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GAMMA GRAPHS OF SOME SPECIAL CLASSES OF TREES

Anna Bień

Abstract. A set $S \subset V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex $v \in V$ which does not belong to S has a neighbour in S. The *domination number* $\gamma(G)$ of the graph G is the minimum cardinality of a dominating set in G. A dominating set S is a γ*-set* in G if $|S| = \gamma(G)$.

Some graphs have exponentially many γ-sets, hence it is worth to ask a question if a γ-set can be obtained by some transformations from another γ-set. The study of gamma graphs is an answer to this reconfiguration problem. We give a partial answer to the question which graphs are gamma graphs of trees. In the second section gamma graphs $\gamma.T$ of trees with diameter not greater than five will be presented. It will be shown that hypercubes Q_k are among $\gamma.T$ graphs. In the third section $\gamma.T$ graphs of certain trees with three pendant vertices will be analysed. Additionally, some observations on the diameter of gamma graphs will be presented, in response to an open question, published by Fricke et al., if $\operatorname{diam}(T(\gamma)) = O(n)$?

1. Introduction

We say that $S \subset V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex $v \in V$ which does not belong to S has a neighbour in S. The *domination number* $\gamma(G)$ of the graph G is the minimum cardinality of a dominating set in G. A dominating set S is a γ*-set* in G if $|S| = \gamma(G)$. We use standard notations of graph theory, as in Diestel [1]. For a comprehensive introduction to the

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theory of domination in graphs we refer the reader to Haynes, Hedetniemi and Slater [4].

A *gamma graph* $\gamma.G$ of a graph G, defined by Lakshmanan and Vijayakumar [5], is such a graph that $S \subset V(G)$ is a vertex of $\gamma.G$ if S is a γ-set of G, and γ-sets S_1 and S_2 are adjacent iff there exist two vertices v, u of the graph G such that $S_1 = S_2 \setminus \{u\} \cup \{v\}$. Independently, in 2011, four authors initiated the study of a similar class of graphs. The gamma graph $G(\gamma)$ studied by G.H. Fricke, S.M. Hedetniemi, S.T. Hedetniemi and K.R Hutson [2] is a $\gamma.G$ graph, whose vertices fulfill the following condition: if $S_1 = S_2 \setminus \{u\} \cup \{v\}$, then u and v are adjacent in the graph G. Notice that every graph $G(\gamma)$ is a spanning subgraph of $\gamma.G$, i.e. has the same vertex set. This straightforward observation is worth noting in the context of the research of Lakshmanan and Vijayakumar on the connectivity of a $\gamma.G$ graph. This means that every sufficient condition for $G(\gamma)$ is also valid for $\gamma.G$.

Other class of graphs whose vertices correspond to dominating sets was introduced by Haas and Seyffarth in 2014 [3]. k-dominating graph $D_k(G)$ is a graph, whose vertices are dominating sets of the graph G such that $|S| \leq k$. Two dominating sets S_1 and S_2 are adjacent in $D_k(G)$ iff there exist a vertex $v \in V(G)$ such that $S_1 = S_2 \cup \{v\}$. These authors also studied the conditions under which such graphs are connected. Their results include the following connection between connectivity of k-dominating graphs and gamma graphs: if $D_{\gamma+1}(G)$ is connected, then $\gamma.G$ is connected. Fricke et al. [2] proved that gamma graphs of trees $T(\gamma)$ are connected and asked which graphs are gamma graphs of trees. We will give a partial answer to this question, but with respect to the definition of Lakshmanan Vijayakumar. In the second section gamma graphs $\gamma.T$ of trees with diameter not greater than five will be presented. It will be shown that every gamma graph $\gamma.T$ of such trees is a hypercube Q_k, a hypercube without one vertex Q_k^-, or a cartesian product $Q_k \square Q_l^-$ or $Q_k^- \square Q_l^-$. Vertices of Q_k correspond to binary sequences of length k and edges correspond to such pairs of sequences which differ in exactly one coordinate. The set of vertices of a cartesian product $G_1 \square G_2$ of two graphs G_1 and G_2 is a set $V(G_1) \times V(G_2)$, and (v_1, v_2) is adjacent to (u_1, u_2) in $G_1 \square G_2$ iff v_1 is adjacent to u_1 in G_1, $v_2 = u_2$ or v_2 is adjacent to u_2 in G_2, $v_1 = u_1$. In the third section $\gamma.T$ graphs of certain trees with three pendant vertices will be analysed. Also in response to an open question, if $\operatorname{diam}(T(\gamma)) = O(n)$ (see [2]), some observations on the diameter of graphs presented in this paper are noted.

