l_z(\beta)$ we denote a boost in direction z with velocity β.

$$l_z(\beta) = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix}. \tag{A.3}$$

Now let us assume that $\vec{p}$ has polar angles θ, ϕ, in which case the helicity state is defined by:

$$|\vec{p}, \lambda\rangle = U[h(\vec{p})]|p_0, s, s_z = \lambda/2\rangle, \tag{A.4}$$

with

$$h(\vec{p}) = r(\phi, \theta, 0) l_z(v), \tag{A.5}$$

$v = \frac{|\vec{p}|}{E}$ and $U[\mathrm{A}]$ is unitary operators from representation of the Lorentz group, which corresponds to element A (in our case $\mathrm{A} = h(\vec{p})$). Now we proceed to calculate the effect of the Lorentz transformation on the helicity state

$$|\vec{p}', \lambda'\rangle = U[\Lambda]|\vec{p}, \lambda\rangle. \tag{A.6}$$

The components of $\vec{p'}$ can be calculated easily, since $p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu}$, with p^{ν} being a momentum four-vector with spatial components $\vec{p}$. Let us now multiply Eq. (A.6) by unity $U[h(\vec{p'})]U^{-1}[h(\vec{p'})]$ and use definition (A.4)

$$U[\Lambda]|\vec{p},\lambda\rangle = U[h(\vec{p'})]\mathcal{R}|p_0,s,\lambda\rangle, \tag{A.7}$$

where $\mathcal{R} = U^{-1}[h(\vec{p'})]U[\Lambda]U[h(\vec{p})] = U|h^{-1}(\vec{p'})\Lambda h(\vec{p})|$. $\mathcal{R}$ represents a rotation. To view this, let us start with p_0, following which $h(\vec{p})$ transforms p_0 to p, Λ changes p to p' and finally $h^{-1}(\vec{p'})$ changes p' to p_0. As such, the standard four-vector remains unchanged, and from its form we observe that it is only unaffected by rotations, so $\mathcal{R}$ must be a rotation. We will call this 'Wick helicity' rotation and denote it by

$$r(\Lambda,\vec{p}) = h^{-1}(\vec{p'})\Lambda h(\vec{p}). \tag{A.8}$$

We now define a matrix representation of the rotation group, let r be rotation and then

$$\mathcal{D}^{s}_{\lambda'\lambda}(r) = \langle p_0,s,\lambda'|U[r]|p_0,s,\lambda\rangle. \tag{A.9}$$

Now we can write

$$|\vec{p'},\lambda'\rangle = U[\Lambda]|\vec{p},\lambda\rangle = \sum_{\lambda} \mathcal{D}^{s}_{\lambda\lambda'}(r(\Lambda,\vec{p}))|\vec{p'},\lambda\rangle. \tag{A.10}$$

To acquire a neutrino state in the LAB frame we need to know the Wick helicity rotation, which is connected with a boost from the CM to the LAB frame. We will assume that we have performed the boost in z direction with velocity β, and a neutrino with mass m has a momentum $\vec{p}$ in CM and velocity v. While in the LAB, the neutrino has momentum $\vec{p'}$[1] and velocity v'. Then, according to (A.8), the Wick helicity rotation is given by:

$$r(l_z(\beta),\vec{p}) = h^{-1}(\vec{p'})l_z(\beta)h(\vec{p}), \tag{A.11}$$

with

$$h(\vec{p}) = r(\phi,\theta,0)l_z(v), \quad h(\vec{p'}) = r(\phi,\theta',0)l_z(v'). \tag{A.12}$$

By performing an explicit calculation we find that $r(l_z(\beta),\vec{p})$ is a rotation around the y axis, so we can simply write

$$r(l_z(\beta),\vec{p}) = r_y(\theta_{Wick}), \tag{A.13}$$

with

$$\sin\theta_{Wick} = \frac{m\beta\sin(\theta)}{\sqrt{\frac{1}{2}\beta^2\left(2m^2+p^2\cos(2\theta)+p^2\right)+2p\beta\sqrt{m^2+p^2}\cos(\theta)+p^2}}, \tag{A.14}$$

