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On the $\mathbb{K}$-Riemann integral and Hermite–Hadamard inequalities for $\mathbb{K}$-convex functions

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Abstract. In the present paper we introduce a notion of the $\mathbb{K}$-Riemann integral as a natural generalization of a usual Riemann integral and study its properties. The aim of this paper is to extend the classical Hermite–Hadamard inequalities to the case when the usual Riemann integral is replaced by the $\mathbb{K}$-Riemann integral and the convexity notion is replaced by $\mathbb{K}$-convexity.

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1. Introduction

Throughout this paper $I \subseteq \mathbb{R}$ stands for an interval and $\mathbb{K}$ denotes a subfield of the field of real numbers $\mathbb{R}$. Clearly, $\mathbb{Q} \subseteq \mathbb{K}$, where $\mathbb{Q}$ denotes the field of rational numbers. We denote the set of the positive elements of $\mathbb{K}$ by $\mathbb{K}_+$. In the sequel the symbol $[a, b]_A$ will denote an A-convex hull of the set $\{a, b\}$, where $A \subseteq \mathbb{R}$ i.e.

$$[a, b]_A = \{\alpha a + (1 - \alpha)b : \alpha \in A \cap [0, 1]\}.$$

In the case when $A = \mathbb{R}$ we will use the standard symbol $[a, b]$ instead of $[a, b]_{\mathbb{R}}$.

Definition 1. A mapping $f : \mathbb{R} \to \mathbb{R}$ is called additive if it satisfies Cauchy's functional equation

$$f(x + y) = f(x) + f(y),$$

for every $x, y \in \mathbb{R}$. A mapping f is called $\mathbb{K}$-linear if f is additive and $\mathbb{K}$-homogeneous i.e.

$$f(\alpha x) = \alpha f(x),$$

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is fulfilled for every $x \in \mathbb{R}$ and $\alpha \in \mathbb{K}$.

It is well-known that every additive function is $\mathbb{Q}$-homogeneous.

Definition 2. A function $f : I \to \mathbb{R}$ is said to be Jensen-convex if

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2},$$

for every $x, y \in I$. A map f is called $\mathbb{K}$-convex if

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y),$$

for every $x, y \in I$ and $\alpha \in \mathbb{K} \cap (0,1)$.

It is known that a given function f is Jensen-convex if and only if it is $\mathbb{Q}$-convex (see [2,9]). On the other hand, if f is $\mathbb{K}$-convex then it is also $\mathbb{Q}$-convex.

In this place we introduce the following definitions

Definition 3. A function $f : I \to \mathbb{R}$ is called *radially* $\mathbb{K}$-*continuous* at a point $x_0 \in I$ if for every $u \in I$

$$\lim_{\mathbb{K}_+ \ni \alpha \to 0} f((1-\alpha)x_0 + \alpha u) = f(x_0).$$

We say that f is *radially* $\mathbb{K}$-*continuous* if it is *radially* $\mathbb{K}$-*continuous* at every point from the domain.

Definition 4. We say that a function $f : I \to \mathbb{R}$ is *uniformly radially* $\mathbb{K}$-*continuous* if for any $x_0 \in I$ and $u \in I$ the mapping

$$[0,1] \cap \mathbb{K} \ni \alpha \longrightarrow f(x_0 + \alpha(u - x_0))$$

is uniformly continuous.

It is easy to see that any continuous and any uniformly continuous function $f : I \to \mathbb{R}$ in the usual sense is *radially* $\mathbb{K}$-*continuous,* and *uniformly radially* $\mathbb{K}$-*continuous*, respectively. However, it can happen that a *uniformly radially* $\mathbb{K}$-*continuous* function is discontinuous at every point in the usual sense. An easy example is provided by any discontinuous $\mathbb{K}$-linear map. On the other hand, every *uniformly radially* $\mathbb{K}$-*continuous* function is also *radially* $\mathbb{K}$-*continuous,* but the converse is not true. We start with the following easy-to-prove propositions.

Proposition 5. *A function* $f : I \to \mathbb{R}$ *is radially* $\mathbb{K}$-*continuous if and only if for every* $a, b \in I$ *the function* $f_{|_{[a,b]_{\mathbb{K}}}}$ *is continuous.*

Proposition 6. *A function* $f : I \to \mathbb{R}$ *is uniformly radially* $\mathbb{K}$-*continuous if and only if for any* $a, b \in I$ *the map* $f_{|_{[a,b]_{\mathbb{K}}}}$ *is uniformly continuous.*

2. Construction of the $\mathbb{K}$-Riemann integral

Now, we introduce a notion of the $\mathbb{K}$-Riemann integral as a natural generalization of the classical Riemann integral. For the theory of the classical Riemann integral see for instance [10,14,15].

Let $\mathcal{P}_{[a,b]}$ denote the set of partitions of the interval $[a,b]$ i.e.

$$\mathcal{P}_{[a,b]} := \bigcup_{n=1}^{\infty} \{(t_0, t_1, \ldots, t_n) : a = t_0 < t_1 < \cdots < t_n = b\}.$$

Following Zs. Páles [12] we define the set of $\mathbb{K}$-partitions of the interval $[a,b]$ in the following way

$$\begin{aligned} \mathcal{P}^{\mathbb{K}}_{[a,b]} &:= \left\{(t_0, t_1, \ldots, t_n) \in \mathcal{P}_{[a,b]} : \frac{t_i - a}{b-a} \in \mathbb{K},\ i = 1, 2, \ldots, n\right\} \\ &= \Big\{(t_0, t_1, \ldots, t_n) \in \mathcal{P}_{[a,b]} : t_i = a + \alpha_i(b-a) : \alpha_i \\ &\quad \in \mathbb{K} \cap [0,1], i = 1, 2, \ldots, n\Big\} \\ &= \left\{(t_0, t_1, \ldots, t_n) \in \mathcal{P}_{[a,b]} : t_i \in [a,b]_{\mathbb{K}},\ i = 1, 2, \ldots, n\right\}. \end{aligned} \tag{1}$$

