Large doubly transitive orbits on a line

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Abstract

Projective planes of order n with a collineation group admitting a 2-transitive orbit on a line of length at least $n/2$ are investigated and new examples are provided.

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1 Introduction

A classical subject in finite geometries is the investigation of a finite projective plane Π of order n admitting a collineation group G which acts 2-transitively on a point-subset $\mathcal{O}$ of size v of Π. It dates back to 1967 and it is due to Cofman [10]. It is easily seen that either

(i) the structure of a non trivial 2-$(v, k, 1)$ design is induced on $\mathcal{O}$, or

(ii) $\mathcal{O}$ is an arc, or

(iii) $\mathcal{O}$ is a contained in a line.

This paper focus entirely on the case when $\mathcal{O}$ is a contained in a line. Starting form Cofman [11], several papers have been devoted to this case. In [11] Cofman proves that Π is Desarguesian and $SL(2, n) \trianglelefteq G$, under the assumptions $v = n + 1$ (that is $\mathcal{O}$ is the entire line), $n \not\equiv 1 \bmod 8$ and G contains involutory homologies. Some years later, Schulz [57] and Czerwinski [13] essentially proved that the unique translation planes with a collineation group acting 2-transitively on the line at infinity, are either Desarguesian or Lüneburg planes. Actually, they proved this characterization under additional assumptions that ruled out the possibility for G to contain Baer collineations. Later, such additional assumptions were totally dropped with the use of the classification of finite 2-transitive groups. In 1981 Korchmáros [44] investigates the general case $v = n + 1$ when $n = 2^r$. Apart from the Desarguesian case, the author proves that either $G \cong Sz(n)$ or $G \cong PSU(3, n)$.

Also the case $v = n$ has been investigated extensively. In 1986 Ganley and Jha [19] proved that if $v = n$, Π is a translation plane and l is the line at infinity, then Π is actually a semifield plane. The case $v = n$ is investigated by

Hiramine [29] in 1993, without any assumption on the structure of Π. Apart from a few numerical values of n, Hiramine shows that the socle of $\bar{G}$, where $\bar{G}$ denotes the group induced by G on $\mathcal{O}$, is an elementary abelian p-group for some prime p, the plane Π has order $n = p^r$ and either $\bar{G}_O \leq \Gamma L(1, p^r)$ or $SL(2, p^r) \leq \bar{G}_O \leq \Gamma L(2, p^r)$. In 1999 Biliotti, Jha and Johnson classify the translation planes Π for $v = n$, $n \neq 2^6$, when l is the line at infinity and $\bar{G} \leq A\Gamma L(1, p^r)$. In 2000, Ganley, Jha and Johnson [20] classify the triple $(\Pi, \mathcal{O}, G)$ for $v = n$, when Π is a translation plane, l is an affine line and G is non solvable. Recently, Biliotti and Francot [7] investigated the general case $v \geq n$, determining all the possible collineation groups.

The problem of classifying the triple $(\Pi, \mathcal{O}, G)$ when $\mathcal{O} \subset l$ and the length v of $\mathcal{O}$ is smaller than n, but close to n, is open. An initial result in this direction is the paper of Biliotti and Montinaro [8] devoted to the case $v = n - 3$. In that paper no non trivial cases arise.

The aim of this paper is to investigate the finite projective planes Π of order n admitting a collineation group G which acts 2-transitively on a subset $\mathcal{O}$ of a line l of Π, under the assumption $v \geq n/2$. In particular the following results are obtained.

Theorem 1 *Let Π be a projective plane of order n and let $\mathcal{O}$ be a 2-transitive G-orbit of length v on a line. If $v \geq n/2$ and G is almost simple, then one of the following occurs:*

(1) $v = n + 1$, and one of the following occurs:

- *(a) $n = q$, $\Pi \cong PG(2, q)$ and $SL(2, q) \trianglelefteq G$;*
- *(b) $n = q^2$, $q = 2^{2s+1}$, $s \geq 1$, and $Sz(q) \trianglelefteq G$;*
- *(c) $n = q^3$, $q = 2^{2s}$, $s \geq 1$, $PSU(3, q) \trianglelefteq G$ and G fixes a point of Π^l.*

(2) $v = (n + 1)/2$, n odd, and one of the following occurs:

- *(a) Π is the Hall plane of order 9 or its dual, $|\mathcal{O}| = 5$ and $SL(2, 5) \trianglelefteq G$;*
- *(b) $n = 2q + 1$, $q \equiv 3 \bmod 4$, $q \neq 7$, $|\mathcal{O}| = q + 1$ and $SL(2, q) \trianglelefteq G$.*

(3) $v = n/2$, n even, and one of the following occurs:

- *(a) Π is the Johnson-Walker translation plane of order 16 or its dual, and $PSL(2, 7) \trianglelefteq G$;*
- *(b) $n = 2(q + 1)$, $q \equiv 3 \bmod 4$, $|\mathcal{O}| = q + 1$ and $SL(2, q) \trianglelefteq G$.*

We remark that the result (1) is already known (see [7] and its references for related examples). So, our task is to prove the results (2) and (3). We also remark that there are no known examples for the cases (2b) and (3b).

Theorem 2 *Let Π be a projective plane of order n and let $\mathcal{O}$ be a 2-transitive G-orbit of length v on a line. If $v \geq n/2$ and G is of affine type, then one of the followings occurs:*

(I) $v = n+1$, *n even, and either*

(a) $\bar{G}_O \leq \Gamma L(1,v)$, *or*

(b) $v \in \{5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$.

(II) $v = n$ *and either*

(a) $\bar{G}_O \leq \Gamma L(1,v)$, *or*

(b) $SL(2, p^{d/2}) \trianglelefteq \bar{G}_O$, *d even, or*

(c) $v \in \{2^4, 3^2, 3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$.

(III) $n/2 \leq v < n$ *and either*

(a) $\bar{G} \leq A\Gamma L(1,v)$, *or*

(b) $v \in \{2^4, 2^6, 3^2, 3^3, 3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$

We remark that the results (I) and (II) are already known (see [7] and [29] for related examples). So, we have to prove the result (III). We stress that, while there are no known examples for the case (IIIb), examples of type (IIIa) occur in the Desarguesian plane of order 8, 9, in the Lorimer-Rahilly plane of order 16 and in the Johnson-Walker plane of order 16 and in their duals. A complete description of these examples is given in section 3.

Clearly, theorems 1 and 2 together cover all possibilities for a 2-transitive collineation group G.

The present paper is structured as follows. In section 2 we fix notation and the background of the problem, and we recall some results which are useful to prove theorems 1 and 2. In section 3 a complete description of the examples provided in the paper is given. In section 4 we give some preliminary reductions for the structure of the 2-transitive collineation group G. Sections 5 and 6 are devoted to the proofs of the theorems 1 and 2, respectively. Finally, in section 7, our main problem is investigated under the additional assumption that Π is the projective extension of a translation plane.

2 Background

In the paper group-theoretical and geometrical notation is standard. For the required background concerning finite groups the reader is referred to [1], [23] and [34]. In particular for the finite groups admitting a 2-transitive permutation representation we have the following classification.

Theorem 3 *Let H be a finite group with a 2-transitive permutation representation of degree v and let $S = \mathrm{soc}(H)$ be the socle of H. Then one of the following occurs:*

1. *S is non abelian simple, and $S \leq H \leq \mathrm{Aut}S$ where S and v are as follows:*

(a) A_v *with* $v \geq 5$;

(b) $PSL(d,q)$, $d \geq 2$, $v = (q^d - 1)/(q-1)$ *and* $(d,q) \neq (2,2),(2,3)$;

(c) $PSU(3,q)$, $v = q^3 + 1$, $q > 2$;

(d) $Sz(q)$, $v = q^2 + 1$, $q = 2^{2e+1} > 2$;

(e) ${}^2G_2(q)'$, $v = q^3 + 1$, $q = 3^{2e+1}$;

(f) $Sp(2n,2)$, $n \geq 3$, $v = 2^{2n-1} \pm 2^{n-1}$;

(g) $PSL(2,11)$, $v = 11$;

(h) *Mathieu groups* $M_v, v = 11, 12, 22, 23, 24$;

(i) M_{11}, $v = 12$;

(j) A_7, $v = 15$;

(k) HS *(Higman-Sims group)*, $v = 176$;

(l) Co_3 *(Conway's smallest group)*, $v = 276$.

2. *S is an elementary abelian group of order $v = p^d$, where p is a prime. Identify G with a group of affine transformations $x \longmapsto x^g + c$ of $GF(p)^d$, where $g \in G_0$. Then one of the following occurs:*

(a) $G_0 \leq \Gamma L(1, p^d)$;

(b) $SL(a,q) \trianglelefteq G_0$, *where* $a \geq 2$ *and* $q^a = p^d$;

(c) $Sp(a,q) \trianglelefteq G_0$, *where* $a \geq 4$, *a even, and* $q^a = p^d$;

(d) $G_2(q)' \trianglelefteq G_0$ *where* $q^6 = p^d$ *and q is even;*

(e) $G_0 \cong A_6$ *or* A_7, $p^d = 2^4$;

(f) $SL(2,3) \trianglelefteq G_0$ *or* $SL(2,5) \trianglelefteq G_0$, $v = p^2$ *and* $p = 5, 7, 11, 19, 23, 29,$ *or* 59, *or* $v = 3^4$;

(g) G_0 *has a normal extraspecial subgroup R of order* 2^5 *and* $G_0/R \leq S_5$;

(h) $G_0 \cong SL(2,13)$, $p^d = 3^6$.

See for example [40].

A finite 2-transitive group is said either *almost simple* or *of affine type* according to whether its socle is a non abelian simple or an elementary abelian p-group for some prime p, respectively.

The background concerning finite projective planes may be found in [33]. Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a finite projective plane of order n. If G is a collineation group and $P \in \mathcal{P}$ ($l \in \mathcal{L}$), we denote by $G(P)$ (by $G(l)$) the subgroup of G consisting of perspectivities with the centre P (the axis l). Also, $G(P,l) = G(P) \cap L(l)$. Furthermore, we denote by $G(P,P)$ (by $G(l,l)$) the subgroup of G consisting of elations with the centre P (the axis l).

The following theorems deal with projective planes Π of order n with a collineation group G acting 2-transitively either on the points of a line, or on the points of a line minus one.

Theorem 4 *Let Π be a projective plane of order n with a collineation group G acting 2-transitively on the points of a line. Then one of the following occurs:*

(1) $\Pi \cong PG(2,n)$ *and* $SL(2,n) \trianglelefteq G$*;*

(2) $n = q^2$, $q = 2^{2s+1}$, $s \geq 1$, *and* $Sz(q) \trianglelefteq G$*;*

(3) $n = q^3$, $q = 2^{2s}$, $s \geq 1$, $PSU(3,q) \trianglelefteq G$ *and G fixes a point of* Π^l*;*

(4) $n = p^h - 1$, *p an odd prime, and* $G \leq A\Gamma L(1,v)$*;*

(5) $n = p^h - 1$, $p^h \in \{5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$, *and G^l is sharply transitive on l except possibly for* $p^h = 5^2$ *or* 29^2*.*

For a proof see [7], Theorems 5.2. and 5.5.

Note that, while the there no known examples corresponding to the cases (3)-(5), the case (2) really occurs in the projective extensions of the Lüneburg planes.

Now, we consider the case where G fixes an incident point-line pair (L,l) of Π and acts 2-transitively on $l - \{L\}$.

Theorem 5 (Hiramine) *Let Π be a projective plane of order n with a collineation group G that fixes an incident point-line pair (L,l) of Π and acts 2-transitively on $l - \{L\}$. Then $n = p^d$, p prime, and $\bar{G}$ contains a normal elementary abelian p-group acting regularly on $l - \{L\}$. In particular one of the following occurs:*

(1) $\bar{G}_O \leq \Gamma L(1,p^d)$*;*

(2) $SL(2,p^{d/2}) \trianglelefteq \bar{G}_O$, *$d$ even;*

(3) $p^d \in \{2^4, 3^2, 3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$.

In the following result the two previous situations are analyzed under the further assumption that Π is the projective extension of a translation plane.

Theorem 6 *Let Π be the projective extension of a translation plane of order n and let (L,l) be an incident point-line pair of Π, and let G be a collineation group of Π fixing the line l. Then one of the following occurs:*

1. *If G acts 2-transitively on l, then l is the line at infinity and either*
   (a) Π *is Desarguesian, or*
   (b) Π *is a Lüneburg plane.*
2. *If $n \neq 2^6$, G fixes the point L and acts 2-transitively on $l - \{L\}$, where l is the line at infinity, then either*
   (a) Π *is a Desarguesian, or*

(b) Π *is a Generalized Twisted Field plane.*

3. *If* $n \notin \{3^4, 3^6, 11^2, 19^2, 29^2, 59^2\}$*,* G *is non solvable,* G *fixes the point* L *and acts* 2*-transitively on* $l - \{L\}$*, where* l *is an affine line, then either*

   (a) Π *is Desarguesian, or*

   (b) Π *is one of the three Walker planes of order* 25*, or*

   (c) Π *is the Dempwolff plane of order* 16*.*

See for example [38], theorem 4.3.16, for a proof of the case (1), see [5], theorem 8.1, for a proof of the case (2), and see [20], main theorem, for a proof of the case (3). Clearly all these cases really occur. Note that a classification of the projective extensions of translation planes, when l is an affine line and one of the situations (1) or (3) of the Hiramine's theorem occurs, is not available. Nevertheless, there are several examples corresponding to each of these situations (see [6]). In particular in the examples referring to the situation (3), A_5 is involved in $\bar{G}_O$ in many cases.

3 Examples

In this section we provide some examples. It is worth noting that, while examples 7 and 8 are already known, example 9 is new.

Example 7 *Let* Π *be a projective plane of order* n*, with* $n \leq 9$*, and let* G *be a collineation group of* Π*. Suppose that* G *induces a group* $\bar{G}$ *which has a* 2*-transitive point-orbit of length* v *on a line* l*. If* $n > v \geq n/2$*, then one of the following occurs:*

(1) $\Pi \cong PG(2,4)$*,* $\bar{G} \cong AGL(1,3)$ *and there is exactly one* 2*-transitive* $\bar{G}$*-orbit of length* 3 *on* l*;*

(2) $\Pi \cong PG(2,5)$*,* $\bar{G} \cong AGL(1,3)$ *and there are exactly two* 2*-transitive* $\bar{G}$*-orbits of length* 3 *on* l*;*

(3) $\Pi \cong PG(2,7)$*,* $G \cong SL(2,3)$*. In particular* $\bar{G} \cong AGL(1,4)$ *and there are exactly two* 2*-transitive* $\bar{G}$*-orbits of length* 4 *on* l*;*

(4) $\Pi \cong PG(2,8)$*,* $\bar{G} \cong AGL(1,4)$ *and there are exactly two* 2*-transitive* $\bar{G}$*-orbits of length* 4 *on* l*;*

(5) $\Pi \cong PG(2,8)$*,* $\bar{G} \cong AGL(1,7)$ *and there is exactly one* 2*-transitive* $\bar{G}$*-orbit of length* 7 *on* l*;*

(6) $\Pi \cong PG(2,9)$*,* $\bar{G} \cong AGL(1,5)$ *and there are exactly two* 2*-transitive* $\bar{G}$*-orbits of length* 5 *on* l*;*

(7) $\Pi \cong PG(2,9)$*,* $G \cong SL(2,5)$*. In particular* $\bar{G} \cong A_5$ *there are exactly two* 2*-transitive* $\bar{G}$*-orbits of length* 5 *on* l*.*

In particular each of these cases really occurs.

Let Π be a projective plane of order n, with $n \le 9$, and let G be a collineation group of Π inducing a group $\bar{G}$ which has a 2-transitive point-orbit of length v on a line l. Assume that $n > v \ge n/2$. Clearly $v \ge 3$.

If $v = 3$, then $3 < n \le 6$. Actually, $n < 6$ by [33], theorem 3.6. Hence either $n = 4$ or $n = 5$. If $\Pi \cong PG(2,4)$ any subgroup $\bar{G}$ of $P\Gamma L(2,4)$ isomorphic to $D_6 \cong AGL(1,3)$ and containing the involution induced by the Frobenius automorphism of $GF(4)$ fixes 2 points on l and acts 2-transitively on the remaining ones. Thus (1). If $\Pi \cong PG(2,5)$ the group $SL(2,5)$ induces A_5 on l and any $\bar{G} \cong D_6$ inside A_5 has two 2-transitive point-orbits on l both of length 3, and hence (2).

If $v = 4$, then $4 < n \le 8$. That is $n = 5$, 7 or 8. The are no examples in $PG(2,5)$, since the stabilizer in $PGL(2,5)$ of three distinct points on a line is trivial. In $PG(2,7)$ there is exactly one example: the group $G \cong SL(2,3)$ has two 2-transitive point-orbits on l both of length 4. Therefore (3). In $PG(2,8)$ there is exactly one example: a subgroup $\bar{G}$ of $P\Gamma L(2,8)$ isomorphic to $AGL(1,4)$ has two 2-transitive point-orbits on l both of length 4. Hence (4).

If $v = 5$, then $5 < n \le 9$. If $n = 7$ or 8, then $\Pi \cong PG(2,n)$. Nevertheless these cases cannot occur, since $5 \nmid |P\Gamma L(2,n)|$. Therefore $n = 9$. Then either Π is Desarguesian or Π is one of the Hall planes by [59]. It is easily seen that there exists a subgroup $\bar{G} \cong AGL(1,5)$ of $P\Gamma L(2,9)$ splitting l in two 2-transitive orbits both of length 5 when Π is Desarguesian, and hence (6). If Π is the Hall plane of order 9, then the group induced on the line at infinity by the full translation complement of Π is described in the proof of lemma 5.2 of [18]. It is easy to check with [21], by using such a description, that the group induced on the line at infinity does not contain solvable subgroups with a 2-transitive permutation representation of degree 5. Nevertheless, the group $G \cong SL(2,5)$ induces A_5 on the line at infinity and this one is split in two 2-transitive A_5-orbits both of length 5 (see [6]). Thus (7).

If $v = 6$, then $6 < n \le 9$. If $n = 7$ or 8, then $\Pi \cong PG(2,n)$. Thus $\bar{G} \le P\Gamma L(2,n)$ such that $\bar{G}$ is 2-transitive orbit on l of length 6. Clearly $5 \mid \left|\bar{G}\right|$. A contradiction, since $5 \nmid |P\Gamma L(2,n)|$ for $n = 7$ or 8. So $n = 9$. Nevertheless this case cannot occur by [8], theorems 25 and 35, since $n - v = 3$.

If $v = 7$, then $7 < n \le 9$. That is $n = 8$ or $n = 9$. If $n = 9$, then $\Pi \cong PG(2,9)$ by [58], lemma 8.2. So $\bar{G} \le P\Gamma L(2,9)$. A contradiction, since $7 \mid \left|\bar{G}\right|$ while $7 \nmid |P\Gamma L(2,9)|$. Thus $n = 8$ and hence $\Pi \cong PG(2,8)$. Let l be a line of $\Pi \cong PG(2,8)$. Clearly $P\Gamma L(2,8)$ acts on l. Pick any Z_7 in $P\Gamma L(2,8)$. Then Z_7 fixes two points P_1 and P_2 on l and Z_7 acts regularly on $l - \{P_1, P_2\}$. Furthermore, $N_{P\Gamma L(2,8)}(Z_7) \cong D_{14}.\langle\sigma\rangle$, where σ is the collineation of Π "induced" by a Frobenius automorphism of $GF(8)$. Set $\mathcal{O} = l - \{P_1, P_2\}$ and $\bar{G} \cong N_{P\Gamma L(2,8)}(Z_7)$. Then $\bar{G} \cong AGL(1,7)$ acts 2-transitively on $\mathcal{O}$ and $v = 7$. Thus (6).

If $v = 8$, then $n = 9$. Then $\Pi \cong PG(2,9)$ by [58], lemma 8.2, since $7 \mid \left|\bar{G}\right|$ as $\bar{G}$ acts 2-transitively on $\mathcal{O}$ and $v = 8$. A contradiction as above.

Example 8 *In the Johnson-Walker translation plane* Π *of order* 16 *or its dual, there exists an affine line* l *on which a group* G *isomorphic to* $PSL(2,7)$ *has two* 2*-transitive orbits of length* 7 *and* 8*.*

This example follows by [15], Theorems 4.8-4.10 and Section 5.

Example 9 *In the Lorimer-Rahilly translation plane of order* 16*, in the Johnson-Walker translation plane of order* 16 *and in their duals the group* $G \cong AGL(1,8)$ *admits a* 2*-transitive orbit* $\mathcal{O}$ *of length* 8 *on a line.*

Let Π be the Lorimer-Rahilly translation plane of order 16, or the Johnson-Walker translation plane of order 16, or one of their duals. Denote by l_∞ the line at infinity of Π. Let T be the full translation group of Π and let $H \cong Z_7$ be a subgroup of the translation complement fixing the point O of Π. Clearly H fixes a point P on l_∞. Let T_1 be the subgroup of translations of Π of direction P. Then T_1 fixes the line PO and it acts regularly on $PO - \{P\}$. Furthermore, H acts on $T_1 - \{1\}$ and on $PO - \{P, O\}$ in the same way by [52], proposition 4.2. In particular H leaves a subgroup T_2 of T_1 of order 8 invariant, since $T_0 - \{1\} \cong PG(3,2)$ and Z_7 fixes exactly a non incident point-plane pair in $PG(3,2)$ by [51], Table I. Clearly Z_7 is transitive on $T_1 - \{1\}$. Set $\mathcal{O} = O^{T_1}$ and $G \cong T_2.H$. Then $G \cong AGL(1,8)$ acts 2-transitively on $\mathcal{O}$ and $|\mathcal{O}| = 8$.