[1]If momentum $\vec{p}$ in the spherical coordinates has length $|\mathbf{p}|$ and angles θ and ϕ, then $\vec{p'}$ has the same angle ϕ, since the boost is in z direction.

$$\cos\theta_{Wick} = \frac{\beta\sqrt{m^2+p^2}\cos(\theta)+p}{\sqrt{\frac{1}{2}\beta^2\left(2m^2+p^2\cos(2\theta)+p^2\right)+2p\beta\sqrt{m^2+p^2}\cos(\theta)+p^2}}. \quad \text{(A.15)}$$

Now, expanding this relation for a small neutrino mass we obtain

$$\sin\theta_{Wick} = \frac{m\beta\sin(\theta)}{p\beta\cos(\theta)+p} - \frac{m^3\left(\beta^2\sin(\theta)(\beta+\cos(\theta))\right)}{2(p\beta\cos(\theta)+p)^3} + O\left(m^4\right), \quad \text{(A.16)}$$

$$\cos\theta_{Wick} = 1 - \frac{m^2\left(\beta^2\sin^2(\theta)\right)}{2(p\beta\cos(\theta)+p)^2} + O\left(m^4\right), \quad \text{(A.17)}$$

so we can observe that θ_{Wick} is close to zero. Now let us check how this affects the neutrino's density matrix, which, after Lorentz transformation, is given by

$$\varrho' = U[\Lambda]\varrho U^{\dagger}[\Lambda], \quad \text{(A.18)}$$

or explicitly on matrix elements

$$[\varrho'(p')]_{\lambda' i,\chi' k} = \mathcal{D}^{s}_{\lambda\lambda'}(r(\Lambda,\vec{p}))(\mathcal{D}^{s}_{\chi\chi'}(r(\Lambda,\vec{p})))^{\dagger}[\varrho(p)]_{\lambda i,\chi k}. \quad \text{(A.19)}$$

Now, since in our case the Wick helicity rotation is a rotation around the y axis, we write

$$\mathcal{D}^{s}_{\lambda\lambda'}(r(l_z(\beta),\vec{p})) = d^{s}_{\lambda\lambda'}(\theta_{Wick}). \quad \text{(A.20)}$$

In the neutrino's case, what we need is a matrix element

$$d^{1/2}_{\lambda\lambda'}(\theta_{Wick}) = \langle p_0, s, \lambda'|e^{i\hat{s}_y\theta_{Wick}}|p_0, s, \lambda\rangle = [e^{i\sigma_y\theta_{Wick}/2}]_{\lambda\lambda'}. \quad \text{(A.21)}$$

Using the properties of σ matrices this can be written as

$$d^{1/2}_{\lambda\lambda'}(\theta_{Wick}) = \delta_{\lambda\lambda'}\cos(\theta_{Wick}/2) + i[\sigma_y]_{\lambda\lambda'}\sin(\theta_{Wick}/2), \quad \text{(A.22)}$$

so, as expected (because $\mathcal{D}^{s}_{\lambda\lambda'}(r)$ form a representation of the rotation group in the neighbourhood of the unit element), for small angles θ_{Wick} we can use the following approximation:

$$d^{1/2}_{\lambda\lambda'}(\theta_{Wick}) = \delta_{\lambda\lambda'}, \quad \text{(A.23)}$$

which in practice means that the Lorentz boost from the CM frame to the LAB does not affect any discrete variables of the density matrix.

APPENDIX B

Useful algebraic theorems

Some of these theorems and their proof can be found in [67].