One of the simplest trees are paths. It was shown (see [2]) that $P_{3k}(\gamma) \simeq K_1$, $P_{3k+2}(\gamma) \simeq P_{k+2}$ and $P_{3k+1}(\gamma) \simeq SG(k+1)$, where $SG(k)$ denotes spanning subgraph of a cartesian product of two paths $P_k \square P_k$ such that

$$V(SG(k)) = \{(i, j) : 1 \leq i, j \leq k, i + j \leq k + 2\}$$

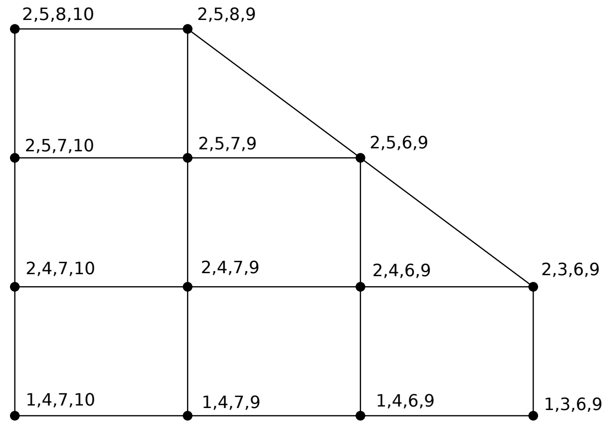

Figure 1. $\gamma.P_{10} \simeq SG^+(4)$

and

$$E(SG(k)) = \{((i,j),(k,l)) : i = k, l = j+1; k = i+1, j = l\}.$$

Notice that $\gamma.P_{3k} \simeq P_{3k}(\gamma)$. If $s = 3k+2$, $k \in \mathbb{N}$, then for every two γ-sets S_1, S_2 which correspond to vertices of $P_s(\gamma)$: if $S_1 = S_2 - \{u\} \cup \{v\}$, then u and v are adjacent in P_s. Hence $\gamma.P_{3k+2} \simeq P_{3k+2}(\gamma)$. $\gamma.P_{3k+1}$ (see Figure 1) can be obtained from $P_{3k+1}(\gamma)$ by adding $k-1$ additional edges. Indeed, if $S = \{g_1, g_2, \ldots, g_{k+1}\}$ is a γ-set of a graph P_{3k+1} such that $g_{i+1} - g_i = 1$ for some $i \leq k$, then distances between other consecutive elements of S equal 3. If $1 < i < k$, then S is adjacent in $\gamma.P_{3k+1}$ to four sets: $S \setminus \{g_i\} \cup \{g_i - 1\}$, $S \setminus \{g_i\} \cup \{g_{i+1}+1\}$, $S \setminus \{g_i\} \cup \{g_i - 2\}$, $S \setminus \{g_i\} \cup \{g_i + 2\}$. If $i = 1$ or $i = k$ then S is adjacent to three such sets. γ-sets of other form have the same degree in both graphs $P_{3k+1}(\gamma)$ and $\gamma.P_{3k+1}$. This means that $\gamma.P_{3k+1} \simeq SG^+(k+1)$, where $SG^+(k)$ denotes such graph that $V(SG^+(k)) = V(SG(k))$ and

$$E(SG^+(k)) = E(SG(k)) \cup \{((i,j)(i+1,j-1)) : 2 \leq i \leq k-1, i+j = k+2\}.$$

For purpose of this article we define a slide graph. Let $Sl(n)$ denote such graph that

$$V(Sl(n)) = V(SG^+(n)) \cup \{(n+1,1)\}$$

and

$$E(Sl(n)) = E(SG^+(n)) \cup \{((n,1)(n+1,1)), ((n,2)(n+1,1))\}.$$

2. Gamma graphs of trees with small diameter

One of the open problems in recent publications on gamma graphs was the question if $\mathrm{diam}(T(\gamma)) = O(n)$. We consider a modified version of this problem: is it true that $\mathrm{diam}(\gamma.T) = O(n)$? For trees with diameter not greater than five the answer to this question is positive. Let T be a tree, $d = \mathrm{diam}(T)$. For $d \in \{0,1\}$ we have $T \simeq K_d$, $\gamma.T \cong K_d$ and $\mathrm{diam}(\gamma.T) = d$. If $d = 2$, then $T = K_{1,n}$, $n > 1$, $\gamma.T \cong K_1$ and $\mathrm{diam}(\gamma.T) = 1 < d$. Since the next lemma can be easily shown by analysing the structure of the graph, we will skip the proof and proceed to more complex cases.