We remark that in the other known projective planes of order 16 there are no examples of 2-transitive orbits of length 8 on a line (see [53], Table I).

4 Preliminaries

Let G be a collineation group having an orbit $\mathcal{O}$ of points of Π on which G acts 2-transitively, then we call $\mathcal{O}$ a 2*-transitive* G*-orbit*, or just a 2*-transitive orbit.* Furthermore, we say that $\mathcal{O}$ is *non trivial*, if $|\mathcal{O}| > 1$. Note that $v \geq 3$, since G is 2-transitive. In the sequel we assume that $v \geq 5$. It is a plain that $n \geq 8$.

The following numerical and group-theoretical lemmas will be useful hereafter.

Lemma 10 *Let* t^j*,* $j \geq 0$*, and* p^r*,* $r \geq 0$*, be two power of primes such that* $p^r \equiv 3 \bmod 4$*. Then the following holds:*

(1) If $t^j = 2p^r + 1$*, then* $j = 1$*.*

(2) If $t^j = 2(p^r + 1)$*, then* $t = 2$*,* $r = 1$ *and* p *is a Mersenne prime.*

Proof. The assertion (2) follows by [55], result (B1.1), since $t = 2$. Hence, assume that $t^j = 2p^r + 1$. Then $t^j \equiv 3 \bmod 4$, since $p^r \equiv 3 \bmod 4$. Thus $t \equiv 3 \bmod 4$ and j is odd. In particular $2p^r = (t-1)\left[(t^j - 1)/(t-1)\right]$. Hence $2p^{r-h} = t - 1$ and $p^h = (t^j - 1)/(t-1)$, $0 \leq h \leq r$, since t and j are odd. Assume that $0 < h < r$. Then $p \mid \gcd(t-1, (t^j - 1)/(t-1))$. Hence $p \mid j$ by [55], result P1.2 (ii). But this contradicts [55], result A8.5 (1). Assume that $h = r$.

Then $t = 3$ and $(3^j - 1)/(3 - 1) = p^r$. Then $(p, r, j) = (11, 2, 3)$ by [61], theorem 3. A contradiction, since $p^r = 11^2$ and $11^2 \not\equiv 3 \bmod 4$. Hence $h = 0$ and $j = 1$. Thus the assertion (1). $\square$

A class of solutions to the first Diophantine equation is furnished by the Sophie-Germain primes ($r = 1$) [56].

Denote by $d_j(H)$, $j \geq 0$, the primitive permutation representation degrees of a group H in increasing order. So, $d_0(H)$ denotes the minimal one. If v is a 2-transitive permutation representation degree of H, then $d_0(H) \leq v$.

Lemma 11 *Let H be a 2-transitive non-abelian simple group such that $d_0(H) = v$. Then either $d_j(L) > d_0(L) + \sqrt{2d_0(L)}$ for $j > 0$, or $d_1(L) = d_0(L) + 1$ and one of the following occurs:*

(1) $H \cong A_5$ and $v = 5$;

(2) $H \cong PSL(2, 7)$ and $v = 7$;

(3) $H \cong PSL(2, 11)$ and $v = 11$;

(5) $H \cong M_{11}$ and $v = 11$.

Proof. The assertion is true when H is sporadic by a direct inspection of [12]. Elementary calculations with [60] and with [42] show that the assertion is true also when H is exceptional of Lie type. When H is alternating the assertion follows by a straightforward calculation by [34], Satz IV.4.6 for $v \geq 9$ and by [12] for $5 \leq v < 9$. Assume that H is simple classical group. Then the assertion follows by [34], haupsatz II.8.27 when $H \cong PSL(2, q)$. Furthermore, the assertion follows by [25] and [49] when $H \cong PSL(3, q)$. It remains to resolve the cases $H \cong PSL(d, q)$, $d \geq 4$, and $H \cong Sp(d, 2)$, $d = 2h$ and $h \geq 3$, by the list given in theorem 3. As a consequence of the structure theorems given in Kleidman and Liebeck [43], every maximal subgroup of H lies in the classes $\cup_{i=1}^{8} \mathcal{C}_i$ or in the class $\mathcal{S}$, where the structure of every member of $\mathcal{C}_i$ is shown in [43], Tables 3.5.A-C. Let M be a maximal subgroup of H such that $[H : M] > d_0(H)$. If $M \in \mathcal{S}$, then $|M| \leq q^{3d}$ by [46]. Then $[H : M] \geq 2v + 1$ by an easy calculation. If $M \in \cup_{i=1}^{8} \mathcal{C}_i$, then a straightforward calculation of $[H : M]$ with [43], Tables 3.5.A-C, with the structure proposition members of $\mathcal{C}_i$ given in [43], Chapter 4, in junction with [9] and with lemma 4 and Table II of [45], shows that the assertion is true also in this case. $\square$

Lemma 12 *Let H be a 2-transitive non-abelian simple group such that $d_0(H) < v$. Then the following holds:*

(1) $d_j(H) > v + \sqrt{2v}$ for $j > 1$;

(3) $2d_0(H) > v + 1$, except

(a) $H \cong A_7$, where $2d_0(H) = v - 1$ and $v = 15$;

(b) *$H \cong A_8$, where $2d_0(H) = v + 1$ and $v = 15$;*

(c) *$H \cong PSU(3,5)$, where $2d_0(H) = v - 26$ and $v = 126$.*

Proof. In the following table the non abelian simple groups such that $d_0(H) < v$ are listed (see theorem 3):

Table I

$PSL(2,5)$	$v = 6$	$d_0(H) = 5$
$PSL(2,7)$	$v = 8$	$d_0(H) = 7$
$PSL(2,9)$	$v = 10$	$d_0(H) = 6$
$PSL(2,11)$	$v = 12$	$d_0(H) = 11$
$PSU(3,5)$	$v = 126$	$d_0(H) = 50$
A_7	$v = 15$	$d_0(H) = 7$
A_8	$v = 15$	$d_0(H) = 8$
M_{11}	$v = 12$	$d_0(H) = 11$
HS	$v = 176$	$d_0(H) = 100$
$Sp(2h,2)$, $h \geq 3$	$v = 2^{h-1}(2^h + 1)$	$d_0(H) = 2^{h-1}(2^h - 1)$

The assertion (1) easily follows by a direct inspection in [12] of the primitive permutation representations of the groups H listed in the Table I and not isomorphic to $Sp(2h,2)$, $h \geq 3$. When $H \cong Sp(2h,2)$, $h \geq 3$, a similar argument to that lemma 11, being H classical, proves the assertion also in this case. Now, the assertion (2) can be easily read off from Table I. □

Let N be the kernel of the action of G on $\mathcal{O}$ and set $\bar{G} = G/N$. We may also assume that G is the minimal preimage of $\bar{G}$. Now, we present some preliminary reductions for the structure of N.

Lemma 13 *$N = \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G.*

Proof. Let S be any Sylow t-subgroup of N. Then $G = N_G(S)N$ by the Frattini's argument. Thus $S \lhd G$ by the minimality of G. Therefore N is nilpotent. Suppose that $N \not\leq \Phi(G)$. Then there exists a maximal subgroup M of G such that $G = NM$ by [34], satz 3.2 (b). Clearly $M < G$ and $M/(M \cap N) \cong \bar{G}$. A contradiction by the minimality of G. Hence, we may assume that $N \leq \Phi(G)$. Note that G_P is maximal in G for each point $P \in \mathcal{O}$, since $N \lhd G_P$ and $\bar{G}$ is primitive on $\mathcal{O}$. Hence $\Phi(G) \lhd G_P$ for each point $P \in \mathcal{O}$. Therefore $N = \Phi(G)$. □

Lemma 14 *If $N \neq \langle 1 \rangle$, then one of the following occurs:*

(1) *G fixes a unique point Q on $\Pi - l$, N is semiregular on $\Pi - (l \cup \{Q\})$ and $|N| \mid n - 1$;*

(2) *N is semiregular on $\Pi - l$ and $|N| \mid n^2$. In particular one of the followings occurs:*

(a) *$|N| \mid n$ and N is semiregular on $[Y] - \{l\}$ for any point $Y \in \mathcal{O}$;*

(b) $n \mid |N|$, $n = u^j$, $j \geq 1$, N *is a* u*-group and* $[N : N_a] = n$ *for any line* a *of* Π *intersecting* l *in* $\mathcal{O}$*;*

(c) $n \mid |N|$, $n = 2^{j+1}$, $j > 1$, N *is a* 2*-group and* $[N : N_a] = n/2$ *for any line* a *of* Π *intersecting* l *in* $\mathcal{O}$*;*

(d) $|N| > n$, $n = 3|S|/2$, S *is a Sylow* 2*-subgroup of* N *and* $S \leq N \leq S \times A$*, where* A *is a group of order a divisor of* 9*.*

Proof. Suppose there exists $\alpha \in N$, $\alpha \neq 1$, such that α is planar on Π. Then $o(\mathrm{Fix}(\alpha)) \geq n/2 - 1$, since α fixes $\mathcal{O}$ pointwise and $v \geq n/2$. So $(n/2-1)^2 \leq n$ by [33], theorem 3.7. A contradiction, since $n \geq 8$. Thus N does not contain any non trivial planar collineation of Π.

Assume there exists an element $\sigma \in N$, $\sigma \neq 1$, such that σ fixes a point P of $\Pi - l$. Actually, P is the unique point on $\Pi - l$ fixed by σ, since N does not contain any non trivial planar collineation of Π. Furthermore, P is the unique point on $\Pi - l$ fixed by $Z(N)$. Thus G fixes P, since $Z(N) \lhd G$ and $Z(N) \neq \langle 1 \rangle$ being N nilpotent by lemma 13. Hence the assertion (1).

Assume that N is semiregular on $\Pi - l$. Hence $|N| \mid n^2$. If N is semiregular on $[Y_0] - \{l\}$, for some point $Y_0 \in \mathcal{O}$, then $|N| \mid n$ and N is semiregular on $[Y] - \{l\}$ for any point $Y \in \mathcal{O}$, since G is transitive on $\mathcal{O}$. Thus the result (2a).

Assume there exists $b_X \in [X] - \{l\}$ such that $N_{b_X} \neq \langle 1 \rangle$ for each $X \in \mathcal{O}$. If $N_e \cap N_c \neq \langle 1 \rangle$ for some couple of lines e and c intersecting l in distinct points of $\mathcal{O}$, then there exists $\gamma \in N$, $\gamma \neq 1$, fixing the point $e \cap c$ which lies in $\Pi - l$. A contradiction, since N is semiregular on $\Pi - l$. Therefore, we may assume that $N_h \cap N_z = \langle 1 \rangle$, with $N_h, N_z \neq \langle 1 \rangle$, for any couple of lines h and z intersecting $\mathcal{O}$ in distinct points. In particular $N_t < N$ for each line t of Π intersecting $\mathcal{O}$. Thus $[P] - \{l\}$ consists of non trivial N-orbits for any point P of $\mathcal{O}$. Let $a \in [O] - \{l\}$, $O \in \mathcal{O}$, such that $N_a \neq \langle 1 \rangle$. Let S_a be the Sylow u-subgroup of N_a, where u is a prime dividing $|N_a|$, and let S be the Sylow u-subgroup of N. Assume that $S = S_a$. Then $S_a \lhd G$ as N is nilpotent. Let $g \in G$ such that $Og \neq O$. Since $S_a \lhd G$, then S_a fixes the line ag and hence the point $a \cap ag$ which lies in $\Pi - l$. A contradiction, since N is semiregular on $\Pi - l$. Therefore $S_a < S$. Furthermore, $|S| - 1 \geq v(|S_a| - 1)$ since $S_a \neq \langle 1 \rangle$, $S \lhd G$ and G is transitive on $\mathcal{O}$. Then $[S : S_a] \geq [v(|S_a| - 1) + 1] / |S_a|$. Hence either $|S_a| \geq 3$ and $[S : S_a] > 2v/3$, or $|S_a| = 2$ and $[S : S_a] \geq (v+1)/2$. Denote by k the number of S-orbits on $[O] - \{l\}$, where $O \in \mathcal{O}$. Arguing as above with S in the role of N, as $S \lhd G$, we see that $[O] - \{l\}$ consists of non trivial S-orbits. Let $x \in [O] - \{l\}$ such that $\left|x^S\right| \leq \left|y^S\right|$ for any $y \in [O] - \{l\}$. Then $k\left|x^S\right| \leq n$.

Assume that $|S_x| \geq 3$. Then $k \leq 2$ as $\left|x^S\right| > 2v/3$ and $n \leq 2v$. If $k = 1$, then $\left|x^S\right| = n$. Hence $n = u^j$, $j \geq 1$, as $\left|x^S\right| = u^j$. Thus $N = S$ as $|N| \mid n^2$, and we have the assertion (2b). Assume that $k = 2$. Then $[O] - \{l\} = x^S \cup b^S$ for some line b of $[O] - (\{l\} \cup x^S)$. Hence $n = \left|x^S\right| + \left|b^S\right|$. Assume that $\left|x^S\right| < \left|b^S\right|$. Then $\left|x^S\right| u \leq \left|b^S\right|$, since S is a u-group. Hence $\left|b^S\right| > 2vu/3$ as $\left|x^S\right| > 2v/3$. Then $2vu/3 + 2v/3 < n$, as $n = \left|x^S\right| + \left|b^S\right|$. A contradiction, since $u \geq 2$ and $n \leq 2v$. As a consequence $\left|x^S\right| = \left|b^S\right|$. Hence $n = 2u^j$ as $\left|x^S\right| = u^j$. Then $u = 2$

and $n = 2^{j+1}$ by [33], theorem 13.18. Then $N = S$ as $|N| \mid n^2$. In particular $\left|x^N\right| = \left|b^N\right| = n/2$. That is the assertion (2c).

Assume that $|S_x| = 2$. Then $k \leq 3$ as $\left|x^S\right| \geq (v+1)/2$ and $n \leq 2v$. Note that $S_x = S(x \cap l, l)$, since $|S_x| = 2$, $S_x < N$ and N cannot contain non trivial planar elements. Thus $|S_y| \geq 2$ and $\left|y^S\right| \leq \left|x^S\right|$ for any $y \in [O] - \{l\}$. Hence $\left|y^S\right| = \left|x^S\right|$, since $\left|x^S\right| \leq \left|y^S\right|$ for any $y \in [O] - \{l\}$. Then $n = k\,|S|/2$, $k \leq 3$. If $k \leq 2$, then $N = S$ as $|N| \mid n^2$, and we have again the assertions (2b) and (2c). If $k = 3$, then $n = 3\,|S|/2$. Hence $S \leq N \leq S \times A$ where A is a group of order a divisor of 9, as $|N| \mid n^2$. Thus the assertion (2d). □

Recall that $\mathrm{soc}(\bar{G})$ denotes the socle of $\bar{G}$. Also, recall that either $\bar{G}$ is almost simple or of affine type, since $\bar{G}$ is 2-transitive on $\mathcal{O}$. We treat these two cases separately.

5 $\bar{G}$ is almost simple.

Assume that $\bar{G}$ is almost simple. We treat the cases $N \neq \langle 1 \rangle$ and $N = \langle 1 \rangle$ separately.

5.1 The unfaithful case.

Assume that $N \neq \langle 1 \rangle$. We continue investigating the structure of N.

Lemma 15 *If $\bar{G}$ is non abelian simple, then one of the following holds:*

(1) G is a covering group for $\bar{G}$;

(2) There exists a Sylow t-subgroup S of N such that $\bar{G} \leq SL(V)$, where $V = S/\Phi(S)$. In particular $|S| \geq 1 + d_0(\bar{G})$.

Proof. Assume that $N \leq Z(G)$. Then G' is a covering group for $\bar{G}$ by [1], theorem 11.33.3, since $\bar{G}$ is a non abelian simple group. Furthermore, $G' = G$ by the minimality of G. Thus the assertion (1).

Assume that $N \nleq Z(G)$. Then there exists a Sylow t-subgroup S of N such that $S \nleq Z(G)$, since N is nilpotent. Set $V = S/\Phi(S)$, where $\Phi(S)$ is the Frattini subgroup of S. Clearly G acts on V. Let R be the kernel of the action of G on V. If U is the Sylow u-subgroup of N, where u is a prime, $u \neq t$, then $[S, U] = \langle 1 \rangle$, since N is nilpotent. This yields $N \trianglelefteq R \trianglelefteq G$, since $S' \leq \Phi(S)$, being S a t-group. If $R = G$, then each Sylow r-subgroup of G, with $r \neq t$, centralizes S by [23], theorem 5.1.4. That is $C_G(S) \nleq N$. Furthermore, $C_G(S) \lhd G$ as $S \lhd G$. Then $N \lhd C_G(S)N \trianglelefteq G$. Hence $G = C_G(S)N$, since $\bar{G}$ is non abelian simple and $C_G(S) \nleq N$. Actually, $G = C_G(S)$ since $N = \Phi(G)$ by lemma 13. A contradiction, since $S \nleq Z(G)$. Hence $R < G$. Then $R = N$ as $\bar{G}$ is non abelian simple. Then $\bar{G} \leq \Gamma L(V)$, since V is a vector space over $GF(t)$. Actually $\bar{G} \leq SL(V)$, since $\bar{G}$ is non abelian simple. In particular G

acts not trivially on the points of $PG(V)$ and hence $|V| \geq 1 + d_0(\bar{G})$. Thus the assertion (2). $\square$

We point out that the condition in (2) of lemma 15 in conjunction with the information contained in the paragraphs 5.3 and 5.4 of [43] furnishes a lower bound for V and hence for N. This lower bound is generally greater than $1 + d_0(\bar{G})$.

Lemma 16 *Let Ω a set of non trivial N-orbits of points (respectively lines) of Π having the same length. If G leaves Ω invariant, then $|\Omega| = \sum_{j\geq 0} \lambda_j d_j(\bar{G})$, where $\lambda_j \geq 0$ for $j \geq 0$, and $\sum_{j\geq 0} \lambda_j > 0$. In particular $|\Omega| \geq d_0(\bar{G})$.*

Proof. Assume that G fixes an element in Ω. Then $G = G_X N$ for some point X of Π such that $X^N \in \Omega$. Actually $G = G_X$, since $N = \Phi(G)$ by lemma 13. This yields $\left|X^N\right| = 1$. A contradiction, since $X^N \in \Omega$ and Ω is a set of non trivial N-orbits of points of Π. Thus G acts on Ω as $\bar{G}$ and this moves each element of Ω. Therefore Ω is union of non trivial $\bar{G}$-orbits. Since each non trivial $\bar{G}$-orbit has length a multiple of some primitive permutation representation degree $d_h(\bar{G})$ of $\bar{G}$, $h \geq 0$, we have that $|\Omega| = \sum_{j\geq 0} \lambda_j d_j(\bar{G})$, where $\lambda_j \geq 0$ for $j \geq 0$, and $\sum_{j\geq 0} \lambda_j > 0$. In particular $|\Omega| \geq d_0(\bar{G})$. $\square$

Lemma 17 *$\bar{G} \not\cong P\Gamma L(2,8)$.*

Proof. Assume that $\bar{G} \cong P\Gamma L(2,8)$. Clearly $28 < n \leq 56$. Set L be the minimal preimage of $PSL(2,8)$ in G and set $H = L \cap N$. Assume that $H = \langle 1 \rangle$. Then $L \cong PSL(2,8)$. It is known that any involution ζ in L fixes exactly four points on $\mathcal{O}$, since $|\mathcal{O}| = 28$. Then ζ is a Baer collineation of Π. Thus either $n = 36$ or $n = 49$, since n must be square and $28 < n \leq 56$. The former is ruled out by [33], theorem 3.6, since $o(\mathrm{Fix}(\zeta)) = 6$, and the latter is ruled out by [30]. Hence, we may assume that $H \neq \langle 1 \rangle$. Then $H = \Phi(L)$ by the argument of lemma 13 with H in the role of N, since $L/H \cong PSL(2,8)$ is primitive on $\mathcal{O}$ as $|\mathcal{O}| = 28$. Now, assume that $H \not\leq Z(L)$. Note that, the assertion of lemma 15 is still true if we replace $\bar{G}$ with L/H, since the 2-transitivity is actually not required in that lemma. Thus there exists a Sylow t-subgroup S of H such that $PSL(2,8) \leq P\Gamma L(V)$, where $V = S/\Phi(S)$, since $L/H \cong PSL(2,8)$. Then either $|V| \geq 3^7$ or $8^2 \mid |V|$ by [43], theorem 5.3.9 and proposition 5.4.13, respectively. Therefore either $|H| \geq 3^7$ or $8^2 \mid |H|$, since $V = S/\Phi(S)$. Then $|H| > n$ in any case, since $n \leq 56$. Thus n and $|H|$ are power of the same prime, $n \mid |H|$ and $|H| \mid n^2$ by lemma 14 with H in the role of N, since H is transitive on $\mathcal{O}$. This rules out the case $|H| \geq 3^7$, since $n \leq 56$. Then $8^2 \mid |H|$ and hence $n = 2^5$, since $28 < n \leq 56$. Let Ω be the set of H-orbits on $\Pi - l$. Then $|\Omega| = 2^4/\theta$, since $n = 2^5$, $|H| = 8^2\theta$, with θ a power of 2, and since H is semiregular on $\Pi - l$. On the other hand, $|\Omega| = \lambda_0 9$, with $\lambda_0 \geq 0$, by lemma 16, with L in the role of G and H in the role of N, since 9 is the unique primitive permutation representation degree of L/H less than 16. A contradiction. Hence L is a covering group for $PSL(2,8)$ by lemma 15. Then $L \cong PSL(2,8)$, since the

Schur multiplier of $PSL(2,8)$ is trivial by [43], theorem 5.1.4. A contradiction by the above argument. $\square$

As $\bar{G} \ncong P\Gamma L(2,8)$, then $\mathrm{soc}(\bar{G})$ is a 2-transitive on $\mathcal{O}$ (this follows by a direct inspection of the list given in theorem 3). Thus, we may assume that $\bar{G} = \mathrm{soc}(\bar{G})$. Hence $\bar{G}$ is a 2-transitive non abelian simple group.