Theorem B.1. *Let B be a matrix and $A = B^\dagger B$. If $Tr[A] = 0$, then $B = 0$.*

Proof. A is positively defined, since

$$\langle x|A|x\rangle = \langle x|B^\dagger B|x\rangle = ||B|x\rangle||^2 \geqslant 0 \tag{B.1}$$

for any $|x\rangle$. Now let $(|i\rangle)_{i=1,n}$ form a base, following which

$$0 = Tr[A] = \sum_{i=1}^{n}\langle i|A|i\rangle, \tag{B.2}$$

but since A is positively defined, then $\langle i|A|i\rangle = ||B|i\rangle||^2 = 0$ for every i. If we now let $|x\rangle = \sum_{i=1}^{n} c_i|i\rangle$ be any vector, then

$$||B|x\rangle|| = ||\sum_{i=1}^{n} c_i B|i\rangle|| \leqslant \sum_{i=1}^{n} |c_i|||B|i\rangle|| = 0, \tag{B.3}$$

so $B|x\rangle = 0$, but since $|x\rangle$ is any vector, then $B = 0$. □

Theorem B.2. *If C is positively defined, all of its square roots are related by unitary transformation.*

Proof. Suppose that $C = A^\dagger A = B^\dagger B$. We can now consider two cases. First, assuming that B is invertible, we define $U = AB^{-1}$. In this case,

$$U^\dagger U = B^{\dagger -1}A^\dagger AB^{-1} = B^{\dagger -1}CB^{-1} = B^{\dagger -1}B^\dagger BB^{-1} = I. \tag{B.4}$$

If B is not invertible, since C is positively defined, we can define a Moore-Penrose pseudo-inverse[1] Let us consider a matrix $\tilde{U} = AB^+$. W will prove that $\tilde{U}$ is partial

[1] A Moore-Penrose pseudo-inverse [68, 69] of matrix A, which is defined as matrix A^+ and satisfies the following conditions:

$$AA^+A = A$$
$$A^+AA^+ = A^+$$
$$(AA^+)^\dagger = AA^+$$
$$(A^+A)^\dagger = A^+A$$

It can be easily proven that A^+ is unique.

isometry, i.e. $\tilde{U}\tilde{U}^+$ is a projector. In this instance we can define $P = \tilde{U}\tilde{U}^+$. The fact that P is a hermitian is obvious, so let us calculate P^2

$$\begin{aligned} P^2 &= AB^+B^{+\dagger}A^\dagger AB^+B^{+\dagger}A^\dagger = AB^+B^{+\dagger}B^\dagger BB^+B^{+\dagger}A^\dagger \\ &= AB^+(BB^+)^\dagger BB^+B^{+\dagger}A^\dagger = AB^+BB^+BB^+B^{+\dagger}A^\dagger = AB^+BB^+B^{+\dagger}A^\dagger \\ &= AB^+B^{+\dagger}A^\dagger = \tilde{U}\tilde{U}^\dagger = P \end{aligned} \tag{B.5}$$

Consequently, P is a projection, so $\tilde{U}$, which is a partial isometry, can be extended to unitary operator U with a domain equal to a whole space [70]. We finally obtain $UB = AB^+B = AA^+A = A$. □

Theorem B.3. *Let matrix A have r non-zero singular values σ_i, $i = 1, ..., r$ and let $\{u_1, ..., u_r\}$ be a orthonormal set of eigenvectors for $AA^\dagger$, i.e. $AA^\dagger u_i = \sigma_i^2 u_i$. If we now define a set of vectors $v_i = \frac{1}{\sigma_i}A^\dagger u_i$, then*

$$A^\dagger A v_i = \sigma_i^2 v_i \tag{B.6}$$

$$(v_i, v_j) = \delta_{ij} \tag{B.7}$$

$$u_i = \frac{1}{\sigma_i} A v_i \tag{B.8}$$

Proof.

$$A^\dagger A v_i = \frac{1}{\sigma_i} A^\dagger AA^\dagger u_i = \sigma_i A^\dagger u_i = \sigma_i^2 v_i \tag{B.9}$$

$$(v_i, v_j) = \frac{1}{\sigma_i\sigma_j}(A^\dagger u_i, A^\dagger u_j) = \frac{1}{\sigma_i\sigma_j}(AA^\dagger u_i, u_j) = \frac{\sigma_i}{\sigma_j}(u_i, u_j) = \delta_{ij} \tag{B.10}$$

by an orthogonality of u_i.

$$u_i = \frac{1}{\sigma_i^2} AA^\dagger u_i = \frac{1}{\sigma_i} A v_i \tag{B.11}$$

□

Theorem B.4 (Singular value decomposition (SVD)). *For any non-zero matrix A, unitary matrices U and V exist such that*

$$\Sigma = U^\dagger A V \tag{B.12}$$

is diagonal with singular values on the diagonal.