Now, suppose that $f : [a,b] \to \mathbb{R}$ is a bounded function on the set $[a,b]_{\mathbb{K}}$ with

$$M := \sup_{x \in [a,b]_{\mathbb{K}}} f(x), \qquad m := \inf_{x \in [a,b]_{\mathbb{K}}} f(x).$$

For a given $\mathbb{K}$-partition $\pi = (t_0, t_1, \ldots, t_n) \in \mathcal{P}^{\mathbb{K}}_{[a,b]}$ let

$$M_i := \sup_{x \in [t_{i-1}, t_i]_{\mathbb{K}}} f(x), \qquad m_i := \inf_{x \in [t_{i-1}, t_i]_{\mathbb{K}}} f(x), \quad i = 1, 2, \ldots, n.$$

These suprema and infima are well-defined, finite real numbers since f is bounded on $[a,b]_{\mathbb{K}}$. Moreover,

$$m \le m_i \le M_i \le M, \quad i = 1, 2, \ldots, n.$$

We define the upper $\mathbb{K}$-Riemann sum of f with respect to the partition π by

$$U_{\mathbb{K}}(f, \pi) := \sum_{i=1}^{n} M_i (t_i - t_{i-1}),$$

and the lower $\mathbb{K}$-Riemann sum of f with respect to the partition π by

$$L_{\mathbb{K}}(f, \pi) := \sum_{i=1}^{n} m_i (t_i - t_{i-1}).$$

Note that

$$m(b-a) \le L_{\mathbb{K}}(f, \pi) \le U_{\mathbb{K}}(f, \pi) \le M(b-a).$$

Now, we define the upper $\mathbb{K}$-Riemann integral of f on $[a,b]$ by

$$\overline{\int_a^b} f(t)d_{\mathbb{K}}t := \inf\left\{U_{\mathbb{K}}(f,\pi):\ \pi\in\mathcal{P}^{\mathbb{K}}_{[a,b]}\right\}$$

and the lower $\mathbb{K}$-Riemann integral by

$$\underline{\int_a^b} f(t)d_{\mathbb{K}}t := \sup\left\{L_{\mathbb{K}}(f,\pi):\ \pi\in\mathcal{P}^{\mathbb{K}}_{[a,b]}\right\}.$$

Definition 7. A function $f:[a,b]\to\mathbb{R}$ bounded on $[a,b]_{\mathbb{K}}$ is said to be $\mathbb{K}$-Riemann integrable on $[a,b]$ if its upper and lower integrals are equal. In that case, the $\mathbb{K}$-Riemann integral of f on $[a,b]$ is denoted by

$$\int_a^b f(t)d_{\mathbb{K}}t.$$

In the case when $\mathbb{K}=\mathbb{R}$ we will use the standard symbol $\int_a^b f(t)dt$ instead of $\int_a^b f(t)d_{\mathbb{R}}t$.

The following theorem gives a criterion for $\mathbb{K}$-Riemann integrability.

Theorem 8. *A function $f:[a,b]\to\mathbb{R}$ is $\mathbb{K}$-Riemann integrable on $[a,b]$ if and only if for every $\varepsilon>0$ there exists a partition $\pi\in\mathcal{P}^{\mathbb{K}}_{[a,b]}$ such that*

$$U_{\mathbb{K}}(f,\pi)-L_{\mathbb{K}}(f,\pi)<\varepsilon.$$

Proof. Let $\varepsilon>0$ and choose a partition $\pi\in\mathcal{P}^{\mathbb{K}}_{[a,b]}$ that satisfies the above condition. Then, since

$$\overline{\int_a^b} f(t)d_{\mathbb{K}}t\le U_{\mathbb{K}}(f,\pi),\quad\text{and}\quad L_{\mathbb{K}}(f,\pi)\le\underline{\int_a^b} f(t)d_{\mathbb{K}}t\le U_{\mathbb{K}}(f,\pi)$$

we have

$$0\le\overline{\int_a^b} f(t)d_{\mathbb{K}}t-\underline{\int_a^b} f(t)d_{\mathbb{K}}t\le U_{\mathbb{K}}(f,\pi)-L_{\mathbb{K}}(f,\pi)<\varepsilon.$$

Since this inequality holds for every $\varepsilon>0$,

$$\overline{\int_a^b} f(t)d_{\mathbb{K}}t=\underline{\int_a^b} f(t)d_{\mathbb{K}}t.$$

Conversely, suppose that f is $\mathbb{K}$-Riemann integrable. Given any $\varepsilon>0$, there are partitions $\pi_1,\pi_2\in\mathcal{P}^{\mathbb{K}}_{[a,b]}$ such that

$$U_{\mathbb{K}}(f,\pi_1)<\overline{\int_a^b} f(t)d_{\mathbb{K}}t+\frac{\varepsilon}{2},\qquad L_{\mathbb{K}}(f,\pi_2)>\underline{\int_a^b} f(t)d_{\mathbb{K}}t-\frac{\varepsilon}{2}.$$