Let K be the kernel of the action of G on $l - \mathcal{O}$. Clearly $K \trianglelefteq G$. Since $KN/N \trianglelefteq \bar{G}$ and since $\bar{G}$ is non abelian simple, then either $G = KN$ or $K \leq N$. Actually $G = K$ in the first case, since $N = \Phi(G)$. So, either $G = K$ and G fixes $l - \mathcal{O}$ pointwise, or $K \leq N$. Now, we investigate the relationship between N and K.

Lemma 18 *If $\bar{G} \ncong PSU(3,5)$, then either $N = N(l,l)$ or $N = N(Q,l)$ for some point $Q \in \Pi - l$. In particular $N \leq K$.*

Proof. Assume that $N \nleq K$. Then there exists a point $P \in l - \mathcal{O}$ such that $\left|P^N\right| > 1$. Let Ω be the set of N-orbits on P^G. Then $|\Omega| \geq d_0(\bar{G})$ by lemma 16, since $N \lhd G$ and $\left|P^N\right| > 1$. Hence $\left|P^N\right| d_0(\bar{G}) \leq \left|P^G\right|$, since $|\Omega| = \left|P^G\right| / \left|P^N\right|$. This yields $2d_0(\bar{G}) \leq v + 1$, since $\left|P^N\right| \geq 2$, $P^G \subset l - \mathcal{O}$ and $|l - \mathcal{O}| \leq v + 1$. Then $d_0(\bar{G}) < v$. In particular $\left|P^N\right| = 2$ and either $\bar{G} \cong PSL(4,2)$ or $\bar{G} \cong A_7$, by lemma 11 (2), since $\bar{G} \ncong PSU(3,5)$ by our assumption.

Assume that $\bar{G} \cong PSL(4,2)$ or $\bar{G} \cong A_7$. Note that $n + 1 \geq v + 2d_0(\bar{G})$ in any of these cases. Thus either $n = 30$ and $\bar{G} \cong PSL(4,2)$, or $n \in \{28, 29, 30\}$ and $\bar{G} \cong A_7$, since $n \leq 2v$. The case $n = 30$ is ruled out by [33], theorem 13.18. Hence $\bar{G} \cong A_7$ and $n \in \{28, 29\}$. Assume that $2 \parallel |N|$. Then N contains a Baer collineation of Π, since N fixes $\mathcal{O}$ pointwise and $\left|P^N\right| = 2$. A contradiction. Thus $2^2 \mid |N|$. Let L be a Sylow 2-subgroup of G. Then $2^5 \mid |L|$, since $2^3 \mid |A_7|$ and $2^2 \mid |N|$. It is easily seen that there exists a non trivial subgroup L_0 of L fixing at least 2 points on $\Pi - l$, since $n^2 \not\equiv 0, 1 \bmod 2^5$ as $n \in \{28, 29\}$. Then $L_0 \cap N = \langle 1 \rangle$, since N fixes at most one point $\Pi - l$ by lemma 14. Then L_0 and hence G contains an involution ξ acting faithfully on $\mathcal{O}$. In particular ζ is Baer collineation of Π, since ζ fixes at least 3 points on $\mathcal{O}$ by [51]. So, n must be a square. A contradiction. Hence $N \leq K$ and the assertion follows by lemma 14. $\square$

At this point we study the cases when either $N = N(l,l)$ or $N = N(Q,l)$ for some point $Q \in \Pi - l$, for $d_0(\bar{G}) = v$ and $d_0(\bar{G}) < v$, separately.

5.1.1 The case $d_0(\bar{G}) = v$.

In this subsection, under the assumption $d_0(\bar{G}) = v$, we prove that for $N = N(l,l)$ or $N = N(Q,l)$, where $Q \in \Pi - l$, the group G is a perfect central extension of $\bar{G}$ and that each involution of G lies in N. This yields that the Sylow 2-subgroups of G are dihedral, and then we use the Gorestein-Walter theorem [24] to complete this case.

Proposition 19 *If $N = N(Q,l)$, $Q \in \Pi - l$, then $N = K$. Furthermore, the followings occur:*

(A) $n = 2v - 1$ and G acts on $l - \mathcal{O}$ as $\bar{G}$ in its 2-transitive permutation representation of degree v;

(B) G is a covering group for $\bar{G}$;

(C) Each involution of G lies in N.

Proof. We proceed in a series of steps.

(A) G **acts on** $l-\mathcal{O}$ **as** $\bar{G}$ **in its** 2**-transitive permutation representation of degree** v**. In particular** $N = K$ **and** $n = 2v - 1$**.**

Assume that G fixes a point A on $l - \mathcal{O}$. Denote by Σ the set of N-orbits of points of $AQ - \{A, Q\}$. Then $|\Sigma| = (n-1)/|N|$, since N is semiregular on $AQ - \{A, Q\}$. In particular $|\Sigma| \geq d_0(\bar{G})$ by lemma 16, since G acts on Σ. This yields $d_0(\bar{G})\,|N| \leq n - 1$, since $|\Sigma| = (n-1)/|N|$. A contradiction, since $d_0(\bar{G}) = v$, $n \leq 2v$ and $|N| \geq 2$. Thus G fixes no points on $l - \mathcal{O}$. Hence $N = K$, where K is the kernel of the action of G on $l - \mathcal{O}$, by lemma 18. Moreover, G acts on $l - \mathcal{O}$ as $\bar{G}$ and $\bar{G}$ fixes no points on $l - \mathcal{O}$. This yields $n + 1 - v = \sum_{j \geq 0} \theta_j d_j(\bar{G})$, where the θ_j, $j \geq 0$, and $\sum_{j \geq 0} \theta_j > 0$, since each $\bar{G}$-orbit on $l-\mathcal{O}$ is a multiple of some $d_h(\bar{G})$, $h \geq 0$, and since $|l - \mathcal{O}| = n+1-v$. Actually, either $\lambda_0 = 1$ and $\lambda_j = 0$ for $j > 0$, or there exists $\bar{j} > 0$, such that $\lambda_{\bar{j}} = 1$, $d_{\bar{j}}(\bar{G}) = v+1$ and $\lambda_j = 0$ for each $j \geq 0$ such that $j \neq \bar{j}$, since $d_0(\bar{G}) = v$. If the latter occurs, then $n = 2v$. Furthermore, $\bar{G}$ is one of the exceptions listed in lemma 1. Nevertheless no one of these exceptions really occurs, since v must be even by [33], theorem 13.18. Hence $\lambda_0 = 1$ and $\lambda_j = 0$ for $j > 0$ for any admissible case. Then $n = 2v-1$ hence G acts on $l - \mathcal{O}$ as $\bar{G}$ in its 2-transitive permutation representation of degree v.

(B) G **is a covering group for** $\bar{G}$**.**

Assume that $N \not\leq Z(G)$. Then there exists a Sylow t-subgroup S of N such that $|S| \geq 1+v$ by lemma 15, since $d_0(\bar{G}) = v$. Furthermore, either $2\,|S| \leq n-1$ or $|S| = n - 1$ since $S \leq N$ and $|N| \mid n - 1$. The former is ruled out, since $2(v+1) \leq n-1$ as $|S| \geq 1 + v$ and $n \leq 2v$. Hence $S = N$ and $|S| = n - 1$. Thus $n - 1 = t^k$, $k \geq 1$, since S is a t-group. This yields $t = 2$ and hence $v = 2^{k-1} + 1$, since $n = 2v - 1$. Assume that G contains a Baer collineation of Π. Thus n must be a square. Then $n = 9$ and $k = 3$ by [55], result A5.1, since $n - 1 = t^k$. A contradiction by [30], theorem A. Hence G contains no Baer collineations of Π. Now, let S be any Sylow 2-subgroup of G. Then S fixes a point C on $\mathcal{O}$, since v is odd and $v > 2$. Thus $S = S_B N$ and $S_B \cap N = \langle 1 \rangle$ for some point $B \in QC - \{Q, C\}$, since N is regular on $QC - \{Q, C\}$ being $N = N(Q, l)$ and $|N| = n-1$. In particular each involution in S_B is a homology of Π with center lying on l and axis distinct from l, since G fixes l, G contains no Baer collineations of Π, n is odd and $S_B \cap N = \langle 1 \rangle$. Moreover, any involution in S_B commutes with some involution in N being $N \leq S$. Then N contains exactly one involution by [39], lemma 2.1 (ii), since $N = N(Q, l)$. Actually, the previous argument yields there exists at most one involutory homology in G with given center and axis. Then $\bar{G} \cong PSL(2,q)$, q odd, by [14], theorem 1, since $\bar{G}$ is non-abelian simple. Then $v = q+1$, q odd, since $d_0(\bar{G}) = v$. A contradiction, since $v = 2^{k-1}+1$.

(C) **Each involution of G lies in N.**

Suppose there exists an involution $\sigma \in G - N$. Assume that σ is a Baer collineation of Π. Then $\sqrt{n}+1 = 2k_\sigma$, where k_σ denotes the number of points of Π fixed by σ on $\mathcal{O}$ and on $l - \mathcal{O}$, since G acts on $\mathcal{O}$ and on $l - \mathcal{O}$ as $\bar{G}$ in the same way by **(A)**. Hence $k_\sigma = (\sqrt{2v-1}+1)/2$. Note that v is known and k_σ is easy to be recovered from the structure of $\bar{G}$ and the action of $\bar{G}$ on $\mathcal{O}$ for each 2-transitive non abelian simple group $\bar{G}$ listed in theorem 3. Hence, we may filter the list given in theorem 3 with respect to $k_\sigma = (\sqrt{2v-1}+1)/2$. So, it remains to investigate the following admissible cases:

(i) $\bar{G} \cong A_v$, $v \geq 5$;

(ii) $\bar{G} \cong PSL(d,q)$, $d \geq 2$, $q = p^r$, $(d,q) \neq (2,2),(2,3)$.

Assume that $\bar{G} \cong A_v$, $v \geq 5$. Note that $v \notin \{6,7\}$, since $n = 2v-1$ must be a square. Thus $N \cong Z_2$ by **(B)** and by [43], theorem 5.1.4, since $N \neq \langle 1 \rangle$. Let $Y \in \mathcal{O}$ and denote by Γ the set of N-orbits on $QY - \{Q,Y\}$. Then $|\Gamma| = (n-1)/|N|$, since N is semiregular on $QY - \{Q,Y\}$. Furthermore G_Y acts on Γ, since $N \lhd G$. In particular $\bar{G}_Y \cong A_{v-1}$. Assume that G_Y fixes an element on Γ. Then $G_Y = G_{Y,O}N$ and $G_{Y,O} \cap N = \langle 1 \rangle$ for some point $O \in QY - \{Q,Y\}$, since N is semiregular on $QY - \{Q,Y\}$. In particular $G_{Y,O}$ acts on $l - \{Y\}$ as $\bar{G}_Y$. Now, pick a 3-cycle ζ in $G_{Y,O}$. Clearly ζ fixes $v-3$ points on $\mathcal{O}$, since $G_{Y,O}$ acts on $l - \{Y\}$ as $\bar{G}_Y$. Then ζ fixes $v-3$ points on $l - \mathcal{O}$, since the action of G on $\mathcal{O}$ and on $l - \mathcal{O}$ is the same. So, ζ fixes exactly $n-6$ points on l, since $n = 2v-1$. Furthermore, ζ fixes the points Q and O, with $O, Q \in \Pi - l$, since ζ lies in $G_{Y,O}$. Therefore, ζ fixes a subplane of Π of order $n-7$. Then $(n-7)^2 \leq n$ by [33], theorem 3.7. This yields either $n = 9$ or $n = 10$, since $n \geq 9$ by our assumption. Nevertheless these cases cannot occur by [30], theorem A, and [33], theorem 13.18, respectively. As a consequence, G_Y moves each element on Γ. Therefore Γ is union of non trivial $\bar{G}_Y$-orbits. Since each $\bar{G}_Y$-orbit on Γ is a multiple of some $d_j(A_{v-1})$, $j \geq 0$, then $\sum_{j\geq 0} \lambda_j d_j(A_{v-1}) = |\Gamma|$. That is $2\sum_{j\geq 0} \lambda_j d_j(A_{v-1}) = n-1$, since $|\Gamma| = (n-1)/|N|$ and $N \cong Z_2$. Then $\lambda_0 = 1$ and $\lambda_j = 0$ for $j > 1$, since $d_o(A_{v-1}) = v-1$ and $n = 2v-1$. Hence $\bar{G}_Y \cong A_{v-1}$ acts in its 2-transitive permutation representation of degree $v-1$ on Γ. Then there still exists a 3-cycle ζ in G fixing $n-6$ points on l and at least 2 points on $QY - \{Y\}$. Hence ζ fixes a subplane of Π of order $n-7$. Again a contradiction.

Assume that $\bar{G} \cong PSL(d,q)$, $d \geq 2$, $q = p^r$, $(d,q) \neq (2,2)$, $(2,3)$. Note that $(d,q) \neq (2,5),(2,9)$, since the cases $PSL(2,5) \cong A_5$ and $v = 5$, $PSL(2,9) \cong A_6$ and $v = 6$ have been ruled out above. Also the case $(d,q) = (2,7)$ and $v = 7$, or $(d,q) = (2,11)$ and $v = 11$ are ruled out since in this cases n is a non square.

Thus $d_0(\bar{G}) = (q^d-1)/(q-1)$ by [9]. Hence $n = 2\left[(q^d-1)/(q-1)\right]-1$. Let E be an elementary abelian subgroup of G of order q^{d-1} which induces on $\mathcal{O}$ of a group projective transvections with the same fixed hyperplane. Let B be a point on $\mathcal{O}$ fixed by E. If there exists a non trivial element δ in E fixing a point on $QB-\{Q,B\}$, then δ is planar on Π. In particular δ fixes exactly $2(q^{d-1}-1)/(q-1)$ points on l, since the action of G on $\mathcal{O}$ and $l-\mathcal{O}$ is the same, and since δ fixes exactly $(q^{d-1}-1)/(q-1)$ points on $\mathcal{O}$. So $\left(2\left[(q^{d-1}-1)/(q-1)\right]-1\right)^2 \leq 2\left[(q^d-1)/(q-1)\right]-1$ by [33], theorem 3.7. Thus $d=2$ and hence $G \cong SL(2,q)$ by [41], theorem 7.1.1(i), since $q \notin \{5,7,9,11\}$. A contradiction, since the unique involution of $G \cong SL(2,q)$ lies in N. Hence E is semiregular on $QB-\{Q,B\}$. Thus $q^{d-1} \mid n-1$. Then either $d=2$ or $(d,q)=(3,2)$, since $n = 2\left[(q^d-1)/(q-1)\right]-1$. Again a contradiction.

The above argument lead us to assert that each involution in $G-N$ must be a homology of Π as n is odd. Assume that N has even order. Any involution in G commutes with N as G is a covering group for $\bar{G}$ by **(B)**. Thus N contains exactly one involution by [39], lemma 2.1 (ii), since $N = N(Q,l)$, N has even order and since $G-N$ contains involutions. Actually, the previous argument yields that exists at most one involutory homology in G with given center and axis. Then $G \cong SL(2,q)$, q odd by theorem 1 of [14] and by [41], theorem 7.1.1(i), since G is a covering group for $\bar{G}$ and since $q \notin \{5,7,9\}$. A contradiction, since there are no involutions in $G-N$. Hence N has odd order. Let α be any involution of $G-N$. Then α fixes exactly 2 points on l, since α is a homology and α fixes l. In particular α fixes exactly one point on $\mathcal{O}$ and one on $l-\mathcal{O}$ since the action of G on $\mathcal{O}$ and $l-\mathcal{O}$ is the same. Thus G_D/N has even order for any $D \in \mathcal{O}$. Moreover, $G_{D_1,D_2}/N$ has odd order for any two distinct points D_1 and D_2 of $\mathcal{O}$, since N has odd order and each involution in $G-N$ is a homology of Π. Then $\bar{G} \cong PSL(2,2^s)$, or $\bar{G} \cong Sz(2^s)$, or $\bar{G} \cong PSU(3,2^s)$ by [3]. Actually, $G \cong SL(2,5)$, or $\bar{G} \cong Sz(8)$, or $\bar{G} \cong PSU(3,2^s)$ and $N \cong Z_3$ by [43], theorem 5.1.4, as $N \neq \langle 1 \rangle$. Nevertheless the case $G \cong SL(2,5)$ cannot occur, since $N \cong Z_2$ and any Sylow 2-subgroup of G is isomorphic to Q_8. The case $\bar{G} \cong Sz(8)$ cannot occur by [33], theorem 3.6, since $n = 129$. Finally, the case $\bar{G} \cong PSU(3,2^s)$ cannot occur, since $n = 2^{3s+1}+1$ while $|N| = 3$ must be a divisor of $n-1$. $\square$

Proposition 20 *If $N = N(l,l)$, then $N = K$. Furthermore, the following occur:*

(A) G fixes exactly one point X on $l-\mathcal{O}$.

(B) $n = 2v$ and G acts on $l-(\mathcal{O}\cup\{X\})$ as $\bar{G}$ in its 2-transitive permutation representation of degree v.

(C) $N = N(X,l)$.

(D) Each involution of G lies in N.

(E) G is a covering group for $\bar{G}$.

Proof. Assume that $N = N(l,l)$. We proceed in a series of steps.

(A) G **fixes at least a point** X **on** $l - \mathcal{O}$**.**

Assume that $\bar{G}$ fixes no points on $l - \mathcal{O}$. Then $l - \mathcal{O}$ is union of non trivial $\bar{G}$-orbits. This yields $n + 1 - v = \sum_{j\geq 0} \theta_j d_j(\bar{G})$, where $\theta_j \geq 0$, $j \geq 0$, and $\sum_{j\geq 0} \theta_j > 0$, since each $\bar{G}$-orbit on $l - \mathcal{O}$ is a multiple of some $d_h(\bar{G})$, $h \geq 0$, and since $|l - \mathcal{O}| = n + 1 - v$. At this point we may use the same argument of part **(A)** of proposition 19 to show that $n = 2v - 1$ and G acts on $l - \mathcal{O}$ as $\bar{G}$ in its 2-transitive permutation representation of degree v. Set $|N(l,l)| = p^h$ with $h > 0$, $|N(C,l)| = p^i$ with $i \geq 0$, for any $C \in \mathcal{O}$, and set $|N(D,l)| = p^j$, $j \geq 0$, for any $D \in l - \mathcal{O}$. Clearly $i + j > 0$, since $N \neq \langle 1 \rangle$. Furthermore $h \geq i, j$. Then

$$(p^i - 1)\frac{(n+1)}{2} + (p^j - 1)\frac{(n+1)}{2} + 1 = p^h, \tag{1}$$

since $\bar{G}$ has the same 2-transitive permutation representation on $\mathcal{O}$ and $l - \mathcal{O}$, and since $v = (n+1)/2$. By managing (1), we have that

$$(p^i + p^j)(n+1) = 2(p^h + n). \tag{2}$$

As $p \mid n$ and $h > 0$, then $p \mid (p^i + p^j)(n+1)$ and hence $i, j > 0$. Furthermore $p^i \mid n$ and $p^j \mid n$, since $N(C,l)$ is semiregular on $[D] - \{l\}$ and $N(D,l)$ is semiregular on $[C] - \{l\}$. Thus $p^f \mid n$, where $f = \max\{i,j\}$. Then $p^f \mid (p^i + p^j)$, since $p^f \mid (p^h + n)$. Hence $i = j$, with $i, j > 0$. Then Π is a translation plane and N is regular on $\Pi - l$ by [33], theorem 4.26. Therefore $G = NG_O$ for some point $O \in \Pi - l$. Then $G = G_O$, since $N = \Phi(G)$ by lemma 13. So, N fixes O. A contradiction, since N is semiregular on $\Pi - l$. Thus N hence G fixes at least a point X on $l - \mathcal{O}$.

(B) G **acts on** $l - (\mathcal{O} \cup \{X\})$ **as** $\bar{G}$ **in its 2-transitive permutation representation of degree** v**. In particular** $N = K$ **and** $n = 2v$**,** v **even.**

Suppose that G fixes $l - \mathcal{O}$ pointwise. Assume there exists a point P of $l - \mathcal{O}$ such that $N(P,l) = \langle 1 \rangle$. Let Λ be the set of the N-orbits on $[P] - \{l\}$. By lemma 16, we have that $|\Lambda| \geq d_0(\bar{G})$. By an argument similar to that used in **(A)** of proposition 19 implies that $n = 2v$ and $N \cong Z_2$, since $|\Lambda| = n/|N|$, $d_0(\bar{G}) = v$ and $v < n \leq 2v$. Thus $N \leq Z(G)$. Let $x \in G$, $x \neq 1$, such that $o(x) \mid v - 1$. Note that $v - 1$ is odd by [33], theorem 13.18, since $n = 2v$. Then x fixes $v + 1$ points on l, since G fixes $l - \mathcal{O}$ pointwise and $|l - \mathcal{O}| = v + 1$. Furthermore x fixes a point R on $\Pi - l$, since $n = 2v$. Let $\sigma \in N$. Then $R\sigma \in \mathrm{Fix}(x)$, since $N \leq Z(G)$. Note that $R\sigma \neq R$, since $\sigma \in N$, $R \in \Pi - l$ and $N = N(l,l)$. Thus x is planar on Π and $o(\mathrm{Fix}(x)) \geq v$. Then $v^2 \leq n \leq 2v$ by [33], theorem 3.7. A contradiction, since $v > 2$. Hence, we may assume that $N(B,l) \neq \langle 1 \rangle$ for each $B \in l - \mathcal{O}$. Let Y be any point of $l - \mathcal{O}$. Let Γ be the set of the N-orbits on $[Y] - \{l\}$. Then $|\Gamma| = n/[N : N(Y,l)]$, since $N(Y,l) < N$ as $N(B,l) \neq \langle 1 \rangle$ for each $B \in l - \mathcal{O}$. A similar argument to that used above yields $n = 2v$ and $[N : N(Y,l)] = 2$ for any $Y \in l - \mathcal{O}$, since $d_0(\bar{G}) = v$ and $v < n \leq 2v$. Thus $N \cong E_4$ and hence $|l - \mathcal{O}| = 3$. That is $n + 1 - v = 3$. A contradiction, since $n = 2v$ and $v \geq 5$.

