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Polynomiography via Ishikawa and Mann Iterations

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Abstract. The aim of this paper is to present some modifications of the complex polynomial roots finding visualization process. In this paper Ishikawa and Mann iterations are used instead of the standard Picard iteration. The name polynomiography was introduced by Kalantari for that visualization process and the obtained images are called polynomiographs. Polynomiographs are interesting both from educational and artistic points of view. By the use of different iterations we obtain quite new polynomiographs that look aestheatically pleasing comparing to the ones from standard Picard iteration. As examples we present some polynomiographs for complex polynomial equation $z^3 - 1 = 0$, permutation and doubly stochastic matrices. We believe that the results of this paper can inspire those who may be interested in created automatically aesthetic patterns. They also can be used to increase functionality of the existing polynomiography software.

1 Introduction

Polynomials are objects that can be met in many mathematical fields. They are interesting not only from the theoretical but also from the practical point of view. The problem of polynomial roots finding has a long history. Sumarians 3000 years B.C. then ancient Greeks faced with practical problems which, formulated in modern mathematical language, can be presented as polynomial roots finding. Next, Newton proposed the method of finding polynomial roots approximately. Cayley in 1879 posed the problem related to the behaviour of Newton's method in the complex plane for equation $z^3 - 1 = 0$. Caley's problem was then solved by Julia in 1919 that led directly to Julia set and then in 1970s to Mandelbrot set and fractals [6]. The last interesting discovery in polynomial roots finding history is Kalantari's contribution [4] who introduced to science the so-called polynomiography. Polynomiography defines visualization process in approximation of the zeros of complex polynomial, using fractal and non-fractal images created via the mathematical convergence properties of iteration functions. An individual image is called a polynomiograph. Polynomiography combines both the art and science aspects. Polynomiography, as a method producing nicely looking graphics that could be widely used, was patented by Kalantari in the USA in 2005 [4].

Both fractal and polynomiograph are generated by iterations. A shape of fractal is completely defined by the input data, e.g. by the coefficients of an IFS (Iterated Function System), and is rather difficult to control efficiently. Fractal is self-similiar, has complicated and not smooth structure and is not dependent on resolution. Polynomiograph is quite different. Its shape can be controlled and designed in a more predictable way in opposition to typical fractal. Generally, fractals and polynomiographs belong to different classes of graphical objects.

Higher flexibility of polynomiography in comparison to fractals can be explained by taking into account the following arguments. It is known that any complex polynomial:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \tag{1}$$

of degree n, according to the Fundamental Theorem of Algebra, has n roots. The polynomial p is well defined by its coefficients $\{a_n, a_{n-1}, \ldots, a_1, a_0\}$ or by its n zeros. So, the degree of polynomial defines the number of basins of attraction. Localization of basins can be controlled by placing roots on the complex plane manually. The chosen roots define the polynomial for which some iteration procedure has to find its zeros. Usually, polynomiographs are coloured based on the number of iterations needed to obtain approximation of some polynomial root with a given accuracy and the iteration method chosen. Description of polynomiography, its theorethical background and artistic applications are described in [3, 4].

Summing up, polynomiography can be treated as a theory and visualization tool based on the roots finding process. It has many possible applications in education, math, sciences, art and design.

In this paper we propose to use Mann and Ishikawa iterations instead of Picard iteration to obtain some modifications of the Newton method and iteration methods from Basic Family of Iterations [5]. Earlier, the other types of iterations were used for superfractals [9] and for fractals generated by an IFS [10].

The paper is organised as follows. In Section 2 the theory of Picard, Mann and Ishikawa iterations is presented. Section 3 is devoted to Newton's method of finding polynomial roots and its generalizations, and presents some iteration formulae. In Section 4 the examples of polynomiographs with different types of iterations (Mann, Ishikawa) for complex equation $z^3 - 1 = 0$, permutation and doubly stochastic matrices are presented. The last section, Section 5, describes some conclusions and plans for future work.