Now, let $\pi := \pi_1 \cup \pi_2$ be the common refinement. Keeping in mind that the $\mathbb{K}$-Riemann integrability of f means $\overline{\int}_a^b f(t)d_{\mathbb{K}}t = \underline{\int}_a^b f(t)d_{\mathbb{K}}t$ we can write

$$\begin{aligned} U_{\mathbb{K}}(f,\pi) - L_{\mathbb{K}}(f,\pi) &\leq U_{\mathbb{K}}(f,\pi_1) - L_{\mathbb{K}}(f,\pi_2) \\ &= \Big(U_{\mathbb{K}}(f,\pi_1) - \overline{\int_a^b} f(t)d_{\mathbb{K}}t\Big) + \Big(\underline{\int_a^b} f(t)d_{\mathbb{K}}t - L_{\mathbb{K}}(f,\pi_2)\Big) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Using the above theorem we can easily obtain the following

Corollary 9. *A function $f : [a,b] \to \mathbb{R}$ is $\mathbb{K}$-Riemann integrable on $[a,b]$ if and only if for every sequence $\{\pi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}^{\mathbb{K}}_{[a,b]}$, $\pi_n = (t_0^{(n)}, t_1^{(n)}, \ldots, t_{k_n}^{(n)})$ such that*

$$\max_{1\leq j\leq k_n}\left(t_j^{(n)} - t_{j-1}^{(n)}\right) \to_{n\to\infty} 0,$$

and for any choice $s_j^{(n)} \in \left[t_{j-1}^{(n)}, t_j^{(n)}\right]_{\mathbb{K}}$ of the partition π_n we have

$$\int_a^b f(t)d_{\mathbb{K}}t = \lim_{n\to\infty}\sum_{j=1}^{k_n} f\left(s_j^{(n)}\right)\left(t_j^{(n)} - t_{j-1}^{(n)}\right).$$

Proposition 10. *Let $\mathbb{K}_1 \subseteq \mathbb{K}_2$ be subfields of $\mathbb{R}$. If a function $f : [a,b] \to \mathbb{R}$ is $\mathbb{K}_2$-Riemann integrable then it is also $\mathbb{K}_1$-Riemann integrable, and*

$$\int_a^b f(t)d_{\mathbb{K}_1}t = \int_a^b f(t)d_{\mathbb{K}_2}t.$$

Proof. Let $\pi_n = \left(t_0^{(n)}, t_1^{(n)}, \ldots, t_{k_n}^{(n)}\right) \in \mathcal{P}^{\mathbb{K}_1}_{[a,b]}$, $n \in \mathbb{N}$ be an arbitrary sequence such that

$$\max_{1\leq j\leq k_n}\left(t_j^{(n)} - t_{j-1}^{(n)}\right) \to_{n\to\infty} 0.$$

By the $\mathbb{K}_2$-Riemann integrability, for any choice $s_j^{(n)} \in [t_{j-1}^{(n)}, t_j^{(n)}]_{\mathbb{K}_1}$ of the partition π_n we have

$$\int_a^b f(t)d_{\mathbb{K}_2}t = \lim_{n\to\infty}\sum_{j=1}^{k_n} f\left(s_j^{(n)}\right)\left(t_j^{(n)} - t_{j-1}^{(n)}\right).$$

Due to the arbitrariness of $\pi_n \in \mathcal{P}^{\mathbb{K}_1}_{[a,b]}$ we infer that

$$\int_a^b f(t)d_{\mathbb{K}_1}t = \lim_{n\to\infty}\sum_{j=1}^{k_n} f\left(s_j^{(n)}\right)\left(t_j^{(n)} - t_{j-1}^{(n)}\right).$$

□

As an immediate consequence of the above proposition we obtain the following.

Corollary 11. *If a function $f : [a,b] \to \mathbb{R}$ is Riemann integrable in the usual sense, then for an arbitrary field $\mathbb{K} \subseteq \mathbb{R}$ f is $\mathbb{K}$-Riemann integrable, moreover,*

$$\int_a^b f(t)d_{\mathbb{K}}t = \int_a^b f(t)dt.$$

Example 1. Let $\mathbb{K}_1 \subseteq \mathbb{K}_2$, $\mathbb{K}_1 \neq \mathbb{K}_2$ be two subfields of $\mathbb{R}$. Consider the following function $f : [a,b] \to \mathbb{R}$

$$f(x) = \begin{cases} 0, & x \in [a,b]_{\mathbb{K}_1} \\ 1, & x \in [a,b]_{\mathbb{K}_2} \setminus [a,b]_{\mathbb{K}_1}. \end{cases}$$

It is easy to observe that f is $\mathbb{K}_1$-Riemann integrable, and $\int_a^b f(t)d_{\mathbb{K}_1}t = 0$. On the other hand for every partition $\pi \in \mathcal{P}_{[a,b]}^{\mathbb{K}_2} \setminus \mathcal{P}_{[a,b]}^{\mathbb{K}_1}$ one can check that

$$S_{\mathbb{K}_2}(\pi, f) = 1, \quad \text{and} \quad L_{\mathbb{K}_2}(\pi, f) = 0.$$

Therefore,

$$0 = \underline{\int_a^b} f(t)d_{\mathbb{K}_2}t \neq \overline{\int_a^b} f(t)d_{\mathbb{K}_2}t = 1.$$

Observe that if we replace in the formula on f the set $\mathbb{K}_1$ by the set D of diadic numbers from the interval $[0,1]$ i.e.

$$D := \left\{x \in [0,1] \mid x = \frac{k}{2^n},\ k \in \mathbb{Z},\ n \in \mathbb{N}\right\},$$

then we obtain an example of a function which is non-$\mathbb{K}$-Riemann integrable for any subfield $\mathbb{K} \subseteq \mathbb{R}$.

3. Properties of the $\mathbb{K}$-Riemann integral

We start our investigation with the following.