(C) $N = N(X, l)$**.**

Assume that $N(X,l) < N$. We may repeat the previous argument on the set of the N-orbits on $[X] - \{l\}$ to show that $[N : N(X,l)] = 2$. Hence N is an elementary abelian 2-group, since $N = N(l,l)$ and $N(X,l) < N$. Set $|N(X,l)| = 2^f$, $f \geq 0$, and set $|N(C,l)| = 2^i$, $i \geq 0$, for any $C \in \mathcal{O}$. Set also $|N(D,l)| = 2^j$, $j \geq 0$, for any $D \in l - (\mathcal{O} \cup \{X\})$ by **(B)**. Then

$$(2^i - 1)v + (2^j - 1)v + 2^f = 2^{f+1}.$$

Furthermore $(i,j) = (1,1), (1,0), (0,1)$, since $[N : N(X,l)] = 2$. If $(i,j) = (1,1)$, then $f = 1$ and $v = 1$ by [33], theorem 4.26. A contradiction. Hence, either $(i,j) = (1,0)$ or $(0,1)$. It is easily seen that $v = 2^f$ and $n = 2^{f+1}$ in any of these two cases. In particular $|N(X,l)| = n/2$. We may assume that $(i,j) = (0,1)$, since the role of i and j can be exchanged in the following argument. Note that $N(X,l) \lhd G$, since G fixes X. Assume that $N(X,l) \nleq Z(G)$. Then $|N(X,l)| \geq 1 + v$, since G acts on $N(X,l)$ as $\bar{G}$ being N abelian and $d_0(\bar{G}) = v$. A contradiction, since $|N(X,l)| = 2^f = v$. Hence $N(X,l) \leq Z(G)$. Let ζ be an element of order a prime dividing $v - 1$. Then ζ must a 2-element by [23], corollary 5.3.3, since $N/N(X,l) \cong Z_2$ and $N(X,l) \leq Z(G)$. A contradiction, since $v - 1$ is odd. Hence $N = N(X,l)$**.**

(D) **Each involution of G lies in N.**

Suppose there exists an involution $\sigma \in G - N$. Assume that σ is a (C_σ, l_σ)-elation of Π. If $C_\sigma = X$, then $N < G(X,X) \lhd G$. A contradiction, since $\bar{G}$ is non-abelian simple. Hence $C_\sigma \neq X$. Furthermore $a_\sigma \neq l$, since $\sigma \notin N$. Denote by R the normal closure of $\langle \sigma \rangle$ in G. Then $G = RN$, since $\bar{G}$ is non-abelian simple. Actually, $G = R$ by the minimality of G. Hence G is generated by involutory elations. Moreover $N = F(G)$, where $F(G)$ denotes the Fitting subgroup of G, since N is nilpotent and $\bar{G}$ is non-abelian simple. Since $4 \mid |\bar{G}|$ by [22], then $\bar{G}$ is isomorphic to $PSL(3,q)$, or to $PSU(3,q)$, or to $SL(2,q)$, or to $Sz(q)$, or to A_6, where $q = 2^r$, by [27]. A contradiction in any case, but A_6, since v must be even by [33], theorem 13.18. Nevertheless the case $\bar{G} \cong A_6$ is ruled out by [36], since $n = 12$. Hence, we may assume that σ is a Baer collineation of Π. Then $\sqrt{n} + 1 = 2k_\sigma + 1$, where k_σ is the number of points of Π fixed by σ both on $\mathcal{O}$ and on $l - (\mathcal{O} \cup \{X\})$. Hence $k_\sigma = \sqrt{v/2}$. Now, arguing similarly to part **(C)** of proposition 19, we may reduce to investigate the following admissible cases:

(i) $\bar{G} \cong A_v$, $v \geq 5$;

(ii) $\bar{G} \cong PSL(d,q)$, $d \geq 2$, $q = p^r$, $(d,q) \neq (2,2)$, $(2,3)$.

Assume that $G \cong A_v$, $v \geq 5$. Let $Y \in \mathcal{O}$ and denote by Ψ the set of N-orbits on $[Y] - \{l\}$. Then $|\Psi| = n/|N|$ as N is semiregular on $[Y] - \{l\}$, being $N = N(X,l)$ with $X \neq Y$. Furthermore, G_Y acts on Ψ, since $N \lhd G$. In particular $\bar{G}_Y \cong A_{v-1}$. Assume that G_Y fixes an element on Ψ. Then

$G_Y = G_{Y,r}N$ and $G_{Y,r} \cap N = \langle 1 \rangle$ for some point $r \in [Y] - \{l\}$. So, σ may be picked in $G_{Y,r}$ as product of two transpositions on $\mathcal{O}$. Then σ fixes exactly $n-7$ points on l, other than X, since $\bar{G}$ acts on $\mathcal{O}$ and on $l - (\mathcal{O} \cup \{X\})$ in the same way. Then $(n-7)^2 = n$ by [33], theorem 3.7. A contradiction. As a consequence, G_Y moves each element on Ψ and Ψ is union of non trivial $\bar{G}_Y$-orbits. Since each $\bar{G}_Y$-orbit on Ψ is a multiple of some $d_j(A_{v-1})$, $j \geq 0$, then $\sum_{j\geq 0} \lambda_j d_j(A_{v-1}) = |\Psi|$. That is $|N| \sum_{j\geq 0} \lambda_j d_j(A_{v-1}) = 2v$ as $|\Psi| = n/|N|$ and $n = 2v$. A contradiction for $v \neq 6, 8$, by lemma 11 applied to $\bar{G}_Y \cong A_{v-1}$ as $d_0(A_{v-1}) = v-1$ and $|N| \geq 2$. Actually, the case $v = 6$ and $\bar{G}_Y \cong A_5$ cannot occur by [36], since $n = 2v$. Hence $\bar{G} \cong A_8$ and $n = 16$. In this case σ fixes 4 points on $\mathcal{O}$ and 4 points on $l - \mathcal{O}$. So σ fixes at least 8 points on l and $\sigma \notin N$. A contradiction by [33], theorem 3.7.

Assume that $G \cong PSL(d,q)$, $d \geq 2$, $q = p^r$, $(d,q) \neq (2,2), (2,3)$. Note that $(d,q) \neq (2,5), (2,7), (2,11)$, since v must be even by [33], theorem 13.18. Also, the case $(d,q) = (2,9)$ is ruled out, since $PSL(2,9) \cong A_6$. Thus $d_0(\bar{G}) = (q^d-1)/(q-1)$ by [9] and hence $n = 2(q^d-1)/(q-1)$. Note that q is odd and d is even, since v must be even. Moreover $n \not\equiv 0, 1 \bmod p$. So, a similar argument to that used in the part **(C)** of proposition 19 shows that G always contains planar p elements fixing $2(q^{d-1}-1)/(q-1)$ points on $l - \{X\}$, as $n \not\equiv 0, 1 \bmod p$. This yields $d = 2$ by [33], theorem 3.7. Hence $G \cong SL(2,q)$ by [41], theorem 7.1.1.(i), since $q \notin \{5, 7, 9, 11\}$. A contradiction, since the unique involution of $G \cong SL(2,q)$ lies in N.

(E) G **is a covering group for** $\bar{G}$**.**

Assume that $N \nleq Z(G)$. Then there exists a Sylow t-subgroup S of N, such that $|S| > 1 + v$ by lemma 15. On the other hand, $|S| \mid 2v$ since $S \leq N$, $|N| \mid n$ by **(C)**, and $n = 2v$ by **(B)**. Thus $|S| = n$, since $|S| > 1 + v$. Hence $S = N$. Then N is a 2-group and n is a power of 2, since $n = 2v$ and $|N| \mid n^2$. Let $A \in \mathcal{O}$. Then $N(A,l) = \langle 1 \rangle$ by lemma 14, since $|N| = n$. Thus N is regular on $[A] - \{l\}$ and hence $G_A = G_{A,s}N$ and $G_{A,s} \cap N = \langle 1 \rangle$ for some point $s \in [A] - \{l\}$. Furthermore $G_{A,s} \cong \bar{G}_A$. Then $G_{A,s}$ must have odd order, since each involution in G actually lies in N by **(D)** and since $N = N(X,l)$ with $X \neq A$. This yields that each involution in $\bar{G}$ fixes no points on $\mathcal{O}$. Then $\bar{G} \cong PSL(2,q)$ with $q \equiv 3 \bmod 4$, by [2]. By lemma 15 (2), we have that $PSL(2,q) \leq PSL(V)$, where $V = S/\Phi(S)$ and this implies $|S| > 1 + v$ as $d_0(\bar{G}) = v$. Actually, in this case $|S| \geq 2^{(q-1)/2}$ by [43], theorem 5.3.9, since $q \equiv 3 \bmod 4$. Then $2^{(q-1)/2} \leq 2(q+1)$, since $|S| \mid 2(q+1)$ as $S \leq N$, $|N| \mid n$ and $n = 2(q+1)$ with $q \equiv 3 \bmod 4$. A contradiction, since $q \neq 7, 11$ as $q \equiv 3 \bmod 4$ and $d_0(\bar{G}) = v$. $\square$

Theorem 21 *Let* Π *be a projective plane of order* n *and let* $\mathcal{O}$ *be a* 2*-transitive* G*-orbit of length* v *on a line with* $n > v \geq n/2$. *If* G *is almost simple and* $d_0(\bar{G}) = v$, *then one of the following occurs:*

(1) Π *is the Hall plane of order* 9 *or its dual,* $G \cong SL(2,5)$ *and* $|\mathcal{O}| = 5$*;*

(2) $n = 2q + 1$, $G \cong SL(2, q)$ *with* $q \equiv 3 \bmod 4$, $q \neq 7$, *and* $|\mathcal{O}| = q + 1$;

(3) $n = 2(q + 1)$, $G \cong SL(2, q)$ *with* $q \equiv 3 \bmod 4$, *and* $|\mathcal{O}| = q + 1$.

Proof. Note that $N \cong Z_2$, unless $\bar{G} \cong PSL(3, 4)$ or $\bar{G} \cong Sz(8)$, by proposition 19 (B) and (C), by proposition 20 (D) and (E), and by [43], theorem 5.1.4, since $d_0(\bar{G}) = v$. Assume that $\bar{G} \cong PSL(3, 4)$ or $\bar{G} \cong Sz(8)$ and assume that $N \ncong Z_2$. Note that the case $n = 2v$ cannot occur by [33], theorem 13.18, since v is odd in both cases. Then $n = 2v - 1$ and $N = N(Q, l)$ for some point $Q \in \Pi - l$ fixed by G by proposition 20. Nevertheless the case $\bar{G} \cong Sz(8)$ cannot occur by [33], theorem 3.6, since $n = 129$ in this case. Hence $\bar{G} \cong PSL(3, 4)$ and $n = 41$. Let U be a Sylow 2-subgroup of G_J, where J is any point of $\mathcal{O}$. Then U must be semiregular on $QJ - \{Q, J\}$, since each involution in G lies in N by proposition 19 (3). Hence $|U| \mid n - 1$. A contradiction, since $2^6 \mid |U|$ and $n = 41$. Hence $N \cong Z_2$ in any admissible case. Thus each Sylow 2-subgroup S of G is isomorphic either to Z_{2^m} or to Q_{2^m} for some positive integer m in any case by proposition 19 (C) and proposition 20 (D). In the first case we have $S/(S \cap N) \cong Z_{2^{m-k}}$, where $0 < k \leq m$. Nevertheless this case is ruled out by [22], theorem 4, applied to $\bar{G}$, since this one is non abelian simple. Hence $S \cong Q_{2^m}$ and hence $S/(S \cap N) \cong D_{2^{m-1}}$, since $S \cap N \cong Z_2$. By [24], either $\bar{G} \cong PSL(2, q)$ with q odd, or A_7. If $\bar{G} \cong A_7$ then $n = 28$ or 29, and this case cannot occur by the same argument of proposition 18. Hence, we may assume that $\bar{G} \cong PSL(2, q)$ with q odd. Assume that $q = 5$. Then $n = 9$ or $n = 10$, since $d_0(\bar{G}) = 5$. Actually the latter is ruled out by [33], theorem 13.18. Hence $n = 9$ and either $\Pi \cong PG(2, 9)$, or Π is the Hall plane of order 9 or Π is the dual of the Hall plane of order 9 by [59]. Thus the assertion (1).

Assume that $q = 7$. Then either $n = 13$ or $n = 14$ since $d_0(\bar{G}) = 7$. The latter is ruled out by [33], theorem 13.18. Hence $n = 13$. Then $\Pi \cong PG(2, 13)$ by [48], since $d_0(\bar{G}) = 7$. Hence $\bar{G} \leq PGL(2, 13)$, since G fixes l. A contradiction. Now, assume that $q = 9$. Then either $n = 11$ or $n = 12$, since $d_0(\bar{G}) = 9$. The latter is ruled out by [36]. Hence $n = 11$ and $\Pi \cong PG(2, 11)$ by [47]. Hence $\bar{G} \leq PGL(2, 13)$, since G fixes l. A contradiction. Hence $q \notin \{7, 9\}$. Thus $G \cong SL(2, q)$ by [41], theorem 7.1.1.(i), for $q \neq 5$, as $q \notin \{7, 9\}$.

Assume that $n = 2v - 1$. Let R be a Sylow 2-subgroup of G_B, where B is any point of $\mathcal{O}$. Then R must be semiregular on $QB - \{Q, B\}$, since each involution in G lies in N. Hence $|R| \mid n - 1$. Then $|R| = 2$ and hence $R = N$, since $n - 1 \equiv 2 \bmod 4$, being $n = 2v - 1$ and $v = q + 1$. Thus $q \equiv 3 \bmod 4$ and we have the assertion (2).

Finally, assume that $n = 2v$. Then v is even by [33], theorem 13.18. Thus q must be odd. Assume that $q \equiv 1 \bmod 4$. Let R be defined as above. In this case $8 \mid 2(q + 1)$, since $R \cong Q_{2^m}$, $m \geq 3$, must be semiregular on $[B] - \{l\}$, $n = 2(q + 1)$ and the unique involution in G is an (X, l)-elation, where X is the unique point on $l - \mathcal{O}$ fixed by G. A contradiction. Hence $q \equiv 3 \bmod 4$. Thus the assertion (3). $\square$

It should be stressed that, if there exist planes of type (2) with n a prime power, then n is actually a prime by lemma 10 (1). Furthermore, if there exist planes of type (3) with n a prime power, then n is a power of 2 and q is a Mersenne prime by lemma 10 (2). Nevertheless, as we will see in section 7, in these cases Π cannot be the projective extension of a translation plane.

5.1.2 The case $d_0(\bar{G}) < v$.

In the following we assume that $\bar{G} \ncong PSU(3,5)$. Then either $N = N(Q,l)$, $Q \in \Pi - l$ or $N = N(l,l)$ by lemma 18. We treat these two cases separately. In particular, for each of them, we show that G is a perfect central extension of $\bar{G}$. Now, since the groups satisfying $d_0(\bar{G}) < v$ are listed in table I of lemma 12, we complete this subsection with a case by case investigation.

Lemma 22 *If $N = N(Q,l)$, $Q \in \Pi - l$, then $n = 23$ and $G \cong SL(2,11)$.*

Proof. Assume that $N = N(Q,l)$, where Q is a point of $\Pi - l$ fixed by G. Assume also that G fixes a point P of $l - \mathcal{O}$ and let Ω the set of N-orbits on $QP - \{Q,P\}$. Then $|\Omega| = (n-1)/|N|$, since N is semiregular on $QP - \{Q,P\}$. Hence

$$n = 1 + |N| \sum_{j \geq 0} \lambda_j d_0(\bar{G}), \qquad (3)$$

where $\lambda_j \geq 0$, $j \geq 0$, and $\sum_{j\geq 0} \lambda_j > 0$, by lemma 16. Assume that $N \nleq Z(G)$. Then $|N| \geq 1 + d_0(\bar{G})$ by lemma 15. By composing the previous inequality with (3) and by bearing in mind that $n \leq 2v$, we obtain $1 + d_0(\bar{G}) + d_0(\bar{G})^2 \leq 2v$. Now, filtering the groups of the Table I with respect to the previous inequality, it is easily seen that no cases arise. Hence G is a covering group for $\bar{G}$ by lemma 15. Then the groups $\bar{G} \cong M_{11}$ or $\bar{G} \cong Sp(2h,2)$, $h > 3$ are ruled out by [43], theorem 5.1.4, since $N \neq \langle 1 \rangle$ by our assumption. For the remaining groups, we have $\lambda_0 \leq 2$ and $\lambda_j = 0$ for $j > 0$ by lemma 12, since $N \neq \langle 1 \rangle$. In particular, $|N| \leq 3$ again by lemma 12 (3). Thus either $N \cong Z_2$ or $N \cong Z_3$ for the groups of the Table I by [43], theorem 5.1.4. Hence $n = 1 + |N| \lambda_0 d_0(\bar{G})$, $\lambda_0 \in \{1,2\}$ and $|N| \in \{2,3\}$. By lemma 12 and since $v < n \leq 2v$, it is easily seen that the admissible cases of the Table I are $G/Z_2 \cong PSL(2,5)$ and $n = 11$, $G/Z_2 \cong PSL(2,7)$ and $n = 15$, or $G/Z_2 \cong SL(2,9)$ and $n = 13$ or $G/Z_3 \cong PSL(2,9)$ and $n = 19$, or $G/Z_2 \cong PSL(2,11)$ and $n = 23$, or $G/Z_2 \cong PSL(4,2)$ and $n = 17$, or $G/Z_3 \cong A_7$ and $n = 21$ or $G/Z_2 \cong A_7$ and $n = 29$, or $G/Z_2 \cong HS$ and $n = 201$, or $G/Z_2 \cong Sp(6,2)$ and $n = 57$. Actually the cases $G/Z_3 \cong A_7$ and $n = 21$, $G/Z_2 \cong Sp(6,2)$ and $n = 57$, or $G \cong HS/Z_2$ and $n = 201$ cannot occur by [33], theorem 3.6. The case $G/Z_2 \cong PSL(2,7)$ and $n = 15$ cannot occur by [31]. If $G/Z_2 \cong PSL(2,5)$ and $n = 11$, then $\Pi \cong PG(2,11)$ and hence $\bar{G} \leq PSL(2,11)$ by [47]. Nevertheless, this case cannot occur, since $\bar{G} \cong PSL(2,5)$ contains involutions fixing a point on $\mathcal{O}$ and hence on l, while $PSL(2,11)$ does not. Assume that $G/Z_2 \cong SL(2,9)$ and $n = 13$. Then $\bar{G} \leq PGL(2,13)$ by [48]. A contradiction. Hence $G/Z_3 \cong PSL(2,9)$ and $n = 19$. In this case there exists an involution δ in G fixing at least 4 points on l, since $n+1 = 20$ and $|\mathcal{O}| = 10$. Clearly $\delta \notin N$. Therefore δ is a Baer collineation of Π. A contradiction. Assume that $G/Z_2 \cong PSL(2,11)$ and $n = 23$. Then $G \cong SL(2,11)$ by [41], theorem 7.1.1.(i). Since $8 \nmid n-1$, then there exists an involution fixing a point on $PQ - \{P,Q\}$. Such a involution must lie outside N. A contradiction. Assume that $G/Z_2 \cong PSL(4,2)$ and $n = 17$, or $\bar{G} \cong A_7$ and $n = 29$. Let J be a Sylow 2-subgroup of G. Then $2^6 \mid |J|$ for $G/Z_2 \cong PSL(4,2)$ and $|J| = 2^4$ for $G/Z_2 \cong A_7$. Furthermore, J fixes a point X of $\mathcal{O}$, since $|\mathcal{O}| = 15$. Then $J_Z \neq \langle 1 \rangle$ for some $Z \in XQ - \{X,Q\}$, since $|J| \nmid n-1$. In particular $J_Z \cap N = \langle 1 \rangle$ and $[J : J_Z] \leq 2^3$. Hence J_Z acts faithfully on $\mathcal{O}$. In particular J_Z contains involutions fixing at least 3 points on $\mathcal{O}$ by [51]. Such involutions are Baer collineations of Π, since $J_Z \cap N = \langle 1 \rangle$. Hence n must be a square. A contradiction. As a consequence, G cannot fix points on $l - \mathcal{O}$. Then $l - \mathcal{O}$ consists of non trivial $\bar{G}$-orbits, since G acts on $l - \mathcal{O}$ as $\bar{G}$. Since each $\bar{G}$-orbit is a multiple of some $d_i(\bar{G})$, $i \geq 0$, we have that $n+1 = v + \sum_{j \geq 0} \mu_j d_j(\bar{G})$, where $\mu_j \geq 0$, $j \geq 0$, and $\sum_{j \geq 0} \mu_j > 0$. Hence either $n = v + d_0(\bar{G}) - 1$ or $n = 2v - 1$ or $n = 28$ and $\bar{G} \cong A_7$ or $n = 30$ and $\bar{G} \cong PSL(4,2)$ by lemma 12, since $n \leq 2v$. The latter is ruled out by [33], theorem 13.18. Assume that $n = 28$ and $\bar{G} \cong A_7$. Then N has odd order as $|N| \mid n-1$. So, there exists an involution fixing 3 points on $\mathcal{O}$ by [51] and not lying in N. Such a involution must be a Baer collineation of Π and hence n must be a square. A contradiction. Hence, assume that $n = v + d_0(\bar{G}) - 1$. Then all the groups of the Table I, but $\bar{G} \cong PSL(2,9)$ and $n = 15$, $\bar{G} \cong A_7$ and $n = 21$, $\bar{G} \cong Sp(2h,2)$ and $n = 2^{2h} - 1$, $G/Z_2 \cong HS$ and $n = 275$, are ruled out by [33], theorem 13.18. Nevertheless the groups $\bar{G} \cong PSL(2,9)$, $\bar{G} \cong A_7$ and $\bar{G} \cong Sp(2h,2)$ cannot occur by [31], by [33], theorem 3.6 and by [26], respectively. Hence $G/Z_2 \cong HS$ and $n = 275$. Let X be a point on $\mathcal{O}$. Then $\bar{G}_X \cong PSU(3,5).Z_2$ by [17], Appendix B. Now, let S be a Sylow 2-subgroup of G_X. Clearly $|S| = 2^6$, since $N \cong Z_2$. Then S fixes a point B on $\mathcal{O} - \{X\}$, since $|\mathcal{O}| = 176$. Furthermore there exists a non trivial subgroup of S_0 of S, such that $[S : S_0] \leq 2^2$, which fixes 3 points on $\mathcal{O}$ and a point on $\Pi - (l \cup \{Q\})$. Thus $S_0 \cap N = \langle 1 \rangle$. Therefore, S_0 contains a Baer collineation of Π and hence n must be a square. A contradiction.