2 Iterations

Let $w : X \to X$ be a mapping on a metric space (X, d), where d is a metric. Further, let $u_0 \in X$ be a starting point. Following [1] we recall some popular iterative procedures:

– Picard iteration:

$$u_{n+1} = w(u_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

– Mann iteration:

$$u_{n+1} = \alpha_n w(u_n) + (1-\alpha_n)u_n, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $0 < \alpha_n \leq 1$.

– Ishikawa iteration:

$$\begin{aligned} u_{n+1} &= \alpha_n w(v_n) + (1-\alpha_n)u_n, \\ v_n &= \beta_n w(u_n) + (1-\beta_n)u_n, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{4}$$

where $0 < \alpha_n \leq 1$ and $0 \leq \beta_n \leq 1$.

The standard Picard iteration is used in the Banach Fixed Point Theorem [1] to obtain the existence of the fixed point x^* such that $x^* = w(x^*)$ and its approximation under additional assumptions on the space X that should be a Banach one and the mapping w should be contractive. The Mann [7] and Ishikawa [2] iterations allow to weak the assumptions on the mapping w. Further, our considerations will be conducted in the space $X = \mathbb{R}^2$ or $X = \mathbb{C}$ that are obviously Banach ones. We take $u_0 = (x_0, y_0) \in \mathbb{R}^2$ and $\alpha_n = \alpha$, $\beta_n = \beta$, such that $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$. It is easily seen that the Ishikawa iteration with $\beta = 0$ is Mann iteration, and for $\beta = 0$, $\alpha = 1$ is Picard iteration. The Mann iteration with $\alpha = 1$ is the Picard iteration.

3 Newton roots finding method and its generalizations

In this section we recall the well-known Newton method for finding roots of a complex polynomial p. The Newton procedure is given by the formula:

$$z_{n+1} = z_n - \frac{p(z_n)}{\dot{p}(z_n)}, \quad n = 0, 1, 2, \ldots, \tag{5}$$

where $z_0 \in \mathbb{C}$ is a starting point.

Applying the Mann iteration (3) in (5) we obtain the following formula:

$$z_{n+1} = \alpha \left(z_n - \frac{p(z_n)}{\dot{p}(z_n)} \right) + (1-\alpha)z_n, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $0 < \alpha \leq 1$.

Using the Ishikawa iteration (4) in (5) we get:

$$\begin{aligned} z_{n+1} &= \alpha \left(v_n - \frac{p(v_n)}{\dot{p}(v_n)} \right) + (1-\alpha)z_n, \\ v_n &= \beta \left(z_n - \frac{p(z_n)}{\dot{p}(z_n)} \right) + (1-\beta)z_n, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{7}$$

where $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$.

The sequence $\{z_n\}_{n=0}^{\infty}$ (or orbit of the point z_0) converges or not to a root of p. If the sequence converges to a root z^* then we say that z_0 is attracted to z^*. A

set of all starting points z_0 for which $\{z_n\}_{n=0}^{\infty}$ converges to z^* is called the basin of attraction of z^*. Boundaries between basins usually are fractals in nature. In [11] some generalizations of the classic Newton formula (5) are discussed. The formulae given above are used in the next section to obtain polynomiographs for complex polynomials that visualize the roots finding process.

Further generalization procedures for roots finding of complex polynomial are given in [4, 5]. They are introduced in the following way. First, define $D_0(z) = 1$ and for $m > 0$ let

$$D_m(z) = \det \begin{bmatrix} \dot{p}(z) & \frac{\ddot{p}}{2!}(z) & \cdots & \frac{p^{m-1}}{(m-1)!}(z) & \frac{p^m}{m!}(z) \\ p(z) & \dot{p}(z) & \cdots & \cdots & \frac{p^{m-1}}{(m-1!)}(z) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \dot{p}(z) & \ddot{p}(z) \\ 0 & 0 & \cdots & p(z) & \dot{p}(z) \end{bmatrix}. \tag{8}$$

The elements of the so-called Basic Family of Iterations are then defined as:

$$B_n(z) = z - p(z)\frac{D_{m-2}(z)}{D_{m-1}(z)}, \quad n = 2, 3, \ldots. \tag{9}$$

It is easily seen that B_2 is the Newton method, whereas B_3 is the Halley method. Iterations from the Basic Family can be modified using Mann or Ishikawa iterations because those iterations produce only different orbits in comparison to Picard iteration. Only the character of convergence is different and the basins of attraction to roots of complex polynomial p are looking differently for different kinds of iteration used.