Lemma 2.1. *If T is a tree such that* $\mathrm{diam}(T) = 3$ *and $T \ncong P_4$, then $\gamma.T \cong K_1$ or $\gamma.T \cong K_2$.*

Lemma 2.2. *If T is a tree such that* $\mathrm{diam}(T) = 4$, *then there exist $n \in \mathbb{N}$, such that $\gamma.T \cong Q_n$ or $\gamma.T \cong Q_n^-$.*

Proof. Let S be an arbitrary γ-set of T. Note that T contains an induced path P_5. Let v_0 be the vertex of this path which is not adjacent to pendant vertices of this path. Note that $d(v, v_0) \leq 2$, for every vertex $v \in V(T)$. Hence T can be obtained from a graph H, which is an independent sum of stars $K_{1,n}$ and isolated vertices, by joining a vertex v_0 to every component of H with one such vertex of this component, which has the maximal degree.

If H has at least two isolated vertices, then v_0 belongs to S, and every vertex not dominated by v_0 is pendant in T and is dominated by itself or its neighbour. Hence, every element of S different from v_0 is a vertex of a component of H which is a star $K_{1,n}$. Moreover, S contains exactly one vertex from every star which has a maximal degree in this component. Two γ-sets S_1 and S_2 are adjacent iff $S_2 = S_1 - \{v\} \cup \{u\}$, where u and v are two vertices of one component $K_{1,1}$ of H. Hence $\gamma.T \simeq Q_k$, where k is the number of $K_{1,1}$s.

If H has one isolated vertex u, then v_0 or u belongs to S. Neither v_0 nor u dominate any vertex v such that $d(v_0, v) = 2$. Analogously to the previous case we infer that S contains exactly one vertex of the maximal degree from every component of H which is a star $K_{1,n}$. Also in this case $\gamma.T \simeq Q_k$, where $k - 1$ is the number of $K_{1,1}$ components of H.

If H has no isolated vertices, then $|S|$ is equal to the number of all components of H. Note that there is a bijection between the γ-sets from $V(\gamma.T)$ and binary sequences of length k, where k is the number of all $K_{1,1}$ components of H. Let enumerate components $K_{1,1}$ of H, and define a mapping such that $S \mapsto (a_i)_{i=1}^k$, where $a_i = 1$ whenever $s \in S$ which lies in the i-th component is a pendant vertex in T, otherwise $a_i = 0$. If H contains at least one component $K_{1,n}$, where $n > 1$, then every γ-set contains the vertex of this star which

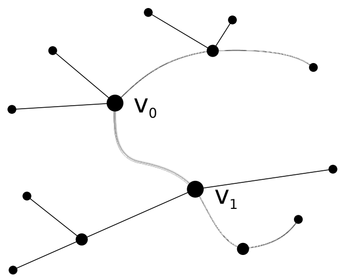

Figure 2. $\mathrm{diam}(T) = 5$

dominates v_0. If H is an independent sum of copies $K_{1,1}$, then the set which contains only pendant vertices and corresponds to the sequence $(1, 1, 1, \ldots, 1)$ does not dominate v_0. In this case $\gamma.T \simeq Q_k^-$. □

Since $\mathrm{diam}(Q_k) = k$, we draw the following conclusion from the lemma above.

COROLLARY 2.3. *If T is a tree with n vertices such that* $\mathrm{diam}(T) = 4$*, then* $\mathrm{diam}(\gamma.T) \leq n/2$.