Assume that $n = 2v - 1$. The above arguments rule out the cases $\bar{G} \cong PSL(2,5)$ and $n = 11$, $\bar{G} \cong PSL(2,7)$ and $n = 15$, $\bar{G} \cong Sp(2h,2)$ and $n = 2^h(2^h - 1) - 1$ and $\bar{G} \cong HS$ and $n = 351$. Furthermore, the same argument of theorem 21 rules out the case $\bar{G} \cong PSL(2,9)$ and $n = 19$, and the above argument used to rule out $\bar{G} \cong A_7$ and $n = 29$, maybe applied to rule out also the case $\bar{G} \cong PSL(4,2)$ and $n = 29$. Finally the case $\bar{G} \cong M_{11}$ and $n = 23$ cannot occur by [43], theorem 5.1.4, since $N \leq Z_2 \times Z_{11}$ in this case. A contradiction. Thus $G \cong SL(2,11)$ and $n = 23$. $\square$

Lemma 23 *If $N = N(l,l)$, then either $n = 16$ and $G \cong SL(2,7)$, or $n = 24$ and $G \cong SL(2,11)$.*

Proof. Assume that $\bar{G}$ does not fix any point on $l - \mathcal{O}$. Then $l - \mathcal{O}$ consists of non trivial $\bar{G}$-orbits. At this point we may use the same argument of lemma 22 to show that either $n = 2v - 1$, or $n = v + d_0(\bar{G}) - 1$, or $n = 28$ and $\bar{G} \cong A_7$. Actually, the case $n = 2v - 1$ is ruled out by the same argument of proposition 20 part **(A)**. Hence, either $n = v + d_0(\bar{G}) - 1$, or $n = 28$ and $\bar{G} \cong A_7$. Assume that $n = 28$ and $\bar{G} \cong A_7$. If $|N| \geq 4$, we may apply the same argument of lemma 18 to ruled out this case. Hence $N \cong Z_2$. Then $|J| = 2^4$, where J is any Sylow 2-subgroup of G. Furthermore, J fixes a point O of $\mathcal{O}$, since $|\mathcal{O}| = 15$. Then $J_m \neq \langle 1 \rangle$ for some $m \in [O] - \{l\}$, since $|J| \nmid n$. In particular $J_m \cap N = \langle 1 \rangle$ and $[J : J_m] \leq 2^3$. Hence J_m acts faithfully on $\mathcal{O}$. In particular J_m contains involutions fixing at least 3 points on $\mathcal{O}$ by [51]. Such involutions are Baer collineations of Π, since $J_m \cap N = \langle 1 \rangle$. Hence n must be a square. A contradiction. Therefore $n = v + d_0(\bar{G}) - 1$. Then each case of Table I, but $\bar{G} \cong HS$ and $n = 275$, is ruled out by the same argument of lemma 22. In the remaining case N has odd order as $|N| \mid n^2$ and n is odd. So, it is easily seen that there exists an involution in G which is Baer collineation of Π. A contradiction, since $n = 275$. Hence, we may assume that G fixes at least a point X on $l - \mathcal{O}$. Assume that $|N| > n$. Then $N(X, l) < N$. Let Ψ be the set of N-orbits on $[X] - \{l\}$. Note that each N-orbit on $[X] - \{l\}$ has length $[N : N(X, l)]$, since $N = N(l, l)$ and $N(X, l) < N$. Clearly, G acts on Ψ as $N \lhd G$. Then $|\Psi| \geq d_0(\bar{G})$ by lemma 16. A contradiction, since $|\Psi| \leq 3$ by lemma 14, as $|N| > n$. Thus $|N| \mid n$. Assume that $N \nleq Z(G)$. There exists a Sylow t-subgroup S of N such that $\bar{G} \leq SL(V)$, where $V = S/\Phi(S)$ by lemma 15. We have $|V| \geq b$, where $b = t^{r(\bar{G})}$ and $r(\bar{G})$ is a suitable lower bound for $\dim_{GF(t)}(V)$. Indeed, such a lower bound can be easily recovered from [43], in particular it can be recovered from the theorem 5.3.9 and the proposition 5.4.13 when $\bar{G}$ is classical, from the proposition 5.3.7 and when $\bar{G} \cong A_7$, and from the proposition 5.3.8 when $\bar{G} \cong HS$ or $\bar{G} \cong M_{11}$. This information must be combined in some cases with [12] in order to determine b as follows: pick $\bar{G} \cong M_{11}$ for example, then $r(M_{11}) = 5$ by [43], proposition 5.3.8. Hence $|V| \geq t^5$. From [12], we see that $t > 2$. Hence $b = 3^5$ and $|V| \geq 3^5$. The same argument can be repeated for each group listed in the Table I. Then $|V| \geq b$ and hence $|N| \geq b$ as $V = S/\Phi(S)$ and $S \leq N$. On the other hand, it must be $b \leq n$, since $|N| \mid n$. By a direct inspection of the Table I, we have that $\bar{G} \cong PSL(2, 7)$ and $|N| \in \{8, 16\}$, $\bar{G} \cong PSL(4, 2)$ or $\bar{G} \cong A_7$ and $|N| = 16$, $\bar{G} \cong Sp(2h, 2)$, $h \geq 3$, and $2^{2h} \mid |N|$ are the unique cases satisfying the inequality $b \leq n$. Assume that $\bar{G} \cong PSL(2, 7)$. Then $n = 16$, since $9 \leq n \leq 16$, $|N| \mid n$ and $|N| \in \{8, 16\}$. Let $C \leq G$ such that $C \cong Z_7$. Then $\mathrm{Fix}(C)$ fixes a subplane of Π of order at least 2, since $n \equiv 2 \bmod 7$. Actually, $\mathrm{Fix}(C) \cong PG(2, 2)$ by [33], theorem 3.7, since $n = 16$. Thus $\bar{G}$ cannot fix $l - \mathcal{O}$ pointwise. Hence $l - \mathcal{O}$ consists of either a $\bar{G}$-orbit of length 7 plus 2 points fixed by $\bar{G}$, or a $\bar{G}$-orbit of length 8 plus 1 point fixed by $\bar{G}$ by [17], Appendix B, since $\bar{G} \cong PSL(2, 7)$. Let $T \leq N_G(C)$, such that $T \cong Z_3$. Then $\mathrm{Fix}(C) \subset \mathrm{Fix}(T)$, since T fixes 2 points $l \cap \mathrm{Fix}(C)$ at least in any of the two possible orbital configurations of $l - \mathcal{O}$. This yields that T must fixes a further point on l, since $|l - (l \cap \mathrm{Fix}(C))| = 14$. So, $\mathrm{Fix}(T) \cong PG(2, 4)$ by [33], theorem 3.7, since $n = 16$. Since a Z_3 normalizes exactly one Z_7 in $\bar{G}$, there exists a point $A \in \mathrm{Fix}(T) - (\mathrm{Fix}(C) \cup l)$ such that $Z_3 \leq G_A \nleq Z_7.Z_3$. Then $Z_3 \leq G_{A^N} \nleq Z_7.Z_3$. Let Ω be the set of N orbits on $\Pi - l$. Then $16^2/|N| = \lambda_1 7 + \lambda_2 8$, with $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 > 0$, by lemma 16, since $|\Omega| = 16^2/|N|$. Note that $\lambda_1 > 0$ since $Z_3 \leq G_{A^N} \nleq Z_7.Z_3$.

Also $\lambda_2 > 0$, since $G_{B^N} \cong Z_7.Z_3$ for any $B \in \text{Fix}(C) - l$. It is a straightforward calculation to show that the above Diophantine equation has solutions only for $|N| \leq 4$. A contradiction.

Assume that $\bar{G} \cong PSL(4,2)$ or $\bar{G} \cong A_7$ and $|N| = 16$. Then $n = 16$ as $|N| \mid n$ and $15 < n \leq 30$. Hence there exists $H < G_O$, where $O \in \mathcal{O}$, such that $H \cong PSL(2,7)$ and $H \cap N = \langle 1 \rangle$, since N is regular on $[O] - \{l\}$ as $|N| = 16$. Clearly Π cannot be Desarguesian, since the full collineation group induced on a line is $P\Gamma L(2,16)$ and $A_7 \nleq P\Gamma L(2,16)$. Then Π is either the Lorimer-Rahilly plane of Johnson-Walker plane or they duals by [15]. A contradiction, since the full collineation group induced on a line in any of these planes is isomorphic to $PSL(2,7) \times S_3$ by [37].

Assume that $\bar{G} \cong Sp(2h,2)$, $h \geq 3$, and $2^{2h} \mid |N|$. Then $n = 2^{2h}$, since $|N| \mid n$ and $v < n \leq 2v$ with $v = 2^{h-1}(2^h+1)$. Then $G_O = G_{O,e}N$ and $G_{O,e} \cap N = \langle 1 \rangle$ for some line $e \in [O] - \{l\}$, since N is regular on $[O] - \{l\}$ as $N = N(X,l)$ with $X \in l - \mathcal{O}$. Then there exists an involution in $G_{O,e}$ fixing 2^{2h-2} points on $\mathcal{O}$ by [17], Example 5.4.3. A contradiction by [33], theorem 3.7. Hence, G is a covering group for G by lemma 15.

Note that the groups $\bar{G} \cong M_{11}$ or $\bar{G} \cong Sp(2h,2)$, $h > 3$ are ruled out by [43], theorem 5.1.4, since $N \neq \langle 1 \rangle$ by our assumption. In particular N is cyclic and $|N| \leq 3$ for the remaining groups of the Table I by [43], theorem 5.1.4. As a consequence, $N = N(X,l)$. Assume that G fixes a further point Y on $l - \mathcal{O}$. Then G acts on the set Ψ of N-orbits on $[Y] - \{l\}$. Then $|\Psi| = n/|N|$ as $N(Y,l) = \langle 1 \rangle$. Then $n = |N| \sum_{j \geq 0} \theta_j d_j(\bar{G})$, with $\theta_j \geq 0$, $j \geq 0$, and $\sum_{j \geq 0} \theta_j > 0$ by lemma 16. Actually, $\theta_j = 0$ for $j > 1$ and $(|N|, \theta_0, \theta_1) = (2,0,1), (2,1,0)$, or $(|N|, \theta_0, \theta_1) = (3,1,0)$ and $\bar{G} \cong PSL(2,9)$, or $\bar{G} \cong A_7$, or $(|N|, \theta_0, \theta_1) = (2,2,0)$ and $\bar{G} \cong A_7$ by lemma 12, since $N \neq \langle 1 \rangle$ and $v < n \leq 2v$. Let $r \in [Y] - \{l\}$. Then $G_{r^N} = G_r N$ and $G_r \cap N = \langle 1 \rangle$. In particular $G_r \cong \bar{G}_{r^N}$, where $\bar{G}_{r^N}$ is the stabilizer in a $\bar{G}$-orbit on Ψ of length $d_0(\bar{G})$, or $2d_0(\bar{G})$, or v. So, if $|G_r|$ is even, then G_r contains involution which are Baer collineations Π, since they fix the points X and Y on l and n is even. So $n = |N| (\theta_0 d_0(\bar{G}) + \theta_1 v)$ must be a square. It is a plain to see that unique groups of the Table I satisfying one the previous numerical conditions are either $G \cong SL(2,7)$ or $G/Z_2 \cong PSL(4,2)$, and $n = 16$. Nevertheless, the latter is ruled out by the same argument as above, since $PSL(2,7) \leq G_r$. Therefore $G \cong SL(2,7)$ and $n = 16$. Let C and T be defined as above. The above argument still works to show that $\text{Fix}(C) \subset \text{Fix}(T)$, with $\text{Fix}(C) \cong PG(2,2)$ and $\text{Fix}(T) \cong PG(2,4)$. This, in particular, still forces $l - \mathcal{O}$ to consist of a $\bar{G}$-orbits of length 7 plus 2 points fixed by G again by the above argument. Now, Let $D \leq N_G(T)$ such that $D \cong Z_4$ as $G \cong SL(2,7)$. Clearly D acts on $\text{Fix}(T)$ and D fixes exactly 3 points on $l \cap \text{Fix}(T)$. In particular $D \leq P\Gamma L(3,4)$ as $\text{Fix}(T) \cong PG(2,4)$. A contradiction, since $P\Gamma L(3,4)$ contains no cyclic subgroups of order 4 fixing exactly 3 points on a line. Assume that $|G_r|$ is odd. Then $G \cong SL(2,11)$ and $n = 24$ by a direct inspection of the Table I. Let $L \leq G$, such that $L \cong Z_{11}$. Then L fixes a subplane of Π of order at least 2, since $n + 1 = 25$ and since G fixes the points X and Y on $l - \mathcal{O}$. Actually, $o(\text{Fix}(L)) = 2$ by [33], theorem 3.7, since $n = 24$. Let $T \leq N_G(L)$ such that

$T \cong Z_5$. Clearly T acts not trivially on $\mathrm{Fix}(L)$. Hence $T \leq PSL(3,2)$, since $\mathrm{Fix}(L) \cong PG(2,2)$. A contradiction. Hence X is the unique point on $l - \mathcal{O}$ which is fixed by G. Thus either $n = v + d_0(\bar{G})$ or $n = 2v$ or $n = 29$ and $\bar{G} \cong A_7$. Assume the latter occurs. Then $G - N$ contains an involution, as $|N| \mid n$ and n is odd. This involution is a Baer collineation of Π, since it fixes 3 points on $\mathcal{O}$ by [51]. A contradiction.

Assume that $n = v + d_0(\bar{G})$. The case $\bar{G} \cong A_7$ and $n = 22$ and $\bar{G} \cong A_8$ and $n = 23$ are ruled out by [33], theorem 13.18, and by [26], respectively. Again, we may apply a similar argument to that of lemma 22 in order to rule out the cases $G \cong SL(2,11)$ and $n = 23$ or $\bar{G} \cong HS$ and $n = 276$ (in this case n is even and there exists also a 2-subgroup of G_A, $A \in \mathcal{O}$, of order 4 at least, fixing 2 points on $\mathcal{O}$ and 2 on $\Pi - l$). Hence $G \cong SL(2,9)$ and $n = 16$. Let $U \leq G$, such that $U \cong E_9$. Clearly $N_G(U) \cong U.T$, where $T \cong Z_8$, since $G \cong SL(2,9)$. In particular $N_G(U)$ fixes a point on R on $\mathcal{O}$, since $|\mathcal{O}| = 10$. Then U fixes a least a line r of $[R] - \{l\}$, since $n = 16$ and $U \cong E_9$. Note that T acts semiregularly on $[R] - \{l\}$, since the unique involution of T generates N and $N = N(X,l)$ with $X \in l - \mathcal{O}$. Thus $\left|r^T\right| = 8$. Moreover, $r^T \subset \mathrm{Fix}(U)$ since $T \leq N_G(U)$ and $r \in \mathrm{Fix}(U)$. Hence T fixes at least a point on $s - \{R\}$ for each $s \in r^T$, since $n = 16$. In particular there exists a non trivial subgroup U_0 of U, such that $[U : U_0] \leq 3$ fixing at least 3 points on l. Therefore $\mathrm{Fix}(U_0)$ is subplane of Π, since $\mathrm{Fix}(U) \subset \mathrm{Fix}(U_0)$. In particular $o(\mathrm{Fix}(U_0)) \geq 7$, since $r^T \cup \{l\} \subset [R] \cap \mathrm{Fix}(U_0)$. A contradiction by [33], theorem 3.7, since $n = 16$.

Assume that $n = 2v$. Note that any admissible case of Table I, but $G \cong SL(2,7)$ and $n = 16$, or $G \cong SL(2,11)$ and $n = 24$, is ruled out by the similar arguments to that used above. Thus the assertion. □

Now assume that the case $\bar{G} \cong PSU(3,5)$ is admissible. The following theorem completes this subsection and shows that the assertions (2) and (3) of theorem 1 are true when $d_0(\mathrm{soc}(\bar{G})) < v$.

Theorem 24 *One of the following occurs:*

(1) $n = 16$ *and* $G \cong SL(2,7)$*;*

(2) $n = 23$ *or* 24 *and* $G \cong SL(2,11)$*.*

Proof. It remains to rule out the group $\bar{G} \cong PSU(3,5)$ in order to prove this theorem by propositions 22 and 23. Assume that $N \nleq Z(G)$. Then there exists a Sylow t-subgroup S of N such that $\bar{G} \leq SL(V)$, where $V = S/\Phi(S)$ by lemma 15. Then either $|V| \geq 2^{20}$ by [43], theorem 5.3.9 when $5 \nmid |V|$, or $5^6 \mid |V|$ by [43], proposition 5.4.13. Hence either $|N| \geq 2^{20}$ or $5^6 \mid |N|$. On the other hand, either $|N| \mid n-1$ or $|N| \mid n^2$ by lemma 14. By composing all these bounds on the order of N, we see that the unique admissible case is $5^6 \mid |N|$ and $|N| \mid n^2$, since $n = 2v$ and $v = 126$. This yields $n = 250$, since $5^3 \mid n$ and $126 < n \leq 252$. A contradiction by [33], theorem 13.18. Hence, G is a covering group for $PSU(3,5)$. Thus $N \cong Z_3$ by [43], theorem 5.1.4, since $N \neq \langle 1 \rangle$ by our assumption. Thus any involution ϕ in G actually lies in $G - N$. So, it is well known that ϕ fixes exactly 6 points on $\mathcal{O}$. Hence n is a square. Therefore $\sqrt{n} \in \{12, 13, 15\}$, since $126 < n \leq 252$ and since $\sqrt{n} = 14$ cannot occur by [33], theorem 3.6. Note that $C_G(\phi)$ is non solvable, since it has a section which is isomorphic to $PGL(2,5)$. Thus the cases $\sqrt{n} = 12$ or 15 are ruled out by [36] and [31], respectively. Hence $\sqrt{n} = 13$. Then $\mathrm{Fix}(\phi) \cong PG(2,13)$ by [48]. Denote by $\bar{C}_G(\phi)$ the group induced on $\mathrm{Fix}(\phi)$ by $C_G(\phi)$. Then $\bar{C}_G(\phi)$ acts trivially on $\mathrm{Fix}(\phi)$, since $5 \mid \left|\bar{C}_G(\phi)\right|$ while $5 \nmid |PGL(3,13)|$. Thus there exists an element of order 5 in $PSU(3,5)$ fixing the same 6 points on $\mathcal{O}$ fixed by ϕ. A contradiction. $\square$

5.2 The faithful case.

In this subsection we deal with case $N = \langle 1 \rangle$. Since G is simple, either $K = \langle 1 \rangle$ and hence G has a non trivial orbit on $l - \mathcal{O}$, or $G = K$ and hence G fixes $l - \mathcal{O}$ pointwise.

This subsection is structured as follows. If G has a non trivial orbit on $l - \mathcal{O}$, we reduce to the case $d_0(G) < v$ by using the arguments of parts (C) and (D) of the propositions 19 and 20, respectively. Then we show that the involutions in G are Baer collineations of Π by using the results of Ho-Gonçalves [32]. Finally, a case by case investigation, show that Π is the Johnson-Walker translation plane of order 16 or its dual, and $G \cong PSL(2,7)$. If G fixes $l - \mathcal{O}$ pointwise. We reduce to the cases $n = v + 1$ by using lemma 11. At this point, we show that G admits another 2-transitive orbit of length v not contained in a line which is in contrast with the order n of Π.

Proposition 25 *If there exists a non trivial G-orbit on $l - \mathcal{O}$, then* Π *is the Johnson-Walker translation plane of order* 16 *or its dual, and* $G \cong PSL(2,7)$.