Proposition 12. *If $f : [a,b] \to \mathbb{R}$ is a function such that $f_{|[a,b]_{\mathbb{K}}}$ is monotone then it is $\mathbb{K}$-Riemann integrable on $[a,b]$.*

Proof. Assume that $f_{|[a,b]_{\mathbb{K}}}$ is monotonic increasing, meaning that

$$f(x) \leq f(y), \quad \text{for } x \leq y, \quad x, y \in [a,b]_{\mathbb{K}}.$$

Fix an arbitrary sequence of partitions

$$\pi_n = \left(t_0^{(n)}, t_1^{(n)}, \ldots, t_{k_n}^{(n)}\right) \in \mathcal{P}_{[a,b]}^{\mathbb{K}}, \quad n \in \mathbb{N},$$

where

$$\max_{1\leq j\leq k_n}\left(t_j^{(n)}-t_{j-1}^{(n)}\right)\rightarrow_{n\to\infty} 0.$$

Since $f_{|[a,b]_{\mathbb{K}}}$ is increasing, for all $j\in\{1,\dots,k_n\}$

$$M_j:=\sup_{t\in[t_{j-1},t_j]_{\mathbb{K}}} f(t)=f(t_j),\qquad m_j:=\inf_{t\in[t_{j-1},t_j]_{\mathbb{K}}} f(t)=f(t_{j-1}).$$

Hence, summing a telescoping series, we get

$$\begin{aligned}U(f,\pi_n)-L(f,\pi_n)&=\sum_{j=1}^{k_n}(M_j-m_j)(t_j^{(n)}-t_{j-1}^{(n)})\\&\leq \max_{1\leq j\leq k_n}\left(t_j^{(n)}-t_{j-1}^{(n)}\right)\sum_{j=1}^{k_n}[f(t_j)-f(t_{j-1})]\\&=\max_{1\leq j\leq k_n}\left(t_j^{(n)}-t_{j-1}^{(n)}\right)[f(b)-f(a)].\end{aligned}$$

It follows that $U(f,\pi_n)-L(f,\pi_n)\to 0$ as $n\to\infty$ and Corollary 9 implies that f is $\mathbb{K}$-Riemann integrable. The proof for a monotonic decreasing function f is similar. □

In our next result we use a well-known fact from mathematical analysis that every uniformly continuous function on a set $A\subset\mathbb{R}^n$ can be uniquely extended onto clA to a continuous function (see for instance [4] p. 206).

Proposition 13. *If $f:[a,b]\to\mathbb{R}$ is uniformly radially $\mathbb{K}$-continuous, then it is $\mathbb{K}$-Riemann integrable on any subset $[c,d]\subset[a,b]$.*

Proof. Fix arbitrary $c,d\in[a,b],\ c<d$. From Proposition 6 we infer that $f_{|[c,d]_{\mathbb{K}}}$ is uniformly continuous. Since $cl([c,d]_{\mathbb{K}})=[c,d]$, there exists a unique continuous function $g_{cd}:[c,d]\to\mathbb{R}$ such that

$$g_{cd}(t)=f(t),\quad t\in[c,d]_{\mathbb{K}}.$$

On account of Corollary 11 f is $\mathbb{K}$-Riemann integrable, moreover,

$$\int_c^d f(t)d_{\mathbb{K}}t=\int_c^d g_{cd}(t)dt.$$

□

In the sequel we will use the following well-known theorems (see [8] p.147) (actually these theorems were proved for Jensen-convex functions, but the proof in our case runs without any essential changes).

Theorem 14. *Let $I\subseteq\mathbb{R}$ be an open interval, and let $f:I\to\mathbb{R}$ be a $\mathbb{K}$-convex function. Then for arbitrary $a,b\in I,\ a<b$ the function $f_{|[a,b]_{\mathbb{K}}}$ is uniformly continuous.*

Theorem 15. *Let $I \subseteq \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a $\mathbb{K}$-convex function. Then for arbitrary $a, b \in I$, $a < b$ there exists a unique continuous function $g_{ab} : [a, b] \to \mathbb{R}$ such that*

$$g_{ab}(x) = f(x), \quad x \in [a, b]_{\mathbb{K}}.$$

The function g_{ab} satisfies the inequality

$$g_{ab}\left(\frac{a+b}{2}\right) \leq \frac{g_{ab}(x) + g_{ab}(y)}{2},$$

for every $x, y \in [a, b]$, in particular g_{ab} is a convex function.

Now, we calculate an integral of a $\mathbb{K}$-linear function. Note that such a function can be discontinuous at every point and non-measurable in the Lebesgue sense (see [8]), so the usual Riemann integral may not exist.

Proposition 16. *Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathbb{K}$-linear function. Then it is a $\mathbb{K}$-Riemann integrable on every interval $[a, b]$, moreover,*

$$\int_a^b f(t) d_{\mathbb{K}} t = f\left(\frac{a+b}{2}\right)(b-a).$$

Proof. Suppose that f is a $\mathbb{K}$-linear function. On account of Proposition 13 and Theorem 14 it is $\mathbb{K}$-Riemann integrable on every interval $[a, b]$. Consider the following sequence of partitions:

$$\pi_n := \left(t_0^{(n)}, t_1^{(n)}, \ldots, t_n^{(n)}\right), \text{ where, } \quad t_j^{(n)} := a + \frac{j}{n}(b-a), \quad j = 0, 1, \ldots, n.$$

From Corollary 9 we obtain

$$\begin{aligned}
\int_a^b f(t) d_{\mathbb{K}} t &= \lim_{n\to\infty} \sum_{j=1}^{n} f\left(t_j^{(n)}\right) \frac{1}{n}(b-a) \\
&= \lim_{n\to\infty} f\left(na + \frac{n(n+1)}{2n}(b-a)\right) \frac{1}{n}(b-a) \\
&= \lim_{n\to\infty} f\left(a + \frac{n+1}{2n}(b-a)\right)(b-a) \\
&= \lim_{n\to\infty} \left[f(a) + \frac{n+1}{2n} f(b-a)\right](b-a) \\
&= \left[f(a) + f\left(\frac{b-a}{2}\right)\right](b-a) = f\left(\frac{a+b}{2}\right)(b-a).
\end{aligned}$$

□

Now, we record some basic properties of $\mathbb{K}$-Riemann integration. We omit the proofs of these properties because they run in a similar way as for the usual Riemann integral.