LEMMA 2.4. *If T is a tree such that* $\mathrm{diam}(T) = 5$*, then there exist $n, m \in \mathbb{N}$ such that $\gamma.T \cong Q_n$ or $\gamma.T \cong Q_n \Box Q_m^-$ or $\gamma.T \cong Q_n^- \Box Q_m^-$.*

PROOF. Let S be an arbitrary γ-set of T, and let v_0, v_1 be the vertices of an induced P_6 subpath of T, whose distance from the pendant vertices of the path P_6 is greater then one (see Figure 2). Let T_0 and T_1 denote two components of the graph $T - v_0v_1$. Assume that T_0 denotes the component which contains v_0. Note that $d(v, v_0) \leq 2$ for every vertex $v \in V(T_0)$, and $d(v, v_1) \leq 2$ for every vertex $v \in V(T_1)$. This implies that both graphs T_1 and T_0 can be constructed like the tree in the proof of the previous lemma. Let l be the the number of stars $K_{n,1}$, in the graph $H = T - \{v_0, v_1\}$, where $n > 1$. Notice that all l vertices whose degree in H is greater than 1 belong to S. Moreover, S contains exactly one vertex of every $K_{1,1}$ component of H. This means that $|S| \geq l + k$. We enumerate components $K_{1,1}$ of H, and consider the mapping $S \mapsto (a_i^S)_{i=1}^k$ defined in the proof of the previous lemma. If $a^S = (0, 0, \ldots, 0)$, then v_0 and v_1 are dominated by vertices which are not isolated in H. It implies that if H does not have isolated vertices, then $\gamma(T) = l + k$.

Let a^{Si} be the subsequence of a^S corresponding to the components of H which are connected to v_i, for $i \in \{0, 1\}$. Assume that H has no isolated vertices, and S contains at least one neighbour of v_0, and one neighbour of v_1.

This means that a^S is a binary sequence such that a^{S0}, and a^{S1} have at least one element which equals 0. In this case S is adjacent in $\gamma.T$ to S' if $a^{S'0} = a^{S0}$, and $a^{S'1} \neq (1, 1 \ldots, 1)$ and differs from a^{S1} only at one coordinate, or symmetrically if $a^{S'1} = a^{S1}$, and $a^{S'0} \neq (1, 1 \ldots, 1)$ and differs from a^{S0} only at one coordinate. In this case $\gamma.T \simeq Q_n^- \Box Q_m^-$, $n + m = k$.

Let H have at least one isolated vertex v. Assume that $v \in V(T_0)$. If there are no pendant vertices adjacent to v_1, then $|S| = k + l + 1$. If there are more than two pendant vertices adjacent to v_0, then $v_0 \in S$ dominates v_1. Furthermore, it is possible that a^S is any binary sequence of length k, hence $\gamma.T \simeq Q_n$. If v is the only pendant vertex adjacent to v_0, then $\gamma.T \simeq Q_n$ or $\gamma.T \simeq Q_n \Box Q_m^-$. $\gamma.T \simeq Q_n$ under the condition that v_1 is adjacent to a vertex u of T_1 with degree greater than 2. In this case u belongs to every γ-set of T. In the other case $\gamma.T \simeq Q_n \Box Q_m^-$, because S contains exactly one vertex from every $K_{1,1}$ component of H and one vertex from the set $\{v, v_0\}$, and a^{S0} can be any binary sequence of an appropriate length, but $a^{S1} * a_{k+1} \neq (1, 1, \ldots, 1)$, where $a_{k+1} = 1$ iff $v \in S$.

If also v_1 is adjacent to a pendant vertex, then $|S| = k + l + 2$. Analogous reasoning lets us deduce that $\gamma.T \simeq Q_n$. □

Note that if $\gamma.T \simeq Q_n^- \Box Q_m$ or $\gamma.T \simeq Q_n^- \Box Q_m^-$, then $m + n \leq |V|/2$. This proves the following fact.

COROLLARY 2.5. *If T is a tree with n vertices such that* $\operatorname{diam}(T) = 5$, *then* $\operatorname{diam}(\gamma.T) \leq n/2$.

3. Gamma graphs of caterpillars with one leg

In this section we consider certain trees with three pendant vertices. These trees can be obtained from a path by joining a new vertex to such vertex v of this path which is not pendant. We will use the following notation. Let a *caterpillar with one leg* Cat_n^s be a graph defined in the following way: $V(\mathrm{Cat}_n^s) = V(P_n) \cup \{0\}$, and $E(\mathrm{Cat}_n^s) = E(P_n) \cup \{(0, s)\}$. For a caterpillar with one leg $G = \mathrm{Cat}_n^s$ let G^- denote the graph $\mathrm{Cat}_n^s - \{0, s\}$, and let $G^{--} = G^- - \{s-1, s+1\}$. Further we partition the class of caterpillars with one leg into the following six classes:

$$\mathcal{C}_{!!} = \{\mathrm{Cat}_{3k}^s : s \equiv 2 (\mathrm{mod}\, 3), 1 < s < 3k, k \in \mathbb{N}\},$$