Proof. Let us take vectors u_i, $i = 1, ..., r$ from theorem B.3, which forms the orthonormal basis of $range(A)$, and let us take some basis of $range(A)^\perp$, which we denote as $\{u_{r+1}, ...u_n\}$. Naturally,

$$A^\dagger u_k = 0, \quad k = r+1, ..., n \tag{B.13}$$

We define matrix U as $U = [u_1, ...u_r, u_{r+1}, ..., u_n]$, while similarly we take vectors v_i from theorem B.3, which forms the basis of $range(A^\dagger)$, and we choose some basis in $range(A^{\dagger})^{\perp}$ $\{v_{r+1}, ...v_m\}$. As a result, matrix V is defined as $V = [v_1, ...v_r, v_{r+1}, ..., v_m]$ and matrix $\Sigma = U^{\dagger}AV$ satisfies

$$(\Sigma)_{ij} = u_i^{\dagger}Av_j = 0 \quad \text{for} \quad i > r \quad \text{or} \quad j > r \tag{B.14}$$

$$(\Sigma)_{ij} = u_i^{\dagger}Av_j = \frac{1}{\sigma_j}u_i^{\dagger}AA^{\dagger}u_j = \sigma_j u_i^{\dagger}u_j = \sigma_j\delta_{ij} \quad \text{for} \quad i, j = 1, ..., r \tag{B.15}$$

□

Matter oscillation potential

Here, we present an effective potential in matter [53] for general types of interactions, and we define an effective matter potential

$$V_a^f = \frac{G_F}{\sqrt{2}} \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \rho_f(p,\lambda) \mathcal{J}_a^f, \tag{C.1}$$

with

$$\mathcal{J}_a^f = \langle f,p,\lambda | \bar{\psi}_f \Gamma_a (g_a + g_a' \gamma^5) \psi_f | f,p,\lambda \rangle.$$

Then, [53] for each fermion f

$$V^S = \frac{G_F}{\sqrt{2}} n_f\, g_S \left\langle \frac{m_f}{E_f} \right\rangle, \tag{C.2}$$

$$V^P = \frac{G_F}{\sqrt{2}} n_f\, g_P' \left\langle \frac{m_f}{E_f} \right\rangle, \tag{C.3}$$

$$V_\mu^V = \frac{G_F}{\sqrt{2}} n_f \left[g_V \left\langle \frac{p_\mu}{E_f} \right\rangle + g_V' \, m_f \left\langle \frac{s_\mu}{E_f} \right\rangle \right], \tag{C.4}$$

$$V_\mu^A = \frac{G_F}{\sqrt{2}} n_f \left[g_A' \left\langle \frac{p_\mu}{E_f} \right\rangle + g_A \, m_f \left\langle \frac{s_\mu}{E_f} \right\rangle \right], \tag{C.5}$$

$$V_{\mu\nu}^T = \frac{G_F}{\sqrt{2}} n_f \left[-g_T \, \varepsilon_{\mu\nu\rho\sigma} \left\langle \frac{p^\rho s^\sigma}{E_f} \right\rangle + i g_T' \left\langle \frac{p_\mu s_\nu - p_\nu s_\mu}{E_f} \right\rangle \right], \tag{C.6}$$

where s_μ is a spin four-vector

$$s \equiv \left(\frac{p \cdot \lambda}{m_f}, \lambda + \frac{p\,(p \cdot \lambda)}{m_f\,(m_f + E_f)} \right),$$

while $\langle Z \rangle$ denotes an average Z:

$$\langle Z \rangle = \frac{1}{n_f} \sum_\lambda \int Z(p,\lambda) p_f(p,\lambda) d^3p.$$