4 Examples of polynomiographs

In this section we present some polynomiographs for complex polynomial equation $z^3 - 1 = 0$, permutation and doubly stochastic matrices. They are obtained for different parameters α and β via Newton method using Picard, Mann or Ishikawa iterations.

In all examples the colour of each point in the image is determined with the help of Algorithm 1. $I_{\alpha,\beta}$ in the algorithm is the Ishikawa iteration method given by (7), but as we mentioned earlier, for particular values of α and β we obtain Picard or Mann iteration method.

Let us start from equation $z^3 - 1 = 0$ having three roots: 1, $-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$. In Fig.1 nine images with three distinct basins of attraction to the three roots of polynomial $z^3 - 1$ are presented. The colours of different image areas depend on the number of iterations needed to reach a root with the given accuracy $\varepsilon = 0.001$. The upper bound of the number of iterations was fixed as $k = 15$. By changing parameters $\alpha, \beta, \varepsilon$ and k one can obtain infinitely many polynomiographs.

Algorithm 1: Colour determination

Input: $z_0 \in \mathbb{C}$ – starting point, k – maximum number of iterations, ε – accuracy, α, β – parameters of iteration $I_{\alpha,\beta}$

Output: colour c of z_0

```
i = 0
while i ≤ k do
    z_{i+1} = I_{α,β}(z_i)
    if |z_{i+1} − z_i| < ε then
        break
    i = i + 1
c = i
```