Proof. Assume there exists a non trivial G-orbit $\mathcal{O}'$ on $l - \mathcal{O}$. So $n \geq v + d_0(G)$. If $d_0(G) = v$, then either $n = 2v - 1$ or $n = 2v$. At this point we may use the arguments of parts **(C)** and **(D)** of the propositions 19 and 20, respectively, to rule out these cases. Hence $d_0(G) < v$. Assume that G contains an involutory perspectivity. If there exists a point $P \in \Pi - l$ such that $G_P = \langle 1 \rangle$, then $|G| \leq |\Pi - l|$. This yields $v(v-1)\theta \leq 4v^2$, since $n \leq 2v$ and since $|G| = v(v-1)\theta$, $\theta \geq 1$. Thus either $\theta = 5$ and $v = 5$, or $\theta \leq 4$. Then either $G \cong PSL(2,5)$ and $n = 11$ or $n = 12$, or $G \cong PSL(2,7)$ and $n = 15$ or $n = 16$ by a direct inspection of the Table I. Nevertheless the former cannot occur, respectively, by [47], since $|\mathcal{O}| = 6$, and by [36]. Also the latter cannot occur, respectively, by [31], and by [15], since G contains involutory perspectivities and $|\mathcal{O}| = 8$. Hence G is totally irregular on Π. Then $G \cong PSL(2,5)$ and $n = 11$ or $n = 12$, or $G \cong PSL(2,7)$ and $n = 15$ or $n = 16$ by [32], theorems 1 and 2. Again a contradiction. Hence each involution is a Baer collineation of Π. So n must be a square. Then either $G \cong PSL(2,7)$ or $G \cong PSL(2,9)$ when $n = 16$, or $G \cong A_7$ or $G \cong A_8$ and $n = 25$, or $G \cong PSU(3,5)$ and $n \in \left\{13^2, 14^2, 15^2\right\}$, or $G \cong HS$ and $n \in \left\{17^2, 18^2\right\}$, or $G \cong Sp(2h,2)$, $h \geq 3$ and $n = 2^{2h}$. Actually, the cases $G \cong PSU(3,5)$ and

$n \in \{13^2, 14^2, 15^2\}$ cannot occur by the same argument of theorem 24. The case $G \cong Sp(2h, 2)$, $h \geq 3$, and $n = 2^{2h}$ cannot occur by [33], theorem 3.7, since G contains Baer involutions fixing 2^{2h-2} points on $\mathcal{O}$ by [17], Example 5.4.3. Assume that $G \cong A_7$ and $n = 25$. Then there exists an involution in G fixing at least 7 points on l by [51] and since $|l - (\mathcal{O} \cup \mathcal{O}')| = 4$. A contradiction by [33], theorem 3.7. Assume that $G \cong A_8$ and $n = 25$. Then there exists an involution in G fixing at least 10 points on l by [51] and since $|l - (\mathcal{O} \cup \mathcal{O}')| = 3$. A contradiction by [33], theorem 3.7. Assume that $G \cong PSL(2, 9)$ and $n = 16$. Set $\{X\} = l - (\mathcal{O} \cup \mathcal{O}')$. Let S be a Sylow 2-subgroup of G. Then $S = \langle \alpha, \beta \rangle$ with $\alpha^4 = 1$, $\beta^2 = 1$ and $\alpha^\beta = \alpha^{-1}$. Note that $|\mathrm{Fix}(\alpha) \cap l| = 3$, $\left|\mathrm{Fix}(\alpha^2) \cap l\right| = 5$ and $|\mathrm{Fix}(\beta) \cap l| = 5$, since $l = \mathcal{O} \cup \mathcal{O}' \cup \{X\}$, and since $G \cong PSL(2, 9)$ acts in its 2-transitive permutation representations of degree 10 and 6 on $\mathcal{O}$ and on $\mathcal{O}'$, respectively. Furthermore, $\left|\mathrm{Fix}(\alpha^2) \cap \mathrm{Fix}(\beta) \cap l\right| = 3$. This yields $\mathrm{Fix}(\alpha^2) \cong \mathrm{Fix}(\beta) \cong PG(2, 4)$ and $\mathrm{Fix}(\alpha) \cong PG(2, 2)$ with $\mathrm{Fix}(\alpha) \subset \mathrm{Fix}(\alpha^2)$. Moreover, $\mathrm{Fix}(\alpha^2) \cap \mathrm{Fix}(\beta) \cong PG(2, 2)$ and $\mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$ consists of 3 collinear points of $\mathrm{Fix}(\alpha^2)$ including X. Thus $\left|\mathrm{Fix}(\alpha^2) - (\mathrm{Fix}(\alpha) \cup \mathrm{Fix}(\beta) \cup l)\right| = 10$. Let $U \leq G$ such that $U \cong E_9$. Is is easily seen that $\mathrm{Fix}(U)$ exactly 2 points on l, since the $\gamma = (123)(456)$ lies in U and γ is f.p.f. on $\mathcal{O}'$. Thus $\mathrm{Fix}(U)$ cannot be a subplane of Π. Then there exists a line r of Π such that $\mathrm{Fix}(U) - l \subset r$. In particular $\mathrm{Fix}(G) \subset \mathrm{Fix}(U)$ and $|\mathrm{Fix}(U) \cap \mathrm{Fix}(\alpha) - l| \leq 3$. Hence, there are at least 2 points of $\Pi - l$ (lying in $\mathrm{Fix}(\alpha) - l$), say X_1 and X_2, such that $G_{X_1} \cong Z_2$ and $G_{X_2} \cong Z_4$, since $\mathrm{Fix}(G) \subset \mathrm{Fix}(U)$, since $\mathrm{Fix}(U) - l \subset r$, since $\mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$ consists of 3 collinear points of $\mathrm{Fix}(\alpha^2)$ including X, and since the are no proper subgroups of G of order divisible by 20. Then $|\Pi - l| \geq 270$, since $X_1^G \cup X_2^G \subset \Pi - l$ with $\left|X_1^G\right| = 180$ and $\left|X_2^G\right| = 90$. A contradiction, since $n = 16$. Finally, if $G \cong PSL(2, 7)$ and $n = 16$, then Π is the Johnson-Walker or its dual translation plane of order 16 and the G-orbits on l have lengths 8, 7, 1 and 1 by [15]. Thus the assertion. □

Theorem 26 *Let Π be a projective plane of order n and let $\mathcal{O}$ be a 2-transitive G-orbit of length v on a line with $n > v \geq n/2$. If G is almost simple and G is faithful on $\mathcal{O}$, then Π is the Johnson-Walker translation plane of order 16 or its dual, and $G \cong PSL(2, 7)$.*

Proof. Assume that G fixes $l - \mathcal{O}$ pointwise. Assume also that $n \geq v + 2$. Thus any involution in G is a Baer collineation of Π, since $|l - \mathcal{O}| \geq 3$. Then $n + 1 - v \leq \sqrt{n} + 1$ and hence $v + 2 \leq n \leq v + \sqrt{2v}$, as $n \leq 2v$. Suppose there exists a point Y on $l - \mathcal{O}$ such that G admits an orbit $\mathcal{O}^*$ of length v on $[Y] - \{l\}$. If G admits also a further non trivial G-orbit on $[Y] - \{l\}$, then Π has order 16, $G \cong PSL(2, 7)$ and $v = 8$ by the dual of the proposition 25. A contradiction by [15], since G fixes $l - \mathcal{O}$ pointwise and $|l - \mathcal{O}| = 9$ in this case. Then G fixes $[Y] - (\{l\} \cup \mathcal{O}^*)$ linewise. If there exists a line r in $[Y] - (\{l\} \cup \mathcal{O}^*)$ such that G fixes two points on $r - \{Y\}$, then G is planar on Π. In particular $o(\mathrm{Fix}(G)) = |l - \mathcal{O}| - 1$, since G is transitive on $\mathcal{O}$ and G fixes $l - \mathcal{O}$ pointwise. Then G_O is planar on Π, since $\mathrm{Fix}(G) \subset \mathrm{Fix}(G_O)$. Furthermore, $o(\mathrm{Fix}(G_O)) = |l - \mathcal{O}|$, since G is 2-transitive on $\mathcal{O}$. So $o(\mathrm{Fix}(G_O) = o(\mathrm{Fix}(G)) + 1$. A contradiction by [33], theorem 3.7. Note that G and G_O are still planar if G fixes a point, other than Y, on at least two distinct lines of $[Y]-(\{l\}\cup\mathcal{O}^*)$. So, also this case cannot occur. Hence, there exists at least a line m in $[Y]-(\{l\}\cup\mathcal{O}^*)$ on which G does not fix any point, since $|[Y]-(\{l\}\cup\mathcal{O}^*)|\geq 2$ as $n\geq v+2$. Then $m-\{Y\}$ consists of non trivial G-orbits. It should be stressed that G cannot admit orbits of length v on $m-\{Y\}$, otherwise Π has order 16, $G\cong PSL(2,7)$ and $v=8$ by the proposition 25, in contrast with the above argument. Let $P\in m-\{Y\}$. Then $\left|P^G\right|>1$, since $m-\{Y\}$ consists of non trivial G-orbits. Clearly $\left|P^G\right|=\lambda d_k(G)$ for some primitive permutation representation degree $d_k(G)$, $k\geq 0$, of G. Assume that $d_0(G)=v$. Clearly $G\ncong Sp(2h,2)$, $h\geq 3$, by the same argument of the proposition 25. Then $\lambda v\leq\left|P^G\right|\leq v+\sqrt{2v}$, since $P^G\subset m-\{Y\}$, $n\leq v+\sqrt{2v}$ and $d_k(G)>v$. Thus $\lambda=1$ and $\left|P^G\right|=v+1$ by lemma 11. Since $n\geq v+2$, then there exists $Q\in m-(\{Y\}\cup P^G)$. Then $\left|Q^G\right|>1$, since $m-\{Y\}$ consists of non trivial G-orbits. Then $\left|Q^G\right|=v+1$ by he previous argument with Q in the role of P. So $n\geq 2v+2$, since $P^G\cup Q^G\subset m-\{Y\}$. A contradiction. Thus $d_0(G)<v$. Note that the above argument yields $\left|P^G\right|=\lambda d_0(G)$ by lemma 12 and since G cannot admit orbits of length v on $m-\{Y\}$. So, each admissible non trivial G-orbit on $m-\{Y\}$ must be a multiple of $d_0(G)$. This implies $n=\theta d_0(G)$, $\theta\geq 1$. Moreover, n must be a square and $v+2\leq n\leq v+\sqrt{2v}$. Now by a direct inspection of the Table I, it is easily seen that no cases arise. Thus $[Y]-\{l\}$ cannot contain orbit of length v for any $Y\in l-\mathcal{O}$. At this point we may use the previous argument to show that for any $Z\in l-\mathcal{O}$, the set $[Z]-\{l\}$ consists of a G-orbit of length $v+1$ plus a line fixed by G, since G cannot be planar on Π. In particular G is one of the exception groups listed in lemma 11. Then $n=9$ and $G\cong PSL(2,7)$, since n must be a square. A contradiction by [30], theorem A, since G contains Baer collineations of Π.

Assume that $n<v+2$. That is $n=v+1$, since $n>v$. Note that G fixes exactly a triangle Δ having l as its side. In particular each side of Δ consists of the vertices of Δ which are fixed by G and of a 2-transitive G-orbit of length v. This implies that G_O fixes a subplane of Π isomorphic to $PG(2,2)$. Then there exists a point $Q\in \mathrm{Fix}(G_O)-(l\cup\Delta)$ such that $\left|Q^G\right|=v$. Clearly Q^G is not contained in a line, since $Q^G\subset\Pi-(l\cup\Delta)$ and $\mathrm{Fix}(G)=\Delta$. If Q^G is a v-arc, then $Q^G\cup\Delta$ is a hyperoval. Then $G\cong PSL(2,2^s)$, $s\geq 2$, or $G\cong Sz(2^s)$, $s\geq 3$, s odd, or $G\cong PSL(2,2^s)$, $s\geq 2$, by [4], Main theorem. Thus $n=2^{is}+2$, $i\in\{1,2,3\}$, respectively, and $s\geq 2$. This yields $n\equiv 2 \bmod 4$. A contradiction by [33], theorem 13.18. Hence Q^G is the set of points of a non trivial 2-$(v,k,1)$ design $\mathcal{D}$ (see the preliminaries of [7]). By [40], theorem 1, we have that either $\mathcal{D}\cong PG(2,q)$, $G\cong PSL(3,q)$ and hence $n=q^2+q+2$, or $\mathcal{D}$ is Hermitian Unital, $G\cong PSU(3,q)$, $q>2$, and hence $n=q^3+2$, or $\mathcal{D}$ is Ree Unital and $G\cong {}^2G_2(q)$, $q=3^{2m+1}$, $m>1$, and hence $n=q^3+2$. If q is even, then $n\equiv 2 \bmod 4$ as $q>2$ (clearly the case $G\cong PSL(3,2)$ and $n=8$ cannot occur). This is impossible by [33], theorem 13.18. Hence q is odd. Now, it easily seen that G contains an involution fixing the 2 points of $l-\mathcal{O}$ and exactly either $q+1$ points on $\mathcal{O}$ when $\mathcal{D}$ is a Unital, or $q+2$ points on $\mathcal{O}$ when $\mathcal{D}\cong PG(2,q)$.

So, either $n = (q+2)^2$ or $n = (q+3)^2$ by [33], theorem 3.7. A contradiction in any case. Thus G cannot fix $l - \mathcal{O}$ pointwise and hence the assertion follows by the proposition 25. □

This completes the proof of that the results (2) and (3) of theorem 1.

6 The affine case

Throughout this section $\mathrm{soc}(\bar{G})$ is assumed to be an elementary abelian p-group for some prime p. Hence $\mathcal{O}$ is endowed with the structure of a $GF(p)$-vector space and the zero vector in $\mathcal{O}$ is denoted by O. Let $|\mathcal{O}| = p^d$, p prime, $d \geq 1$. Then $\bar{G} = \bar{T}\bar{G}_O$, where $\bar{T}$ is the whole translation group of $\mathcal{O}$ and $\bar{G}_O \leq \Gamma L(t,p)$. By [28] a structure of d^*-dimensional vector space V over a field $L \cong GF(p^h)$, $h \mid d$, $d = hd^*$, may be defined on $\mathcal{O}$ in such a way that $\bar{G} \leq A\Gamma L(d^*, p^h)$ and O is identified with the zero-vector of V.

6.1 The faithful case.

Assume that $N = \langle 1 \rangle$. Then $\bar{G} = G$ and hence $G = TG_O$.

In this subsection we prove the following result whose proof relies essentially on theorem 3.

Theorem 27 *If $v \notin \{5^2, 7^2, 11^2, 29^2, 59^2\}$, then $G \leq A\Gamma L(1,v)$ and one of the followings occurs:*

(1) $n = v+1$, $v = 2^d$ or $v \equiv 3 \bmod 4$;

(2) $n = 2v - 1$;

(3) $n = 2v$, $v = 2^d$;

(4) $n - \sqrt{n} + 1 = v$ and v is a prime.

Proof. Let K be the kernel of the representation of G on $l - \mathcal{O}$. Since G is primitive on $\mathcal{O}$, then either $K = \langle 1 \rangle$ or $T \leq K \leq G$ by [17], theorem 4.3B, since $T = \mathrm{soc}(G)$. Assume that $K = \langle 1 \rangle$. Then there exists $X \in l - \mathcal{O}$ such that the kernel of the action of G on X^G is trivial again by [17], theorem 4.3B. Set $\mathcal{O}' = X^G$. Then $v \mid |\mathcal{O}'|$ by the O'Nan-Scott theorem (see, for example, [17], theorem 4.1A.), since $v = |T|$. Thus $n \geq 2p^d - 1$, since $\mathcal{O} \cup \mathcal{O}' \subset l$ and $v = p^d$. Actually, either $n = 2p^d - 1$ or $n = 2p^d$, since $n \leq 2p^d$. In particular, we obtain that the action of G on $\mathcal{O}$ and on $\mathcal{O}'$ is the same. Assume that G contains a Baer involution of Π. Then n is a square. If $n = 2p^d - 1$, then either $(n,p,d) = (239^2, 13, 4)$ or p is odd and $d \leq 2$ by [55], results A11.1 and the result of page 141. Nevertheless, the former is ruled out by [30], theorem A. Thus $d \leq 2$ and hence $d^* \leq 2$. At this point we may use [28], lemma 5.10, to show that $G \leq A\Gamma L(1,v)$. Thus the assertion (2). Assume that $n = 2v$. Then $v = 2^d$ and d is odd by [33], theorem 13.18, and since n is a square.

Thus d^* is odd, since $d^* \mid d$. Therefore, by theorem 3, either $G \leq A\Gamma L(1, 2^d)$ or $SL(d^*, 2^h) \trianglelefteq G_O$, $d^* \geq 3$. Assume that the latter occurs. Let γ be any involution of G_O inducing a transvection on $\mathcal{O}$. Then γ fixes 2^{d-h} points on $\mathcal{O}$ and the point in $l - (\mathcal{O} \cup \mathcal{O}')$. So, $2^{2(d-h)} \leq 2^{d+1}$ by [33], theorem 3.7, since γ is a Baer collineation of Π and $n = 2^{d+1}$. This yields $2(d-h) \leq d+1$. Thus $d^* = 3$ and $h = 1$, since $d = d^*h$ and $d^* \geq 3$. Hence $SL(3, 2) \leq G_O$ and $n = 16$. A contradiction by [15], since $|\mathcal{O}| = 8$ and G is of affine type. Thus $G \leq A\Gamma L(1, 2^d)$ and we obtain the assertion (3).

Assume that each involution in G is a perspectivity of Π. If $n = 2v - 1$, then each involution in G must fix exactly 1 point on $\mathcal{O}$ and 1 point on $\mathcal{O}'$, as n is odd. Thus v must be odd. Then either $G \leq A\Gamma L(1, v)$ or $SL(2, p^h) \trianglelefteq G_O$ or $p^d \in \left\{3^4, 3^6, 19^2, 23^2\right\}$ by [28], theorem 6.7, since $p^d \notin \left\{5^2, 7^2, 11^2, 29^2, 59^2\right\}$ by our assumption. Actually, the cases $p^d \in \left\{3^4, 3^6, 19^2, 23^2\right\}$ cannot occur by [33], theorem 3.6. Assume that $SL(2, p^h) \trianglelefteq G_O$. Then there exists an element ϕ of order p inducing a transvection on $\mathcal{O}$ and on $\mathcal{O}'$. Then ϕ fixes $2p^h$ points on l. Clearly $(n, p) = 1$, since $n = 2v - 1$. Furthermore $(n-1, p) = 1$, since p is odd. Therefore ϕ fixes subplane of Π of order $2p^h - 1$. Then $(2p^h - 1)^2 \leq 2p^{2h}$ by [33], theorem 3.7. A contradiction. Thus $G \leq A\Gamma L(1, v)$ and hence the assertion (2). If $n = 2v$, then $v = 2^d$ by [33], theorem 13.18. Then G is solvable by [2], Satz 1. In particular by $G \leq A\Gamma L(1, v)$ by [35], theorem XII.7.3. That is the assertion (2).

Assume that $T \leq K \leq G$. Assume also that $|l - \mathcal{O}| \leq 2$. Then $|l - \mathcal{O}| = 2$ and hence $n = p^d + 1$, since $n > v$ and $v = p^d$. If $4 \mid |G_O|$, then G_O contains a Baer involution of Π, since $|l - \mathcal{O}| = 2$. Thus n must be a square. Clearly $d \geq 2$. Then $n = 9$ and $v = 8$ by [55], result A5.1. A contradiction by [30], theorem A. Hence $2 \parallel |G_O|$. Then $G \leq A\Gamma L(1, v)$ by [28], theorem 5.15. Furthermore, either $p = 2$ or $p^d \equiv 3 \bmod 4$, as $p^d - 1 \mid |G_O|$. Thus the assertion (1).

Assume that $|l - \mathcal{O}| > 2$. Assume also that v is even. Then each non trivial element in T is a Baer collineation of Π, since $|l - \mathcal{O}| > 2$ and $T \leq K$. Thus $n + 1 - v = \sqrt{n} + 1$, since T fixes $l - \mathcal{O}$ pointwise and T is regular on $\mathcal{O}$. Hence $v = \sqrt{n}(\sqrt{n} - 1)$. A contradiction, since $v = p^d$ and $v > 4$. Hence v is odd.

Assume that G_O contains the involutory O-dilatation α. Suppose that T does not contain planar elements. If there exists $X \in \Pi - l$ such that $T_X \neq \langle 1 \rangle$, then $T_X = T$, since T is abelian, T fixes $l - \mathcal{O}$ pointwise and T does not contain any planar element. Thus T is semiregular on $XY - \{X, Y\}$ for any $Y \in l - \mathcal{O}$ again by the facts that T fixes $l - \mathcal{O}$ pointwise and T does not contain any planar element. Then $v \mid n - 1$ and hence $n = v + 1$, since $v < n \leq 2v$. A contradiction, since $|l - \mathcal{O}| > 2$. Hence T is semiregular on $\Pi - l$. In particular $p \mid n$. Assume that α is a (C_α, a_α)-perspectivity of Π. Let $\gamma \in T$, $\gamma \neq 1$. Then α^γ is the $O\gamma$-involutory dilatation of G. Furthermore, α^γ is a $(C_\alpha\gamma, a_\alpha\gamma)$-perspectivity, where $C_\alpha\gamma \in l$ and $a_\alpha\gamma \neq l$. Clearly $\alpha^\gamma \neq \alpha$, as $O\gamma \neq O$. Then $\langle \alpha, \alpha^\gamma \rangle$ fixes $a_\alpha \cap a_{\alpha^\gamma}$ pointwise. A contradiction, since $\langle \alpha, \alpha^\gamma \rangle \cap T \neq \langle 1 \rangle$ and $(a_\alpha \cap a_{\alpha^\gamma}) \cap (\Pi - l) \neq \emptyset$. Hence α is a Baer collineation of Π. Then $p \mid \sqrt{n}$, since $p \mid n$ and n is a square. Assume there exists a point $P \in l - \mathcal{O}$, such that T is semiregular on $[P] - \{l\}$. Then $v \mid n$ and hence $n = 2v$, since $v < n \leq 2v$.

