Title: On approximate solutions of a non-linear functional equation

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On approximate solutions of a non-linear functional equation

In paper [1] we have considered approximate solutions of some linear functional equations. The role of the approximate solution was played by the n-th term of a suitable functional sequence whose limit was the exact solution. A similar method may be applied to more general equations.

In the present paper we construct approximate solutions, with a preassigned accuracy, for continuous solutions of the functional equation

(1) $$\varphi(x) = h\{x, \varphi[f(x)]\},$$

where $\varphi(x)$ is the required real-valued function of a real variable. The functions $h(x, y)$ and $f(x)$ are given real-valued functions of two, respectively one, real variable.

In the sequel we shall assume that the function $f(x)$ fulfils the following condition:

(I) $f(x)$ is continuous and stricly increasing in an interval $[\xi, a)$, moreover $\xi < f(x) < x$ for $x \in (\xi, a)$.
(In other words, $f \in R^0_\xi([\xi, a])$; cf. [2]).

Let $f^n(x)$ denote the n-th iterate of the function $f(x)$:

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)], \; n = 0, 1, 2, \ldots$$

Under hypothesis (I) we have $f(\xi) = \xi$ and the iterates $f^n(x)$ are defined, continuous and strictly increasing in $[\xi, a)$; furthermore, for every fixed $x \in (\xi, a)$, the sequence $\{f^n(x)\}$ is stricly decreasing and $\lim_{n\to\infty} f^n(x) = \xi$ (cf. [2]).

Let η be a real number such that

(2) $$h(\xi, \eta) = \eta.$$

Let us fix arbitrarily a c, $0 < c < a - \xi$.

We shall make the following assumptions concerning the function $h(x, y)$:

(II) The function $h(x, y)$ is continuous in the rectangle

$$D = [\xi, \xi + c] \times [\eta - d, \eta + d], \quad d > 0,$$

moreover, $h(D) \subset [\eta - d, \eta + d]$.

(III) There exist constants $0 < \Theta < 1$, $0 < \delta \leqslant c$, $0 < \sigma \leqslant d$ such that

(3) $$|h(x,y) - h(x, \bar{y})| \leqslant \Theta |y - \bar{y}| \text{ for } (x, y), (x, \bar{y}) \in [\xi, \xi + \delta] \times [\eta - \sigma, \eta + \sigma]$$

and

(4) $$|h(x, \eta) - h(\xi, \eta)| \leqslant (1 - \Theta)\sigma \quad \text{for} \quad x \in [\xi, \xi + \delta].$$

(IV) The function $h(x, y)$ fulfils in the whole rectangle D the Lipschitz condition with a constant $L > 0$, i. e.

(5) $$|h(x,y) - h(x,\bar{y})| \leqslant L |y - \bar{y}| \quad \text{for} \quad (x,y), (x,\bar{y}) \in D.$$

As has been proved in [2], under hypotheses (I), (II), and (III) equation (1) has exactly one solution φ which is continuous in $[\xi, \xi + c]$ and fulfils the condition $\varphi(\xi) = \eta$. This solution is given by the formula

(6) $$\varphi(x) = \lim_{n \to \infty} \varphi_n(x),$$

where

(7) $$\varphi_{n+1}(x) = h\{x, \varphi_n[f(x)]\}, \quad n = 0, 1, 2, \ldots,$$

and $\varphi_0(x)$ is an arbitrary continuous function defined in $[\xi, \xi + c]$, taking values in $[\eta - d, \eta + d]$ and fulfilling the condition

(8) $$\varphi_0(\xi) = \eta.$$

The functions $\varphi_n(x)$ defined by (7) may be regarded as approximate solutions of equation (1).

Let us fix arbitrarily $\varepsilon > 0$. We are going to find the index n in such a manner that function $\varphi_n(x)$ should differ from the exact solution $\varphi(x)$ in the whole interval $[\xi, \xi + c]$ by less than ε.

Let $\varphi_0(x)$ be an arbitrary continuous function in $[\xi, \xi + c]$ which takes values in $[\eta - d, \eta + d]$, fulfils condition (8) and is such that

(9) $$|\varphi_0(x) - \eta| \leqslant \sigma \quad \text{for} \quad x \in [\xi, \xi + \delta].$$

It follows from (9), (3) and (4) that for $x \in [\xi, \xi + \delta]$ we have

$$|\varphi_1(x) - \eta| \leqslant |h(x, \varphi_0[f(x)]) - h(x, \eta)| + |h(x, \eta) - h(\xi, \eta)| \leqslant \leqslant \Theta |\varphi_0[f(x)] - \eta| + (1 - \Theta)\sigma \leqslant \sigma,$$

and, as is easily proved by induction,

(10) $$|\varphi_n(x) - \eta| \leqslant \sigma \quad \text{for} \quad x \in [\xi, \xi + \delta], \quad n = 0, 1, 2, \ldots$$

Let N be the least non-negative integer such that

$$f^N(\xi + c) \in [\xi, \xi + \delta]. \tag{11}$$

Hence it follows by hypothesis (I) that

$$f^n(x) \in [\xi, \xi + \delta] \quad \text{for} \quad n \geqslant N \quad \text{and} \quad x \in [\xi, \xi + c]. \tag{12}$$

Let us write

$$K = \max_{[\xi, \xi + \delta]} |\varphi_1(x) - \varphi_0(x)|, \tag{13}$$

$$a = \log_\Theta \frac{\varepsilon(1 - \Theta)}{L^N K} + N. \tag{14}$$

We have the following

THEOREM. *Under hypotheses (I), (II), (III) and (IV) for every $n > a$ and every $x \in [\xi, \xi + c]$, we have*

$$|\varphi_n(x) - \varphi(x)| < \varepsilon,$$

where $\varphi_n(x)$ and $\varphi(x)$ are defined by (6) and (7), and $\varphi_0(x)$ fulfils inequality (9).