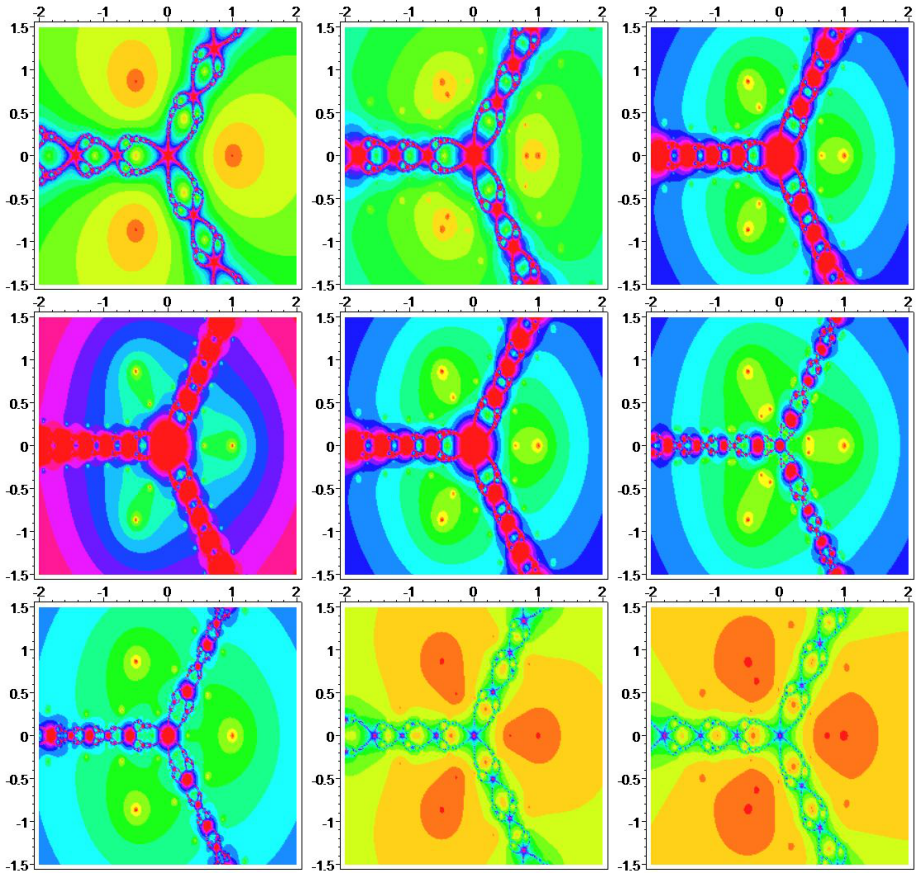

Fig. 1. Polynomiographs of equation $z^3 - 1 = 0$, the top row (from the left): Picard iteration, Mann iterations for $\alpha = 0.8$ and $\alpha = 0.6$, the middle row (from the left): Mann for $\alpha = 0.5$, Ishikawa for $\{\alpha = 0.6, \beta = 0.0\}$, $\{\alpha = 0.6, \beta = 0.1\}$, and the bottom row (from the left): Ishikawa for $\{\alpha = 0.6, \beta = 0.5\}$, $\{\alpha = 1.0, \beta = 0.5\}$, $\{\alpha = 1.0, \beta = 0.7\}$, respectively.

Now recall that a $n \times n$ matrix $\Pi = (\pi_{ij})$ is a matrix whose rows and columns form a permutation of the identity matrix. To each matrix Π we can associate a complex polynomial in the following way. To the location (i, j) in Π we set Θ_{ij}:

$$\Theta_{ij} = i + j\mathbf{i}, \tag{10}$$

where $\mathbf{i} = \sqrt{-1}$.

Next, to the matrix Π we further define a $n \times n$ matrix $\overline{\Pi} = (\overline{\pi}_{ij})$ as $\overline{\pi}_{ij} = \pi_{j,(n+1-i)}$. This matrix is analogous to the transpose, except that i-th row of Π corresponds to the i-th column of $\overline{\Pi}$ but written from the bottom up. Finally, for the matrix $\Pi = (\pi_{ij})$ the complex polynomial p_Π can be defined as [5]:

$$p_\Pi(z) = \prod_{\overline{\pi}_{ij}=1} (z - \Theta_{ij}). \tag{11}$$

As an example take 2×2 permutation matrices Π_1 and Π_2 and create $\overline{\Pi}_1$ and $\overline{\Pi}_2$:

$$\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \overline{\Pi}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \overline{\Pi}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{12}$$

Complex polynomials associated to matrices Π_1 and Π_2 are as follows:

$$p_{\Pi_1}(z) = (z - (1 + 2\mathbf{i}))(z - (2 + \mathbf{i})), \tag{13}$$

$$p_{\Pi_2}(z) = (z - (1 + \mathbf{i}))(z - (2 + 2\mathbf{i})). \tag{14}$$

Their polynomiographs are presented in Fig. 2 and Fig. 3, respectively. It is easily seen that localizations of ones in permutation matrices Π_1 and Π_2 correspond to the images of polynomiographs. Polynoniographs obtained via Mann and Ishikawa iterations for different α, β are quite different in comparison to the Picard iteration. All the images have been obtained for $\varepsilon = 0.001$ and $k = 8$.

It is worth mentioning that permutation matrices have the following obvious properties:

1. If Π_1 and Π_2 are $n \times n$ permutation matrices then $\Pi_1\Pi_2$ is also $n \times n$ permutation matrix.
2. Inverse Π^{-1} to a permutation matrix Π exists and it is also a permutation matrix, it is equal to its transpose, i.e. $\Pi^{-1} = \Pi^T$.
3. Tensor product of $n \times n$ permutation matrices Π_1, Π_2, i.e. $\Pi_1 \otimes \Pi_2$, is $n^2 \times n^2$ permutation matrix.

The number of $n \times n$ permutation matrices is huge and equals $n!$. So, very many nice polynomiographs can be generated.