A contradiction by [33], theorem 13.18, since v is odd. Hence, for each point $B \in l - \mathcal{O}$, there exists a line $r_B \in [B] - \{l\}$ such that $T_{r_B} \neq \langle 1 \rangle$. Assume there exists a point $D \in l - \mathcal{O}$ such that $D\alpha \neq D$. Then T_{r_D} fixes also $r_{D\alpha}$, since α acts as the inversion on T. Thus T_{r_D} fixes the point $r_D \cap r_{D\alpha}$ lying on $\Pi - l$. A contradiction, since T is semiregular on $\Pi - l$. As a consequence, $l \cap \mathrm{Fix}(\alpha) = (l - \mathcal{O}) \cup \{O\}$. Then $n+1-v = \sqrt{n}$, since $|l - \mathcal{O}| = n+1-v$. That is $n - \sqrt{n} + 1 = v$. A contradiction, since $p \mid \sqrt{n}$ and $v = p^d$. Hence T contains a non trivial planar element τ. Assume that α is a (C_α, a_α)-perspectivity, then $C_\alpha \in l$ and $a_\alpha \neq l$, since α is the involutory O-dilatation in G_O and $\mathcal{O} \subset l$. Note that $(C_\alpha, a_\alpha) \in \mathrm{Fix}(\tau)$, since α inverts τ. So τ fixes O, since either $C_\alpha = O$ or $\{O\} = a_\alpha \cap l$. A contradiction, since $O \in \mathcal{O}$, while τ is semiregular on $\mathcal{O}$. Hence, α is a Baer collineation of Π. Then α fixes $l - \mathcal{O}$ pointwise, since τ is planar, τ fixes $l - \mathcal{O}$ pointwise and α fixes only the point O on $\mathcal{O}$. Thus $|l - \mathcal{O}| = \sqrt{n}$. That is $n+1-v = \sqrt{n}$ and hence $(\sqrt{n}-1)^2 + (\sqrt{n}-1) + 1 = v$. Then either $v = p$ or $(\sqrt{n}-1, v) = (18, 7^3)$ by [55], result A7.1. Assume that $(\sqrt{n}-1, v) = (18, 7^3)$. Clearly α acts on $\mathrm{Fix}(\tau)$ trivially by [33], theorem 13.18, as $o(\mathrm{Fix}(\tau)) = 18$. Hence $\mathrm{Fix}(\tau) \subset \mathrm{Fix}(\alpha)$. A contradiction by [33], theorem 3.7, since $o(\mathrm{Fix}(\alpha)) = 19$. Therefore $v = p$ and we have the assertion (4).

Assume that G does not contain involutory dilatations. Then $SL(d^*, p^h) \trianglelefteq G_O$ with d^* odd by theorem 3, since v is odd. Let ζ be the involution in G_O represented by the matrix $A = \mathrm{diag}(-I_2, I_{d^*-2})$. Then ζ is a Baer collineation of Π fixing exactly p^{d-2h} points on $\mathcal{O}$. Then $p^{2(d-2h)} \leq n$, by [33], theorem 3.7. Thus $d^* = 3$ as d^* is odd and $d^* > 1$. Let $L \leq C_{G_O}(\zeta)$, where $L = \langle diag(C,1) : C \in SL(2,q) \rangle$. Let L_0 be the kernel of the action of L on $\mathrm{Fix}(\zeta)$. Clearly $\langle \zeta \rangle \trianglelefteq L_0 \trianglelefteq L$. Actually, $L_0 < L$, otherwise L would contain planar p-elements of Π inducing transvections on $\mathcal{O}$ (for example pick $B = \mathrm{diag}(B_0, 1)$ where $B_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$). Then $L_0 = \langle \zeta \rangle$ and $L/L_0 \cong PSL(2,q)$ acts on $\mathrm{Fix}(\zeta)$ fixing q points on $\mathcal{O}$. It easily seen that L/L_0 contains a Baer collineation $\bar{\beta}$ of $\mathrm{Fix}(\zeta)$ fixing a further point on $\mathrm{Fix}(\zeta) \cap l - \mathcal{O}$, since $L/L_0 \cong PSL(2,q)$ cannot be a group of perspectivities of $\mathrm{Fix}(\zeta)$ with axis $\mathrm{Fix}(\zeta) \cap l$. Thus $o(\mathrm{Fix}(\zeta)) \geq q^2$ by [33], theorem 3.7. Then $n \geq q^4$ again by [33], theorem 3.7. A contradiction, since $n \leq 2q^3$ and q is odd. $\square$

6.2 The unfaithful 2-transitive orbits.

Throughout this subsection we assume that $N \neq \langle 1 \rangle$.

The proof of theorem 29, which is the main theorem in this subsection, is structured as a follows. We firstly show that G can be written in a 'nice' form (see the following lemma). Then we reduce to case $N \leq Z(G)$, otherwise lemma 15 provides a lower bound for $|N|$ which is in contrast with the possible upper bounds given in lemma 14. At this point we essentially use the Schur multipliers (see [41]) to obtain the assertion.

Lemma 28 *The followings hold:*

(1) $G = UG_O$*, where* U *is a normal* p*-subgroup* U *of* G *such that* $U/(U \cap N) \cong \bar{T}$*;*

(2) If $G_O = HN$ *with* $H \cap N = \langle 1 \rangle$ *and* H *quasisimple, then* $N \leq U$*.*

Proof. Let T be the full preimage in G of $\bar{T}$. Then $T = UN$, where U is a Sylow p-subgroup of T. Furthermore, $G = N_G(U)T$ by the Frattini's argument. Actually $G = N_G(U)$, since $T = UN$ and $N = \Phi(G)$. Hence $U \lhd G$. Moreover, UG_O induces $\bar{G}$ on $\mathcal{O}$, since $N < G_O$. Then $G = UG_O$ by the minimality of G. Thus the assertion (1).

Assume there exists $H \leq G_O$ such that $G_O = HN$ with $H \cap N = \langle 1 \rangle$ and H quasisimple. Then $G = UH$ by (1), since $N = \Phi(G)$. Furthermore, $U \cap H = \langle 1 \rangle$, by [23], theorem 3.1.3, since $H \cong \bar{G}_O$ and $U \cap H$ is a normal p-subgroup of H. Clearly UN/U is isomorphic to a normal subgroup of H. Thus $U/(U \cap N)$ is isomorphic to a subgroup of $Z(\bar{G}_O)$, as $H \cong \bar{G}_O$, the group $\bar{G}_O$ is quasisimple and N is nilpotent. In particular $U/(U \cap N)$ has order coprime to p by [23], theorem 3.1.3. Hence $N = P \times Z$, where $P = U \cap N$ and Z is isomorphic to a subgroup of $Z(\bar{G}_O)$, as N is nilpotent. In particular, Z is cyclic as $Z(\bar{G}_O)$ is so by [23], theorem 3.2.1. Let W be the Sylow t-subgroup of N, with $t \neq p$. Then $W \leq Z$ as $N = P \times Z$. Thus W is cyclic. Moreover, $W \lhd G$ as N is nilpotent. Then G acts on W with kernel Q. Then $G/Q \leq \mathrm{Aut}(W)$. Actually, $Q = G$ by [23], theorem 1.3.10 and lemma 5.4.1, since W is cyclic, $N \leq Q$ and $G = UH$ with H quasisimple. Thus Z is central in G and hence $UN = U \times Z$ as $(|U|, |Z|) = 1$. This yields $UN \cap H = \langle 1 \rangle$, since $U \cap H = \langle 1 \rangle$ and $Z \cap H = \langle 1 \rangle$. As a consequence $Z = \langle 1 \rangle$ and $N \leq U$. Thus the assertion (2). □

Theorem 29 *Let* $\mathcal{J} = \{2^4, 2^6, 3^2, 3^3, 3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$*. Then one of the followings occurs:*

(1) $\bar{G} \leq A\Gamma L(1, v)$*, or*

(2) $v \in \mathcal{J}$*.*

Proof. Deny. By theorem 3, we have that if $\bar{G} = \bar{T}\bar{G}_O$ is 2-transitive on $\mathcal{O}$, then $\bar{G} = \bar{T}.\mathrm{soc}(\bar{G}_O)$ is still 2-transitive on $\mathcal{O}$ and $\mathrm{soc}(\bar{G}_O)$ is quasisimple. Thus, we may assume without loss of generality that $\bar{G}_O = \mathrm{soc}(\bar{G}_O)$. Hence $\bar{G}_O$ is quasisimple. In particular, we have the following possibilities for $\bar{G}_O$:

(i) $\bar{G}_O \cong SL(d^*, p^h)$, $d^* \geq 2$;

(ii) $\bar{G}_O \cong Sp(d^*, p^h)$, d^* even and $d^* \geq 4$;

(iii) $\bar{G}_O \cong G_2(p^h)$, $d^* = 6$ and $p = 2$.

We treat the cases $N \nleq Z(G)$ and $N \leq Z(G)$ separately.

(I) The case $N \nleq Z(G)$.

Assume that $N \nleq Z(G)$. Then the same argument of lemma 15 yields that G acts on V with kernel R, where $V = S/\Phi(S)$, S a Sylow t-subgroup of N such that $S \nleq Z(G)$, and $N \trianglelefteq R \trianglelefteq G$. Assume that $R = G$. Then each Sylow r-subgroup T of G, with $r \neq t$, centralizes S by [23], theorem 5.1.4. Thus $C_G(S) \nleq N$ and $[G : C_G(S)] = t^h$, $h \geq 0$. Furthermore, $C_G(S) \lhd G$ as $S \lhd G$. Hence $N \lhd C_G(S)N \trianglelefteq G$. Set $L = C_G(S)N$. Clearly, either $L = G$ or $[G : L] = t^i$ for some $1 \leq i \leq h$. Actually, the former is ruled out by the same argument of lemma 15. Thus $[G : L] = t^i$ and hence $\left[\bar{G} : \bar{L}\right] = t^i$ as $N \lhd L$. Then $\bar{L}/\bar{T} \lhd \bar{G}/\bar{T}$ and $\left[\bar{G}/\bar{T} : \bar{L}/\bar{T}\right] = t^i$ as $\bar{T} \leq \bar{L}$ by [17], theorem 4.3.B. This implies that $\bar{G}_O$ must contain a normal subgroup of index t^i. A contradiction, since $\bar{G}_O$ is quasisimple. Then $R < G$. Hence, either $R = N$ or $\bar{T} \leq \bar{R} < \bar{T}.Z(\bar{G}_O)$, since $\bar{G} = \bar{T}\bar{G}_O$. Assume that $\bar{T} \leq \bar{R} \leq \bar{T}.Z(\bar{G}_O)$. Set $H = G/R$. Clearly $H \leq P\Gamma L(V)$, since V is a vector space over $GF(t)$. Note that $H \cong \bar{G}/\bar{R}$, since $\bar{G}/\bar{R} \cong (\bar{G}/\bar{T})/(\bar{R}/\bar{T})$. This yields that H is isomorphic to a central extension of $\bar{G}_O/Z(\bar{G}_O)$, since $\bar{G}/\bar{T} \cong \bar{G}_O$ and since $\bar{R}/\bar{T}$ is isomorphic to a subgroup of $Z(\bar{G}_O)$. Thus $H \leq PSL(V)$, since $H \leq P\Gamma L(V)$ and H is quasisimple. Recall that $\bar{G}_O$ is one of the groups listed above. So, if the representation is in coprime characteristic then $|V| \geq 2^{R_{p'}(H)}$ by [43], corollary 5.3.3, theorem 5.3.9 and corollary 5.4.14.(i), since $v \notin \mathcal{J}$. This yields $|V| > 4v^2$ for $v > 8$. Hence $|N| > 4v^2$ for $v > 8$ as $V = S/\Phi(S)$ and S a Sylow t-subgroup of N. A contradiction, since $|N| \leq 4v^2$ as $|N| \leq n^2$ by lemma 14 and $n \leq 2v$. Hence $v \leq 8$. Actually $v = 8$, since $v = p^d$, $d \geq 2$ and $v \geq 5$. In particular N is regular on $\Pi - l$ as $|N| = 2^6$. Then $G = G_C N$ for some $C \in \Pi - l$. Actually $G = G_C$, since $N = \Phi(G)$ by lemma 24. A contradiction, since N is semiregular on $\Pi - l$. Thus the representation of H as a subgroup of $PSL(V)$ is in the natural characteristic. Therefore $v \mid |V|$ by [43], corollary 5.3.3. and proposition 5.4.13, since $\bar{G} \nleq A\Gamma L(1, v)$ and $v \notin \mathcal{J}$. As a consequence $v \mid |N|$ for $N < R < G$. Finally, assume that $R = N$. Then $\bar{G} \leq PSL(V)$. Then, by the above argument with $\bar{G}_O$, $O \in \mathcal{O}$, in the role of H, we still obtain $v \mid |N|$. Hence $v \mid |N|$ in any admissible case.

Assume that $|N| \mid n - 1$. Then G fixes a unique point Q on $\Pi - l$ and N is semiregular on $\Pi - (l \cup \{Q\})$ by lemma 14. Then $|N| = n - 1$, since $v \mid |N|$ and $v < n \leq 2v$. Thus $G_O = G_{O,A}N$ with $G_{O,A} \cap N = 1$, for some point $A \in OQ - \{O, Q\}$. Then $G = UG_{O,A}$ by lemma 28, since $G = UG_O$, $G_O = G_{O,A}N$ and $N = \Phi(G)$. Then the same argument of theorem 27, with $\bar{G}$ in the role of G, rules out this case, since $\bar{G} \nleq A\Gamma L(1, v)$, $v \notin \mathcal{J}$ and $v = p^d$.

Assume that $|N| \mid n$. Then $|N| = n$ and $n = 2v$, since $v \mid |N|$ and $v < n \leq 2v$. Therefore $v = 2^d$ and $n = 2^{d+1}$ by [33], theorem 13.18. Furthermore, N is semiregular on $[O] - \{l\}$ where $O \in \mathcal{O}$ by lemma 14. Let Ω be the set of N-orbits on $[O] - \{l\}$. Clearly $|\Omega| \leq 2$, since $v \mid |N|$ and $n = 2v$. Thus G_O fixes Ω elementwise. Then $G_O = G_{O,a}N$ for some line $a \in [O] - \{l\}$. Then the same argument of theorem 27, with $G_{O,a}$ in the role of G_O, rules out the case d odd. Hence d is even and n is a non square as $n = 2^{d+1}$. Thus $G_{O,a}$ must have odd order, since v and n are even and $G_{O,a} \cong \bar{G}_O$. Therefore $\bar{G}_O$ must have odd order. Hence G is solvable by [2], Satz 1. In particular $G \leq A\Gamma L(1, v)$ by [35], theorem XII.7.3. A contradiction.

Assume that $|N| > n$. Then either $n \mid |N|$ and n is a prime power, or $n = 3|J|/2$ where J is the Sylow 2-subgroup of N by lemma 14. Then either $n = 2v$, or $n = 3v/2$ and v even, since $v \mid n^2$ as $|N| \mid n^2$, and since $v < n \leq 2v$, respectively. Note that v is even also in the first case by [33], theorem 13.18. Let Ω be defined as above. Then $|\Omega| \leq 3$, since each N-orbit on $[O] - \{l\}$ has length $n/3$ at least by lemma 14. Thus G_O fixes Ω elementwise, since G_O is quasisimple. Hence $G_O = G_{O,b}N$ for some line $b \in [O] - \{l\}$. We stress that $G_{O,b} \cap N \neq \langle 1 \rangle$, since $N_b \neq \langle 1 \rangle$ being $|N| > n$. Now, we may repeat the above argument, with $G_{O,b}$ in the role of H and N_b in the role of N to assert that either $v \mid |N_b|$ or $N_b \leq Z(G_{O,b})$. Assume that $v \mid |N_b|$. Then $|N| \geq vn/3$, since $[N : N_b] \geq n/3$ as $|\Omega| \leq 3$. Let Ψ be the set of N-orbits of points on $\Pi - l$. Then $|\Psi| \leq 6$, since N is semiregular on $\Pi - l$ and $|N| \geq vn/3$. It is a plain to see that G fixes Ψ elementwise, since $v \geq 8$. So $G = G_A N$ for some $A \in \Pi - l$. Actually $G = G_A$, since $N = \Phi(G)$ by lemma 24. A contradiction, since N is semiregular on $\Pi - l$. Hence $N_b \leq Z(G_{O,b})$. Then $G_{O,b} = G'_{O,b}N$ where $G'_{O,b}$ is a covering group for $\bar{G}_O$ by [1], theorem 11.3.33. Hence, $G'_{O,b} \cap N$ is isomorphic to a subgroup of the Schur multiplier of $\bar{G}_O$. If $\bar{G}_O \cong SL(d^*, p^h)$, $d^* \geq 2$, then $G'_{O,b} \cap N = \langle 1 \rangle$ by [41], theorem 7.1.1 (i), since $v \notin \mathcal{J}$. If $\bar{G}_O \cong Sp(d^*, p^h)$, d^* even, $d^* \geq 4$, then $G'_{O,b} \cap N = \langle 1 \rangle$ by [41], theorem 2.5.12 for p is odd, and by [43], theorem 5.1.4 for $p = 2$, since $\bar{G}_O$ is perfect, $d^* \geq 4$ and $v \notin \mathcal{J}$. Finally, if $\bar{G}_O \cong G_2(p^h)$ and $p = 2$, then $G'_{O,b} \cap N = \langle 1 \rangle$ by [43], theorem 5.1.4, since $v \notin \mathcal{J}$. Thus $G'_{O,b} \cap N = \langle 1 \rangle$ in any admissible case. Then the above argument with $G'_{O,b}$ in the role of $G_{O,a}$ rules out this case. Actually, such a argument works when we replace $n = 2v$ with $n = 3v/2$. So, also the case $n = 3v/2$ cannot occur.

(II) The case $N \leq Z(G)$.

Assume that $N \leq Z(G)$. Then $G_O = G'_O N$ with G'_O a covering group for $\bar{G}_O$ by [1], theorem 11.3.33. Hence, $G'_O \cap N$ is isomorphic to a subgroup of the Schur multiplier of $\bar{G}_O$. Actually, $G'_O \cap N = \langle 1 \rangle$ in any admissible case by the above argument, with G'_O in the role of $G'_{O,b}$. Then $N \leq U$ by lemma 28 (2). Furthermore, there exists $\zeta \in G'_O$, such that $o(\zeta)$ is a primitive prime divisor of $v - 1$ by [43], theorem 5.2.14, since $\bar{G} \nleq A\Gamma L(1, v)$ and $v \notin \mathcal{J}$. Then ζ acts irreducibly on $U/(U \cap N) \cong \bar{T}$ by [28], Section 5 and theorem 3.5. Then each proper ζ-invariant normal subgroup of U lies in N. Therefore ζ acts trivially on each proper ζ-invariant normal subgroup of U, since $N = Z(G_O)$. Then U is special and $N = \Phi(U)$ by [23], theorem 5.3.7, since $U/N \cong \bar{T}$ and $N \neq \langle 1 \rangle$. Recall that K is the kernel of the representation of G on $l - \mathcal{O}$ and $N = \Phi(G)$. If $K < G$, then there exists a non trivial G-orbit $\mathcal{O}^*$ on $l - \mathcal{O}$. Let Λ be the set of N-orbits on $\mathcal{O}^*$ and let F be the kernel of the representation of G on Λ. Clearly $N \trianglelefteq F \trianglelefteq G$. If $F = G$, then $G = G_X N$ for some $X \in \mathcal{O}^*$. That is $G = G_X$, since $N = \Phi(G)$. A contradiction, since $X \in \mathcal{O}^*$ and $\mathcal{O}^*$ is a non trivial G-orbit. Therefore $R < G$. If $R = N$, then G induces $\bar{G}$ on $\mathcal{O}^*$. Then $v \mid |\mathcal{O}^*|$ by the O'Nan-Scott theorem, since $v = |\bar{T}|$. This forces N to fix $l - \mathcal{O}$

pointwise. Then either $n = 2v - 1$ and $N = N(Q, l)$ with $Q \in \Pi - l$, or $n = 2v$ and $N = N(l, l)$ by lemma 14, since $n \leq 2v$. Since $\bar{G} \nleq A\Gamma L(1, v)$ and $v \notin \mathcal{J}$, and since $G'_O \cap N = \langle 1 \rangle$ and $G'_O \cong \bar{G}_O$, then the arguments of theorem 27 still works and hence we may rule out the cases $n = 2v - 1$ and $n = 2v$.