Using spinor normalisation condition $u^\dagger u = 1$, neutrino currents can be evaluated as follows. Up to a first order in neutrino mass [23], we obtain the following expansions:

$$
\begin{aligned}
\bar{u}_i(\lambda)u_j(\lambda) &= \tfrac{m_i+m_j}{2E} + 0((\tfrac{m}{E})^3), \\
\bar{u}_i(\lambda)u_j(-\lambda) &= 0, \\
\bar{u}_i(\lambda)\gamma^5 u_j(\lambda) &= \lambda\, \tfrac{m_i-m_j}{2E} + 0((\tfrac{m}{E})^3), \\
\bar{u}_i(\lambda)\gamma^5 u_j(-\lambda) &= 0, \\
\bar{u}_i(\lambda)\gamma^\mu u_j(\lambda) &= n^\mu + 0((\tfrac{m}{E})^2), \\
\bar{u}_i(-1)\gamma^\mu u_j(+1) &= -[\bar{u}_i(+1)\gamma^\mu u_j(-1)]^* = m^\mu\, \tfrac{m_i-m_j}{2E} + 0((\tfrac{m}{E})^3), \\
\bar{u}_i(\lambda)\gamma^\mu\gamma_5 u_j(\lambda) &= \lambda\, n^\mu \; + 0((\tfrac{m}{E})^2), \\
\bar{u}_i(-1)\gamma^\mu\gamma_5 u_j(+1) &= [\bar{u}_i(+1)\gamma^\mu\gamma_5 u_j(-1)]^* = m^\mu\, \tfrac{m_i+m_j}{2E} + 0((\tfrac{m}{E})^3), \\
\bar{u}_i(\lambda)\sigma^{0k} u_j(\lambda) &= i\, n^k\, \tfrac{m_i-m_j}{2E} + 0((\tfrac{m}{E})^3), \\
\bar{u}_i(\lambda)\sigma^{kl} u_j(\lambda) &= \lambda\, \epsilon^{klr} n^r\, \tfrac{m_i+m_j}{2E} + 0((\tfrac{m}{E})^3), \\
\bar{u}_i(-1)\sigma^{0k} u_j(+1) &= [\bar{u}_i(+1)\sigma^{0k} u_j(-1)]^* = i\, m^k + 0((\tfrac{m}{E})^2), \\
\bar{u}_i(-1)\sigma^{kl} u_j(+1) &= [\bar{u}_i(+1)\sigma^{kl} u_j(-1)]^* = \epsilon^{klr} m^r + 0((\tfrac{m}{E})^2),
\end{aligned}
\tag{C.7}
$$

where $n^\mu = (1, \vec{n})$, $\vec{n} = (\sin\theta\cos\varphi\ , \sin\theta\sin\varphi\ , \cos\theta)$ is a vector describing the neutrino's direction of flight and $m^\mu = (0, \vec{m})$, $\vec{m} = (\cos\theta\cos\varphi - i\sin\varphi\ , \cos\theta\sin\varphi + i\cos\varphi\ , -\sin\theta)$.

Muon decay amplitudes

In this appendix we present 16 helicity amplitudes for muon decay, evaluated in a muon rest frame. We treat all the particles except muon as massless because $m_\nu \lll m_e \ll M$. M stands for muon mass, and G_F is a Fermi constant. Variables with subscript e refer to electrons, while subscript n means a neutrino and m an antineutrino. We assume the following Lagrangian:

$$\mathcal{L}_I = -2\sqrt{2}G_F \times$$
$$\left[(g^S_{\alpha\beta})_{ij}\left(\overline{\nu}_i P_{-\alpha} e\right)\left(\overline{\mu} P_\beta \nu_j\right) + (g^V_{\alpha\beta})_{ij}\left(\overline{\nu}_i \gamma^\delta P_\alpha e\right)\left(\overline{\mu}\gamma_\delta P_\beta \nu_j\right)\right] + h.c., \quad \text{(D.1)}$$

