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# ORTHOGONALLY PEXIDER FUNCTIONS MODULO A DISCRETE SUBGROUP

Wirginia Wyrobek-Kochanek

**Abstract.** Under appropriate conditions on abelian topological groups $G$ and $H$, an orthogonality $\perp \subset G^2$ and a $\sigma$-algebra $\mathfrak{M}$ of subsets of $G$ we prove that if at least one of the functions $f, g, h\colon G \to H$ satisfying

$$f(x+y) - g(x) - h(y) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y,$$

where $K$ is a discrete subgroup of $H$, is continuous at a point or $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and constants $a, b \in H$ such that

$$\begin{cases} f(x) - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b \in K \end{cases}$$

for $x \in G$ and

$$B(x,y) = 0 \quad \text{for } x, y \in G \text{ such that } x \perp y.$$

Let $G$ and $H$ be groups and $\perp \subset G^2$ an orthogonality. We say that a function $f\colon G \to H$ is orthogonally additive, if

$$f(x+y) = f(x) + f(y) \quad \text{for } x, y \in G \text{ such that } x \perp y.$$

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In the paper [3] J. Brzdęk considers the Rätz orthogonality (cf.[5]) and, under some assumptions, gives a description of orthogonally additive functions modulo a discrete subgroup, i.e. functions $f\colon G\to H$ such that

$$f(x+y)-f(x)-f(y)\in K \quad \text{for } x,y\in G \text{ such that } x\perp y,$$

where $K$ is a discrete subgroup of $H$. In the papers [7] and [4] authors prove similar theorems (for continuous or measurable functions), but for the orthogonality defined by K. Baron and P. Volkmann in [2], which includes the Rätz orthogonality.

Now we would like to obtain some similar results for the Pexider difference instead of the Cauchy difference, i.e. we assume that functions $f,g,h\colon G\to H$ are orthogonally Pexider modulo a discrete subgroup, which means that they satisfy

$$f(x+y)-g(x)-h(x)\in K \quad \text{for } x,y\in G \text{ such that } x\perp y,$$

where $K$ is a discrete subgroup of $H$. We start with the following result.

Lemma. *Let $G$ be a groupoid with a neutral element, $H$ an abelian group, $K$ a subgroup of $H$. Let $\Delta\subset G\times G$ be a set with*

$$(0,x),(x,0)\in\Delta \quad \text{for all } x\in G. \tag{1}$$

*If functions $f,g,h\colon G\to H$ satisfy*

$$f(x+y)-g(x)-h(y)\in K \quad \text{for } (x,y)\in\Delta, \tag{2}$$

*then the following are true:*

(a) *There are functions $k_1,l_1\colon G\to K$, $\varphi_1\colon G\to H$ and constants $a,b\in H$ such that*

$$\varphi_1(x+y)-\varphi_1(x)-\varphi_1(y)\in K \quad \text{for } (x,y)\in\Delta$$

*and*

$$\begin{cases} f(x)=\varphi_1(x)+a,\\ g(x)=\varphi_1(x)+k_1(x)+b,\\ h(x)=\varphi_1(x)-k_1(x)+l_1(x)+a-b \end{cases} \tag{3}$$

*for all $x\in G$.*

(b) *There are functions* $k_2, l_2\colon G \to K$, $\varphi_2\colon G \to H$ *and constants* $a, b \in H$ *such that*

$$\varphi_2(x+y) - \varphi_2(x) - \varphi_2(y) \in K \quad \text{for } (x,y) \in \Delta$$

*and*

$$\begin{cases} f(x) = \varphi_2(x) + k_2(x) + a, \\ g(x) = \varphi_2(x) + b, \\ h(x) = \varphi_2(x) + l_2(x) + a - b \end{cases}$$

*for all* $x \in G$.

(c) *There are functions* $k_3, l_3\colon G \to K$, $\varphi_3\colon G \to H$ *and constants* $a, b \in H$ *such that*

$$\varphi_3(x+y) - \varphi_3(x) - \varphi_3(y) \in K \quad \text{for } (x,y) \in \Delta$$

*and*

$$\begin{cases} f(x) = \varphi_3(x) + k_3(x) + a, \\ g(x) = \varphi_3(x) + l_3(x) + b, \\ h(x) = \varphi_3(x) + a - b \end{cases}$$

*for all* $x \in G$.

*Moreover, each of assertions* (a), (b), (c) *gives a complete description of solutions of* (2), *that is, every triple* $(f, g, h)$, *being of one of the forms described above, is a solution of* (2).