$$\mathcal{C}_{!0} = \{\mathrm{Cat}_{3k+2}^s : s \not\equiv 0 (\mathrm{mod}\, 3), 1 < s < 3k+2, k \in \mathbb{N}\},$$

$$\mathcal{C}_{!2} = \{\mathrm{Cat}_{3k+1}^s : s \not\equiv 1 (\mathrm{mod}\, 3), 1 < s < 3k+1, k \in \mathbb{N}\},$$

$$\mathcal{C}_{00} = \{\mathrm{Cat}^s_{3k+1} : s \equiv 1 (\mathrm{mod}\, 3), 1 < s < 3k+1, k \in \mathbb{N}\},$$
$$\mathcal{C}_{02} = \{\mathrm{Cat}^s_{3k} : s \not\equiv 2 (\mathrm{mod}\, 3), 1 < s < 3k, k \in \mathbb{N}\},$$
$$\mathcal{C}_{22} = \{\mathrm{Cat}^s_{3k+2} : s \equiv 0 (\mathrm{mod}\, 3), 1 < s < 3k+2, k \in \mathbb{N}\}.$$

Note that G^- is a sum of two independent paths and that indices $!, 0$ and 2 indicate how many vertices belong to each component of G^-: the index ! corresponds to a component with $3t+1$ vertices, 0 corresponds to a component with $3t$ vertices, and 2 corresponds to a component with $3t+2$ vertices. We use the index ! instead of 1 to stress that the component corresponding to this index can be omitted in finding $\gamma.G$.

COROLLARY 3.1. *If $v \in G$ is a pendant vertex adjacent to a cutvertex c of $G - v$ and there exists a γ-set S such that $v \in S$, then the subgraph H of $\gamma.G$ induced by γ-sets containing v is a graph $H \simeq H_1 \Box H_2 \Box \ldots \Box H_k$, where H_i for $i \in \{1, \ldots, k\}$ are gamma graphs of components of $G - \{v, c\}$.*

PROOF. It suffices to prove that if S is a γ-set of G such that $v \in S$, then $S = \{v\} \cup S_1 \cup S_2 \cup \cdots \cup S_k$, where S_i is a vertex of H_i, $i \in \{1, 2, \ldots, k\}$. Note that $c \notin S$, otherwise $S \setminus \{v\} \subset S$ would be a dominating set such that $|S| < \gamma(G)$. Since $G - \{v, c\}$ is not connected and $c \notin S$, then S contains a γ-set S_i of every component of $G - \{v, c\}$. On the other hand every set S such that $S = \{v\} \cup S_1 \cup S_2 \cup \cdots \cup S_k$, where $S_1, S_2 \ldots S_k$ are gamma sets of components of $G - \{v, c\}$ dominates G. □

Further we will use this corollary for caterpillars with one leg. Below we explicitly formulate the special case which is essential in proofs of final theorems.

LEMMA 3.2. *Let H be a subgraph of $G = \gamma.\mathrm{Cat}^s_t$ induced by all γ-sets of G which contain 0. Then $H = H_1 \Box H_2$, where vertices of H_1 and H_2 correspond to dominating sets of components of G^-.*

A similar observation can be made for a subgraph H of $G = \gamma.\mathrm{Cat}^s_t$ induced by all γ-sets of G which contain s. Notice that $H = H_1 \Box H_2$, where vertices of H_1 and H_2 correspond to subsets of components of G^- which dominate components of G^{--}. These observations will be helpful in describing the structure of $\gamma.\mathrm{Cat}^s_t$.

THEOREM 3.3. *Let $G = \gamma.\mathrm{Cat}^s_t$, and let H be a subgraph of G induced by all γ-sets of G which contain 0.*
If $G \in \mathcal{C}_{00}$, then $H \simeq K_1$,
if $G \in \mathcal{C}_{02}$, then $H \simeq P_n$,
if $G \in \mathcal{C}_{22}$, then $H \simeq P_n \Box P_m$.

It suffices to apply Lemma 3.2 to the appropriate caterpillar with one leg. The graph H is a product of $\gamma.G$ graphs of two paths.