where we assume summation over neutrino mass indices i, j and different chiral structures $\alpha = L, R$ and $\beta = L, R$ while $-L = R$ and $-R = L$. We do not include tensor interactions because for phenomenological reasons we do not expect it to be relevant at tree-level in any NP model. We can relate this parameter with those previously used in simpler Lagrangians (2.21) and (2.38):

$$g^V = g^V_{LL} \quad \text{(D.2)}$$
$$g^S = g^S_{LL} \quad \text{(D.3)}$$
$$r^V = g^V_{RR} \quad \text{(D.4)}$$
$$r^S = g^S_{RR} \quad \text{(D.5)}$$

(from this point we use a matrix notation, i.e. we do not write explicitly neutrino mass indices). ε denotes particle energy, and φ and θ are the particle's angles in spherical coordinates. We now introduce the following quantity:

$$\zeta = G_F e^{-\frac{1}{2}i(\varphi_e+\varphi_m+\varphi_n)}\sqrt{M\varepsilon_e\varepsilon_m\varepsilon_n} \quad \text{(D.6)}$$

which simplifies notation. We use the following order of helicities appearing as arguments of an amplitude $A[\lambda_\mu, \lambda_e, \lambda_m, \lambda_n]$. We also use coupling constant matrices as $g^X_{\alpha\beta} = X_{\alpha\beta}$ to simplify notation. The amplitudes for the Dirac neutrinos are given below:

$$\begin{aligned}
&\mathrm{A}[-1,-1,-1,-1]= \\
&8\zeta\cos\left[\tfrac{\theta_m}{2}\right]\left(e^{i\varphi_e}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]S_{\mathrm{RL}}-e^{i\varphi_n}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]S_{\mathrm{RL}}\right)
\end{aligned}\tag{D.7}$$

$$\begin{aligned}
&\mathrm{A}[-1,-1,-1,1]= \\
&8\zeta\sin\left[\tfrac{\theta_m}{2}\right]\left(e^{i\varphi_e}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]-e^{i\varphi_n}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)S_{\mathrm{RR}}
\end{aligned}\tag{D.8}$$

$$\begin{aligned}
&\mathrm{A}[-1,-1,1,-1]= \\
&16\zeta\left(-e^{i\varphi_e}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]+e^{i\varphi_m}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]\right)\sin\left[\tfrac{\theta_n}{2}\right]V_{\mathrm{LL}}
\end{aligned}\tag{D.9}$$

$$\begin{aligned}
&\mathrm{A}[-1,-1,1,1]= \\
&16\zeta\cos\left[\tfrac{\theta_e}{2}\right]\left(-e^{i\varphi_n}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]+e^{i\varphi_m}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)V_{\mathrm{LR}}
\end{aligned}\tag{D.10}$$

$$\begin{aligned}
&\mathrm{A}[-1,1,-1,-1]= \\
&16\zeta\sin\left[\tfrac{\theta_e}{2}\right]\left(e^{i\varphi_m}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]-e^{i\varphi_n}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)V_{\mathrm{RL}}
\end{aligned}\tag{D.11}$$

$$\begin{aligned}
&\mathrm{A}[-1,1,-1,1]= \\
&16\zeta\cos\left[\tfrac{\theta_n}{2}\right]\left(e^{i\varphi_m}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]-e^{i\varphi_e}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]\right)V_{\mathrm{RR}}
\end{aligned}\tag{D.12}$$

$$\begin{aligned}
&\mathrm{A}[-1,1,1,-1]= \\
&-8\zeta\cos\left[\tfrac{\theta_m}{2}\right]\left(e^{i\varphi_n}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]-e^{i\varphi_e}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)S_{\mathrm{LL}}
\end{aligned}\tag{D.13}$$