Proof. Setting $y = 0$ in (2), by (1) we get

$$\mu(x) := f(x) - g(x) - h(0) \in K \quad \text{for } x \in G \tag{4}$$

and setting $x = 0$ we have

$$\nu(y) := f(y) - g(0) - h(y) \in K \quad \text{for } y \in G. \tag{5}$$

In particular,

$$f(0) - g(0) - h(0) \in K. \tag{6}$$

Denote $a = f(0)$, $b = g(0)$ and define $\varphi_i, k_i, l_i \colon G \to H$ for $i = 1, 2, 3$ by

$$\begin{aligned} \varphi_1 &= f - a, & k_1 &= g - \varphi_1 - b, & l_1 &= h + k_1 - \varphi_1 - a + b, \\ \varphi_2 &= g - b, & k_2 &= f - \varphi_2 - a, & l_2 &= h - \varphi_2 - a + b, \\ \varphi_3 &= h - a + b, & k_3 &= f - \varphi_3 - a, & l_3 &= g - \varphi_3 - b. \end{aligned}$$

Using (4), (5), (2) and (6) for every $(x, y) \in \Delta$ we get

$$\begin{aligned} \varphi_1(x+y) - \varphi_1(x) - \varphi_1(y) &= f(x+y) - a - f(x) + a - f(y) + a \\ &= f(x+y) - \mu(x) - g(x) - h(0) - \nu(y) - g(0) - h(y) + a \in K; \\ \varphi_2(x+y) - \varphi_2(x) - \varphi_2(y) &= g(x+y) - b - g(x) + b - g(y) + b \\ &= f(x+y) - \mu(x+y) - h(0) - g(x) + \mu(y) - f(y) + h(0) + b \\ &= f(x+y) - \mu(x+y) - g(x) + \mu(y) - \nu(y) - g(0) - h(y) + b \in K; \\ \varphi_3(x+y) - \varphi_3(x) - \varphi_3(y) &= h(x+y) - a + b - h(x) + a - b - h(y) + a - b \\ &= f(x+y) - g(0) - \nu(x+y) + \nu(x) - f(x) + g(0) - h(y) + a - b \\ &= f(x+y) - \nu(x+y) + \nu(x) - \mu(x) - g(x) - h(0) - h(y) + a - b, \\ &\in K. \end{aligned}$$

We also have

$$\begin{aligned} k_1(x) &= g(x) - f(x) + a - b = -\mu(x) - h(0) + a - b \in K, \\ k_2(x) &= f(x) - g(x) + b - a = \mu(x) + h(0) + b - a \in K, \\ k_3(x) &= f(x) - h(x) + a - b - a = \nu(x) + g(0) - b \in K, \\ l_1(x) &= h(x) + k_1(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_1(x) + b \in K, \\ l_2(x) &= h(x) + k_2(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_2(x) + b \in K, \\ l_3(x) &= g(x) + k_3(x) - f(x) + a - b = -\mu(x) - h(0) + k_3(x) + a - b \in K \end{aligned}$$

for $x \in G$. □

The part (b) of this lemma in the case when $\Delta = G^2$ was also obtained by K. Baron and PL. Kannappan in [1], even under some weaker assumptions. Some variations of (2) for functions with values in groupoids were studied by J. Sikorska in [6].

We work with the orthogonality proposed by K. Baron and P. Volkmann in [2], assuming additionally that the last condition in the following definition holds:

Let $G$ be a group such that the mapping

$$x \mapsto 2x, \quad x \in G, \tag{7}$$

is a bijection onto the group $G$. A relation $\perp \subset G^2$ is called *orthogonality* if it satisfies the following three conditions:

(i) $0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

(ii) If an orthogonally additive function from $G$ to an abelian group is odd, then it is additive; if it is even, then it is quadratic.

(iii) $x \perp 0$ and $0 \perp x$ for every $x \in G$.

For a subset $U$ of a given group and for $n \in \mathbb{N}$ the symbol $nU$ denotes the set $\{nx : x \in U\}$.