Theorem 17. *Let f, g be $\mathbb{K}$-Riemann integrable on $[a, b]$ and let $c, d \in \mathbb{R}$. Then*

(i) *the function $cf + dg$ is $\mathbb{K}$-Riemann integrable on $[a, b]$, moreover,*

$$\int_a^b [cf(t) + dg(t)]d_{\mathbb{K}}t = c\int_a^b f(t)d_{\mathbb{K}}t + d\int_a^b g(t)d_{\mathbb{K}}t.$$

(ii) *If $f(t) \geq 0,\ t \in [a, b]_{\mathbb{K}}$, then $\int_a^b f(t)d_{\mathbb{K}}t \geq 0$, moreover, if f is radially $\mathbb{K}$-continuous on $[a, b]$ and $f(t) > 0,\ t \in [a, b]_{\mathbb{K}}$ then*

$$\int_a^b f(t)d_{\mathbb{K}}t > 0.$$

(iii) *The absolute value $|f|$ is $\mathbb{K}$-Riemann integrable on $[a, b]$ and*

$$\Big|\int_a^b f(t)d_{\mathbb{K}}t\Big| \leq \int_a^b |f(t)|d_{\mathbb{K}}t.$$

Theorem 18. *Suppose that $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]_{\mathbb{K}}$. Then f is $\mathbb{K}$-Riemann integrable on $[a, b]$ if and only if it is $\mathbb{K}$-Riemann integrable on $[a, c]$ and $[c, b]$. Moreover, in that case,*

$$\int_a^b f(t)d_{\mathbb{K}}t = \int_a^c f(t)d_{\mathbb{K}}t + \int_c^b f(t)d_{\mathbb{K}}t.$$

Proof. Assume that f is $\mathbb{K}$-Riemann integrable on $[a, b]$. Then, given $\varepsilon > 0$ there is a partition $\pi \in \mathcal{P}^{\mathbb{K}}_{[a,b]}$ such that

$$U_{\mathbb{K}}(f, \pi) - L_{\mathbb{K}}(f, \pi) < \varepsilon.$$

Let $\overline{\pi}$ be the refinement of π obtained by adding c to the endpoints of π. Then $\overline{\pi} = \pi_1 \cup \pi_2$, where

$$\pi_1 := \overline{\pi} \cap [a, c]_{\mathbb{K}}, \qquad \pi_2 := \overline{\pi} \cap [c, b]_{\mathbb{K}}.$$

Obviously, $\pi_1 \in \mathcal{P}^{\mathbb{K}}_{[a,c]}$ and $\pi_2 \in \mathcal{P}^{\mathbb{K}}_{[c,b]}$, moreover,

$$U_{\mathbb{K}}(f, \overline{\pi}) = U_{\mathbb{K}}(f, \pi_1) + U_{\mathbb{K}}(f, \pi_2), \qquad L(f, \overline{\pi}) = L_{\mathbb{K}}(f, \pi_1) + L_{\mathbb{K}}(f, \pi_2).$$

It follows that

$$\begin{aligned} U_{\mathbb{K}}(f, \pi_1) - L_{\mathbb{K}}(f, \pi_1) &= U_{\mathbb{K}}(f, \overline{\pi}) - L_{\mathbb{K}}(f, \overline{\pi}) - [U_{\mathbb{K}}(f, \pi_2) - L_{\mathbb{K}}(f, \pi_2)] \\ &\leq U_{\mathbb{K}}(f, \pi) - L_{\mathbb{K}}(f, \pi) < \varepsilon, \end{aligned}$$

which proves that f is $\mathbb{K}$-Riemann integrable on $[a, c]$. Exchanging π_1 and π_2, we get the proof for $[c, b]$.

Conversely, if f is $\mathbb{K}$-Riemann integrable on $[a, c]$ and $[c, b]$ then there are partitions $\pi_1 \in \mathcal{P}^{\mathbb{K}}_{[a,c]}$ and $\pi_2 \in \mathcal{P}^{\mathbb{K}}_{[c,b]}$ such that

$$U_{\mathbb{K}}(f, \pi_1) - L_{\mathbb{K}}(f, \pi_1) < \frac{\varepsilon}{2}, \qquad U_{\mathbb{K}}(f, \pi_2) - L_{\mathbb{K}}(f, \pi_2) < \frac{\varepsilon}{2}.$$

Let $\pi := \pi_1 \cup \pi_2$. Then

$$U_{\mathbb{K}}(f,\pi) - L_{\mathbb{K}}(f,\pi) = U_{\mathbb{K}}(f,\pi_1) - L_{\mathbb{K}}(f,\pi_1) + U_{\mathbb{K}}(f,\pi_2) - L_{\mathbb{K}}(f,\pi_2) < \varepsilon,$$

which proves that f is $\mathbb{K}$-Riemann integrable on $[a,b]$.