Now, assume that $N < R < G$. That is $\langle 1 \rangle < \bar{R} \lhd \bar{G}$. Hence $U \leq R$, by [17], theorem 43.B, since $N \leq U$, $\bar{U} = \bar{T}$ and $\bar{T} = \mathrm{soc}(\bar{G})$. Thus U fixes Λ elementwise. Then $U = U_Y N$ for each $Y \in \mathcal{O}^*$, since $N \leq U$. Furthermore, since $N = \Phi(U)$, then U fixes $\mathcal{O}^*$ pointwise. Hence $N < U \leq K \leq G$ and hence $N = N(l, l)$ or $N(Q, l)$ with $Q \in \Pi - l$ by lemma 14. Assume that $N = N(l, l)$. Clearly $N(X, l) = \langle 1 \rangle$ for any $X \in \mathcal{O}$, since $N \leq Z(G)$. Hence there exists a point $E \in l - \mathcal{O}$ such that $N(E, l) \neq \langle 1 \rangle$, since $N \neq \langle 1 \rangle$. Assume that $N(E, l) < N$. Then G acts on the set Σ of N-orbits on $[E] - \{l\}$, since $N \leq Z(G)$. If U fixes some element in Σ, then $U = U_r N$ for some $r \in [E] - \{l\}$, since $N \leq U$. As a consequence, $U = U_r$ since $N = \Phi(U)$. A contradiction, since $N_r = N(E, l)$ and $N(E, l) < N$. Thus U moves each element in Σ. In particular, G induces $\bar{G}$ on Σ, and $\bar{G}$ does not fix any element in Σ. Then $v \mid |\Sigma|$ by the O'Nan-Scott theorem, since $v = \left|\bar{T}\right|$. Then $n = v\,[N : N(E, l)]$, since $|\Sigma| = n / [N : N(E, l)]$. Thus $[N : N(E, l)] = 2$ and $n = 2v$, since $v < n \leq 2v$. Since $\bar{G} \nleq A\Gamma L(1, v)$, $v \notin \mathcal{J}$ and since $G'_O \cap N = \langle 1 \rangle$ and $G'_O \cong \bar{G}_O$, it is easily seen that the argument of theorem 27 still works and we may again rule out the case $n = 2v$. Therefore $N = N(E, l)$. Assume that v is even. If there exists $\sigma \in U - N$, σ involution, then σ is a Baer collineation of Π fixing the $v + 1$ points of $l - \mathcal{O}$, since $n = 2v$ and since U fixes $l - \mathcal{O}$ pointwise. Then $v^2 \leq n$ by [33], theorem 3.7. A contradiction, since $n = 2v$ and $v > 2$. As a consequence $U - N$ does not contain involutions. Thus U must be semiregular on $[Y] - \{l\}$ for any point Y on $l - (\mathcal{O} \cup \{E\})$, since U fixes $l - \mathcal{O}$ pointwise, $N \lhd U$ and $N = N(E, l)$. Thus $|U| \mid n$. Then $n = 2v$ and $N \cong Z_2$, since $|U| = v\,|N|$ and $N \neq \langle 1 \rangle$. Hence v is even as $N \leq U$ and U is a p-group. A contradiction by the same argument as above. Assume that v is odd. Assume also U fixes a line f of $[E] - \{l\}$. If U is semiregular on $f - \{E\}$, then $n = 2v$, since $|U| = v\,|N|$. A contradiction by [33], theorem 13.18, since v is odd. Thus $U_D \neq \langle 1 \rangle$ for some $D \in f - \{E\}$. Then U_D fixes D^N pointwise, since $N \leq Z(G)$. Then there exists a non trivial element τ in $U - N$ fixing $D^N \cup (l - \mathcal{O})$ pointwise. Thus τ is planar on Π, since $|l - \mathcal{O}| \geq 2$, $\left|D^N\right| \geq 2$ and $D^N \subset \Pi - l$. So $o(\mathrm{Fix}(\tau)) = n - v$. If $n + 1 - v = \sqrt{n}$, then v is a prime by arguing as in theorem 27. A contradiction, since $\bar{G} \nleq A\Gamma L(1, v)$. Thus v is odd and $n + 1 - v < \sqrt{n}$. In particular, $n \leq v + \sqrt{2v}$ by [33], theorem 3.7, since $n \leq 2v$. Recall that $G'_O \cap N = \langle 1 \rangle$. The same argument of theorem 27 implies that G'_O contains an involution α inducing an involutory O-dilatation on $\mathcal{O}$. So v is a square and hence n cannot be a square, since $v < n \leq v + \sqrt{2v}$. Clearly α cannot be a Baer collineation of Π. Thus α is an involutory (C_α, a_α) perspectivity. Clearly $C_\alpha \in l$ and $a_\alpha \neq l$ since $\alpha \notin N$ and α fixes l. Thus there exists a point W on $l - \mathcal{O}$ such that $\left|W^H\right| > 1$. Let Γ be the set of α-orbits on W^H. Then $|\Gamma| = \left|W^H\right|/2$, since α is central in H, $a_\alpha \neq l$ and H is transitive on W^H. Then $|\Gamma| \geq d_0(H/Z(H))$ by [43], proposition 5.2.1. Hence $2d_0(H/Z(H)) \leq n - v + 1$, since $|\Gamma| = |W^H|/2$ and $|W^H| \leq n-v+1$. Actually, $(2d_0(H/Z(H))-1)^2 \leq 2v$, since $v < n \leq v + \sqrt{2v}$. At this point, it is a straightforward calculation to show that no one case satisfies the previous inequality by bearing in mind that $d_0(H/Z(H))$ is given in [9]. Hence U does not fix lines of $[E] - \{l\}$. Therefore $[E] - \{l\}$ is union of non trivial $\bar{G}$-orbits. Thus $v \mid n$, since the length of each these orbits is a multiple of v by the O'Nan-Scott theorem, as $v = |\bar{T}|$. Hence $n = 2v$. Again a contradiction.

Assume $N = N(Q, l)$ for some $Q \in \Pi - l$. Then N is semiregular on $QB - \{Q, B\}$, where B is any point of $l - \mathcal{O}$. If U is semiregular on $QB - \{Q, B\}$, then $|U| \mid n-1$. Then $2v \leq n-1$, since $|U| = v|N|$ and $N \neq \langle 1 \rangle$. A contradiction, since $n \leq 2v$. In particular there exists $\tau_1 \in U - N$ fixing a point C of $QB - \{Q, B\}$. Then τ_1 is planar on Π, since τ_1 fixes the points C and Q on $\Pi - l$ and since U fixes $l - \mathcal{O}$ pointwise. At this point the same argument used for the case $N = N(E, l)$, with τ_1 in the role of τ, still works and we may rule out this case. This completes the proof. □

7 Translation Planes

In this section we investigate what theorems 1 and 2 say when Π is the projective extension of a translation plane of order n and $\mathcal{O}$ is a 2-transitive G-orbit of length v on a line l, with $n > v \geq n/2$. The case when $\mathcal{O}$ has length v with $v \geq n$ is already contained in section 2.

Theorem 30 *Let Π be the projective extension of a translation plane of order n and let $\mathcal{O}$ be a 2-transitive G-orbit of length v on a line l. If $n > v \geq n/2$ and G is almost simple, then one of the following occurs:*

1. *Π is the Hall plane of order 9 or its dual, $|\mathcal{O}| = 5$ and $SL(2,5) \trianglelefteq G$. In particular l is the line at infinity;*
2. *Π is the Johnson-Walker translation plane of order 16 or its dual, and $PSL(2,7) \trianglelefteq G$. In particular l is an affine line.*

Proof. Suppose that Π is the projective extension of a translation plane of order n. Then $n = t^j$ where t is a prime and $j \geq 1$. Assume that $n = 2q+1$, $q \equiv 3 \bmod 4$, $q \neq 7$, $|\mathcal{O}| = q+1$ and $SL(2,q) \trianglelefteq G$. Then $j = 1$ by lemma 10 (1). So Π is Desarguesian. Then $PSL(2,q) \leq PGL(2,n)$, since G induces the group $PSL(2,q)$ on l. Since $PSL(2,q)$ contains non trivial elements fixing 4 points on l, namely 2 points on $\mathcal{O}$ and 2 points on $\mathcal{O}'$, we have a contradiction. Now, assume that $n = 2(q+1)$, $q \equiv 3 \bmod 4$, $|\mathcal{O}| = q+1$ and $SL(2,q) \trianglelefteq G$. Then $n = 2^j$ and q is a Mersenne prime by lemma 10 (1). Clearly $G \leq P\Gamma L(2h, 2^{j_1})$, with $j = j_1 h$, since G leaves invariant the line at infinity. Thus $SL(2,q) \leq PSL(2h, 2^{j_1})$, since $SL(2,q) \trianglelefteq G$. Then $2h > (q-1)/2$ by [43], proposition 5.3.2 and theorem 5.3.9, since $q \neq 5, 9$ as q is a Mersenne prime. Hence $2(q+1) \geq 2^{(q-1)/4}$, since $n = 2^j$ and $n = 2(q+1)$. An easy computation shows that the previous inequality is impossible for $q > 19$, since q

is odd. Hence $q \leq 19$. Actually $q = 7$, since q must be a Mersenne prime. Thus $SL(2,7) \trianglelefteq G$ and $n = 16$. A direct inspection of the list of the full collineation groups of all translation planes of order 16 given in [16] (see also [54]) rules out this case. So, the cases (2b) and (3b) of theorem 1 cannot occur when Π is the projective extension of a translation plane of order n. Thus the assertion follows by theorem 1, since $n > v \geq n/2$. □

Theorem 31 *Let Π be the projective extension of a translation plane of order n and let $\mathcal{O}$ be a 2-transitive G-orbit of length v on a line l, with $n > v \geq n/2$. If G is of affine type, G acts faithfully on $\mathcal{O}$ and $v \notin \{5^2, 7^2, 11^2, 29^2, 59^2\}$, then $G \leq A\Gamma L(1,v)$. Furthermore one of the following occurs:*

1. *$n = 2v - 1$, $v = p^d$. In particular, either $(p^d, n) = (13^4, 239^2)$ or $n = t^{2^s}$ and $d \leq 2$;*
2. *$n = 2v$, $v = 2^d$.*

Proof. Suppose that Π is the projective extension of a translation plane. Then $n = t^j$, $j \geq 1$, for some prime t. Assume that $\mathcal{O}$ is a 2-transitive G-orbit of length v on a line, with $n > v \geq n/2$ and $v \notin \{5^2, 7^2, 11^2, 29^2, 59^2\}$. Assume also that G is a collineation group of Π of affine type acting faithfully on $\mathcal{O}$. Then $G \leq A\Gamma L(1,v)$ and n satisfies one of the relations (1)-(4) given in theorem 27. Assume that $n = v + 1$ with $v = 2^d$ or $v \equiv 3 \bmod 4$. If $v = 2^d$ then either $d = 3$ and $n = 9$ or n is a Fermat prime by [55], result (B1.1). If $n = 9$, then Π is either Desarguesian or one of the Hall planes. Nevertheless these planes cannot occur since $7 \mid |G|$, as G is 2-transitive on $\mathcal{O}$, $\mathcal{O} \subset l$ and $v = 8$. Thus n is a Fermat prime and hence Π is Desarguesian. Then $G \leq PGL(2,n)$. Thus $2^d(2^d - 1) \mid 2^d(2^d + 1)(2^d + 2)$, since G is 2-transitive on $\mathcal{O}$ and $n = 2^d + 1$. A contradiction, since $n > 5$ by our assumptions.

Assume that $n - \sqrt{n} + 1 = v$ and $v = p$. Then $G = AGL(1,p)$. In this case the group G fixes $l - \mathcal{O}$ pointwise and the element ρ in G of order p is planar by theorem 27 (see its proof). Hence ρ fixes a subplane of Π of order $t^j - p$ as $n = t^j$, $j \geq 1$. Then $t^j - p \mid t^j$ as Π is a translation plane. So $t = p = 2$ and $n = t^j = 4$. A contradiction, since $n - \sqrt{n} + 1 = p$. Thus either $n = 2v - 1$ with $v = p^d$, or $n = 2v$ with $v = 2^d$ by theorem 27. The second case leads to the assertion (2). Hence, assume that $n = 2v - 1$ with $v = p^d$. Then $2p^d = t^j + 1$ as $n = t^j$. Assume that j is not a power of 2. Then $t^h + 1 \mid t^j + 1$ for some integer $1 \leq h < j$. Then $p \mid t^h + 1$ and hence $t^j + 1$ has not primitive prime divisors for $j > 1$, as $2p^d = t^j + 1$. Then $t^j = 8$ by [55], result (P1.7)(ii). A contradiction, since t must be odd. Therefore $j = 1$ and hence $t = 2p^d - 1$. So, Π is Desarguesian. Then $G \leq PGL(2,t)$, since G leaves invariant l. Since $p^d = (t+1)/2$ and the Sylow p-subgroup of G is normal in G, then $G \leq N_{PGL(2,t)}(Z_{t+1})$. So $p^d(p^d - 1) \mid 2(t+1)$, since G is 2-transitive on $\mathcal{O}$. Then $p^d - 1 \mid 4$, since $t = 2p^d - 1$. Hence $p^d = 5$ and $n = 9$, since $v \geq 5$. A contradiction, since $n = t$ with t prime. Hence $j = 2^h$, $h \geq 0$. If $d > 2$, then $(p^d, t^j) = (13^4, 239^2)$ by [55], result (A11.1) and result of page 141. Thus the assertion (1). □

We remark that, while there are no known examples corresponding to the case (1) of the previous theorem, the example 9 is an example for the case (2). Indeed (2) and (6) in example 7 does not correspond to the case (1) of the previous theorem, but correspond to particular cases of theorem 29. Finally, we remark that there are no improvements of theorem 29 when Π is the projective extension of a translation plane.

References

[1] M. Aschbacher, Finite group theory, (Cambridge University Press 1996).

[2] H. Bender, 'Endliche zweifach transitive permutationsgruppen, deren involutionen keine fixpunkte haben', *Math. Z.* **104** (1968), 175-204.

[3] H. Bender, 'Transitive gruppen gerader ordnung, in denen jede involution genau einen puntk festläßt', *J. Algebra* **17** (1971), 527-554.

[4] M. Biliotti, G. Korchmáros, Some new results on collineation groups preserving an oval of a finite projective plane, Combinatorics '88, Vol. 1 (Ravello, 1988), 159-170, in Res. Lecture Notes Math., Mediterranean, Rende, 1991.

[5] M. Biliotti, V. Jha, N. L. Johnson, 'The collineation group of generalized twisted fields planes', *Geom. Dedicata* **76** (1999), 97-126.

[6] M. Biliotti, N. L. Johnson, 'The Non-Solvable Rank 3 Affine Planes', *J. Combin. Theory. Ser. A* **93** (2000), 201-230.

[7] M. Biliotti, E. Francot, 'Two-transitive orbits in finite projective planes', *J. Geom.* **82** (2005), 1-24.

[8] M. Biliotti, A. Montinaro, 'Finite projective planes of order n with a 2-transitive orbit of length $n-3$', *Adv. Geom.* **6** (2005), 15-37.

[9] B.N. Cooperstein, 'Minimal degree for a permutation representation of a classical group', *Israel J. Math.* **30** (1978), 213-235.

[10] J. Cofman, 'Double transitivity in finite affine and projective planes', *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* (8) 43 (1967), 317-320.

[11] J. Cofman, 'On a conjecture of Hughes', *Proc. Camb. Phil. Soc.* **63** (1967), 647-652.

[12] J. H. Conway, R. T. Curtis, R. A. Parker, R. A. Wilson, Atlas of Finite Groups. Maximal subgroups and ordinary characters for simple groups (Oxford University Press 1985).

[13] T. Czerwinski, 'Finite translation planes with a collineation groups doubly transitive on the points at infinity', *J. Algebra* **22** (1972), 428-441.

[14] T. Czerwinski, 'On collineation groups that fix a line of a finite projective plane', *Illinois J. Math.* **16** (1977), 221-230.

[15] U. Dempwolff, 'The projective planes of order 16 admitting $SL(3,2)$', *Rad. Mat.* **7** (1991), 123-134.

[16] U. Dempwolff, A. Reifart, 'The classification of the translation planes of order 16. I', *Geom. Dedicata* **15** (1984), 137-153.

[17] J. D. Dixon, B. Mortimer, Permutation groups (Springer Verlag, New York, 1966).

[18] D. A. Foulser, 'Solvable flag transitive affine groups', *Math. Z.* **86** (1964), 191-204.

[19] M. J. Ganley, V. Jha, 'On translation planes with a 2-transitive orbit on the line at infinity', *Arch. Math. (Basel)* **47** (1986), 379-384.

[20] M. J. Ganley, V. Jha, N. L. Johnson, 'The translation planes admitting a nonsolvable doubly transitive line-sized orbit', *J. Geom.* 69 (2000), 88-109.

[21] The GAP Group, 'GAP - Groups, Algorithms, and Programming', Version 4.3; (2002) (http://www.gap-system.org).

[22] G. Glauberman, 'Central elements in Core-free groups', *J. Algebra* **4** (1966), 403-420.

[23] D. Gorestein, Finite groups (Chelsea Publishing Company, New York, 1980).

[24] D. Gorestein, J. H. Walter, 'The characterization of finite groups with dihedral Sylow 2-subgroups I', *J. Algebra* **2** (1965), 85-151.

[25] R. W. Hartley, 'Determination of the ternary collineation groups whose coefficients lie in $GF(2^n)$', *Ann. Math.* ***27*** (1926), 140-158.

[26] C. Hering, 'Eine Bemerkung über Automorphismengruppen von endlichen projektiven Ebenen und Möbiusebenen', *Arch. Math. (Basel)* **18** (1967), 107-110.

[27] C. Hering, 'On Involutorial Elations of Projective Planes', *Math. Z.* **132** (1973), 91-97.

[28] C. Hering, 'Transitive linear groups and linear groups which contain irreducible subgroups of prime order', *Geom. Dedicata* ***2*** (1974), 425-460.

[29] Y. Hiramine, 'On finite affine planes with a 2-transitive orbit on l_∞', *J. Algebra* **162** (1993), 392-409.

[30] C. Y. Ho, 'Involutory collineations of finite planes', *Math. Z.* **193** (1986), 235-240.

[31] C. Y. Ho, 'Projective planes of order 15 and other odd composite orders', *Geom. Dedicata* **27** (1988), 49-64.

[32] C. Y. Ho, A. Gonçalves, 'On totally irregular simple collineation groups', *Advances in finite geometries and designes* (Chewood gate, Oxford Univ. Press, New York, 1991), 177-193.

[33] D. R. Hughes, F. C. Piper, Projective Planes (Springer Verlag, New York - Berlin, 1973).

[34] B. Huppert, Endliche Gruppen I (Springer Verlag, New York - Berlin, 1967).

[35] B. Huppert, N. Blackburn, Finite Groups III (Springer Verlag, Berlin - Heidelberg - New York, 1982).

[36] Z. Janko, T. Van Trung, 'The full collineation group of any projective plane of order 12 is a $\{2,3\}$-group', *Geom. Dedicata* **12** (1982), 101-110.

[37] N. L. Johnson, 'A note on the derived semifield planes of order 16', *Aequationes Math.* **18** (1978), 103-111.

[38] M. Kallaher, Translation Planes', *Handbook Of Incidence Geometry* (Ed. F. Buekenhout), Elsevier (1995),137-192.

[39] W. M. Kantor, 'On unitary polarities of finite projective planes', *Canad. J. Math.* **23** (1971), 1060-1077.

[40] W. M. Kantor, 'Homogeneus designs and geometric lattices', *J. Combin. Theory. Ser. A* **38** (1985), 66-74.

[41] G. Karpilovsky, The Schur Multiplier (Clarendon Press, Oxford, 1987).

[42] P. B. Kleidman, 'The Maximal Subgroups of the Chevalley Groups $G_2(q)$ with q odd, the Ree Groups ${}^2G_2(q)$, and their Automorphism Groups', *J. Algebra* **117** (1988), 30-71.

[43] P. B. Kleidman, M. Liebeck, The subgroup structure of the finite classical groups (Cambridge University Press, Cambridge, 1990).

[44] G. Korchmáros, 'Collineation groups doubly transitive on the points at infinity in an affine plane of order 2^r', *Arch. Math. (Basel)* **37** (1981), 572-576.

[45] X. Li, 'A characterization of the finite simple groups', *J. Algebra* **245** (2001), 620-649.

[46] M. W. Liebeck, On the order of maximal subgroups of the finite classical groups, *Proc. London Math. Soc.* (3) **50** (1985), 426-446.

[47] I. Matuli'c-Bedeni'c, 'The classification of projective planes of order 11 which possess an involution', *Rad. Mat.* **1** (1985), 149-157.

[48] I. Matuli'c-Bedeni'c, 'The classification of projective planes of order 13 which possess an involution', *Rad. Hrvatske Akad. Znam. Umjet.* **456** (1991), 9-13.

[49] H. H. Mitchell, 'Determination of ordinary and modular ternary linear groups', *Trans. Amer. Math. Soc.* ***12*** (1911), 207-242.

[50] A. Montinaro, The Bounded Cofman's Problem, Ph. D. Thesis, Università degli Studi di Lecce (Italy), June 2005.

[51] B. Mwene, 'On the subgroups of the group $PSL_4(2^m)$', *J. Algebra* **41** (1976), 79-107.

[52] D. S. Passman, Permutation Groups (W. A. Benjamin, Inc., New York -Amsterdam 1968).

[53] T. Penttila, G. F. Royle, M. K. Simpson, 'Hyperovals in the Known Projective Planes of order 16', *J. Combin. Des.* ***4*** (1996), 59-65.

[54] A. Reifart, 'The classification of the translation planes of order 16. II', *Geom. Dedicata* **17** (1984), 1-9.

[55] P. Ribenboim, Catalan's conjecture (Acad. Press, Boston, 1994).

[56] P. Ribenboim, Fermat's last theorem for amateurs, (Springer-Verlag, New York, 1999).

[57] R. H. Schulz, 'Über Translationsebenen mit Kollineationsgruppen, die die Punkte der ausgezeichneten Geraden zweinfach transitiv permutieren', *Math. Z.* **122** (1971), 246-266.

[58] R. Shull, 'Collineations of Projective Planes of Order 9', *J. Combin. Theory. Ser. A* **37** (1984), 99-120.

[59] R. Shull, 'The classification of projective planes of order 9 possessing a collineation group of order 5', *Algebras Groups Geom.* **2** (1985), 365-379.

[60] M. Suzuki, 'On a Class of Doubly Transitive Groups', *Ann. of Math.* (2) **75** (1962), 105-145.

[61] L. Yu, M. Le, 'On the diophantine equation $(x^n - 1)/(x - 1) = y^m$', *Acta Arith.* **21** (1972), 299-301.

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