Doubly stochastic matrices have all non-negative elements and the sum of the entries of each row and column equals 1. According to Birkhoff-von Neumann theorem [8] any double stochastic matrix A can be represented as a convex combination of permutation matrices:

$$A = \sum_{i=1}^{k} \alpha_i \Pi_i, \tag{15}$$

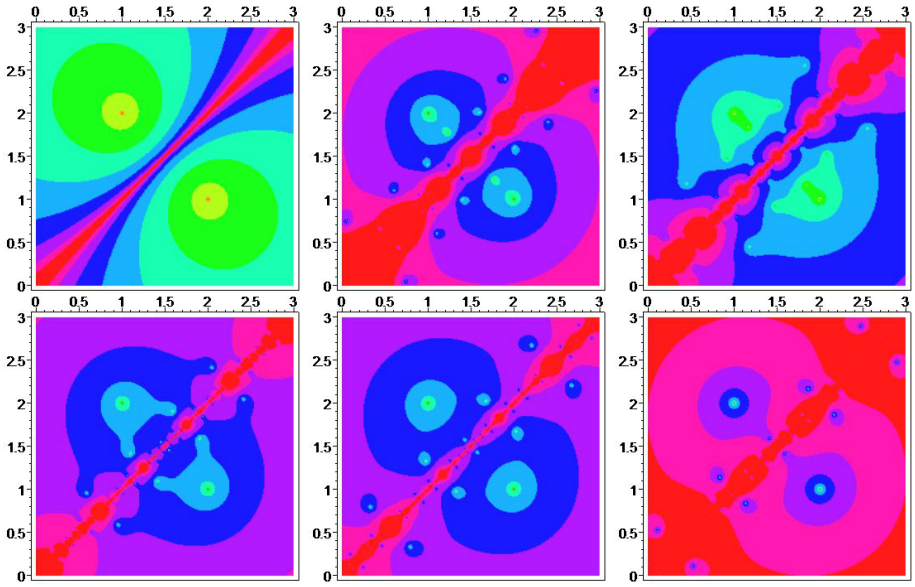

Fig. 2. Polynomiographs of matrix Π_1, the top row (from the left): Picard iteration, Mann iterations for $\alpha = 0.7$ and $\alpha = 0.8$, the bottom row (from the left): Ishikawa iterations for $\{\alpha = 0.7, \beta = 0.4\}$, $\{\alpha = 0.7, \beta = 0.6\}$, $\{\alpha = 0.6, \beta = 0.9\}$, respectively.

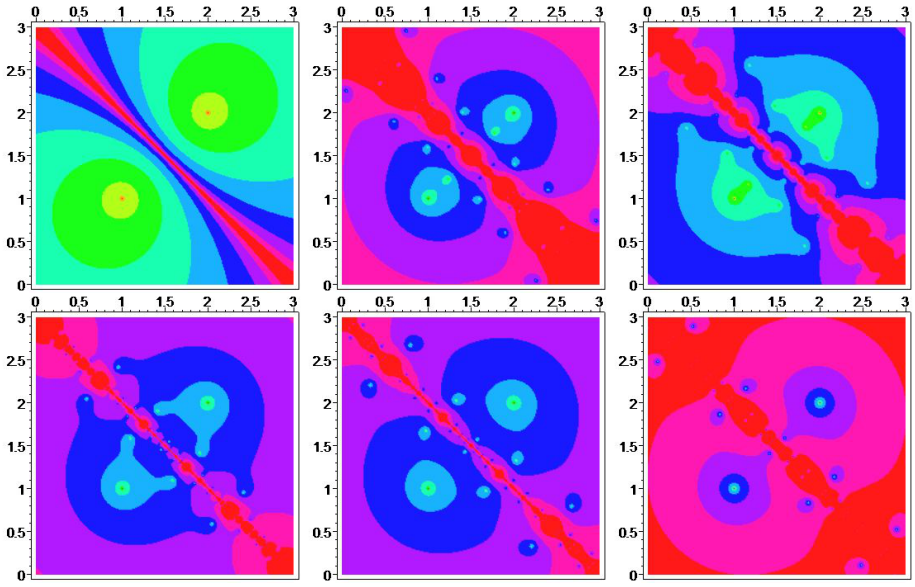

Fig. 3. Polynomiographs of matrix Π_2, the top row (from the left): Picard iteration, Mann iterations for $\alpha = 0.7$ and $\alpha = 0.8$, the bottom row (from the left): Ishikawa iterations for $\{\alpha = 0.7, \beta = 0.4\}$, $\{\alpha = 0.7, \beta = 0.6\}$, $\{\alpha = 0.6, \beta = 0.9\}$, respectively.

where $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i \geq 0$ for $i = 1, \ldots, k$.

