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## STABILITY OF THE PEXIDER FUNCTIONAL EQUATION

ROMAN BADORA, BARBARA PRZEBIERACZ, AND PETER VOLKMANN

**Abstract.** A stability result for the Pexider equation will be derived from a stability theorem published in [9] for the Cauchy functional equation. Then we discuss the quality of some constants occurring in this context; as a model case we consider functions defined on the multiplicative semigroup  $\{1, 0\}$ .

### 1. Introduction

In Theorem 1 below we describe the stability result for the Cauchy equation, which had been mentioned in the Abstract.

Let  $S$  be a groupoid, i.e.,  $S$  is a set and for all  $x, y \in S$  we have a product  $xy \in S$ . For  $x \in S$  and  $k = 0, 1, 2, \dots$  the powers  $x^{2^k}$  are recursively defined by

$$x^{2^0} = x^1 = x, \quad x^{2^{k+1}} = x^{2^k} x^{2^k}.$$

Józef Tabor [8] pointed out the usefulness of the following condition for stability investigations:

(T) For  $x, y \in S$  there always is an entire  $k \geq 1$  such that

$$(1) \quad (xy)^{2^k} = x^{2^k} y^{2^k}.$$

In the present paper, groupoids  $S$  satisfying (T) are called *Tabor groupoids*. Three examples are particular cases of them; they are ordered in decreasing generality:

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1. Groupoids  $S$  with a square-symmetric operation, i.e.,

$$(xy)^2 = x^2y^2, \quad x, y \in S$$

(cf. the joint paper with Zsolt Páles and R. Duncan Luce [6]). Then (1) is true for all  $x, y \in S$  with the same  $k$ , viz.  $k = 1$ .

2. Groupoids  $S$  with a bisymmetric operation, i.e.,

$$(xy)(\bar{x}\bar{y}) = (x\bar{x})(y\bar{y}), \quad x, y, \bar{x}, \bar{y} \in S.$$

Here  $\bar{x} = x$ ,  $\bar{y} = y$  leads to square-symmetry.

3. Commutative semigroups  $S$ .

Let us mention that Zbigniew Gajda and Zygfryd Kominek [1] considered semigroups satisfying condition (T). Inspired by Józef Tabor [8], they call them weakly commutative.

Now let  $E$  be a Banach space. A subset  $V$  of  $E$  is called *ideally convex* (E. A. Lifšic [3]), if for every bounded sequence  $d_1, d_2, d_3, \dots$  in  $V$  and for every numerical sequence  $\alpha_1, \alpha_2, \alpha_3, \dots \geq 0$  such that  $\sum_{k=1}^{\infty} \alpha_k = 1$  we get

$$\sum_{k=1}^{\infty} \alpha_k d_k \in V.$$

The following theorem is taken from [9]; in the case of a commutative semigroup  $S$  it goes back to Jacek Tabor [7].

**THEOREM 1.** *Let  $S$  be a Tabor groupoid, and let  $V$  be a bounded and ideally convex subset of the Banach space  $E$ . For  $f: S \rightarrow E$  we suppose*

$$f(xy) - f(x) - f(y) \in V, \quad x, y \in S.$$

*Then there exists a (unique) function  $F: S \rightarrow E$  such that*

$$F(xy) = F(x) + F(y), \quad F(x) - f(x) \in V, \quad x, y \in S.$$

## 2. The Pexider equation

**THEOREM 2.** *Let  $S$  be a Tabor groupoid having a neutral element  $n$ , i.e.,  $n \in S$  and*

$$nx = xn = x, \quad x \in S.$$

Let  $V$  be a symmetric, bounded, and ideally convex subset of a Banach space  $E$  (the symmetry means  $-V = V$ ). For  $f, g, h: S \rightarrow E$  we suppose

$$(2) \quad f(xy) - g(x) - h(y) \in V, \quad x, y \in S.$$

Then there are  $F, G, H: S \rightarrow E$  satisfying the Pexider equation

$$(3) \quad F(xy) = G(x) + H(y), \quad x, y \in S,$$

as well as the conditions

$$(4) \quad F(x) - f(x) \in 3V, \quad G(x) - g(x) \in 4V, \quad H(x) - h(x) \in 4V, \quad x \in S.$$

PROOF. With  $y = n$  and with  $x = n$  in (2) we get

$$(5) \quad f(x) - g(x) - h(n) \in V, \quad f(y) - g(n) - h(y) \in V,$$

hence  $f(x) \in g(x) + h(n) + V$ ,  $f(y) \in h(y) + g(n) + V$ , thus

$$\begin{aligned} f(xy) - f(x) - f(y) + g(n) + h(n) &\in f(xy) - g(x) - h(y) + V + V \\ &\subseteq V + V + V = 3V, \end{aligned}$$

the last equality being true, since  $V$  is convex. For

$$(6) \quad \tilde{f}(x) := f(x) - g(n) - h(n), \quad x \in S,$$

this means

$$\tilde{f}(xy) - \tilde{f}(x) - \tilde{f}(y) \in 3V, \quad x, y \in S,$$

and by Theorem 1 there is a function  $\Phi: S \rightarrow E$  such that

$$\Phi(xy) = \Phi(x) + \Phi(y), \quad \Phi(x) - \tilde{f}(x) \in 3V, \quad x, y \in S.$$