Theorem. *Assume $G$ is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that*

$$U \subset 2U \quad \textit{and} \quad G = \bigcup\{2^n U : \ n \in \mathbb{N}\}. \tag{8}$$

*Let $\perp \subset G^2$ be an orthogonality, $H$ an abelian topological group and $K$ a discrete subgroup of $H$. Assume that functions $f, g, h \colon G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in K \quad \textit{for } x, y \in G \textit{ such that } x \perp y. \tag{9}$$

(i) *If at least one of the functions $f, g, h$ is continuous at a point, then there exist: a continuous additive function $A \colon G \to H$, a continuous biadditive and symmetric function $B \colon G \times G \to H$ and constants $a, b \in H$ such that*

$$\begin{cases} f(x) - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b \in K \end{cases} \tag{10}$$

*for $x \in G$ and*

$$B(x,y) = 0 \quad \textit{for } x, y \in G \textit{ such that } x \perp y. \tag{11}$$

(ii) *Let $\mathfrak{M}$ be a $\sigma$-algebra of subsets of $G$ such that*

$$x \pm 2A \in \mathfrak{M} \quad \textit{for all } x \in G \textit{ and } A \in \mathfrak{M} \tag{12}$$

*and there is a proper $\sigma$-ideal $\mathfrak{I}$ of subsets of $G$ with*

$$0 \in \operatorname{Int}(A - A) \quad \textit{for } A \in \mathfrak{M} \setminus \mathfrak{I}. \tag{13}$$

*Assume moreover that $H$ is separable metric and the following condition* (G) *is fulfilled:*

(G) *either $G$ is a first countable Baire group, or $G$ is metric separable, or $G$ is metric and $\mathfrak{M}$ contains all Borel subsets of $G$.*

*If at least one of the functions $f, g, h$ is $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and constants $a, b \in H$ such that* (10) *and* (11) *hold.*

*Moreover, each of assertions* (i), (ii) *gives a complete description of solutions of* (9).

Proof. (i): Case 1. Assume that $f$ is continuous at a point. Let $k_1, l_1 : G \to K$, $\varphi_1\colon G \to H$ be as in Lemma (a). Then the function $\varphi_1$ is continuous at a point. According to Theorem 1 from [7] we get a continuous additive function $A\colon G \to H$ and a continuous biadditive and symmetric function $B\colon G \times G \to H$ such that

$$\varphi_1(x) - B(x,x) - A(x) \in K \quad \text{for } x \in G$$

and (11) hold. Then, according to (3),

$$\begin{aligned} f(x) - B(x,x) - A(x) - a &= \varphi_1(x) + a - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b &= \varphi_1(x) + k_1(x) + b - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b &= \varphi_1(x) - k_1(x) + l_1(x) + a - b \\ &\qquad - B(x,x) - A(x) - a + b \in K \end{aligned}$$

for all $x \in G$.

Case 2. If the function $g$ is continuous at a point then instead of Lemma (a) we use Lemma (b).

Case 3. If the function $h$ is continuous at a point then we use Lemma (c).

(ii): If one of the functions $f, g, h$ is $\mathfrak{M}$-measurable then we use Theorem 1 from [4] instead of Theorem 1 from [7]. □

For $\perp = G^2$ some special cases were obtained in [1] (cf. Corollaries 6 and 7 there).

If in the Theorem $G$ is Baire and we consider the Baire measurability, then we do not need to assume the first countability of $G$ in order to get the factorization with a separately continuous biadditive term only (cf. Corollary 2 in [4]).

Corollary 1. *Assume $G$ is an abelian topological group such that the mapping* (7) *is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that* (8) *holds. Let $\perp \subset G^2$ be an* *orthogonality, $H$ an abelian separable metric group, $K$ a discrete subgroup of $H$ and functions $f, g, h\colon G \to H$ satisfy* (9)*. If $G$ is Baire and at least one of the functions $f, g, h$ is Baire measurable, then there exist: a continuous additive function $A\colon G \to H$, a function $B\colon G \times G \to H$ biadditive, symmetric and continuous in each variable, and constants $a, b \in H$ such that* (10) *and* (11) *hold.*

If we take $\perp = G^2$, then our Theorem gives us Corollary 2 below. It also leads to another conclusions in the case when we consider Baire or Christensen measurability.

Corollary 2. *Assume $G$ is an abelian topological group such that the mapping* (7) *is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that* (8) *holds. Let $H$ be an abelian separable metric group, $K$ a discrete subgroup of $H$, $\mathfrak{M}$ a $\sigma$-algebra of subsets of $G$ satisfying* (12) *and such that there is a proper $\sigma$-ideal $\mathfrak{I}$ of subsets of $G$ with property* (13)*. If functions $f, g, h\colon G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in K \quad \textit{for } x, y \in G$$

*and at least one of them is $\mathfrak{M}$-measurable, then there exist a continuous additive function $A\colon G \to H$ and constants $a, b \in H$ such that*

$$\begin{cases} f(x) - A(x) - a \in K, \\ g(x) - A(x) - b \in K, \\ h(x) - A(x) - a + b \in K \end{cases}$$

*for $x \in G$.*

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