Lemma 3.4. *If $G \in \mathcal{C}_{!0} \cup \mathcal{C}_{!2} \cup \mathcal{C}_{!!}$, then the vertex 0 does not belong to any γ-set of G.*

Proof. If $G = \mathrm{Cat}^s_{3k+i} \in \mathcal{C}_{!0} \cup \mathcal{C}_{!2} \cup \mathcal{C}_{!!}$, then P_{3t+1} is a component of G^-. Suppose that S is a γ-set such that $0 \in S$. Notice that $s \notin S$ because S is minimal. It means that $|S| = 1 + (t+1) + (k-t) = k+2$. On the other hand if S' is a gamma set such that $0 \notin S'$, then $s \in S'$. No vertex of the graph G^{--} is dominated by s, and domination numbers of its components are $\Gamma(P_{3t}) = t$, and $\Gamma(P_{3(k-t-1)+i}) \leq k-t$, hence $|S'| \leq 1+t+k-t = k+1$. Since $|S'| < |S|$, then any gamma set of Cat^s_{3k+i} cannot contain the vertex 0. □

Notice that if S is a gamma set of Cat^s_t and $0 \in S$, then $S \setminus \{0\} \cup \{s\}$ is also a γ-set. Therefore if H_s is a subgraph of G induced by all γ-sets of G which contain s and H_0 is a subgraph of G induced by all γ-sets of G which contain 0, then H_0 is isomorfic to an induced subgraph of H_s. This proves the following proposition.

Proposition 3.5. *If $G = \mathrm{Cat}^s_t$, H_s is a subgraph of $\gamma.G$ induced by all γ-sets of G which contain s and H_0 is a subgraph of $\gamma.G$ induced by all γ-sets of G which contain 0, then $V(\gamma.G) = V(H_s) \cup V(H_0)$ and*

$$E(\gamma.G) = E(H_s) \cup E(H_0) \cup \{(S, S \setminus \{0\} \cup \{s\}) : S \in V(H_0)\}.$$

Notice that for $\mathrm{Cat}^s_t \in \mathcal{C}_{!0} \cup \mathcal{C}_{!2} \cup \mathcal{C}_{!!}$ the graph H_0 is empty. In this case $\gamma.\mathrm{Cat}^s_t \simeq H_s$. In order to complete the description of $\gamma.\mathrm{Cat}^s_t$ graphs it suffices to prove the following theorem.

Theorem 3.6. *Let $G = \gamma.\mathrm{Cat}^s_t$. If H is a subgraph of G induced by all γ-sets of G which contain s, then there exist $n, m \in \mathbb{N}$ that:*
if $G \in \mathcal{C}_{00}$, then $H \simeq P_n \square P_m$,
if $G \in \mathcal{C}_{02}$, then $H \simeq P_n \square Sl(n)$,
if $G \in \mathcal{C}_{22}$, then $H \simeq Sl(n) \square Sl(m)$,
if $G \in \mathcal{C}_{!0}$, then $H \simeq P_n$,
if $G \in \mathcal{C}_{!2}$, then $H \simeq Sl(n)$,
if $G \in \mathcal{C}_{!!}$, then $H \simeq K_1$.

Proof. Let S be a γ-set of G. If $G \in \mathcal{C}_{!0} \cup \mathcal{C}_{!2} \cup \mathcal{C}_{!!}$, then P_{3n+1} is a component of G^-. We can assume that $n > 1$, and then P_{3n} is one of the components of G^{--}. Additionally, n vertices of S which dominate this component belong to its γ-set. Otherwise one of the pendant vertices of P_{3n} would be dominated by a vertex of $G^- - G^{--}$, but the vertices of P_{3n-1} cannot be dominated by a set of $n-1$ vertices. In this case $S = \Gamma \cup \{s\} \cup \Gamma'$, where Γ is the only γ-set of P_{3n}, and $\Gamma' \subset V(G^- - P_{3n+1})$ dominates vertices of the other component of G^{--}.

If $G \in \mathcal{C}_{!0} \cup \mathcal{C}_{00} \cup \mathcal{C}_{02}$, then P_{3n} is a component of G^-, and then $P_{3(n-1)+2}$ is one of the components of G^{--}. Similarly, like in the previous case, n vertices of S which dominate this component belong to its γ-set. Hence $S = \Gamma \cup \{s\} \cup \Gamma'$, where Γ is a γ-set of $P_{3(n-1)+2}$, and $\Gamma' \subset V(G^- - P_{3n+1})$ dominates the other component of G^{--}. In other words, Γ corresponds to a vertex of the graph $\gamma.P_{3(n-1)+2} \simeq P_{n-1+2}$.