Now it is easily seen that for  $F(x) := \Phi(x) + g(n) + h(n)$ ,  $G(x) := \Phi(x) + g(n)$ ,  $H(x) := \Phi(x) + h(n)$ ,  $x \in S$ , we get (3) and (4):

(3) is obvious;  $F(x) - f(x) \in 3V$  follows from (6) and  $\Phi(x) - \tilde{f}(x) \in 3V$ ; the remaining formulae in (4) are consequences of (5), (6), and  $\Phi(x) - \tilde{f}(x) \in 3V$ .  $\square$

REMARK 1. Theorem 2 should be compared to other stability results for the Pexider equation, e.g. to those of Kazimierz Nikodem [5] and Zygfryd Kominek [2], where the target space for the functions is more general than a Banach space.

When choosing  $V = \{x \mid x \in E, \|x\| \leq \varepsilon\}$ , then we get from Theorem 2 the following Corollary, which had been obtained by Nikodem [5] in the case of a commutative semigroup  $S$ .

COROLLARY 1. *Let  $S$  be a Tabor groupoid having a neutral element, and let  $E$  be a Banach space. For  $f, g, h: S \rightarrow E$  we suppose*

$$\|f(xy) - g(x) - h(y)\| \leq \varepsilon, \quad x, y \in S.$$

*Then there are  $F, G, H: S \rightarrow E$  such that*

$$F(xy) = G(x) + H(y), \quad x, y \in S,$$

$$(7) \quad \|F(x) - f(x)\| \leq 3\varepsilon, \quad \|G(x) - g(x)\| \leq 4\varepsilon, \quad \|H(x) - h(x)\| \leq 4\varepsilon, \quad x \in S.$$

REMARK 2. If  $S$  is a commutative semigroup, then according to Zenon Moszner's survey [4], the constants  $4\varepsilon$  in (7) can be replaced by  $3\varepsilon$ . We do not know, whether this also holds for arbitrary Tabor groupoids.

REMARK 3. In the next paragraph we shall consider the commutative semigroup  $S = \{1, 0\}$ . It will follow that  $3\varepsilon$  in (7) cannot be replaced by a number less than  $2\varepsilon$ . It also will follow that, when having in (7) the better inequality  $\|F(x) - f(x)\| \leq 2\varepsilon$ , then the constants  $4\varepsilon$  cannot be replaced by numbers less than  $3\varepsilon/2$ .

REMARK 4. By calculations similar to those in the next paragraph, it can be shown that for the cyclic groups  $S = Z_2$ ,  $S = Z_3$  of two and of three elements, respectively, all the numbers  $3\varepsilon$ ,  $4\varepsilon$  in (7) can be replaced by  $\varepsilon$ .

### 3. The semigroup $S = \{1, 0\}$

In  $S = \{1, 0\}$  we have  $1 \cdot 1 = 1$ ,  $1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$ . It is easily seen that in this case solutions of the Pexider equation (3) necessarily are constant functions:

$$G(1) = G(0) = a, \quad H(1) = H(0) = b, \quad F(1) = F(0) = a + b.$$

THEOREM 3. Consider  $S = \{1, 0\}$ , let  $N$  be a normed space, and let  $f, g, h: S \rightarrow N$  satisfy

$$(8) \quad \|f(xy) - g(x) - h(y)\| \leq \varepsilon, \quad x, y \in S.$$

Then there exist  $a, b \in N$  such that

$$(9) \quad \|g(1) - a\| \leq \frac{1}{2}\varepsilon, \quad \|h(1) - b\| \leq \frac{1}{2}\varepsilon,$$

$$(10) \quad \|g(0) - a\| \leq \frac{3}{2}\varepsilon, \quad \|h(0) - b\| \leq \frac{3}{2}\varepsilon,$$

$$(11) \quad \|f(x) - a - b\| \leq 2\varepsilon, \quad x = 0, 1.$$

PROOF. Indeed, (8) means

$$(12) \quad \begin{aligned} f(1) - g(1) - h(1) &= r_1, \\ f(0) - g(1) - h(0) &= r_2, \\ f(0) - g(0) - h(1) &= r_3, \\ f(0) - g(0) - h(0) &= r_4, \end{aligned}$$