$$\begin{aligned}
&\mathrm{A}[-1,1,1,1]= \\
&-8\zeta\sin\left[\tfrac{\theta_m}{2}\right]\left(e^{i\varphi_n}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]S_{\mathrm{LR}}-e^{i\varphi_e}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]S_{\mathrm{LR}}\right)
\end{aligned}\tag{D.14}$$

$$\begin{aligned}
&\mathrm{A}[1,-1,-1,-1]= \\
&8\zeta\sin\left[\tfrac{\theta_m}{2}\right]e^{i\varphi_m}\left(e^{i\varphi_n}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]S_{\mathrm{RL}}-e^{i\varphi_e}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]S_{\mathrm{RL}}\right)
\end{aligned}\tag{D.15}$$

$$\begin{aligned}
&\mathrm{A}[1,-1,-1,1]= \\
&8e^{i\varphi_m}\zeta\cos\left[\tfrac{\theta_m}{2}\right]\left(e^{i\varphi_e}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]-e^{i\varphi_n}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)S_{\mathrm{RR}}
\end{aligned}\tag{D.16}$$

$$\begin{aligned}
&\mathrm{A}[1,-1,1,-1]= \\
&16e^{i\varphi_n}\zeta\cos\left[\tfrac{\theta_n}{2}\right]\left(-e^{i\varphi_e}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]+e^{i\varphi_m}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]\right)V_{\mathrm{LL}}
\end{aligned}\tag{D.17}$$

$$\begin{aligned}
&\mathrm{A}[1,-1,1,1]= \\
&16e^{i\varphi_e}\zeta\sin\left[\tfrac{\theta_e}{2}\right]\left(e^{i\varphi_n}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]-e^{i\varphi_m}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)V_{\mathrm{LR}}
\end{aligned}\tag{D.18}$$

$$\begin{aligned}
&\mathrm{A}[1,1,-1,-1]= \\
&16e^{i\varphi_e}\zeta\cos\left[\tfrac{\theta_e}{2}\right]\left(e^{i\varphi_m}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]-e^{i\varphi_n}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)V_{\mathrm{RL}}
\end{aligned}\tag{D.19}$$

$$\begin{aligned}
&\mathrm{A}[1,1,-1,1]= \\
&16e^{i\varphi_n}\zeta\left(-e^{i\varphi_m}\cos\left[\tfrac{\theta_m}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]+e^{i\varphi_e}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_m}{2}\right]\right)\sin\left[\tfrac{\theta_n}{2}\right]V_{\mathrm{RR}}
\end{aligned}\tag{D.20}$$

$$\begin{aligned}
&\mathrm{A}[1,1,1,-1]= \\
&8e^{i\varphi_m}\zeta\sin\left[\tfrac{\theta_m}{2}\right]\left(e^{i\varphi_n}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]-e^{i\varphi_e}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]\right)S_{\mathrm{LL}}
\end{aligned}\tag{D.21}$$

$$\begin{aligned}
&\mathrm{A}[1,1,1,1]= \\
&8e^{i\varphi_m}\zeta\cos\left[\tfrac{\theta_m}{2}\right]\left(-e^{i\varphi_n}\cos\left[\tfrac{\theta_n}{2}\right]\sin\left[\tfrac{\theta_e}{2}\right]S_{\mathrm{LR}}+e^{i\varphi_e}\cos\left[\tfrac{\theta_e}{2}\right]\sin\left[\tfrac{\theta_n}{2}\right]S_{\mathrm{LR}}\right)
\end{aligned}\tag{D.22}$$

For Majorana neutrinos, the situation is more complicated because two identical particles can appear in a final state, which requires amplitudes to be antisymmetric with respect to the interchange of two neutrinos. Therefore, amplitudes for Majorana neutrinos are obtained by taking $\frac{1}{\sqrt{2}}(A[\lambda_\mu, \lambda_e, \lambda_m, \lambda_n] - A^T[\lambda_\mu, \lambda_e, \lambda_n, \lambda_m])$, where T denotes taking a transpose in mass indices.

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