If $G \in \mathcal{C}_{!2} \cup \mathcal{C}_{02} \cup \mathcal{C}_{22}$, then P_{3n+2} is a component of G^- and P_{3n+1} is one of the components of G^{--}. We can assume that $V(P_{3n+2}) = \{1, 2, \ldots, 3n+2\}$ and $3n+3 = s$. In this case $n+1$ vertices of S which dominate P_{3n+1} either belong to its γ-set or $3n+2$ is one of these vertices and n other vertices dominate the path P_{3n}, to which they belong. Assume that the second case is true and notice that the n vertices which dominate P_{3n} is the unique γ-set S_0 of this path, where $S_0 = \{3i+2 : 0 \leq i < n\}$. Hence $S = \Gamma \cup \{s\} \cup \Gamma'$, where Γ is a γ-set of P_{3n+1} or $\Gamma = S_0 \cup \{3n+2\}$, and $\Gamma' \subset G^- - P_{3n+2}$ dominates the other component of G^{--}. $S_0 \cup \{3n+1\}$ and $S_0 \cup \{3n\}$ are the only γ-sets of P_{3n+1} which have only one element different from the set $S_0 \cup \{3n+2\}$. This proves that subsets of γ-sets of G which dominate the component P_{3n+1} correspond to vertices of the graph obtained from $SG^+(n)$ by adding one new vertex and two edges. This graph is isomorfic to $Sl(n)$.

If $G \in \mathcal{C}_{!0} \cup \mathcal{C}_{!!} \cup \mathcal{C}_{!2}$, then one component of the graph G^- is a path P_{3t+1}. If additionally $t = 0$, then G^{--} has only one component $P_{3(k-1)+i}$, $i \in \{0, 1, 2\}$.

If G^{--} has exactly two components G_1 and G_2, then every γ-set of G which contains s is a set $\{s\} \cup \Gamma_1 \cup \Gamma_2$, where Γ_1 dominates G_1 and Γ_2 dominates G_2. Two such gamma sets $\{s\} \cup \Gamma_1 \cup \Gamma_2$ and $\{s\} \cup \Gamma_1' \cup \Gamma_2'$ are adjacent iff $\Gamma_1 = \Gamma_1'$ and Γ_2 has only one element different from Γ_2', or Γ_1 has exactly one element different from Γ_1' and $\Gamma_2 = \Gamma_2'$. Hence $\gamma.G \cong H_1 \Box H_2$, where H_1 and H_2 are isomorfic to a path, slide or a graph K_1. □

We finish with observations on the size of diameter of $\gamma.\operatorname{Cat}_t^s$. Let H_0 and H_s be induced subgraphs of $\gamma.\operatorname{Cat}_t^s$ defined as in Proposition 3.5. It is obvious that $\operatorname{diam}(H_0) < \operatorname{diam}(H_s)$. Since every vertex of H_0 is adjacent in $\gamma.\operatorname{Cat}_t^s$ to some vertex of H_s, we immediately conclude that $\operatorname{diam}(\gamma.\operatorname{Cat}_t^s) \leq \operatorname{diam}(H_s)+1$. Furthermore it is easy to show that $\operatorname{diam}(Sl(n)) = n$, $\operatorname{diam}(G_1 \Box G_2) = \operatorname{diam}(G_1) + \operatorname{diam}(G_2)$. The proof of the theorem shows that the ratio between $|V(H_s)|$ and $|V(\operatorname{Cat}_t^s)|$ is the biggest when $\operatorname{Cat}_t^s \in \mathcal{C}_{00}$. In this case H_s is a product of two paths and for every γ-set $S = \Gamma_1 \cup \{s\} \cup \Gamma_2 \in V(H_s)$ the sets Γ_1 and Γ_2 correspond to some vertices of $\gamma.P_{3k+2} \simeq P_{k+2}$ and $\gamma.P_{3l+2} \simeq P_{l+2}$, where $|V(\operatorname{Cat}_t^s)| = (3k+2) + (3l+2) + 3 + 1 = 3(k+l) + 8$. Then

$|V(G_1)| + |V(G_2)| = l + k + 4$ and hence $\mathrm{diam}(H_s) = (n + 4)/3$. Therefore, the following proposition holds.

PROPOSITION 3.7. *If $G = \mathrm{Cat}_t^s$ is a caterpillar with one leg then*

$$\mathrm{diam}(\gamma.G) \leq (t + 5)/3 + 1.$$

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