Finally, if f is $\mathbb{K}$-Riemann integrable, then with partitions π, π_1, π_2 as above, we have

$$\begin{aligned}\int_a^b f(t)d_{\mathbb{K}}t &\le U_{\mathbb{K}}(f,\pi) = U_{\mathbb{K}}(f,\pi_1) + U_{\mathbb{K}}(f,\pi_2)\\ &< L_{\mathbb{K}}(f,\pi_1) + L_{\mathbb{K}}(f,\pi_2) + \varepsilon\\ &< \int_a^c f(t)d_{\mathbb{K}}t + \int_c^b f(t)d_{\mathbb{K}}t + \varepsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b f(t)d_{\mathbb{K}}t &\ge L_{\mathbb{K}}(f,\pi) = L_{\mathbb{K}}(f,\pi_1) + L_{\mathbb{K}}(f,\pi_2)\\ &> U_{\mathbb{K}}(f,\pi_1) + U_{\mathbb{K}}(f,\pi_2) - \varepsilon\\ &> \int_a^c f(t)d_{\mathbb{K}}t + \int_c^b f(t)d_{\mathbb{K}}t - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we see that

$$\int_a^b f(t)d_{\mathbb{K}}t = \int_a^c f(t)d_{\mathbb{K}}t + \int_c^b f(t)d_{\mathbb{K}}t.$$

□

Remark 19. Observe that for a $\mathbb{K}$-linear function $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{K} \neq \mathbb{R}$ and a point $c = \alpha a + (1-\alpha)b \in (a,b)$, where $\alpha \in (0,1)$ we have

$$\int_a^b f(t)d_{\mathbb{K}}t - \int_a^c f(t)d_{\mathbb{K}}t - \int_c^b f(t)d_{\mathbb{K}}t = \frac{1}{2}\Big(f(\alpha(a-b)) - \alpha f(a-b)\Big)(a-b).$$

Therefore, it can happen that for some $\alpha \in (0,1) \setminus \mathbb{K}$ the above expression is different from zero.

4. Connections between the radial $\mathbb{K}$-derivative and the $\mathbb{K}$-Riemann integral

In 2006 Z. Boros and Zs. Páles in [1] introduced and examined the notion of radial $\mathbb{K}$-derivative of a map at a point in the given direction.

Definition 20. A map $f : I \to \mathbb{R}$ (I stands for an open interval) is said to have a *radial* $\mathbb{K}$-*derivative* at a point $x \in I$ in the direction $u \in \mathbb{R}$ provided that there exists a finite limit

$$D_{\mathbb{K}} f(x,u) := \lim_{\mathbb{K}_+ \ni r \to 0} \frac{f(x+ru)-f(x)}{r}.$$

We will say that f is *radially* $\mathbb{K}$*-differentiable* at a point x whenever $D_{\mathbb{K}} f(x,u)$ does exist for every $u \in \mathbb{R}$. A function $f : I \to \mathbb{R}$ is termed *radially* $\mathbb{K}$*-differentiable* if f is *radially* $\mathbb{K}$*-differentiable* at every point $x \in \mathbb{R}$.

It is known that each $\mathbb{K}$-convex function $f : I \to \mathbb{R}$ is radially $\mathbb{K}$-differentiable. In particular, such is every $\mathbb{K}$-linear function $a : \mathbb{R} \to \mathbb{R}$ with

$$D_{\mathbb{K}} a(x,u) = a(u), \quad x,u \in \mathbb{R}.$$

On the other hand, if a function $f : I \to \mathbb{R}$ is differentiable in the usual sense at a point $x \in I$ then it is radially $\mathbb{K}$-differentiable at x with

$$D_{\mathbb{K}} f(x,u) = f'(x)u, \quad \text{for } u \in \mathbb{R}.$$

We have the following relationship between the radial $\mathbb{K}$-derivative and the $\mathbb{K}$-Riemann integral.

Theorem 21. *Suppose that $f : [a,b] \to \mathbb{R}$ is $\mathbb{K}$-Riemann integrable on each subset of the form $[a,x]_{\mathbb{K}}$, for any $x \in (a,b]$. Let us define the function $F : [a,b] \to \mathbb{R}$ by the formula*

$$F(x) := \int_a^x f(t) d_{\mathbb{K}} t.$$

Then, if f is radially $\mathbb{K}$-continuous at a point $x \in (a,b]$ then F is radially $\mathbb{K}$-differentiable at x in the direction $x-a$, moreover,

$$D_{\mathbb{K}} F(x, x-a) = f(x)(x-a).$$

Proof. Fix $x \in [a,b]$ and $\varepsilon > 0$ arbitrarily. For $\alpha \in \mathbb{K}_+$, since $x \in [a, x+\alpha(x-a)]_{\mathbb{K}}$, on account of Theorem 18 and condition (iii) from Theorem 17 we obtain

$$\begin{aligned}
&\left| \tfrac{F(x+\alpha(x-a))-F(x)}{\alpha} - f(x)(x-a) \right| \\
&\quad = \left| \tfrac{1}{\alpha} \Big(\int_a^{x+\alpha(x-a)} f(t) d_{\mathbb{K}} t - \int_a^x f(t) d_{\mathbb{K}} t \Big) - \tfrac{1}{\alpha} \int_x^{x+\alpha(x-a)} f(x) d_{\mathbb{K}} t \right| \\
&\quad = \left| \tfrac{1}{\alpha} \int_x^{x+\alpha(x-a)} f(t) d_{\mathbb{K}} t - \tfrac{1}{\alpha} \int_x^{x+\alpha(x-a)} f(x) d_{\mathbb{K}} t \right| \\
&\quad = \tfrac{1}{\alpha} \left| \int_x^{x+\alpha(x-a)} (f(t)-f(x)) d_{\mathbb{K}} t \right| \le \tfrac{1}{\alpha} \int_x^{x+\alpha(x-a)} |f(t)-f(x)| d_{\mathbb{K}} t.
\end{aligned}$$