where

$$(13) \quad \|r_j\| \leq \varepsilon, \quad j = 1, 2, 3, 4.$$

We easily get

$$(14) \quad g(0) - g(1) = r_2 - r_4,$$

$$(15) \quad h(0) - h(1) = r_3 - r_4,$$

$$(16) \quad f(0) - g(1) - h(1) = r_2 + r_3 - r_4.$$

We define

$$a = g(1) + \frac{1}{2}r_2, \quad b = h(1) + \frac{1}{2}r_3,$$

then (13) already leads to (9). From (14), (15) we now get

$$(17) \quad g(0) = g(1) + r_2 - r_4 = a + \frac{1}{2}r_2 - r_4,$$

$$(18) \quad h(0) = h(1) + r_3 - r_4 = b + \frac{1}{2}r_3 - r_4,$$

and this gives (10). From (12), (16) we finally have

$$\begin{aligned} f(1) - a - b &= r_1 + g(1) - a + h(1) - b = r_1 - \frac{1}{2}r_2 - \frac{1}{2}r_3, \\ f(0) - a - b &= f(0) - g(1) - \frac{1}{2}r_2 - h(1) - \frac{1}{2}r_3 = \frac{1}{2}r_2 + \frac{1}{2}r_3 - r_4, \end{aligned}$$

and these two lines prove (11).  $\square$

EXAMPLE. The following example shows that

- I)  $2\varepsilon$  in (11) is best possible,
- II) having  $2\varepsilon$  in (11), then also  $\frac{1}{2}\varepsilon$  in (9) and  $\frac{3}{2}\varepsilon$  in (10) are best possible:

We define  $f, g, h: S \rightarrow \mathbb{R}$  by

$$f(1) = 2, \quad f(0) = -2, \quad g(1) = 1, \quad g(0) = -1, \quad h(1) = 0, \quad h(0) = -2.$$

Then

$$\begin{aligned} f(1) - g(1) - h(1) &= 2 - 1 - 0 = 1, \\ f(0) - g(1) - h(0) &= -2 - 1 + 2 = -1, \\ f(0) - g(0) - h(1) &= -2 + 1 - 0 = -1, \\ f(0) - g(0) - h(0) &= -2 + 1 + 2 = 1, \end{aligned}$$

hence (8) holds for

$$(19) \quad \varepsilon = 1$$

(with absolute value in  $\mathbb{R}$  being the norm).

PROOF OF I). Suppose (11) to hold for some  $a+b \in \mathbb{R}$  and with  $2\varepsilon$  replaced by some  $\eta$ :

$$(20) \quad |2 - a - b| = |f(1) - a - b| \leq \eta, \quad |-2 - a - b| = |f(0) - a - b| \leq \eta.$$

Then  $4 \leq |2 - a - b| + |-2 - a - b| \leq 2\eta$ , hence (cf. (19))  $2\varepsilon = 2 \leq \eta$ .

PROOF OF II). Inequality (11) with  $2\varepsilon = 2$  leads to (20) with  $\eta = 2$ , hence to  $a + b = 0$ , i.e.,  $b = -a$ . Then (10) with  $\eta$  instead of  $\frac{3}{2}\varepsilon$  leads to

$$|-1 - a| = |g(0) - a| \leq \eta, \quad |-2 + a| = |h(0) - b| \leq \eta,$$

which implies  $3 = (2 - a) + (1 + a) \leq 2\eta$ , hence  $\frac{3}{2}\varepsilon = \frac{3}{2} \leq \eta$ .

In the same way we get from (9) with  $\frac{1}{2}\varepsilon$  replaced by  $\eta$  that

$$|1 - a| = |g(1) - a| \leq \eta, \quad |0 + a| = |h(1) - b| \leq \eta,$$

which implies  $1 = (1 - a) + (0 + a) \leq 2\eta$ , hence  $\frac{1}{2}\varepsilon = \frac{1}{2} \leq \eta$ .

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## References

- [1] Gajda Z., Kominek Z., *On separation theorems for subadditive and superadditive functionals*, Studia Math. **100** (1991), 25–38.
- [2] Kominek Z., *On Hyers–Ulam stability of the Pexider equation*, Demonstratio Math. **37** (2004), 373–376.
- [3] Lifšic E.A., *Ideal'no vypuklye množestva*, Funkcional'. Analiz Priložen. **4** (1970), no. 4, 76–77.
- [4] Moszner Z., *On the stability of functional equations*, Aequationes Math. **77** (2009), 33–88.
- [5] Nikodem K., *The stability of the Pexider equation*, Ann. Math. Sil. **5** (1991), 91–93.
- [6] Páles Z., Volkman P., Luce R.D., *Hyers–Ulam stability of functional equations with a square-symmetric operation*, Proc. Nat. Acad. Sci. U.S.A. **95** (1998), 12772–12775.
- [7] Tabor Jacek, *Ideally convex sets and Hyers theorem*, Funkcial. Ekvac. **43** (2000), 121–125.
- [8] Tabor Józef, *Remark 18* (at the 22nd International Symposium on Functional Equations, Oberwolfach 1984), Aequationes Math. **29** (1985), 96.
- [9] Volkman P., *O stabilności równań funkcyjnych o jednej zmiennej*, Sem. LV, no. 11 (2001), 6 pp., Errata ibid. no. 11 bis (2003), 1 p., <http://www.math.us.edu.pl/smdk>

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