PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 30

PRACE MATEMATYCZNE III, 1973

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A REMARK CONCERNING MEASURE AND CATEGORY

In [1] the authors prove that if a metric space X is dense in itself and contains a dense subset the potency of which has measure zero, and if μ is a σ-finite Borel measure on X, then X admits a decomposition $X = H \cup K$, where $\mu(H) = 0$ and K is of the first category ([1], proposition (vi) modified according to the subsequent remarks). They also show by a suitable example that the condition that μ is σ-finite is essential.

Further (p. 17) they derive hence the following theorem.

Let ν be a σ-finite measure defined on a σ-additive field $\mathbf{F}$ of subsets of a set X_0, and let f be a mapping of X_0 into a metric space X dense in itself and containing a dense subset the potency of which is of measure zero. If f is measurable (i.e. if $f^{-1}(X) \in \mathbf{F}$ for every Borel set $X \subset X$), then there is a set $X_0 \in \mathbf{F}$ such that $\nu(X_0) = 0$ and $f(X_0 \setminus X_0)$ is of the first category.

In order to prove this theorem the authors write

$$\mu(X) = \nu(f^{-1}(X)) \tag{1}$$

and assert that μ is a σ-finite Borel measure in X. Now, this observation is invalid. To see this, take $X = X_0 = Q^+$, the set of positiv rational numbers (with the ordinary metric) and let $\mathbf{F}$ be the set of all subsets of Q^+. Let $\{r_n\}_{n=0,1,2,\ldots}$ be an ordering of Q^+ into a sequnce and define $\nu(Z)$ for every $Z \subset X_0 = Q^+$ as the number of the elements of Z. Finally. let

$$f(x) = r_n \text{ for } x \in Q^+ \cap [n, n+1), \; n = 0, 1, 2, \ldots$$

Then f is certainly measurable, X is dense in itself and has the potency aleph zero of measure zero, and ν is σ-finite. But the measure μ defined on X by formula (1) is not σ-finite. In fact, we have

$$\mu(X) = \begin{cases} 0 & \text{if } X = \Phi, \\ \infty & \text{if } X \neq \Phi. \end{cases}$$

On the other hand, the theorem itself is true, and may be proved by a slight modification of the argument given in [1]. Let

$$X_0 = \bigcup_{n=1}^{\infty} A_n \tag{2}$$

with $A_n \in \mathbf{F}$, $\nu(A_n) < \infty$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and put

$$\mu_n(X) = \nu(A_n \cap f^{-1}(X)) \tag{3}$$

for Borel sets $X \subset \boldsymbol{X}$. By (2) μ_n are finite Borel measures on $\boldsymbol{X}$ and consequently there exist decompositions

$$\boldsymbol{X} = H_n \cup K_n, \quad n = 1, 2, \ldots$$

such that $\mu_n(H_n) = 0$ and K_n are of the first category. We may assume also that $H_n \cap K_n = \emptyset$ for $n = 1, 2, \ldots$ Write

$$H = \bigcap_{n=1}^{\infty} H_n, \quad K = \bigcup_{n=1}^{\infty} K_n. \tag{4}$$

Then K is of the first category and $K = \boldsymbol{X} \setminus H$. Moreover, we have by (2) for $X_o = f^{-1}(H)$

$$X_o = f^{-1}(H) = f^{-1}(H) \cap \boldsymbol{X}_o = \bigcup_{n=1}^{\infty} (A_n \cap f^{-1}(H)),$$

whence by (4) and (3)

$$\nu(X_o) = \sum_{n=1}^{\infty} \nu(A_n \cap f^{-1}(H)) = \sum_{n=1}^{\infty} \mu_n(H) \leqslant \sum_{n=1}^{\infty} \mu_n(H_n) = 0.$$

Further,

$$f(\boldsymbol{X}_o \setminus X_o) = f(\boldsymbol{X}_o \setminus f^{-1}(H)) = f(\boldsymbol{X}_o) \cap K \subset K,$$

and thus $f(\boldsymbol{X}_o \setminus X_o)$ is of the first category.

REFERENCE

[1] E. Marczewski and R. Sikorski, *Remarks on measure and category*, Colloquium Math. 2 (1951), 13—19.

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Streszczenie

Autor wskazuje na pewną niedokładność w dowodzie jednego z twierdzeń w [1] i pokazuje, jak można lukę tę uzupełnić.

Oddano do Redakcji 6. 4. 71.