The corresponding complex polynomial p_A to a doubly stochastic matrix A can be defined as follows:

$$p_A(z) = \prod_{\overline{a}_{ij}>0} (z - \overline{a}_{ij}\Theta_{ij}), \tag{16}$$

where matrix $\overline{A}$ to A is constructed in a similar way as matrix $\overline{\Pi}$ to Π.

As an example take the following double stochastic matrix A:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{17}$$

The corresponding complex polynomial p_A to the matrix A has the following form:

$$p_A(z) = \left(z - \frac{1+\mathbf{i}}{2}\right)\left(z - \frac{1+2\mathbf{i}}{2}\right)\left(z - \frac{2+\mathbf{i}}{2}\right)\left(z - \frac{2+2\mathbf{i}}{2}\right). \tag{18}$$

In Fig.4 polynomiographs for a double stochastic matrix A are presented.

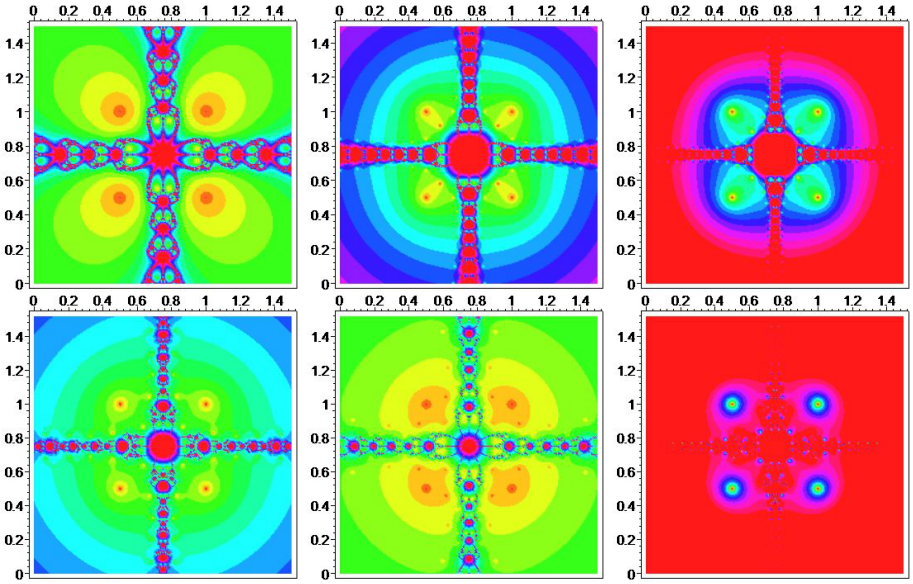

Fig. 4. Polynomiographs of doubly stochastic matrix A, the top row (from the left): Picard iteration, Mann iterations for $\alpha = 0.5$ and $\alpha = 0.3$, the bottom row (from the left): Ishikawa iterations for $\{\alpha = 0.5, \beta = 0.6\}$, $\{\alpha = 0.8, \beta = 0.6\}$, $\{\alpha = 0.2, \beta = 0.7\}$, respectively.

5 Conclusions

In this paper we presented some generalizations of the classic Newton method obtained by the use of Mann or Ishikawa iterations instead of the standard Picard iteration. The obtained polynomiographs for complex equation $z^3 - 1 = 0$, permutation and doubly stochastic matrices look quite different in comparison to Picard iteration. Mann and Ishikawa iterations can be used to generalize Basic Family of Iteration. Further experiments will be carried out to check how polynomiographs look after replacing Picard iteration by Mann or Ishikawa iterations. We believe that the results of this paper can be interesting for those whose works or hobbies are related to automatically created nicely looking graphics. We also think that using Mann and Ishikawa iterations can be applied to increase the functionality of the existing polynomiography software.

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