Let $\alpha \in \mathbb{K}_+$ be so small that

$$|f(t)-f(x)| < \frac{\varepsilon}{x-a}, \quad \text{for } t \in [x, x+\alpha(x-a)]_{\mathbb{K}}.$$

Then,

$$\frac{1}{\alpha}\int_x^{x+\alpha(x-a)} |f(t)-f(x)|d_{\mathbb{K}}t$$
$$\le \frac{1}{\alpha}\int_x^{x+\alpha(x-a)} \frac{\varepsilon}{x-a}d_{\mathbb{K}}t = \frac{1}{\alpha}\cdot\alpha(x-a)\cdot\frac{\varepsilon}{x-a} = \varepsilon.$$

□

Now, we are in a position to prove the following characterization of $\mathbb{K}$-convex functions.

Theorem 22. *Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a $\mathbb{K}$-convex function. Then, for every $a, x \in I$, we have*

$$f(x) = f(a) + \frac{1}{x-a}\int_a^x D_{\mathbb{K}}f(t, x-a)d_{\mathbb{K}}t.$$

Proof. Take arbitrary $a, x \in I$, $a \neq x$, say $a < x$. Since f is a $\mathbb{K}$-convex function, on account of Theorem 15 there exists a uniformly continuous, convex function $g : [a, x] \to \mathbb{R}$ such that

$$f(t) = g(t), \qquad t \in [a, x]_{\mathbb{K}}.$$

Therefore, for $t \in [a, x]_{\mathbb{K}}$ we get

$$D_{\mathbb{K}}f(t, x-a) = \lim_{\mathbb{K}_+\ni\alpha\to 0}\frac{f(t+\alpha(x-a)) - f(t)}{\alpha}$$
$$= (x-a)\cdot\lim_{\mathbb{K}_+\ni\alpha\to 0}\frac{g(t+\alpha(x-a)) - g(t)}{\alpha(x-a)} = (x-a)g'_+(t).$$

It follows from the above formula and from the fundamental theorem of calculus for the usual Riemann integral that

$$\frac{1}{x-a}\int_a^x D_{\mathbb{K}}f(t, x-a)d_{\mathbb{K}}t = \int_a^x g'_+(t)dt = g(x) - g(a) = f(x) - f(a),$$

which was to be proved. □

5. Hermite–Hadamard inequalities

There are many inequalities valid for convex functions. Probably two of the most well-known ones are the Hermite–Hadamard [3,5–8,11,16] inequalities.

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}, \quad a < b. \tag{2}$$

They play an important role in convex analysis. In the literature one can find their various generalizations and applications. For more information on this type of inequalities see the book [3] and the references therein. We just note here that first Hermite [7] published these inequalities with some important applications and then, 10 years later, Hadamard [5] rediscovered their left-hand side.

It turns out that each of the two sides of (2) in fact characterizes convex functions. More precisely, if I is an interval and $f : I \to \mathbb{R}$ a continuous function whose restriction to every compact subinterval $[a, b]$ verifies the left-hand side then f is convex. The same works when the left-hand side is replaced by the right-hand side. More general results are given by Rado [13].

Now, we are in a position to prove our main result. The following theorem establishes the Hermite–Hadamard inequalities for $\mathbb{K}$-convex functions.

Theorem 23. *Let $I \subseteq \mathbb{R}$ be a nonempty open interval and let $f : I \to \mathbb{R}$ be a $\mathbb{K}$-convex function. Then for arbitrary $a, b \in I, a < b$ the inequalities*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(t)d_{\mathbb{K}}t \le \frac{f(a)+f(b)}{2}, \quad a < b, \tag{3}$$

hold.

Proof. Let $a, b \in I, a < b$ be arbitrarily fixed. It follows from Theorem 15 that there exists a unique continuous and convex function $g_{ab} : [a, b] \to \mathbb{R}$ such that

$$g_{ab}(x) = f(x), \quad x \in [a, b]_{\mathbb{K}}.$$

Since g_{ab} is convex, it satisfies the classical Hermite–Hadamard inequalities, namely

$$f\Big(\frac{a+b}{2}\Big) = g_{ab}\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b g_{ab}(t)dt \le \frac{g_{ab}(a)+g_{ab}(b)}{2} = \frac{f(a)+f(b)}{2}.$$

However, on account of Corollary 11

$$\int_a^b g_{ab}(t)dt = \int_a^b f(t)d_{\mathbb{K}}t,$$

which finishes the proof. □

Since in the proof of the above theorem we used the classical Hermite–Hadamard inequalities, we can not say that it is a more general result. Therefore, now we give another proof without using these inequalities.