Proof. By (5), (3), (10), (12) and (13) we get the estimation of $|\varphi_{n+1}(x) - \varphi_n(x)|$, valid for $n \geqslant N$ and $x \in [\xi, \xi + c]$

$$|\varphi_{n+1}(x) - \varphi_n(x)| \leqslant L^N |\varphi_{n-N+1}[f^N(x)] - \varphi_{n-N}[f^N(x)]| \leqslant$$
$$\leqslant L^N \Theta^{n-N} |\varphi_1[f^n(x)] - \varphi_0[f^n(x)]| \leqslant \left(\frac{L}{\Theta}\right)^N \Theta^n K.$$

For the difference $\varphi_n(x) - \varphi(x)$ we get hence for $n \geqslant N$ and $x \in [\xi, \xi + c]$

$$|\varphi_n(x) - \varphi(x)| \leqslant \sum_{s=n}^{\infty} |\varphi_{s+1}(x) - \varphi_s(x)| \leqslant \sum_{s=n}^{\infty} \left(\frac{L}{\Theta}\right)^N \cdot K \cdot \Theta^s =$$
$$= \left(\frac{L}{\Theta}\right)^N \cdot K \cdot \frac{\Theta^n}{1 - \Theta}.$$

For $n > a$, where a is defined by (14), and for every $x \in [\xi, \xi + c]$

$$|\varphi_n(x) - \varphi(x)| < \varepsilon$$

which was to be proved.

If $\delta = c$, $\sigma = d$, then for a we have the simpler formula

$$a = \log_\Theta \frac{\varepsilon(1 - \Theta)}{K}.$$

It results from the estimation obtained in the proof that the convergence of the sequence $\{\varphi_n(x)\}$ is the more rapid the smaller is the number Θ, but if we reduce Θ then N increases and a need not decrease, This may be seen from the following

EXAMPLE. We consider the functional equation

$$\varphi(x) = \frac{(x+2)\,\varphi\left(\dfrac{x}{2}\right)}{(x+2)\,\varphi\left(\dfrac{x}{2}\right)+1} \quad \text{for} \quad x \in [0,1].$$

We have here

$$h(x, y) = \frac{(x+2)\,y}{(x+2)\,y+1}, \quad f(x) = \frac{x}{2}, \; \xi = 0, \quad c = 1,$$

We take $\eta = \frac{1}{2}$ and $d = \frac{1}{2}$. Then $D = [0,1] \times [0,1]$ and

$$|h\,(x, y) - h\,(x, \bar{y})| \leqslant 3\,|y - \bar{y}| \quad \text{for} \quad (x, y),\; (x, \bar{y}) \in D\,.$$

For an arbitrarily chosen Θ with $\frac{1}{2} < \Theta < 1$ inequalities (3) and (4) are fulfilled for

$$\delta = \frac{4(1-\Theta)\,(2\Theta-1)}{2\Theta^2 - \Theta + 1}, \qquad \sigma = \frac{2\Theta - 1}{4\Theta}\,.$$

If we take $\varphi_0(x) \equiv \frac{1}{2}$, then

$$K = \max_{[0,\ \delta]} |\varphi_1(x) - \varphi_0(x)| = (1 - \Theta)\sigma$$

and for α we have the formula

$$\alpha = \log_{\Theta} \frac{\varepsilon}{3^N \sigma} + N$$

Since $f^n(x) = \frac{x}{2^n}$, N is the least non-negative integer such that $2^N \geqslant \frac{1}{\delta}$.

Let us take $\varepsilon = 10^{-3}$. The exact determination of the least number α is too complicated, but trying various values we can find that the possible best estimation we have for $\Theta = \frac{3}{5}$. Then we have

$$14 < \alpha < 15.$$

Therefore for the equation considered

$$|\varphi_{15}(x) - \varphi(x)| < 10^{-3} \quad \text{for} \quad x \in [0,1].$$

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O PRZYBLIŻONYCH ROZWIĄZANIACH NIELINIOWEGO RÓWNANIA FUNKCYJNEGO

Streszczenie

W pracy rozważane są przybliżenia ciągłego rozwiązania równania funkcyjnego $\varphi(x) = h\{x,\varphi[f(x)]\}$, gdzie φ jest funkcją szukaną. Podobnie jak w pracy [1] jako rozwiązanie przybliżone przyjmuje się n-ty wyraz odpowiedniego ciągu funkcyjnego, którego granicą jest dokładne rozwiązanie.

Celem pracy jest dobór liczby rzeczywistej a w ten sposób, by n-te przybliżenie $\varphi_n(x)$ rozwiązania różniło się od dokładnego rozwiązania o z góry zadaną wielkość w góry zadanym przedziale, gdy tylko $n > a$.

Praca zawiera 1 twierdzenie podające sposób doboru liczby a, przy czym sposób ten jest również zilustrowany na odpowiednim przykładzie.

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