Proof. To prove the right-hand side of (3) observe that

$$f(x) \le f(a) + \frac{f(b)-f(a)}{b-a}(x-a), \qquad x \in [a, b]_{\mathbb{K}},$$

so, integrating the above inequality over $[a, b]$ and dividing by $b - a$ we get

$$\frac{1}{b-a}\int_a^b f(x)d_{\mathbb{K}}x \le \frac{f(a)+f(b)}{2}.$$

To obtain the left-hand side of (2) we use the following easy-to-prove expression

$$\int_a^b f(t)d_{\mathbb{K}}t = (b-a)\int_0^1 f(sa+(1-s)b)d_{\mathbb{K}}s.$$

Using the above formula and the Jensen-convexity of f we get

$$\begin{aligned}\tfrac{1}{b-a}\textstyle\int_a^b f(t)d_{\mathbb{K}}t &= \tfrac{1}{b-a}\Big(\textstyle\int_a^{\frac{a+b}{2}} f(t)d_{\mathbb{K}}t + \int_{\frac{a+b}{2}}^b f(t)d_{\mathbb{K}}t\Big)\\ &= \tfrac{1}{2}\textstyle\int_0^1 \Big[f\Big(\tfrac{a+b-t(b-a)}{2}\Big)+f\Big(\tfrac{a+b+t(b-a)}{2}\Big)\Big]d_{\mathbb{K}}t \geq f\Big(\tfrac{a+b}{2}\Big).\end{aligned}$$

□

It turns out, that as with convex functions, in the class of uniformly radially $\mathbb{K}$-continuous functions each of the inequalities (3) is equivalent to $\mathbb{K}$-convexity. Namely, the following theorem holds true.

Theorem 24. *If a function $f : I \to \mathbb{R}$ is uniformly radially $\mathbb{K}$-continuous and, for all elements $a < b$ of I, satisfies either the inequality*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)d_{\mathbb{K}}t,$$

or

$$\frac{1}{b-a}\int_a^b f(t)d_{\mathbb{K}}t \leq \frac{f(a)+f(b)}{2},$$

then it is $\mathbb{K}$-convex.

Proof. Suppose that f satisfies the first inequality (for the second inequality the proof runs in a similar way). It is enough to prove that for every $a, b \in I, a < b$ a unique extension g_{ab} of $f_{|[a,b]_{\mathbb{K}}}$ onto $[a, b]$ to a continuous function is convex. To see it, fix arbitrarily $c, d \in [a, b], c < d$. There exist sequences $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ such that $c_n, d_n \in [a, b]_{\mathbb{K}},\ c_n < d_n,\ n \in \mathbb{N}$, and

$$\lim_{n\to\infty} c_n = c, \qquad \lim_{n\to\infty} d_n = d.$$

Since $c_n, d_n \in [a, b]_{\mathbb{K}}$, the extension $g_{c_n d_n}$ onto $[c_n, d_n]$ to a continuous function by virtue of uniqueness satisfies the condition

$$g_{ab}(t) = g_{c_n d_n}(t) = f(t), \quad t \in [c_n, d_n]_{\mathbb{K}}, \quad n \in \mathbb{N}.$$

By the assumption for all $n \in \mathbb{N}$

$$g_{ab}\Big(\frac{c_n+d_n}{2}\Big) = f\Big(\frac{c_n+d_n}{2}\Big) \leq \frac{1}{d_n-c_n}\int_{c_n}^{d_n} f(t)d_{\mathbb{K}}t = \frac{1}{d_n-c_n}\int_{c_n}^{d_n} g_{ab}(t)dt.$$

Taking limits as $n \to \infty$ gives

$$g_{ab}\Big(\frac{c+d}{2}\Big) \leq \frac{1}{d-c}\int_c^d g_{ab}(t)dt.$$

We have shown that a continuous function g_{ab} satisfies the left-hand side of the classical Hermite–Hadamard inequalities, so as we know, it is convex. Due to the arbitrariness of $a, b \in I$ we infer that f is $\mathbb{K}$-convex, which finishes the proof. $\square$

References

[1] Boros, Z.: Zs. Páles, $\mathbb{Q}$-subdifferential of Jensen-convex functions. J. Math. Anal. Appl. **321**, 99–113 (2006)
[2] Daróczy, Z., Páles, Z.: Convexity with given infinite weight sequences. Stochastica **11**, 5–12 (1987)
[3] Dragomir, S.S., Pearce, C.E.M.: Selected Topics on Hermite–Hadamard Inequalities and Applications. Victoria University, RGMIA Monographs (2002)
[4] Giaguinta, M., Modica, G.: Mathematical Analysis. Linear and Metric Structures and Continuity. Springer, New York (2007)
[5] Hadamard, J.: Étude sur les properiétés entiéres et en particulier dúne fonction considerée par Riemann. J. Math. Pures Appl. **58**, 171–215 (1893)
[6] Hardy, G., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press, Cambridge (1951)
[7] Hermite, C.: Sur deux limites dúne intégrale définie. Mathesis **3**, 82 (1883)
[8] Kuczma, M.: An introduction to the theory of functional equations and inequalities. Birkhäuser, Basel (2009)
[9] Kuhn, N.: A note on t-convex functions. Gen. Inequal. **4**, 269–276 (1984)
[10] Kurtz, D.S., Kurzweil, J., Swartz, C.: Theories of Integration: The Integrals of Riemann, Lebesgue, Henstock–Kurzweil, and McShane, Volume 9 of Series in Real Analysis. World Scientific (2004)
[11] Niculescu, C.P., Persson, L.E.: Convex Functions and their Applications, A Contemporary Approach CMS Books in Mathematics, vol. 23. Springer, New York (2006)
[12] Páles, Zs: Problem 2, Report of Meeting, The Thirteenth Katowice-Debrecen Winter Seminar. Ann. Math. Sil. **27**, 107–125 (2013)
[13] Rado, T.: On convex functions. Trans. Am. Math. Soc. **37**, 266–285 (1935)
[14] Riemann, G.F.B.: In: H. Weber (ed.) Gesammelte Mathematische Werke. Dover Publications, New York (1953)
[15] Riemann, G.F.B.: In: K. Hattendorff (ed.) Partielle Differentialgleichungen und deren Anwendung auf physikalische Fragen. Vieweg, Branschweig (1869)
[16] Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York (1973)

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